MATHS IN FOCUS 12 EXTENSION 2

WORKED SOLUTIONS

Chapter 1: Complex numbers

Exercise 1.01 Complex numbers

Question 1

a

$$\sqrt{-4} = \sqrt{i^2 \times 4}$$
$$= 2i$$

b

$$\sqrt{-7} = \sqrt{i^2 \times 7}$$
$$= i\sqrt{7}$$

c

$$\sqrt{-\frac{1}{9}} = \sqrt{i^2 \times \frac{1}{9}}$$
$$= \frac{i}{3}$$

d

$$\sqrt{-12} = \sqrt{i^2 \times 4 \times 3}$$
$$= 2i\sqrt{3}$$

e

$$\sqrt{-\frac{6}{25}} = \sqrt{i^2 \times \frac{6}{25}}$$
$$= \frac{i\sqrt{6}}{5}$$

f

$$\sqrt{(-2) - 4 \times 3 \times 3} = \sqrt{4 - 36}$$
$$= \sqrt{-32}$$
$$= \sqrt{i^2 \times 16 \times 2}$$
$$= 4i\sqrt{2}$$

g $i^7 = (i^2)^3 i = -i$

h $i^{13} = (i^2)^6 i = i$

i $i^{99} = (i^2)^{49} i = -i$

j $i + i^2 + i^3 + i^4 + i^5 + \ldots\ldots\ldots + i^{149} + i^{150}$

Grouping the first 4 terms we get

$i - 1 - i + 1 = 0$

so each grouping of 4 equals 0 which leaves us with the last two terms

$i^{149} + i^{150} = i - 1$

k

$$\frac{i^4}{i} = i^3 = -i$$

l

$$\frac{1}{i^3} = \frac{-1}{i}$$
$$= \frac{-1}{i} \times \frac{i}{i}$$
$$= \frac{-i}{i^2}$$
$$= i$$

Question 2

a

$$x^2 = -4$$
$$x = \pm\sqrt{-4} = \pm 2i$$

b

$$x^2 + 9 = 0$$
$$x^2 = -9$$
$$x = \pm\sqrt{-9} = \pm 3i$$

c

$$z^2 = -\frac{1}{36}$$
$$z = \pm\sqrt{-\frac{1}{36}} = \pm\frac{i}{6}$$

d

$$5z^2 + 100 = 0$$
$$5z^2 = -100$$
$$z^2 = -20$$
$$x = \pm\sqrt{-20} = \pm 2i\sqrt{5}$$

Question 3

a

$$x^2 + 2x + 3 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-2 \pm \sqrt{2^2 - 4\times 1\times 3}}{2\times 1}$$
$$= \frac{-2 \pm \sqrt{-8}}{2} = \frac{-2 \pm 2i\sqrt{2}}{2}$$
$$= -1 \pm i\sqrt{2}$$

b

$$x^2 - x + 6 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{1 \pm \sqrt{(-1)^2 - 4\times 1\times 6}}{2\times 1}$$
$$= \frac{1 \pm \sqrt{-23}}{2}$$
$$= \frac{1 \pm i\sqrt{23}}{2}$$

c

$$z^2 + 3z + 3 = 0$$
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-3 \pm \sqrt{3^2 - 4\times 1\times 3}}{2\times 1}$$
$$= \frac{-3 \pm \sqrt{-3}}{2}$$
$$= \frac{-3 \pm i\sqrt{3}}{2}$$

d

$$3z^2 - 5z + 9 = 0$$
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{5 \pm \sqrt{5^2 - 4\times 3\times 9}}{2\times 3}$$
$$= \frac{5 \pm \sqrt{-83}}{6}$$
$$= \frac{5 \pm i\sqrt{83}}{6}$$

Question 4

a

$$x^2 - 2x + 3 = 0$$
$$x^2 - 2x + 1 + 2 = 0$$
$$(x-1)^2 + 2 = 0$$
$$(x-1)^2 = -2$$
$$x - 1 = \pm\sqrt{-2}$$
$$x - 1 = \pm i\sqrt{2}$$
$$x = 1 \pm i\sqrt{2}$$

b

$$x^2 - 4x + 11 = 0$$
$$x^2 - 4x + 4 + 7 = 0$$
$$(x-2)^2 + 7 = 0$$
$$(x-2)^2 = -7$$
$$x - 2 = \pm\sqrt{-7}$$
$$x - 2 = \pm i\sqrt{7}$$
$$x = 2 \pm i\sqrt{7}$$

c

$$z^2 + 8z + 20 = 0$$
$$z^2 + 8z + 16 + 4 = 0$$
$$(z+4)^2 + 4 = 0$$
$$(z+4)^2 = -4$$
$$z + 4 = \pm\sqrt{-4}$$
$$z + 4 = \pm 2i$$
$$z = -4 \pm 2i$$

d

$$z^2 - 2z + 4 = 0$$
$$z^2 - 2z + 1 + 3 = 0$$
$$(z-1)^2 + 3 = 0$$
$$(z-1)^2 = -3$$
$$z - 1 = \pm\sqrt{-3}$$
$$z - 1 = \pm i\sqrt{3}$$
$$z = 1 \pm i\sqrt{3}$$

Question 5

a

$$x^2-2x+2=0$$
$$x^2-2x+1+1=0$$
$$(x-1)^2-i^2=0$$
$$(x-1+i)(x-1-i)=0$$
$$x=1-i,1+i$$

b

$$v^2-6v+12=0$$
$$v^2-6v+9+3=0$$
$$(v-3)^2-3i^2=0$$
$$\left(v-3+i\sqrt{3}\right)\left(v-3-i\sqrt{3}\right)=0$$
$$v=3-i\sqrt{3},3+i\sqrt{3}$$

c

$$w^2+4w+10=0$$
$$w^2+4w+4+6=0$$
$$(w+2)^2-6i^2=0$$
$$\left(w+2+i\sqrt{6}\right)\left(w+2-i\sqrt{6}\right)=0$$
$$w=-2-i\sqrt{6},-2+i\sqrt{6}$$

d

$$z^2+2z+7=0$$
$$z^2+2z+1+6=0$$
$$(z+1)^2-6i^2=0$$
$$\left(z+1+i\sqrt{6}\right)\left(z+1-i\sqrt{6}\right)=0$$
$$z=-1-i\sqrt{6},-1+i\sqrt{6}$$

e

$$z^2+z+1=0$$
$$z^2+z+\frac{1}{4}+\frac{3}{4}=0$$
$$\left(z+\frac{1}{2}\right)^2-\frac{3i^2}{4}=0$$
$$\left(z+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\left(z+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)=0$$
$$z=-\frac{1}{2}+\frac{i\sqrt{3}}{2}\quad -\frac{1}{2}-\frac{i\sqrt{3}}{2}$$

f

$$z^2-3z+4=0$$
$$z^2-3z+\frac{9}{4}+\frac{7}{4}=0$$
$$\left(z-\frac{3}{2}\right)^2-\frac{7i^2}{4}=0$$
$$\left(z-\frac{3}{2}+\frac{i\sqrt{7}}{2}\right)\left(z-\frac{1}{2}-\frac{i\sqrt{7}}{2}\right)=0$$
$$z=\frac{3}{2}+\frac{i\sqrt{7}}{2}\quad \frac{3}{2}-\frac{i\sqrt{7}}{2}$$

Question 6

a $z = \sqrt{3} + i$

$\text{Re}(z) = \sqrt{3}$

$\text{Im}(z) = 1$

b $z = \dfrac{5 - i\sqrt{2}}{2}$

$\text{Re}(z) = \dfrac{5}{2}$

$\text{Im}(z) = \dfrac{-\sqrt{2}}{2}$

c $z = 6i - 3$

$\text{Re}(z) = -3$

$\text{Im}(z) = 6$

d $z = x - iy + 3 + 2i$

$\text{Re}(z) = x + 3$

$\text{Im}(z) = 2 - y$

e $z = \dfrac{a + 2ib}{a^2 + 4b^2}$

$\text{Re}(z) = \dfrac{a}{a^2 + 4b^2}$

$\text{Im}(z) = \dfrac{2b}{a^2 + 4b^2}$

f $z = \dfrac{x - i - 4 + ix - 6y + iy}{x^2 + y^2}$

$\text{Re}(z) = \dfrac{x - 4 - 6y}{x^2 + y^2}$

$\text{Im}(z) = \dfrac{-1 + x + y}{x^2 + y^2}$

Question 7

a $\bar{z} = \sqrt{3} - i$

b $\bar{z} = \dfrac{5 + i\sqrt{2}}{2}$

c $\bar{z} = -3 - 6i$

d $\bar{z} = x + 3 - i(2 - y)$

e $\bar{z} = \dfrac{a - 2ib}{a^2 + 4b^2}$

f $\bar{z} = \dfrac{x - 4 - 6y + i - ix - iy}{x^2 + y^2}$

Question 8

a $(2 + 3i)(2 - 3i) = 4 + 9 = 13$

b $\left(1 - i\sqrt{2}\right)\left(1 + i\sqrt{2}\right) = 1 + 2 = 3$

c $(5i + 4)(5i - 4) = -25 - 16 = -41$

d $\left(\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}\right)\left(\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}\right) = \dfrac{1}{4} + \dfrac{3}{4} = 1$

e $\left(\dfrac{4 + i}{3}\right)\left(\dfrac{4 - i}{3}\right) = \dfrac{16 + 1}{9} = \dfrac{17}{9}$

f $\left(\dfrac{\sqrt{2} + i2\sqrt{2}}{8}\right)\left(\dfrac{\sqrt{2} - i2\sqrt{2}}{8}\right) = \dfrac{2 + 8}{64} = \dfrac{10}{64} = \dfrac{5}{32}$

Question 9

a

$$\begin{aligned} z &= 5+6i \\ \bar{z} &= 5-6i \\ z\bar{z} &= (5+6i)(5-6i) \\ &= 25+36 \\ &= 61 \end{aligned}$$

b

$$\begin{aligned} z &= \sqrt{3}-i \\ \bar{z} &= \sqrt{3}+i \\ z\bar{z} &= \left(\sqrt{3}-i\right)\left(\sqrt{3}+i\right) \\ &= 3+1 \\ &= 4 \end{aligned}$$

c

$$\begin{aligned} z &= 4i-3 \\ \bar{z} &= -4i-3 \\ z\bar{z} &= (4i-3)(-4i-3) \\ &= 16+9 \\ &= 25 \end{aligned}$$

d

$$\begin{aligned} z &= \frac{1+i\sqrt{3}}{2} \\ \bar{z} &= \frac{1-i\sqrt{3}}{2} \\ z\bar{z} &= \left(\frac{1+i\sqrt{3}}{2}\right)\left(\frac{1-i\sqrt{3}}{2}\right) \\ &= \frac{1+3}{4} \\ &= 1 \end{aligned}$$

e

$$\begin{aligned} z &= \frac{1}{17}-\frac{4i}{17} \\ \bar{z} &= \frac{1}{17}+\frac{4i}{17} \\ z\bar{z} &= \left(\frac{1}{17}-\frac{4i}{17}\right)\left(\frac{1}{17}+\frac{4i}{17}\right) \\ &= \frac{1}{289}+\frac{16}{289} \\ &= \frac{17}{289} \\ &= \frac{1}{17} \end{aligned}$$

f

$$\begin{aligned} z &= \sqrt{5}+i\sqrt{3} \\ \bar{z} &= \sqrt{5}-i\sqrt{3} \\ z\bar{z} &= \left(\sqrt{5}+i\sqrt{3}\right)\left(\sqrt{5}-i\sqrt{3}\right) \\ &= 5+3 \\ &= 8 \end{aligned}$$

g

$$\begin{aligned} z &= 2a-3bi \\ \bar{z} &= 2a+3bi \\ z\bar{z} &= (2a-3bi)(2a+3bi) \\ &= 4a^2+9b^2 \end{aligned}$$

h

$$\begin{aligned} z &= x+y+i(x-y) \\ \bar{z} &= x+y-i(x-y) \\ z\bar{z} &= \left[x+y+i(x-y)\right]\left[x+y-i(x-y)\right] \\ &= x^2+2xy+y^2+x^2-2xy+y^2 \\ &= 2x^2+2y^2 \end{aligned}$$

Question 10

$$\begin{aligned} z &= w-iv \\ \bar{z} &= w+iv \\ z\bar{z} &= (w-iv)(w+iv) \\ &= w^2+v^2 \in \mathbb{R} \end{aligned}$$

Question 11

$$z = a + ib$$
$$w = c + id$$
$$\overline{z + w} = \overline{a + ib + c + id}$$
$$= \overline{a + c + ib + id}$$
$$= a + d - i(b + d)$$
$$= a - ib + c - id$$
$$= \bar{z} + \bar{w}$$

Question 12

a $2x + 8i – 4 + iy = 0$

Equating real and imaginary parts

$2x – 4 = 0$

$2x = 4$

$x = 2$

$8 + y = 0$

$y = –8$

b $3x + 2iy = 9 – 8i$

Equating real and imaginary parts

$3x = 9$

$x = 3$

$2y = –8$

$y = –4$

c $x + y + 2xi – iy = 7 + 8i$

Equating real and imaginary parts

$x + y = 7$

$2x – y = 8$

$3x = 15$

$x = 5$

$5 + y = 7$

$y = 2$

d $3x – 2y – 8 + ix + 3iy – 10i = 0$

Equating real and imaginary parts

$3x – 2y – 8 = 0$

$x + 3y – 10 = 0$

$3x + 9y – 30 = 0$

$11y – 22 = 0$

$11y = 22$

$y = 2$

$3x – 4 – 8 = 0$

$3x – 12 = 0$

$3x = 12$

$x = 4$

Question 13

$z = 3y - 6i + xi - 8 + yi$

$\text{Im}(z) = 0$

$\Rightarrow \quad -6 + x + y = 0$

$x + y = 6$

Question 14

a $4 - 3i + 7i - 8 = -4 + 4i$

b $2(3 + i) - i(7 - 2i) = 6 + 2i - 7i - 2$

$= 4 - 5i$

c $(2 - 9i)^2 = (2 - 9i)\times(2 - 9i)$

$= 4 - 2\times 2\times(-9i) - 81$

$= 4 - 36i - 81$

$= -77 - 36i$

d $(4+i)(5-3i) = 20 - 12i + 5i + 3$

$= 23 - 7i$

e $\left(\sqrt{5} - 4i\right)\left(\sqrt{5} + 4i\right) = \left(\sqrt{5}\right)^2 - (4i)^2$

$= 5 + 16$

$= 21$

f $3(8i - 1)(2 + i) = 3(16i - 8 - 2 - i)$

$= 3(-10 + 15i)$

$= -30 + 45i$

g $\left(\sqrt{2} + i\right)\left(\sqrt{2} - i\sqrt{3}\right) - i\left(\sqrt{6} + 2i\right) = 2 - i\sqrt{6} + i\sqrt{2} + \sqrt{3} - i\sqrt{6} + 2$

$= 4 + \sqrt{3} + i\sqrt{2} - 2i\sqrt{6}$

h $(x - iy)^2 - (x + iy)^2 = x^2 - 2xyi + y^2 - (x^2 + 2xyi + y^2)$

$= -4xyi$

Question 15

a $\left(1+i\sqrt{3}\right)\left(\sqrt{3}+i\right)=\sqrt{3}+i+3i-\sqrt{3}=4i$

$\therefore\left(1+i\sqrt{3}\right)\left(\sqrt{3}+i\right)$ is purely imaginary

b $\left(\sqrt{2}+i\sqrt{2}\right)\left(-\sqrt{2}+i\sqrt{2}\right)=-2+2i+2i-2=-4$

$\therefore\left(\sqrt{2}+i\sqrt{2}\right)\left(-\sqrt{2}+i\sqrt{2}\right)$ is real

Question 16

a **i** $(z-2+i)(\mathrm{z}-2-i)=0$

$z^2+\mathrm{z}(-2-i)+\mathrm{z}(-2+i)+(-2-i)(-2+i)=0$

$z^2-2\mathrm{z}-ix-2\mathrm{z}+iz+4+1=0$

$z^2-4\mathrm{z}+5=0$

Real coefficients

ii $\left(z-\left(\sqrt{3}+5i\right)\right)\left(z-\left(\sqrt{3}-5i\right)\right)=0$

$z^2-z\left(\sqrt{3}-5i\right)-z\left(\sqrt{3}+5i\right)+\left(\sqrt{3}-5i\right)\left(\sqrt{3}+5i\right)=0$

$z^2-\sqrt{3}z-5iz-\sqrt{3}z+5iz+3+25=0$

$z^2-2\sqrt{3}z+28=0$

Real coefficients

iii $\left(z-\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\right)\left(z-\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\right)=0$

$z^2-z\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)-z\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)+\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)=0$

$z^2-\frac{1}{2}z+\frac{iz\sqrt{3}}{2}-\frac{1}{2}z-\frac{iz\sqrt{3}}{2}+\frac{1}{4}+\frac{3}{4}=0$

$z^2-z+1=0$

Real coefficients

iv $\left[x-\left(-4+i\sqrt{5}\right)\right]\left[x-\left(-4-i\sqrt{5}\right)\right]=0$

$x^2-x\left(-4-i\sqrt{5}\right)-x\left(-4+i\sqrt{5}\right)+\left(-4+i\sqrt{5}\right)\left(-4-i\sqrt{5}\right)=0$

$x^2+4x+ix\sqrt{5}+4x-ix\sqrt{5}+16+5=0$

$x^2+8x+21=0$

Real coefficients

b A quadratic equation with complex conjugate roots will have real coefficients.

Question 17

a

$$\frac{1}{2-i}=\frac{1}{2-i}\times\frac{2+i}{2+i}$$
$$=\frac{2+i}{4+1}$$
$$=\frac{2+i}{5}$$

$\text{Im}(z) = \frac{1}{5}$

b

$$\frac{1+i}{1-2i}=\frac{1+i}{1-2i}\times\frac{1+2i}{1+2i}$$
$$=\frac{1+2i+i-2}{1+4}$$
$$=\frac{-1+3i}{5}$$

$\text{Im}(z) = \frac{3}{5}$

c

$$\frac{5-7i}{3+4i}=\frac{5-7i}{3+4i}\times\frac{3-4i}{3-4i}$$
$$=\frac{15-20i-21i-28}{9+16}$$
$$=\frac{-13-41i}{25}$$

$\text{Im}(z) = -\frac{41}{25}$

d

$$\frac{\sqrt{3}-i\sqrt{2}}{\sqrt{3}+i\sqrt{2}}=\frac{\sqrt{3}-i\sqrt{2}}{\sqrt{3}+i\sqrt{2}}\times\frac{\sqrt{3}-i\sqrt{2}}{\sqrt{3}-i\sqrt{2}}$$
$$=\frac{3-i2\sqrt{6}-2}{3+2}$$
$$=\frac{1-i2\sqrt{6}}{5}$$

$\text{Im}(z) = -\frac{2\sqrt{6}}{5}$

Question 18

a

$$\frac{-2+2i}{1+i}=\frac{-2+2i}{1+i}\times\frac{1-i}{1-i}$$
$$=\frac{-2+2i+2i+2}{1+1}$$
$$=\frac{4i}{2}$$
$$=2i$$

$\therefore \frac{-2+2i}{1+i}$ is purely imaginary

b

$$\frac{1+i\sqrt{3}}{\left(1-i\sqrt{3}\right)^2}=\frac{1+i\sqrt{3}}{\left(1-i\sqrt{3}\right)\left(1-i\sqrt{3}\right)}$$

$$=\frac{1+i\sqrt{3}}{1-i\sqrt{3}-i\sqrt{3}-3}$$

$$=\frac{1+i\sqrt{3}}{-2-2i\sqrt{3}}$$

$$\frac{1+i\sqrt{3}}{-2-2i\sqrt{3}}\times\frac{-2+2i\sqrt{3}}{-2+2i\sqrt{3}}$$

$$=\frac{-2+2i\sqrt{3}-2i\sqrt{3}-6}{4+12}$$

$$=-\frac{8}{16}$$

$$=-\frac{1}{2}$$

$$\therefore \frac{1+i\sqrt{3}}{\left(-1-i\sqrt{3}\right)^2} \text{ is real}$$

Question 19

a

$$\frac{1}{(1+2i)^2}=\frac{1}{1+4i-4}=\frac{1}{-3+4i}$$

$$=\frac{1}{-3+4i}\times\frac{-3-4i}{-3-4i}$$

$$=\frac{-3-4i}{9+16}$$

$$=\frac{-3-4i}{25}$$

b $$\frac{1}{\left(\sqrt{3}+i\sqrt{3}\right)^2}-\frac{1}{\left(\sqrt{3}-i\sqrt{3}\right)^2}=\frac{1}{3+6i-3}-\frac{1}{3-6i-3}$$

$$=\frac{1}{6i}+\frac{1}{6i}$$

$$=\frac{1}{3i}$$

$$=\frac{1}{3i}\times\frac{i}{i}$$

$$=-\frac{i}{3}$$

Question 20

a $[z-(3-i)][z-(2+9i)]=0$

$z^2-z(2+9i)-z(3-i)+(3-i)(2+9i)=0$

$z^2-2z-9iz-3z+iz+6+27i-2i+9=0$

$z^2+z(-5-8i)+15+25i=0$

$a=1,\ b=-5-8i,\ c=15+25i$

b A quadratic equation with complex non-conjugate roots will have some coefficients that are not real.

Question 21

a

$$\frac{2-3i}{3+4i}=\frac{2-3i}{3+4i}\times\frac{3-4i}{3-4i}$$
$$=\frac{6-8i-9i-12}{9+16}$$
$$=\frac{-6-17i}{25}$$
$$x=-\frac{6}{25}\quad y=-\frac{17}{25}$$

b

$(x + iy)(1 - 5i) = 2 + i$

$x - 5ix + iy + 5y = 2 + i$

Equating real and imaginary parts

$x + 5y = 2$

$-5x + y = 1$

$5x + 25y = 10$

$26y = 11$

$$y=\frac{11}{26}$$
$$-5x+\frac{11}{26}=1$$
$$-5x=\frac{15}{26}$$
$$x=-\frac{3}{26}$$
$$x=-\frac{3}{26}\quad y=\frac{11}{26}$$

Question 22

$z = 5 - 2i$

$w = -3 + i$

a

$zw = (5 - 2i)(-3 + i)$

$= -15 + 5i + 6i + 2$

$= -13 + 11i$

b

$z - w = 5 - 2i - (-3 + i)$

$= 5 + 3 - 2i - i$

$= 8 - 3i$

c

$$\frac{z}{w}=\frac{5-2i}{-3+i}$$
$$=\frac{5-2i}{-3+i}\times\frac{-3-i}{-3-i}$$
$$=\frac{-15-5i+6i-2}{9+1}$$
$$=\frac{-17+i}{10}$$

d

$z^2 - w^2 = (5 - 2i)^2 - (-3 + i)^2$

$= 25 - 20i - 4 - (9 - 6i - 1)$

$= 25 - 20i - 4 - 9 + 6i + 1$

$= 13 - 14i$

Exercise 1.02 Square root of a complex number

Question 1

a $\sqrt{3+4i}$

Let $3 + 4i = (x + iy)^2$

$3 + 4i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$3 = x^2 - y^2$

$4 = 2xy$

$y = \frac{2}{x}$

$3 = x^2 - \left(\frac{2}{x}\right)^2$

$3x^2 = x^4 - 4$

$x^4 - 3x^2 - 4 = 0$

$(x^2 - 4)(x^2 + 1) = 0$

As $x \in \mathbb{R}$, $x^2 + 1 = 0$ has no real solutions

$x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2$

$y = \frac{2}{x}$

$y = \pm 1$

$x = 2, y = 1;\ x = -2, y = -1$

$\sqrt{3+4i} = \pm(2+i)$

b $\sqrt{5-12i}$

Let $5 - 12i = (x + iy)^2$

$5 - 12i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$5 = x^2 - y^2$

$-12 = 2xy$

$y = \frac{-6}{x}$

$5 = x^2 - \left(\frac{-6}{x}\right)^2$

$5x^2 = x^4 - 36$

$x^4 - 5x^2 - 36 = 0$

$(x^2 - 9)(x^2 + 4) = 0$

As $x \in \mathbb{R}$, $x^2 + 4 = 0$ has no real solutions

$x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3$

$y = \frac{-6}{x}$

$y = \mp 2$

$x = 3, y = -2;\ x = -3, y = 2$

$\sqrt{5-12i} = \pm(3-2i)$

c $\sqrt{8+6i}$

Let 8 + 6i = $(x + iy)^2$

8 + 6i = $x^2 + 2xyi - y^2$

Equating real and imaginary parts

$8 = x^2 - y^2$

$6 = 2xy$

$$y = \frac{3}{x}$$

$$8 = x^2 - \left(\frac{3}{x}\right)^2$$

$$8x^2 = x^4 - 9$$

$$x^4 - 8x^2 - 9 = 0$$

$$\left(x^2 - 9\right)\left(x^2 + 1\right) = 0$$

As $x \in \mathbb{R},\ x^2 + 1 = 0$ has no real solutions

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

$$y = \frac{3}{x}$$

$$y = \pm 1$$

$$x = 3, y = 1;\ x = -3, y = -1$$

$$\sqrt{8+6i} = \pm\left(3+i\right)$$

d $\sqrt{4i}$

Let 4i = $(x + iy)^2$

4i = $x^2 + 2xyi - y^2$

Equating real and imaginary parts

$0 = x^2 - y^2$

$4 = 2xy$

$$y = \frac{2}{x}$$

$$0 = x^2 - \left(\frac{2}{x}\right)^2$$

$$0 = x^4 - 4$$

$$x^4 = 4$$

$$x = \pm\sqrt{2}$$

$$y = \frac{2}{x}$$

$$y = \pm\frac{2}{\sqrt{2}}$$

$$y = \pm\sqrt{2}$$

$$x = \sqrt{2}, y = \sqrt{2};\quad x = -\sqrt{2}, y = -\sqrt{2}$$

$$\sqrt{4i} = \pm\sqrt{2}\left(1+i\right)$$

Question 2

a Let $15 - 8i = (x + iy)^2$

$15 - 8i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$15 = x^2 - y^2$

$-8 = 2xy$

$y = \frac{-4}{x}$

$15 = x^2 - \left(\frac{-4}{x}\right)^2$

$15x^2 = x^4 - 16$

$x^4 - 15x^2 - 16 = 0$

$(x^2 - 16)(x^2 + 1) = 0$

As $x \in \mathbb{R}$, $x^2 + 1 = 0$ has no real solutions

$x^2 - 16 = 0$

$x^2 = 16$

$x = \pm 4$

$y = \frac{-4}{x}$

$y = \mp 1$

$x = 4, y = -1;\ x = -4, y = 1$

$\sqrt{15 - 8i} = \pm(4 - i)$

b Let $-3 - 4i = (x + iy)^2$

$-3 - 4i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$-3 = x^2 - y^2$

$-4 = 2xy$

$y = \frac{-2}{x}$

$-3 = x^2 - \left(\frac{-2}{x}\right)^2$

$-3x^2 = x^4 - 4$

$x^4 + 3x^2 - 4 = 0$

$(x^2 + 4)(x^2 - 1) = 0$

As $x \in \mathbb{R}$, $x^2 + 4 = 0$ has no real solutions

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$

$y = \frac{-2}{x}$

$y = \mp 2$

$x = 1, y = -2 \quad x = -1, y = 2$

$\sqrt{-3 - 4i} = \pm(1 - 2i)$

c Let $21 - 20i = (x + iy)^2$

$21 - 20i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$21 = x^2 - y^2$

$-20 = 2xy$

$y = \frac{-10}{x}$

$21 = x^2 - \left(\frac{-10}{x}\right)^2$

$21x^2 = x^4 - 100$

$x^4 - 21x^2 - 100 = 0$

$\left(x^2 - 25\right)\left(x^2 + 4\right) = 0$

As $x \in \mathbb{R}$, $x^2 + 4 = 0$ has no real solutions

$x^2 - 25 = 0$

$x^2 = 25$

$x = \pm 5$

$y = \frac{-10}{x}$

$y = \mp 2$

$x = 5, y = -2;\ x = -5, y = 2$

$\sqrt{21 - 20i} = \pm\left(5 - 2i\right)$

d Let $-24 + 10i = (x + iy)^2$

$-24 + 10i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$-24 = x^2 - y^2$

$10 = 2xy$

$y = \frac{5}{x}$

$-24 = x^2 - \left(\frac{5}{x}\right)^2$

$-24x^2 = x^4 - 25$

$x^4 + 24x^2 - 25 = 0$

$\left(x^2 + 25\right)\left(x^2 - 1\right) = 0$

As $x \in \mathbb{R}$, $x^2 + 25 = 0$ has no real solutions

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$

$y = \frac{5}{x}$

$y = \pm 5$

$x = 1, y = 5;\ x = -1, y = -5$

$\sqrt{-24 + 10i} = \pm\left(1 + 5i\right)$

e Let $-9i = (x + iy)^2$

$-9i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$0 = x^2 - y^2$

$-9 = 2xy$

$$y = \frac{-9}{2x}$$

$$0 = x^2 - \left(\frac{-9}{2x}\right)^2$$

$$0 = x^2 - \frac{81}{4x^2}$$

$$x^4 = \frac{81}{4}$$

$$x = \pm\frac{3}{\sqrt{2}}$$

$$y = \frac{-9}{2x}$$

$$y = \pm\frac{-9}{\frac{2\times 3}{\sqrt{2}}}$$

$$y = \mp\frac{3}{\sqrt{2}}$$

$$x = \frac{3}{\sqrt{2}}, y = -\frac{3}{\sqrt{2}};\ x = -\frac{3}{\sqrt{2}}, y = \frac{3}{\sqrt{2}}$$

$$\sqrt{-9i} = \pm\frac{3}{\sqrt{2}}(1-i)$$

f Let $i = (x + iy)^2$

$i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$0 = x^2 - y^2$

$1 = 2xy$

$$y = \frac{1}{2x}$$

$$0 = x^2 - \left(\frac{1}{2x}\right)^2$$

$$0 = x^4 - \frac{1}{4}$$

$$4x^4 = 1$$

$$x = \pm\frac{1}{\sqrt{2}}$$

$$x = \pm\frac{\sqrt{2}}{2}$$

$$y = \frac{1}{2x}$$

$$y = \pm\frac{1}{2\frac{\sqrt{2}}{2}}$$

$$y = \pm\frac{1}{\sqrt{2}}$$

$$y = \pm\frac{\sqrt{2}}{2}$$

$$\sqrt{i} = \pm\frac{\sqrt{2}}{2}(1+i)$$

Question 3

a $z^2 = 9 + 40i$

Let $9 + 40i = (x + iy)^2$

$9 + 40i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$9 = x^2 - y^2$

$40 = 2xy$

$y = \frac{20}{x}$

$9 = x^2 - \left(\frac{20}{x}\right)^2$

$9x^2 = x^4 - 400$

$x^4 - 9x^2 - 400 = 0$

$\left(x^2 - 25\right)\left(x^2 + 16\right) = 0$

As $x \in \mathbb{R}$, $x^2 + 16 = 0$ has no real solutions

$x^2 - 25 = 0 \quad \Rightarrow \quad x = \pm 5$

$y = \frac{20}{x}$

$y = \pm 4$

$x = 5, y = 4; x = -5, y = -4$

$z = \pm\left(5 + 4i\right)$

b $z^2 = -7 + 24i$

Let $-7 + 24i = (x + iy)^2$

$-7 + 24i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$-7 = x^2 - y^2$

$24 = 2xy$

$y = \frac{12}{x}$

$-7 = x^2 - \left(\frac{12}{x}\right)^2$

$-7x^2 = x^4 - 144$

$x^4 + 7x^2 - 144 = 0$

$\left(x^2 - 9\right)\left(x^2 + 16\right) = 0$

As $x \in \mathbb{R}$, $x^2 + 16 = 0$ has no real solutions

$x^2 - 9 = 0 \quad \Rightarrow \quad x = \pm 3$

$y = \frac{12}{x}$

$y = \pm\frac{12}{3} = \pm 4$

$x = 3, y = 4; \ x = -3, y = -4$

$z = \pm\left(3 + 4i\right)$

c $z^2 = 12 - 16i$

Let $12 - 16i = (x + iy)^2$

$12 - 16i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$12 = x^2 - y^2$

$-16 = 2xy$

$y = \frac{-8}{x}$

$12 = x^2 - \left(\frac{-8}{x}\right)^2$

$12x^2 = x^4 - 64$

$x^4 - 12x^2 - 64 = 0$

$\left(x^2 - 16\right)\left(x^2 + 4\right) = 0$

As $x \in \mathbb{R},\ x^2 + 4 = 0$ has no real solutions

$x^2 - 16 = 0 \quad \Rightarrow \quad x = \pm 4$

$y = \frac{-8}{x}$

$y = \mp 2$

$x = 4, y = -2;\ x = -4, y = 2$

$z = \pm\left(4 - 2i\right)$

Question 4

a $\sqrt{-3+4i}$

Let $-3 + 4i = (x + iy)^2$

$-3 + 4i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$-3 = x^2 - y^2$

$4 = 2xy$

$$y = \frac{2}{x}$$

$$-3 = x^2 - \left(\frac{2}{x}\right)^2$$

$$-3x^2 = x^4 - 4$$

$$x^4 + 3x^2 - 4 = 0$$

$$\left(x^2 - 1\right)\left(x^2 + 4\right) = 0$$

As $x \in \mathbb{R},\ x^2 + 4 = 0$ has no real solutions

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

$$y = \frac{2}{x}$$

$$y = \pm 2$$

$$x = 1, y = 2 \qquad x = -1, y = -2$$

$$z = \pm\left(1 + 2i\right)$$

b $z^2 - 3z + 3 - i$

$a = 1,\ b = -3,\ c = 3 - i$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{3 \pm \sqrt{9 - 4\left(3 - i\right)}}{2}$$

$$= \frac{3 \pm \sqrt{9 - 12 + 4i}}{2}$$

$$= \frac{3 \pm \sqrt{-3 + 4i}}{2}$$

From part **a**,

$$\sqrt{-3+4i} = \pm\left(1 + 2i\right)$$

$$\therefore z = \frac{3 \pm \left(1 + 2i\right)}{2}$$

$$z = \frac{4 + 2i}{2} \quad \frac{2 - 2i}{2}$$

$$z = 2 + i, 1 - i$$

Question 5

a $x^2 - (2 + 3i)x + (-5 + i) = 0$

$a = 1,\ b = -2 - 3i,\ c = -5 + i$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 + 3i \pm \sqrt{(2+3i)^2 - 4(-5+i)}}{2}$$

$$x = \frac{2 + 3i \pm \sqrt{-5 + 12i + 20 - 4i}}{2}$$

$$x = \frac{2 + 3i \pm \sqrt{15 + 8i}}{2}$$

$\sqrt{15 + 8i}$

Let $15 + 8i = (x + iy)^2$

$15 + 8i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$15 = x^2 - y^2$

$8 = 2xy$

$$y = \frac{4}{x}$$

$$15 = x^2 - \left(\frac{4}{x}\right)^2$$

$15x^2 = x^4 - 16$

$x^4 - 15x^2 - 16 = 0$

$(x^2 - 16)(x^2 + 1) = 0$

As $x \in \mathbb{R}$ $x^2 + 1 = 0$ has no real solutions

$x^2 - 16 = 0 \quad \Rightarrow \quad x = \pm 4$

$$y = \frac{4}{x}$$

$y = \pm 1$

$x = 4, y = 1 \qquad x = -4, y = -1$

$z = \pm(4 + i)$

$$x = \frac{2 + 3i \pm (4 + i)}{2}$$

$$x = \frac{6 + 4i}{2}\ \ \frac{-2 + 2i}{2}$$

$x = 3 + 2i, -1 + i$

b $v^2 - (3 + 2i)v + 6i = 0$

$a = 1,\ b = -3 - 2i,\ c = -6i$

$$v = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$v = \frac{3 + 2i \pm \sqrt{(3+2i)^2 - 4(6i)}}{2}$$

$$v = \frac{3 + 2i \pm \sqrt{5 + 12i - 24i}}{2}$$

$$v = \frac{2 + 3i \pm \sqrt{5 - 12i}}{2}$$

From 1 **b**,

$\sqrt{5 - 12i} = \pm 3 - 2i$

$$v = \frac{3 + 2i \pm (3 - 2i)}{2}$$

$$v = \frac{6}{2}\ \ \frac{4i}{2}$$

$v = 3, 2i$

c $iz^2 - z + 2i = 0$

$a = i,\ b = -1,\ c = 2i$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{1 \pm \sqrt{(-1)^2 - 4(i)(2i)}}{2i}$$

$$z = \frac{1 \pm \sqrt{9}}{2i}$$

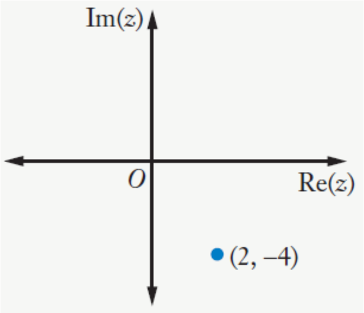
$$z = \frac{1 \pm 3}{2i}$$

$$z = \frac{2}{i}\ \ -\frac{1}{i}$$

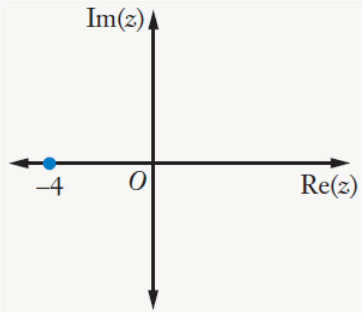
$z = -2i, i$

Question 1

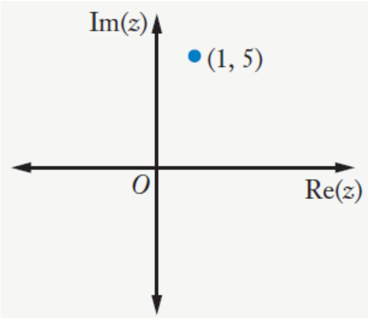
a



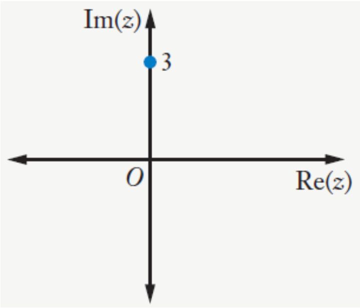
c



b



d



Question 2

A $-3 + i$

B $4 + 2i$

C $5 - 3i$

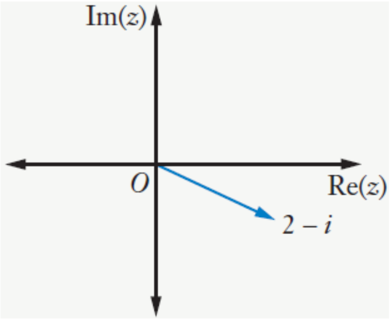
D $-4 - 5i$

E 2

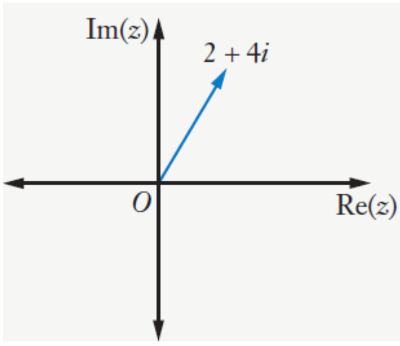
F $-3i$

Question 3

a $4 - 3i + (-2 + 2i) = 2 - i$

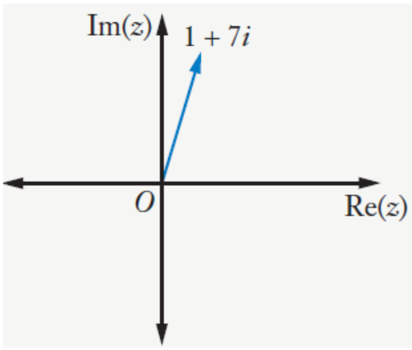


b $(5 - i) - (3 - 5i) = 2 + 4i$

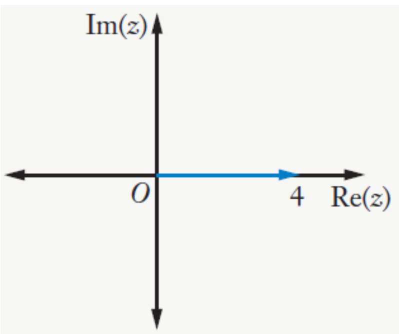


c $(3 + i)(1 + 2i) = 3 + 6i + i - 2$

$= 1 + 7i$

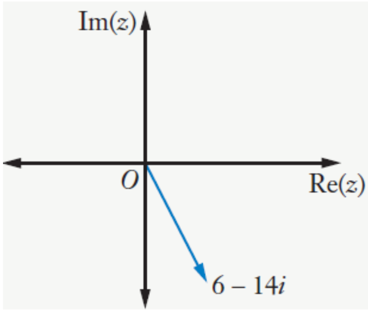


d $\left(1+i\sqrt{3}\right)\left(1-i\sqrt{3}\right) = 1 + 3 = 4$

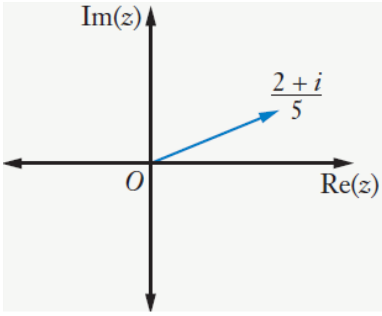


e $4(2-5i)-2i(-3-i)=8-20i+6i-2$

$=6-14i$



f $\dfrac{1}{2-i}=\dfrac{1}{2-i}\times\dfrac{2+i}{2+i}=\dfrac{2+i}{3}$

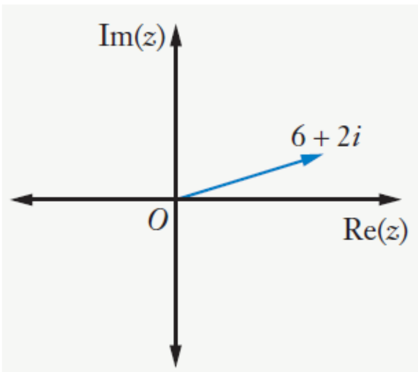


g $\dfrac{4+8i}{1+i}=\dfrac{4+8i}{1+i}\times\dfrac{1-i}{1-i}$

$=\dfrac{4-4i+8i+8}{2}$

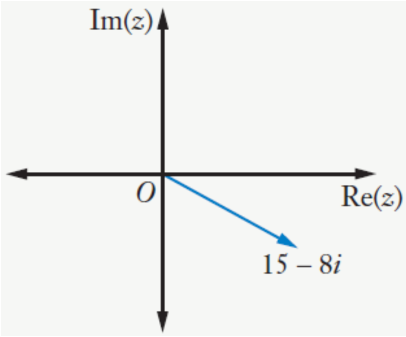
$=\dfrac{12+4i}{2}$

$=6+2i$

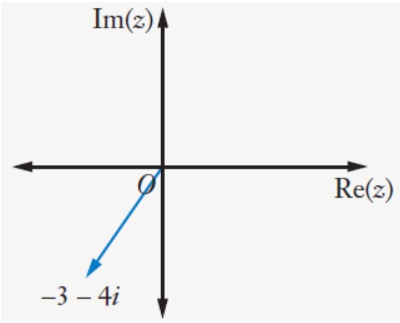


Question 4

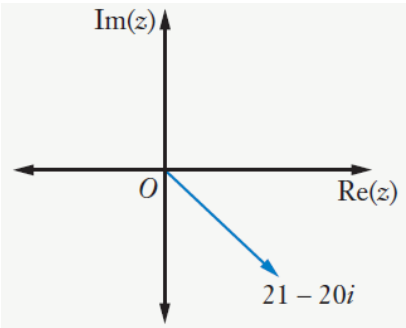
a $15 - 8i$



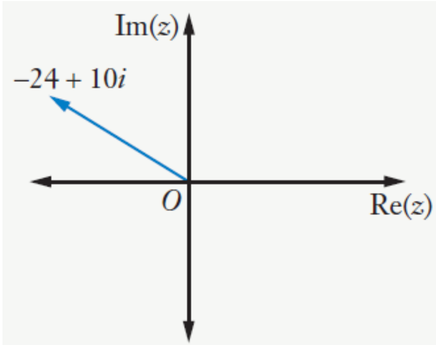
b $-3 - 4i$



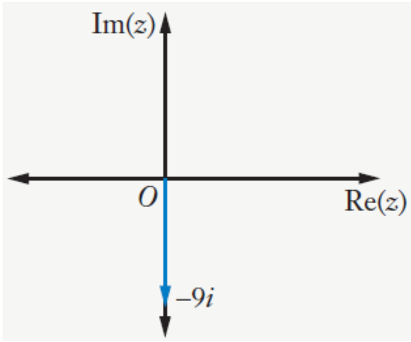
c $21 - 20i$



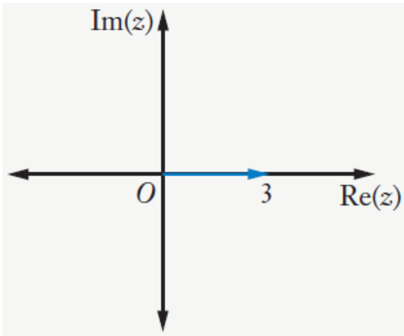
d $-24 + 10i$



e $-9i$

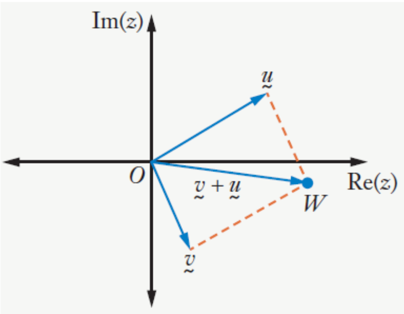


f 3

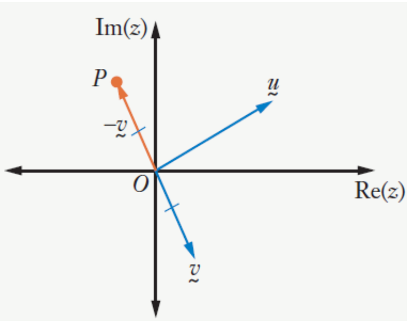


Question 5

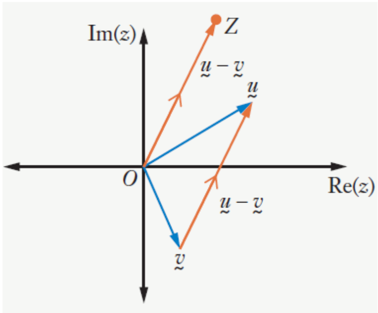
a **v** + **u**



b –**v**

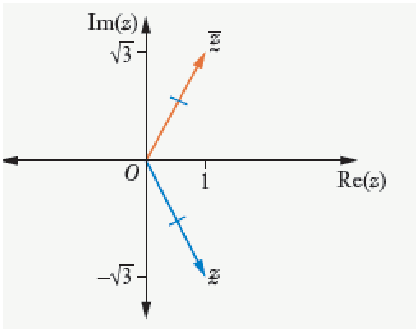


c **u** – **v**

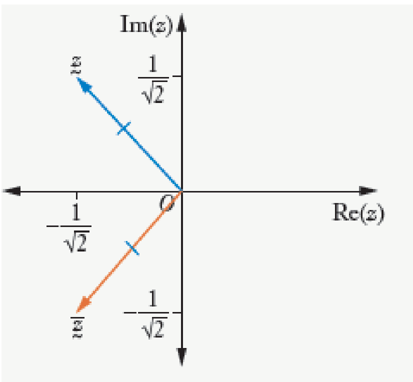


Question 6

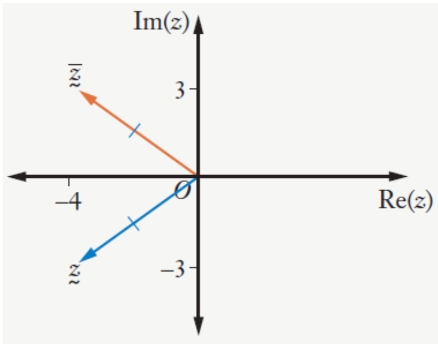
a $z = 1 - i\sqrt{3},\ \overline{z} = 1 + i\sqrt{3}$



b $z = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \quad \overline{z} = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$



c $z = -4 - 3i,\ \overline{z} = -4 + 3i$

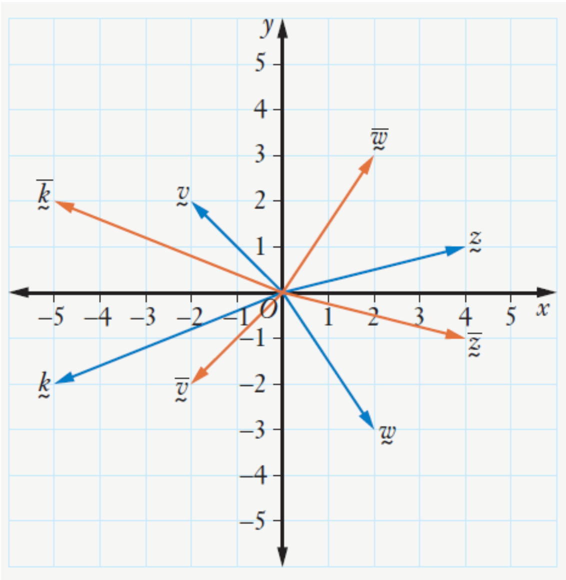


Question 7

a $z = 4 + i$, $v = -2 + 2i$, $k = -5 - 2i$, $w = 2 - 3i$

b $\overline{z} = 4 - i$, $\overline{v} = -2 - 2i$, $\overline{k} = -5 + 2i$, $\overline{w} = 2 + 3i$

c

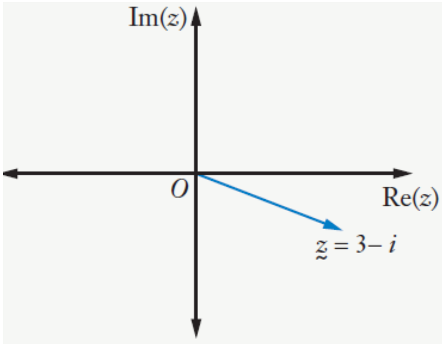


Question 8

a $z = (2 + i)(1 - i)$

$= 2 - 2i + i + 1$

$= 3 - i$



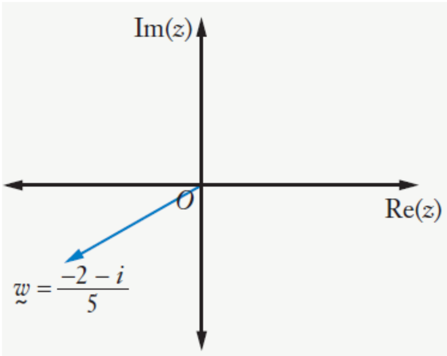
b

$$w = \frac{1}{-2+i}$$

$$= \frac{1}{-2+i} \times \frac{-2-i}{-2-i}$$

$$= \frac{-2-i}{4+1}$$

$$= \frac{-2-i}{5}$$

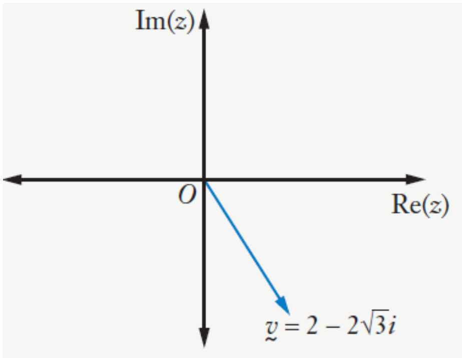


c

$$v = \left(\sqrt{3} - i\right)^2$$

$$= 3 - 2i\sqrt{3} - 1$$

$$= 2 - 2i\sqrt{3}$$



Question 9

$z = 2 + 3i$

a **i** $w = iz$

$= i(2 + 3i)$

$= 2i - 3$

$= -3 + 2i$

ii $v = i^2 z$

$= -1(2 + 3i)$

$= -2 - 3i$

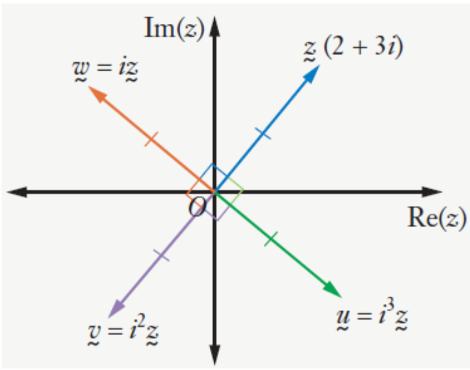
iii $u = i^3 z$

$= -i(2 + 3i)$

$= -2i + 3$

$= 3 - 2i$

b



c It rotates the vector about the origin by 90° counterclockwise.

Exercise 1.04 Modulus and argument

Question 1

a $1 + i$

$r = \sqrt{1^2 + 1^2}$

$r = \sqrt{2}$

b $2 + 4i$

$r = \sqrt{2^2 + 4^2}$

$r = \sqrt{20}$

$r = 2\sqrt{5}$

c $7 - 2i$

$r = \sqrt{7^2 + 2^2}$

$r = \sqrt{53}$

d $\sqrt{3} + i\sqrt{2}$

$r = \sqrt{\left(\sqrt{3}\right)^2 + \left(\sqrt{2}\right)^2}$

$r = \sqrt{5}$

e $-\frac{1}{7} + \frac{6i}{7}$

$r = \sqrt{\left(\frac{1}{7}\right)^2 + \left(\frac{6}{7}\right)^2}$

$r = \sqrt{\frac{1}{49} + \frac{36}{49}}$

$r = \frac{\sqrt{37}}{7}$

f $-5 - i\sqrt{2}$

$r = \sqrt{(5)^2 + \left(\sqrt{2}\right)^2}$

$r = \sqrt{27}$

$r = 3\sqrt{3}$

g i

$r = 1$

Question 2

a $1 + i$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right)$$

$$\theta = \frac{\pi}{4}$$

b $\sqrt{3} + i$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\theta = \frac{\pi}{6}$$

c $\sqrt{2} - i\sqrt{2}$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{2}}{\sqrt{2}}\right)$$

$$\theta = \frac{-\pi}{4} \quad \theta \text{ is in the fourth quadrant}$$

d $-\frac{\sqrt{3}}{2} + \frac{i}{2}$

$$\theta = \tan^{-1}\left(\frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}\right)$$

$$\theta = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$$

$$\theta = \frac{5\pi}{6} \quad \theta \text{ is in the second quadrant}$$

e $-1 - i$

$$\theta = \tan^{-1}\left(\frac{-1}{-1}\right)$$

$$\theta = -\frac{3\pi}{4} \quad \theta \text{ is in the third quadrant}$$

f 4

$\theta = 0$

g i

$$\theta = \frac{\pi}{2}$$

Question 3

a $z = 1 - i$

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{-1}{1}\right)$$

$$\theta = -\frac{\pi}{4} \quad \theta \text{ is in the fourth quadrant}$$

$$z = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$$

b $z = -1 + i\sqrt{3}$

$$r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$$

$$\theta = \frac{\pi}{3} \quad \theta \text{ is in the second quadrant}$$

$$z = 2\text{cis}\left(\frac{2\pi}{3}\right)$$

c $z = \dfrac{-2-2i}{3} = -\dfrac{2}{3} - \dfrac{2i}{3}$

$$r = \sqrt{\left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

$$\theta = \tan^{-1}\left(\frac{-\frac{2}{3}}{-\frac{2}{3}}\right) = \tan^{-1}(1)$$

$$= \frac{\pi}{4} \quad \theta \text{ is in the third quadrant}$$

$$z = \frac{2\sqrt{2}}{3}\text{cis}\left(\frac{-3\pi}{4}\right)$$

d $z = \dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

$$\theta = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

$$z = \text{cis}\left(\frac{\pi}{3}\right)$$

e $z = \dfrac{\sqrt{2} + i\sqrt{2}}{7} = \dfrac{\sqrt{2}}{7} + \dfrac{i\sqrt{2}}{7}$

$$r = \sqrt{\left(\tfrac{\sqrt{2}}{7}\right)^2 + \left(\tfrac{\sqrt{2}}{7}\right)^2} = \sqrt{\frac{2}{49} + \frac{2}{49}}$$

$$= \sqrt{\frac{4}{49}} = \frac{2}{7}$$

$$\theta = \tan^{-1}\left(\frac{\frac{\sqrt{2}}{7}}{\frac{\sqrt{2}}{7}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$z = \frac{2}{7}\text{cis}\left(\frac{\pi}{4}\right)$$

f $z = -2\sqrt{3} - 2i$

$$r = \sqrt{\left(-2\sqrt{3}\right)^2 + (-2)^2} = \sqrt{12 + 4}$$

$$= \sqrt{16} = 4$$

$$\theta = \tan^{-1}\left(\frac{-2}{-2\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{-5\pi}{6} \quad \theta \text{ is in the third quadrant}$$

$$z = 4\,\text{cis}\left(-\frac{5\pi}{6}\right)$$

g $z = -\sqrt{6}$, $r = \sqrt{6}$, $\theta = \pi$

$$z = \sqrt{6}\,\text{cis}\,\pi$$

Question 4

a

$$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$x = 2\times\cos\frac{\pi}{3} = 2\times\frac{1}{2} = 1$$

$$y = 2\times i\sin\frac{\pi}{3} = 2\times i\frac{\sqrt{3}}{2} = i\sqrt{3}$$

$$z = 1 + i\sqrt{3}$$

b

$$z = \frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$x = \frac{1}{2}\times\cos\frac{\pi}{6} = \frac{1}{2}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$$

$$y = \frac{1}{2}\times i\sin\frac{\pi}{6} = \frac{1}{2}\times i\frac{1}{2} = \frac{i}{4}$$

$$z = \frac{\sqrt{3}}{4} + \frac{i}{4}$$

c

$$z = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$x = 3\times\cos\frac{\pi}{4} = 3\times\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$$

$$y = 3\times i\sin\frac{\pi}{4} = 3\times i\frac{\sqrt{2}}{2} = \frac{i3\sqrt{2}}{2}$$

$$z = \frac{3\sqrt{2}}{2} + \frac{i3\sqrt{2}}{2}$$

d

$$z = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$$

$$x = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$$

$$y = i\sin\left(-\frac{\pi}{3}\right) = -i\frac{\sqrt{3}}{2}$$

$$z = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

e

$$z = \sqrt{2}\left[\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right]$$

$$x = \sqrt{2}\times\cos\left(\frac{-2\pi}{3}\right) = \sqrt{2}\times\left(-\frac{1}{2}\right) = -\frac{\sqrt{2}}{2}$$

$$y = \sqrt{2}\times i\sin\left(\frac{-2\pi}{3}\right) = \sqrt{2}\times i\left(\frac{-\sqrt{3}}{2}\right)$$

$$= -\frac{i\sqrt{6}}{2}$$

$$z = -\frac{\sqrt{2}}{2} - \frac{i\sqrt{6}}{2}$$

f

$$z = 2\left[\cos\left(\frac{-5\pi}{6}\right) + i\sin\left(\frac{-5\pi}{6}\right)\right]$$

$$x = 2\times\cos\left(\frac{-5\pi}{6}\right) = 2\times\frac{-\sqrt{3}}{2} = -\sqrt{3}$$

$$y = 2\times i\sin\left(\frac{-5\pi}{6}\right) = 2\times i\frac{-1}{2} = -1$$

$$z = -\sqrt{3} - i$$

Question 5

a

$$z=\frac{1}{\sqrt{3}}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)$$

$$x=\frac{1}{\sqrt{3}}\times\cos\frac{\pi}{4}=\frac{1}{\sqrt{3}}\times\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2\sqrt{3}}=\frac{\sqrt{6}}{6}$$

$$y=-\frac{1}{\sqrt{3}}\times i\sin\frac{\pi}{4}=-\frac{1}{\sqrt{3}}\times i\frac{\sqrt{2}}{2}=-\frac{i\sqrt{2}}{2\sqrt{3}}=-\frac{i\sqrt{6}}{6}$$

$$z=\frac{\sqrt{6}}{6}-\frac{i\sqrt{6}}{6}$$

b

$$z=\sqrt{3}\left(\cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6}\right)$$

$$x=\sqrt{3}\times\cos\frac{5\pi}{6}=\sqrt{3}\times\frac{-\sqrt{3}}{2}=-\frac{3}{2}$$

$$y=-\sqrt{3}\times i\sin\frac{5\pi}{6}=-\sqrt{3}\times i\frac{1}{2}=-\frac{i\sqrt{3}}{2}$$

$$z=-\frac{3}{2}-\frac{i\sqrt{3}}{2}$$

c

$$z=-2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$$x=-2\times\cos\frac{\pi}{2}=0$$

$$y=-2\times i\sin\frac{\pi}{2}=-2i$$

$$z=-2i$$

d

$$z=-\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$$

$$x=-1\times\cos\left(-\frac{\pi}{4}\right)=-1\times\frac{\sqrt{2}}{2}=-\frac{\sqrt{2}}{2}$$

$$y=-1\times i\sin\left(-\frac{\pi}{4}\right)=-1\times i\frac{-\sqrt{2}}{2}=\frac{i\sqrt{2}}{2}$$

$$z=-\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}$$

e

$$z = 3\sqrt{2}\left[\cos(-\pi) + i\sin(-\pi)\right]$$
$$x = 3\sqrt{2} \times \cos(-\pi) = 3\sqrt{2} \times -1 = -3\sqrt{2}$$
$$y = 3\sqrt{2} \times i\sin(-\pi) = 0$$
$$z = -3\sqrt{2}$$

f

$$z = i\left[\cos\left(-\frac{\pi}{3}\right) - i\sin\left(-\frac{\pi}{3}\right)\right]$$
$$x = i\cos\left(-\frac{\pi}{3}\right) = \frac{i}{2}$$
$$y = i \times -i\sin\left(-\frac{\pi}{3}\right) = -1 \times \frac{\sqrt{3}}{2}$$
$$z = -\frac{\sqrt{3}}{2} + \frac{i}{2}$$

g

$$z = \sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{3}\right)$$
$$x = \sqrt{3} \times \cos\frac{\pi}{6} = \sqrt{3} \times \frac{\sqrt{3}}{2} = \frac{3}{2}$$
$$y = \sqrt{3} \times i\sin\frac{\pi}{3} = \sqrt{3} \times i\frac{\sqrt{3}}{2} = \frac{i3}{2}$$
$$z = \frac{3}{2} + \frac{3i}{2}$$

h

$$z = 2\left[\cos\left(\frac{-5\pi}{4}\right) - i\sin\frac{\pi}{4}\right]$$
$$x = 2 \times \cos\left(\frac{-5\pi}{4}\right) = 2 \times \cos\left(\frac{3\pi}{4}\right) = 2 \times \frac{-\sqrt{2}}{2} = -\sqrt{2}$$
$$y = -2 \times i\sin\frac{\pi}{4} = -2 \times i\frac{\sqrt{2}}{2} = -i\sqrt{2}$$
$$z = -\sqrt{2} - i\sqrt{2}$$

Question 6

a $z = 3\,\text{cis}\left(\frac{\pi}{3}\right)$

b $w = 5\,\text{cis}\left(\frac{-\pi}{4}\right)$

c $u = \sqrt{3}\,\text{cis}\left(\frac{5\pi}{6}\right)$

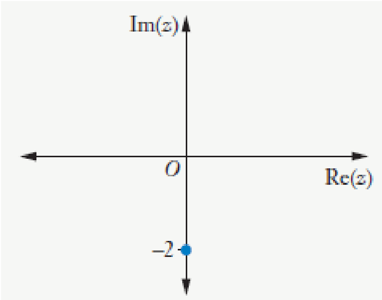
d $v = 2\,\text{cis}\left(\frac{-3\pi}{5}\right)$

e $z = 3\,\text{cis}(\pi)$

f $w = 6\,\text{cis}\left(\frac{-\pi}{2}\right)$

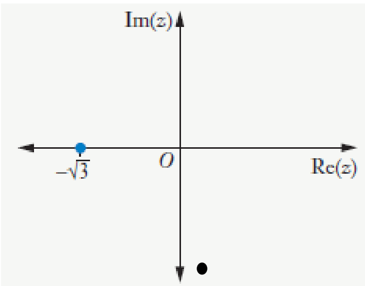
Question 7

a $z = -2i$



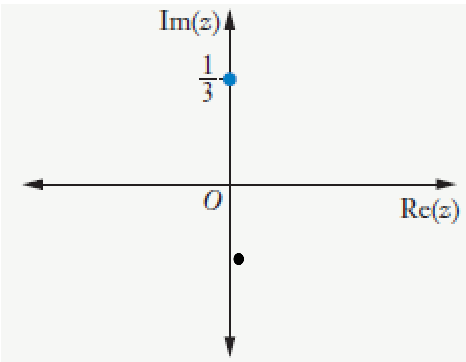
$$z = 2\,\text{cis}\left(\frac{-\pi}{2}\right)$$

b $z = -\sqrt{3}$



$$z = \sqrt{3}\,\text{cis}(\pi)$$

c $z = \frac{i}{3}$



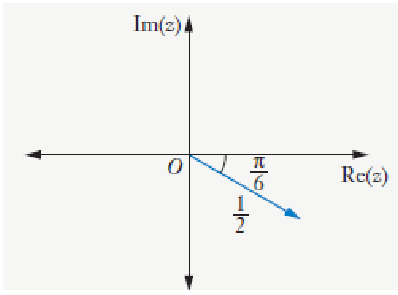
$$z = \frac{1}{3}\text{cis}\left(\frac{\pi}{2}\right)$$

d $z = \frac{1}{2}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

$$x = \frac{1}{2} \times \cos\frac{\pi}{6} = \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$$

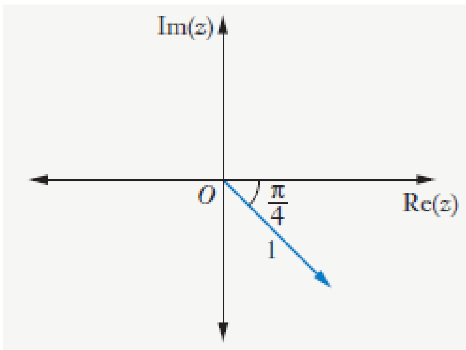
$$y = -\frac{1}{2} \times i\sin\frac{\pi}{6} = -\frac{1}{2} \times i\frac{1}{2} = -\frac{i}{4}$$

$$z = \frac{\sqrt{3}}{4} - \frac{i}{4}$$



$$z = \frac{1}{2}\text{cis}\left(\frac{-\pi}{6}\right)$$

e $z = -\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$



$$z = \text{cis}\left(\frac{-\pi}{4}\right)$$

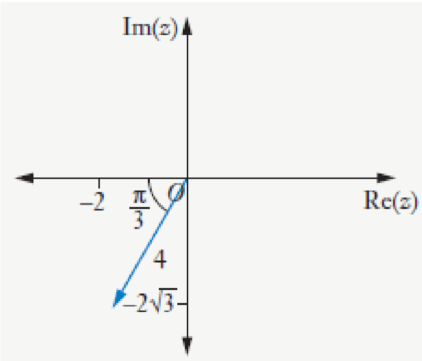
f $z = -2 - 2\sqrt{3}i$

$r = \sqrt{(-2)^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{4+12} = \sqrt{16} = 4$

$\theta = \tan^{-1}\left(\dfrac{-2\sqrt{3}}{-2}\right) = \tan^{-1}\left(\sqrt{3}\right) = \dfrac{\pi}{3}$

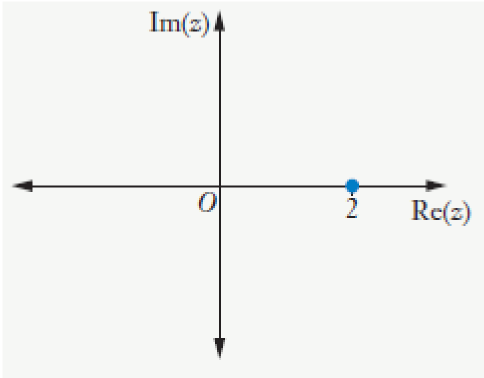
$= -\dfrac{2\pi}{3}$ θ is in the third quadrant

$z = 4\,\text{cis}\left(-\dfrac{2\pi}{3}\right)$



$z = 4\,\text{cis}\left(-\dfrac{2\pi}{3}\right)$

g $z = 2$



$z = 2\,\text{cis}(0)$

Question 8

a $z = 2 + i$

$r = \sqrt{(2)^2 + (1)^2} = \sqrt{4+1} = \sqrt{5}$

$\theta = \tan^{-1}\left(\dfrac{1}{2}\right)$

$= 0464$ θis in the first quadrant

$z = \sqrt{5}\,\text{cis}(0.464)$

b $z = -5 + 7i$

$r = \sqrt{(-5)^2 + (7)^2} = \sqrt{25+49} = \sqrt{74}$

$\theta = \tan^{-1}\left(\dfrac{7}{-5}\right)$

$\theta = -0951$ θis in the second quadrant

$\theta = \pi - 0.951 = 2.191$

$z = \sqrt{74}\,\text{cis}(2.191)$

c

$$z=\frac{4-i}{5}$$

$$r=\sqrt{\left(\frac{4}{5}\right)^2+\left(\frac{-1}{5}\right)^2}=\sqrt{\frac{16}{25}+\frac{1}{25}}=\frac{\sqrt{17}}{5}$$

$$\theta=\tan^{-1}\left(\frac{\frac{-1}{5}}{\frac{4}{5}}\right)$$

$$\theta=-0245 \qquad \theta \text{ is in the fourth quadrant}$$

$$z=\frac{\sqrt{17}}{5}\operatorname{cis}(-0.245)$$

d

$$z=4\operatorname{cis}\left(\frac{7\pi}{6}\right)=4\operatorname{cis}\left(\frac{-5\pi}{6}\right)$$

e

$$z=2\operatorname{cis}\left(\frac{9\pi}{4}\right)=2\operatorname{cis}\left(\frac{\pi}{4}\right)$$

f

$$z=-\sqrt{5}-i\sqrt{10}$$

$$r=\sqrt{\left(-\sqrt{5}\right)^2+\left(-\sqrt{10}\right)^2}=\sqrt{5+10}=\sqrt{15}$$

$$\theta=\tan^{-1}\left(\frac{-\sqrt{10}}{-\sqrt{5}}\right)$$

$$=0955 \qquad \theta \text{ is in the third quadrant}$$

$$\theta=-\pi+0.955\ =-2.186$$

$$z=\sqrt{15}\operatorname{cis}(-2.186)$$

g

$$z=-\left[\cos\left(\frac{4\pi}{3}\right)-i\sin\left(\frac{4\pi}{3}\right)\right]=-\left[\cos\left(\frac{-4\pi}{3}\right)+i\sin\left(\frac{-4\pi}{3}\right)\right]$$

$$=-\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]=\left[\cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right]$$

$$=\operatorname{cis}\left(-\frac{\pi}{3}\right)$$

h

$$z=\sqrt{2}\left[\sin\left(\frac{\pi}{3}\right)+i\cos\left(\frac{\pi}{3}\right)\right]=\sqrt{2}\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]$$

$$=\sqrt{2}\operatorname{cis}\left(\frac{\pi}{6}\right)$$

i

$$z = -\frac{i}{4} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4} - \frac{i}{4}$$

$$r = \sqrt{\left(\frac{\sqrt{3}}{4}\right)^2 + \left(-\frac{1}{4}\right)^2} = \sqrt{\frac{3}{16} + \frac{1}{16}} = \sqrt{\frac{4}{16}} = \frac{1}{2}$$

$$\theta = \tan^{-1}\left(\frac{-\frac{1}{4}}{\frac{\sqrt{3}}{4}}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

$$z = \frac{1}{2}\text{cis}\left(-\frac{\pi}{6}\right)$$

j

$$z = -\left[\sin\left(\frac{\pi}{4}\right) - i\cos\left(\frac{3\pi}{4}\right)\right] = -\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$

$$= \cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right) = \text{cis}\left(\frac{-3\pi}{4}\right)$$

Question 1

Let $z = -1 + 2i$

a

$$|z| = \sqrt{(-1)^2 + (2)^2} = \sqrt{5}$$

$$|z|^2 = 5$$

$$z \cdot \overline{z} = (-1+2i)(-1-2i) = 1+4 = 5$$

$$\therefore |z|^2 = z \cdot \overline{z}$$

b

$$z^2 = (-1+2i)(-1+2i)$$

$$= 1-4i-4 = -3-4i$$

$$|z^2| = \sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5$$

$$\therefore |z|^2 = |z^2|$$

Question 2

$z_1 = 3-i,\ \ z_2 = 2+5i$

a

$$z_1z_2 = (3-i)(2+5i) = 6+13i+5 = 11+13i$$

$$|z_1z_2| = \sqrt{(11)^2 + (13)^2} = \sqrt{290}$$

$$|z_1| = \sqrt{(3)^2 + (-1)^2} = \sqrt{10}$$

$$|z_2| = \sqrt{(2)^2 + (5)^2} = \sqrt{29}$$

$$|z_1||z_2| = \sqrt{10} \times \sqrt{29} = \sqrt{290}$$

$$\therefore |z_1z_2| = |z_1||z_2|$$

b

$$\frac{z_1}{z_2} = \frac{(3-i)}{(2+5i)} = \frac{(3-i)}{(2+5i)} \times \frac{(2-5i)}{(2-5i)}$$

$$= \frac{6-15i-2i-5}{29} = \frac{1-17i}{29}$$

$$\left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{1}{29}\right)^2 + \left(\frac{-17}{29}\right)^2} = \sqrt{\frac{290}{841}} = \sqrt{\frac{10}{29}}$$

$$\frac{|z_1|}{|z_2|} = \frac{\sqrt{10}}{\sqrt{29}} = \sqrt{\frac{10}{29}}$$

$$\therefore \left|\frac{z}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

c

$$\overline{z_1 + z_2} = \overline{(3-i)+(2+5i)}$$

$$= \overline{5+4i} = 5-4i$$

$$\overline{z_1} + \overline{z_2} = \overline{(3-i)} + \overline{(2+5i)}$$

$$= 3+i+2-5i = 5-4i$$

$$\therefore \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

d

$$\overline{z_1z_2} = \overline{(3-i)(2+5i)}$$

$$= \overline{11+13i} = 11-13i$$

$$\overline{z_1}\,\overline{z_2} = \overline{(3-i)}\,\overline{(2+5i)} = (3+i)(2-5i)$$

$$= 6-15i+2i+5 = 11-13i$$

$$\therefore \overline{z_1z_2} = \overline{z_1}\,\overline{z_2}$$

Question 3

$z_1 = 1 - i,\ z_2 = \sqrt{3} + i,\ \ z_1 = 1 - i$

$r = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-1}{1}\right)$

$= -\frac{\pi}{4}$ θ is in the fourth quadrant

$z_1 = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

$z_2 = \sqrt{3} + i$

$r = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$ θ is in the fourth quadrant

$z_2 = 2\,\text{cis}\left(\frac{\pi}{6}\right)$

a $\arg(z_1 z_2) = \arg(z\ \ ) + \arg(z_2) = \left(\frac{-\pi}{4}\right) + \left(\frac{\pi}{6}\right) = \left(\frac{-\pi}{12}\right)$

b $\arg\left(\frac{z}{z_2}\right) = \arg(z_1) - \arg(z_2) = \left(\frac{-\pi}{4}\right) - \left(\frac{\pi}{6}\right) = \left(\frac{-5\pi}{12}\right)$

c $\arg\left(z_2^5\right) = 5\arg(z_2) = 5\left(\frac{\pi}{6}\right) = \left(\frac{5\pi}{6}\right)$

d $\arg\left(z_1^{-2}\right) = -2\arg(z_1) = -2\left(\frac{-\pi}{4}\right) = \left(\frac{2\pi}{4}\right) = \left(\frac{\pi}{2}\right)$

Question 4

$$z_1 = \sqrt{3}\,\text{cis}\left(\frac{\pi}{3}\right),\ z_2 = 2\,\text{cis}\left(\frac{\pi}{5}\right)$$

a $$z_1 z_2 = \sqrt{3}\,\text{cis}\left(\frac{\pi}{3}\right)\times\left[2\,\text{cis}\left(\frac{\pi}{5}\right)\right] = 2\sqrt{3}\,\text{cis}\left(\frac{\pi}{3}+\frac{\pi}{5}\right) = 2\sqrt{3}\,\text{cis}\left(\frac{8\pi}{15}\right)$$

b $$\frac{z_1}{z_2} = \frac{\sqrt{3}\,\text{cis}\left(\frac{\pi}{3}\right)}{2\,\text{cis}\left(\frac{\pi}{5}\right)} = \frac{\sqrt{3}}{2}\text{cis}\left(\frac{\pi}{3}-\frac{\pi}{5}\right) = \frac{\sqrt{3}}{2}\text{cis}\left(\frac{2\pi}{15}\right)$$

c $$\frac{1}{z_2} = \frac{1}{2\,\text{cis}\left(\frac{\pi}{5}\right)} = \frac{1}{2}\text{cis}\left(-\frac{\pi}{5}\right)$$

d $$(z_1)^5 = \left[\sqrt{3}\,\text{cis}\left(\frac{\pi}{3}\right)\right]^5 = \left(\sqrt{3}\right)^5\text{cis}\left(5\times\frac{\pi}{3}\right) = 9\sqrt{3}\,\text{cis}\left(\frac{5\pi}{3}\right) = 9\sqrt{3}\,\text{cis}\left(-\frac{\pi}{3}\right)$$

Question 5

a $$(\text{cis}\,\alpha)(\text{cis}\,2\alpha) = (\text{cis}\,3\alpha)$$

b
$$\begin{aligned}\frac{\cos 3\beta - i\sin 3\beta}{\cos 2\lambda + i\sin 2\lambda} &= \frac{1}{(\cos 3\beta + i\sin 3\beta)(\cos 2\lambda + i\sin 2\lambda)}\\ &= \frac{1}{\cos(3\beta+2\lambda)+i\sin(3\beta+2\lambda)}\\ &= \cos(-3\beta-2\lambda)+i\sin(-3\beta-2\lambda)\\ &= \text{cis}(-3\beta-2\lambda)\end{aligned}$$

c
$$\begin{aligned}\frac{(\cos 5 + i\sin 5)(\cos 3 + i\sin 3)}{\cos 2 - i\sin 2} &= \frac{(\cos 5 + i\sin 5)(\cos 2 + i\sin 2)}{\cos 3 + i\sin 3}\\ &= \cos(5+2-3)+i\sin(5+2-3)\\ &= \cos(4)+i\sin(4)\\ &= \text{cis}(4)\end{aligned}$$

Question 6

$z = a + ib \quad \overline{z} = a - ib$

$z^2 = a^2 - b^2 + 2iab$

a

$$|z| = \sqrt{a^2 + b^2}$$

$$|z|^2 = a^2 + b^2$$

$$z\overline{z} = (a + ib)(a - ib) = a^2 + b^2$$

$$\therefore |z|^2 = z\overline{z}$$

b

$$|z|^2 = a^2 + b^2$$

$$|z^2| = |a^2 - b^2 + 2iab|$$

$$= \sqrt{(a^2 - b^2)^2 + (2ab)^2}$$

$$= \sqrt{a^4 + b^4 - 2a^2b^2 + 4a^2b^2}$$

$$= \sqrt{a^4 + b^4 + 2a^2b^2}$$

$$= \sqrt{(a^2 + b^2)^2}$$

$$= (a^2 + b^2)$$

$$\therefore |z|^2 = |z^2|$$

Question 7

$z_1 = a + ib \quad z_2 = c + id$

a $z_1z_2 = (a+ib)(c+id) = ac - bd + adi + cbi = (ac-bd) + (ad+cb)i$

$$\begin{aligned}|z_1z_2| &= \sqrt{(ac-bd)^2 + (ad+cb)^2}\\ &= \sqrt{a^2c^2 - 2acbd + b^2d^2 + a^2d^2 + 2acbd + c^2b^2}\\ &= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + c^2b^2}\\ &= \sqrt{a^2(c^2+d^2) + b^2(c^2+d^2)}\\ &= \sqrt{(a^2+b^2)(c^2+d^2)}\\ &= \sqrt{(a^2+b^2)}\sqrt{(c^2+d^2)}\\ &= |z_1||z_2|\end{aligned}$$

b $\dfrac{z_1}{z_2} = \dfrac{(a+ib)}{(c+id)} = \dfrac{(a+ib)}{(c+id)} \times \dfrac{(c-id)}{(c-id)} = \dfrac{ac+bd+(bc-ad)i}{c^2+d^2}$

$$\begin{aligned}\left|\frac{z_1}{z_2}\right| &= \sqrt{\left(\frac{ac+bd}{c^2+d^2}\right)^2 + \left(\frac{bc-ad}{c^2+d^2}\right)^2} = \frac{\sqrt{(ac+bd)^2 + (bc-ad)^2}}{c^2+d^2}\\ &= \frac{\sqrt{a^2c^2 + 2abcd + b^2d^2 + b^2c^2 - 2abcd + a^2d^2}}{c^2+d^2}\\ &= \frac{\sqrt{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2}}{c^2+d^2} = \frac{\sqrt{(a^2+b^2)(c^2+d^2)}}{c^2+d^2}\\ &= \frac{\sqrt{(a^2+b^2)}}{\sqrt{c^2+d^2}} = \frac{|z_1|}{|z_2|}\end{aligned}$$

c $\overline{z_1 + z_2} = \overline{a + ib + c + id} = \overline{a + c + ib + id}$

$$\begin{aligned}&= a + c - ib - id = a - ib + c - id\\ &= \overline{z_1} + \overline{z_2}\end{aligned}$$

d $\overline{z_1z_2} = \overline{(a+ib)(c+id)} = \overline{ac - bd + ibc + iad}$

$$\begin{aligned}&= ac - bd - ibc - iad = a(c-id) - ib(c-id)\\ &= (a-ib)(c-id)\\ &= \overline{z_1}\,\overline{z_2}\end{aligned}$$

Question 8

$z = r(\cos\theta + i\sin\theta)$

a $\quad z + \overline{z} = r(\cos\theta + i\sin\theta) + r(\cos\theta - i\sin\theta)$

$$= r\cos\theta + ri\sin\theta + r\cos\theta - ri\sin\theta$$

$$= 2r\cos\theta \in \mathbb{R}$$

b $\quad z\overline{z} = r(\cos\theta + i\sin\theta) \times r(\cos\theta - i\sin\theta)$

$$= r^2(\cos^2\theta + \sin^2\theta)$$

$$= r^2 \in \mathbb{R}$$

Question 9

From **6 b**

$|z|^2 = |z^2|$

$\therefore \dfrac{1}{|z|^2} = \dfrac{1}{|z^2|}$

Question 10

$z = r(\cos\theta + i\sin\theta)$

a

$$\begin{aligned} z + \frac{1}{z} &= z + z^{-1} \\ &= \cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta + i\sin\theta + \cos(\theta) - i\sin(\theta) \\ &= 2\cos\theta \end{aligned}$$

b

$$\begin{aligned} z^2 + \frac{1}{z^2} &= z^2 + z^{-2} \\ &= \cos 2\theta + i\sin 2\theta + \cos(-2\theta) + i\sin(-2\theta) \\ &= \cos 2\theta + i\sin 2\theta + \cos(2\theta) - i\sin(2\theta) \\ &= 2\cos 2\theta \end{aligned}$$

c

$$\begin{aligned} z^3 + \frac{1}{z^3} &= z^3 + z^{-3} \\ &= \cos 3\theta + i\sin 3\theta + \cos(-3\theta) + i\sin(-3\theta) \\ &= \cos 3\theta + i\sin 3\theta + \cos(3\theta) - i\sin(3\theta) \\ &= 2\cos 3\theta \end{aligned}$$

d

$$\begin{aligned} z^2 - \frac{1}{z^2} &= z^2 - z^{-2} \\ &= \cos 2\theta + i\sin 2\theta - \cos(-2\theta) - i\sin(-2\theta) \\ &= \cos 2\theta + i\sin 2\theta - \cos(2\theta) + i\sin(2\theta) \\ &= 2i\sin 2\theta \end{aligned}$$

e

$$\begin{aligned} z^3 - \frac{1}{z^3} &= z^3 - z^{-3} \\ &= \cos 3\theta + i\sin 3\theta - \cos(-3\theta) - i\sin(-3\theta) \\ &= \cos 3\theta + i\sin 3\theta - \cos(3\theta) + i\sin(3\theta) \\ &= 2i\sin 3\theta \end{aligned}$$

Question 11

a

$$\begin{aligned}\left|(7-4i)(3+i)\right| &= \left|21-12i+7i+4\right| \\ &= \left|25-5i\right| \\ &= \sqrt{25^2+5^2} \\ &= \sqrt{650} \\ &= 5\sqrt{26}\end{aligned}$$

b

$$\begin{aligned}\left|\left(1+i\sqrt{3}\right)^2\left(\sqrt{2}+i\sqrt{2}\right)\right| &= \left|1+i\sqrt{3}\right|^2\left|\sqrt{2}+i\sqrt{2}\right| \\ &= \left(\sqrt{1^2+\sqrt{3}^2}\right)^2\left(\sqrt{\sqrt{2}^2+\sqrt{2}^2}\right) \\ &= \left(\sqrt{4}\right)^2\left(\sqrt{4}\right) \\ &= 8\end{aligned}$$

c

$$\left|\frac{3-2i}{5+i}\right| = \frac{|3-2i|}{|5+i|} = \frac{\sqrt{3^2+2^2}}{\sqrt{5^2+1^2}} = \frac{\sqrt{13}}{\sqrt{26}} = \frac{1}{\sqrt{2}}$$

d

$$\left|\frac{(2+i)^2}{(4-3i)^2}\right| = \frac{|(2+i)|^2}{|(4-3i)|^2} = \frac{\left(\sqrt{2^2+1^2}\right)^2}{\left(\sqrt{4^2+3^2}\right)^2} = \frac{5}{25} = \frac{1}{5}$$

e

$$\left|\frac{1}{(1-i)^6}\right| = \frac{1}{|(1-i)|^6} = \frac{1}{\left(\sqrt{1^2+1^2}\right)^6} = \frac{1}{2^3} = \frac{1}{8}$$

Question 12

a

$$\begin{aligned}\arg\left[(1+i)\left(1-i\sqrt{3}\right)\right] &= \arg(1+i)+\arg\left(1-i\sqrt{3}\right)\\ &= \frac{\pi}{4}+\left(-\frac{\pi}{3}\right)\\ &= -\frac{\pi}{12}\end{aligned}$$

b

$$\begin{aligned}\arg\left[\left(\sqrt{3}-i\right)\left(-\sqrt{2}-i\sqrt{2}\right)\right] &= \arg\left(\sqrt{3}-i\right)+\arg\left(-\sqrt{2}-i\sqrt{2}\right)\\ &= -\frac{\pi}{6}+\left(-\frac{3\pi}{4}\right)\\ &= -\frac{11\pi}{12}\end{aligned}$$

c

$$\arg\left[i(3+3i)\right] = \arg(i)+\arg(3+3i) = \frac{\pi}{2}+\frac{\pi}{4} = \frac{3\pi}{4}$$

d

$$\begin{aligned}\arg\left[\frac{(2-2i)}{\left(\sqrt{3}+i\right)}\right] &= \arg(2-2i)-\arg\left(\sqrt{3}+i\right)\\ &= \left(-\frac{\pi}{4}\right)-\left(\frac{\pi}{6}\right)\\ &= -\frac{5\pi}{12}\end{aligned}$$

e

$$\begin{aligned}\arg\left(\frac{-1-i}{\sqrt{2}+i\sqrt{2}}\right) &= \arg(-1-i)-\arg\left(\sqrt{2}+i\sqrt{2}\right)\\ &= \left(\frac{3\pi}{4}\right)-\left(\frac{\pi}{4}\right)\\ &= \frac{\pi}{2}\end{aligned}$$

Question 13

$z = 3\,\text{cis}\left(\frac{\pi}{6}\right),\ \ w = 2\,\text{cis}\left(\frac{\pi}{4}\right)$

a $z = 3\,\text{cis}\left(\frac{\pi}{6}\right) = 3\cos\left(\frac{\pi}{6}\right) + 3i\sin\left(\frac{\pi}{6}\right) = 3\times\frac{\sqrt{3}}{2} + 3\times i\,\frac{1}{2} = \frac{3\sqrt{3}}{2} + \frac{3i}{2}$

$w = 2\,\text{cis}\left(\frac{\pi}{4}\right) = 2\cos\left(\frac{\pi}{4}\right) + 2i\sin\left(\frac{\pi}{4}\right) = 2\times\frac{\sqrt{2}}{2} + 2\times i\,\frac{\sqrt{2}}{2} = \sqrt{2} + i\sqrt{2}$

b $\frac{w}{z} = \frac{\sqrt{2} + i\sqrt{2}}{\frac{3\sqrt{3}}{2} + \frac{3i}{2}} = \frac{2\sqrt{2} + i2\sqrt{2}}{3\sqrt{3} + 3i} = \frac{2\sqrt{2} + i2\sqrt{2}}{3\sqrt{3} + 3i} \times \frac{3\sqrt{3} - 3i}{3\sqrt{3} - 3i}$

$= \frac{6\sqrt{6} - i6\sqrt{2} + i6\sqrt{6} + 6\sqrt{2}}{36} = \frac{\sqrt{6} + \sqrt{2} - i\left(\sqrt{2} - \sqrt{6}\right)}{6}$

$\frac{w}{z} = \frac{2\,\text{cis}\left(\frac{\pi}{4}\right)}{3\,\text{cis}\left(\frac{\pi}{6}\right)} = \frac{2}{3}\text{cis}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{2}{3}\text{cis}\left(\frac{\pi}{12}\right)$

c Equating real and imaginary parts

i

$\frac{2}{3}\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6} + \sqrt{2}}{6}$

$\cos\left(\frac{\pi}{12}\right) = \frac{3\left(\sqrt{6} + \sqrt{2}\right)}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$

ii

$\frac{2}{3}\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{6}$

$\sin\left(\frac{\pi}{12}\right) = \frac{3\left(\sqrt{6} - \sqrt{2}\right)}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$

Question 14

$$\begin{aligned}|z-w|^2 &= (z-w)\overline{(z-w)} = (z-w)(\overline{z}-\overline{w})\\ &= z\overline{z} - z\overline{w} - w\overline{z} + w\overline{w}\\ &= |z|^2 + |w|^2 - z\overline{w} - w\overline{z}\\ &= |z|^2 + |w|^2 - 2\,\mathrm{Re}(z\overline{w})\\ &\geq |z|^2 + |w|^2 - 2|z\overline{w}| = |z|^2 + |w|^2 - 2|z||w| = (|z|-|w|)^2\end{aligned}$$

Therefore

$$\begin{aligned}|z-w|^2 &\geq (|z|-|w|)^2\\ |z-w| &\geq (|z|-|w|)\end{aligned}$$

Exercise 1.06 Euler's formula

Question 1

a **i** $e^{2i\pi} = \text{cis}(2\pi) = \text{cis}(0)$

ii 1

b **i** $\sqrt{2}e^{\frac{-\pi}{3}} = \sqrt{2}\,\text{cis}\left(\frac{-\pi}{3}\right)$

$= \sqrt{2}\left[\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right]$

ii $\sqrt{2}\cos\left(\frac{-\pi}{3}\right) + i\sqrt{2}\sin\left(\frac{-\pi}{3}\right)$

$= \sqrt{2} \times \frac{1}{2} + i\sqrt{2} \times \left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{2}}{2} - i\frac{\sqrt{6}}{2}$

c **i** $5e^{3i} = 5\,\text{cis}(3)$

$= 5\left[\cos(3) + i\sin(3)\right]$

ii $5\cos(3) + i5\sin(3)$

$= -4.95 + 0.71i$

d **i** $-e^{\frac{-\pi}{2}} = -\text{cis}\left(\frac{\pi}{2}\right)$

$= \text{cis}\left(\frac{-\pi}{2}\right)$

ii $\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)$

$= 0 - i$

$= -i$

Question 2

a $\sqrt{3}\,\text{cis}\left(\frac{3\pi}{4}\right) = \sqrt{3}e^{\frac{3i\pi}{4}}$

b $2\,\text{cis}\left(\frac{-\pi}{3}\right) = 2e^{\frac{-\pi}{3}}$

c $\frac{1}{2}\left[\cos\left(\frac{\pi}{5}\right) - i\sin\left(\frac{\pi}{5}\right)\right] = \frac{1}{2}\left[\cos\left(\frac{-\pi}{5}\right) + i\sin\left(\frac{-\pi}{5}\right)\right]$

$= \frac{1}{2}e^{\frac{-\pi}{5}}$

d $\sqrt{3}\,\text{cis}\left(\frac{7\pi}{6}\right) = \sqrt{3}\,\text{cis}\left(\frac{-5\pi}{6}\right) = \sqrt{3}e^{\frac{-5\pi}{6}}$

e $6\,\text{cis}(1) = 6e^{i}$

f $4-4i$

$r=\sqrt{(4)^2+(4)^2}=\sqrt{16+16}=4\sqrt{2}$

$\theta=\tan^{-1}\left(\frac{-4}{4}\right)=\tan^{-1}(-1)=\frac{-\pi}{4}$

$4\sqrt{2}\,\text{cis}\left(\frac{-\pi}{4}\right)=4\sqrt{2}e^{\frac{-\pi}{4}}$

g $-\sqrt{3}+i$

$r=\sqrt{\left(\sqrt{3}\right)^2+(1)^2}=\sqrt{3+1}=2$

$\theta=\tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)=\frac{5\pi}{6}$

$2\,\text{cis}\left(\frac{5\pi}{6}\right)=2e^{\frac{i5\pi}{6}}$

h $\frac{-1+i\sqrt{3}}{2}$

$r=\sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{1}=1$

$\theta=\tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)=\tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)=\frac{2\pi}{3}$

$\text{cis}\left(\frac{2\pi}{3}\right)=e^{\frac{i2\pi}{3}}$

i $i=e^{\frac{i\pi}{2}}$

j $\frac{1}{2}=\frac{1}{2}e^{0i}=\frac{1}{2}e^{2\pi i}$

Exercise 1.07 Applying Euler's formula

Question 1

a $\text{cis}\left(\frac{\pi}{6}\right)\text{cis}\left(\frac{5\pi}{6}\right) = e^{\frac{i\pi}{6}}e^{\frac{i5\pi}{6}} = e^{\frac{i6\pi}{6}}$

$= e^{i\pi} = -1$

b $\left[3\,\text{cis}\left(\frac{\pi}{4}\right)\right]\left[\sqrt{2}\,\text{cis}\left(\frac{\pi}{3}\right)\right] = 3e^{\frac{i\pi}{4}}\sqrt{2}e^{\frac{i\pi}{3}}$

$= 3\sqrt{2}e^{\frac{i7\pi}{12}}$

c $\dfrac{\text{cis}\left(\frac{-7\pi}{8}\right)}{\text{cis}\left(\frac{3\pi}{4}\right)} = \dfrac{e^{\frac{-7\pi}{8}}}{e^{\frac{i3\pi}{4}}} = e^{\frac{3i\pi}{8}}$

d $\dfrac{-5\,\text{cis}\left(\frac{\pi}{2}\right)}{\sqrt{5}\,\text{cis}\left(\frac{2\pi}{3}\right)} = \dfrac{-5e^{\frac{i\pi}{2}}}{\sqrt{5}e^{\frac{i2\pi}{3}}} = \dfrac{5e^{-\frac{i\pi}{2}}}{\sqrt{5}e^{\frac{i2\pi}{3}}} = \sqrt{5}e^{\frac{5i\pi}{6}}$

Question 2

a $\text{cis}(\theta_1)\text{cis}(\theta_2) = e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

b $\dfrac{\text{cis}(\theta_1)}{\text{cis}(\theta_2)} = \dfrac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i(\theta_1-\theta_2)}$

c $\left\{\frac{1}{2}\left[\cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right)\right]\right\}^2 = \frac{1}{4}\left[\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right]^2 = \frac{1}{4}\left(e^{-\frac{3\pi}{4}}\right)^2$

$= \frac{1}{4}e^{\frac{-i3\pi}{2}} = \frac{1}{4}e^{\frac{i\pi}{2}}$

d $\left[\sqrt{2}\,\text{cis}\left(\frac{2\pi}{5}\right)\right]^{-1} = \left(\sqrt{2}e^{\frac{-i2\pi}{5}}\right)^{-} = \frac{1}{\sqrt{2}}e^{\frac{-i2\pi}{5}}$

e $\dfrac{\sqrt{10}\left[\cos(2\alpha) - i\sin(2\alpha)\right]}{\sqrt{2}\left[\cos(5\lambda) + i\sin(5\lambda)\right]} = \dfrac{\sqrt{10}\left[\cos(-2\alpha) + i\sin(-2\alpha)\right]}{\sqrt{2}\left[\cos(5\lambda) + i\sin(5\lambda)\right]} = \dfrac{\sqrt{10}e^{-2\alpha}}{\sqrt{2}e^{i5\lambda}}$

$= \sqrt{5}e^{-\ 2\alpha+5\lambda)}$

f $(-2+2i)\left(1-i\sqrt{3}\right) = 2\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)2\,\text{cis}\left(\frac{-\pi}{3}\right) = 2\sqrt{2}e^{\frac{i3\pi}{4}}\,2e^{\frac{-\pi}{3}} = 4\sqrt{2}e^{\frac{i5\pi}{12}}$

g $i\left(\frac{-\sqrt{3}+i}{2}\right) = e^{\frac{-\pi}{2}}2e^{\frac{i5\pi}{6}} = e^{\frac{i2\pi}{6}} = e^{\frac{i\pi}{3}}$

h

$$r=\sqrt{\left(\frac{2\sqrt{2}}{5}\right)^2+\left(\frac{2\sqrt{2}}{5}\right)^2}=\sqrt{\frac{8}{25}+\frac{8}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

$$\theta=\tan^{-1}\left(\frac{\frac{2\sqrt{2}}{5}}{\frac{2\sqrt{2}}{5}}\right)=\frac{\pi}{4}$$

$$\frac{2\sqrt{2}+i2\sqrt{2}}{5}=\frac{4}{5}e^{\frac{i\pi}{4}}$$

Question 3

$z_1=e^{i\theta_1} \quad z_2=e^{i\theta_2}$

a

$$\arg\left(z_1z_2\right)=\arg\left(e^{i\theta_1}e^{i\theta_2}\right)=\arg\left(e^{i\theta_1+i\theta_2}\right)=\arg\left(e^{i(\theta_1+\theta_2)}\right)=\theta_1+\theta_2$$
$$=\arg\left(z_1\right)+\arg\left(z_2\right)$$

b

$$\arg\left(\frac{z}{z_2}\right)=\arg\left(\frac{e^{\theta}}{e^{\theta_2}}\right)=\arg\left(e^{i\theta_1-i\theta_2}\right)=\arg\left(e^{i(\theta_1-\theta_2)}\right)=\theta_1-\theta_2$$
$$=\arg\left(z_1\right)-\arg\left(z_2\right)$$

Question 4

$e^{\theta}=\cos\theta+i\sin\theta$

a

$$e^{-\theta}=e^{i(-\theta)}$$
$$=\cos(-\theta)+i\sin(-\theta)$$
$$=\cos\theta-i\sin\theta$$

b **i**

$$e^{i\theta}-e^{-i\theta}=\cos\theta+i\sin\theta-\left(\cos\theta-i\sin\theta\right)$$
$$=i2\sin\theta$$
$$i2\sin\theta=e^{i\theta}-e^{-i\theta}$$
$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$

ii

$$e^{i\theta}+e^{-i\theta}=\cos\theta+i\sin\theta+\left(\cos\theta-i\sin\theta\right)$$
$$=2\cos\theta$$
$$2\cos\theta=e^{i\theta}+e^{-i\theta}$$
$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$$

Test yourself 1

Question 1

a $\sqrt{-25} = 5i$

b $\sqrt{-18} = i3\sqrt{2}$

c $\sqrt{-\frac{8}{9}} = \frac{i2\sqrt{2}}{3}$

Question 2

a $i^5 = i$

b $\sqrt{(-4)^2 - 4(2)(7)} = \sqrt{16-56} = \sqrt{-40} = i2\sqrt{10}$

c $\frac{1}{i^6} = \frac{1}{i^2} = \frac{1}{-1} = -1$

d $i^{91} = i^3 \times i^{88} = -i$

e $\frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

Question 3

a $x^2 + 49 = 0$

$x^2 = -49$

$x = \pm 7i$

b $(x + 3)^2 + 4 = 0$

$(x + 3)^2 = -4$

$x + 3 = \pm 2i$

$x = -3 \pm 2i$

Question 4

a

$$x^2-4x+9=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{4\pm\sqrt{(-4)^2-4\times1\times9}}{2\times1}=\frac{4\pm\sqrt{-20}}{2}=\frac{4\pm i2\sqrt{5}}{2}$$
$$x=2\pm i\sqrt{5}$$

b

$$x^2+6x+15=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-6\pm\sqrt{(6)^2-4\times1\times15}}{2\times1}=\frac{-6\pm\sqrt{-24}}{2}=\frac{-6\pm i2\sqrt{6}}{2}$$
$$x=-3\pm i\sqrt{6}$$

c

$$2x^2+3x+9=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-3\pm\sqrt{(3)^2-4\times2\times9}}{2\times1}=\frac{-3\pm\sqrt{-63}}{4}=\frac{-3\pm i3\sqrt{7}}{4}$$

Question 5

a

$$x^2-2x+2=0$$
$$x^2-2x+1=-1$$
$$(x-1)^2=-1$$
$$(x-1)=\pm i$$
$$x=1\pm i$$

c

$$x^2-x+3=0$$
$$x^2-x+\frac{1}{4}=\frac{1}{4}-3$$
$$\left(x-\frac{1}{2}\right)^2=-\frac{11}{4}$$
$$x-\frac{1}{2}=\pm\frac{i\sqrt{11}}{2}$$
$$x=\frac{1}{2}\pm\frac{i\sqrt{11}}{2}$$

b

$$x^2+8x+20=0$$
$$x^2+8x+16=-4$$
$$(x+4)^2=-4$$
$$x+4=\pm 2i$$
$$x=-4\pm 2i$$

Question 6

a

$$x^2+4x+8=0$$
$$x^2+4x+4+4=0$$
$$(x+2)^2-(2i)^2=0$$
$$(x+2+2i)(x+2-2i)=0$$
$$x=-2-2i,-2+2i$$

b

$$x^2-8x+25=0$$
$$x^2-8x+16+9=0$$
$$(x-4)^2-(3i)^2=0$$
$$(x-4+3i)(x-4-3i)=0$$
$$x=4-3i,4+3i$$

c

$$x^2+10x+41=0$$
$$x^2+10x+25+16=0$$
$$(x+5)^2-(4i)^2=0$$
$$(x+5+4i)(x+5-4i)=0$$
$$x=-5-4i,-5+4i$$

Question 7

a

$$z=\frac{\sqrt{3}-2i}{4}$$
$$\mathrm{Re}(z)=\frac{\sqrt{3}}{4},\quad \mathrm{Im}(z)=\frac{-1}{2}$$

b

$$z=(-3-7i)+(5-2i)=2-9i$$
$$\mathrm{Re}(z)=2,\quad \mathrm{Im}(z)=-9$$

c

$$z=\frac{(4x-3iy)+(x+iy)}{x^2+y^2}=\frac{5x-2iy}{x^2+y^2}$$
$$\mathrm{Re}(z)=\frac{5x}{x^2+y^2},\quad \mathrm{Im}(z)=\frac{-2y}{x^2+y^2}$$

Question 8

a $z=-6+11i,\ \overline{z}=-6-11i$

b

$$w=\frac{3-i\sqrt{2}}{2}\quad \overline{w}=\frac{3+i\sqrt{2}}{2}$$

c

$$u=\frac{-a+2i+ai-7b}{a^2+b^2}=\frac{-a-7b+i(2+a)}{a^2+b^2}$$
$$\overline{u}=\frac{-a-7b-i(2+a)}{a^2+b^2}$$

Question 9

$z = p - 3iq \qquad p, q \in \mathbb{R}$

a $\quad z\overline{z} = (p - 3iq)(p + 3iq) = p^2 + 9q^2 \in \mathbb{R}$

b $\quad z + \overline{z} = (p - 3iq) + (p + 3iq) = 2p \in \mathbb{R}$

c $\quad (\text{Re}(z))^2 \text{ and } (\text{Im}(z))^2 \in \mathbb{R}^+, \ \therefore \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2} \in \mathbb{R}^+$

Question 10

a $4x - 6i - 3yi + 2 = 0$

Equating real and imaginary parts

$4x - 12 = 0$

$4x = 12$

$x = 3$

$-6 - 3y = 0$

$-3y = 6$

$y = -2$

$x = 3, y = -2$

b $5x + 2xi + i - 3y + yi + 20 - 4i = 0$

Equating real and imaginary parts

$5x - 3y + 20 = 0$

$2x + y - 3 = 0$

$\Rightarrow y = 3 - 2x$

$5x - 3(3 - 2x) + 20 = 0$

$5x - 9 + 6x + 20 = 0$

$11x = -11$

$x = -1$

$y = 3 - 2x = 3 - 2(-1) = 5$

$x = -1, y = 5$

Question 11

$v = \dfrac{2x + 2yi - 5 + 3ix - 2y + 7i}{x^2 + y^2}$

For v to be real

$\dfrac{2yi + 3ix + 7i}{x^2 + y^2} = 0$

$\Rightarrow 2yi + 3ix + 7i = 0$

$3x = -7 - 2y$

$x = \dfrac{-7 - 2y}{3}$

Question 12

a $8i+5-4i+10=15+4i$

b $-3(2-7i)+2i(6-i)$
$=-6+21i+12i+2$
$=-4+33i$

c $(1-3i)(4+9i)=4+9i-12i+27$
$=31-3i$

d $(2-5i)^2-(-3+4i)(-3-4i)$
$=4-20i-25-(9+16)$
$=-46-20i$

Question 13

a $\alpha=1-i\sqrt{2},\ \beta=1+i\sqrt{2}$
$\alpha+\beta=1-i\sqrt{2}+1+i\sqrt{2}=2$
$\alpha\beta=(1-i\sqrt{2})(1+i\sqrt{2})=1+2=3$
$x^2-2x+3=0$

b $\alpha=-3-5i,\ \beta=-3+5i$
$\alpha+\beta=-3-5i+-3+5i=-6$
$\alpha\beta=(-3-5i)(-3+5i)=9+25=34$
$x^2+6x+34=0$

c $\alpha=\sqrt{7}+3i,\ \beta=\sqrt{7}-3i$
$\alpha+\beta=\sqrt{7}+3i+\sqrt{7}-3i=2\sqrt{7}$
$\alpha\beta=(\sqrt{7}+3i)(\sqrt{7}-3i)=7+9=16$
$x^2-2\sqrt{7}x+16=0$

d $\alpha=\frac{-1}{2}+\frac{i\sqrt{3}}{2}\quad \beta=\frac{-1}{2}-\frac{i\sqrt{3}}{2}$
$\alpha+\beta=\frac{-1}{2}+\frac{i\sqrt{3}}{2}+\frac{-1}{2}-\frac{i\sqrt{3}}{2}=-1$
$\alpha\beta=\left(\frac{-1}{2}+\frac{i\sqrt{3}}{2}\right)\left(\frac{-1}{2}-\frac{i\sqrt{3}}{2}\right)=\frac{1}{4}+\frac{3}{4}=1$
$x^2+x+1=0$

Question 14

a $\dfrac{1}{1-2i}=\dfrac{1}{1-2i}\times\dfrac{1+2i}{1+2i}=\dfrac{1+2i}{4+1}=\dfrac{1+2i}{5}$

b $\dfrac{\sqrt{2}-i\sqrt{2}}{1+i}=\dfrac{\sqrt{2}-i\sqrt{2}}{1+i}\times\dfrac{1-i}{1-i}=\dfrac{\sqrt{2}-i\sqrt{2}-i\sqrt{2}-\sqrt{2}}{1+1}=\dfrac{-i2\sqrt{2}}{2}=-i\sqrt{2}$

c $\dfrac{1}{2\sqrt{3}+i}-\dfrac{1}{5+i}=\dfrac{1}{2\sqrt{3}+i}\times\dfrac{2\sqrt{3}-i}{2\sqrt{3}-i}-\dfrac{1}{5+i}\times\dfrac{5-i}{5-i}=\dfrac{2\sqrt{3}-i}{13}-\dfrac{5-i}{26}=\dfrac{4\sqrt{3}-2i}{26}-\dfrac{5-i}{26}$

$=\dfrac{4\sqrt{3}-5-i}{26}$

d $\dfrac{\sqrt{3}+2i}{\sqrt{3}-2i}=\dfrac{\sqrt{3}+2i}{\sqrt{3}-2i}\times\dfrac{\sqrt{3}+2i}{\sqrt{3}+2i}=\dfrac{3-4+i4\sqrt{3}}{3+4}=\dfrac{-1+i4\sqrt{3}}{7}$

Question 15

a Let $3-4i=(x+iy)^2$

$3-4i=x^2+2xyi-y^2$

Equating real and imaginary parts

$3=x^2-y^2,\quad -4=2xy$

$y=\dfrac{-2}{x}$

$3=x^2-\left(\dfrac{-2}{x}\right)^2$

$3x^2=x^4-4$

$x^4-3x^2-4=0$

$\left(x^2-4\right)\left(x^2+1\right)=0$

As $x\in\mathbb{R}$, $x^2+1=0$ has no real solutions

$x^2-4=0\quad\Rightarrow\quad x=\pm2$

$y=\dfrac{-2}{x}$

$y=\mp1$

$x=2, y=-1\qquad x=-2, y=1$

$\sqrt{3-4i}=\pm\left(2-i\right)$

b Let $8+6i=(x+iy)^2$

$8+6i=x^2+2xyi-y^2$

Equating real and imaginary parts

$8=x^2-y^2,\quad 6=2xy$

$y=\dfrac{3}{x}$

$8=x^2-\left(\dfrac{3}{x}\right)^2$

$8x^2=x^4-9$

$x^4-8x^2-9=0$

$\left(x^2-9\right)\left(x^2+1\right)=0$

As $x\in\mathbb{R}$, $x^2+1=0$ has no real solutions

$x^2-9=0\quad\Rightarrow\quad x=\pm3$

$y=\dfrac{3}{x}$

$y=\pm1$

$x=3, y=1\qquad x=-3, y=-1$

$\sqrt{6+8i}=\pm\left(3+i\right)$

c Let $15 - 8i = (x + iy)^2$

$15 - 8i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$15 = x^2 - y^2$, $-8 = 2xy$

$$y = \frac{-4}{x}$$

$$15 = x^2 - \left(\frac{-4}{x}\right)^2$$

$$15x^2 = x^4 - 16$$

$$x^4 - 15x^2 - 16 = 0$$

$$\left(x^2 - 16\right)\left(x^2 + 1\right) = 0$$

As $x \in \mathbb{R}$, $x^2 + 1 = 0$ has no real solutions

$$x^2 - 16 = 0 \quad \Rightarrow \quad x = \pm 4$$

$$y = \frac{-4}{x}$$

$$y = \mp 1$$

$$x = 4, y = -1 \qquad x = -4, y = 1$$

$$\sqrt{15 - 8i} = \pm\left(4 - i\right)$$

d Let $12 + 5i = (x + iy)^2$

$12 + 5i = x^2 + 2xyi - y^2$

Equating real and imaginary parts

$12 = x^2 - y^2$, $5 = 2xy$

$$y = \frac{5}{2x}$$

$$12 = x^2 - \left(\frac{5}{2x}\right)^2$$

$$12x^2 = x^4 - \frac{25}{4}$$

$$4x^4 - 48x^2 - 25 = 0$$

$$\left(2x^2 - 25\right)\left(2x^2 + 1\right) = 0$$

As $x \in \mathbb{R}$, $2x^2 + 1 = 0$ has no real solutions

$$2x^2 - 25 = 0$$

$$x^2 = \frac{25}{2} \quad \Rightarrow \quad x = \pm\frac{5}{\sqrt{2}}$$

$$y = \frac{5}{2x}$$

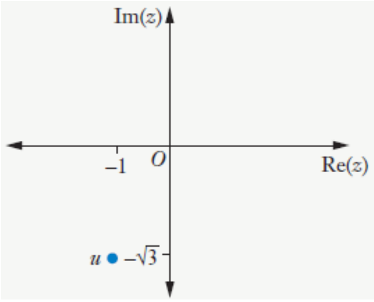
$$y = \pm\frac{1}{\sqrt{2}}$$

$$x = \frac{5}{\sqrt{2}}, y = \frac{1}{\sqrt{2}} \qquad x = -\frac{5}{\sqrt{2}}, y = -\frac{1}{\sqrt{2}}$$

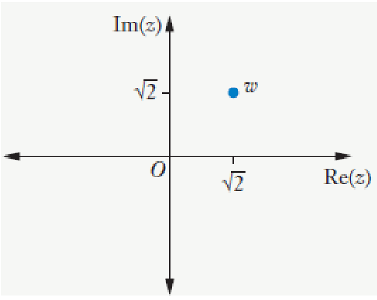
$$\sqrt{12 + 5i} = \pm\left(\frac{5}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

Question 16

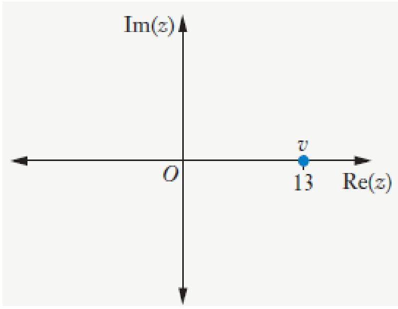
a $u = -1 - i\sqrt{3}$



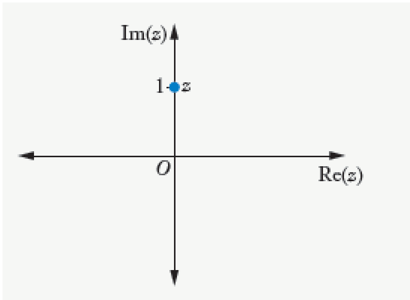
b $u = \sqrt{2} + i\sqrt{2}$



c $v = (2 - 3i)(2 + 3i) = 4 + 9 = 13$



d $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{1+2i-1}{1+1} = \frac{2i}{2} = i$



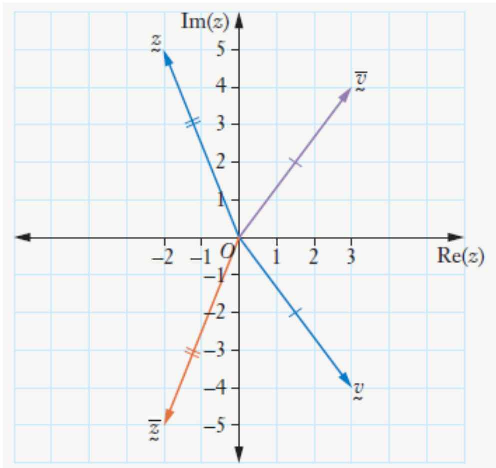
Question 17

$z = -2 + 5i$

$\overline{z} = -2 - 5i$

$v = 3 - 4i$

$\overline{v} = 3 + 4i$



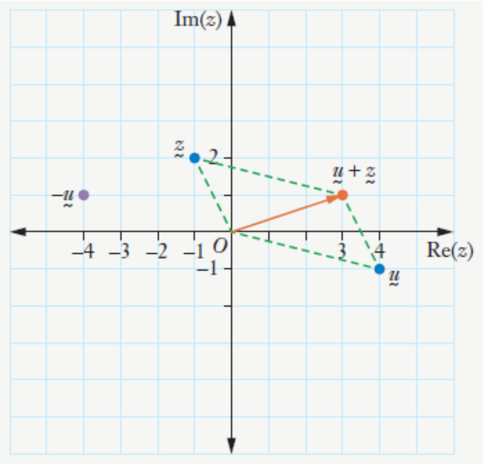
Question 18

$z = -1 + 2i$

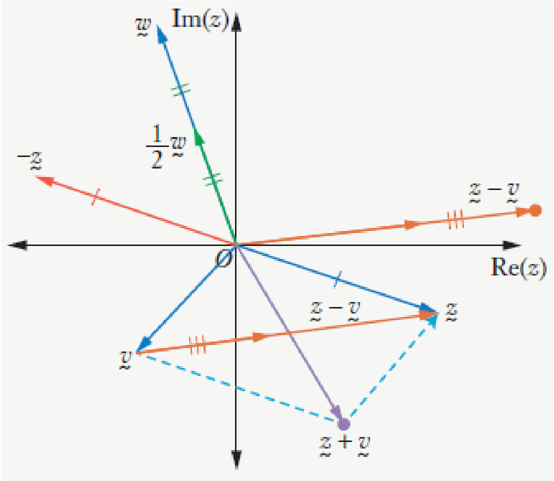
$u = 4 - i$

$-u = -4 + i$

$$u + z = 4 - i + (-1 + 2i)$$
$$= 3 + i$$



Question 19



Question 20

a

$z=\sqrt{2}-i\sqrt{2}$

$r=\sqrt{\left(\sqrt{2}\right)^2+\left(-\sqrt{2}\right)^2}=2$

$\theta=\tan^{-1}\left(\frac{-\sqrt{2}}{\sqrt{2}}\right)$

$\theta=-\frac{\pi}{4}$ $\quad\theta$ is in the fourth quadrant

$\arg(z)=-\frac{\pi}{4}$

b

$z=\sqrt{3}+i$

$r=\sqrt{\left(\sqrt{3}\right)^2+(1)^2}=2$

$\theta=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\theta=\frac{\pi}{6}$ $\quad\theta$ is in the first quadrant

$\arg(z)=\frac{\pi}{6}$

c

$z=-2+i2\sqrt{3}$

$r=\sqrt{(-2)^2+\left(2\sqrt{3}\right)^2}=\sqrt{4+12}=4$

$\theta=\tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right)=\tan^{-1}\left(-\sqrt{3}\right)$

$\theta=\frac{2\pi}{3}$ $\quad\theta$ is in the second quadrant

$\arg(z)=\frac{2\pi}{3}$

d

$z=\frac{-1-i}{2}$

$r=\sqrt{\left(\frac{-1}{2}\right)^2+\left(\frac{-1}{2}\right)^2}=\sqrt{\frac{2}{4}}=\frac{\sqrt{2}}{2}$

$\theta=\tan^{-1}\left(\frac{\frac{-1}{2}}{\frac{-1}{2}}\right)=\tan^{-1}(1)$

$\theta=-\frac{3\pi}{4}$ $\quad\theta$ is in the third quadrant

$\arg(z)=-\frac{3\pi}{4}$

Question 21

a

$$z = \frac{-\sqrt{3}+i}{2}$$

$$r = \sqrt{\left(\frac{-\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = \sqrt{1} = 1$$

$$\theta = \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{-\sqrt{3}}{2}}\right) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

$$\theta = \frac{5\pi}{6} \qquad \theta \text{ is in the third quadrant}$$

$$z = \text{cis}\left(\frac{5\pi}{6}\right)$$

b

$$z = \sqrt{2}\left[\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)\right]$$

$$= \sqrt{2}\left[\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right]$$

$$= \sqrt{2}\,\text{cis}\left(-\frac{2\pi}{3}\right)$$

c

$$z = -\frac{1}{2}\left[\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right]$$

$$= -\frac{1}{2}\left[\cos\left(\frac{-5\pi}{6}\right) + i\sin\left(\frac{-5\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\text{cis}\left(\frac{\pi}{6}\right)$$

Question 22

a $$z = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = 2\times\frac{\sqrt{2}}{2} + i2\times\frac{\sqrt{2}}{2} = \sqrt{2} + i\sqrt{2}$$

b $$z = \frac{1}{2}\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{6}\right)\right] = \frac{1}{2}\times\frac{1}{2} + i\frac{1}{2}\times\frac{1}{2} = \frac{1}{4} + \frac{i}{4}$$

c $$z = \sqrt{2}\left[i\cos\left(\frac{5\pi}{6}\right) + \sin\left(\frac{5\pi}{6}\right)\right] = i\sqrt{2}\times\left(\frac{-\sqrt{3}}{2}\right) + \sqrt{2}\times\frac{1}{2} = \frac{\sqrt{2}}{2} - \frac{i\sqrt{6}}{2}$$

Question 23

$$z_1 = \frac{1}{\sqrt{3}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \quad z_2 = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

a

$$z_1 z_2 = \frac{1}{\sqrt{3}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\times 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{3}{\sqrt{3}}\left[\cos\left(\frac{\pi}{6}+\frac{3\pi}{4}\right) + i\sin\left(\frac{\pi}{6}+\frac{3\pi}{4}\right)\right]$$

$$= \sqrt{3}\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right]$$

$$= \sqrt{3}\,\text{cis}\left(\frac{11\pi}{12}\right)$$

b

$$\frac{z_1}{z_2} = \frac{\frac{1}{\sqrt{3}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)}{3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)} = \frac{1}{3\sqrt{3}}\left[\cos\left(\frac{3\pi}{4}-\frac{\pi}{6}\right) + i\sin\left(\frac{3\pi}{4}-\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{3\sqrt{3}}\left[\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right]$$

$$= \frac{1}{3\sqrt{3}}\,\text{cis}\left(\frac{7\pi}{12}\right)$$

c

$$\frac{z_1}{z_2} = \frac{1}{3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)} = \frac{1}{3}\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{3}\,\text{cis}\left(-\frac{\pi}{6}\right)$$

d

$$(z)^{11} = \left[\frac{1}{\sqrt{3}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^{11} = \left(\frac{1}{\sqrt{3}}\right)^{11}\left[\cos\left(11\times\frac{3\pi}{4}\right) + i\sin\left(11\times\frac{3\pi}{4}\right)\right]$$

$$= \frac{1}{243\sqrt{3}}\left[\cos\left(\frac{33\pi}{4}\right) + i\sin\left(\frac{33\pi}{4}\right)\right]$$

$$= \frac{1}{243\sqrt{3}}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{243\sqrt{3}}\,\text{cis}\left(\frac{\pi}{4}\right)$$

Question 24

a $z=\dfrac{\cos\theta+i\sin\theta}{\cos 2\beta+i\sin\theta}=\cos(\theta-2\beta)+i\sin(\theta-2\beta)$

$\arg z = \theta - 2\beta$

b
$$
\begin{aligned}
z &= (\cos 3\alpha - i\sin 3\alpha)(\cos 2\lambda + i\sin 2\lambda)\\
&= \left[\cos(-3\alpha)+i\sin(-3\alpha)\right](\cos 2\lambda + i\sin 2\lambda)\\
&= \cos(2\lambda-3\alpha)+i\sin(2\lambda-3\alpha)
\end{aligned}
$$

$\arg z = 2\lambda - 3\alpha$

c
$$
\begin{aligned}
z &= \frac{\left[\cos\left(\frac{\delta}{2}\right)+i\sin\left(\frac{\delta}{2}\right)\right]\left[\cos\left(\frac{\alpha}{4}\right)-i\sin\left(\frac{\alpha}{4}\right)\right]}{\cos(2\phi)-i\sin(2\phi)}\\
&= \frac{\left[\cos\left(\frac{\delta}{2}\right)+i\sin\left(\frac{\delta}{2}\right)\right]\left[\cos\left(-\frac{\alpha}{4}\right)+i\sin\left(-\frac{\alpha}{4}\right)\right]}{\cos(-2\phi)+i\sin(-2\phi)}\\
&= \cos\left(\frac{\delta}{2}-\frac{\alpha}{4}+2\phi\right)+i\sin\left(\frac{\delta}{2}-\frac{\alpha}{4}+2\phi\right)
\end{aligned}
$$

$\arg z = \dfrac{\delta}{2}-\dfrac{\alpha}{4}+2\phi$

d
$$
\begin{aligned}
z &= \left[\cos(3\varepsilon)-i\sin(3\varepsilon)\right]^{-} = \cos(-3\varepsilon)-i\sin(-3\varepsilon)\\
&= \cos(3\varepsilon)+i\sin(3\varepsilon)
\end{aligned}
$$

$\arg z = 3\varepsilon$

Question 25

a $z=(1+7i)(2-3i)$

$|z|=|(1+7i)(2-3i)|=|(1+7i)||(2-3i)|=\sqrt{1^2+7^2}\times\sqrt{2^2+(-3)^2}$

$=\sqrt{50}\times\sqrt{13}=5\sqrt{26}$

b $z=\dfrac{2+i}{\sqrt{3}-4i}$

$|z|=\left|\dfrac{2+i}{\sqrt{3}-4i}\right|=\dfrac{|(2+i)|}{|(\sqrt{3}-4i)|}=\dfrac{\sqrt{2^2+1^2}}{\sqrt{(\sqrt{3})^2+(-4)^2}}=\dfrac{\sqrt{5}}{\sqrt{19}}=\sqrt{\dfrac{5}{19}}$

c $z=\dfrac{1}{(2-i)^2}$

$|z|=\left|\dfrac{1}{(2-i)^2}\right|=\dfrac{1}{|(2-i)|^2}=\dfrac{1}{\left(\sqrt{(2)^2+(-1)^2}\right)^2}=\dfrac{1}{5}$

Question 26

$z=2\operatorname{cis}\left(\dfrac{\pi}{4}\right),\quad w=4\operatorname{cis}\left(\dfrac{\pi}{3}\right)$

a $z=2\operatorname{cis}\left(\dfrac{\pi}{4}\right)=2\cos\left(\dfrac{\pi}{4}\right)+2i\sin\left(\dfrac{\pi}{4}\right)$

$=2\times\dfrac{\sqrt{2}}{2}+2\times i\dfrac{\sqrt{2}}{2}=\sqrt{2}+i\sqrt{2}$

$w=4\operatorname{cis}\left(\dfrac{\pi}{3}\right)=4\cos\left(\dfrac{\pi}{3}\right)+4i\sin\left(\dfrac{\pi}{3}\right)$

$=4\times\dfrac{1}{2}+4\times i\dfrac{\sqrt{3}}{2}=2+i2\sqrt{3}$

b $zw=2\operatorname{cis}\left(\dfrac{\pi}{4}\right)\times4\operatorname{cis}\left(\dfrac{\pi}{3}\right)=8\operatorname{cis}\left(\dfrac{\pi}{4}+\dfrac{\pi}{3}\right)=8\operatorname{cis}\left(\dfrac{7\pi}{12}\right)$

$zw=\left(\sqrt{2}+i\sqrt{2}\right)\left(2+i2\sqrt{3}\right)$

$=2\sqrt{2}+i2\sqrt{6}+i2\sqrt{2}-2\sqrt{6}$

$=2\left(\sqrt{2}-\sqrt{6}\right)+2i\left(\sqrt{2}+\sqrt{6}\right)$

c Equating real parts

$$8\cos\left(\frac{7\pi}{12}\right) = 2\left(\sqrt{2}-\sqrt{6}\right)$$
$$\cos\left(\frac{7\pi}{12}\right) = \frac{2\left(\sqrt{2}-\sqrt{6}\right)}{8} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

Equating imaginary parts

$$8\sin\left(\frac{7\pi}{12}\right) = 2\left(\sqrt{2}+\sqrt{6}\right)$$
$$\sin\left(\frac{7\pi}{12}\right) = \frac{2\left(\sqrt{2}+\sqrt{6}\right)}{8} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

Question 27

a $z = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} = e^{\frac{3i\pi}{5}}$

b $z = \sqrt{2}\left(\cos 3 - i\sin 3\right) = \sqrt{2}\left(\cos(-3) + i\sin(-3)\right) = \sqrt{2}e^{-3i}$

Question 28

a $z = e^{\frac{\pi i}{4}} = \text{cis}\left(\frac{\pi}{4}\right)$

b $z = \frac{e^{-2i}}{2} = \frac{1}{2}\text{cis}(-2)$

Question 29

$$z_1 = r_1 e^{i\theta_1}$$
$$z_2 = r_2 e^{i\theta_2}$$
$$\arg\left(\frac{z_1}{z_2}\right) = \arg\left(\frac{r_1 e^{\theta}}{r_2 e^{\theta_2}}\right) = \arg\left(\frac{r_1}{r_2} e^{\theta_1 - \theta_2}\right) = \theta_1 - \theta_2$$
$$= \arg(z_1) - \arg(z_2)$$

Question 30

a $e^{\frac{5\pi i}{6}} \times e^{-\frac{7\pi i}{6}} = e^{-\frac{2\pi i}{6}} = e^{-\frac{\pi i}{3}}$

b $\left(e^{-\frac{2\pi}{3}}\right)^{12} = e^{12\times\left(-\frac{2\pi}{3}\right)} = e^{-8\pi} = e^{0} = 1$

Question 31

$$\frac{\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}}{\cos\left(-\frac{3\pi}{14}\right) + i\sin\left(-\frac{3\pi}{14}\right)} = \frac{e^{\frac{2i\pi}{7}}}{e^{\frac{-3\pi}{14}}} = e^{\left(\frac{2i\pi}{7}+\frac{3i\pi}{14}\right)} = e^{\frac{7i\pi}{14}} = e^{\frac{i\pi}{2}} = i$$

Question 32

Let $z = re^{i\theta}$

$$(z)^{-n} = \left(re^{\theta}\right)^{-n} = r^{-n}e^{-ni\theta}$$

$$\arg\left(z^{-n}\right) = \arg\left(r^{-n}e^{-ni\theta}\right) = -n\theta = -n\arg z$$

MATHS IN FOCUS 12
MATHEMATICS EXTENSION 2

WORKED SOLUTIONS

Chapter 2: Mathematical proof

Exercise 2.01 The language of proof

Question 1

a P: There are crumbs.

Q: ants will come.

$P \Rightarrow Q$

b P: A quadrilateral has equal diagonals.

Q: The quadrilateral is a square.

$P \Rightarrow Q$

c P: People are unemployed.

Q: People are bored.

$P \Rightarrow Q$

Question 2

a If you go skiing, you live in Cooma.

b If you have friends, you like maths.

c If you can debate, you are a politician.

d If an animal is a bird, it can fly.

Question 3

a If you eat meat, then you ae a carnivore.

False – Omnivores

b If you are on a boat, you are seasick.

False – There are lots of people on a boat who are not seasick.

c If a shape has equal sides, it is a square.

False – Rhombus or any regular polygon of side $n > 4$

d If an animal can sting, it is a honeybee.

False – Wasps

Question 4

a If $x - 5 = 4$, then $x = 9$

Converse

If $x = 9$, then $x - 5 = 4$

True, so it is an equivalence.

$\therefore\ x - 5 = 4$ iff $x = 9$

b If a quadrilateral has diagonals that are perpendicular, then it is a rhombus.

False, could be a kite.

c If $\frac{1}{a} < \frac{1}{b}$ then $a > b > 0$

False, e.g. $a = -2$, $b = -10$

d If you passed a driving test, then you have a driver's licence.

True, so it is an equivalence.

$\therefore$ you have a driver's licence iff you passed a driving test.

Question 5

a It is not white.

b I do not know everything.

c Not all fish swim in the ocean.

d Not all babies are cute.

e There is not more than five.

f There are some.

g Someone passed the test.

h Teachers are not mean.

i The potatoes are more than or equal to 3 kg.

j Cassie is not small.

Question 6

a If you are not rich, you do not live in a mansion.

b If you do not have boots, you are not in the army.

c If you are not wise, you are not old.

d If $x^2 \neq 9$, then $x \neq 3$

e If an animal does not have four legs, it is not a horse.

f If you are not superior, you are not a woman.

Question 7

a If there is global warming the water is rising.

b If you have accidents then you speed.

c If the animals die there is a drought.

d If a number is a fraction it is rational.

e If Sam is not lazy then he will pass his exams.

f If a number has a square root it is not negative.

Question 8

a If $\frac{1}{n} < \frac{1}{n+1}$, then $n < 1$.

b If the gradient of a line is not zero, then the line is not horizontal.

c If the bulldust is not red, then it is not in the outback.

d If they are not mammals, then they are not blue whales.

Question 9

a If you exercise your heart rate increases.

True

Contrapositive

If your heart rate does not increase, you do not exercise.

True

b If a plant does not get water then it dies.

True

Contrapositive

If a plant does not die then it gets sufficient water.

True

c If a triangle is isosceles then it has two equal angles.

True

Contrapositive

If a triangle does not have two equal angles it is not isosceles.

True

d If a number is an integer then it is real.

True

Contrapositive

If a number is not real it is not an integer.

True

e If $x > 2$ then $x^2 > 4$

True

Contrapositive

If $x^2 \leq 4$ then $x \leq 2$

True

Question 10

a If an animal does not have a beak it is not a bird.

True

b If a quadrilateral is not a rhombus it does not have two pairs of opposite angles equal.

False – rectangle

c If an animal does not have fins, then it is not a fish.

True

d If $x^2 > 25$ then $x > 5$.

False: $x = -15$

e If a number is not prime then it is not odd.

False: 15

Question 11

This is not the contrapositive. She did negate the statement but did not reverse their order.

Question 12

D

If ~$A \Rightarrow$ B, then the contrapositive is $B \Rightarrow A$.

Question 13

a $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})\ x < y$

b $(\forall x \in \mathbb{Q})(\exists p \in \mathbb{Z})(\exists q \in \mathbb{Z}, q \neq 0) : x = \frac{p}{q}$

c $(\forall a \in \mathbb{Z}, a \neq 0)(\exists b \in \mathbb{Q}) : b = \frac{1}{a}$

d $(\forall (x, y)\, and\, (w, v), x, y, w, v \in \mathbb{R})(\exists (c, d) \in \mathbb{R}) : x < c < w \text{ and } y < d < v$

e $(\forall x \in \mathbb{R}^+, x \geq 0)(\exists y \in \mathbb{R}^+, y \geq 0) : y = \sqrt{x}$

Question 14

a For all natural numbers *m*, there exists an integer *n* such than $n + m = 0$.

b For all integers *a* and *b* where *b* is non-zero, there exist rational numbers *p* and *q* such that $\frac{1}{a+b\sqrt{2}} = p + q\sqrt{2}$

Question 15

C

M iff *N* can be written as $N \Leftrightarrow M$.

Exercise 2.02 Proof by contradiction

Question 1

a Suppose $a, b \in \mathbb{Z}$ and $b \neq 0$, with a and b having no common factors such that $\frac{a}{b} = \sqrt{2}$.

$\frac{a}{b} = \sqrt{2} \Rightarrow \left(\frac{a}{b}\right)^2 = 2 \Rightarrow \frac{a^2}{b^2} = 2$

$\Rightarrow a^2 = 2b^2 \Rightarrow a^2$ is even

$\Rightarrow a$ is even

$\therefore$ we can write $a = 2m$.

$\frac{a^2}{b^2} = 2 \Rightarrow \frac{(2m)^2}{b^2} = 2 \Rightarrow \frac{4m^2}{b^2} = 2$

$\Rightarrow 4m^2 = 2b^2 \Rightarrow 2m^2 = b^2$

$\Rightarrow b^2$ is even $\Rightarrow b$ is even

$\therefore$ we can write $b = 2n$

This is a contradiction as both a and b have no common factors

$\therefore \sqrt{2} \notin \mathbb{Q}$

b Suppose $a, b \in \mathbb{Z}$ and $b \neq 0$, with a and b having no common factor such that $\frac{a}{b} = \sqrt{5}$.

$\frac{a}{b} = \sqrt{5} \Rightarrow \left(\frac{a}{b}\right)^2 = 5 \Rightarrow \frac{a^2}{b^2} = 5$

$\Rightarrow a^2 = 5b^2 \Rightarrow a^2$ is divisible by 5

$\Rightarrow a$ is divisible by 5

$\therefore$ we can write $a = 5m$

$\frac{a^2}{b^2} = 5 \Rightarrow \frac{(5m)^2}{b^2} = 5 \Rightarrow \frac{25m^2}{b^2} = 5$

$\Rightarrow 25m^2 = 5b^2 \Rightarrow 5m^2 = b^2$

$\Rightarrow b^2$ is divisible by 5

$\Rightarrow b$ is divisible by 5

$\therefore$ we can write $b = 5n$

This is a contradiction as both a and b have no commn factors.

$\therefore \sqrt{5} \notin \mathbb{Q}$

c Suppose $a, b \in \mathbb{Z}$ and $b \neq 0$, with a and b having no common factors such that $\frac{a}{b} = \sqrt{7}$.

$\frac{a}{b} = \sqrt{7} \Rightarrow \left(\frac{a}{b}\right)^2 = 7 \Rightarrow \frac{a^2}{b^2} = 7$

$\Rightarrow a^2 = 7b^2 \Rightarrow a^2$ is divisible by 7

$\Rightarrow a$ is divisible by 7

$\therefore$ we can write $a = 7m$

$\frac{a^2}{b^2} = 7 \Rightarrow \frac{(7m)^2}{b^2} = 7 \Rightarrow \frac{49m^2}{b^2} = 7$

$\Rightarrow 49m^2 = 7b^2 \Rightarrow 7m^2 = b^2$

$\Rightarrow b^2$ is divisible by 7

$\Rightarrow b$ is divisible by 7

$\therefore$ we can write $b = 7n$

This is a contradiction as both a and b have no common factors

$\therefore \sqrt{7} \notin \mathbb{Q}$

d Suppose $a, b \in \mathbb{Z}$ and $b \neq 0$, with a and b having no common factors such that $\frac{a}{b} = \sqrt[3]{2}$.

$\frac{a}{b} = \sqrt[3]{2} \Rightarrow \left(\frac{a}{b}\right)^3 = 2 \Rightarrow \frac{a^3}{b^3} = 2$

$\Rightarrow a^3 = 2b^3 \Rightarrow a^3$ is even

$\Rightarrow a$ is even

$\therefore$ we can write $a = 2m$

$\frac{a^3}{b^3} = 2 \Rightarrow \frac{(2m)^3}{b^3} = 2 \Rightarrow \frac{8m^3}{b^3} = 2$

$\Rightarrow 8m^3 = 2b^3 \Rightarrow 4m^3 = b^3$

$\Rightarrow b^3$ is even $\Rightarrow b$ is even

$\therefore$ we can write $b = 2n$.

This is a contradiction as both a and b have no commonfactors.

$\therefore \sqrt[3]{2} \notin \mathbb{Q}$

e Suppose $a,b\in\mathbb{Z}$ and $b\neq 0$, with a and b having no common factors such that $\frac{a}{b}=\sqrt[3]{5}$.

$\frac{a}{b}=\sqrt[3]{5}\Rightarrow\left(\frac{a}{b}\right)^3=5\Rightarrow\frac{a^3}{b^3}=5\Rightarrow a^3=5b^3\Rightarrow a^3$ is divisible by 5

$\Rightarrow a$ is divisible by 5

$\therefore$ we can write $a=5m$.

$\frac{a^3}{b^3}=5\Rightarrow\frac{(5m)^3}{b^3}=5\Rightarrow\frac{125m^3}{b^3}=5$

$\Rightarrow 125m^3=5b^3\Rightarrow 25m^3=b^3$

$\Rightarrow b^3$ is divisible by 5

$\Rightarrow b$ is divisible by 5

$\therefore$ we can write $b=5n$.

This is a contradiction as both a and b have no common factors.

$\therefore\sqrt[3]{5}\notin\mathbb{Q}$

Question 2

a Suppose $a,b\in\mathbb{Z}$ and $b\neq 0$, such that $\frac{a}{b}=\log_2 5$.

$\frac{a}{b}=\log_2 5\Rightarrow 2^{\frac{a}{b}}=5\Rightarrow 2^a=5^b$

LHS is always even and RHS is always odd.

An even number cannot be an odd number so it is a contradiction.

$\therefore\log_2 5\notin\mathbb{Q}$

b Suppose $a,b\in\mathbb{Z}$ and $b\neq 0$, such that $\frac{a}{b}=\log_2 7$.

$\frac{a}{b}=\log_2 7\Rightarrow 2^{\frac{a}{b}}=7\Rightarrow 2^a=7^b$

LHS is always even and RHS is always odd.

An even number cannot be an odd number so it is a contradiction.

$\therefore\log_2 7\notin\mathbb{Q}$

c Suppose $a,b\in\mathbb{Z}$ and $b\neq 0$, such that $\frac{a}{b}=\log_3 8$.

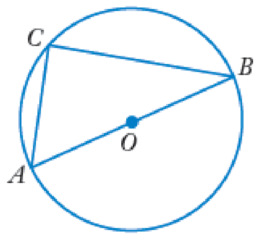
$\frac{a}{b}=\log_3 8\Rightarrow 3^{\frac{a}{b}}=8\Rightarrow 3^a=8^b$

LHS is always odd and RHS is always even.

An even number cannot be an odd number, so it is a contradiction.

$\therefore\log_3 8\notin\mathbb{Q}$

Question 3



Suppose $\angle ACB \neq 90°$

An angle subtended at the centre of the circle by an arc is twice any angle at the circumference on the same arc.

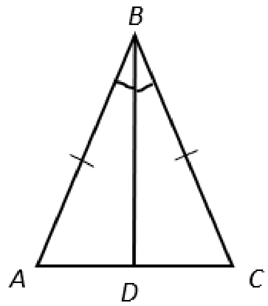
$\Rightarrow \angle AOB = 2\angle ACB$

But $2\angle ACB \neq 180°$, which is a contradiction.

$\therefore$ the angle in a semicircle is 90°.

Question 4

a Assume $AD \neq DC$



$AB = BC$ (Isosceles triangle has equal lengths)

$\angle ABD = \angle DBC$ (Angle ABC is bisected)

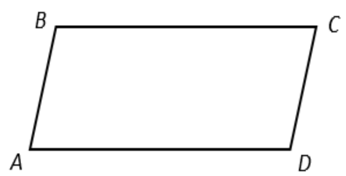
BD is common

$\therefore \triangle ABD \equiv \triangle BDC$ (by SAS)

This is a contradiction

$\therefore$ The bisector of the angle between the equal sides of an isosceles triangle bisects the third side.

b Assume $\angle A \neq \angle C$ and $\angle B \neq \angle D$



Suppose $\angle A = \alpha$

$\angle B = 180° - \alpha$ (Co-interior angles in a transversal of parallel lines are supplementary)

$\angle C = 180° - \angle B$ (Co-interior angles in a transversal of parallel lines are supplementary)

$= \alpha = \angle A$

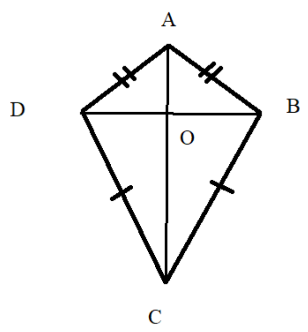
$\angle D = 180° - \angle C$ (Co-interior angles in a transversal of parallel lines are supplementary)

$= 180° - \alpha = \angle B$

Both of these are contradictory.

$\therefore$ Opposite angles of a parallelogram are equal.

c Suppose *ABCD* is a kite and that the diagonals do not intersect at right angles.



As *ABCD* is a kite *AC* bisects $\angle DAB$

$\angle DAC = \angle CAB$

$DA = AB$ (Properties of a kite)

AO is common

$\therefore \triangle DAO \equiv \triangle BAO$ (SAS)

$\therefore \angle DOA = \angle AOB$ (Corresponding angles of congruent triangles are equal)

As *DOB* is a straight line $\therefore \angle DOA + \angle AOB = 180$

$\therefore \angle DOA = \angle AOB = 90$

Which is a contradiction

$\therefore$ The diagonals of a kite intersect at right angles.

Question 5

a Assume there is a triangle with side 8, 15 and 17, which is not a right-angled triangle.

For a right-angled triangle with *c* as the hypotenuse, Pythagoras' theorem applies: $a^2 + b^2 = c^2$.

$\Rightarrow c^2 \neq a^2 + b^2$

$17^2 \neq 8^2 + 15^2$

$289 \neq 64 + 225$

$289 \neq 289$

Which is a contradiction, therefore a triangle with sides 8, 15, 17 is a right-angled triangle.

b Assume there is a triangle with side 4, 5 and 6, which is a right-angled triangle.

For a right-angled triangle with *c* as the hypotenuse, Pythagoras' theorem applies: $a^2 + b^2 = c^2$.

$\Rightarrow c^2 = a^2 + b^2$

$6^2 = 4^2 + 5^2$

$36 = 16 + 25$

$36 = 41$

Which is a contradiction, therefore a triangle with sides 4, 5, 6 is not a right-angled triangle.

c Suppose $a, b \in \mathbb{Z}$ and $b \neq 0,$ with a and b having no common factors.

Let $I = 5 + \sqrt{2} = \frac{5b + a}{b}$

$\therefore \sqrt{2} = \frac{a}{b}$

$\Rightarrow \left(\frac{a}{b}\right)^2 = 2$

$\Rightarrow \frac{a^2}{b^2} = 2$

$\Rightarrow a^2 = 2b^2$

$\Rightarrow a^2$ is even

$\Rightarrow a$ is even

$\therefore$ we can write $a = 2m$

$\frac{a^2}{b^2} = 2$

$\Rightarrow \frac{(2m)^2}{b^2} = 2$

$\Rightarrow \frac{4m^2}{b^2} = 2$

$\Rightarrow 4m^2 = 2b^2$

$\Rightarrow 2m^2 = b^2$

$\Rightarrow b^2$ is even

$\Rightarrow b$ is even

$\therefore$ we can write $b = 2n$

This is a contradiction as both a and b have no common actors.

$\therefore 5 + \sqrt{2} \notin \mathbb{Q}$

d Let $I = \sqrt{2}+\sqrt{3}$, then $I^2 = \left(\sqrt{2}+\sqrt{3}\right)^2 = 2+2\sqrt{6}+3 = 5+2\sqrt{6}$.

Suppose $a, b \in \mathbb{Z}$ and $b \neq 0$, with a and b having no common factors such that

$I^2 = 5+2\sqrt{6} = \dfrac{5b+2a}{b}$.

$\sqrt{6} = \dfrac{a}{b}$

$\Rightarrow \left(\dfrac{a}{b}\right)^2 = 6$

$\Rightarrow \dfrac{a^2}{b^2} = 6$

$\Rightarrow a^2 = 6b^2$

$\Rightarrow a^2$ is even

$\Rightarrow a$ is even

$\therefore$ we can write $a = 2m$

$\dfrac{a^2}{b^2} = 6 \Rightarrow \dfrac{(2m)^2}{b^2} = 6$

$\Rightarrow \dfrac{4m^2}{b^2} = 6$

$\Rightarrow 4m^2 = 6b^2$

$\Rightarrow 2m^2 = 3b^2$

$\Rightarrow b^2$ is even

$\Rightarrow b$ is even

$\therefore$ we can write $b = 2n$.

This is a contradiction as both a and b have no common factors.

$\therefore I^2 = 5+2\sqrt{6} \notin \mathbb{Q}$

If $I^2 \notin \mathbb{Q} \Rightarrow I \notin \mathbb{Q}$

e

Suppose $p, q \in \mathbb{Z}$ and $q \neq 0$, with p and q having no common factors such that

$a + b\sqrt{2} = \dfrac{qa + bp}{q}$.

$\therefore \dfrac{bp}{q} = b\sqrt{2}$

$\Rightarrow \dfrac{p}{q} = \sqrt{2}$

$\Rightarrow \left(\dfrac{p}{q}\right)^2 = 2$

$\Rightarrow \dfrac{p^2}{q^2} = 2$

$\Rightarrow p^2 = 2q^2$

$\Rightarrow p^2$ is even

$\Rightarrow p$ is even

$\therefore$ we can write $p = 2m$

$\dfrac{p^2}{q^2} = 2$

$\Rightarrow \dfrac{(2m)^2}{q^2} = 2$

$\Rightarrow \dfrac{4m^2}{q^2} = 2$

$\Rightarrow 4m^2 = 2q^2$

$\Rightarrow 2m^2 = q^2$

$\Rightarrow q^2$ is even

$\Rightarrow q$ is even

$\therefore$ we can write $q = 2n$

This is a contradiction as both p and q have no common factors.

$\therefore a + b\sqrt{2} \notin \mathbb{Q}$

This is a contradiction because $\sqrt{2} \notin \mathbb{Q}$.

Question 6

a Suppose $(\forall a, b \in \mathbb{N})(\exists p, q \in \mathbb{Q}) : \frac{1}{a+b\sqrt{2}} \neq p+q\sqrt{2}.$

$$\frac{1}{a+b\sqrt{2}} = \frac{1}{a+b\sqrt{2}} \times \frac{a-b\sqrt{2}}{a-b\sqrt{2}}$$

$$= \frac{a-b\sqrt{2}}{a^2-2b^2}$$

$$= \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}$$

but $\frac{a}{a^2-2b^2}, \frac{b}{a^2-2b^2} \in \mathbb{Q},$ which is a contradiction.

$\therefore (\forall a, b \in \mathbb{N})(\exists p, q \in \mathbb{Q}) : \frac{1}{a+b\sqrt{2}} = p+q\sqrt{2}$

b Suppose $(\forall b \in \mathbb{N})(\exists a \in \mathbb{Z}) : a+b \neq 0$

As $a \in \mathbb{N},$

$a+(-a)=0$

$-a \in \mathbb{Z}$

By setting $b=-a$

$a+b=0,$ which is a contradiction

$\therefore (\forall a \in \mathbb{N})(\exists b \in \mathbb{Z}) : a+b=0$

Exercise 2.03 Proof by counterexample

Question 1

a

$$(\forall n \in \mathbb{R})\ n^2 \geq n$$

Counterexample

$$n = \frac{1}{2}$$

$$n^2 = \frac{1}{4}$$

$$n^2 < n$$

b

$$(\forall n \in \mathbb{R}), n^2 + n \geq 0$$

Counterexample

$$n = -\frac{1}{2}$$

$$n^2 + n = \frac{1}{4} - \frac{1}{2}$$

$$= -\frac{1}{4} < 0$$

c All prime numbers are odd.

p = 2 is an even prime number.

d

$$(\forall x\ \ y \in \mathbb{Z})(n \in \mathbb{N})$$

$$x^n + y^n = (x+y)\left(x^{n-1} - x^{n-2}y + x^{n-3}y - \quad - x^2y^{n-2} + y^{n-1}\right)$$

Counterexample

$$n = 2$$

$$x^2 + y^2 \neq (x+y)(x-y)$$

e

$$(\forall x \in \mathbb{Z}), x + \frac{1}{x} \geq 2$$

Counterexample

$x = -1$

$x + \frac{1}{x}$

$= -1 + \frac{1}{-1}$

$= -2 < 2$

Question 2

a If $n^2 = 100$ then $n = 10$.

Counterexample

$n = -10$

b

$$(\forall x \in \mathbb{Z}^+) : x^3 - 6x^2 + 11x - 6 = 0$$

Counterexample

$x = 4$

$4^3 - 6 \times 4^2 + 11 \times 4 - 6 = 6 \neq 0$

c All lines that never meet are parallel.

Counterexample

Segments, secants, rays and skew lines (depending on definition and geometry being used)

d If an animal lays eggs then it is a bird.

Counterexample

Snakes or lizards.

Question 3

a If a quadrilateral has diagonals that are perpendicular then it is a square.

False – Kite

b If $p \le 3$ then $\frac{1}{p} \ge \frac{1}{3}$.

False

$p = -1$

$-1 < \frac{1}{3}$

c

$(\forall x, y \in \mathbb{R}):(x+y)^2 \ge x^2 + y^2$

False $x = 2, y = -1$

$(x+y)^2 = 1$

$x^2 + y^2 = 5$

d If $pq = rq$ then $r = p$.

False when $q = 0$. Then r and p can be different values.

e All rectangles are similar.

False

A rectangle 3 cm × 1 cm is not similar to a rectangle 4 cm × 1 cm.

Question 4

a The counterexample does show the statement is false.

b This does not show the statement is false. To show the statement is false you need an example that shows an animal is a dog AND it is not domesticated.

Question 5

If $a > b$ then $\frac{1}{a} < \frac{1}{b}$.

False.

$a = 1$, $b = -2$

Question 6

$$\frac{1}{(x-1)(x-2)} + \frac{1}{(x-2)(x-3)}$$

$$\frac{(x-3)}{(x-1)(x-2)(x-3)} + \frac{(x-1)}{(x-1)(x-2)(x-3)}$$

$$= \frac{2x-4}{(x-1)(x-2)(x-3)}$$

Yes it is true for $x = 4, 5, 6, \ldots$

Question 7

$$\frac{1}{n} < \frac{1}{n-1}$$

No it is not true.

$n = 1$

Question 8

All squares are rhombuses.

True.

All rhombuses are squares.

False.

Question 9

A circle can always be drawn through the four vertices of a rectangle. Is this true for all quadrilaterals?

False

3 points determine a unique circle. A 4th point can be chosen not on the circle to make a quadrilateral.

Question 10

The angle sum of all polygons with *n* sides is $S_n = 180°(n-2)$.

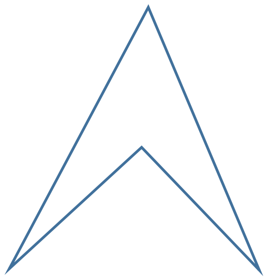
True.

Question 11

The diagonals of a kite always intersect inside the kite.

False.

Try arrowhead shape.



Question 12

Is it always true that $|x-y| = |y-x|$?

True.

Question 13

Is it always true that if $n > m$ that $nk > mk$?

False.

$k = -1$

Exercise 2.04 proofs involving numbers

Question 1

Let $M = 2m - 1$ and $N = 2n - 1$ for some $m, n \in \mathbb{N}$..

Then

$$\begin{aligned} M \times N &= (2m-1)\times(2n-1) \\ &= 4mn - 2m - 2n + 1 \\ &= 2(2mn - m - n) + 1 \\ &= 2P + 1 \end{aligned}$$

where $P \in \mathbb{N}$..

Since $2P$ is even then $2P+1$ is odd.

Therefore the product of 2 odd numbers *M* and *N* is odd. QED.

Question 2

a Let the 2 even numbers be *M* and *N*.

Let $M = 2m$ and $N = 2n$ for some $m, n \in \mathbb{N}$.

Then

$$\begin{aligned} M + N &= 2m + 2n \\ &= 2(m+n) \\ &= 2P \end{aligned}$$

where $P \in \mathbb{N}$.

Therefore 2*P* is even.

Therefore the sum of 2 even numbers *M* and *N* is even. QED.

b Let the 2 even numbers be M and N.

Let $M = 2m$ and $N = 2n$ for some $m, n \in \mathbb{N},\ m, n > 0$.

Then

$$\begin{aligned} M \times N &= 2m \times 2n \\ &= 2(2mn) \\ &= 2P \end{aligned}$$

where $P \in \mathbb{N}$.

Therefore $2P$ is even.

Therefore the product of 2 even numbers M and N is even. QED.

c Let the 2 even numbers be M and N.

Let $M = 2m$ and $N = 2n$ for some $m, n \in \mathbb{N},\ m > n.$.

Then

$$\begin{aligned} M - N &= 2m - 2n \\ &= 2(m - n) \\ &= 2P \end{aligned}$$

where $P \in \mathbb{N}$.

Therefore $2P$ is even.

Therefore the difference between 2 even numbers M and N is even. QED.

d Let the 2 odd numbers be *M* and *N*.

Let $M = 2m-1$ and $N = 2n-1$ for some $m, n \in \mathbb{N}$, $m > n$.

Then

$$\begin{aligned} M - N &= (2m-1)-(2n-1) \\ &= 2m-2n \\ &= 2(m-n) \\ &= 2P \end{aligned}$$

where $P \in \mathbb{N}$..

Therefore 2*P* is even.

Therefore the difference between 2 odd numbers *M* and *N* is even. QED.

e Let the odd and even numbers be *M* and *N* respectively.

Let $M = 2m+1$ and $N = 2n$ for some $m, n \in \mathbb{N}$..

Then

$$\begin{aligned} M + N &= (2m+1)+2n \\ &= 2(m+n)+1 \\ &= 2P+1 \end{aligned}$$

where $P \in \mathbb{N}$..

Since *2P* is even 2*P* + 1 is odd.

Therefore the sum of an odd number and an even number is odd. QED.

f Let the odd and even numbers be *M* and *N* respectively.

Let $M = 2m+1$ and $N = 2n$ for some $m, n \in \mathbb{N}$.

Then

$$\begin{aligned} M \times N &= (2m+1) \times 2n \\ &= 4mn + 2n \\ &= 2(2mn + n) \\ &= 2P \end{aligned}$$

where $P \in \mathbb{N}$.

Therefore 2*P* is even.

Therefore the product of an odd number and an even number is even. QED.

g Let the odd and even numbers be *M* and *N* respectively.

Let $M = 2m+1$ and $N = 2n$ for some $m, n \in \mathbb{N}, m > n$.

Then

$$\begin{aligned} M - N &= (2m+1) - 2n \\ &= 2(m-n) + 1 \\ &= 2P + 1 \end{aligned}$$

where $P \in \mathbb{N}$.

Since 2*P* is even, 2*P* + 1 is odd.

Therefore the difference between an odd number and an even number is odd. QED.

h Let the odd number be *M*.

Let $M = 2m - 1$ for some $m \in \mathbb{N}, m > 0$.

Then

$$\begin{aligned} M^2 &= (2m-1)\times(2m-1) \\ &= 4m^2 - 4m + 1 \\ &= 2\left(2m^2 - 2m\right) + 1 \\ &= 2P + 1 \end{aligned}$$

where $P \in \mathbb{N}$.

Since $2P$ is even then $2P + 1$ is odd.

Therefore the square of an odd number *M* is odd. QED.

i Let the even number be *M*.

Let $M = 2m$ for some $m \in \mathbb{N}$.

Then

$$\begin{aligned} M^2 &= (2m)\times(2m) \\ &= 4m^2 \\ &= 2\left(2m^2\right) \\ &= 2P \end{aligned}$$

where $P \in \mathbb{N}$.

Therefore 2*P* is even.

Therefore the square of an even number *M* is even. QED.

Question 3

a Required to prove

$$\left(\forall a,b \in \mathbb{N}\right)\left(\exists p,q \in \mathbb{Q}\right)$$

$$:\frac{1}{a+b\sqrt{3}} = p+q\sqrt{3}$$

Let $a,b \in \mathbb{N}$

$$\frac{1}{a+b\sqrt{3}}$$

$$=\frac{1}{a+b\sqrt{3}}\times\frac{a-b\sqrt{3}}{a-b\sqrt{3}}$$

$$=\frac{a-b\sqrt{3}}{a^2-3b^2}$$

$$=\frac{a}{a^2-3b^2}-\frac{b\sqrt{3}}{a^2-3b^2}$$

$$\frac{a}{a^2-3b^2}\quad \frac{-b}{a^2-3b^2}\in\mathbb{Q}$$

$$\therefore p=\frac{a}{a^2-3b^2}\quad q=\frac{-b}{a^2-3b^2}$$

$$\therefore \frac{1}{a+b\sqrt{3}} = p+q\sqrt{3}$$

b Required to prove

$$\left(\forall a,b,c,d \in \mathbb{N}\right)\left(\exists p,q \in \mathbb{Q}\right)$$

$$:\frac{a+b\sqrt{2}}{c+d\sqrt{2}} = p+q\sqrt{2}$$

Let $a,b,c,d \in \mathbb{N}$

$$\frac{a+b\sqrt{2}}{c+d\sqrt{2}}$$

$$=\frac{a+b\sqrt{2}}{c+d\sqrt{2}}\times\frac{c-d\sqrt{2}}{c-d\sqrt{2}}$$

$$=\frac{ac-ad\sqrt{2}+bc\sqrt{2}-2bd}{c^2-2d^2}$$

$$=\frac{ac-2bd+\left(bc-ad\right)\sqrt{2}}{c^2-2d^2}$$

$$=\frac{ac-2bd}{c^2-2d^2}+\frac{\left(bc-ad\right)\sqrt{2}}{c^2-2d^2}$$

$$\frac{ac-2bd}{c^2-2d^2}\ \frac{\left(bc-ad\right)}{c^2-2d^2}\in\mathbb{Q}$$

$$\therefore\ p=\frac{ac-2bd}{c^2-2d^2}\ \ q=\frac{bc-ad}{c^2-2d^2}$$

$$\therefore\frac{a+b\sqrt{2}}{c+d\sqrt{2}} = p+q\sqrt{2}$$

Question 4

a Required to prove

$n \in \mathbb{N}$

If $S_n = n^2+n$ then $S_n - S_{n-} = 2n$

$$S_n - S_{n-} = n^2+n-\left(\left(n-1\right)^2+\left(n-1\right)\right)$$

$$= n^2+n-\left(n^2-2n+1+n-1\right)$$

$$= n^2+n-n^2+2n-n$$

$= 2n$ QED

b Required to prove

$n \in \mathbb{N}$

If $S_n = n^2$ then $S_n - S_{n-} = 2n - 1$

$S_n - S_{n-} = n^2 - (n-1)^2$

$= n^2 - (n^2 - 2n + 1)$

$= n^2 - n^2 + 2n - 1$

$= 2n - 1$ QED

c Required to prove

$n \in \mathbb{N}$

If $S_n = 3n - n^2$ then $S_n - S_{n-} = 4 - 2n$

$S_n - S_{n-} = 3n - n^2 - \left(3(n-1) - (n-1)^2\right)$

$= 3n - n^2 - (3n - 3 - n^2 + 2n - 1)$

$= 3n - n^2 - 5n + 4 + n^2$

$= 4 - 2n$ QED

d Required to prove

$n \in \mathbb{N}$

If $S_n = 2n^2 + n$ then $S_n - S_{n-} = 4n - 1$

$S_n - S_{n-} = 2n^2 + n - \left(2(n-1)^2 + (n-1)\right)$

$= 2n^2 + n - (2n^2 - 4n + 2 + n - 1)$

$= 2n^2 + n - 2n^2 + 3n - 1$

$= 4n - 1$ QED

e Required to prove

$n \in \mathbb{N}$

If $S_n = \frac{(n+1)(n+3)}{2}$ then $S_n - S_{n-} = \frac{2n+3}{2}$

$S_n - S_{n-1} = \frac{(n+1)(n+3)}{2} - \frac{((n-1)+1)((n-1)+3)}{2}$

$= \frac{(n+1)(n+3)}{2} - \frac{n(n+2)}{2}$

$= \frac{n^2+4n+3-n^2-2n}{2}$

$= \frac{2n+3}{2}$ QED

Question 5

a Required to prove

If $f(x) = x^2$ then $f(x) = f(-x)$

$f(-x) = (-x)^2$

$= x^2$

$= f(x)$ QED

b Required to prove

If $f(x) = x^3$ then -$f(x) = f(-x)$

$f(-x) = (-x)^3$

$= -x^3$

$= -f(x)$ QED

c Required to prove

If $f(x)=\frac{x^3}{x^2-1}$ then $-f(x)=f(-x)$

$f(-x)=\frac{(-x)^3}{(-x)^2-1}$

$=\frac{-x^3}{x^2-1}$

$=-\frac{x^3}{x^2-1}$

$=-f(x)$ *QED*

d Required to prove

If $f(x)=x\sin x$ then $f(x)=f(-x)$

$f(-x)=(-x)\sin(-x)$

$=-x(-\sin x)$

$=x\sin x$

$=f(x)$ QED

e Required to prove

If $f(x)=x^2\cos x$ then $f(x)=f(-x)$

$f(-x)=(-x)^2\cos(-x)$

$=x^2\cos x$

$=f(x)$ QED

f Required to prove

If $f(x)=xe^{-x^2}$ then $-f(x)=f(-x)$

$f(-x)=(-x)e^{-\,-x)^2}$

$=-xe^{-x^2}$

$=-f(x)$ QED

Question 6

a Required to prove

$$\frac{n(n+1)(2n+1)}{6}+(n+1)^2=\frac{(n+1)(n+2)(2n+3)}{6}$$

$$\text{L.H.S.} = \frac{n(n+1)(2n+1)}{6}+(n+1)^2$$

$$=\frac{n(n+1)(2n+1)}{6}+\frac{6(n+1)^2}{6}$$

$$=\frac{(n+1)\left[n(2n+1)+6(n+1)\right]}{6}$$

$$=\frac{(n+1)\left(2n^2+7n+6\right)}{6}$$

$$=\frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \text{R.H.S.}$$

b Required to prove

$$\frac{n(n+1)(n+2)}{3}+(n+1)(n+2)=\frac{(n+1)(n+2)(n+3)}{3}$$

$$\text{L.H.S.} = \frac{n(n+1)(n+2)}{3}+(n+1)(n+2)$$

$$=\frac{n(n+1)(n+2)}{3}+\frac{3(n+1)(n+2)}{3}$$

$$=\frac{(n+1)(n+2)(n+3)}{3}$$

$$= \text{R.H.S.}$$

c Required to prove

$$2^{k+1}-1+2^{k+1}=2^{k+2}-1$$

$$\text{L.H.S.} = 2^{k+1}-1+2^{k+1}$$

$$=2\times 2^{k+}-1$$

$$=2^{k+2}-1$$

$$= \text{R.H.S.}$$

d Required to prove

$k\times 2^k+(k+2)2^k=(k+1)2^{k+1}$

L.H.S. $= k\times 2^k+(k+2)2^k$

$= k\times 2^k+k\times 2^k+2\times 2^k$

$= 2k\times 2^k+2^{k+1}$

$= k\times 2^{k+1}+2^{k+1}$

$= (k+1)2^{k+1}$

$=$ R.H.S.

e Required to prove

$$\frac{m}{m+1}+\frac{1}{(m+1)(m+2)}=\frac{m+1}{m+2}$$

L.H.S. $= \dfrac{m}{m+1}+\dfrac{1}{(m+1)(m+2)}$

$$=\frac{m(m+2)}{(m+1)(m+2)}+\frac{1}{(m+1)(m+2)}$$

$$=\frac{m^2+2m+1}{(m+1)(m+2)}$$

$$=\frac{(m+1)^2}{(m+1)(m+2)}$$

$$=\frac{m+1}{m+2}$$

$=$ R.H.S.

Question 7

a Required to prove

If $3^n - 1 = 2X$ for some $X \in \mathbb{N}$, then $3^{n+} - 1 = 2Y$ for some $Y \in \mathbb{N}$

Let $3^n - 1 = 2X$

$3\left(3^n - 1\right) = 3 \times 2X$

$3 \times 3^n - 3 = 6X$

$3^{n+} = 6X + 3$

$3^{n+} - 1 = 6X + 3 - 1$

$3^{n+} - 1 = 6X + 2$

$3^{n+} - 1 = 2\left(3X + 1\right)$

$3^{n+} - 1 = 2Y$

b Required to prove

If $4^n - 1 = 3X$ for some $X \in \mathbb{N}$, then $4^{n+} - 1 = 3Y$ for some $Y \in \mathbb{N}$

Let $4^n - 1 = 3X$

$4\left(4^n - 1\right) = 4 \times 3X$

$4 \times 4^n - 4 = 12X$

$4^{n+} = 12X + 4$

$4^{n+} - 1 = 12X + 4 - 1$

$4^{n+} - 1 = 12X + 3$

$4^{n+} - 1 = 3\left(4X + 1\right)$

$4^{n+} - 1 = 3Y$

c Required to prove

If $n^3 + 2n = 3X$ for some $X \in \mathbb{N}$, then $\left(n+1\right)^3 + 2\left(n+1\right) = 3Y$ for some $Y \in \mathbb{N}$

Let $n^3 + 2n = 3X$

$n^3 + 3n^2 + 3n + 1 + 2n + 2 = 3X + 3n^2 + 3n + 1 + 2$

$\left(n+1\right)^3 + 2\left(n+1\right) = 3X + 3\left(n^2 + n + 1\right)$

$\left(n+1\right)^3 + 2\left(n+1\right) = 3\left(X + n^2 + n + 1\right)$

$\left(n+1\right)^3 + 2\left(n+1\right) = 3Y$

d Required to prove

For n being an even number:

If $n^2+2n=8X$ for some $X\in\mathbb{N}$, then $(n+2)^2+2(n+2)=8Y$ for some $Y\in\mathbb{N}$

Let $n^2+2n=8X$

$n^2+4n+4+2n+4=8X+4n+4+4$

$(n+2)^2+2(n+2)=8X+4n+8$

As n is even we canwrite it as 2 m

$(n+2)^2+2(n+2)=8X+8m+8$

$(n+2)^2+2(n+2)=8(X+m+1)$

$(n+2)^2+2(n+2)=8Y$

Question 8

a Required to prove

$(x+5)^3(2x-1)^2+(x+5)^4(2x-1)=(2x-1)(x+5)^3(3x+4)$

L.H.S.

$(x+5)^3(2x-1)^2+(x+5)^4(2x-1)$

$=(x+5)^3\left[(2x-1)^2+(x+5)(2x-1)\right]$

$=(x+5)^3(2x-1)[2x-1+x+5]$

$=(2x-1)(x+5)^3(3x+4)$

$=$ R.H.S.

b Required to prove

$$\frac{4x^3(2-3x)^5+15x^4(2-3x)^4}{(2-3x)^{10}}=\frac{x^3(8+3x)}{(2-3x)^6}$$

L.H.S.

$$\frac{4x^3(2-3x)^5+15x^4(2-3x)^4}{(2-3x)^{10}}$$

$$=\frac{(2-3x)^4\left(4x^3(2-3x)+15x^4\right)}{(2-3x)^{10}}$$

$$=\frac{\left(8x^3-12x^4+15x^4\right)}{(2-3x)^6}$$

$$=\frac{\left(8x^3+3x^4\right)}{(2-3x)^6}$$

$$=\frac{x^3(8+3x)}{(2-3x)^6}$$

$=$ R.H.S.

Question 9

For $x>0$

$|x|=x$

$\therefore \frac{|x|}{x}=1$

For $x<0$

$|x|=-x$

$\therefore \frac{|x|}{x}=-1$

Exercise 2.05 Proofs involving inequalities

Question 1

$$a^2+b^2-2ab$$
$$=(a-b)^2 \ge 0$$
$$\therefore a^2+b^2 \ge 2ab$$

Question 2

a From **1**

$$a^2+b^2 \ge 2ab$$
Let $x^2=a, y^2=b$
$$\Rightarrow x^4+y^4 \ge 2x^2y^2$$

b From **1**

$$a^2+b^2 \ge 2ab$$
Let $xy=a, wv=b$
$$\Rightarrow x^2y^2+w^2v^2 \ge 2xywv$$

c From **1**

$$a^2+b^2 \ge 2ab$$
Let $\frac{1}{x}=a, \frac{1}{y}=b$
$$\Rightarrow \left(\frac{1}{x}\right)^2+\left(\frac{1}{y}\right)^2 \ge \frac{2}{xy}$$

d From **1**

$$a^2+b^2 \ge 2ab$$
Let $\frac{1}{x^2}=a, \frac{1}{y^2}=b$
$$\Rightarrow \left(\frac{1}{x}\right)^4+\left(\frac{1}{y}\right)^4 \ge \frac{2}{x^2y^2}$$

Question 3

$\left(\sqrt{x}-\sqrt{y}\right)^2 \geq 0$

$\Rightarrow x-2\sqrt{xy}+y \geq 0$

$\Rightarrow x+y \geq 2\sqrt{xy}$

$\Rightarrow \dfrac{x+y}{2} \geq \sqrt{xy}$

Question 4

a Required to prove

If $a > b, b > c$ then $a > c$.

Let $a > b$

$b > c$

adding both gives

$a+b>b+c$

$\Rightarrow a>c$ QED

b Required to prove

If $a > b, c > 0$ then $ac > bc$.

Let $a > b$

$a-b > 0$

$c(a-b) > 0$ as $c > 0$

$ac-bc > 0$

$\Rightarrow ac > bc$ QED

c Required to prove

If $a > b, c < 0$ then $ac > bc$.

Let $a > b$

$a-b > 0$

$c(a-b) < 0$ as $c < 0$

$ac-bc < 0$

$\Rightarrow ac < bc$ QED

d Required to prove

If $a > b, b > 0$ then $ab > b^2$.
Let $a > b$
$a - b > 0$
$b(a-b) > 0$ as $b > 0$
$ab - b^2 > 0$
$\Rightarrow ab > b^2$ QED

e Required to prove

If $a > b > 0, c > d > 0$ then $ac > bd$.
Let $a > b$
$c > d$
multiplying both (as they are positive it does not change the inequality) gives
$ac > bd$ QED

f Required to prove

If $a > b, c > d$ then $ac + bd > ad + bc$.
Let $a > b$
$c > d$
$\Rightarrow a - b > 0$ and $c - d > 0$
$\therefore a - b$ and $c - d$ are positive
$\therefore (a-b)(c-d) > 0$
$\Rightarrow ac - ad - bc + bd > 0$
$\Rightarrow ac + bd > ad + bc$ QED

g Required to prove

If $a > b, b > 0$ then $\frac{1}{a} < \frac{1}{b}$.
Let $a > b$
$\Rightarrow \frac{a}{b} > 1$
$\Rightarrow \frac{1}{b} > \frac{1}{a}$
$\Rightarrow \frac{1}{a} < \frac{1}{b}$ QED

Question 5

a Required to prove

If $T_n = \frac{1}{n}$ then $T_n > T_{n+1}$

$T_n - T_{n+1} = \frac{1}{n} - \frac{1}{n+1}$

$= \frac{(n+1)-n}{n(n+1)}$

$= \frac{1}{n(n+1)}$

As n and n + 1 is greater than zero $n(n+1) > 0$.

$\therefore \frac{1}{n(n+1)} > 0$

$\therefore T_n > T_{n+1}$

b Required to prove

If $T_n = \frac{n}{n+1}$ then $T_n < T_{n+1}$

$T_{n+} - T_n = \frac{n+1}{(n+1)+1} - \frac{n}{n+1}$

$= \frac{n+1}{n+2} - \frac{n}{n+1}$

$= \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)}$

$= \frac{n^2+2n+1-n^2-2n}{(n+2)(n+1)}$

$= \frac{1}{(n+2)(n+1)}$

As n + 1 and $n+2$ are greater than zero $(n+1)(n+2) > 0$.

$\therefore \frac{1}{(n+2)(n+1)} > 0$

$\therefore T_n < T_{n+1}$

c Required to prove

If $T_n = \dfrac{n^2 - 1}{n^2 + 1}$ then $T_n < T_{n+1}$

$$T_{n+} - T_n = \frac{(n+1)^2 - 1}{(n+1)^2 + 1} - \frac{n^2 - 1}{n^2 + 1}$$

$$= \frac{n^2 + 2n + 1 - 1}{n^2 + 2n + 1 + 1} - \frac{n^2 - 1}{n^2 + 1}$$

$$= \frac{n^2 + 2n}{n^2 + 2n + 2} - \frac{n^2 - 1}{n^2 + 1}$$

$$= \frac{(n^2 + 2n)(n^2 + 1) - (n^2 - 1)(n^2 + 2n + 2)}{(n^2 + 2n + 2)(n^2 + 1)}$$

$$= \frac{n^4 + n^2 + 2n^3 + 2n - (n^4 - n^2 + 2n^3 - 2n + 2n^2 - 2)}{(n^2 + 2n + 2)(n^2 + 1)}$$

$$= \frac{4n + 2}{(n^2 + 2n + 2)(n^2 + 1)}$$

As n is greater than zero $(4n + 2)$ and $(n^2 + 2n + 2)(n^2 + 1) > 0$.

$$\therefore \frac{4n + 2}{(n^2 + 2n + 2)(n^2 + 1)} > 0$$

$$\therefore T_n < T_{n+1}$$

Question 6

a

$$a^2 + b^2 \geq 2ab$$
$$\Rightarrow a^2 + b^2 + 2ab \geq 2ab + 2ab$$
$$\Rightarrow (a+b)^2 \geq 4ab$$

b

$$a^2 + b^2 \geq 2ab$$
Let $\sqrt{ab} = a,\ \sqrt{cd} = b$
$$\Rightarrow \left(\sqrt{ab}\right)^2 + \left(\sqrt{cd}\right)^2 \geq 2\sqrt{abcd}$$
$$\Rightarrow ab + cd \geq 2\sqrt{abcd}$$

c

$$a^2 + b^2 \geq 2ab$$
$$\Rightarrow \frac{a^2 + b^2}{ab} \geq 2$$
$$\Rightarrow \frac{a}{b} + \frac{b}{a} \geq 2$$
$$\Rightarrow \frac{a}{b} + \frac{b}{a} + 2 \geq 4$$
$$\Rightarrow (a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$$

d

$$a^2 + b^2 \geq 2ab$$
Let $\sqrt[4]{ab} = a,\ \sqrt[4]{cd} = b$
$$\Rightarrow \left(\sqrt[4]{ab}\right)^2 + \left(\sqrt[4]{cd}\right)^2 \geq 2\sqrt[4]{abcd}$$
$$\Rightarrow \frac{\sqrt{ab} + \sqrt{cd}}{2} \geq \sqrt[4]{abcd}$$

Question 7

a Required to prove

$$a+\frac{1}{a}\geq 2$$

$$\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2\geq 0$$

$$a-2+\frac{1}{a}\geq 0$$

$$a+\frac{1}{a}\geq 2 \quad \text{QED}$$

b Required to prove

$$\frac{a}{b}+\frac{b}{a}\geq 2$$

$$(a-b)^2\geq 0$$

$$a^2-2ab+b^2\geq 0$$

$$a^2+b^2\geq 2ab$$

$$\frac{a^2+b^2}{ab}\geq 2$$

$$\frac{a^2}{ab}+\frac{b^2}{ab}\geq 2$$

$$\frac{a}{b}+\frac{b}{a}\geq 2 \quad \text{QED}$$

c Required to prove

$$\frac{1}{a^2}+\frac{1}{b^2}\ge\frac{4}{a^2+b^2}$$

$$\left(\frac{1}{a}-\frac{1}{b}\right)^2\ge 0$$

$$\frac{1}{a^2}-\frac{2}{ab}+\frac{1}{b^2}\ge 0$$

$$\frac{1}{a^2}+\frac{1}{b^2}\ge\frac{2}{ab}$$

$$\frac{1}{a^2}+\frac{1}{b^2}\ge\frac{4}{2ab}$$

$$2ab < a^2+b^2 \quad \therefore \quad \frac{1}{2ab}>\frac{1}{a^2+b^2}$$

$$\therefore \frac{1}{a^2}+\frac{1}{b^2}\ge\frac{4}{a^2+b^2} \text{ QED}$$

d From **6d**

$$\frac{\sqrt{ab}+\sqrt{cd}}{2}\ge\sqrt[4]{abcd}$$

$$\Rightarrow \frac{\sqrt{ab}}{2}+\frac{\sqrt{cd}}{2}\ge\sqrt[4]{abcd}$$

From AM – GM inequality

$$\frac{a+b}{2}\ge\sqrt{ab} \text{ and } \frac{c+d}{2}\ge\sqrt{cd}$$

$$\therefore \frac{\sqrt{ab}}{2}+\frac{\sqrt{cd}}{2}\ge\sqrt[4]{abcd}$$

$$\Rightarrow \frac{\frac{a+b}{2}}{2}+\frac{\frac{c+d}{2}}{2}\ge\sqrt[4]{abcd}$$

$$\Rightarrow \frac{a+b+c+d}{4}\ge\sqrt[4]{abcd}$$

Question 8

a Required to prove

$x\sqrt{x}+1 \geq x+\sqrt{x}, \forall x \in \mathbb{R}, x \geq 0$

Consider the difference

$x\sqrt{x}+1-\left(x+\sqrt{x}\right)$

$= x\sqrt{x}+1-x-\sqrt{x}$

$=\left(\sqrt{x}-1\right)(x-1)$

For $x>1$

$\left(\sqrt{x}-1\right)$and$(x-1)>0$

$\therefore\left(\sqrt{x}-1\right)(x-1)>0$

For $x<1$

$\left(\sqrt{x}-1\right)$and$(x-1)<0$

$\therefore\left(\sqrt{x}-1\right)(x-1)>0$

For $x=1$

$\left(\sqrt{x}-1\right)$ and$(x-1)=0$

$\therefore\left(\sqrt{x}-1\right)(x-1)=0$

Hence

$x\sqrt{x}+1-\left(x+\sqrt{x}\right)\geq 0$

$\therefore x\sqrt{x}+1 \geq x+\sqrt{x}$

b **i** required to prove

$1 \leq p \leq q$, where $p\ q \in \mathbb{N}$ that $p(q-p+1) \geq q$

Consider the difference

$p(q-p+1)-q$

$pq-p^2+p-q$

$p(q-p)-(q-p)$

$(p-1)(q-p)$

As $p \geq 1,\ p-1 \geq 0$

As $q \geq p,\ q-p \geq 0$

$\therefore (p-1)(q-p) \geq 0$

$\therefore p(q-p+1) \geq q$

ii Required to prove

For $1 \le r \le s$ where $r\ s \in \mathbb{N}$ that $\sqrt{s} \le \sqrt{r(s-r+1)} \le \dfrac{s+1}{2}$

From **i**.

$s \le r(s-r+1)$

$\Rightarrow \sqrt{s} \le \sqrt{r(s-r+1)}$

For the other half of the inequality

$\dfrac{(s+1-2r)^2}{4} \ge 0$

$\Rightarrow \dfrac{(s+1)^2 - 4r(s+1) + 4r^2}{4} \ge 0$

$\Rightarrow \dfrac{(s+1)^2}{4} - r(s+1) + r^2 \ge 0$

$\Rightarrow \dfrac{(s+1)^2}{4} \ge r(s+1) - r^2$

$\Rightarrow \dfrac{(s+1)^2}{4} \ge r(s+1-r)$

$\Rightarrow \dfrac{s+1}{2} \ge \sqrt{r(s+1-r)}$

$\therefore \sqrt{s} \le \sqrt{r(s-r+1)} \le \dfrac{s+1}{2}$

Question 9

Required to prove

$$\left(kn-mp\right)^2 \geq \left(k^2-m^2\right)\left(n^2-p^2\right) \quad \forall k, m, n, p \in \mathbb{R}$$

$$\left(kn-mp\right)^2 \geq \left(kn-mp\right)^2 - \left(kp-mn\right)^2$$

As $\left(kp-mn\right)^2 \geq 0$

$$\therefore \left(kn-mp\right)^2 \geq k^2n^2 - 2kmnp + m^2p^2 - k^2p^2 + 2kmnp - m^2n^2$$

$$\left(kn-mp\right)^2 \geq k^2n^2 + m^2p^2 - k^2p^2 - m^2n^2$$

$$\left(kn-mp\right)^2 \geq k^2\left(n^2-p^2\right) - m^2\left(n^2-p^2\right)$$

$$\left(kn-mp\right)^2 \geq \left(k^2-m^2\right)\left(n^2-p^2\right) \quad \text{QED}$$

Let $n = k^2$, $p = m^2$

Using

$$\left(kn-mp\right)^2 \geq \left(k^2-m^2\right)\left(n^2-p^2\right)$$

$$\left(kk^2-mm^2\right)^2 \geq \left(k^2-m^2\right)\left(\left(k^2\right)^2-\left(m^2\right)^2\right)$$

$$\left(k^3-m^3\right)^2 \geq \left(k^2-m^2\right)\left(k^4-m^4\right) \quad \text{QED}$$

Question 10

a **i** Required to prove $x^4+y^4+w^4+z^4 \geq 4xywz$

Using $a^2+b^2 \geq 2ab$

We can write

$$x^4+y^4 \geq 2x^2y^2$$

$$w^4+z^4 \geq 2w^2z^2$$

Adding gives

$$x^4+y^4+w^4+z^4 \geq 2x^2y^2+2w^2z^2$$

$$x^4+y^4+w^4+z^4 \geq 2\left(x^2y^2+w^2z^2\right)$$

$$x^4+y^4+w^4+z^4 \geq 2\left(2xywz\right) \qquad \text{using } \left(xy\right)^2+\left(wz\right)^2 \geq 2xywz$$

$$x^4+y^4+w^4+z^4 \geq 4xywz \quad \text{QED}$$

ii Required to prove

If $x^4+y^4+w^4+z^4<4$ and $x, y, z, w>0$ then $\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{w^4}+\frac{1}{z^4}>4$.

Let $x^4+y^4+w^4+z^4<4$

From **i**, this implies

$$4xywz<4 \Rightarrow xywz<1$$

$$\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{w^4}+\frac{1}{z^4}$$

$$=\frac{x^4y^4w^4+x^4y^4z^4+x^4w^4z^4+y^4w^4z^4}{x^4y^4w^4z^4}$$

$$\geq\frac{4(xyw)(xyz)(xwz)(ywz)}{x^4y^4w^4z^4}$$

$$=\frac{4x^3y^3w^3z^3}{x^4y^4w^4z^4}$$

$$=\frac{4}{xywz}$$

As $xywz<1 \Rightarrow \frac{4}{xywz}>4$

$\therefore \frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{w^4}+\frac{1}{z^4}>4$ QED

iii Required to prove $(x+y+z+w)^4 \geq 256xywz$

From **10 a i**

$$x^4+y^4+w^4+z^4 \geq 4xywz$$

$$\frac{x^4+y^4+w^4+z^4}{4} \geq xywz$$

Let

$$x=\sqrt[4]{x}, y=\sqrt[4]{y}, w=\sqrt[4]{w}, z=\sqrt[4]{z}$$

$$\frac{x+y+w+z}{4} \geq \sqrt[4]{xywz}$$

$$x+y+w+z \geq 4\sqrt[4]{xywz}$$

$(x+y+w+z)^4 \geq 256xywz$ QED

b Required to prove

$(a+b)(b+c)(c+a)=c(a-b)^2+b(c-a)^2+a(b-c)^2+8abc$

L.H.S.

$(a+b)(b+c)(c+a)$

$=(ab+ac+b^2+bc)(c+a)$

$=abc+ac^2+cb^2+bc^2+a^2b+a^2c+ab^2+abc$

$=ac^2+cb^2+bc^2+a^2b+a^2c+ab^2+2abc$

$=cb^2-2abc+a^2c+bc^2-2abc+a^2b+ab^2-2abc+ac^2+2abc+2abc+2abc+2abc$

$=c(b-a)^2+b(c-a)^2+a(b-c)^2+8abc$ QED

$(a+b)(b+c)(c+d)=c(a-b)^2+b(c-a)^2+a(b-c)^2+8abc$

$c,(a-b)^2,b,(c-a)^2,a,(b-c)^2$ are all greater than zero.

$\therefore (a+b)(b+c)(c+d)>8abc$

c Required to prove

$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$

$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

$=1+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+1+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+1$

$=3+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}$

Using $\frac{a}{b}+\frac{b}{a}\geq 2$

$\geq 3+2+2+2$

$=9$

$\therefore (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$ QED

d **i**

$$a^3+b^3+c^3-3abc$$
$$=a^3+ab^2-ab^2+ac^2-ac^2-abc$$
$$+b^3+a^2b-a^2b+cb^2-cb^2-abc$$
$$+c^3+a^2c-a^2c+cb^2-cb^2-abc$$
$$=(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right) \text{ QED}$$

ii

$$(a-b)^2+(c-a)^2+(b-c)^2$$
$$a^2-2ab+b^2+c^2-2ac+a^2+b^2-2bc+c^2$$
$$=2\left(a^2+b^2+c^2-ab-bc-ca\right) \quad \text{QED}$$

iii Required to prove

$$a^3+b^3+c^3 \geq 3abc$$
$$a^3+b^3+c^3-3abc$$
$$=(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right)$$
$$=(a+b+c)\times\frac{1}{2}\left((a-b)^2+(c-a)^2+(b-c)^2\right)$$
$$\geq 0$$
$$\therefore a^3+b^3+c^3-3abc \geq 0$$
$$\Rightarrow a^3+b^3+c^3 \geq 3abc \text{ QED}$$

iv Required to prove $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$

From **iii**

$$a^3+b^3+c^3 \geq 3abc$$
$$\text{Let } a=\sqrt[3]{a}, b=\sqrt[3]{b}, c=\sqrt[3]{c}$$
$$a+b+c \geq 3\sqrt[3]{abc}$$
$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \quad \text{QED}$$

v Required to prove $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$

Using

$$\frac{a+b+c}{3}\geq \sqrt[3]{abc}$$

Let $a = a^2c, b = b^2a, c = c^2b$

$$\frac{a^2c+b^2a+c^2b}{3}\geq \sqrt[3]{a^2cb^2ac^2b}$$

$$\frac{a^2c+b^2a+c^2b}{3}\geq \sqrt[3]{a^3b^3c^3}$$

$$\frac{a^2c+b^2a+c^2b}{3}\geq abc$$

$$\frac{a^2c+b^2a+c^2b}{abc}\geq \frac{3abc}{abc}$$

$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$ QED

e Required to prove $(a+b+c)^3\geq 27abc$

From **10 d iii**

$$a^3+b^3+c^3\geq 3abc$$

$$\Rightarrow \frac{a^3+b^3+c^3}{3}\geq abc$$

Let $a = \sqrt[3]{a}, b = \sqrt[3]{b}, c = \sqrt[3]{c}$

$$\frac{a+b+c}{3}\geq \sqrt[3]{abc}$$

$$a+b+c\geq 3\sqrt[3]{abc}$$

$(a+b+c)^3\geq 27abc$ QED

f Required to prove $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\geq 4$

From **7 f**

$$\frac{a+b+c+d}{4}>\sqrt[4]{abcd}$$

Let $a = a^2cd, b = ab^2d, c = abc^2, d = bcd^2$

$$\Rightarrow \frac{a^2cd+ab^2d+abc^2+bcd^2}{4}>\sqrt[4]{a^2cdab^2dabc^2bcd^2}$$

$$\Rightarrow \frac{a^2cd+ab^2d+abc^2+bcd^2}{4}>\sqrt[4]{a^4b^4c^4d^4}$$

$$\Rightarrow \frac{a^2cd+ab^2d+abc^2+bcd^2}{abcd}>\frac{4abcd}{abcd}$$

$$\Rightarrow \frac{a^2cd+ab^2d+abc^2+bcd^2}{abcd}>\frac{4abcd}{abcd}$$

$$\Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\geq 4 \quad \text{QED}$$

The equality holds when $a = b = c = d$.

Question 11

a

$$|z+w|+|z-w|\ge|z+w+z-w|$$
$$=|2z|$$
$$\therefore|z+w|+|y-z|\ge|2z|$$

b

$$|x+z|+|y-z|\ge|x+z+y-z|$$
$$=|x+y|$$
$$\therefore|x+z|+|y-z|\ge|x+y|$$

c

$$|x-z|=|x-y+y-z|$$
$$=|x-y-(z-y)|$$
$$\ge\big||x-y|-|z-y|\big|$$
$$\ge|x-y|-|z-y|$$

Test yourself 2

Question 1

a P: I get a lot of sleep.

Q: I am healthy.

$P \Rightarrow Q$

b P: a polygon has 5 sides.

Q: It is a pentagon.

$P \Rightarrow Q$

c P: The teacher is nice.

Q: I will learn.

$P \Rightarrow Q$

Question 2

a $A \Rightarrow B$

Converse

$B \Rightarrow A$

b $\neg P \Rightarrow Q$

Converse

$Q \Rightarrow \neg P$

c $N \Rightarrow \neg M$

Converse

$\neg M \Rightarrow N$

d $\neg B \Rightarrow \neg F$

Converse

$\neg F \Rightarrow \quad B$

e If I can save money then I can buy a car.

Converse

If I can buy a car then I can save money.

f If my computer is broken then I am bored.

Converse

If I am bored then my computer is broken.

g If $a = b$ then $a^3 = b^3$.

Converse

If $a^3 = b^3$ then $a = b$.

Question 3

Iff means a statement and its converse is true.

Eg.

If x is even then x^2 is even.

If x^2 is even then x is even.

$\therefore$ x is even iff x^2 is even.

Question 4

a If a quadrilateral has equal diagonals it is a square.

(False – rectangles)

Converse

If a quadrilateral is a square it has equal diagonals.

True.

Not an equivalence statement as the original is false and the converse is true.

b If $x > 1$ then $\frac{1}{x} < 1$

True

Converse

If $\frac{1}{x} < 1$ then $x > 1$

False because $\frac{1}{x} < 1$ when $x < 0$ also.

Not an equivalence statement as the original is true and the converse is false.

c If I pass my exams, then I study hard.

Not necessarily true.

Converse

If I study hard then I pass my exams.

Not necessarily true.

Not an equivalence statement as the original and the converse are not necessarily true.

d If $a = 3$ then $a^2 = 9$.

True

Converse

If $a^2 = 9$ then $a = 3$.

False: $a = –3$ also.

Not an equivalence statement as the original is true and the converse is false.

e If a triangle has 2 equal sides it is isosceles.

True

Converse

If a triangle is isosceles then it has 2 equal sides.

True

A triangle has two equal sides iff it is isosceles.

Question 5

a It is raining

Negation

It is not raining

b The apple is not ripe.

Negation

The apple is ripe.

c Koalas are cute.

Negation

Not all koalas are not cute.

d Some people are sexist.

Negation

No people are sexist.

e They are all correct.

Negation

Some are not correct.

f $x \leq 4$

Negation

$x > 4$

g $p \in \mathbb{N}$

Negation

$p \notin \mathbb{N}$

Question 6

No he is not correct. The correct statement should be:

There are less than or equal to 10.

Question 7

a $A \Rightarrow B$

Contrapositive

$\neg B \Rightarrow \neg A$

b $\neg P \Rightarrow Q$

Contrapositive

$\neg Q \Rightarrow P$

c $N \Rightarrow \neg M$

Contrapositive

$M \Rightarrow \neg N$

d $\neg B \Rightarrow \neg F$

Contrapositive

$F \Rightarrow B$

e If the boy has red hair then he has blue eyes.

Contrapositive

If the boy does not have blue eyes then he does not have red hair.

f If the country is rich then the citizens have money.

Contrapositive

If the citizens do not have money then the country is not rich.

g If a quadrilateral is a kite then the adjacent sides are equal in length.

Contrapositive

If a quadrilateral has adjacent sides that are not equal in length then it is not a kite.

h If $x = y$ then $x^2 = y^2$.

Contrapositive

If $x^2 \neq y^2$ then $x \neq y$.

i If $a \in \mathbb{N}$ then $a \in \mathbb{Z}$.

Contrapositive

If $a \notin \mathbb{Z}$ then $a \notin \mathbb{N}$.

Question 8

A statement and its contrapositive are equivalent because the truth tables for them are exactly the same.

Eg.

If x is odd then x^2 is odd.

Compared with:

If x^2 is not odd then x is not odd.

Question 9

a If $a^2 \neq b^2$ then $a \neq b$.

Contrapositive

$a = b$ then $a^2 = b^2$.

This is true.

$\therefore$ If $a^2 \neq b^2$ then $a \neq b$ is true.

b If the battery is flat then the car does not start.

Contrapositive

If the car does start then the battery is not flat

True

$\therefore$ If the battery is flat then the car does not start is true.

c If a number is an integer then it is rational.

Contrapositive

If a number is not rational then it is not an integer.

True

$\therefore$ If a number is an integer then it is rational is true.

d If a quadrilateral has diagonals that bisect each other at right angles then it is a rhombus.

Contrapositive

If a quadrilateral is not a rhombus then its diagonals do not bisect each other at right angles.

True.

$\therefore$ If a quadrilateral has diagonals that bisect each other at right angles then it is a rhombus is true.

e If $a > b$ then $ab > b^2$.

Contrapositive

If $ab \le b^2$ then $a \le b$.

False $a = 2$, $b = -3$

f If an animal lives in the water then it is a fish.

Contrapositive

If an animal is not a fish then it does not live in the water.

False: Whales

Question 10

a

$$\forall x, y \in \mathbb{R}, x, y > 0$$
$$x > y \Rightarrow x^2 > y^2$$

b

$$(\exists c \in \mathbb{Q})(\forall a, b \in \mathbb{Z}, a < b)\, a < c < b$$
$$c = \frac{a+b}{2}$$

c

$$(\forall n \in \mathbb{N})$$
$$: 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$$

Question 11

a For all integers *n* and *m* greater than zero, if *n* is less than *m* then $\frac{1}{n}$ is greater than $\frac{1}{m}$.

b For all real numbers *a* and *b*, $a^2 + b^2 \geq 2ab$

c For all rational numbers *p* and *q* such that *p* is less than *q*, there exists a real number *r* such that $p < r < q$.

Question 12

a

Suppose $a, b \in \mathbb{Z}$ with a and b having no common factors.

$\frac{a}{b} = \sqrt{11}$

$\Rightarrow \left(\frac{a}{b}\right)^2 = 11$

$\Rightarrow \frac{a^2}{b^2} = 11$

$\Rightarrow a^2 = 11b^2$

$\Rightarrow a^2$ is divisible by 11

$\Rightarrow a$ is divisible by 11

$\therefore$ we can write $a = 11m$

$\frac{a^2}{b^2} = 11$

$\Rightarrow \frac{(11m)^2}{b^2} = 11$

$\Rightarrow \frac{121m^2}{b^2} = 1$

$\Rightarrow 121m^2 = 11b^2$

$\Rightarrow 11m^2 = b^2$

$\Rightarrow b^2$ is divisible by 11

$\Rightarrow b$ is divisible by 11

$\therefore$ we can write $b = 11n$

This is a contradiction as both a and b have no common factors.

$\therefore \sqrt{11} \notin \mathbb{Q}$

b

Suppose $a, b \in \mathbb{Z}$ with a and b having no common factors.

$\frac{a}{b} = \log_3 4$

$\Rightarrow 3^{\frac{a}{b}} = 4$

$\Rightarrow 3^a = 4^b$

LHS is always odd and RHS is always even.

This is a contradiction.

$\therefore \log_3 4 \notin \mathbb{Q}$

c

Suppose $a, b \in \mathbb{N}$ with a and b having no common factors

$$:\frac{2b+a}{b} = 2+\sqrt{5}$$

$$\Rightarrow \left(\frac{a}{b}\right)^2 = 5$$

$$\Rightarrow \frac{a^2}{b^2} = 5$$

$\Rightarrow a^2 = 5b^2$

$\Rightarrow a^2$ is divisible by 5

$\Rightarrow a$ is divisible by 5

$\therefore$ we can write $a = 5m$

$$\frac{a^2}{b^2} = 5$$

$$\Rightarrow \frac{(5m)^2}{b^2} = 5$$

$$\Rightarrow \frac{25m^2}{b^2} = 5$$

$\Rightarrow 25m^2 = 5b^2$

$\Rightarrow 5m^2 = b^2$

$\Rightarrow b^2$ is divisible by 5

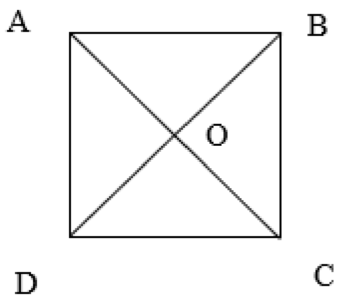
$\Rightarrow b$ is divisible by 5

$\therefore$ we can write $b = 5n$

This is a contradiction as both a and b have no common factors

$\therefore 2+\sqrt{5} \notin \mathbb{Q}$

d



Suppose *ABCD* is a square whose diagonals intersect at *O* and are not perpendicular.

$\angle BAO$ = 45° (diagonals of a square bisect the angles of the square)

$\angle ABO$ = 45° (diagonals of a square bisect the angles of the square)

$\angle AOB$ = 90° (sum of angles of a triangle add up to 180°)

Contradiction

e Let there be a triangle with sides

$2t, t^2-1, t^2+1$

Such that they do not form a right angled triangle

$$(2t)^2+(t^2-1)^2 \neq (t^2+1)^2$$
$$4t^2+t^4-2t^2+1 \neq t^4+2t^2+1$$
$$t^4+2t^2+1 \neq t^4+2t^2+1$$

Which is a contradiction.

Question 13

a $\forall x, y \in \mathbb{Q}, x, y \neq 0$, if $x^2 = y^2$ then $x = y$.

False: $x = 3$, $y = -3$

b $\forall n \in \mathbb{N},\ n > \frac{1}{n}$.

False: $n = 1$

c If an animal sheds its skin then it is a snake.

False – lizards or cicadas

d If $a^2 + b^2 = c^2$ then a, b, c form a right-angled triangle.

False: $a = -3$, $b = 4$, $c = 5$, or $a = 6$, $b = 0$, $c = 6$.

e $\forall k \in \mathbb{N}, k \geq 1$, $k(k-1)+17$ is prime.

False: $k = 17$ gives the number 289 which is 17^2.

f $\forall c \in \mathbb{R},\ (c \leq 1) \Rightarrow (c^2 \leq 1)$.

False: $c = -10$, $c^2 = 100$

Question 14

$a \geq b$ and $c \geq d$ then $ac \geq bd$, $\forall a, b, c, d \in \mathbb{R}$

False

Let $a = 3$, $b = -1$, $c = -2$, $d = -3$

$a \geq b$ and $c \geq d$ but $3 \times (-2) < -1 \times (-3)$.

Question 15

If $x > y \quad \forall x, y \in \mathbb{R}, x, y \neq 0$, then $\dfrac{1}{x^2} < \dfrac{1}{y^2}$

False

$$x = -5, y = -6$$
$$x > y$$
$$\frac{1}{x^2} = \frac{1}{25}$$
$$\frac{1}{y^2} = \frac{1}{36}$$
$$\frac{1}{x^2} > \frac{1}{y^2}$$

Question 16

a If $m \in \mathbb{N}$, then $m(m+1)$ is always even.

Let *m* be an even number.

From **2.04 2 f** an even number times an odd number is always even.

So $m(m + 1)$ is even

If *m* is odd, $m + 1$ must be even.

As was stated an even number times an odd number is always even,

so $m(m + 1)$ is even.

$\therefore$ $m(m + 1)$ is always even.

b If $n \in \mathbb{N}$, then $n(n+1)(n+2)$ is always divisible by 6.

For any 3 consecutive natural numbers, one must be a multiple of 3.

Hence $n(n+1)(n+2)$ is divisible by 3.

For any 3 consecutive natural numbers, must be even.

Hence $n(n+1)(n+2)$ is divisible by 2.

As $n(n+1)(n+2)$ is divisible by both 2 and 3 it must be divisible by 6. QED

c If n is odd, then $n(n+2)+(n+2)(n+4)$ is always even.

$$\begin{aligned} n(n+2)+(n+2)(n+4) &= (n+2)(n+n+4) \\ &= (n+2)(2n+4) \\ &= 2(n+2)(n+2) \end{aligned}$$

If n is odd then $n + 2$ is also odd.

$2(n + 2)(n + 2)$ is even as the sum of 2 odd numbers is even.

$\therefore\ n(n+2)+(n+2)(n+4)$ is always even.

Question 17

$$\begin{aligned} &k^6 - m^6 \\ &= (k^3+m^3)(k^3-m^3) \\ &= (k+m)(k^2-km+m^2)(k-m)(k^2+km+m^2) \\ &= (k+m)(k-m)(k^2-km+m^2)(k^2+km+m^2) \end{aligned}$$

$$\begin{aligned} &k^6 - m^6 \\ &= (k^2-m^2)(k^4+k^2m^2+m^4) \\ &= (k+m)(k-m)(k^4+k^2m^2+m^4) \end{aligned}$$

Equating both results gives

$$(k+m)(k-m)(k^2-km+m^2)(k^2+km+m^2) = (k+m)(k-m)(k^4+k^2m^2+m^4)$$

$$\Rightarrow k^4+k^2m^2+m^4 = (k^2-km+m^2)(k^2+km+m^2)$$

Question 18

a Required to prove

$n \in \mathbb{N}$,

If $S_n = \dfrac{4^n - 1}{3}$ then $S_n - S_{n-} \;= 4^{n-1}$

$S_n - S_{n-1} = \dfrac{4^n - 1}{3} - \dfrac{4^{n-} \;- 1}{3}$

$= \dfrac{4^n - 1 - 4^{n-} \;+ 1}{3}$

$= \dfrac{4^{n-} \;(4 - 1)}{3}$

$= \dfrac{3 \times 4^{n-1}}{3}$

$= 4^{n-}$ QED

b Required to prove

$n \in \mathbb{N}$,

If $S_n = 2^{n+1} - n - 2$ then $S_{n+} \; - S_n = 2^{n+1} - 1$

$S_{n+} \; - S_n = 2^{n+1+1} - (n+1) - 2 - \left(2^{n+} \; - n - 2\right)$

$= 2 \times 2^{n+1} - 1 - 2^{n+1}$

$= 2^{n+} \; - 1$ QED

c Required to prove

$$\forall k \in \mathbb{N},\ \frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}=\frac{k+1}{3k+4}$$

$$\begin{aligned}\text{LHS} &= \frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}\\ &= \frac{k(3k+4)}{(3k+1)(3k+4)}+\frac{1}{(3k+1)(3k+4)}\\ &= \frac{3k^2+4k+1}{(3k+1)(3k+4)}\\ &= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}\\ &= \frac{k+1}{3k+4}\\ &= \text{RHS}\end{aligned}$$

d

$$f(x)=x^2\sin x$$

$$f(-x)=(-x)^2\sin(-x)$$

$$=x^2\sin(-x)$$

$$=-x^2\sin x$$

$$=-f(x)$$

Question 19

a

Let

$a > b$

$ab^2 > bb^2 \quad \text{as } b^2 > 0$

$ab^2 > b^3 \quad \text{QED}$

b

Let

$a - b > b - c$

$a - b + c > b$

$a + c > b + b$

$a + c > 2b$

$\frac{a+c}{2} > b \quad \text{QED}$

c

Let

$x \geq 0$

$|x| = x$

$\therefore |x| \geq x \text{ for } x \geq 0$

$x < 0$

$|x| = -x$

$\therefore |x| > x \text{ for } x < 0$

$\therefore |x| > x \text{ for all } x \in \mathbb{R} \quad \text{QED}$

Question 20

a

$$\frac{a^2+b^2}{2}-ab$$
$$=\frac{1}{2}\left(a^2+b^2-2ab\right)$$
$$=\frac{1}{2}\left(a-b\right)^2>0$$
$$\therefore\frac{a^2+b^2}{2}\geq ab$$

b

Let $T_k=\dfrac{k}{2k+1}$

$$T_{k+}\ -T_k$$
$$=\frac{k+1}{2\left(k+1\right)+1}-\frac{k}{2k+1}$$
$$=\frac{k+1}{2k+2+1}-\frac{k}{2k+1}$$
$$=\frac{k+1}{2k+3}-\frac{k}{2k+1}$$
$$=\frac{\left(k+1\right)\left(2k+1\right)}{\left(2k+3\right)\left(2k+1\right)}-\frac{k\left(2k+3\right)}{\left(2k+3\right)\left(2k+1\right)}$$
$$=\frac{\left(k+1\right)\left(2k+1\right)-k\left(2k+3\right)}{\left(2k+3\right)\left(2k+1\right)}$$
$$=\frac{2k^2+3k+1-2k^2-3k}{\left(2k+3\right)\left(2k+1\right)}$$
$$=\frac{1}{\left(2k+3\right)\left(2k+1\right)}$$

As $\left(2k+3\right)\text{and}\left(2k+1\right)>0$

$$\Rightarrow\frac{1}{\left(2k+3\right)\left(2k+1\right)}>0$$
$$\therefore T_{k+}\ >T_k$$

Question 21

a

Required to prove

$$\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$$

$$(a-b)^2 \geq 0$$

$$\Rightarrow a^2 - 2ab + b^2 \geq 0$$

$$\Rightarrow a^2 + b^2 \geq 2ab$$

$$\Rightarrow 2a^2 + 2b^2 \geq 2ab + a^2 + b^2$$

$$\Rightarrow \frac{a^2+b^2}{2} \geq \frac{2ab+a^2+b^2}{4}$$

$$\Rightarrow \frac{a^2+b^2}{2} \geq \frac{(a+b)^2}{4}$$

$$\Rightarrow \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2} \quad \text{QED}$$

b

Required to prove

$$\sqrt{\frac{a^2+b^2+c^2+d^2}{4}} \geq \sqrt[4]{abcd}$$

From **2.05 10 a**

$$a^4 + b^4 + c^4 + d^4 \geq 4abcd$$

$$\Rightarrow \frac{a^4+b^4+c^4+d^4}{4} \geq abcd$$

Let $a = \sqrt{a}, b = \sqrt{b}, c = \sqrt{c}, d = \sqrt{d}$

$$\Rightarrow \frac{a^2+b^2+c^2+d^2}{4} \geq \sqrt{abcd}$$

$$\Rightarrow \sqrt{\frac{a^2+b^2+c^2+d^2}{4}} \geq \sqrt[4]{abcd} \quad \text{QED}$$

c Let $a + b = 1$

i From **Example 17 b**

$$\frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b}$$

As $a + b = 1$

$$\frac{1}{a}+\frac{1}{b} \geq 4 \quad \text{QED}$$

ii From part **i**

$$\frac{1}{a}+\frac{1}{b} \geq 4$$

$$\frac{a+b}{ab} \geq 4$$

As $a + b = 1$

$$\frac{1}{ab} \geq 4$$

$$\frac{2}{ab} \geq 8$$

Proving the required statement:

$$\frac{1}{a}+\frac{1}{b} \geq 4$$

$$\Rightarrow \left(\frac{1}{a}+\frac{1}{b}\right)^2 \geq 16$$

$$\Rightarrow \frac{1}{a^2}+\frac{2}{ab}+\frac{1}{b^2} \geq 16$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2} \geq 16-\frac{2}{ab}$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2} \geq 16-8$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2} \geq 8 \quad \text{QED}$$

Question 22

From **2.05 10 c**

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$$
$$\Rightarrow\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq\frac{9}{a+b+c}$$

Question 23

a **From 2.05 10 c**

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$$
$$a+b+c=1$$
$$\Rightarrow\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$$

b From **2.05 10 d iii**

$$a^3+b^3+c^3\geq 3abc$$

Let $a=\sqrt[3]{a}, b=\sqrt[3]{b}, c=\sqrt[3]{c}$

$$a+b+c\geq 3\sqrt[3]{abc} \quad (*)$$
$$(a+b+c)^2\geq 9\sqrt[3]{(abc)^2}$$

Let $a=\dfrac{1}{a^2}, b=\dfrac{1}{b^2}, c=\dfrac{1}{c^2}$

Substituting into (*) gives:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq 3\sqrt[3]{\frac{1}{a^2b^2c^2}}$$

Multiplying the 2 inequalities gives:

$$(a+b+c)^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\geq 9\sqrt[3]{(abc)^2}\,3\sqrt[3]{\frac{1}{a^2b^2c^2}}$$

As $a+b+c=1$

$$\Rightarrow\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq 27 \quad \text{QED}$$

Question 24

a Required to prove

$$|x|-|y|\le|x-y|,\forall x,y\in\mathbb{R}$$

$$|x|-|y|=|x-0|-|y-0|$$

$$\le|(x-0)-(y-0)|$$

$$=|x-y|\ \text{ QED}$$

b Required to prove

$$|x|+|y|+|z|\ge|x+y+z|,\forall x,y,z\in\mathbb{R}$$

$$|x|+|y|+|z|\ge|x+y|+|z|$$

$$\ge|x+y+z|\ \text{ QED}$$

c Required to prove

$$|x-y|-|z-y|\le|x-z|,\forall x,y,z\in\mathbb{R}$$

$$|x-y|-|z-y|\le|(x-y)-(z-y)|$$

$$=|x-z|\ \text{ QED}$$

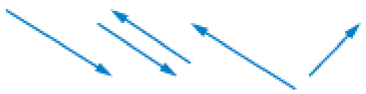
MATHS IN FOCUS 12
MATHEMATICS EXTENSION 2

WORKED SOLUTIONS

Chapter 3: Vectors

Exercise 3.01 Review of 2D vectors

Question 1



There are 5 different vectors.

Question 2

a Geometrically, the sum of the vectors is the longer diagonal of the parallelogram of the vectors.

b Geometrically, the difference of the vectors is the shorter diagonal of the parallelogram of the vectors.

c Two vectors are parallel iff one of the vectors is a scalar multiple of the other.

d Two vectors are perpendicular iff the dot product is zero.

Question 3

a

$\underset{\sim}{v} = 2\underset{\sim}{j}$

$|\underset{\sim}{v}| = 2$ By inspection

$\theta = 90°$ By inspection

b

$\underset{\sim}{v} = 5\underset{\sim}{i}$

$|\underset{\sim}{v}| = 5$ By inspection

$\theta = 0°$ By inspection

c

$\underset{\sim}{v} = 10\underset{\sim}{i} - 5\underset{\sim}{j}$

$|\underset{\sim}{v}| = \sqrt{10^2 + (-5)^2} = 5\sqrt{5}$

$\tan\theta = \dfrac{-5}{10}$

$\theta = \tan^{-}\left(\dfrac{-5}{10}\right) = 333°26'$

d

$\underset{\sim}{v} = -4\underset{\sim}{i} + 4\underset{\sim}{j}$

$|\underset{\sim}{v}| = \sqrt{(-4)^2 + (4)^2} = 4\sqrt{2}$

$\tan\theta = \dfrac{4}{-4}$

$\theta = \tan^{-}(-1) = 135°$

e

$\underset{\sim}{v} = 2\underset{\sim}{i} + 2\sqrt{3}\underset{\sim}{j}$

$|\underset{\sim}{v}| = \sqrt{(2)^2 + \left(2\sqrt{3}\right)^2} = 4$

$\tan\theta = \dfrac{2\sqrt{3}}{2}$

$\theta = \tan^{-}\left(\sqrt{3}\right) = 60°$

Question 4

a

$x = 6\cos 45° = 3\sqrt{2}\underset{\sim}{i}$

$y = 6\sin 45° = 3\sqrt{2}\underset{\sim}{j}$

$\underset{\sim}{v} = 3\sqrt{2}\underset{\sim}{i} + 3\sqrt{2}\underset{\sim}{j}$

b

$x = 8\cos 30° = 4\sqrt{3}\underset{\sim}{i}$

$y = 8\sin 30° = 4\underset{\sim}{j}$

$\underset{\sim}{v} = 4\sqrt{3}\underset{\sim}{i} + 4\underset{\sim}{j}$

c

$x = 2\cos 135° = -\sqrt{2}\underset{\sim}{i}$

$y = 2\sin 135° = \sqrt{2}\underset{\sim}{j}$

$\underset{\sim}{v} = -\sqrt{2}\underset{\sim}{i} + \sqrt{2}\underset{\sim}{j}$

d

$x = 10\cos(-60°) = 5\underset{\sim}{i}$

$y = 10\sin(-60°) = -5\sqrt{3}\underset{\sim}{j}$

$\underset{\sim}{v} = 5\underset{\sim}{i} - 5\sqrt{3}\underset{\sim}{j}$

e

$x = 6\cos(-150°) = -3\sqrt{3}\underset{\sim}{i}$

$y = 6\sin(-150°) = -3\underset{\sim}{j}$

$\underset{\sim}{v} = -3\sqrt{3}\underset{\sim}{i} - 3\underset{\sim}{j}$

Question 5

a $\begin{pmatrix} 3\sqrt{2} \\ 3\sqrt{2} \end{pmatrix}$

b $\begin{pmatrix} 4\sqrt{3} \\ 4 \end{pmatrix}$

c $\begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix}$

d $\begin{pmatrix} 5 \\ -5\sqrt{3} \end{pmatrix}$

e $\begin{pmatrix} -3\sqrt{3} \\ -3 \end{pmatrix}$

Question 6

a $$\begin{aligned}(3,0)\bullet(5,0) &= 3\times 5+0\times 0\\ &=15\end{aligned}$$

b $$\begin{aligned}(2,0)\bullet(0,7) &= 2\times 0+0\times 7\\ &=0\end{aligned}$$

c $$\begin{aligned}(\underset{\sim}{i}+2\underset{\sim}{j})\bullet(2\underset{\sim}{i}+\underset{\sim}{j}) &= 1\times 2+2\times 1\\ &=4\end{aligned}$$

d $$\begin{aligned}(\underset{\sim}{i}-2\underset{\sim}{j})\bullet(\underset{\sim}{i}+2\underset{\sim}{j}) &= 1\times 1+(-2)\times 2\\ &=-3\end{aligned}$$

e $$\begin{aligned}(6\underset{\sim}{i}+2\underset{\sim}{j})\bullet(3\underset{\sim}{i}-4\underset{\sim}{j}) &= 6\times 3+2\times(-4)\\ &=10\end{aligned}$$

Question 7

a

$\underset{\sim}{v} = (3,4)$

$|\underset{\sim}{v}| = \sqrt{3^2 + 4^2} = 5$

$\underset{\sim}{u} = (4,-3)$

$|\underset{\sim}{u}| = \sqrt{4^2 + (-3)^2} = 5$

$\underset{\sim}{v} \bullet \underset{\sim}{u} = 3 \times 4 + 4 \times (-3) = 0$

As the dot product is zero the vectors must be perpendicular

$\therefore \theta = 90°$

b

$\underset{\sim}{v} = (2,8)$

$|\underset{\sim}{v}| = \sqrt{2^2 + 8^2} = 2\sqrt{17}$

$\underset{\sim}{u} = (1,4)$

$|\underset{\sim}{u}| = \sqrt{1^2 + 4^2} = \sqrt{17}$

$\underset{\sim}{v} \bullet \underset{\sim}{u} = 2 \times 1 + 8 \times 4 = 34$

$\underset{\sim}{v} \bullet \underset{\sim}{u} = |\underset{\sim}{v}||\underset{\sim}{u}|\cos\theta$

$\therefore 2\sqrt{17} \times \sqrt{17}\cos\theta = 34$

$\cos\theta = 1$

$\therefore \theta = 0°$

c

$\underset{\sim}{v} = (-1,1)$

$|\underset{\sim}{v}| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\underset{\sim}{u} = (3,-3)$

$|\underset{\sim}{u}| = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$

$\underset{\sim}{v} \bullet \underset{\sim}{u} = (-1) \times 3 + 1 \times (-3) = -6$

$\underset{\sim}{v} \bullet \underset{\sim}{u} = |\underset{\sim}{v}||\underset{\sim}{u}|\cos\theta$

$\therefore \sqrt{2} \times 3\sqrt{2}\cos\theta = -6$

$\cos\theta = -1$

$\therefore \theta = 180°$

d

$\underset{\sim}{v} = (-5,-4)$

$|\underset{\sim}{v}| = \sqrt{(-5)^2 + (-4)^2} = \sqrt{41}$

$\underset{\sim}{u} = (4,0)$

$|\underset{\sim}{u}| = \sqrt{4^2 + 0^2} = 4$

$\underset{\sim}{v} \bullet \underset{\sim}{u} = (-5) \times 4 + (-4) \times 0 = -20$

$\underset{\sim}{v} \bullet \underset{\sim}{u} = |\underset{\sim}{v}||\underset{\sim}{u}|\cos\theta$

$\therefore \sqrt{41} \times 4\cos\theta = -20$

$\cos\theta = \dfrac{-20}{4\sqrt{41}}$

$\theta = \cos^{-1}\left(\dfrac{-20}{4\sqrt{41}}\right)$

$\therefore \theta = 141°20 \approx 141°$

e

$\underset{\sim}{v} = (-2,3)$

$|\underset{\sim}{v}| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$

$\underset{\sim}{u} = (1,-2)$

$|\underset{\sim}{u}| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$

$\underset{\sim}{v} \bullet \underset{\sim}{u} = (-2) \times 1 + 3 \times (-2) = -8$

$\underset{\sim}{v} \bullet \underset{\sim}{u} = |\underset{\sim}{v}||\underset{\sim}{u}|\cos\theta$

$\therefore \sqrt{13} \times \sqrt{5}\cos\theta = -8$

$\cos\theta = \dfrac{-8}{\sqrt{65}}$

$\theta = \cos^{-1}\left(\dfrac{-8}{\sqrt{65}}\right)$

$\therefore \theta = 172°52 \approx 173°$

Question 8

a $\underset{\sim}{v}$ and $\underset{\sim}{w}$ By inspection

b $\underset{\sim}{u} \bullet \underset{\sim}{w} = 5 \times (-2) + 2 \times 5 = 0$

$\therefore \underset{\sim}{u}$ and $\underset{\sim}{w}$ are perpendicular

Question 9

a $\underset{\sim}{v}$ and $\underset{\sim}{w}$

$-2 \times \underset{\sim}{w} = -2\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \underset{\sim}{v}$

d $\underset{\sim}{u}$ and $\underset{\sim}{w}$

$2 \times \underset{\sim}{u} = 2 \times \left(3\underset{\sim}{i} + \underset{\sim}{j}\right) = 6\underset{\sim}{i} + 2\underset{\sim}{j} = \underset{\sim}{w}$

Question 10

$\overrightarrow{PQ} = (2, 4)$ $\overrightarrow{PR} = (4, 8)$

As $\overrightarrow{PR} = 2\overrightarrow{PQ}$ and there is a common point, *P*, *P*, *Q* and *R* are collinear.

Exercise 3.02 3D vectors

Question 1

a $\underset{\sim}{v} = 4\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{4^2 + 3^2 + (-1)^2} = \sqrt{26}$

b $\underset{\sim}{v} = 8\underset{\sim}{i} - 6\underset{\sim}{j} + 5\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{8^2 + (-6)^2 + 5^2} = \sqrt{125} = 5\sqrt{5}$

c $\underset{\sim}{v} = -4\underset{\sim}{i} + 5\underset{\sim}{j} - 2\sqrt{2}\,\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{(-4)^2 + 5^2 + (-2\sqrt{2})^2} = \sqrt{49} = 7$

d $\underset{\sim}{v} = -2\underset{\sim}{i} + 2\sqrt{3}\underset{\sim}{j} + 3\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{(-2)^2 + (2\sqrt{3})^2 + 3^2} = \sqrt{25} = 5$

Question 2

a $\underset{\sim}{v} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$\underset{\sim}{v} = \frac{1}{\sqrt{3}}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

b $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$

$\underset{\sim}{v} = \frac{1}{3}(2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})$

c $\underset{\sim}{v} = 3\underset{\sim}{i} + 4\underset{\sim}{j} - 12\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{3^2 + 4^2 + (-12)^2} = \sqrt{169} = 13$

$\underset{\sim}{v} = \frac{1}{13}(3\underset{\sim}{i} + 4\underset{\sim}{j} - 12\underset{\sim}{k})$

d $\underset{\sim}{v} = \underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$

$\underset{\sim}{v} = \frac{1}{\sqrt{14}}(\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k})$

Question 3

a $\underset{\sim}{v} = (3,0,0)-(0,4,0) = (3,-4,0)$

$|\underset{\sim}{v}| = \sqrt{3^2+(-4)^2+0^2} = 5$

b $\underset{\sim}{v} = (-1,1,1)-(1,1,1) = (-2,0,0)$

$|\underset{\sim}{v}| = \sqrt{2^2+0^2+0^2} = 2$

c $\underset{\sim}{v} = (1,1,2)-(2,2,3) = (-1,-1,-1)$

$|\underset{\sim}{v}| = \sqrt{(-1)^2+(-1)^2+(-1)^2} = \sqrt{3}$

d $\underset{\sim}{v} = (3,4,5)-(-2,-2,-3) = (5,6,8)$

$|\underset{\sim}{v}| = \sqrt{5^2+6^2+8^2} = \sqrt{125} = 5\sqrt{5}$

Question 4

a $\overline{OA} = \underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$

$\overline{OB} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

$\overline{AB} = \overline{OB} - \overline{OA}$

$= 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k} - (\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k})$

$= \underset{\sim}{i} + 2\underset{\sim}{j}$

$|\overline{AB}| = \sqrt{1^2+2^2} = \sqrt{5}$

b $\overline{OC} = 2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$

$\overline{OD} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

$\overline{CD} = \overline{OD} - \overline{OC}$

$= \underset{\sim}{0}$ As they are the same point

$|\overline{CD}| = 0$

c $\overline{OX} = \underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$

$\overline{OY} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$

$\overline{XY} = \overline{OY} - \overline{OX}$

$= \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} - (\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k})$

$= 4\underset{\sim}{j} + \underset{\sim}{k}$

$|\overline{AB}| = \sqrt{4^2+1^2} = \sqrt{17}$

Question 5

a $2\underset{\sim}{i}$ By inspection

b

$$\underset{\sim}{v}=\frac{3}{7}\underset{\sim}{i}-\frac{2}{7}\underset{\sim}{j}+\frac{6}{7}\underset{\sim}{k}$$

$$|\underset{\sim}{v}|=\sqrt{\left(\frac{3}{7}\right)^2+\left(\frac{2}{7}\right)^2+\left(\frac{6}{7}\right)^2}=\sqrt{\frac{49}{49}}=1$$

$$\therefore 21\underset{\sim}{v}=21\left(\frac{3}{7}\underset{\sim}{i}-\frac{2}{7}\underset{\sim}{j}+\frac{6}{7}\underset{\sim}{k}\right)=9\underset{\sim}{i}-6\underset{\sim}{j}+18\underset{\sim}{k}$$

c

$$\underset{\sim}{v}=\frac{3}{5}\underset{\sim}{i}-\frac{4}{5}\underset{\sim}{j}$$

$$|\underset{\sim}{v}|=\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{-4}{5}\right)^2+\left(\frac{6}{7}\right)^2}=\sqrt{\frac{25}{25}}=1$$

$$\therefore 5\underset{\sim}{v}=5\left(\frac{3}{5}\underset{\sim}{i}-\frac{4}{5}\underset{\sim}{j}\right)=3\underset{\sim}{i}-4\underset{\sim}{j}$$

d

$$\underset{\sim}{v}=\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}$$

$$|\underset{\sim}{v}|=\sqrt{1^2+(-1)^2+1^2}=\sqrt{3}$$

$$\therefore \sqrt{12}\underset{\sim}{v}=\sqrt{12}\,\frac{1}{\sqrt{3}}\left(\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}\right)=2\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}$$

Question 6

$$\underset{\sim}{u}=\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}$$

$$\underset{\sim}{v}=\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}$$

a

$$\begin{aligned}2\underset{\sim}{u}-\underset{\sim}{v}&=2\left(\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}\right)-\left(\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}\right)\\&=\underset{\sim}{i}-3\underset{\sim}{j}-\underset{\sim}{k}\end{aligned}$$

b

$$\begin{aligned}\underset{\sim}{u}+2\underset{\sim}{v}&=\left(\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}\right)+2\left(\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}\right)\\&=3\underset{\sim}{i}-4\underset{\sim}{j}+7\underset{\sim}{k}\end{aligned}$$

c

$$\begin{aligned}\underset{\sim}{u}+\underset{\sim}{v}&=\left(\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}\right)+\left(\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}\right)\\&=2\underset{\sim}{i}-3\underset{\sim}{j}+4\underset{\sim}{k}\end{aligned}$$

d

$$\begin{aligned}\underset{\sim}{u}-\underset{\sim}{v}&=\left(\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}\right)-\left(\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}\right)\\&=-\underset{\sim}{j}-2\underset{\sim}{k}\end{aligned}$$

Question 7

a $\underset{\sim}{v} = (0,4,4) - (2,1,2) = (-2,3,2)$

$|\underset{\sim}{v}| = \sqrt{(-2)^2 + 3^2 + 2^2} = \sqrt{17}$

b $\underset{\sim}{v} = (2,5,0) - (-1,1,2) = (3,4,-2)$

$|\underset{\sim}{v}| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{29}$

c $\underset{\sim}{v} = (-1,2,-1) - (1,-2,1) = (-2,4,-2)$

$|\underset{\sim}{v}| = \sqrt{(-2)^2 + 4^2 + (-2)^2} = \sqrt{24} = 2\sqrt{6}$

d $\underset{\sim}{v} = (-2,-2,2) - (3,1,4) = (-5,-3,-2)$

$|\underset{\sim}{v}| = \sqrt{(-5)^2 + (-3)^2 + (-2)^2} = \sqrt{38}$

Question 8

$\underset{\sim}{u} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{v} = 3\underset{\sim}{i} - 4\underset{\sim}{k}$

a $|\underset{\sim}{u}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$

b $|\underset{\sim}{v}| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$

c $\underset{\sim}{u} = \frac{1}{3}\left(2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

d $\underset{\sim}{v} = \frac{1}{5}\left(3\underset{\sim}{i} - 4\underset{\sim}{k}\right)$

e $\underset{\sim}{u} + \underset{\sim}{v} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k} + 3\underset{\sim}{i} - 4\underset{\sim}{k} = 5\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

f $\underset{\sim}{u} - \underset{\sim}{v} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k} - \left(3\underset{\sim}{i} - 4\underset{\sim}{k}\right) = -\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}$

Question 9

a $\underset{\sim}{v} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$

$\therefore \sqrt{7}\underset{\sim}{v} = \sqrt{7}\dfrac{1}{\sqrt{14}}\left(3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right) = \dfrac{1}{\sqrt{2}}\left(3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

b $\underset{\sim}{v} = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{4^2 + (-4)^2 + 4^2} = \sqrt{48} = 4\sqrt{3}$

$\therefore 3\underset{\sim}{v} = 3\dfrac{1}{4\sqrt{3}}4\left(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) = \sqrt{3}\left(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right)$

c $\underset{\sim}{v} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$

$\therefore 6\underset{\sim}{v} = 6\dfrac{1}{3}\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right) = 2\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right) = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$

Question 10

$\underset{\sim}{u} = -\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{v} = \underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$

$\underset{\sim}{u} = -\underset{\sim}{v}$

$\therefore$ The angle between the two vectors is 180°.

The significance is that they point in opposite directions.

Exercise 3.03 Angle between vectors

Question 1

a $(2,5,-1)\bullet(4,1,1)=2\times4+5\times1+(-1)\times1=12$

b $2\underset{\sim}{i}\bullet6\underset{\sim}{j}=(2,0)\bullet(0,6)=2\times0+0\times6=0$

These two vectors are perpendicular hence their dot product is zero.

c $4\underset{\sim}{k}\bullet(2\underset{\sim}{i}+\underset{\sim}{k})=0\times1+4\times1=4$

d $(2,0,4)\bullet(-3,1,3)=2\times(-3)+0\times1+4\times3=6$

e $(2\underset{\sim}{i}+3\underset{\sim}{k})\bullet(6\underset{\sim}{i}+2\underset{\sim}{j}-4\underset{\sim}{k})=2\times6+0\times2+3\times(-4)=0$

Question 2

$\text{A}\bullet\text{B}=(5,2,3)\bullet(0,1,-1)=5\times0+2\times1+3\times(-1)=-1$

$\therefore$ They are not perpendicular.

$\text{A}\bullet\text{C}=(5,2,3)\bullet(-2,2,2)=5\times(-2)+2\times2+3\times2=0$

$\therefore$ They are perpendicular.

$\text{A}\bullet\text{D}=(5,2,3)\bullet(-1,0,2)=5\times(-1)+2\times0+3\times2=-1$

$\therefore$ They are not perpendicular.

$\text{B}\bullet\text{C}=(0,1,-1)\bullet(-2,2,2)=0\times(-2)+1\times2+(-1)\times2=0$

$\therefore$ They are perpendicular.

$\text{B}\bullet\text{D}=(0,1,-1)\bullet(-1,0,2)=0\times(-1)+1\times0+(-1)\times2=-2$

$\therefore$ They are not perpendicular.

$\text{C}\bullet\text{D}=(-2,2,2)\bullet(-1,0,2)=(-2)\times(-1)+2\times0+2\times2=6$

$\therefore$ They are not perpendicular.

A and **C**. **B** and **C** are perpendicular.

Question 3

a

$$\underset{\sim}{u} = (4, -1, 0)$$
$$|\underset{\sim}{u}| = \sqrt{4^2 + (-1)^2 + 0^2} = \sqrt{17}$$
$$\underset{\sim}{v} = (1, 2, 3)$$
$$|\underset{\sim}{v}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$
$$\underset{\sim}{u} \bullet \underset{\sim}{v} = 4 \times 1 + (-1) \times 2 + 0 \times 3 = 2$$
$$\cos\theta = \frac{\underset{\sim}{u} \bullet \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \frac{2}{\sqrt{17}\sqrt{14}}$$
$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{17}\sqrt{14}}\right) = 82°33 \approx 83°$$

b

$$\underset{\sim}{u} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$$
$$|\underset{\sim}{u}| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$$
$$\underset{\sim}{v} = \underset{\sim}{i} + 5\underset{\sim}{j} - \underset{\sim}{k}$$
$$|\underset{\sim}{v}| = \sqrt{1^2 + 5^2 + (-1)^2} = \sqrt{27} = 3\sqrt{3}$$
$$\underset{\sim}{u} \bullet \underset{\sim}{v} = 2 \times 1 + 1 \times 5 + (-2) \times (-1) = 9$$
$$\cos\theta = \frac{\underset{\sim}{u} \bullet \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \frac{9}{3 \times 3\sqrt{3}}$$
$$\theta = \cos^{-}\left(\frac{1}{\sqrt{3}}\right) = 54°44 \approx 55°$$

c

$$\underset{\sim}{u} = (0, 5, 1)$$
$$|\underset{\sim}{u}| = \sqrt{0^2 + 5^2 + 1^2} = \sqrt{26}$$
$$\underset{\sim}{v} = (1, 5, -1)$$
$$|\underset{\sim}{v}| = \sqrt{1^2 + 5^2 + (-1)^2} = \sqrt{27} = 3\sqrt{3}$$
$$\underset{\sim}{u} \bullet \underset{\sim}{v} = 0 \times 1 + 5 \times 5 + 1 \times (-1) = 24$$
$$\cos\theta = \frac{\underset{\sim}{u} \bullet \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \frac{24}{3\sqrt{3}\sqrt{26}}$$
$$\theta = \cos^{-}\left(\frac{8}{\sqrt{78}}\right) = 25°04 \approx 25°$$

d

$$\underset{\sim}{u} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$$
$$|\underset{\sim}{u}| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$$
$$\underset{\sim}{v} = 4\underset{\sim}{j}$$
$$|\underset{\sim}{v}| = \sqrt{0^2 + 4^2 + 0^2} = \sqrt{16} = 4$$
$$\underset{\sim}{u} \bullet \underset{\sim}{v} = 2 \times 0 + 1 \times 4 + (-2) \times 0 = 4$$
$$\cos\theta = \frac{\underset{\sim}{u} \bullet \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \frac{4}{3 \times 4}$$
$$\theta = \cos^{-}\left(\frac{1}{3}\right) = 70°32 \approx 71°$$

Question 4

$$\underset{\sim}{u} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$$
$$|\underset{\sim}{u}| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$$
$$\underset{\sim}{v} = 3\underset{\sim}{j} + 4\underset{\sim}{k}$$
$$|\underset{\sim}{v}| = \sqrt{0^2 + 3^2 + 4^2} = \sqrt{25} = 5$$
$$\underset{\sim}{u} \bullet \underset{\sim}{v} = 2 \times 0 + 1 \times 3 + (-2) \times 4 = -5$$
$$\cos\theta = \frac{\underset{\sim}{u} \bullet \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \frac{-5}{3 \times 5} = -\frac{1}{3}$$

Question 5

$\underset{\sim}{u} = (2,2,2), \quad \underset{\sim}{v} = (3,2,-1), \quad \underset{\sim}{w} = (-1,4,1)$

a $\underset{\sim}{u} \cdot \underset{\sim}{v} = 2\times 3 + 2\times 2 + 2\times(-1) = 8$

$\underset{\sim}{u} \cdot \underset{\sim}{w} = 2\times(-1) + 2\times 4 + 2\times 1 = 8$

$\therefore \underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{u} \cdot \underset{\sim}{w}$

b $\underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{u} \cdot \underset{\sim}{w} \Rightarrow \underset{\sim}{u} \cdot \underset{\sim}{v} - \underset{\sim}{u} \cdot \underset{\sim}{w} = 0$

$\Rightarrow \underset{\sim}{u} \cdot (\underset{\sim}{v} - \underset{\sim}{w}) = 0$

$\therefore \underset{\sim}{u}$ and $\underset{\sim}{v} - \underset{\sim}{w}$ are perendicular.

Question 6

$\underset{\sim}{u} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}, \quad \underset{\sim}{v} = 6\underset{\sim}{i} + 8\underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = 2\times 6 + (-3)\times 8 + 4\times 3 = 0$

$\therefore \underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular because their scalar product is zero..

Question 7

Let $\underset{\sim}{u} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{v} = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$,

and the required vector $\underset{\sim}{w} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

such that $\underset{\sim}{v} \cdot \underset{\sim}{w} = 0$ and $\underset{\sim}{u} \cdot \underset{\sim}{w} = 0$

$\underset{\sim}{u} \cdot \underset{\sim}{w} = 0 \Rightarrow 2\times x + 1\times y + (-1)\times z = 0$

$2x + y - z = 0 \quad [1]$

$\underset{\sim}{v} \cdot \underset{\sim}{w} = 0 \Rightarrow 1\times x + (-2)\times y + 1\times z = 0$

$x - 2y + z = 0 \quad [2]$

$[1]+[2] \quad 3x - y = 0$

$y = 3x$

In [1] $2x + 3x - z = 0$

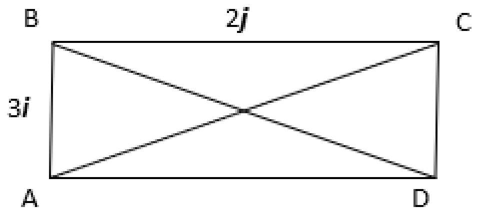
$z = 5x$

Let $x = 1, \Rightarrow (\underset{\sim}{i}, 3\underset{\sim}{j}, 5\underset{\sim}{k})$

So any vectors similar to $c(\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k})$ where c is a constant,

are perpendiular to $\underset{\sim}{u}$ and $\underset{\sim}{v}$

Question 8



a $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = 2\underset{\sim}{j} - 3\underset{\sim}{i} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$

b $\overrightarrow{AC} \bullet \overrightarrow{BD} = 3 \times (-3) + 2 \times 2 = -5$

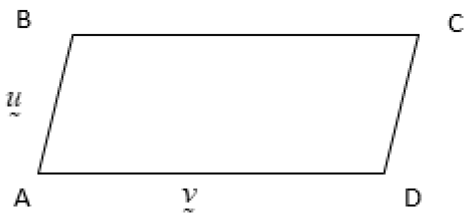
$\left|\overrightarrow{AC}\right| = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\left|\overrightarrow{BD}\right| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$

$\cos\theta = \dfrac{\overrightarrow{AC} \bullet \overrightarrow{BD}}{\left|\overrightarrow{AC}\right|\left|\overrightarrow{BD}\right|} = \dfrac{-5}{\sqrt{13}\sqrt{13}}$

$\theta = \cos^{-}\left(-\dfrac{5}{13}\right) = 112°37'$

Question 9



As the parallelogram has sides of equal length

$|\underset{\sim}{u}| = |\underset{\sim}{v}|$

$\overrightarrow{AC} = \underset{\sim}{u} + \underset{\sim}{v}$

$\overrightarrow{BC} = \underset{\sim}{v} - \underset{\sim}{u}$

$\overrightarrow{AC} \bullet \overrightarrow{BC} = (\underset{\sim}{u} + \underset{\sim}{v}) \bullet (\underset{\sim}{v} - \underset{\sim}{u}) = |\underset{\sim}{v}|^2 - |\underset{\sim}{u}|^2$

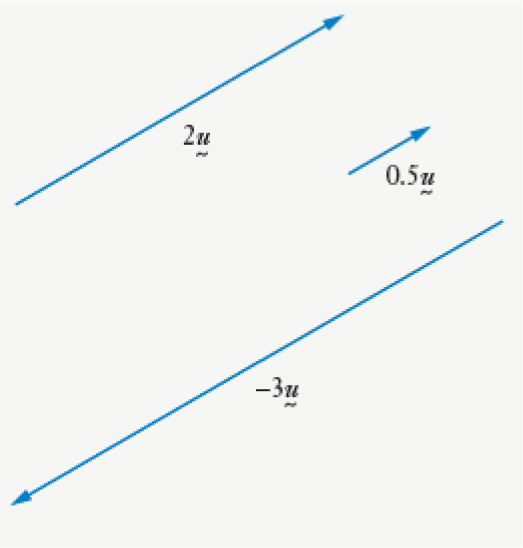
As $|\underset{\sim}{v}| = |\underset{\sim}{u}|$

$|\underset{\sim}{v}|^2 - |\underset{\sim}{u}|^2 = 0$

$\therefore$ The diagonals are perpendicular.

Exercise 3.04 Geometry proofs using vectors

Question 1

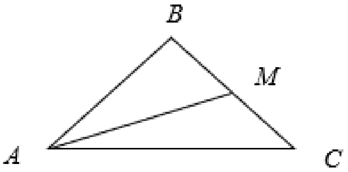


Question 2

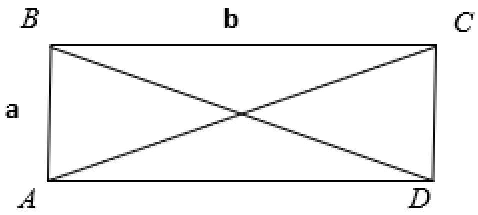
$$\overrightarrow{BC} = -\overrightarrow{AB} + \overrightarrow{AC}$$

$$\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\left(-\overrightarrow{AB} + \overrightarrow{AC}\right)$$

$$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} = \overrightarrow{AB} + \frac{1}{2}\left(-\overrightarrow{AB} + \overrightarrow{AC}\right) = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right)$$



Question 3



$\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{BD} = \underset{\sim}{b} - \underset{\sim}{a}$

As the diagonals are perpendicular:

$(\underset{\sim}{a} + \underset{\sim}{b}) \bullet (\underset{\sim}{b} - \underset{\sim}{a}) = 0$

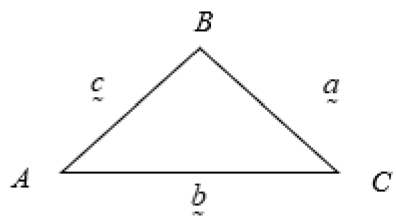
$|\underset{\sim}{b}|^2 - |\underset{\sim}{a}|^2 = 0$

$|\underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2$

$\therefore$ The lengths are the same, therefore it is a square.

Question 4

$$\begin{aligned}\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} &= \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} \\ &= \overrightarrow{BC} + \overrightarrow{CB} \\ &= \overrightarrow{BB} \\ &= \underset{\sim}{0}\end{aligned}$$



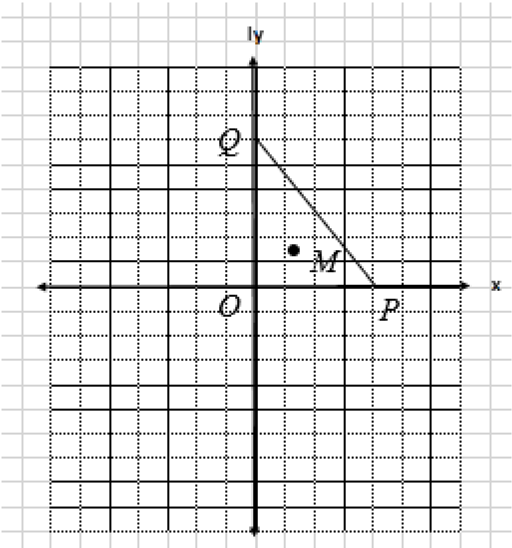
Question 5

Let

$\overrightarrow{OM} = x\underset{\sim}{i} + y\underset{\sim}{j}$

$\overrightarrow{QM} = -\overrightarrow{OQ} + \overrightarrow{OM}$

$\overrightarrow{PM} = -\overrightarrow{OP} + \overrightarrow{OM}$



Using

$\left|\overrightarrow{QM}\right| = \left|\overrightarrow{OM}\right|$

$\sqrt{x^2 + (y-6)^2} = \sqrt{x^2 + y^2}$

$x^2 + y^2 - 12y + 36 = x^2 + y^2$

$-12y + 36 = 0$

$12y = 36$

$y = 3$

Using

$\left|\overrightarrow{pM}\right| = \left|\overrightarrow{OM}\right|$

$\sqrt{(x-4)^2 + y^2} = \sqrt{x^2 + y^2}$

$x^2 - 8x + 16 + y^2 = x^2 + y^2$

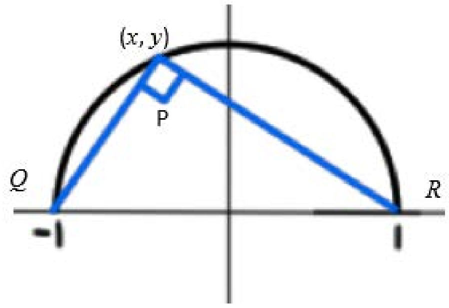
$-8x + 16 = 0$

$8x = 16$

$x = 2$

$\therefore M = (2,3)$

Question 6



$\overrightarrow{QP} = \underset{\sim}{i} + x\underset{\sim}{i} + y\underset{\sim}{j}$

$\overrightarrow{RP} = -\underset{\sim}{i} + x\underset{\sim}{i} + y\underset{\sim}{j}$

$\overrightarrow{QP} \bullet \overrightarrow{RP} = (1+x)(x-1) + y \times y$

$= -1 + x^2 + y^2$

As P is on the unit circle

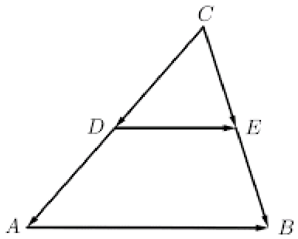
$x^2 + y^2 = 1$

$\therefore \overrightarrow{QP} \bullet \overrightarrow{RP} = 0$

$\therefore \overrightarrow{QP}$ is perpendicuar to $\overrightarrow{RP}$.

$\therefore$ The lines connecting any point on a semicircle are perpendicular.

Question 7



$$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{CB} - \overrightarrow{CA}$$

$$\overrightarrow{CD} = \frac{1}{2}\overrightarrow{CA}$$

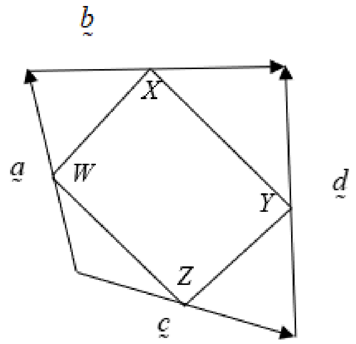
$$\overrightarrow{CE} = \frac{1}{2}\overrightarrow{CB}$$

$$\overrightarrow{DE} = -\overrightarrow{CD} + \overrightarrow{CE} = -\frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB} = \frac{1}{2}\left(\overrightarrow{CB} - \overrightarrow{CA}\right) = \frac{1}{2}\overrightarrow{AB}$$

$\therefore$ $\overrightarrow{DE}$ is parallel to $\overrightarrow{AB}$ and half the magnitude.

Question 8

From the diagram



$$\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$$

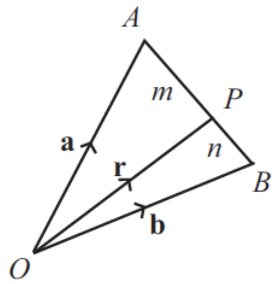
$$\Rightarrow \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right) = \frac{1}{2}\left(\underset{\sim}{c} + \underset{\sim}{d}\right)$$

$$\Rightarrow \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} = \frac{1}{2}\underset{\sim}{c} + \frac{1}{2}\underset{\sim}{d}$$

WX is parallel and equal in length to *ZY*.

WXYZ is a parallelogram.

Question 9



$$\overrightarrow{AB} = -\underset{\sim}{a} + \underset{\sim}{b}$$

$$\begin{aligned}
\overrightarrow{OP} &= \overrightarrow{OA} + \frac{m}{m+n}\overrightarrow{AB} \\
&= \underset{\sim}{a} + \frac{m}{m+n}\left(-\underset{\sim}{a} + \underset{\sim}{b}\right) \\
&= \frac{\underset{\sim}{a}(m+n) + m\left(-\underset{\sim}{a} + \underset{\sim}{b}\right)}{m+n} \\
&= \frac{\underset{\sim}{a}m + \underset{\sim}{a}n - m\underset{\sim}{a} + m\underset{\sim}{b}}{m+n} \\
&= \frac{\underset{\sim}{a}n + m\underset{\sim}{b}}{m+n} \\
&= \frac{n}{m+n}\underset{\sim}{a} + \frac{m}{m+n}\underset{\sim}{b}
\end{aligned}$$

Exercise 3.05 3D space

Question 1

a $\underset{\sim}{v} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$

b $\underset{\sim}{v} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{3^2 + (-4)^2 + 12^2} = \sqrt{169} = 13$

c $\underset{\sim}{v} = 2\underset{\sim}{i} + 5\underset{\sim}{j} + 14\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{2^2 + 5^2 + 14^2} = \sqrt{225} = 15$

d $\underset{\sim}{v} = 4\underset{\sim}{i} + 7\underset{\sim}{j} - 32\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{4^2 + 7^2 + (-32)^2} = \sqrt{1089} = 33$

e $\underset{\sim}{v} = -3\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{(-3)^2 + (-2)^2 + 6^2} = \sqrt{49} = 7$

Question 2

a $\hat{\underset{\sim}{v}} = \frac{1}{3}\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

b $\hat{\underset{\sim}{v}} = \frac{1}{13}\left(3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}\right)$

c $\hat{\underset{\sim}{v}} = \frac{1}{15}\left(2\underset{\sim}{i} + 5\underset{\sim}{j} + 14\underset{\sim}{k}\right)$

d $\underset{\sim}{v} = \frac{1}{33}\left(4\underset{\sim}{i} + 7\underset{\sim}{j} - 32\underset{\sim}{k}\right)$

e $\hat{\underset{\sim}{v}} = \frac{1}{7}\left(-3\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k}\right)$

Question 3

a

$\underset{\sim}{u} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$|\underset{\sim}{u}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

$\underset{\sim}{v} = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$

$\underset{\sim}{u} \bullet \underset{\sim}{v} = 2 \times 1 + 1 \times 1 + 1 \times 2 = 5$

$\cos\theta = \dfrac{\underset{\sim}{u} \bullet \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \dfrac{5}{\sqrt{6}\sqrt{6}}$

$\theta = \cos^{-}\left(\dfrac{5}{6}\right) = 33.6°$

b

$\underset{\sim}{u} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$|\underset{\sim}{u}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

$\underset{\sim}{v} = -\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{(-1)^2 + 1^2 + 2^2} = \sqrt{6}$

$\underset{\sim}{u} \bullet \underset{\sim}{v} = 1 \times (-1) + 2 \times 1 + 3 \times 2 = 7$

$\cos\theta = \dfrac{\underset{\sim}{u} \bullet \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \dfrac{7}{\sqrt{14}\sqrt{6}}$

$\theta = \cos^{-}\left(\dfrac{7}{\sqrt{84}}\right) = 40.2°$

c

$\underset{\sim}{u} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$

$|\underset{\sim}{u}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$

$\underset{\sim}{v} = \underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$

$\underset{\sim}{u} \bullet \underset{\sim}{v} = 2 \times 1 + 2 \times (-2) + 1 \times (-2) = -4$

$\cos\theta = \dfrac{\underset{\sim}{u} \bullet \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \dfrac{-4}{3 \times 3}$

$\theta = \cos^{-}\left(\dfrac{-4}{9}\right) = 116.4°$

d

$\underset{\sim}{u} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$|\underset{\sim}{u}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

$\underset{\sim}{v} = \underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$

$\underset{\sim}{u} \bullet \underset{\sim}{v} = 2 \times 1 + 1 \times 1 + 1 \times (-2) = 1$

$\cos\theta = \dfrac{\underset{\sim}{u} \bullet \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \dfrac{1}{\sqrt{6}\sqrt{6}}$

$\theta = \cos^{-}\left(\dfrac{1}{6}\right) = 80.4°$

e

$\underset{\sim}{u} = -3\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k}$

$|\underset{\sim}{u}| = \sqrt{(-3)^2 + (-2)^2 + 6^2} = \sqrt{49} = 7$

$\underset{\sim}{v} = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

$|\underset{\sim}{v}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$

$\underset{\sim}{u} \bullet \underset{\sim}{v} = (-3) \times 1 + (-2) \times 1 + 6 \times 2 = 7$

$\cos\theta = \dfrac{\underset{\sim}{u} \bullet \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \dfrac{7}{7\sqrt{6}}$

$\theta = \cos^{-}\left(\dfrac{1}{\sqrt{6}}\right) = 65.9°$

Question 4

a $\underset{\sim}{v} = 2\underset{\sim}{i} - y\underset{\sim}{j} + 14\underset{\sim}{k}$

$|\underset{\sim}{v}| = 15 = \sqrt{2^2 + (-y)^2 + 14^2}$

$225 = 2^2 + (-y)^2 + 14^2$

$225 = 2^2 + y^2 + 14^2$

$y^2 = 25$

$y = \pm 5$

b $\underset{\sim}{v} = 2\underset{\sim}{i} + 9\underset{\sim}{j} + z\underset{\sim}{k}$

$|\underset{\sim}{v}| = 43 = \sqrt{2^2 + 9^2 + z^2}$

$1849 = 4 + 81 + z^2$

$z^2 = 1764$

$z = \pm 42$

c $\underset{\sim}{v} = x\underset{\sim}{i} - 11\underset{\sim}{j} - 110\underset{\sim}{k}$

$|\underset{\sim}{v}| = 111 = \sqrt{x^2 + (-11)^2 + (-110)^2}$

$12321 = x^2 + 121 + 1210$

$x^2 = 100$

$x = \pm 10$

Question 5

$\underset{\sim}{v} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + m\underset{\sim}{k}$

$\hat{\underset{\sim}{v}} = \frac{1}{\sqrt{29}}\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + m\underset{\sim}{k}\right)$

$\Rightarrow |\underset{\sim}{v}| = \sqrt{29}$

$\therefore \sqrt{29} = \sqrt{2^2 + (-3)^2 + m^2}$

$29 = 4 + 9 + m^2$

$m^2 = 16$

$m = \pm 4$

Question 6

$\underset{\sim}{u} = (2, -3, 4)$

Let $\underset{\sim}{v} = (x, y, z)$ such that $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular.

$\underset{\sim}{u} \bullet \underset{\sim}{v} = 2 \times x + (-3) \times y + 4 \times z = 0 \quad \Rightarrow 2x - 3y + 4z = 0$

$\Rightarrow 2x + 4z = 3y$

$\underset{\sim}{v}$ is any vector such that its components (x, y, z) satisfy the condition $2x + 4z = 3y$

e.g. $(1, 2, 1)$

Question 7

$$\underset{\sim}{u} = (1, m, -1)$$

$$|\underset{\sim}{u}| = \sqrt{1^2 + m^2 + (-1)^2} = \sqrt{2 + m^2}$$

$$\underset{\sim}{v} = (1, -1, 1)$$

$$|\underset{\sim}{v}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

$$\underset{\sim}{u} \bullet \underset{\sim}{v} = 1 \times 1 + m \times (-1) + (-1) \times 1 = -m$$

$$\cos\theta = \frac{\underset{\sim}{u} \bullet \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$$

$$\cos 60° = \frac{-m}{\sqrt{2 + m^2}\sqrt{3}}$$

$$\frac{1}{2} = \frac{-m}{\sqrt{2 + m^2}\sqrt{3}}$$

$$-2m = \sqrt{6 + 3m^2}$$

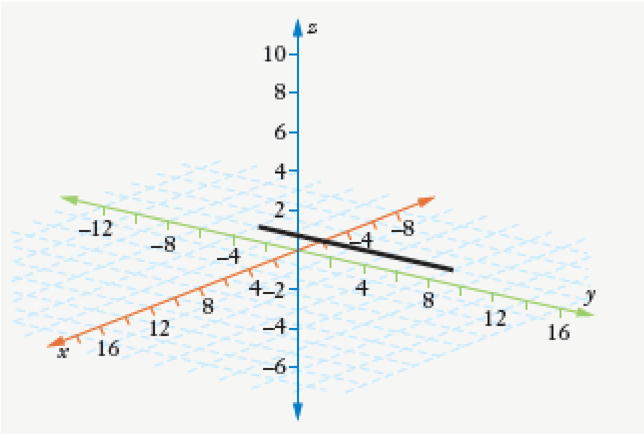
$$4m^2 = 6 + 3m^2$$

$$m^2 = 6$$

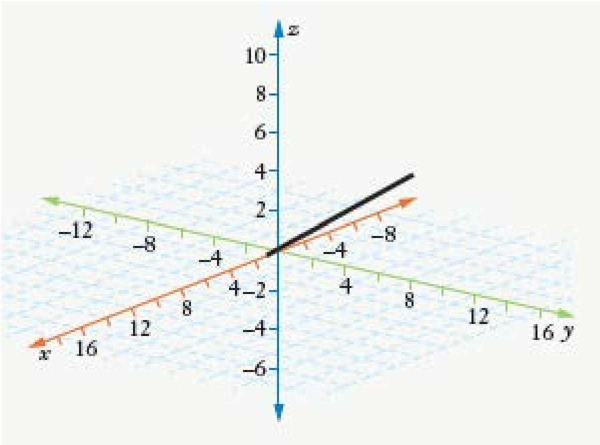
$$m = \pm\sqrt{6}$$

Question 1

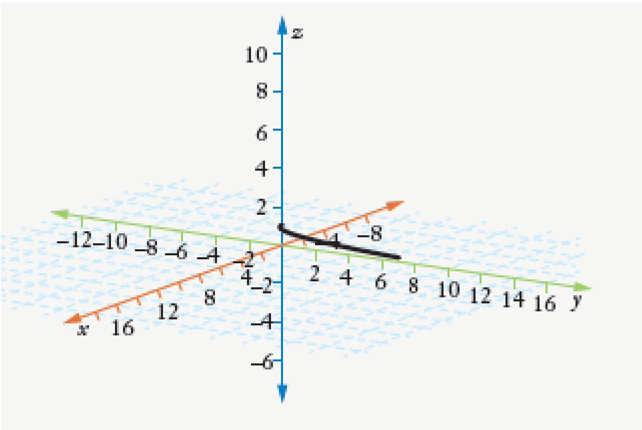
a



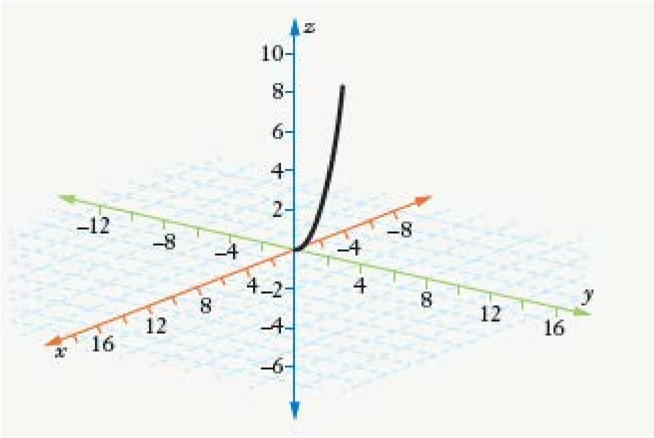
b



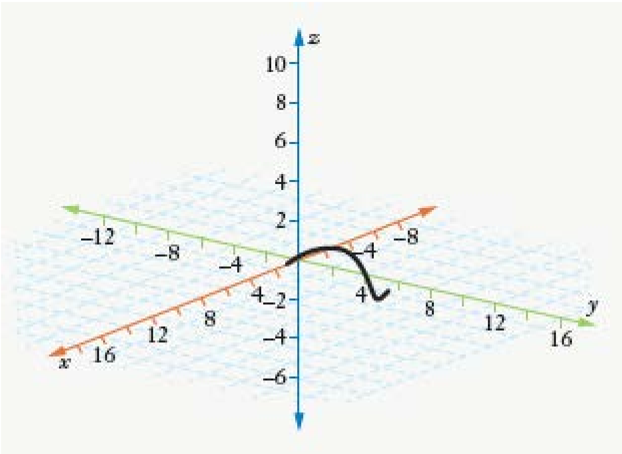
c



d



e

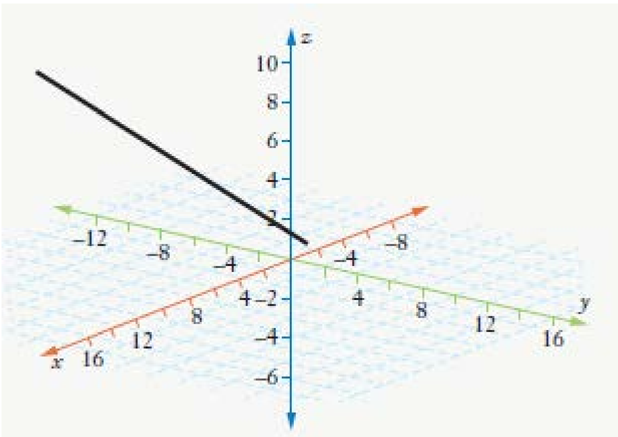


A helix (spiral) of radius 1 unit revolving around the *y*-axis,

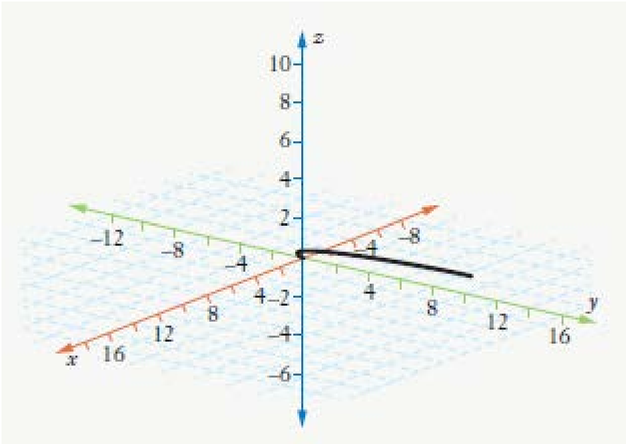
with endpoints (–1, 0, 0) and (–1, π, 0)

Question 2

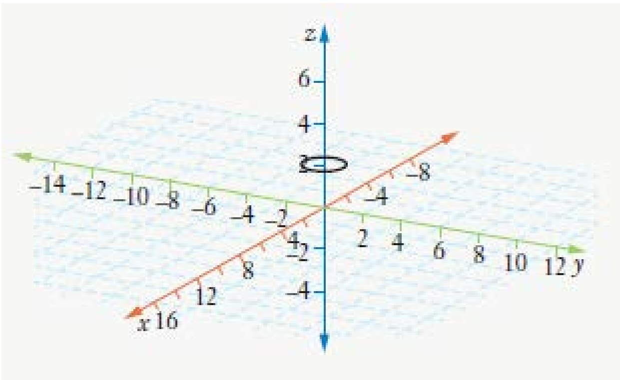
a



b


z
10
8
6
4
2
–12
–8
–4
4
8
12
16
x
–4
–8
4
8
12
16
y
–2
–4
–6

c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–10
–8
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–4
–2
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y
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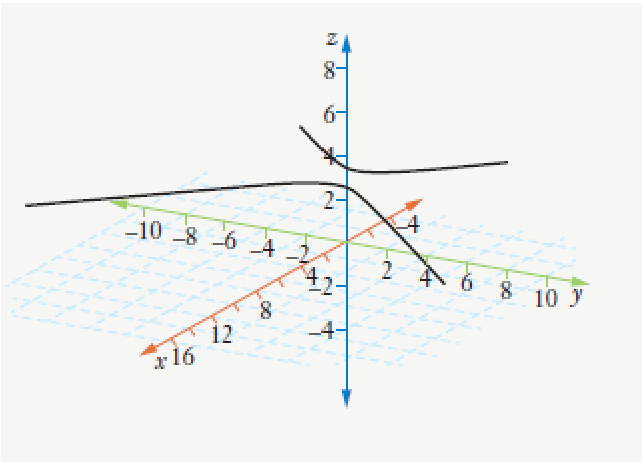
–4
4
8
12
x 16
–2
–4

d


z
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y
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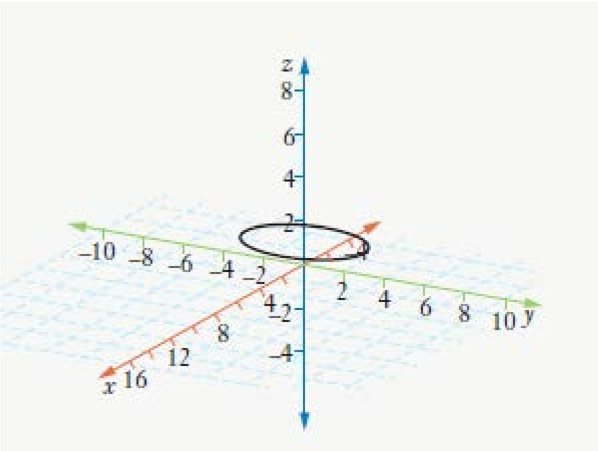
12
x 16
–2
–4

e



Question 3

a vertical line going through (1, 1, 0).

b helix (spiral) starting at (1, 0, 0) in both downward and anticlockwise directions around the negative *z*-axis.

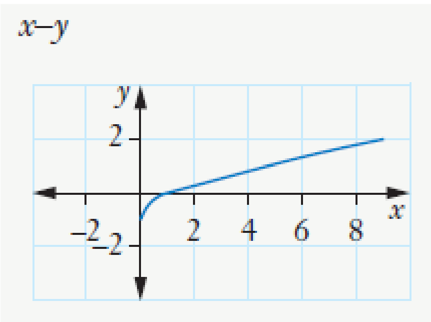
c ellipse (oval) 1 unit in front of the *y*–*z* plane, centred on (1, 0, 0), 4 units long on the *y*-axis, 2 units high on the *z*-axis.

d helix (spiral) in *z* direction starting at (0, 0, 0) and increasing in radius and height

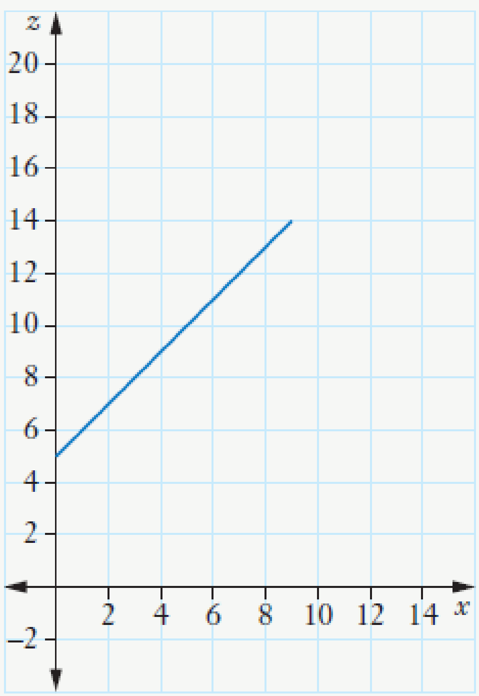
Question 4

$x = 3\cos(t),\ \ y = 3\sin(t),\ \ z = 2 - 3\sin(t)$

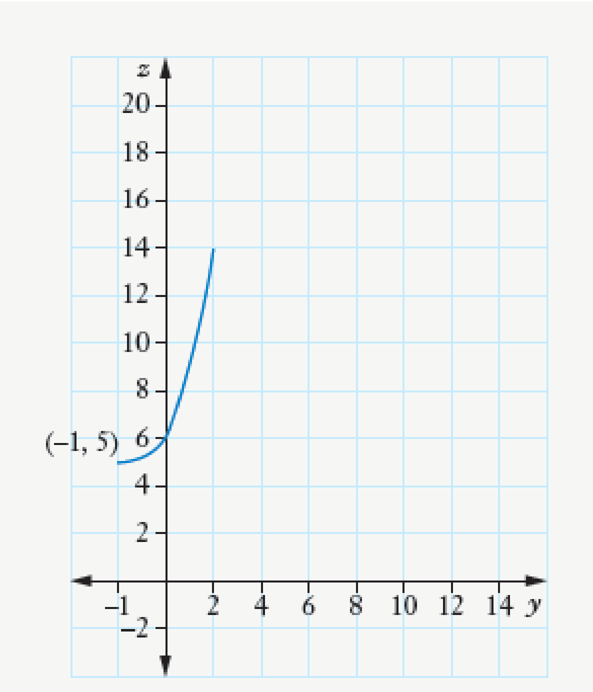
Question 5



x–*z*



y–*z*



Question 6

Given that $y = x$, we can write $z = 2x^2$ or $z = 2y^2$. So we get $(t, t, 2t^2)$ or in parametric form $x = t$, $y = t$ and $z = 2t^2$.

Question 7

Let

$x = t$

$\Rightarrow z = 1 + y = \sqrt{t^2 + y^2}$

$t^2 + y^2 = (1 + y)^2$

$y^2 + 2y + 1 = y^2 + t^2$

$2y + 1 = t^2$

$2y = t^2 - 1$

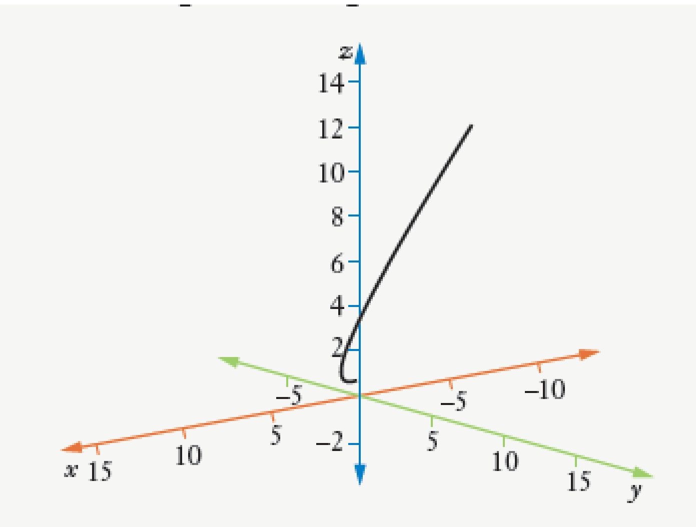
$y = \dfrac{t^2 - 1}{2}$

Using $z = 1 + y$ gives

$z = \dfrac{t^2 + 1}{2}$

which gives

$\left(t, \dfrac{t^2 - 1}{2}, \dfrac{t^2 + 1}{2}\right)$



Question 8

a $(x - 1)^2 + (y + 1)^2 + (z - 1)^2 = 1$

Generally $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ represents a sphere with centre (a, b, c) and radius r.

So centre $(1, -1, 1)$ and radius $= \sqrt{1} = 1$.

b $(x + 2)^2 + (y - 3)^2 + (z - 1)^2 = 4$

Applying the general equation for a sphere, centre $(-2, 3, 1)$ and radius $= \sqrt{4} = 2$.

c $(x - 3)^2 + (y + 1)^2 + (z + 1)^2 = 9$

Applying the general equation for a sphere, centre $(3, -1, -1)$ and radius $= \sqrt{9} = 3$.

d $x^2 + 2x + y^2 + 2y + z^2 - 2z = 6$

$x^2 + 2x + 1 + y^2 + 2y + 1 + z^2 - 2z + 1 = 6 + 1 + 1 + 1$

$(x + 1)^2 + (y + 1)^2 + (z - 1)^2 = 9$

Applying the general equation for a sphere, centre $(-1, -1, 1)$ and radius $= \sqrt{9} = 3$.

e $x^2 - 4x + y^2 - 6y + z^2 + 2z = 11$

$x^2 - 4x + 4 + y^2 - 6y + 9 + z^2 + 2z + 1 = 11 + 4 + 9 + 1$

$(x - 2)^2 + (y - 3)^2 + (z + 1)^2 = 25$

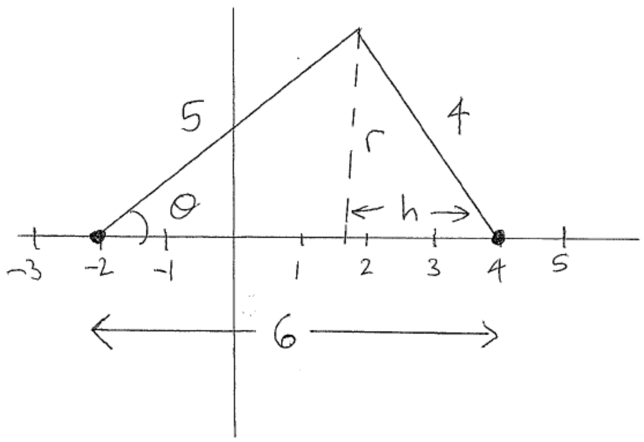
Applying the general equation for a sphere, centre $(2, 3, -1)$ and radius $= \sqrt{25} = 5$.

Question 9

First sphere: $(x + 2)^2 + (y + 3)^2 + (z - 4)^2 = 16$, centre (–2, –3, 4), radius 4.

Second spere: $(x + 2)^2 + (y + 3)^2 + (z + 2)^2 = 25$, centre (–2, –3, –2), radius 5.

The line joining the centres of the spheres is in the z-plane and is 6 units long, and forms a triangle other sides of length 4 and 5 (the radii of the spheres).



For the angle between the centre joining line and the 5 radius line,

$4^2 = 5^2 + 6^2 - 2(5)(6)\cos\theta$

$16 = 25 + 36 - 60\cos\theta$

$\cos\theta = \dfrac{-45}{-60} = \dfrac{3}{4}$

Radius of intersection circle = $r = 5\sin\theta$

$\sin\theta = \dfrac{\sqrt{7}}{4}$

$r = \dfrac{5\sqrt{7}}{4}$

Distance between circle centres = $z = 5\cos\theta$

$z = 5 \times \dfrac{3}{4} = \dfrac{15}{4}$

Centre of intersection circle has z-value $-2 + \dfrac{15}{4} = 1\dfrac{3}{4}$

Centre $(-2, -3, 1\frac{3}{4})$ and radius $\dfrac{5\sqrt{7}}{4}$.

Exercise 3.07 Vector equation of a straight line

This section will have multiple answers.

Question 1

a

$$(x,y,z)=(1,1,0)+\lambda(2,-3,0)$$
$$x=1+2\lambda$$
$$y=1-3\lambda$$
$$z=0$$

b

$$(x,y,z)=(11,2,0)+\lambda(3,0,0)$$
$$x=11+3\lambda$$
$$y=2$$
$$z=0$$

c

$$(x,y,z)=(3,0,-1)+\lambda(6,-9,1)$$
$$x=3+6\lambda$$
$$y=-9\lambda$$
$$z=-1+\lambda$$

d

$$(x,y,z)=(5,-2,1)+\lambda(7,-4,2)$$
$$x=5+7\lambda$$
$$y=-2-4\lambda$$
$$z=1+2\lambda$$

Question 2

a

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{d}$$
$$\underset{\sim}{r} = (2, -1, 3) + \lambda(1, 2, -1)$$

b

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{d}$$
$$\underset{\sim}{r} = (-2, 1, -3) + \lambda(-1, 2, 3)$$

c

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{d}$$
$$\underset{\sim}{r} = (1, -1, 1) + \lambda(2, -2, 1)$$

Question 3

a

$$(-2,-8)-(3,-5)=(-5,-3)$$
$$\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$$
$$\underset{\sim}{r}=(-2,-8)+\lambda(-5,-3)$$

b

$$(6,2,5)-(9,2,8)=(-3,0,-3)$$
$$\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$$
$$\underset{\sim}{r}=(6,2,5)+\lambda(-3,0,-3)$$

c

$$(1,1,-3)-(1,-1,-5)=(0,2,2)$$
$$\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$$
$$\underset{\sim}{r}=(1,1,-3)+\lambda(0,2,2)$$

d

$$(1,0,3)-(1,2,4)=(0,-2,-1)$$
$$\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$$
$$\underset{\sim}{r}=(1,0,3)+\lambda(0,-2,-1)$$

Question 4

a

$(1,3,-1)+\lambda\ (1,1,0)$

and

$(0,0,0)+\lambda_2(1,4,5)$

If they intesect then then for some a,b $\in \mathbb{R}$ such that

$(1,3,-1)+a(1,1,0)=(0,0,0)+b(1,4,5)$

$\therefore 1+a=b$

$3+a=4b$

$-1=5b$

$b=-\frac{1}{5}$

Using $1+a=b$

$a=-\frac{6}{5}$

Using $3+a=4b$

$3+\left(-\frac{6}{5}\right)=4\times\left(-\frac{1}{5}\right)$

Which is a contradiction.

$\therefore$ The two lines are skew.

b

$(1,0,2)+\lambda\ (-1,-1,2)$

and

$(4,4,2)+\lambda_2(2,2,-4)$

These lines are parallel as setting $\lambda_1(-1,-1,2)=\lambda_2(2,2,-4)$

for $\lambda_1=-2\lambda_1$

c

$(1,2,-1)+\lambda\ (1,2,3)$

and

$(1,0,1)+\lambda_2\left(\frac{2}{3},2,\frac{4}{3}\right)$

If they intesect then then for some a,b $\in \mathbb{R}$ such that

$(1,2,-1)+a(1,2,3)=(1,0,1)+b\left(\frac{2}{3},2,\frac{4}{3}\right)$

$\therefore 1+a=1+\frac{2}{3}b$

$2+2a=2b$

$-1+3a=1+\frac{4}{3}b$

Using $1+a=1+\frac{2}{3}b$

$a=\frac{2}{3}b$

Using $2+2a=2b$

$2+2\left(\frac{2}{3}b\right)=2b$

$2=\frac{2}{3}b$

$b=3$

$a=2$

Using $-1+3a=1+\frac{4}{3}b$

$-1+3\times 2=1+\frac{4}{3}\times 3$

Which is consistent.

$\therefore$ The two lines meet when $\lambda_1=2$ and $\lambda_2=3$, at $(3,6,5)$.

d

$(1,1,2)+\lambda\ (1,2,-3)$

and

$(2,3,-1)+\lambda_2(2,4,-6)$

These lines are parallel as setting $\lambda_1(1,2,-3)=\lambda_2(2,4,-6)$

for $\lambda_1=2\lambda_1$

Setting λ = 1 gives is the point $(2,3,-1)$

$\therefore$ They are the same line.

Question 5

$(3,2,6)-(-1,0,4)$

$=(4,2,2)$

Vector form

$\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$

$\underset{\sim}{r}=(3,2,6)+\lambda(4,2,2)$

Parametric form

$x=3+4\lambda$

$y=2+2\lambda$

$z=6+2\lambda$

Cartesian form

$\frac{x-3}{4}=\frac{y-2}{2}=\frac{z-6}{2}$

Question 6

$(2,7,4)-(0,2,1)=(2,5,3)$

$\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$

$\underset{\sim}{r}=(2,7,4)+\lambda(2,5,3)$

For $(a_1,1,a_3)$ to lie on the line $(2,7,4)+\lambda(2,5,3)$

We need to solve 7 + 5λ = 1

$5\lambda=-6$

$\lambda=\frac{-6}{5}$

$a_1=2+\left(\frac{-6}{5}\right)\times 2=-\frac{2}{5}$

$a_3=4+\left(\frac{-6}{5}\right)\times 3=\frac{2}{5}$

$a_1=-\frac{2}{5}\quad a_3=\frac{2}{5}$

Question 7

$(2,1,3)+\lambda\ (1,1,2)$

and

$(3,2,5)+\lambda_2(2,2,4)$

These lines are parallel as setting $\lambda_1(1,1,2)=\lambda_2(2,2,4)$

for $\lambda_1 = 2\lambda_1$

Setting λ = 1 gives the point $(3,2,5)$.

$\therefore$ They are the same line.

Question 8

$$\frac{-x+2}{7}=\frac{3y-1}{5}=\frac{2z+1}{3}$$

$-x=-2+7t$

$x=2-7t$

$3y=1+5t$

$y=\frac{1}{3}+\frac{5}{3}t$

$2z=-1+3t$

$z=-\frac{1}{2}+\frac{3}{2}t$

Vector form

$$\left(2,\frac{1}{3},-\frac{1}{2}\right)+\lambda\left(-7,\frac{5}{3},\frac{3}{2}\right)$$

Question 9

$(2,2,1)-(5,1,-2)$

$=(-3,1,3)$

$\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$

$\underset{\sim}{r}=(2,2,1)+\lambda(-3,1,3)$

For a point with x ordinate 4 to lie on the line $(2,2,1)+\lambda(-3,1,3)$

We need to solve $2-3\lambda=4$

$-3\lambda=2$

$\lambda=\dfrac{-2}{3}$

$z=1+\left(\dfrac{-2}{3}\right)\times 3$

$=-1$

Exercise 3.08 Parallel and perpendicular lines

Question 1

The vectors are parallel if $\lambda_2 = c\lambda_1$.

a $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}$

$\lambda_2 \begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} = \lambda_2 \left[-2 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right]$ so $\lambda_2 = -2\lambda_1$ and they are parallel.

b $\left(2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) + \lambda \left(\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$ and $\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right) + \lambda_2 \left(-3\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}\right)$

$\lambda_2 \left(-3\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}\right) = \lambda_2 \left[-3\left(\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)\right]$ so $\lambda_2 = -3\lambda_1$ and they are parallel.

c $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -0.5 \\ 1 \end{pmatrix}$

$\lambda_2 \begin{pmatrix} 1 \\ -0.5 \\ 1 \end{pmatrix} = \lambda_2 \left[-0.5 \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} \right]$ so $\lambda_2 = -0.5\lambda_1$ and they are parallel.

d $\left(3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}\right) + \lambda \left(2\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)$ and $\left(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right) + \lambda_2 \left(4\underset{\sim}{i} + 4\underset{\sim}{j} - 6\underset{\sim}{k}\right)$

$\lambda_2 \left(4\underset{\sim}{i} + 4\underset{\sim}{j} - 6\underset{\sim}{k}\right) = \lambda_2 \left[2\left(2\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)\right]$ so $\lambda_2 = 2\lambda_1$ and they are parallel.

Question 2

Vectors of the form $(a_1, a_2, a_3) + \lambda(b_1, b_2, b_3)$ and $(c_1, c_2, c_3) + \lambda(d_1, d_2, d_3)$ are perpendicular if $b_1d_1 + b_2d_2 + b_3d_3 = 0$

a $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$b_1d_1 + b_2d_2 + b_3d_3 = 1 \times 0 + 0 \times 2 + 1 \times 0 = 0$, so they are perpendicular.

b $\left(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) + \lambda\left(2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}\right)$ and $\left(3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right) + \lambda\left(2\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)$

$b_1d_1 + b_2d_2 + b_3d_3 = 2 \times 2 + (-1) \times (-2) + 2 \times (-3) = 0$, so they are perpendicular.

c $\begin{pmatrix} 7 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$b_1d_1 + b_2d_2 + b_3d_3 = (-1) \times (-2) + 0 \times 0 + (-2) \times 1 = 0$, so they are perpendicular.

d $\left(\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}\right) + \lambda\left(-\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}\right)$ and $\left(4\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right) + \lambda\left(-\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}\right)$

$b_1d_1 + b_2d_2 + b_3d_3 = (-1) \times (-1) + (-2) \times (-3) + 5 \times 1 = 0$, so they are perpendicular.

Question 3

a The vector part must be parallel to $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ but it can go through any point, so

$(x_1, y_1, z_1) + \lambda(2, -1, 3)$

b The vector part must be parallel to $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ but it can go through any point, so

$(x_1, y_1, z_1) + \lambda(3, 1, -2)$

c The vector part must be parallel to $\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$ but it can go through any point, so

$(x_1, y_1, z_1) + \lambda(4, 3, -1)$

d The vector part must be parallel to $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ but it can go through any point, so

$(x_1, y_1, z_1) + \lambda(2, 2, -1)$

Question 4

For the vectors to be perpendicular $b_1d_1 + b_2d_2 + b_3d_3 = 0$, so the required vectors will be of the form $(x_1, y_1, y_2) + \lambda(x_2, y_2, z_2)$, and $d_1x_2 + d_2y_2 + d_3z_2 = 0$.

a $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $d_1 = 2$, $d_2 = -1$, $d_3 = 3$, so $2x_2 - y_2 + 3z_2 = 0$.

b $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$, $d_1 = 3$, $d_2 = 1$, $d_3 = -2$, so $3x_2 + y_2 - 2z_2 = 0$.

c $4\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$, $d_1 = 4$, $d_2 = 4$, $d_3 = -1$, so $4x_2 + 3y_2 - z_2 = 0$.

d $\left(3\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) + \lambda\left(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$, $d_1 = 2$, $d_2 = 2$, $d_3 = -1$, so $2x_2 + 2y_2 - z_2 = 0$.

Question 5

Check if each pair is parallel or skew, and if they are not, find the point of intersection. These solutions do not show the lines that were parallel or skew:

A intersects **D:**

$7 + 4\lambda_1 = 4 + \lambda_4$

$4\lambda_1 - \lambda_4 = -3$ [1]

$-2 + 0\lambda_1 = -11 + 2\lambda_4$

$2\lambda_4 = 9$

$\lambda_4 = \frac{9}{2}$ [2]

Substitute into [1]:

$4\lambda_1 - \frac{9}{2} = -3$

$4\lambda_1 = \frac{3}{2}$

$\lambda_1 = \frac{3}{8}$

Checking for consistency:

$17 - 8\lambda_1 = 5 + 2\lambda_4$

$\text{LHS} = 17 - 8\left(\frac{3}{8}\right) = 14$

$\text{RHS} = 5 + 2\left(\frac{9}{2}\right) = 14$

$\therefore$ Point of intersection is: $\left(7+\frac{3}{8}\times 4, -2+\frac{3}{8}\times 0, 17-\frac{3}{8}\times 8\right) \equiv \left(8\frac{1}{2}, -2, 14\right)$

B intersects **D:**

$5 - 10\lambda_2 = 4 + \lambda_4$

$10\lambda_2 + \lambda_4 = 1$ [1]

$-9 + 5\lambda_2 = -11 + 2\lambda_4$

$5\lambda_2 - 2\lambda_4 = -2$ [2]

[1] × 2:

$20\lambda_2 + 2\lambda_4 = 2$ [3]

[2] + [3]:

$25\lambda_2 = 0$

$\lambda_2 = 0$

Substitute into [1]:

$10(0) + \lambda_4 = 1$

$\lambda_4 = 1$

Checking for consistency:

$7 - 9\lambda_2 = 5 + 2\lambda_4$

LHS = 7 – 9(0) = 7

RHS = 5 + 2(1) = 7

$\therefore$ Point of intersection is: $\left(5-0\times\left[-10\right],-9+0\times5,7-0\times\left[-9\right]\right)\equiv\left(5,-9,7\right)$

C intersects **D:**

$6 + 6\lambda_3 = 4 + \lambda_4$

$6\lambda_3 - \lambda_4 = -2$ [1]

$3 + 2\lambda_3 = -11 + 2\lambda_4$

$2\lambda_3 - 2\lambda_4 = -14$

$\lambda_3 - \lambda_4 = -7$ [2]

[1] – [2]:

$5\lambda_3 = 5$

$\lambda_3 = 1$

Substitute into [2]:

$1 - \lambda_4 = -7$

$1 + 7 = \lambda_4$

$\lambda_4 = 8$

Checking for consistency:

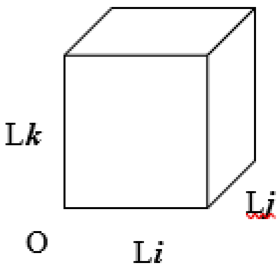
$16 + 5\lambda_3 = 5 + 2\lambda_4$

LHS = 16 + 5(1) = 21

RHS = 5 + 2(8) = 21

$\therefore$ Point of intersection is: $(6+1\times 6, 3+1\times 2, 16+1\times 5) \equiv (12, 5, 21)$

Question 6



For the point $L\left(\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}\right)$ an adjacent edge is $L\underset{\sim}{i}$

$$\underset{\sim}{u}=L\left(\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}\right)$$

$$\left|\underset{\sim}{u}\right|=L\sqrt{1^2+1^2+1^2}$$

$$=L\sqrt{3}$$

$$\underset{\sim}{v}=L\underset{\sim}{i}$$

$$\left|\underset{\sim}{v}\right|=L\sqrt{1^2+0^2+0^2}$$

$$=L$$

$$\underset{\sim}{u}\bullet\underset{\sim}{v}=L\times L+L\times 0+L\times 0$$

$$=L^2$$

$$\cos\theta=\frac{\underset{\sim}{u}\bullet\underset{\sim}{v}}{\left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|}$$

$$\cos\theta=\frac{L^2}{L\sqrt{3}\times L}$$

$$\theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\theta=547\,^\circ$$

Question 7

a $A(2, -3, 3)$ and $C(-2, 3, -1)$:

The line AC has direction $\begin{pmatrix} -2-2 \\ 3-(-3) \\ -1-3 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ -4 \end{pmatrix}$ and equation $(2, -3, 3) + \lambda(-4, 6, -4)$.

$B(-3, 2, 1)$ and $D(3, -2, 1)$:

The line BD has direction $\begin{pmatrix} 3-(-3) \\ -2-2 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 0 \end{pmatrix}$ and equation $(-3, 2, 1) + \lambda(6, -4, 0)$.

The direction vectors $\begin{pmatrix} -4 \\ 6 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -4 \\ 0 \end{pmatrix}$ are different and so the lines are not parallel.

If they intersect there must be 2 values a and b such that

$$\begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix} + a\begin{pmatrix} -4 \\ 6 \\ -4 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} 6 \\ -4 \\ 0 \end{pmatrix}$$

So $2 - 4a = -3 + 6b$ [1]

$-3 + 6a = 2 - 4b$ [2]

$3 - 4a = 1$ [3]

From [3] $a = \frac{1}{2}$, from [2] $b = \frac{1}{2}$,

and in [1] LHS = $2 - 4(\frac{1}{2}) = 0$ and RHS = $-3 + 6(\frac{1}{2}) = 0$,

so a and b satisfy all 3 equations.

Hence, the 2 lines intersect.

b Angle is θ where:

$$\begin{aligned}\cos\theta &= \frac{\overrightarrow{AC}\cdot\overrightarrow{BD}}{\left|\overrightarrow{AC}\right|\left|\overrightarrow{BD}\right|}\\ &= \frac{(-4)\times 6 + 6\times(-4) + (-4)\times 0}{\sqrt{(-4)^2+6^2+(-4)^2}\sqrt{6^2+(-4)^2+0^2}}\\ &= \frac{-48}{\sqrt{68}\sqrt{52}}\\ &= -0.8072...\end{aligned}$$

$$\begin{aligned}\theta &= 1438239\ ^\circ\\ &\approx 1438\ ^\circ\end{aligned}$$

Question 8

The line will be of the form (6, –2, 1) + λ(*a*, *b*, *c*), such that (*a*, *b*, *c*) is perpendicular to (3, –1, 1) and (1, –3, 7).

From the perpendicular conditions:

$3a - b + c = 0$

$a - 3b + 7c = 0$

Let $a = 1$, so $-b + c = -3$ [1] and $-3b + 7c = -1$ [2].

[2] – 7[1] gives $4b = 20$, $b = 5$.

So $3 - 5 + c = 0$, so $c = 2$.

(6, –2, 1) + λ(1, 5, 2)

Question 9

$$\frac{x-4}{8}=\frac{y-12}{5}=\frac{z-15}{2}$$

$x=4+8\lambda$

$y=12+5\lambda$

$z=15+2\lambda$

$\underset{\sim}{r}=(4,12,15)+\lambda(8,5,2)$

Question 10

Any 2 lines of the form $\begin{pmatrix}2\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}a\\b\\c\end{pmatrix}$ and $\begin{pmatrix}2\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}$ where $a_1a_2 + b_1b_2 + c_1c_2 = 0$, for example, $\begin{pmatrix}2\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}$.

Test yourself 3

Question 1

$\underset{\sim}{u} = (2,1)$

$|\underset{\sim}{u}| = \sqrt{2^2 + 1^2}$

$= \sqrt{5}$

$\underset{\sim}{v} = (1,2)$

$|\underset{\sim}{v}| = \sqrt{1^2 + 2^2}$

$= \sqrt{5}$

$\underset{\sim}{u} \bullet \underset{\sim}{v} = 2 \times 1 + 1 \times 2$

$= 4$

$\underset{\sim}{u} \bullet \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}| \cos \theta$

$\therefore \sqrt{5} \times \sqrt{5} \cos \theta = 4$

$\cos \theta = \dfrac{4}{5}$

$\theta = \cos^{-1}\left(\dfrac{4}{5}\right)$

$\therefore \theta = 36°52'$

Question 2

$\underset{\sim}{u} = (2, -3)$

$\underset{\sim}{v} = (4.5, 3)$

These vectors do not appear to be a multiple of each other, so they are not parallel.

$\underset{\sim}{u} \bullet \underset{\sim}{v} = 2 \times 4.5 + (-3) \times 3$

$= 0$

$\therefore$ They are perpendicular.

Question 3

$\underset{\sim}{v} = 8(\cos 60, \sin 60)$

$= \left(4, 4\sqrt{3}\right)$

$= 4\underset{\sim}{i} + 4\sqrt{3}\underset{\sim}{j}$

$\underset{\sim}{v} = \frac{1}{8}\left(4\underset{\sim}{i} + 4\sqrt{3}\underset{\sim}{j}\right)$

Question 4

$\underset{\sim}{v} = -2\underset{\sim}{i} + 5\underset{\sim}{j}$

$|\underset{\sim}{v}| = \sqrt{(-2)^2 + (5)^2}$

$= \sqrt{29}$

$\tan\theta = \frac{5}{-2}$

$\theta = \tan^{-}\left(-\frac{5}{2}\right)$

$= 180° - 68°12' = 111°48'$

Question 5

$(0,3,1) - (-1,2,-3)$

$= (1,1,4)$

Question 6

$\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$

$\underset{\sim}{v} = \underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$

$\underset{\sim}{u} \bullet \underset{\sim}{v} = 2\times1 + (-1)\times(-2) + 4\times(-1)$

$= 0$

The vectors are perpendicular.

Question 7

Let

$\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{v} = \underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$

$\underset{\sim}{w} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

such that $\underset{\sim}{v} \bullet \underset{\sim}{w} = 0$ and $\underset{\sim}{u} \bullet \underset{\sim}{w} = 0$

$x - 2y - z = 0$

$2x - y + z = 0$

Let $x = 1, 2y + z = 1$ and $-y + z = -2$.

So $3y = 3, y = 1$ and $z = -1$.

$\therefore$ Any vector such as $\lambda(1,1,-1)$

will be perpendicular to both $\underset{\sim}{v}$ and $\underset{\sim}{u}$

Such a unit vector would be $\frac{1}{\sqrt{3}}(1,1,-1)$

Question 8

$\overline{OX} = 2\underset{\sim}{a} + \underset{\sim}{b}$

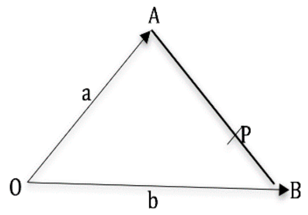
$\overline{OY} = 3\underset{\sim}{a} + 4\underset{\sim}{b}$

$\overline{XY} = -\overline{OX} + \overline{OY}$

$= 3\underset{\sim}{a} + 4\underset{\sim}{b} - (2\underset{\sim}{a} + \underset{\sim}{b})$

$= \underset{\sim}{a} + 3\underset{\sim}{b}$

Question 9



a

$$\overline{AB} = -\overline{OA} + \overline{OB}$$
$$= -\underset{\sim}{a} + \underset{\sim}{b}$$
$$= \underset{\sim}{b} - \underset{\sim}{a}$$

b

$$\overline{AP} = \frac{3}{5}\overline{AB}$$
$$\overline{OP} = \overline{OA} + \overline{AP}$$
$$= \overline{OA} + \frac{3}{5}\overline{AB}$$
$$= \underset{\sim}{a} + \frac{3}{5}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$$
$$= \underset{\sim}{a} + \frac{3}{5}\underset{\sim}{b} - \frac{3}{5}\underset{\sim}{a}$$
$$= \frac{1}{5}\left(2\underset{\sim}{a} + 3\underset{\sim}{b}\right)$$

Question 10

$\underset{\sim}{u} = (2, -2, 1)$

$|\underset{\sim}{u}| = \sqrt{2^2 + (-2)^2 + 1^2}$

$= \sqrt{9}$

$= 3$

$\underset{\sim}{v} = \left(-3, 1, 2\sqrt{2}\right)$

$|\underset{\sim}{v}| = \sqrt{(-3)^2 + 1^2 + \left(2\sqrt{2}\right)^2}$

$= \sqrt{18}$

$= 3\sqrt{2}$

$\underset{\sim}{u} \bullet \underset{\sim}{v} = 2 \times (-3) + (-2) \times 1 + 1 \times \left(2\sqrt{2}\right)$

$= 2\sqrt{2} - 8$

$\cos\theta = \dfrac{\underset{\sim}{u} \bullet \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$

$\cos\theta = \dfrac{2\sqrt{2} - 8}{3 \times 3\sqrt{2}}$

$\theta = \cos^{-1}\left(\dfrac{2\sqrt{2} - 8}{3 \times 3\sqrt{2}}\right)$

$\theta = 11397\,°$

Question 11

$\underset{\sim}{u} = \underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$

Let

$\underset{\sim}{v} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ such that $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular.

$\underset{\sim}{u} \bullet \underset{\sim}{v} = 1 \times x + (-2) \times y + (-1) \times z = 0$

$\Rightarrow x - 2y - z = 0$

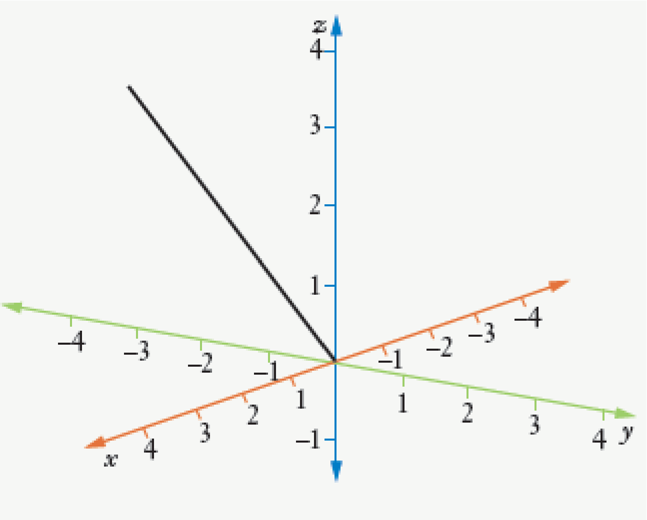
$x = 2y + z$

$\underset{\sim}{v}$ is any vector such that its components (x, y, z)

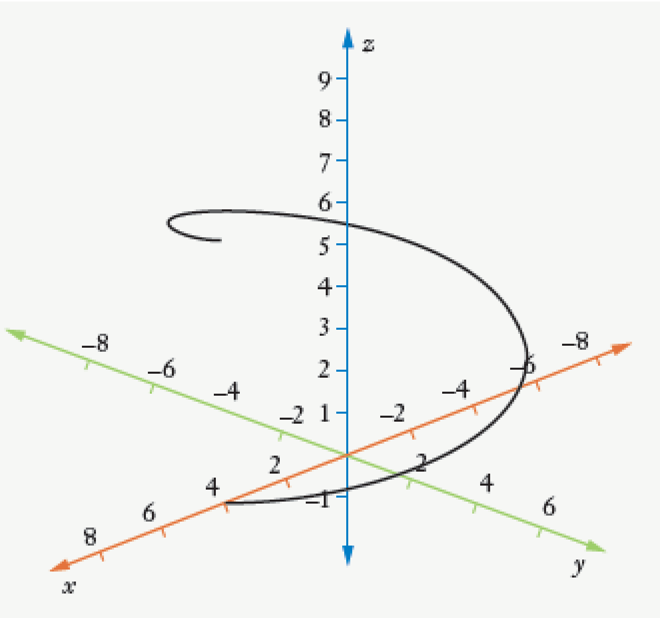
satisfy the condition $x = 2y + z$

e.g. $(2, -1, 4)$

Question 12



Question 13



Question 14

$x^2 + (y - 4)^2 + (z + 1)^2 = 5$

Generally $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ represents a sphere with centre (a, b, c) and radius r.

So centre $(0, 4, -1)$ and radius = $\sqrt{5}$

Question 15

$$\frac{(x+3)}{2} = \frac{(y-2)}{1} = \frac{(z+1)}{3}$$

Parametric equation.

$x = -3 + 2\lambda$

$y = 2 + \lambda$

$z = -1 + 3\lambda$

Vector equation

$\underset{\sim}{r} = (-3, 2, -1) + \lambda(2, 1, 3)$

Question 16

$(-2,2,3)-(1,5,-2)$
$=(-3,-3,5)$
$\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$
$\underset{\sim}{r}=(-2,2,3)+\lambda(-3,-3,5)$
For $(a_1,-3,a_3)$ to lie on the line $(-2,2,3)+\lambda(-3,-3,5)$
We need to solve $2-3\lambda=-3$
$-5=-3\lambda$
$\lambda=\frac{5}{3}$
$a_1=-2+\left(\frac{5}{3}\right)\times(-3)=-7$
$a_3=3+\left(\frac{5}{3}\right)\times 5=11\frac{1}{3}$
x coordinate = -7
z coordinate = $11\frac{1}{3}$

Question 17

Let the vectors be $\lambda_1\begin{pmatrix}1\\0\\-2\end{pmatrix}$ and $\lambda_2\begin{pmatrix}-2\\0\\4\end{pmatrix}$

The vectors are parallel if $\lambda_2 = c\lambda_1$.

$\lambda_2\begin{pmatrix}-2\\0\\4\end{pmatrix}=\lambda_2\left[-2\begin{pmatrix}1\\0\\-2\end{pmatrix}\right]$ so $\lambda_2 = -2\lambda_1$ and they are parallel.

Question 18

Vectors of the form $(a_1, a_2, a_3) + \lambda(b_1, b_2, b_3)$ and $(c_1, c_2, c_3) + \lambda(d_1, d_2, d_3)$ are perpendicular if $b_1d_1 + b_2d_2 + b_3d_3 = 0$

$$\lambda\begin{pmatrix}7\\1\\-2\end{pmatrix} \text{ and } \lambda\begin{pmatrix}3\\-3\\9\end{pmatrix}$$

$b_1d_1 + b_2d_2 + b_3d_3 = 7 \times 3 + 1 \times (-3) + (-2) \times 9 = 0$, so they are perpendicular.

Question 19

$$\begin{aligned}
\underset{\sim}{u} &= ((-2),1,0) \\
|\underset{\sim}{u}| &= \sqrt{(-2)^2 + 1^2 + 0^2} \\
&= \sqrt{5} \\
\underset{\sim}{v} &= (0,1,-2) \\
|\underset{\sim}{v}| &= \sqrt{0^2 + 1^2 + (-2)^2} \\
&= \sqrt{5} \\
\underset{\sim}{u} \bullet \underset{\sim}{v} &= (-2)\times 0 + 1\times 1 + 0\times(-2) \\
&= 1 \\
\cos\theta &= \frac{\underset{\sim}{u} \bullet \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} \\
\cos\theta &= \frac{1}{\sqrt{5}\sqrt{5}} \\
\theta &= \cos^{-1}\left(\frac{1}{5}\right) \\
\theta &= 785\ ^\circ
\end{aligned}$$

Question 20

The line will be of the form (1, –3, 2) + λ(a, b, c), such that (a, b, c) is perpendicular to (1, –2, 3) and (0, 1, –5).

From the perpendicular conditions:

$a - 2b + 3c = 0$

$b - 5c = 0$

Let $c = 1$, so $b - 5 = 0$, $b = 5$ and $a - 10 + 3 = 0$, so $a = 7$.

(1, –3, 2) + λ(7, 5, 1)

MATHS IN FOCUS 12 MATHEMATICS EXTENSION 2

WORKED SOLUTIONS

Chapter 4: Applying complex numbers

Exercise 4.01 De Moivre's theorem

Question 1

a

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$$

b

$$\begin{aligned}&(\cos\theta + i\sin\theta)^{-3}\\&= \cos(-3\theta) + i\sin(-3\theta)\\&= \cos 3\theta - i\sin 3\theta\end{aligned}$$

c

$$\begin{aligned}&(\cos\theta - i\sin\theta)^{7}\\&= \cos(7\theta) + i\sin(-7\theta)\\&= \cos 7\theta - i\sin 7\theta\end{aligned}$$

d

$$\begin{aligned}&\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-5}\\&= \cos\left(-\frac{5\theta}{2}\right) + i\sin\left(-\frac{5\theta}{2}\right)\\&= \cos\left(\frac{5\theta}{2}\right) - i\sin\left(\frac{5\theta}{2}\right)\end{aligned}$$

e

$$(\cos 4\theta + i\sin 4\theta)^{\frac{3}{4}} = \cos 3\theta + i\sin 3\theta$$

Question 2

a $z = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

i

$$z^5 = 2^5\left(\cos\left(5\times\frac{\pi}{6}\right) + i\sin\left(5\times\frac{\pi}{6}\right)\right)$$
$$= 32\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$$

ii

$$z = 32\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$$
$$z = 32\times\left(-\frac{\sqrt{3}}{2}\right) + i\times 32\times\frac{1}{2}$$
$$z = -16\sqrt{3} + 16i$$

b $z = \sqrt{3}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

i

$$z^5 = \sqrt{3}^5\left(\cos\left(5\times\frac{\pi}{4}\right) + i\sin\left(5\times\frac{\pi}{4}\right)\right)$$
$$= 9\sqrt{3}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)$$
$$= 9\sqrt{3}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$$

ii

$$z = 9\sqrt{3}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$$
$$= 9\sqrt{3}\cos\left(-\frac{3\pi}{4}\right) + 9\sqrt{3}i\sin\left(-\frac{3\pi}{4}\right)$$
$$= -\frac{9\sqrt{6}}{2} - \frac{9\sqrt{6}}{2}i$$
$$= -\frac{9\sqrt{6}}{2}(1+i)$$

c $z = \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

i

$$z^5 = \left(\frac{1}{\sqrt{2}}\right)^5\left(\cos\left(5\times\frac{\pi}{3}\right) + i\sin\left(5\times\frac{\pi}{3}\right)\right)$$
$$= \frac{1}{4\sqrt{2}}\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$$
$$= \frac{1}{4\sqrt{2}}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

ii

$$z = \frac{\sqrt{2}}{8}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$
$$= \frac{\sqrt{2}}{8}\times\frac{1}{2} + \frac{\sqrt{2}}{8}\times\frac{-\sqrt{3}}{2}i$$
$$= \frac{1}{8\sqrt{2}} - \frac{\sqrt{3}}{8\sqrt{2}}i$$

d $z=-3\left(\cos\left(\frac{7\pi}{10}\right)+i\sin\left(\frac{7\pi}{10}\right)\right)$

i

$$\begin{aligned}z^5&=(-3)^5\left(\cos\left(5\times\frac{7\pi}{10}\right)+i\sin\left(5\times\frac{7\pi}{10}\right)\right)\\&=-243\left(\cos\left(\frac{35\pi}{10}\right)+i\sin\left(\frac{35\pi}{10}\right)\right)\\&=-243\left(\cos\left(\frac{7\pi}{2}\right)+i\sin\left(\frac{7\pi}{2}\right)\right)\\&=-243\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)\\&=-243\left(-\cos\left(\frac{\pi}{2}\right)-i\sin\left(\frac{\pi}{2}\right)\right)\\&=243\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)\end{aligned}$$

ii

$$\begin{aligned}z&=243\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)\\&=243\big(0+i(1)\big)\\&=243i\end{aligned}$$

Question 3

$$\begin{aligned}&\left(\cos\left(-\frac{3\pi}{5}\right)+i\sin\left(-\frac{3\pi}{5}\right)\right)^{-6}\\&=\cos\left(-6\times-\frac{3\pi}{5}\right)+i\sin\left(-6\times-\frac{3\pi}{5}\right)\\&=\cos\left(\frac{18\pi}{5}\right)+i\sin\left(\frac{18\pi}{5}\right)\\&=\operatorname{cis}\left(\frac{8\pi}{5}\right)\\&=\operatorname{cis}\left(-\frac{2\pi}{5}\right)\\&=\cos\left(-\frac{2\pi}{5}\right)+i\sin\left(-\frac{2\pi}{5}\right)\end{aligned}$$

Question 4

a

$$(1-i)^3$$

$$z = 1 - i$$

$$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan\theta = \left(\frac{(-1)}{1}\right)$$

$$\theta = \tan^{-}(-1) = -\frac{\pi}{4}$$

$$z^3 = \left(\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)\right)^3$$

$$= \sqrt{2}^3\operatorname{cis}\left(3 \times -\frac{\pi}{4}\right)$$

$$= 2\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$$

$$= 2\sqrt{2}\cos\left(-\frac{3\pi}{4}\right) + i2\sqrt{2}\sin\left(-\frac{3\pi}{4}\right)$$

$$= 2\sqrt{2} \times \left(\frac{-\sqrt{2}}{2}\right) + i2\sqrt{2} \times \left(\frac{-\sqrt{2}}{2}\right)$$

$$= -2 - 2i$$

b

$$\left(1+i\sqrt{3}\right)^3$$

$$z = 1 + i\sqrt{3}$$

$$|z| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\tan\theta = \left(\frac{\sqrt{3}}{1}\right)$$

$$\theta = \tan^{-}\ \sqrt{3} = \frac{\pi}{3}$$

$$z^4 = \left(2\text{cis}\left(\frac{\pi}{3}\right)\right)^4$$

$$= 2^4\text{cis}\left(4\times\frac{\pi}{3}\right)$$

$$= 16\text{cis}\left(\frac{4\pi}{3}\right)$$

$$= 16\text{cis}\left(\frac{-2\pi}{3}\right)$$

$$= 16\cos\left(-\frac{2\pi}{3}\right) + i16\sin\left(-\frac{2\pi}{3}\right)$$

$$= 16\times\left(-\frac{1}{2}\right) + i16\times\left(-\frac{\sqrt{3}}{2}\right)$$

$$= -8 - 8\sqrt{3}i$$

c

$$\left(-\sqrt{2}+i\sqrt{2}\right)^5$$

$$z=-\sqrt{2}+i\sqrt{2}$$

$$|z|=\sqrt{\left(-\sqrt{2}\right)^2+\sqrt{2}^2}=2$$

$$\tan\theta=\left(\frac{\sqrt{2}}{\left(-\sqrt{2}\right)}\right)$$

$$\theta=\tan^{-}\left(-1\right)=\frac{3\pi}{4}$$

$$z^5=\left(2\text{cis}\left(\frac{3\pi}{4}\right)\right)^5$$

$$=2^5\text{cis}\left(5\times\frac{3\pi}{4}\right)$$

$$=32\text{cis}\left(\frac{15\pi}{4}\right)$$

$$=32\text{cis}\left(\frac{7\pi}{4}\right)$$

$$=32\text{cis}\left(\frac{-\pi}{4}\right)$$

$$=32\cos\left(-\frac{\pi}{4}\right)+i32\sin\left(-\frac{\pi}{4}\right)$$

$$=32\times\left(\frac{\sqrt{2}}{2}\right)+i32\times\left(-\frac{\sqrt{2}}{2}\right)$$

$$=16\sqrt{2}-16\sqrt{2}i$$

d

$$\left(\frac{1+i}{\sqrt{2}}\right)^{-3}$$

$$z=\frac{1+i}{\sqrt{2}}$$

$$|z|=\sqrt{\left(\frac{1}{\sqrt{2}}\right)^{2}+\left(\frac{1}{\sqrt{2}}\right)^{2}}=1$$

$$\tan\theta=\left(\frac{\left(\frac{1}{\sqrt{2}}\right)}{\left(\frac{1}{\sqrt{2}}\right)}\right)$$

$$\theta=\tan^{-}(1)=\frac{\pi}{4}$$

$$z^{-3}=\left(\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^{-3}$$

$$=\operatorname{cis}\left(-3\times\frac{\pi}{4}\right)$$

$$=\operatorname{cis}\left(-\frac{3\pi}{4}\right)$$

$$=\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)$$

$$=-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$$

e

$$\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{\frac{1}{2}}$$

$$z=\frac{1}{2}+\frac{i\sqrt{3}}{2}$$

$$|z|=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=1$$

$$\tan\theta=\left(\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)}\right)$$

$$\theta=\tan^{-}\left(\sqrt{3}\right)=\frac{\pi}{3}$$

$$z^{\frac{1}{2}}=\left(\text{cis}\left(\frac{\pi}{3}\right)\right)^{\frac{1}{2}}$$

$$=\text{cis}\left(\frac{1}{2}\times\frac{\pi}{3}\right)$$

$$=\text{cis}\left(\frac{\pi}{6}\right)$$

$$=\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)$$

$$=\frac{\sqrt{3}}{2}+\frac{i}{2}$$

Question 5

a

$$\frac{1}{(3+3i)^4}$$

$$(3+3i)^{-4}$$

$$z = 3+3i$$

$$|z| = \sqrt{3^2+3^2} = 3\sqrt{2}$$

$$\tan\theta = \left(\frac{3}{3}\right) = 1$$

$$\theta = \tan^{-}(1) = \frac{\pi}{4}$$

$$z^{-4} = \left(3\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)\right)^{-4}$$

$$= \left(3\sqrt{2}\right)^{-4}\text{cis}\left(-4\times\frac{\pi}{4}\right)$$

$$= \frac{1}{324}\text{cis}(\pi)$$

$$= \frac{1}{324}(\cos\pi + i\sin\pi)$$

b

$$\frac{1}{\left(\sqrt{3}-i\right)^{9}}$$

$$\left(\sqrt{3}-i\right)^{-9}$$

$$z=\sqrt{3}-i$$

$$|z|=\sqrt{\sqrt{3}^{2}+\left(-1\right)^{2}}=2$$

$$\tan\theta=\left(-\frac{1}{\sqrt{3}}\right)$$

$$\theta=\tan^{-}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$$

$$z^{-9}=\left(2\text{cis}\left(\frac{-\pi}{6}\right)\right)^{-9}$$

$$=\left(2\right)^{-9}\text{cis}\left(-9\times\frac{-\pi}{6}\right)$$

$$=\frac{1}{512}\text{cis}\left(\frac{9\pi}{6}\right)$$

$$=\frac{1}{512}\text{cis}\left(\frac{3\pi}{2}\right)$$

$$=\frac{1}{512}\text{cis}\left(-\frac{\pi}{2}\right)$$

$$=\frac{1}{512}\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$$

Question 6

$$\left(\frac{1}{\sqrt{3}}+i\frac{1}{\sqrt{3}}\right)^{2}\left(\frac{3}{2}+i\frac{3\sqrt{3}}{2}\right)^{6}$$

$$z=\frac{1}{\sqrt{3}}+i\frac{1}{\sqrt{3}}$$

$$|z_1|=\sqrt{\left(\frac{1}{\sqrt{3}}\right)^{2}+\left(\frac{1}{\sqrt{3}}\right)^{2}}=\sqrt{\frac{2}{3}}$$

$$\tan\theta=\left(\frac{\frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{3}}}\right)=1$$

$$\theta=\tan^{-}(1)=\frac{\pi}{4}$$

$$z_1=\sqrt{\frac{2}{3}}\text{cis}\left(\frac{\pi}{4}\right)$$

$$z_1^{12}=\left[\sqrt{\frac{2}{3}}\text{cis}\left(\frac{\pi}{4}\right)\right]^{2}=\left[\sqrt{\frac{2}{3}}\right]^{12}\text{cis}\left(12\times\frac{\pi}{4}\right)$$

$$=\left(\frac{2}{3}\right)^{6}\text{cis}(\pi)=\left(\frac{2}{3}\right)^{6}\times(-1)$$

$$z_2=\frac{3}{2}+i\frac{3\sqrt{3}}{2}$$

$$|z_2|=\sqrt{\left(\frac{3}{2}\right)^{2}+\left(\frac{3\sqrt{3}}{2}\right)^{2}}=\sqrt{\frac{36}{4}}=\frac{6}{2}=3$$

$$\tan\theta_2=\left(\frac{\frac{3\sqrt{3}}{2}}{\frac{3}{2}}\right)=\sqrt{3}$$

$$\theta_2=\tan^{-}\left(\sqrt{3}\right)=\frac{\pi}{3}$$

$$z_2=3\text{cis}\left(\frac{\pi}{3}\right)$$

$$z_2^{6}=\left[3\text{cis}\left(\frac{\pi}{3}\right)\right]^{6}$$

$$=3^{6}\text{cis}\left(6\times\frac{\pi}{3}\right)=3^{6}\text{cis}(2\pi)=3^{6}$$

$z_1 \times z_2$

$= -\left(\frac{2}{3}\right)^6 \times 3^6$

$= -2^6$

$= -64$

As required

Question 7

a

$$\left(\text{cis}\,\theta\right)^3 \times \left(\text{cis}\,\theta\right)^{-7}$$
$$= \left(\text{cis}\,\theta\right)^{-4}$$
$$= \text{cis}\left(-4\theta\right)$$
$$= \cos\left(-4\theta\right) + i\sin\left(-4\theta\right)$$
$$= \cos\left(4\theta\right) - i\sin\left(4\theta\right)$$

b

$$\left(\text{cis}\,\alpha\right)^4 \times \left[\text{cis}\left(-\beta\right)\right]^6$$
$$= \text{cis}\left(4\alpha\right) \times \text{cis}\left(-6\beta\right)$$
$$= \text{cis}\left(4\alpha - 6\beta\right)$$
$$= \cos\left(4\alpha - 6\beta\right) + i\sin\left(4\alpha - 6\beta\right)$$

c

$$\frac{\left(\text{cis}\,3\delta\right)^8}{\left(\text{cis}\,2\delta\right)^3}$$
$$= \frac{\left(\text{cis}\,24\delta\right)}{\left(\text{cis}\,6\delta\right)}$$
$$= \text{cis}\left(18\delta\right)$$
$$= \cos\left(18\delta\right) + i\sin\left(18\delta\right)$$

d

$$\frac{(\text{cis}\,\beta)^3\times(\cos 2\beta-i\sin 2\beta)^{-2}}{(\text{cis}\,\beta)^5}$$

$$=\frac{(\text{cis}\,\beta)^3\times\left[\text{cis}(-2\beta)\right]^{-2}}{(\text{cis}\,\beta)^5}$$

$$=\frac{(\text{cis}\,\beta)^3\times(\text{cis}\,\beta)^4}{(\text{cis}\,\beta)^5}$$

$$=(\text{cis}\,\beta)^2$$

$$=\text{cis}(2\beta)$$

$$=\cos(2\beta)+i\sin(2\beta)$$

e

$$=\frac{\left[\text{cis}\left(\frac{\pi}{2}\right)\right]^{\frac{1}{4}}\times\left[\text{cis}\left(\frac{\pi}{2}\right)\right]^{\frac{2}{3}}}{\left[\text{cis}\left(\frac{\pi}{2}\right)\right]^{-\frac{1}{12}}}$$

$$=\left[\text{cis}\left(\frac{\pi}{2}\right)\right]^{\frac{1}{4}+\frac{2}{3}+\frac{1}{12}}$$

$$=\text{cis}\left(\frac{\pi}{2}\right)$$

$$=i$$

Question 8

$$(\cos\alpha + i\sin\alpha)^2$$
$$= \cos^2\alpha + 2i\cos\alpha\sin\alpha + i^2\sin^2\alpha$$
$$= \cos^2\alpha + 2i\cos\alpha\sin\alpha - \sin^2\alpha$$
$$= \cos^2\alpha - \sin^2\alpha + 2i\cos\alpha\sin\alpha$$
$$(\cos\alpha + i\sin\alpha)^2 = \cos 2\alpha + i\sin 2\alpha$$

Equating real and imaginary parts

$$\cos^2\alpha - \sin^2\alpha = \cos 2\alpha$$
$$2\cos\alpha\sin\alpha = \sin 2\alpha$$
$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha}$$
$$= \frac{2\cos\alpha\sin\alpha}{\cos^2\alpha - \sin^2\alpha}$$
$$= \frac{\frac{2\cos\alpha\sin\alpha}{\cos^2\alpha}}{\frac{\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha}}$$
$$= \frac{\frac{2\sin\alpha}{\cos\alpha}}{1 - \frac{\sin^2\alpha}{\cos^2\alpha}}$$
$$= \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

Question 9

a

$$\int_0^{\frac{\pi}{2}} 4\cos^3\theta\, d\theta$$

Using $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$4\cos^3\theta = \cos 3\theta + 3\cos\theta$

$$\int_0^{\frac{\pi}{2}} 4\cos^3\theta\, d\theta = \int_0^{\frac{\pi}{2}} \left(\cos 3\theta + 3\cos\theta\right) d\theta$$

$$= \left[\frac{1}{3}\sin 3\theta + 3\sin\theta\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{1}{3}\sin 3\times\frac{\pi}{2} + 3\sin\frac{\pi}{2}\right) - \left(\frac{1}{3}\sin 0 + 3\sin 0\right)$$

$$= \left(\frac{1}{3}\sin\frac{3\pi}{2} + 3\sin\frac{\pi}{2}\right)$$

$$= -\frac{1}{3} + 3$$

$$= 2\frac{2}{3}$$

b

Using $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$4\sin^3\theta = 3\sin\theta - \sin 3\theta$

$$\sin^3\theta = \frac{1}{4}\left(3\sin\theta - \sin 3\theta\right)$$

$$\int_0^{\frac{\pi}{3}} \sin^3\theta\; d\theta = \int_0^{\frac{\pi}{3}} \frac{1}{4}\left(3\sin\theta - \sin 3\theta\right) d\theta$$

$$= \frac{1}{4}\left[-3\cos\theta + \frac{1}{3}\cos 3\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{4}\left[\left(-3\cos\frac{\pi}{3} + \frac{1}{3}\cos 3\times\frac{\pi}{3}\right) - \left(-3\cos 0 + \frac{1}{3}\cos 0\right)\right]$$

$$= \frac{1}{4}\left[\left(-\frac{3}{2} + \frac{1}{3}\times(-1)\right) - \left(-3 + \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left(-\frac{11}{6} + \frac{16}{6}\right)$$

$$= \frac{5}{24}$$

Question 10

$$\left(\cos\alpha + i\sin\alpha\right)^4$$
$$= \cos^4\alpha + 4i\cos^3\alpha\sin\alpha + 6i^2\cos^2\alpha\sin^2\alpha + 4i^3\cos\alpha\sin^3\alpha + i^4\sin^4\alpha$$
$$= \cos^4\alpha + 4i\cos^3\alpha\sin\alpha - 6\cos^2\alpha\sin^2\alpha - 4i\cos\alpha\sin^3\alpha + \sin^4\alpha$$
$$\left(\cos\alpha + i\sin\alpha\right)^4 = \cos 4\alpha + i\sin 4\alpha$$

a Equating real parts

$$\cos 4\alpha = \cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha$$
$$= \cos^4\alpha - 6\cos^2\alpha\left(1-\cos^2\alpha\right) + \left(1-\cos^2\alpha\right)^2$$
$$= \cos^4\alpha - 6\cos^2\alpha + 6\cos^4\alpha + 1 - 2\cos^2\alpha + \cos^4\alpha$$
$$= 8\cos^4\alpha - 8\cos^2\alpha + 1$$

b Equating imaginary parts

$$i\sin 4\alpha = 4i\cos^3\alpha\sin\alpha - 4i\cos\alpha\sin^3\alpha$$
$$\sin 4\alpha = 4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha$$
$$= 4\sin\alpha\cos\alpha\left(\cos^2\alpha - \sin^2\alpha\right)$$

Question 11

a

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}$$

b

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}$$

From question **9**

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\tan 3\theta = \frac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta}$$

$$= \frac{\sin\theta\left(3 - 4\sin^2\theta\right)}{\cos\theta\left(4\cos^2\theta - 3\right)}$$

$$= \frac{\tan\theta\left(3 - 4\sin^2\theta\right)}{4\cos^2\theta - 3}$$

$$= \frac{\tan\theta\left(3\sec^2\theta - 4\tan^2\theta\right)}{4 - 3\sec^2\theta}$$

$$= \frac{\tan\theta\left(3\left(1+\tan^2\theta\right) - 4\tan^2\theta\right)}{4 - 3\left(1+\tan^2\theta\right)}$$

$$= \frac{\tan\theta\left(3 - \tan^2\theta\right)}{1 - 3\tan^2\theta}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

Question 12

$\text{cis}\,\theta + \text{cis}\,2\theta + \text{cis}\,3\theta + \text{cis}\,4\theta + ... + cisn\theta = \text{cis}\,\theta + \text{cis}^2\,\theta + \text{cis}^3\,\theta + \text{cis}^4\,\theta + ... + \text{cis}^n\,\theta$

by De Moivre's theorem.

This is a geometric series with $a = \text{cis}\,\theta, r = \text{cis}\,\theta$

$\therefore \text{cis}\,\theta + \text{cis}\,2\theta + \text{cis}\,3\theta + \text{cis}\,4\theta + ... + \text{cis}\,n\theta$

$$= \frac{\text{cis}\,\theta\left(\left(\text{cis}\,\theta\right)^n - 1\right)}{\text{cis}\,\theta - 1}$$

$$= \frac{\text{cis}\,\theta\left(\text{cis}\,n\theta - 1\right)}{\text{cis}\,\theta - 1}$$

$$= \frac{\left(\cos\theta + i\sin\theta\right)\left(\cos n\theta + i\sin n\theta - 1\right)}{\cos\theta + i\sin\theta - 1}$$

Question 13

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \cos^4 x\ dx$$

Using $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

$$8\cos^4 x = \cos 4x + 8\cos^2 x - 1$$

$$= \cos 4x + 8\cos^2 x - 4 + 3$$

$$= \cos 4x + 4\cos 2x + 3$$

$$\cos^4 x = \frac{1}{8}\left(\cos 4x + 4\cos 2x + 3\right)$$

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \cos^4 x\ dx = \frac{1}{8}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left(\cos 4x + 4\cos 2x + 3\right)dx$$

$$= \frac{1}{8}\left[\frac{1}{4}\sin 4x + 2\sin 2x + 3x\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$$

$$= \frac{1}{8}\left[\left(\frac{1}{4}\sin 4\times\frac{2\pi}{3} + 2\sin 2\times\frac{2\pi}{3} + 3\times\frac{2\pi}{3}\right) - \left(\frac{1}{4}\sin 4\times\frac{\pi}{3} + 2\sin 2\times\frac{\pi}{3} + 3\times\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{8}\left[\left(\frac{1}{4}\times\frac{\sqrt{3}}{2} + 2\times\left(-\frac{\sqrt{3}}{2}\right) + 2\pi\right) - \left(\frac{1}{4}\times\left(-\frac{\sqrt{3}}{2}\right) + 2\times\frac{\sqrt{3}}{2} + \pi\right)\right]$$

$$= \frac{1}{8}\left[2\pi - \frac{7\sqrt{3}}{8} - \left(\frac{7\sqrt{3}}{8} + \pi\right)\right]$$

$$= \frac{1}{8}\left[\pi - \frac{14\sqrt{3}}{8}\right]$$

$$= \frac{1}{8}\left[\pi - \frac{7\sqrt{3}}{4}\right]$$

$$= \frac{4\pi - 7\sqrt{3}}{32}$$

$$= -\frac{7\sqrt{3}}{32} + \frac{\pi}{8}$$

Question 14

a Required to prove

$\left(\text{cis}\,\theta\right)^n = \text{cis}\left(n\theta\right)$

Prove true for $n = 1$

$\text{LHS} = \left(\text{cis}\,\theta\right) = \text{cis}\,\theta$

$\text{RHS} = \text{cis}\,1\theta = \text{cis}\,\theta = \text{LHS}$

True for $n = 1$

Assume formula is true for $n = k$:

$\left(\text{cis}\,\theta\right)^k = \text{cis}\,k\theta$

Prove true for $n = k+1$, that is:

$\left(\text{cis}\,\theta\right)^{k+} = \text{cis}\left[\left(k+1\right)\theta\right]$

$$
\begin{aligned}
\text{LHS} &= \left(\text{cis}\,\theta\right)^{k+1} \\
&= \left(\text{cis}\,\theta\right)^k\left(\text{cis}\,\theta\right) \\
&= \text{cis}\,k\theta\,\text{cis}\,\theta \\
&= \left(\cos k\theta + i\sin k\theta\right)\left(\cos\theta + i\sin\theta\right) \\
&= \cos k\theta\cos\theta + \cos k\theta i\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta \\
&= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i\left[\cos k\theta\sin\theta + \sin k\theta\cos\theta\right] \\
&= \cos\left(k\theta+\theta\right) + i\sin\left(k\theta+\theta\right) \\
&= \cos\left[\left(k+1\right)\theta\right] + i\sin\left[\left(k+1\right)\theta\right] \\
&= \text{cis}\left[\left(k+1\right)\theta\right] \\
&= \text{RHS}
\end{aligned}
$$

So the formula is true for $n = k+1$.

The formula is true for $n = 1$ so by mathematical induction it is true $\forall n \in \mathbb{N}$.

b Required to prove

$\left|z^n\right| = \left|z\right|^n$

Prove true for $n = 1$

$\text{LHS} = \left|z^1\right| = \left|z\right|$

$\text{RHS} = \left|z\right| \ = \left|z\right| = \text{LHS}$

True for $n = 1$

Assume formula is true for $n = k$, that is:

$\left|z^k\right| = \left|z\right|^k$

Prove true for $n = k + 1$, that is:

$\left|z^{k+1}\right| = \left|z\right|^{k+1}$

$\text{LHS} = \left|z^{k+} \ \right| = \left|z^k \times z\right|$

$= \left|z^k\right| \times \left|z\right|$ as $\left|a \times b\right| = \left|a\right| \times \left|b\right|$

$= \left|z\right|^k \times \left|z\right|$

$= \left|z\right|^{k+1}$

= RHS

So the formula is true for $n = k + 1$.

The formula is true for $n = 1$ so by mathematical induction it is true $\forall n \in \mathbb{N}$.

c Required to prove

$\arg\left(z^n\right) = n\arg z$

Prove true for $n = 1$

$\text{LHS} = \arg\left(z\ \right) = \arg z$

$\text{RHS} = 1\arg z = \arg z = \text{LHS}$

True for $n = 1$

Assume formula is true for $n = k$:

$\arg\left(z^k\right) = k\arg z$

Prove true for $n = k+1$, that is:

$\arg\left(z^{k+}\ \right) = \left(k+1\right)\arg z$

$\text{LHS} = \arg\left(z^{k+1}\right)$

$= \arg\left(z^k \times z\right)$

$= \arg\left(z^k\right) + \arg z$ (property of arguments)

$= k\arg z + \arg z$

$= \left(k+1\right)\arg z$

$= \text{RHS}$

So the formula is true for $n = k+1$.

The formula is true for $n = 1$ so by mathematical induction it is true $\forall n \in \mathbb{N}$.

Question 15

Let $z = \text{cis}\,\theta$

a

$$
\begin{aligned}
& z-\frac{1}{z} \\
& =\operatorname{cis}\theta-(\operatorname{cis}\theta)^{-1} \\
& =(\cos\theta+i\sin\theta)-[\cos(-\theta)+i\sin(-\theta)] \\
& =\cos\theta-\cos(-\theta)+i\sin\theta-i\sin(-\theta) \\
& =\cos\theta-\cos\theta+i\sin\theta+i\sin\theta \\
& =2i\sin\theta
\end{aligned}
$$

b

$$
\begin{aligned}
& z^2-\frac{1}{z^2} \\
& =\operatorname{cis}(2\theta)-(\operatorname{cis}\theta)^{-2} \\
& =[\cos(2\theta)+i\sin(2\theta)]-[\cos(-2\theta)+i\sin(-2\theta)] \\
& =\cos(2\theta)-\cos(-2\theta)+i\sin(2\theta)-i\sin(-2\theta) \\
& =\cos(2\theta)-\cos(2\theta)+i\sin(2\theta)+i\sin(2\theta) \\
& =2i\sin(2\theta)
\end{aligned}
$$

c

$$
\begin{aligned}
& z^n-\frac{1}{z^n} \\
& =(\operatorname{cis}\theta)^n-(\operatorname{cis}\theta)^{-n} \\
& =[\cos(n\theta)+i\sin(n\theta)]-[\cos(-n\theta)+i\sin(-n\theta)] \\
& =\cos(n\theta)-\cos(-n\theta)+i\sin(n\theta)-i\sin(-n\theta) \\
& =\cos(n\theta)-\cos(n\theta)+i\sin(n\theta)+i\sin(n\theta) \\
& =2i\sin(n\theta)
\end{aligned}
$$

Question 16

a

$$\text{cis}\left(\frac{\pi}{12}\right)+\frac{1}{\text{cis}\left(\frac{\pi}{12}\right)}$$

$$=\text{cis}\left(\frac{\pi}{12}\right)+\left[\text{cis}\left(\frac{\pi}{12}\right)\right]^{-1}$$

$$=\left[\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)\right]+\left[\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right]$$

$$=\cos\left(\frac{\pi}{12}\right)+\cos\left(-\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)$$

$$=\cos\left(\frac{\pi}{12}\right)+\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)-i\sin\left(\frac{\pi}{12}\right)$$

$$=2\cos\left(\frac{\pi}{12}\right)$$

b

$$\text{cis}\left(\frac{\pi}{6}\right)^{6}-\frac{1}{\text{cis}\left(\frac{\pi}{6}\right)^{6}}$$

$$=\text{cis}\left(\frac{\pi}{6}\right)^{6}-\text{cis}\left(\frac{\pi}{6}\right)^{-6}$$

$$=2i\sin\left(\frac{4\pi}{6}\right)$$

$$=2i\sin\left(\frac{2\pi}{3}\right)$$

$$=2i\times\frac{\sqrt{3}}{2}$$

$$=i\sqrt{3}$$

c

$$\left[\operatorname{cis}\left(\frac{5\pi}{7}\right)\right]^7+\frac{1}{\left[\operatorname{cis}\left(\frac{5\pi}{7}\right)\right]^7}$$
$$=\operatorname{cis}(5\pi)+\operatorname{cis}(-5\pi)$$
$$=\left[\cos(5\pi)+i\sin(5\pi)\right]+\left[\cos(-5\pi)+i\sin(-5\pi)\right]$$
$$=\cos(5\pi)+\cos(-5\pi)+i\sin(5\pi)+i\sin(-5\pi)$$
$$=\cos(5\pi)+\cos(5\pi)+i\sin(5\pi)-i\sin(5\pi)$$
$$=2\cos(5\pi)$$
$$=2\cos\pi$$
$$=-2$$

Question 17

a

$$z^2+\frac{1}{z^2}$$
$$=\operatorname{cis}(2\theta)+(\operatorname{cis}\theta)^{-2}$$
$$=\left[\cos(2\theta)+i\sin(2\theta)\right]+\left[\cos(-2\theta)+i\sin(-2\theta)\right]$$
$$=\cos(2\theta)+\cos(-2\theta)+i\sin(2\theta)+i\sin(-2\theta)$$
$$=\cos(2\theta)+\cos(2\theta)+i\sin(2\theta)-i\sin(2\theta)$$
$$=2\cos(2\theta)\in\mathbb{R}$$

b

$$z^3-\frac{1}{z^3}$$
$$=\operatorname{cis}(3\theta)-(\operatorname{cis}\theta)^{-3}$$
$$=\left[\cos(3\theta)+i\sin(3\theta)\right]-\left[\cos(-3\theta)+i\sin(-3\theta)\right]$$
$$=\cos(3\theta)-\cos(-3\theta)+i\sin(3\theta)-i\sin(-3\theta)$$
$$=\cos(3\theta)-\cos(3\theta)+i\sin(3\theta)+i\sin(3\theta)$$
$$=2i\sin(3\theta)\text{ which is purely imaginary}$$

c

$$z^n + \frac{1}{z^n}$$
$$= \text{cis}(n\theta) + (\text{cis}\,\theta)^{-n}$$
$$= \left[\cos(n\theta) + i\sin(n\theta)\right] + \left[\cos(-n\theta) + i\sin(-n\theta)\right]$$
$$= \cos(n\theta) + \cos(-n\theta) + i\sin(n\theta) + i\sin(-n\theta)$$
$$= \cos(n\theta) + \cos(n\theta) + i\sin(n\theta) - i\sin(n\theta)$$
$$= 2\cos(n\theta) \in \mathbb{R}$$

Question 18

$$\left(z - \frac{1}{z}\right)^3$$
$$= z^3 - 3z + \frac{3}{z} - \frac{1}{z^3}$$
$$= \text{cis}(3\theta) - 3\,\text{cis}\,\theta + 3\,\text{cis}(-\theta) - \text{cis}(-3\theta)$$
$$= \text{cis}(3\theta) - \text{cis}(-3\theta) - 3\left[\text{cis}\,\theta - \text{cis}(-\theta)\right]$$
$$= 2i\sin(3\theta) - 3(2i\sin\theta)$$
$$= 2i\sin(3\theta) - 6i\sin\theta$$
$$z - \frac{1}{z} = 2i\sin\theta$$
$$\left(z - \frac{1}{z}\right)^3 = (2i\sin\theta)^3$$
$$= -8i\sin^3\theta$$
$$\therefore -8i\sin^3\theta = 2i\sin(3\theta) - 6i\sin\theta$$
$$4\sin^3\theta = 3\sin\theta - \sin(3\theta)$$
$$\sin^3\theta = \frac{3\sin\theta - \sin(3\theta)}{4}$$

Exercise 4.02 Quadratic equations with complex coefficients

Question 1

a

$$x^2 - 2ix + 3 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2i \pm \sqrt{(-2i)^2 - 4\times1\times3}}{2\times1}$$

$$= \frac{2i \pm \sqrt{-4-12}}{2}$$

$$= \frac{2i \pm \sqrt{-16}}{2}$$

$$= \frac{2i \pm 4i}{2}$$

$$= -i, 3i$$

b

$$x^2 + 6ix = 5$$

$$z^2 + 6ix - 5 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6i \pm \sqrt{(6i)^2 - 4\times1\times(-5)}}{2\times1}$$

$$= \frac{-6i \pm \sqrt{-36+20}}{2}$$

$$= \frac{-6i \pm \sqrt{-16}}{2}$$

$$= \frac{-6i \pm 4i}{2}$$

$$= -5i, -i$$

c

$$x^2-(3+2i)x+(1+3i)=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{(3+2i)\pm\sqrt{(-(3+2i))^2-4\times1\times(1+3i)}}{2\times1}$$

$$=\frac{3+2i\pm\sqrt{5+12i-4-12i}}{2}$$

$$=\frac{3+2i\pm\sqrt{1}}{2}$$

$$=\frac{3+2i\pm1}{2}$$

$$=2+i,1+i$$

d

$$3x^2-5ix+2=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{5i\pm\sqrt{(-5i)^2-4\times3\times2}}{2\times3}$$

$$=\frac{5i\pm\sqrt{-25-24}}{6}$$

$$=\frac{5i\pm\sqrt{-49}}{6}$$

$$=\frac{5i\pm7i}{6}$$

$$=-\frac{i}{3},2i$$

Question 2

a

$$|i| = 1$$

$$\arg i = \frac{\pi}{2}$$

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = \text{cis}\,\frac{\pi}{2}$$

$$z^2 = \text{cis}\,\frac{\pi}{2}$$

2 roots z_1 and z_2 :

$$z_1 = \left(\text{cis}\,\frac{\pi}{2}\right)^{\frac{1}{2}}$$

$$= \text{cis}\,\frac{\pi}{4}$$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$= \frac{1+i}{\sqrt{2}}$$

z_2 will be equally spaced around the origin, so at an angle of π from z_1:

$$z_2 = \text{cis}\left(\frac{\pi}{4} - \pi\right)$$

$$= \text{cis}\left(-\frac{3\pi}{4}\right)$$

$$= \cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)$$

$$= -\frac{1}{\sqrt{2}} + i\left(-\frac{1}{\sqrt{2}}\right)$$

$$= \frac{-1-i}{\sqrt{2}}$$

So $z = \dfrac{\pm(1+i)}{\sqrt{2}}$

b

$$|-9i| = 9$$

$$\arg(-9i) = -\frac{\pi}{2}$$

$$-9i = 9\left(\cos\left[-\frac{\pi}{2}\right] + i\sin\left[-\frac{\pi}{2}\right]\right) = \text{cis}\left[-\frac{\pi}{2}\right]$$

$$z^2 = \text{cis}\left[-\frac{\pi}{2}\right]$$

2 roots z_1 and z_2 :

$$z_1 = \left(9\text{cis}\left[-\frac{\pi}{2}\right]\right)^{\frac{1}{2}}$$

$$= 3\text{cis}\left[-\frac{\pi}{4}\right]$$

$$= 3\left(\cos\left[-\frac{\pi}{4}\right] + i\sin\left[-\frac{\pi}{4}\right]\right)$$

$$= 3\left(\frac{1}{\sqrt{2}} + i\left[-\frac{1}{\sqrt{2}}\right]\right)$$

$$= \frac{3(1-i)}{\sqrt{2}}$$

z_2 will be equally spaced around the origin, so at an angle of π from z :

$$z_2 = 3\text{cis}\left(-\frac{\pi}{4} + \pi\right)$$

$$= 3\text{cis}\left(\frac{3\pi}{4}\right)$$

$$= 3\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$$

$$= 3\left[-\frac{1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{-3(1-i)}{\sqrt{2}}$$

So $z = \dfrac{\pm 3(1-i)}{\sqrt{2}}$

c

$$\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\text{cis}\frac{\pi}{3}$$

$$z^2=\text{cis}\frac{\pi}{3}$$

2 roots z_1 and z_2 :

$$z_1=\left(\text{cis}\frac{\pi}{3}\right)^{\frac{1}{2}}$$
$$=\text{cis}\frac{\pi}{6}$$
$$=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$$
$$=\frac{\sqrt{3}}{2}+i\frac{1}{2}$$
$$=\frac{\sqrt{3}+i}{2}$$

z_2 will be equally spaced around the origin, so at an angle of π from z_1:

$$z_2=\text{cis}\left(\frac{\pi}{6}-\pi\right)$$
$$=\text{cis}\left(-\frac{5\pi}{6}\right)$$
$$=\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)$$
$$=-\frac{\sqrt{3}}{2}+i\left(-\frac{1}{2}\right)$$
$$=\frac{-\sqrt{3}-i}{2}$$

So $z=\dfrac{\pm\left(\sqrt{3}+i\right)}{2}$

d

$$z^2 - 1 + i = 0$$
$$z^2 = 1 - i$$
$$|1-i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
$$\arg(1-i) = -\frac{\pi}{4}$$
$$1 - i = \sqrt{2}\left(\cos\left[-\frac{\pi}{4}\right] + i\sin\left[-\frac{\pi}{4}\right]\right) = \text{cis}\left[-\frac{\pi}{4}\right]$$
$$z^2 = \sqrt{2}\text{cis}\left[-\frac{\pi}{4}\right]$$

2 roots z_1 and z_2 :

$$z_1 = \left(\sqrt{2}\text{cis}\left[-\frac{\pi}{4}\right]\right)^{\frac{1}{2}}$$
$$= 2^{\frac{1}{4}}\text{cis}\left[-\frac{\pi}{8}\right]$$
$$= \sqrt[4]{2}\text{cis}\left(-\frac{\pi}{8}\right)$$

z_2 will be equally spaced around the origin, so at an angle of π from z :

$$z_2 = \sqrt[4]{2}\text{cis}\left(-\frac{\pi}{8} + \pi\right)$$
$$= \sqrt[4]{2}\text{cis}\left(\frac{7\pi}{8}\right)$$
$$= \sqrt[4]{2}\left[\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right]$$
$$= \sqrt[4]{2}\left[-\cos\left(-\frac{\pi}{8}\right) - i\sin\left(-\frac{\pi}{8}\right)\right]$$
$$= -\sqrt[4]{2}\left[\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right]$$
$$= -\sqrt[4]{2}\text{cis}\left(-\frac{\pi}{8}\right)$$

So $z = \pm\sqrt[4]{2}\text{cis}\left(-\frac{\pi}{8}\right)$

e

$$z^2 = e^{3i\pi}$$

2 roots z_1 and z_2 :

$$\begin{aligned} z &= \left(e^{3i\pi}\right)^{\frac{1}{2}} \\ &= e^{\frac{3i\pi}{2}} \\ &= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} \\ &= 0 + i(-1) \\ &= -i \end{aligned}$$

z_2 will be equally spaced around the origin, so at an angle of π from z_1:

$$\begin{aligned} z_2 &= e^{\left(\frac{3\pi}{2} - \pi\right)i} \\ &= e^{\frac{i\pi}{2}} \\ &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ &= 0 + i(1) \\ &= i \end{aligned}$$

So $z = \pm i$

f

$$z^2 = 16e^{-\frac{2i\pi}{3}}$$

2 roots z_1 and z_2 :

$$\begin{aligned} z &= \left(16e^{-\frac{2\pi}{3}}\right)^{\frac{1}{2}} \\ &= 4e^{-\frac{i\pi}{3}} \\ &= 4\left(\cos\left[-\frac{\pi}{3}\right] + i\sin\left[-\frac{\pi}{3}\right]\right) \\ &= 4\left(\frac{1}{2} + i\left[-\frac{\sqrt{3}}{2}\right]\right) \\ &= 2\left(1 - i\sqrt{3}\right) \end{aligned}$$

z_2 will be equally spaced around the origin, so at an angle of π from z_1:

$$\begin{aligned} z_2 &= 4e^{\left(-\frac{\pi}{3}+\pi\right)i} \\ &= 4e^{\frac{2i\pi}{3}} \\ &= 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \\ &= 4\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= -2\left(1 - i\sqrt{3}\right) \end{aligned}$$

So $z = \pm 2\left(1 - i\sqrt{3}\right)$

g

$$z^2 = \cos 4 + i\sin 4$$

2 roots z_1 and z_2 :

$$\begin{aligned} z &= \left(\cos 4 + i\sin 4\right)^{\frac{1}{2}} \\ &= \cos 2 + i\sin 2 \end{aligned}$$

z_2 will be equally spaced around the origin, so at an angle of π from z_1:

$$\begin{aligned} z_2 &= \cos\left(2-\pi\right) + i\sin\left(2-\pi\right) \\ &= \cos\left(\pi-2\right) - i\sin\left(\pi-2\right) \\ &= -\cos 2 - i\sin 2 \end{aligned}$$

So $z = \pm\left(\cos 2 + i\sin 2\right)$

Question 3

$z^2-6z+iz+7+3i$

$z^2-(6-i)z+7+3i$

Let $z=5-2i$

$(5-2i)^2-(6-i)(5-2i)+7+3i$

$=25-20i-4-(30-12i-5i-2)+7+3i$

$=21-20i-28+17i+7+3i$

$=0$

$\therefore 5-2i$ is a root

$$
\begin{array}{r l}
 & z-(1+i) \\
z-(5-2i) \big) & z^2-(6-i)z+7+3i \\
 & z^2-(5-2i)z \\
\hline
 & (-1-i)z+7+3i \\
 & (-1-i)z-(1+i)(-5+2i) \\
\hline
 & 0
\end{array}
$$

The other root is $1+i$

Question 4

a

$$3-i, 1+7i$$
$$0=\left[x-(3-i)\right]\left[x-(1+7i)\right]$$
$$0=x^2+(3-i)x-(1+7i)x-(3-i)(1+7i)$$
$$0=x^2+(-4-6)x-3+21i-i+7$$
$$0=x^2-(4+6i)x+10+20i$$

b

$$-4i, 3+5i$$
$$0=\left[x-(-4i)\right]\left[x-(3+5i)\right]$$
$$0=x^2+4ix-(3+5i)x-4i(3+5i)$$
$$0=x^2-(3+i)x+20-12i$$

c

$$\frac{2+i}{3}, 1-i$$
$$0=\left[x-\left(\frac{2+i}{3}\right)\right]\left[x-(1-i)\right]$$
$$0=x^2+\left(\frac{2+i}{3}\right)x-(1-i)x-\left(\frac{2+i}{3}\right)(1-i)$$
$$0=x^2+\left(\frac{5-2i}{3}\right)x+\frac{6-6i+3+3i}{3}$$
$$0=3x^2-(5-2i)x+3-i$$

Question 5

$$\left[z-\left(\frac{1}{2}+\frac{i}{2}\right)\right]\left[z-\left(\frac{3}{2}-\frac{i}{2}\right)\right]=0$$
$$\frac{1}{2}\left[2z-(1+i)\right]\frac{1}{2}\left[2z-(3-i)\right]=0$$
$$\left[2z-(1+i)\right]\left[2z-(3-i)\right]=0$$
$$4z^2-(1+i)2z-(3-i)2z+(1+i)(3-i)=0$$
$$4z^2-8z+4+2i=0$$
$$2z^2-4z+2+i=0$$
$$a=2, p=-4, q=2+i$$

Question 6

a

Let $z^2 = 8+6i$

$(x+iy)^2 = 8+6i$

$x^2 - y^2 + 2ixy = 8+6i$

Equate real and imaginary parts

$x^2 - y^2 = 8$

$2xy = 6$

$xy = 3$

$y = \frac{3}{x}$

$x^2 - y^2 = 8$

$x^2 - \left(\frac{3}{x}\right)^2 = 8$

$x^4 - 9 = 8x^2$

$x^4 - 8x^2 - 9 = 0$

$(x^2-9)(x^2+1) = 0$

$(x^2+1) = 0$ has no solution as $x \in \mathbb{R}$

$(x^2-9) = 0$

$x^2 = 9$

$x = \pm 3,\quad \text{use positive value so } x = 3$

$y = \frac{3}{3} = 1$

$z = 3+i$

b

$$z^2 + 2z + 4iz = 11 + 2i$$
$$z^2 + (2+4i)z - (11+2i) = 0$$
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(2+4i) \pm \sqrt{(2+4i)^2 - 4\times 1\times(-(11+2i))}}{2\times 1}$$
$$= \frac{-2-4i \pm \sqrt{4+16i-16+44+8i}}{2}$$
$$= \frac{-2-4i \pm \sqrt{32+24i}}{2}$$
$$= \frac{-2-4i \pm 2\sqrt{8+6i}}{2}$$
$$\text{from part } \mathbf{a}\ \sqrt{8+6i} = 3+i$$
$$= \frac{-2-4i \pm 2(3+i)}{2}$$
$$= -1-2i \pm (3+i)$$
$$= 2-i, -4-3i$$

Question 7

a

$$x^2-(1+i)x+i=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{(1+i)\pm\sqrt{(1+i)^2-4\times1\times i}}{2\times1}$$

$$=\frac{1+i\pm\sqrt{2i-4i}}{2}$$

$$=\frac{1+i\pm\sqrt{-2i}}{2}$$

Let $z^2=-2i$

$$(x+iy)^2=-2i$$

$$x^2-y^2+2ixy=-2i$$

Equate real and imaginary parts

$$x^2-y^2=0$$

$$2xy=-2$$

$$xy=-1$$

$$y=-\frac{1}{x}$$

$$x^2-y^2=0$$

$$x^2-\left(-\frac{1}{x}\right)^2=0$$

$$x^4-1=0$$

$$x^4=1$$

$$x=\pm1$$

$$y=\mp\frac{1}{1}=\mp1$$

$$z=\pm(1-i)$$

$$x=\frac{1+i\pm\sqrt{-2i}}{2}$$

$$=\frac{1+i\pm(1-i)}{2}$$

$$=\frac{2}{2}\ \ \frac{2i}{2}$$

$$=1,i$$

b

$$x^2 - 2x + 1 - 2i = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4\times1\times(1-2i)}}{2\times1}$$
$$= \frac{2 \pm \sqrt{8i}}{2}$$
$$= 1 \pm \sqrt{2i}$$

Let $z^2 = 2i$

$$(x+iy)^2 = 2i$$
$$x^2 - y^2 + 2ixy = 2i$$

Equate real and imaginary parts

$$x^2 - y^2 = 0$$
$$2xy = 2$$
$$xy = 1$$
$$y = \frac{1}{x}$$
$$x^2 - y^2 = 0$$
$$x^2 - \left(\frac{1}{x}\right)^2 = 0$$
$$x^4 - 1 = 0$$
$$x^4 = 1$$
$$x = \pm1$$
$$y = \pm\frac{1}{1} = \pm1$$
$$z = \pm(1+i)$$

$$x = 1 \pm \sqrt{2i}$$
$$= 1 \pm (1+i)$$
$$= 2+i, -i$$

c

$$x^2-3x+3ix-5i=0$$
$$x^2-(3-3i)x-5i=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-(-(3-3i))\pm\sqrt{(-(3-3i))^2-4\times1\times(-5i)}}{2\times1}$$
$$=\frac{3-3i\pm\sqrt{-18i+20i}}{2}$$
$$=\frac{3-3i\pm\sqrt{2i}}{2}$$

Let $z^2=2i$

$$(x+iy)^2=2i$$
$$x^2-y^2+2ixy=2i$$

Equate real and imaginary parts

$$x^2-y^2=0$$
$$2xy=2$$
$$xy=1$$
$$y=\frac{1}{x}$$
$$x^2-y^2=0$$
$$x^2-\left(\frac{1}{x}\right)^2=0$$
$$x^4-1=0$$
$$x^4=1$$
$$x=\pm1$$
$$y=\pm\frac{1}{1}=\pm1$$
$$z=\pm(1+i)$$
$$x=\frac{3-3i\pm\sqrt{2i}}{2}$$
$$=\frac{3-3i\pm(1+i)}{2}$$
$$=\frac{4-2i}{2}\ \frac{2-4i}{2}$$
$$=2-i,1-2i$$

d

$$x^2-(4+3i)x=2-8i \Rightarrow x^2-(4+3i)x-(2-8i)=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-(4+3i))\pm\sqrt{(-(4+3i))^2-4\times1\times(2-8i)}}{2\times1}$$

$$=\frac{4+3i\pm\sqrt{7+24i+8-32i}}{2}$$

$$=\frac{4+3i\pm\sqrt{15-8i}}{2}$$

Let $z^2=15-8i$

$(x+iy)^2=15-8i$

$x^2-y^2+2ixy=15-8i$

Equate real and imaginary parts

$x^2-y^2=15$

$2xy=-8 \Rightarrow xy=-4$

$y=-\frac{4}{x}$

$x^2-y^2=15$

$x^2-\left(-\frac{4}{x}\right)^2=15$

$x^4-16=15x^2$

$x^4-15x^2-16=0$

$(x^2-16)(x^2+1)=0$

$(x^2+1)=0$ has no solution as $x\in\mathbb{R}$

$(x^2-16)=0$

$x^2=16$

$x=\pm4$

$y=\mp\frac{4}{4}=\mp1$

$z=\pm(4-i)$

$$x=\frac{4+3i\pm(4-i)}{2}$$

$$=\frac{8+2i}{2}\ \frac{4i}{2}$$

$=4+i,2i$

e

$$ix^2-(1+3i)x+(2+2i)=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-(1+3i))\pm\sqrt{(-(1+3i))^2-4\times i\times(2+2i)}}{2\times i}$$

$$=\frac{1+3i\pm\sqrt{-8+6i-8i+8}}{2i}$$

$$=\frac{1+3i\pm\sqrt{-2i}}{2i}$$

Let $z^2=-2i$

$$(x+iy)^2=-2i$$

$$x^2-y^2+2ixy=-2i$$

Equate real and imaginary parts

$$x^2-y^2=0$$

$$2xy=-2$$

$$xy=-1$$

$$y=-\frac{1}{x}$$

$$x^2-y^2=0$$

$$x^2-\left(-\frac{1}{x}\right)^2=0$$

$$x^4-1=0$$

$$x^4=1$$

$$x=\pm1$$

$$y=\mp\frac{1}{1}=\mp1$$

$$z=\pm(1-i)$$

$$x=\frac{1+3i\pm\sqrt{-2i}}{2i}$$

$$=\frac{1+3i\pm(1-i)}{2i}$$

$$=\frac{2+2i}{2i}\ \frac{4i}{2i}$$

$$=\frac{1+i}{i},2$$

$$=1-i,2$$

Exercise 4.03 Polynomial equations

Question 1

a $z^3 + z$

i $z\left(z^2+1\right)$

ii $z\left(z+i\right)\left(z-i\right)$

b $z^3 - 6z^2 + 10z$

i $z\left(z^2-6z+10\right)$

ii

$$z\left(z^2-6z+10\right)$$
$$z\left(z^2-6z+9+1\right)$$
$$z\left(\left(z-3\right)^2+1\right)$$
$$z\left(z-3+i\right)\left(z-3-i\right)$$

c $z^3 + 1$

i $\left(z+1\right)\left(z^2-z+1\right)$

ii

$$\left(z+1\right)\left(z^2-z+1\right)$$
$$\left(z+1\right)\left(z^2-z+\frac{1}{4}+\frac{3}{4}\right)$$
$$\left(z+1\right)\left(\left(z-\frac{1}{2}\right)^2+\frac{3}{4}\right)$$
$$\left(z+1\right)\left(z-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\left(z-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)$$

d z^3-8

i $(z-2)(z^2+2z+4)$

ii

$$(z-2)(z^2+2z+4)$$
$$(z-2)(z^2+2z+1+3)$$
$$(z-2)((z+1)^2+3)$$
$$(z-2)(z+1+i\sqrt{3})(z+1-i\sqrt{3})$$

e z^4+3z^2-4

i

$$(z^2+4)(z^2-1)$$
$$(z^2+4)(z+1)(z-1)$$

ii

$$(z^2+4)(z+1)(z-1)$$
$$(z+2i)(z-2i)(z+1)(z-1)$$

f $z^4+10z^2 9$

i $(z^2+9)(z^2+1)$

ii

$$(z+3i)(z-3i)(z+i)(z-i)$$

g z^3+z^2+z+1

i

$$z^2(z+1)+z+1$$
$$(z^2+1)(z+1)$$

ii

$$(z^2+1)(z+1)$$
$$(z+1)(z+i)(z-i)$$

h $z^3 - z^2 + 2z - 2$

i

$$z^2(z-1) + 2(z-1)$$
$$(z^2+2)(z-1)$$

ii

$$(z^2+2)(z-1)$$
$$(z-1)\left(z+i\sqrt{2}\right)\left(z-i\sqrt{2}\right)$$

Question 2

a

$$P(z) = z^3 - 2z^2 - 3z + 10$$

$$\begin{array}{r} z^2 - 4z + 5 \\ z+2 \overline{\big) z^3 - 2z^2 - 3z + 10} \\ \underline{z^3 + 2z^2} \qquad\qquad \\ -4z^2 - 3z \qquad \\ \underline{-4z^2 - 8z} \qquad \\ 5z + 10 \\ \underline{5z + 10} \\ 0 \end{array}$$

$$P(z) = z^3 - 2z^2 - 3z + 10$$
$$= (z+2)(z^2 - 4z + 5)$$
$$= (z+2)(z^2 - 4z + 4 + 1)$$
$$= (z+2)\left((z-2)^2 + 1\right)$$
$$= (z+2)(z-2+i)(z-2-i)$$

roots

$-2, 2-i, 2+i$

b

$P(z)=z^4-2z^3-2z^2-2z-3$

If $z=i$ is a root then $z=-i$ must also be a root

$(z-i)(z+i)=z^2+1$

$$
\begin{array}{r}
z^2-2z-3 \\
z^2+1\overline{\big)z^4-2z^3-2z^2-2z-3} \\
\underline{z^4+0z^3+z^2} \qquad\qquad \\
-2z^3-3z^2-2z \quad \\
\underline{-2z^3-0z^2-2z} \quad \\
-3z^2-3 \\
\underline{-3z^2-3} \\
0 \quad
\end{array}
$$

$P(z)=z^4-2z^3-2z^2-2z-3$

$=(z^2+1)(z^2-2z-3)$

$=(z+1)(z-3)(z+i)(z-i)$

roots

$-1,3,\pm i$

c

$z=e^{\frac{i\pi}{4}}$ is a root therefore $e^{\frac{-i\pi}{4}}$ is also a root.

$e^{\frac{i\pi}{4}}\times e^{\frac{-i\pi}{4}}\times\gamma=-(-1)$

$\gamma=1$

roots are

$e^{\frac{i\pi}{4}},e^{\frac{-i\pi}{4}},1$

d

$P(z) = z^4 - 6z^3 + 6z^2 - 2z - 15$

$z^2 - 4z - 5$ is a root

$$\begin{array}{r|l} & z^2 - 2z + 3 \\ z^2 - 4z - 5 & z^4 - 6z^3 + 6z^2 - 2z - 15 \\ & \underline{z^4 - 4z^3 - 5z^2} \\ & -2z^3 + 11z^2 - 2z \\ & \underline{-2z^3 + 8z^2 + 10z} \\ & 3z^2 - 12z - 15 \\ & \underline{3z^2 - 12z - 15} \\ & 0 \end{array}$$

$P(z) = z^4 - 6z^3 + 6z^2 - 2z - 15$

$= (z^2 - 4z - 5)(z^2 - 2z + 3)$

$= (z+1)(z-5)(z^2 - 2z + 1 + 2)$

$= (z+1)(z-5)((z-1)^2 + 2)$

$= (z+1)(z-5)(z - 1 + \sqrt{2})(z - 1 - \sqrt{2})$

roots

$-1, 5, 1 - \sqrt{2}, 1 + \sqrt{2}$

e

$z - 2 + i\sqrt{5}$ is a factor therefore $z - 2 - i\sqrt{5}$ is also a factor.

$(z - 2 + i\sqrt{5})(z - 2 - i\sqrt{5})$

$= z^2 - z(2 - i\sqrt{5}) - z(2 + i\sqrt{5}) + (2 - i\sqrt{5})(2 + i\sqrt{5})$

$= z^2 - 4z + 9$

$P(z) = z^3 - 2z^2 + z + 18$

Looking at the product of roots taken three at a time

$\alpha\beta\gamma = -d$

$\alpha\beta\gamma = -18$

$\alpha\beta = 9$

$\gamma = -2$

Roots are

$2 + i\sqrt{5}, 2 - i\sqrt{5}, -2$

Question 3

As all coefficients are real, complex roots must occur in conjugate pairs.

$1+3i$ is a root therefore $1-3i$ is also a root.

$(z-1-3i)(z-1+3i)$

$=z^2-2z+10$

$P(z)=z^4-6z^3+pz^2+qz+70$

$$
\begin{array}{r l}
 & z^2-4z+7 \\
z^2-2z+10 \,\big) & z^4-6z^3+pz^2+qz+70 \\
 & \underline{z^4-2z^3+10z^2} \\
 & -4z^3+(p-10)z^2+qz \\
 & \underline{-4z^3+8z^2-40z} \\
 & (p-18)z^2+(q+40)z+70 \\
 & \underline{7z^2-14z+70} \\
 & (p-25)z^2+(q+54)z=0
\end{array}
$$

$\therefore\ p-25=0$

$p=25$

$q+54=0$

$q=-54$

$P(z)=z^4-6z^3+25z^2-54z+70$

$=(z^2-2z+10)(z^2-4z+7)$

$=(z-1+3i)(z-1-3i)(z^2-4z+4+3)$

$=(z-1+3i)(z-1-3i)((z-2)^2+3)$

$=(z-1+3i)(z-1-3i)(z-2+i\sqrt{3})(z-2-i\sqrt{3})$

roots

$1\pm 3i, 2\pm i\sqrt{3}$

Question 4

a

$$P(z) = z^3 - 3z^2 + z - 3$$

$$P(3) = 0$$

$z - 3$ is a root

$$\begin{array}{r} z^2 + 1 \\ z-3\,\overline{)\,z^3 - 3z^2 + z - 3} \\ \underline{z^3 - 3z^2} \qquad\quad \\ z - 3 \\ \underline{z - 3} \\ 0 \end{array}$$

$$P(z) = z^3 - 3z^2 + z - 3$$

$$= (z-3)(z^2+1)$$

$$= (z-3)(z+i)(z-i)$$

roots

$3, -i, i$

b

$$P(z) = z^4 - z^3 - 3z^2 + 4z - 4$$

$P(2) = 0 \Rightarrow z - 2$ is a root

$$\begin{array}{r} z^3 + z^2 - z + 2 \\ z - 2 \overline{\big)\, z^4 - z^3 - 3z^2 + 4z - 4} \\ \underline{z^4 - 2z^3} \quad\quad\quad\quad\quad \\ z^3 - 3z^2 \quad\quad\quad\quad \\ \underline{z^3 - 2z^2} \quad\quad\quad\quad \\ -z^2 + 4z \quad\quad \\ \underline{-z^2 + 2z} \quad\quad \\ 2z - 4 \\ \underline{2z - 4} \\ 0 \end{array}$$

$$P(z) = z^4 - z^3 - 3z^2 + 4z - 4 = (z - 2)\left(z^3 + z^2 - z + 2\right)$$

$$Q(z) = z^3 + z^2 - z + 2$$

$Q(-2) = 0 \Rightarrow z + 2$ is a root

$$\begin{array}{r} z^2 - z + 1 \\ z + 2 \overline{\big)\, z^3 + z^2 - z + 2} \\ \underline{z^3 + 2z^2} \quad\quad\quad \\ -z^2 - z \quad\quad \\ \underline{-z^2 - 2z^2} \quad\quad \\ z + 2 \\ \underline{z + 2} \\ 0 \end{array}$$

$$= (z + 2)\left(z^2 - z + 1\right)$$

$$= (z + 2)\left(z^2 - z + \frac{1}{4} + \frac{3}{4}\right)$$

$$= (z + 2)\left(\left(z - \frac{1}{2}\right)^2 + \frac{3}{4}\right)$$

$$= (z + 2)\left(z - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left(z - \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$$

$$P(z) = z^4 - z^3 - 3z^2 + 4z - 4 = (z + 2)(z - 2)\left(z - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left(z - \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$$

roots $-2, 2, \frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{2}$

Question 5

a

$$\left(z-(1+i)\right)\left(z-(1-i)\right)$$
$$=z^2-2z+2$$
$$(z-5)\left(z^2-2z+2\right)$$
$$z^3-2z^2+2z-5z^2+10z-10$$
$$=z^3-7z^2+12z-10$$

The equation is

$$0=z^3-7z^2+12z-10$$

b

$$\left(z-\left(1+i\sqrt{3}\right)\right)\left(z-\left(1-i\sqrt{3}\right)\right)$$
$$=z^2-2z+4$$
$$(z+2)\left(z^2-2z+4\right)$$
$$z^3-2z^2+4z+2z^2-4z+8$$
$$=z^3+8$$

The equation is

$$0=z^3+8$$

c

$$\left(z-\left(3+i\sqrt{5}\right)\right)\left(z-\left(3-i\sqrt{5}\right)\right)$$
$$=z^2-6z+44$$
$$\left(z-(-1-4i)\right)\left(z-(-1+4i)\right)$$
$$=z^2+2z+17$$
$$\left(z^2-6z+44\right)\left(z^2+2z+17\right)$$
$$z^4+2z^3+17z^2-6z^3-12z^2-102z+14z^2+28z+238$$
$$=z^4-4z^3+19z^2-74z+238$$

The equation is

$$0=z^4-4z^3+19z^2-74z+238$$

d

$$\left(z-\left(\frac{1}{3}+\frac{i\sqrt{3}}{3}\right)\right)\left(z-\left(\frac{1}{3}-\frac{i\sqrt{3}}{3}\right)\right)$$

$$=z^2-\frac{2}{3}z+\frac{4}{9}$$

$$(z-4)\left(z^2-\frac{2}{3}z+\frac{4}{9}\right)$$

$$z^3-\frac{2}{3}z^2+\frac{4}{9}z-4z^2+\frac{8}{3}z-\frac{16}{9}$$

$$z^3-\frac{14}{3}z^2+\frac{28}{9}z-\frac{16}{9}$$

$$=9z^3-42z^2+28z-16$$

The equation is

$$0=9z^3-42z^2+28z-16$$

e

$$\left(z-e^{\frac{-i\pi}{3}}\right)\left(z-e^{\frac{i\pi}{3}}\right)$$

$$=z^2-z+1$$

$$(z+3)(z^2-z+1)$$

$$z^3-z^2+z+3z^2-3z+3$$

$$z^3+2z^2-2z+3$$

The equation is

$$0=z^3+2z^2-2z+3$$

Question 6

a As coefficients are real, any complex root must also have its conjugate pair as another root.

Roots will be 4, –2, –3i, 3i

So there will be 4 roots, and the minimum degree of the polynomial will be 4.

b As coefficients are real, any complex root must also have its conjugate pair as another root.

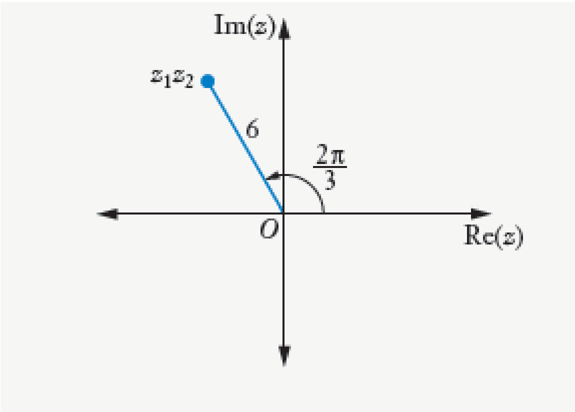
Roots will be $\sqrt{2}+i\sqrt{2},\sqrt{2}-i\sqrt{2},-2-5i,-2+5i,-1$

So there will be 5 roots, and the minimum degree of the polynomial will be 5.

Question 1

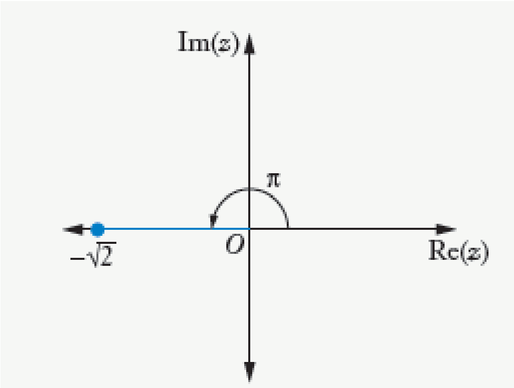
a

$$z_1 = 2\text{cis}\frac{\pi}{6}, z_2 = 3\text{cis}\frac{\pi}{2}$$
$$|z_1||z_2| = 2\times 3 = 6$$
$$\arg(z_1 z_2) = \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3}$$



b

$$z_1 = \sqrt{2}\text{cis}\frac{\pi}{3}, z_2 = \text{cis}\frac{2\pi}{3}$$
$$|z_1||z_2| = \sqrt{2}\times 1 = \sqrt{2}$$
$$\arg(z_1 z_2) = \frac{\pi}{3} + \frac{2\pi}{3} = \pi$$

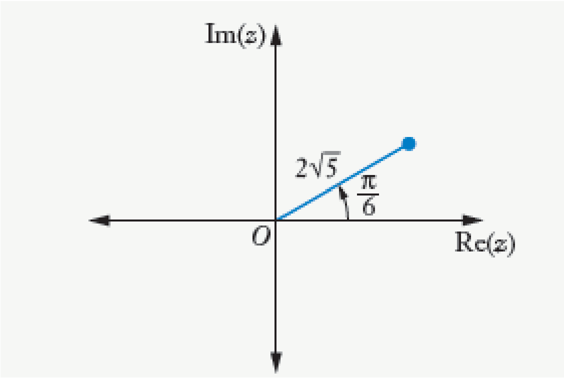


c

$$z_1 = \sqrt{5}\text{cis}\frac{-\pi}{6}, z_2 = 2\text{cis}\frac{\pi}{3}$$

$$|z_1||z_2| = \sqrt{5} \times 2 = 2\sqrt{5}$$

$$\arg(z_1 z_2) = \frac{-\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6}$$

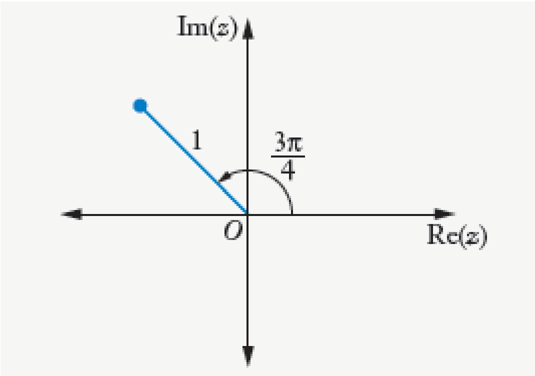


d

$$z_1 = \frac{1}{\sqrt{3}}\text{cis}\frac{-3\pi}{4}, z_2 = \sqrt{3}\text{cis}\frac{-\pi}{2}$$

$$|z_1||z_2| = \frac{1}{\sqrt{3}} \times \sqrt{3} = 1$$

$$\arg(z_1 z_2) = \frac{-3\pi}{4} + \frac{-\pi}{2} = \frac{-5\pi}{4} = \frac{3\pi}{4}$$



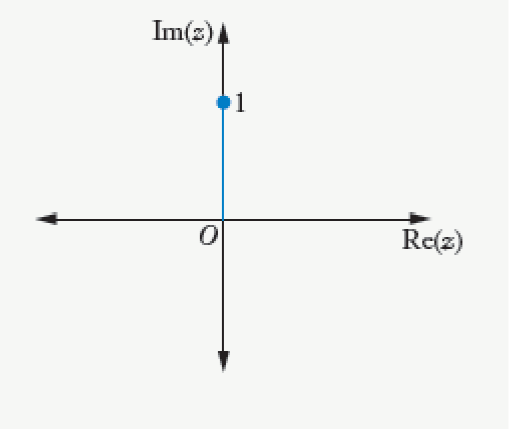
Question 2

a

$$z_1 = \text{cis}\frac{2\pi}{3}, z_2 = \text{cis}\frac{\pi}{6}$$

$$\frac{|z_1|}{|z_2|} = \frac{1}{1} = 1$$

$$\arg\left(\frac{z}{z_2}\right) = \frac{2\pi}{3} - \frac{\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$$

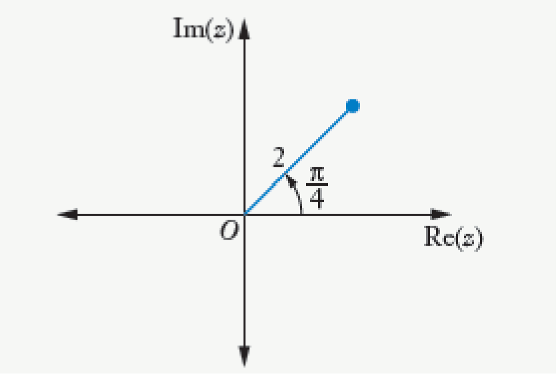


b

$$z_1 = 4\text{cis}\frac{3\pi}{4}, z_2 = 2\text{cis}\frac{\pi}{2}$$

$$\frac{|z_1|}{|z_2|} = \frac{4}{2} = 2$$

$$\arg\left(\frac{z}{z_2}\right) = \frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$$

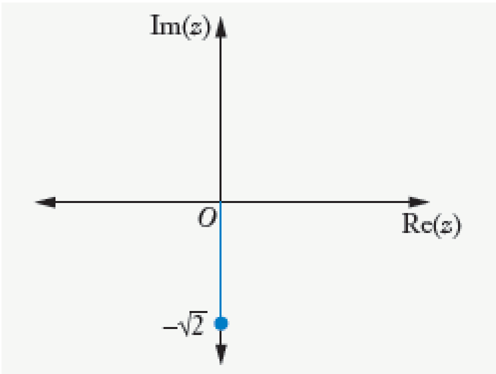


c

$z_1 = 2\text{cis}\frac{-\pi}{3}, z_2 = \sqrt{2}\text{cis}\frac{\pi}{6}$

$\frac{|z_1|}{|z_2|} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\arg\left(\frac{z}{z_2}\right) = \frac{-\pi}{3} - \frac{\pi}{6} = \frac{-3\pi}{6} = -\frac{\pi}{2}$

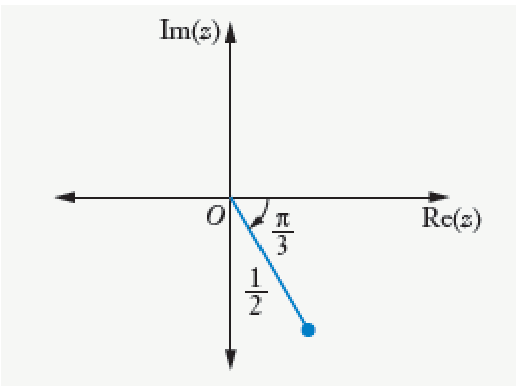


d

$z_1 = 3\text{cis}\frac{-7\pi}{12}, z_2 = 6\text{cis}\frac{-\pi}{4}$

$\frac{|z_1|}{|z_2|} = \frac{3}{6} = \frac{1}{2}$

$\arg\left(\frac{z}{z_2}\right) = \frac{-7\pi}{12} - \left(\frac{-\pi}{4}\right) = \frac{-4\pi}{12} = -\frac{\pi}{3}$



Question 3

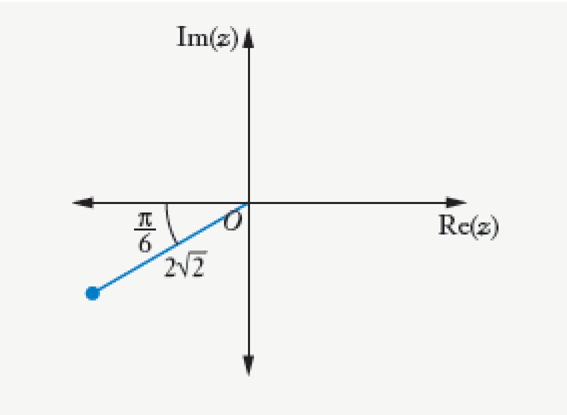
$$z = 2\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right) = 2\text{cis}\left(\frac{-2\pi}{3}\right)$$

$$z_2 = \sqrt{2}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = \sqrt{2}cis\left(\frac{-\pi}{6}\right)$$

$$|z_1||z_2| = 2\times\sqrt{2} = 2\sqrt{2}$$

$$\arg(z_1z_2) = \frac{-2\pi}{3} + \left(\frac{-\pi}{6}\right) = -\frac{5\pi}{6}$$

$$z_1z_2 = 2\sqrt{2}\text{cis}\left(-\frac{5\pi}{6}\right)$$



Question 4

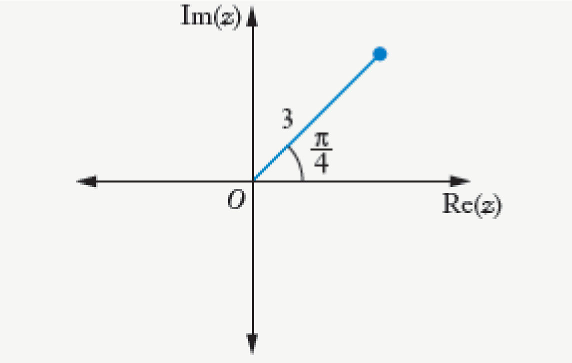
$$z = 6\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right) = 6\text{cis}\left(-\frac{\pi}{2}\right)$$

$$z_2 = 2\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right) = 2\text{cis}\left(-\frac{3\pi}{4}\right)$$

$$\frac{|z_1|}{|z_2|} = \frac{6}{2} = 3$$

$$\arg\left(\frac{z}{z_2}\right) = \left(\frac{-\pi}{2}\right) - \left(\frac{-3\pi}{4}\right) = \frac{\pi}{4}$$

$$\frac{z_1}{z_2} = 3\text{cis}\left(\frac{\pi}{4}\right)$$



Question 5

$$z_1 = \sqrt{3}\text{cis}\left(-\frac{\pi}{6}\right)$$

$$z_2 = \sqrt{6}\text{cis}\left(-\frac{2\pi}{3}\right)$$

$$z_3 = -3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 3\text{cis}\left(-\frac{2\pi}{3}\right)$$

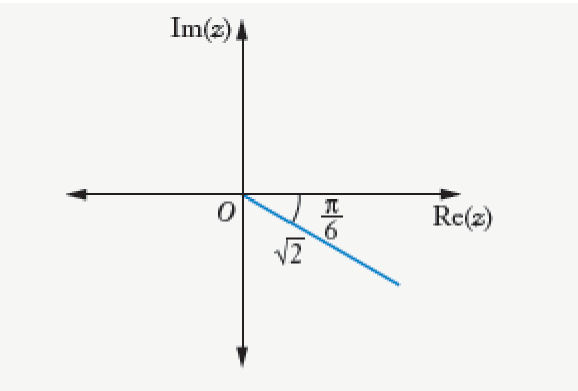
$$\frac{z_1 z_2}{z_3} = \frac{\sqrt{3}\text{cis}\left(-\frac{\pi}{6}\right)\sqrt{6}\text{cis}\left(-\frac{2\pi}{3}\right)}{3\text{cis}\left(-\frac{2\pi}{3}\right)}$$

$$= \frac{\sqrt{18}\text{cis}\left(-\frac{\pi}{6} - \frac{2\pi}{3}\right)}{3\text{cis}\left(-\frac{2\pi}{3}\right)}$$

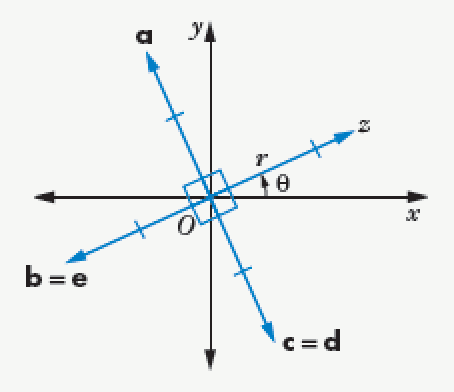
$$= \frac{3\sqrt{2}\text{cis}\left(-\frac{5\pi}{6}\right)}{3\text{cis}\left(-\frac{2\pi}{3}\right)}$$

$$= \sqrt{2}\text{cis}\left(-\frac{5\pi}{6} - \frac{2\pi}{3}\right)$$

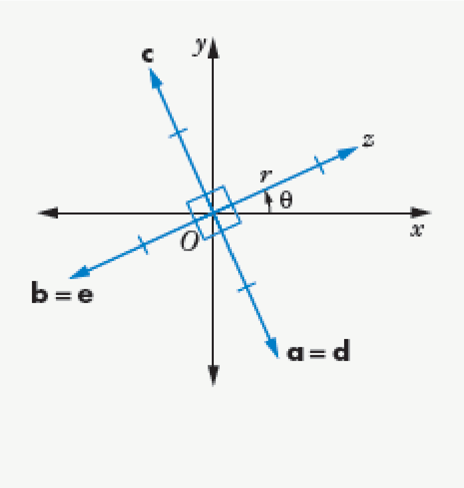
$$= \sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right)$$



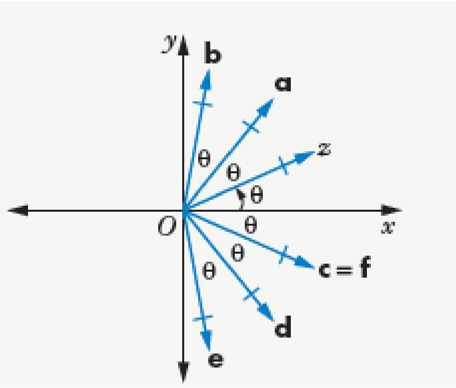
Question 6



Question 7



Question 8



Let $z = \text{cis}\theta$

$\frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}}$

$= \frac{\bar{z}}{(\bar{z})^2}$

$= \frac{\bar{z}}{|z|^2}$

$= \bar{z}$ as $|z|^2 = 1$

It is only true when $|z| = 1$.

Question 9

a $v = -iu$ or $\frac{u}{i}$

b $v = -u = i^2u$

c $v = iu$

Question 10

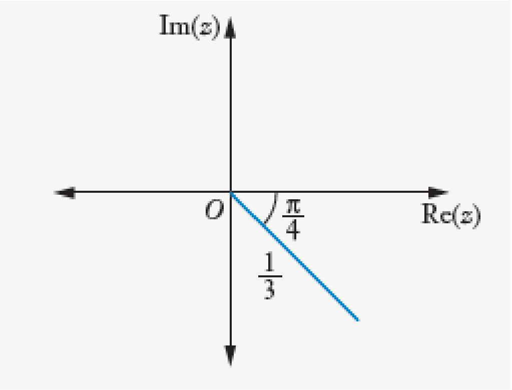
a

$$z = 3\operatorname{cis}\frac{\pi}{4}$$

$$\frac{1}{z} = \frac{\bar{z}}{|z||\bar{z}|}$$

$$= \frac{3\operatorname{cis}\left(-\frac{\pi}{4}\right)}{9}$$

$$= \frac{1}{3}\operatorname{cis}\left(-\frac{\pi}{4}\right)$$

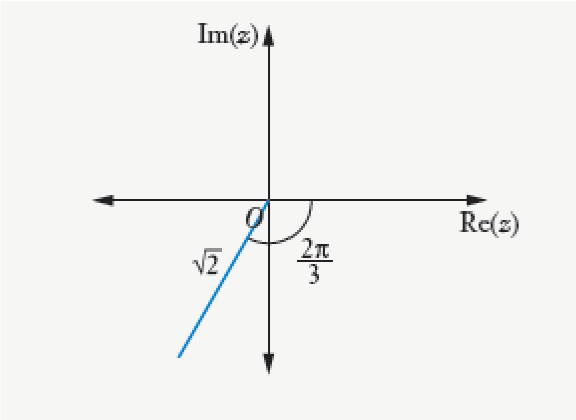


b

$$z = \frac{1}{\sqrt{2}} \operatorname{cis} \frac{2\pi}{3}$$

$$\frac{1}{z} = \frac{\bar{z}}{|z||\bar{z}|} = \frac{\frac{1}{\sqrt{2}} \operatorname{cis}\left(-\frac{2\pi}{3}\right)}{\frac{1}{2}}$$

$$= \sqrt{2} \operatorname{cis}\left(-\frac{2\pi}{3}\right)$$

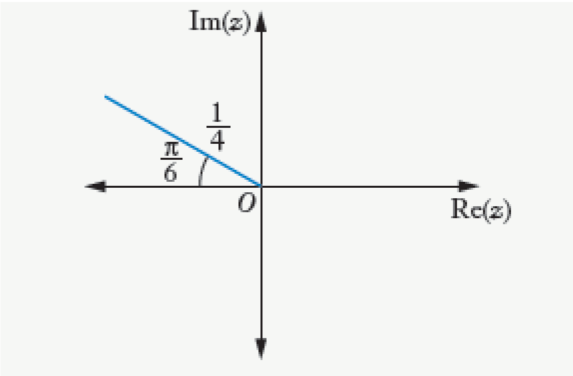


c

$$z = 4 \operatorname{cis} \frac{-5\pi}{6}$$

$$\frac{1}{z} = \frac{\bar{z}}{|z||\bar{z}|} = \frac{4 \operatorname{cis}\left(\frac{5\pi}{6}\right)}{16}$$

$$= \frac{1}{4} \operatorname{cis}\left(\frac{5\pi}{6}\right)$$



Question 11

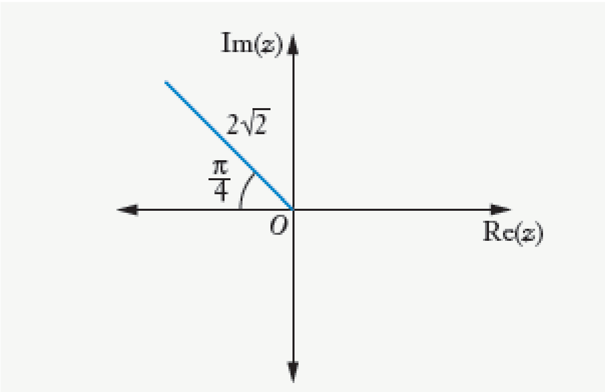
a

$$z = 1 + i = \sqrt{2}\,\text{cis}\frac{\pi}{4}$$

$$z^3 = \left(\sqrt{2}\,\text{cis}\frac{\pi}{4}\right)^3$$

$$= \left(\sqrt{2}\right)^3 \text{cis}\left(\frac{\pi}{4}\times 3\right)$$

$$= 2\sqrt{2}\,\text{cis}\frac{3\pi}{4}$$



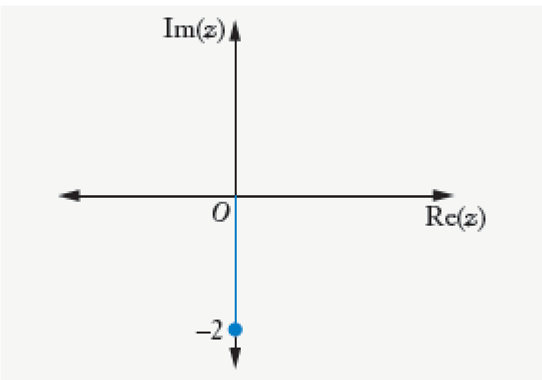
b

$$z = \frac{1}{2} + \frac{i}{2} = \frac{1}{\sqrt{2}}\,\text{cis}\frac{\pi}{4}$$

$$z^{-2} = \left(\frac{1}{\sqrt{2}}\,\text{cis}\frac{\pi}{4}\right)^{-2}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{-2} \text{cis}\left(\frac{\pi}{4}\times(-2)\right)$$

$$= 2\text{cis}\left(-\frac{\pi}{2}\right)$$



c

$$z = 1 - i\sqrt{3}$$

$$|z| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2}$$

$$= 2$$

$$\arg(z) = \tan^{-1}\frac{\left(-\sqrt{3}\right)}{1}$$

$$\theta = -\frac{\pi}{3}$$

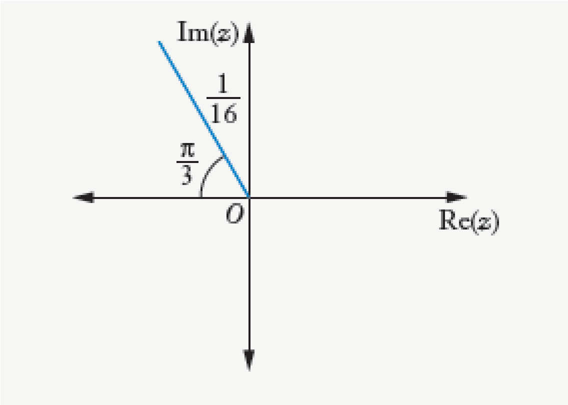
$$= 2\,\text{cis}\left(-\frac{\pi}{3}\right)$$

$$z^{-4} = \left(2\text{cis}\left[-\frac{\pi}{3}\right]\right)^{-4}$$

$$= 2^{-4}\,\text{cis}\left(-\frac{\pi}{3}\times(-4)\right)$$

$$= \frac{1}{16}\text{cis}\frac{4\pi}{3}$$

$$= \frac{1}{16}\text{cis}\left(-\frac{2\pi}{3}\right)$$



d

$$z = -\frac{3}{2} + \frac{i\sqrt{3}}{2}$$

$$|z| = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{3}$$

$$\arg(z) = \tan^{-1}\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(-\frac{3}{2}\right)}$$

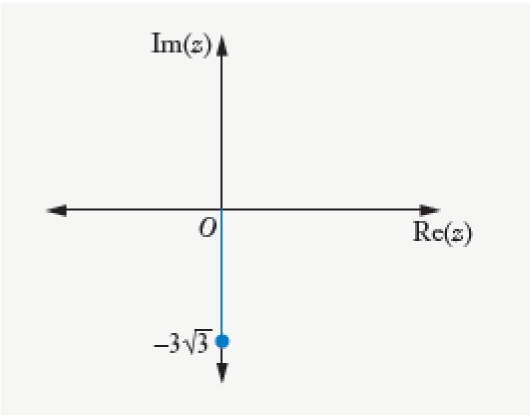
$$\theta = \frac{5\pi}{6}$$

$$z = \sqrt{3}\,\text{cis}\frac{5\pi}{6}$$

$$z^{-3} = \left(2\,\text{cis}\frac{5\pi}{6}\right)^{-3}$$

$$= \sqrt{3}^{-3}\text{cis}\left(\frac{5\pi}{6} \times (-3)\right)$$

$$= \frac{1}{3\sqrt{3}}\text{cis}\left(-\frac{\pi}{2}\right)$$



e

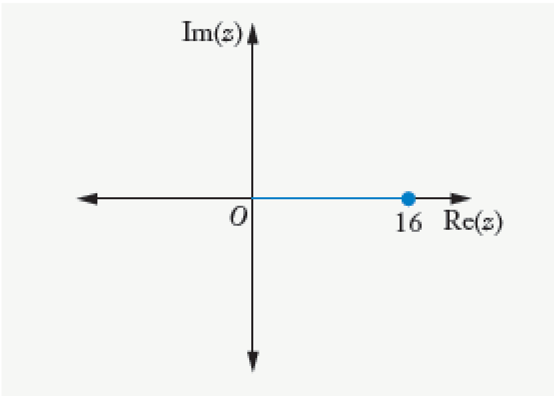
$$\left(\sqrt{2}e^{\frac{\pi}{4}}\right)^8$$

$$=\left(\sqrt{2}\right)^8 e^{\frac{i8\pi}{4}}$$

$$=16e^{2i\pi}$$

$$=16$$

$= 16 \text{ cis } 0$



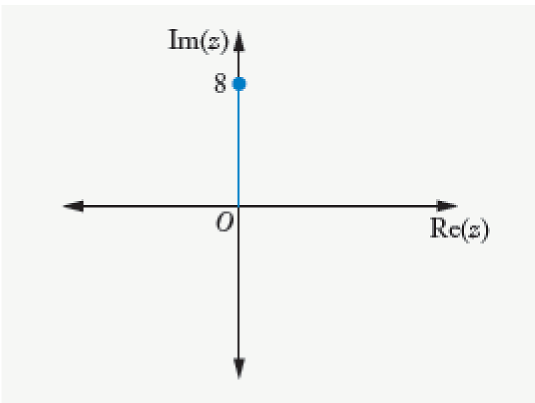
f

$$\left(\frac{1}{2}e^{-\frac{\pi}{6}}\right)^{-3}$$

$$=\left(\frac{1}{2}\right)^{-3} e^{\frac{i3\pi}{6}}$$

$$=8e^{\frac{i\pi}{2}}$$

$= 8 \text{ cis } \frac{\pi}{2}$



Question 12

a Modulus doubled, anticlockwise rotation of $\frac{\pi}{3}$:

$$w = z \times 2\,\text{cis}\frac{\pi}{3}$$

b Modulus halved, clockwise rotation of $\frac{7\pi}{12} + \frac{\pi}{4} = \frac{10\pi}{12} = \frac{5\pi}{6}$:

$$w = z \times \frac{1}{2}\text{cis}\left(-\frac{5\pi}{6}\right) \text{ or } w = \frac{z}{2\text{cis}\left(\frac{5\pi}{6}\right)}$$

c Modulus raised to power of 4, argument multiplied by 4:

$$w = z^4$$

OR Modulus multiplied by 8, anticlockwise rotation of 3θ:

$$w = z \times 8\,\text{cis}3\theta$$

Question 13

a $z_3 = -iz_1 = \frac{z_1}{i}$

b

$$OB = OA + AB$$
$$= z_1 + z_3$$
$$= z_1 - iz_1$$
$$= z\ (1 - i)$$

c

$$OM = \frac{1}{2}OB$$
$$= \frac{1}{2} \times z\ (1 - i)$$
$$= \frac{z}{2}(1 - i)$$

Question 14

a

$$|AD| = 3|AB|$$
$$\therefore |CB| = 3|CD|$$
$$CD = \delta - \gamma$$
$$CB = \beta - \gamma$$
As $CD \perp CB$
$$\frac{3(\delta - \gamma)}{i} = \beta - \gamma$$
$$\frac{\delta - \gamma}{i} = \frac{\beta - \gamma}{3}$$

b From diagram,

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OD} + \overrightarrow{DM} \\ &= \overrightarrow{OD} + \frac{1}{2}\overrightarrow{DB} \\ &= \overrightarrow{OD} + \frac{1}{2}\left(\overrightarrow{CD} + \overrightarrow{DC}\right)\end{aligned}$$

$$\begin{aligned}m &= \delta + \frac{1}{2}(\beta - \delta) \\ &= \delta + \frac{1}{2}\beta - \frac{1}{2}\delta \\ &= \frac{1}{2}\delta + \frac{1}{2}\beta \\ &= \frac{1}{2}(\delta + \beta)\end{aligned}$$

Question 15

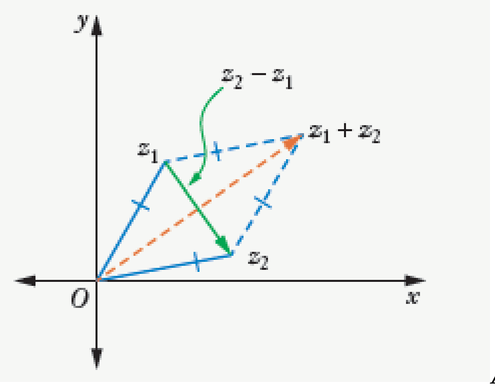
$PQ \perp PR$

$\therefore w_2 - w_1 = i(w_3 - w_1)$

$\frac{w_2 - w_1}{w_3 - w_1} = i$

Question 16

a



$A = z_1$, $B = z_2$, $C = z_1 + z_2$

b It is a rhombus as all 4 sides are equal.

c If $|z_1| = |z_2 - z_1|$, then $OA = AB$, so $\triangle ABC$ is equilateral as all 3 sides are equal.

Question 17

Let $w=\dfrac{u-iv}{1-i}$

consider mods:

$$|w-u|^2=\left|\frac{u-iv}{1-i}-u\right|^2$$

$$=\left|\frac{u-iv-u+iu}{1-i}\right|^2$$

$$=\frac{1\times|u-v|^2}{2}$$

Similarly

$$|w-v|^2=\frac{|u-v|^2}{2}$$

$$|w-u|^2+|w-v|^2=\frac{|u-v|^2}{2}+\frac{|u-v|^2}{2}$$

$$=|u-v|^2$$

$\therefore$ UVW is a right-angled triangle.

Exercise 4.05 Roots of unity

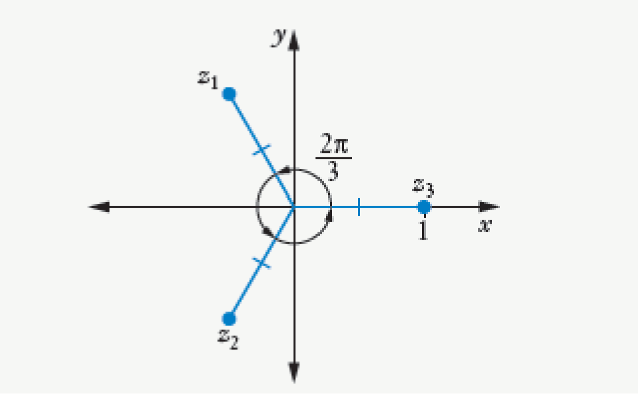
Question 1

a

$z^3 = 1$

The third roots of 1 are equally spaced $\dfrac{2\pi}{3}$ apart from $z = 1$

$z = 1, \text{cis}\dfrac{2\pi}{3}, \text{cis}\left(-\dfrac{2\pi}{3}\right)$



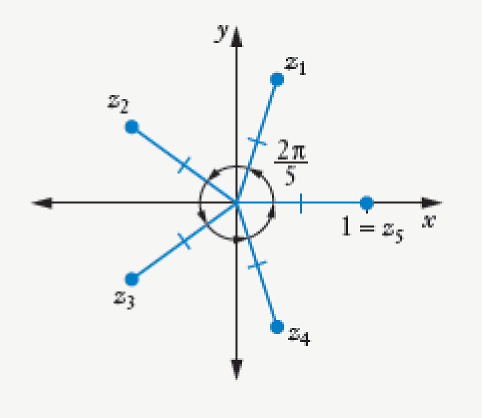
b

$z^5 - 1 = 0$

$z^5 = 1$

The fifth roots of 1 are equally spaced $\dfrac{2\pi}{5}$ apart from $z = 1$

$z = 1, \text{cis}\dfrac{2\pi}{5}, \text{cis}\dfrac{4\pi}{5}, \text{cis}\left(-\dfrac{2\pi}{5}\right), \text{cis}\left(-\dfrac{4\pi}{5}\right)$

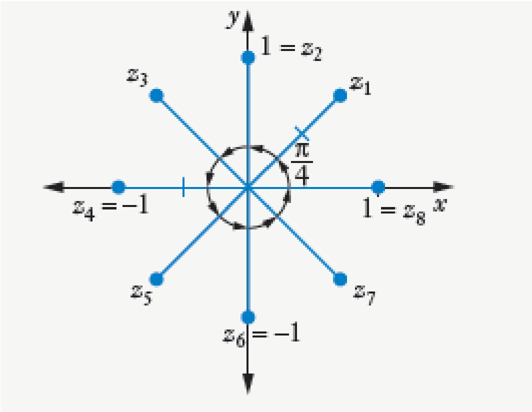


c

$z^8 = 1$

The eighth roots of 1 are equally spaced $\frac{\pi}{4}$ apart from $z = 1$

$$z = 1, -1, \text{cis}\frac{\pi}{4}, \text{cis}\frac{\pi}{2} = i, \text{cis}\frac{3\pi}{4}, \text{cis}\left(-\frac{\pi}{4}\right), \text{cis}\left(-\frac{\pi}{2}\right) = -i, \text{cis}\left(-\frac{3\pi}{4}\right)$$



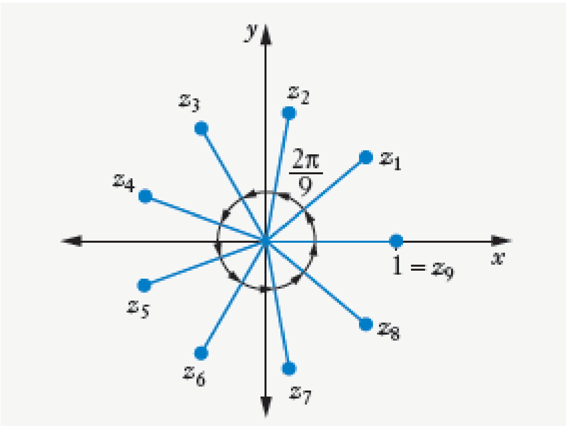
d

$z^9 - 1 = 0$

$z^9 = 1$

The ninth roots of 1 are equally spaced $\frac{2\pi}{9}$ apart from $z = 1$

$$z = 1, \text{cis}\frac{2\pi}{9}, \text{cis}\frac{4\pi}{9}, \text{cis}\frac{2\pi}{3}, \text{cis}\frac{8\pi}{9}, \text{cis}\left(-\frac{2\pi}{9}\right), \text{cis}\left(-\frac{4\pi}{9}\right), \text{cis}\left(-\frac{2\pi}{3}\right), \text{cis}\left(-\frac{8\pi}{9}\right)$$



Question 2

a

$$\alpha = \text{cis}\frac{\pi}{3}$$
$$\alpha^6 = \text{cis}\left(6\times\frac{\pi}{3}\right)$$
$$= \text{cis}(2\pi)$$
$$= 1$$

b

$$\alpha = \text{cis}\left(-\frac{4\pi}{7}\right)$$
$$\alpha^7 = \text{cis}\left(7\times\frac{-4\pi}{7}\right)$$
$$= \text{cis}(-4\pi)$$
$$= 1$$

c

$$\alpha = \text{cis}\frac{\pi}{6}$$
$$\alpha^2 = \text{cis}\left(12\times\frac{\pi}{6}\right)$$
$$= \text{cis}(2\pi)$$
$$= 1$$
$$\alpha^7 = \text{cis}\left(7\times\frac{\pi}{6}\right)$$
$$= \text{cis}\left(\frac{7\pi}{6}\right)$$
$$= \text{cis}\left(-\frac{5\pi}{6}\right)$$
$$\overline{\alpha} = \text{cis}\left(-\frac{\pi}{6}\right)$$
$$\overline{\alpha}^5 = \text{cis}\left(5\times\frac{-\pi}{6}\right)$$
$$= \text{cis}\left(-\frac{5\pi}{6}\right)$$
$$\therefore \alpha^7 = \overline{\alpha}^5$$

d

$$\alpha = \text{cis}\left(-\frac{2\pi}{3}\right)$$
$$\alpha^9 = \text{cis}\left(9\times\frac{-2\pi}{3}\right)$$
$$= \text{cis}(-6\pi)$$
$$= 1$$
$$\alpha^{-4} = \text{cis}\left(-4\times\frac{-2\pi}{3}\right)$$
$$= \text{cis}\left(\frac{8\pi}{3}\right)$$
$$= \text{cis}\left(\frac{2\pi}{3}\right)$$
$$\overline{\alpha} = \text{cis}\left(\frac{2\pi}{3}\right)$$
$$\overline{\alpha}^4 = \text{cis}\left(4\times\frac{2\pi}{3}\right)$$
$$= \text{cis}\left(\frac{8\pi}{3}\right)$$
$$= \text{cis}\left(\frac{2\pi}{3}\right)$$
$$\therefore \alpha^{-4} = \overline{\alpha}^4$$

Question 3

a

$$z^7 - 1 = 0$$
$$z^7 = 1$$
$$\overline{\alpha} = \alpha^6, \overline{\alpha}^2 = \alpha^5, \overline{\alpha}^3 = \alpha^4, \overline{\alpha}^4 = \alpha^3, \overline{\alpha}^5 = \alpha^2, \overline{\alpha}^6 = \alpha$$

b

$$z^{11} = 1$$
$$\overline{\alpha} = \alpha^0, \overline{\alpha}^2 = \alpha^9, \overline{\alpha}^3 = \alpha^8, \overline{\alpha}^4 = \alpha^7, \overline{\alpha}^5 = \alpha^6, \overline{\alpha}^6 = \alpha^5, \overline{\alpha}^7 = \alpha^4, \overline{\alpha}^8 = \alpha^3, \overline{\alpha}^9 = \alpha^2, \overline{\alpha}^{10} = \alpha$$

Question 4

$\beta = \text{cis}\dfrac{2\pi}{5}$

$z^5 - 1 = 0$

$(z-1)(z^4 + z^3 + z^2 + z + 1) = 0$

As $\beta \neq 1$ it must be a root of $(z^4 + z^3 + z^2 + z + 1)$

$\therefore \beta^4 + \beta^3 + \beta^2 + \beta + 1 = 0$

dividing both sides by $\dfrac{1}{\beta^2}$ gives

$\beta^2 + \beta + 1 + \dfrac{1}{\beta} + \dfrac{1}{\beta^2} = 0$

Question 5

$z^3 = 1$

$z^3 - 1 = 0$

$(z-1)(z^2 + z + 1) = 0$

As $w \neq 1$ it must be a root of $(z^2 + z + 1)$

$\therefore w^2 + w + 1 = 0$

a

$w^3 + w^2 + w$

$= w(w^2 + w + 1)$

$= w \times 0$

$= 0$

b

$$w^2+w+1=0$$
$$w=-w^2-1$$
$$w^2+w=-1$$
$$w^2=-w-1$$
$$\left(w^2+w\right)\left(w^2+w^3\right)\left(w+w^3\right)$$
$$=-1\times w\left(w^2+w\right)w\left(w^2+1\right)$$
$$=-1\times w\times(-1)\times w\times(-w)$$
$$=1\times\left(w^3\right)$$
$$=1\times(-1)$$
$$=-1$$

c

$$\left(6w+1\right)\left(6w^2+1\right)$$
$$=36w^3+6w+6w^2+1$$
$$=36+6\left(w^2+w\right)+1$$
$$=36-6+1$$
$$=31$$

d

$$\left(1-w-w^2\right)\left(w-w^2-1\right)\left(w^2-1-w\right)$$
$$=\left(1-\left(w+w^2\right)\right)\left(w-\left(w^2+1\right)\right)\left(w^2-(1+w)\right)$$
$$=\left(1-(-1)\right)\left(w-(-w)\right)\left(w^2-\left(-w^2\right)\right)$$
$$=2\times 2w\times 2w^2$$
$$=8w^3$$
$$=8$$

e

$$w^7+w^8+w^9+w^{10}+w^{11}+w^{12}+w^{13}+w^{14}$$
$$=w^7\left(1+w+w^2+w^3+w^4+w^5+w^6+w^7\right)$$
$$=w\left(1+w+w^2+w^3\left(1+w+w^2\right)+w^6\left(1+w\right)\right)$$
$$=w\left(0+1(0)+1(1+w)\right)$$
$$=w(1+w)$$
$$=w\left(-w^2\right)$$
$$=-w^3$$
$$=-1$$

Question 6

a

$$z^9=1$$
$$z^9-1=0$$
$$(z-1)\left(z^8+z^7+z^6+z^5+z^4+z^3+z^2+z+1\right)=0$$
As $\alpha\neq1$ it must be a root of $\left(z^8+z^7+z^6+z^5+z^4+z^3+z^2+z+1\right)$
$$\therefore \alpha^8+\alpha^7+\alpha^6+\alpha^5+\alpha^4+\alpha^3+\alpha^2+\alpha+1=0$$

b

$$\left(z^3\right)^3=1$$
$$\left(z^3\right)^3-1=0$$
$$\left(z^3-1\right)\left(\left(z^3\right)^2+\left(z^3\right)+1\right)=0$$
$$\left(z^3-1\right)\left(z^6+z^3+1\right)=0$$
As $\alpha\neq1$ it must be a root of $\left(z^6+z^3+1\right)$
$$\therefore \alpha^6+\alpha^3+1=0$$

c

Sum of roots = 0

$$1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6+\alpha^7+\alpha^8=0$$

$$\alpha+\overline{\alpha}+\alpha^2+\overline{\alpha^2}+\alpha^3+\overline{\alpha^3}+\alpha^4+\overline{\alpha^4}=-1$$

$$2\cos\frac{2\pi}{9}+2\cos\frac{4\pi}{9}+2\cos\frac{6\pi}{9}+2\cos\frac{8\pi}{9}=-1$$

$$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{6\pi}{9}+\cos\frac{8\pi}{9}=-\frac{1}{2}$$

Question 1

a

$$u = 1 + i\sqrt{3}$$

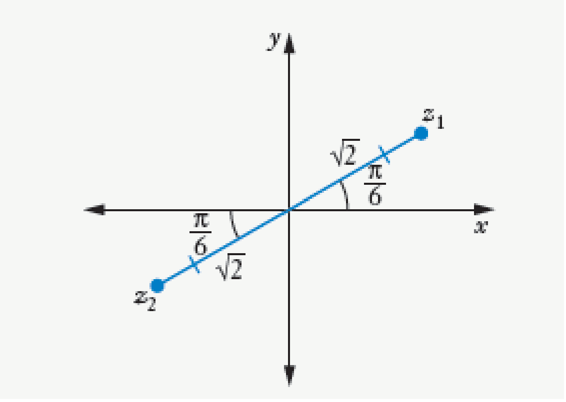
$$|u| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\arg(u) = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$$

$$u = 2\text{cis}\frac{\pi}{3}$$

$$z^2 = 2\text{cis}\frac{\pi}{3}$$

$$z = \sqrt{2}\,\text{cis}\frac{\pi}{6}, \sqrt{2}\,\text{cis}\left(-\frac{5\pi}{6}\right)$$



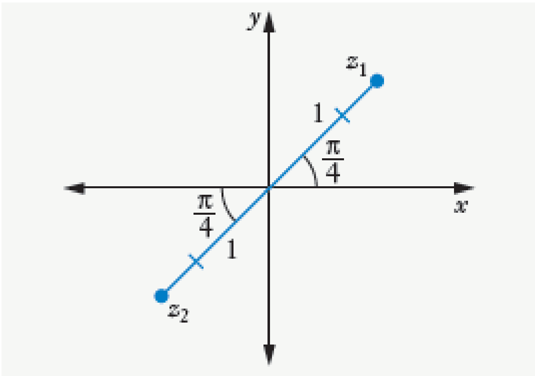
b

$$u = i$$

$$u = \text{cis}\frac{\pi}{2}$$

$$z^2 = \text{cis}\frac{\pi}{2}$$

$$z = \text{cis}\frac{\pi}{4}, \text{cis}\left(\frac{-3\pi}{4}\right)$$



c

$$u = -1 - i\sqrt{3}$$

$$|u| = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = 2$$

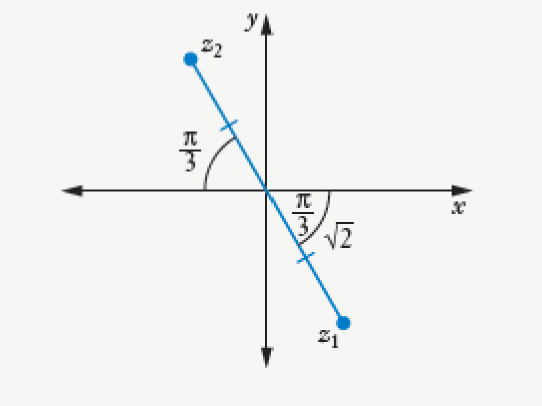
$$\arg(u) = \tan^{-1}\frac{-\sqrt{3}}{-1} = -\frac{2\pi}{3}$$

$$u = 2\text{cis}\left(-\frac{2\pi}{3}\right)$$

$$z^2 = 2\text{cis}\left(-\frac{2\pi}{3}\right)$$

$$z = \sqrt{2}\,\text{cis}\left(-\frac{2\pi}{6}\right)$$

$$z = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{3}\right), \sqrt{2}\,\text{cis}\frac{2\pi}{3}$$



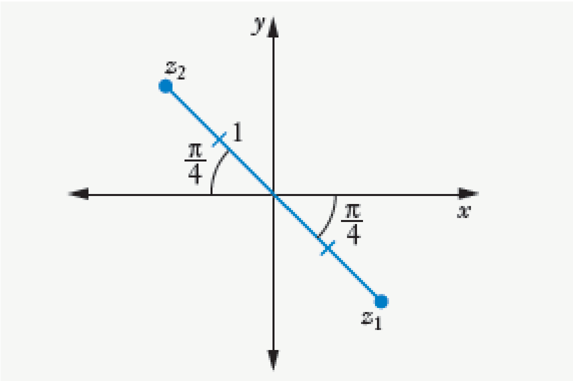
d

$u = -i$

$u = \text{cis}\left(-\frac{\pi}{2}\right)$

$z^2 = \text{cis}\left(-\frac{\pi}{2}\right)$

$z = \text{cis}\left(-\frac{\pi}{4}\right), \text{cis}\left(\frac{3\pi}{4}\right)$



Question 2

a

$$z^3 = -1$$

As roots are equally spaced around the argand diagram

$$z = \text{cis}\,\pi, \text{cis}\frac{\pi}{3}, \text{cis}\left(-\frac{\pi}{3}\right)$$

b

$$z^3 = i$$

$$z^3 = \text{cis}\frac{\pi}{2}$$

$$z = \text{cis}\left(\frac{\pi}{2} \div 3\right)$$

$$z = \text{cis}\frac{\pi}{6}$$

As roots are equally spaced around the argand diagram

$$z = \text{cis}\frac{\pi}{6}, \text{cis}\frac{5\pi}{6}, \text{cis}\left(-\frac{3\pi}{6}\right)$$

$$z = \text{cis}\frac{\pi}{6}, \text{cis}\frac{5\pi}{6}, \text{cis}\left(-\frac{\pi}{2}\right)$$

c

$$z^3 = -i$$

$$z^3 = \text{cis}\left(-\frac{\pi}{2}\right)$$

$$z = \text{cis}\left(-\frac{\pi}{2} \div 3\right)$$

$$z = \text{cis}\left(-\frac{\pi}{6}\right)$$

As roots are equally spaced around the argand diagram

$$z = \text{cis}\left(-\frac{\pi}{6}\right), \text{cis}\frac{3\pi}{6}, \text{cis}\left(-\frac{5\pi}{6}\right)$$

$$z = \text{cis}\left(-\frac{\pi}{6}\right), \text{cis}\frac{\pi}{2}, \text{cis}\left(-\frac{5\pi}{6}\right)$$

Question 3

a

$$z^4 = 16i$$

$$z^4 = 16\,\text{cis}\frac{\pi}{2}$$

$$z = \sqrt[4]{16}\,\text{cis}\left(\frac{\pi}{2} \div 4\right)$$

$$z = 2\,\text{cis}\frac{\pi}{8}$$

As roots are equally spaced around the argand diagram

$$z = 2\,\text{cis}\frac{\pi}{8}, 2\,\text{cis}\frac{5\pi}{8}, 2\text{cis}\left(-\frac{3\pi}{8}\right), 2\text{cis}\left(-\frac{7\pi}{8}\right)$$

b

$$u = -1 - i\sqrt{3}$$

$$|u| = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = 2$$

$$\arg(u) = \tan^{-1}\frac{-\sqrt{3}}{-1} = -\frac{2\pi}{3}$$

$$u = 2\text{cis}\left(-\frac{2\pi}{3}\right)$$

$$z^4 = 2\,\text{cis}\left(-\frac{2\pi}{3}\right)$$

$$z = \sqrt[4]{2}\,\text{cis}\left(-\frac{2\pi}{3} \div 4\right)$$

$$z = \sqrt[4]{2}\,\text{cis}\left(-\frac{\pi}{6}\right)$$

As roots are equally spaced around the argand diagram

$$z = \sqrt[4]{2}\,\text{cis}\left(-\frac{\pi}{6}\right), \sqrt[4]{2}\,\text{cis}\left(-\frac{4\pi}{6}\right), \sqrt[4]{2}\,\text{cis}\frac{2\pi}{6}, \sqrt[4]{2}\,\text{cis}\frac{5\pi}{6}$$

$$z = \sqrt[4]{2}\,\text{cis}\left(-\frac{\pi}{6}\right), \sqrt[4]{2}\,\text{cis}\left(-\frac{2\pi}{3}\right), \sqrt[4]{2}\,\text{cis}\frac{\pi}{3}, \sqrt[4]{2}\,\text{cis}\frac{5\pi}{6}$$

c

$$z^4 = -i$$
$$z^4 = \text{cis}\left(-\frac{\pi}{2}\right)$$
$$z = \text{cis}\left(-\frac{\pi}{2} \div 4\right)$$
$$z = \text{cis}\left(-\frac{\pi}{8}\right)$$

As roots are equally spaced around the argand diagram

$$z = \text{cis}\left(-\frac{\pi}{8}\right), \text{cis}\left(-\frac{5\pi}{8}\right), \text{cis}\frac{3\pi}{8}, \text{cis}\frac{7\pi}{8}$$

Question 4

a

$$z = \text{cis}\left(-\frac{3\pi}{5}\right)$$
$$z^5 = \left(\text{cis}\left(-\frac{3\pi}{5}\right)\right)^5$$
$$= \text{cis}\left(-\frac{3\pi}{5} \times 5\right)$$
$$= \text{cis}(-3\pi)$$
$$= -1$$
$$z^5 + 1 = -1 + 1$$
$$= 0$$
$$\therefore z = cis\left(\frac{-3\pi}{5}\right) \text{ is a fifth root of } z^5 + 1 = 1$$

b

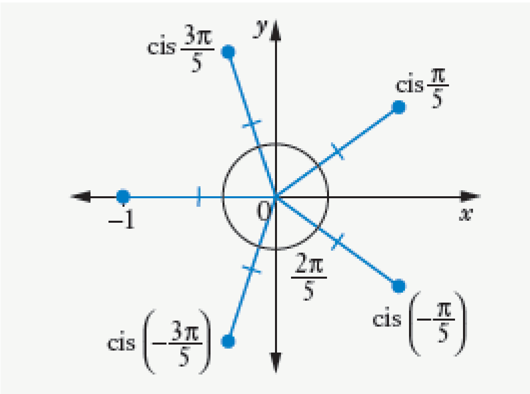
$$z_1 = \text{cis}\,\pi = -1$$

$$z_2 = \text{cis}\left(-\frac{3\pi}{5}\right)$$

$$z_3 = \text{cis}\,\frac{3\pi}{5}$$

$$z_4 = \text{cis}\left(-\frac{\pi}{5}\right)$$

$$z_5 = \text{cis}\,\frac{\pi}{5}$$



c

$$z_2 = \overline{z_3}$$
$$z_4 = \overline{z_5}$$

d

$$z^5 + 1 = 0$$

$$z_1 + z_2 + z_3 + z_4 + z_5 = 0$$

$$\text{cis}\,\pi + \text{cis}\left(-\frac{3\pi}{5}\right) + \text{cis}\left(\frac{3\pi}{5}\right) + \text{cis}\left(-\frac{\pi}{5}\right) + \text{cis}\left(\frac{\pi}{5}\right) = 0$$

$$-1 + 2\cos\left(\frac{3\pi}{5}\right) + 2\cos\left(\frac{\pi}{5}\right) = 0$$

$$2\left(\cos\left(\frac{3\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right)\right) = 1$$

$$\cos\left(\frac{3\pi}{5}\right) + \cos\left(\frac{\pi}{5}\right) = \frac{1}{2}$$

Question 5

$z^7 = -1$

$z^7 = \text{cis}(\pi)$

$z_1 = -1$

As roots are equally spaced around the argand diagram

$$z = -1, \text{cis}\left(-\frac{\pi}{7}\right), \text{cis}\left(-\frac{3\pi}{7}\right), \text{cis}\left(-\frac{5\pi}{7}\right), \text{cis}\frac{\pi}{7}, \text{cis}\frac{3\pi}{7}, \text{cis}\frac{5\pi}{7}$$

a As the complex solutions are vectors of a regular septagon the sum of the vectors is zero.

b

$$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 = 0$$

$$\text{cis}\pi + \text{cis}\left(-\frac{\pi}{7}\right) + \text{cis}\left(\frac{\pi}{7}\right) + \text{cis}\left(-\frac{3\pi}{7}\right) + \text{cis}\left(\frac{3\pi}{7}\right) + \text{cis}\left(-\frac{5\pi}{7}\right) + \text{cis}\left(\frac{5\pi}{7}\right) = 0$$

$$-1 + 2\cos\left(\frac{\pi}{7}\right) + 2\cos\left(\frac{3\pi}{7}\right) + 2\cos\left(\frac{5\pi}{7}\right) = 0$$

$$2\left(\cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right)\right) = 1$$

$$\cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right) = \frac{1}{2}$$

Question 6

a

$$z^6 = -1$$
$$z^6 = \text{cis}(\pi)$$
$$z = \text{cis}\left(-\frac{\pi}{6}\right)$$

As roots are equally spaced around the argand diagram

$$z = \text{cis}\left(-\frac{\pi}{6}\right), \text{cis}\frac{\pi}{6}, \text{cis}\left(-\frac{3\pi}{6}\right), \text{cis}\frac{3\pi}{6}, \text{cis}\left(-\frac{5\pi}{6}\right), \text{cis}\frac{5\pi}{6}$$
$$z = \text{cis}\left(-\frac{\pi}{6}\right), \text{cis}\frac{\pi}{6}, \text{cis}\left(-\frac{\pi}{2}\right), \text{cis}\frac{\pi}{2}, \text{cis}\left(-\frac{5\pi}{6}\right), \text{cis}\frac{5\pi}{6}$$

b

$$z^8 = -1$$
$$z^8 = \text{cis}(\pi)$$
$$z = \text{cis}\left(\frac{\pi}{8}\right)$$

As roots are equally spaced around the argand diagram

$$z = \text{cis}\left(-\frac{\pi}{8}\right), \text{cis}\frac{\pi}{8}, \text{cis}\left(-\frac{3\pi}{8}\right), \text{cis}\frac{3\pi}{8}, \text{cis}\left(-\frac{5\pi}{8}\right), \text{cis}\frac{5\pi}{8}, \text{cis}\left(-\frac{7\pi}{8}\right), \text{cis}\frac{7\pi}{8}$$

c

$$z^5 = i$$
$$z^5 = \text{cis}\frac{\pi}{2}$$
$$z = \text{cis}\frac{\pi}{10}$$

As roots are equally spaced around the argand diagram

$$z = \text{cis}\frac{\pi}{10}, \text{cis}\frac{\pi}{2} = i, \text{cis}\frac{9\pi}{10}, \text{cis}\left(-\frac{7\pi}{10}\right), \text{cis}\left(-\frac{3\pi}{10}\right)$$

Question 7

a

$$\text{Let } z = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$$

$$z^3 - 2 - 2i = \left(\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)\right)^3 - 2 - 2i$$

$$= \left(\sqrt{2}\right)^3 \text{cis}\left(\frac{3\pi}{4} \times 3\right) - 2 - 2i$$

$$= 2\sqrt{2}\,\text{cis}\left(\frac{9\pi}{4}\right) - 2 - 2i$$

$$= 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right) - 2 - 2i$$

$$= 2\sqrt{2}\cos\left(\frac{\pi}{4}\right) + i2\sqrt{2}\sin\left(\frac{\pi}{4}\right) - 2 - 2i$$

$$= 2\sqrt{2} \times \frac{1}{\sqrt{2}} + i2\sqrt{2} \times \frac{1}{\sqrt{2}} - 2 - 2i$$

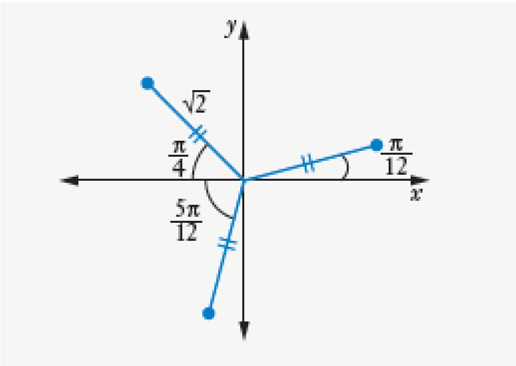
$$= 2 + 2i - 2 - 2i$$

$$= 0$$

So $z = \sqrt{2}\ \text{cis}\frac{3\pi}{4}$ is a factor of $z^3 - 2 - 2i = 0$ as required.

b

The other 2 roots will be length $\sqrt{2}$, and equally spaced around the diagram, $\frac{2\pi}{3}$ from each other.

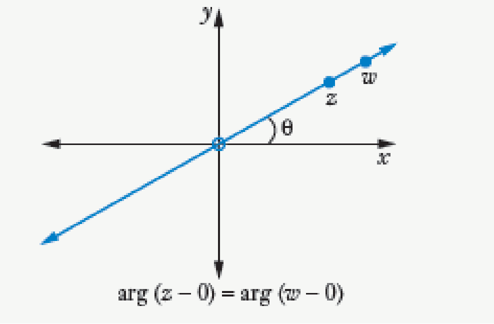


Exercise 4.07 Curves and regions on the complex plane

Question 1

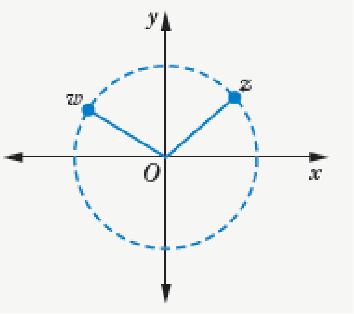
a arg (z) = arg (w)

Means z and w lie on the same line through O (or vector or ray from O) on the same side of O.



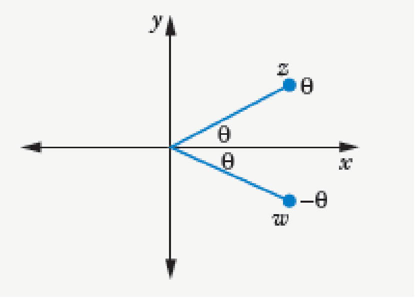
b $|z| = |w|$

Means z and w lie on the same circle of centre O.



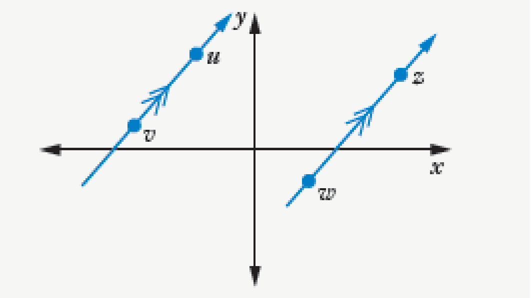
c arg (z) = –arg (w)

Means the line OW is a reflection of the line OZ over the x-axis, but not necessarily the same size.



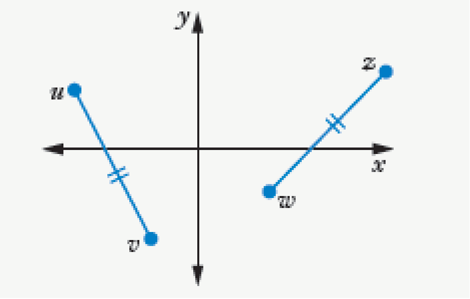
d arg $(z - w)$ = arg $(u - v)$

Means vector $(z - w)$ is parallel to vector $(u - v)$.



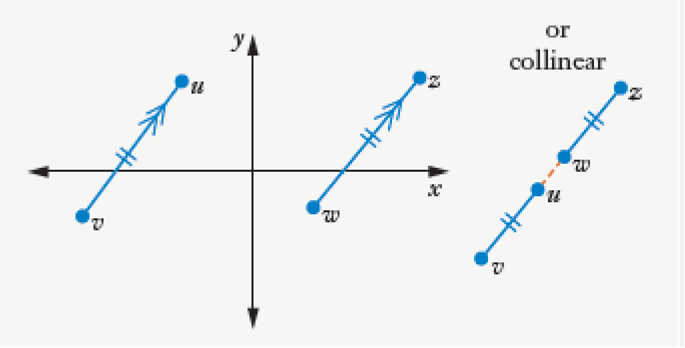
e $|z - w| = |u - v|$

Means the lengths of the vectors $(z - w)$ and $(u - v)$ are equal.



f $z - w = u - v$

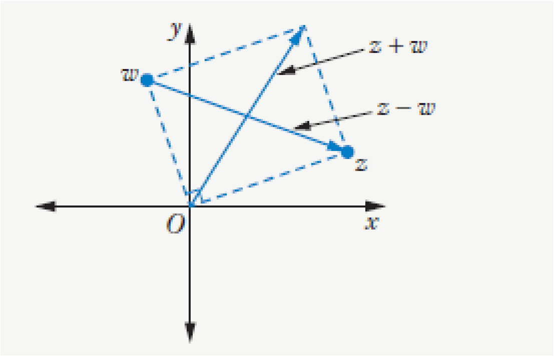
Means the vectors $(z - w)$ and $(u - v)$ are equal in length and parallel (or collinear with the same argument).



g $|z+w| = |z-w|$

Means the lengths of vectors ($z + w$) and ($z – w$) are equal in length.

Hence O, w, z, $z + w$ form a rectangle.



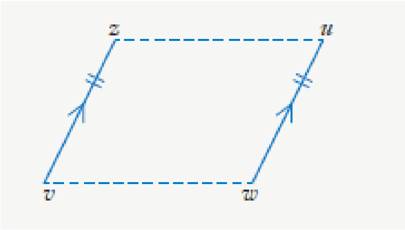
h

$z + w = u + v$

Rearranging gives

$z - v = u - w$

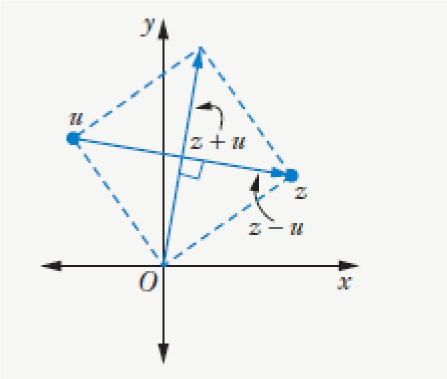
Means the vectors ($z – v$) and ($u – w$) are equal in length and parallel, forming a parallelogram or they are collinear.



i $z - u = i(z + u)$

Means the diagonals of the quadrilateral formed by O, z, u and $z + u$ are equal and perpendicular.

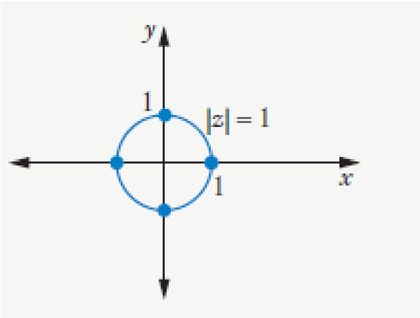
Hence it is a square.



Question 2

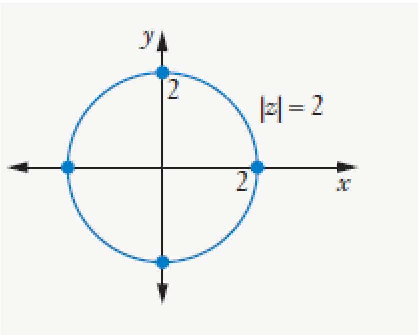
a $|z| = 1$

Circle of radius 1 centred on the origin.



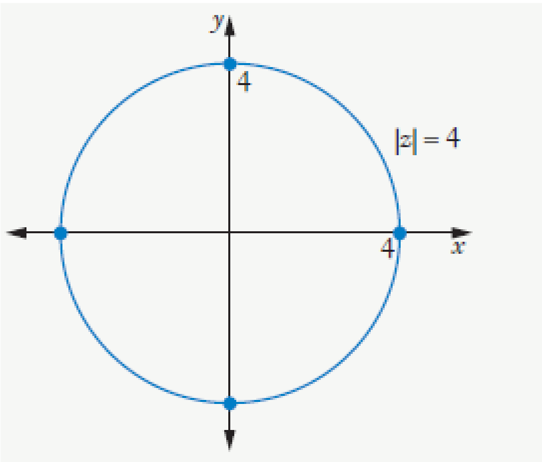
b $|z| = 2$

Circle of radius 2 centred on the origin.



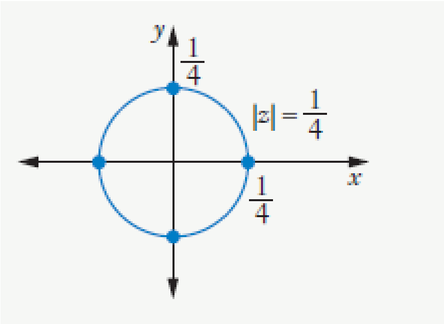
c $|z| = 4$

Circle of radius 4 centred on the origin



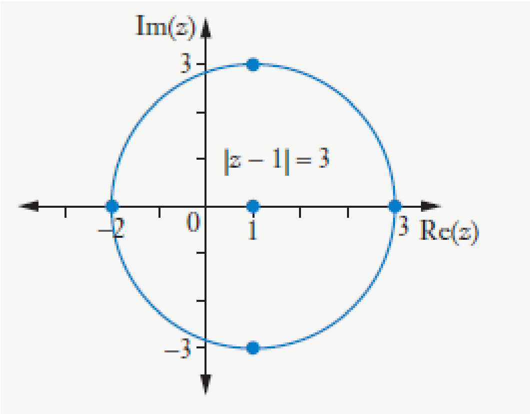
d $|z| = \frac{1}{4}$

Circle of radius $\frac{1}{4}$ centred on the origin.



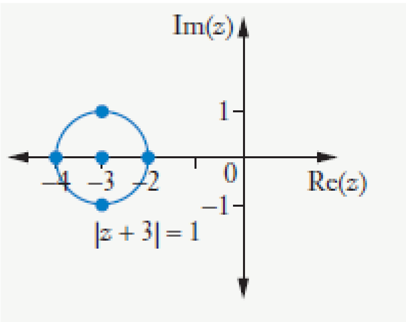
e $|z - 1| = 3$

Circle of radius 3 centred on (0, 1).



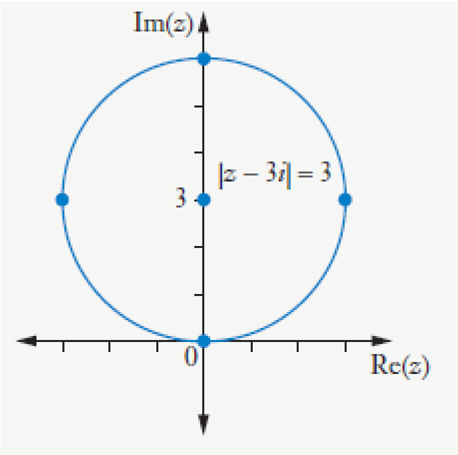
f $|z + 3| = 1$

Circle of radius 1 centred on (–3, 0).



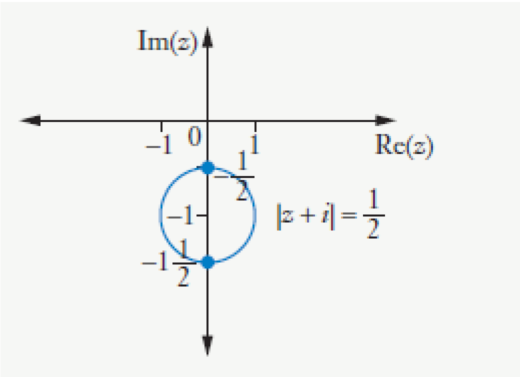
g $|z-3i|=3$

Circle of radius 3 centred on (0, 3).



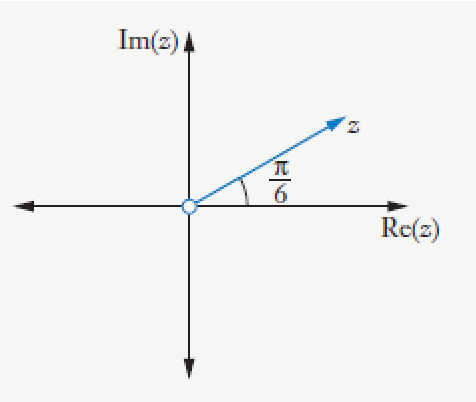
h $|z+i|=\frac{1}{2}$

Circle of radius $\frac{1}{2}$ centred on (0, –1).

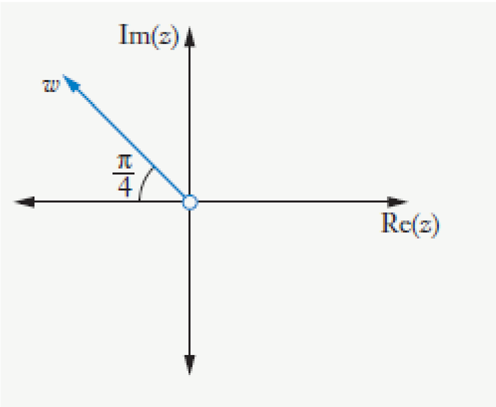


Question 3

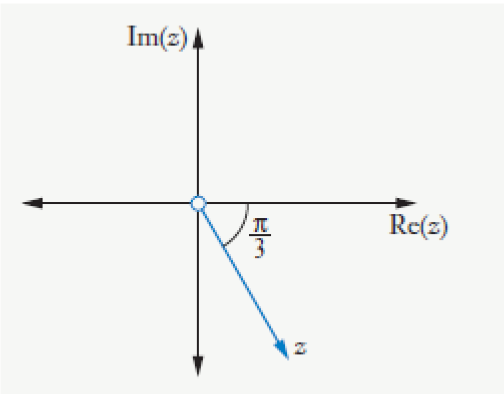
a $\arg(z) = \frac{\pi}{6}$



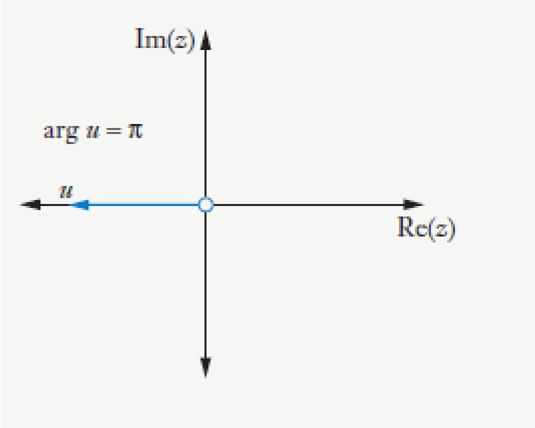
b $\arg(w) = \frac{3\pi}{4}$



c $\arg(z) = -\frac{\pi}{3}$

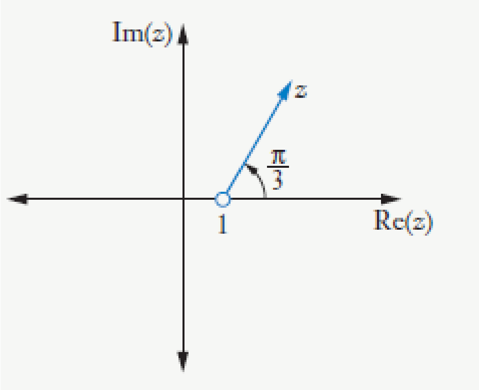


d $\arg(u) = \pi$

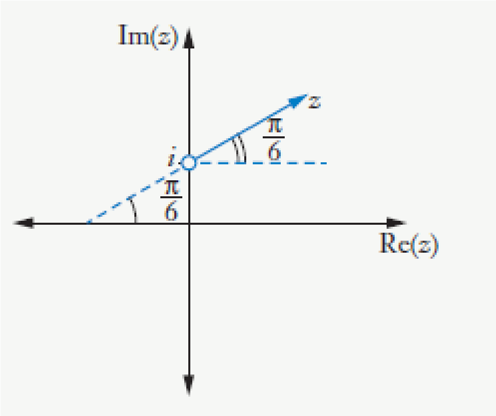


Question 4

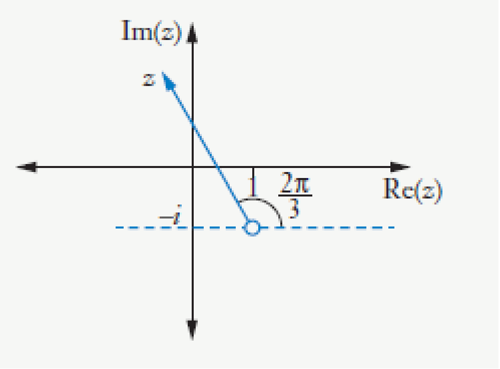
a $\arg(z-1) = \frac{\pi}{3}$



b $\arg(z-i) = \frac{\pi}{6}$



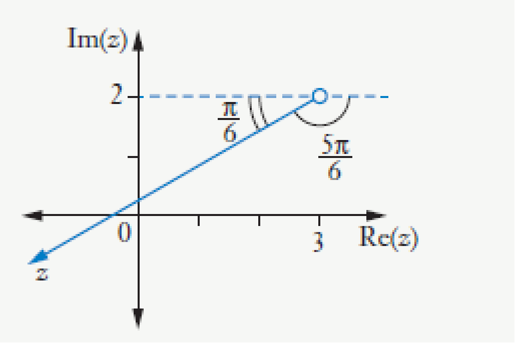
c $\arg\left(z-(1-i)\right)=\frac{2\pi}{3}$



d

$$\arg\left(z-3-2i\right)=-\frac{5\pi}{6}$$

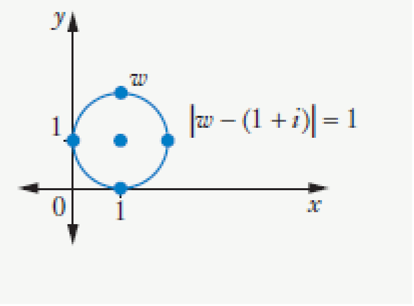
$$\arg\left(z-(3+2i)\right)=-\frac{5\pi}{6}$$



Question 5

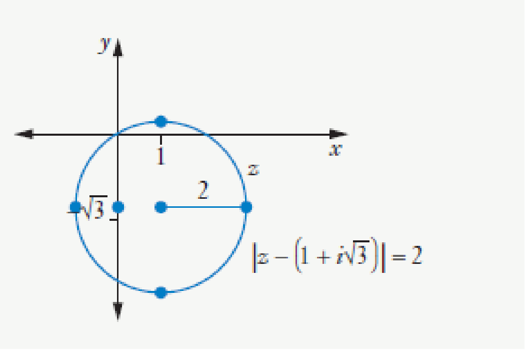
a $\left|w-(1+i)\right|=1$

Circle of radius 1 centred on (1, 1).



b $\left|z-\left(1-i\sqrt{3}\right)\right|=2$

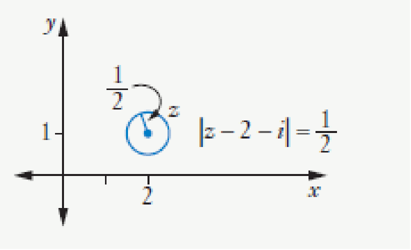
Circle of radius 2 centred on $\left(1,-\sqrt{3}\right)$.



c

$|z-2-i|=\frac{1}{2}\Rightarrow|z-(2+i)|=\frac{1}{2}$

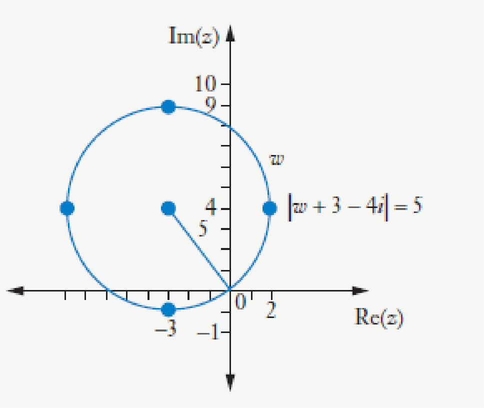
Circle of radius $\frac{1}{2}$ centred on (2, 1).



d

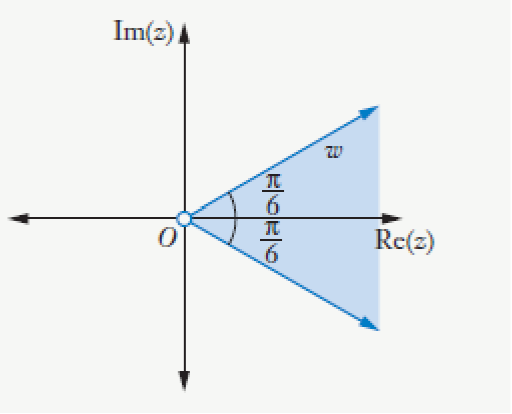
$|w+3-4i|=5\Rightarrow|w-(-3+4i)|=5$

Circle of radius 5 centred on (–3, 4).

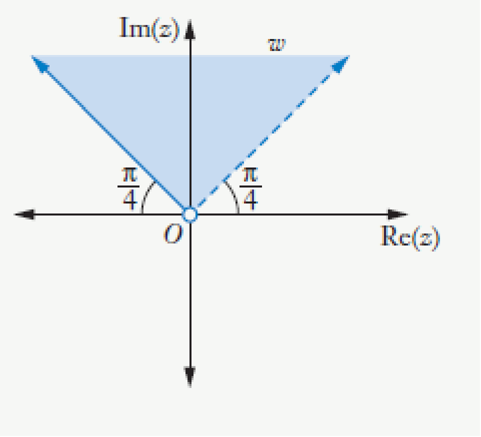


Question 6

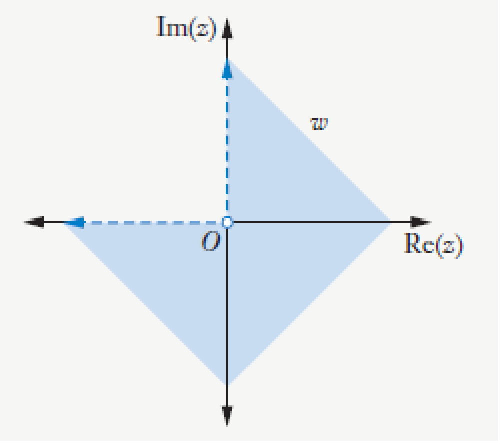
a $-\frac{\pi}{6} \le \arg w \le \frac{\pi}{6}$



b $\frac{\pi}{4} < \arg w < \frac{3\pi}{4}$



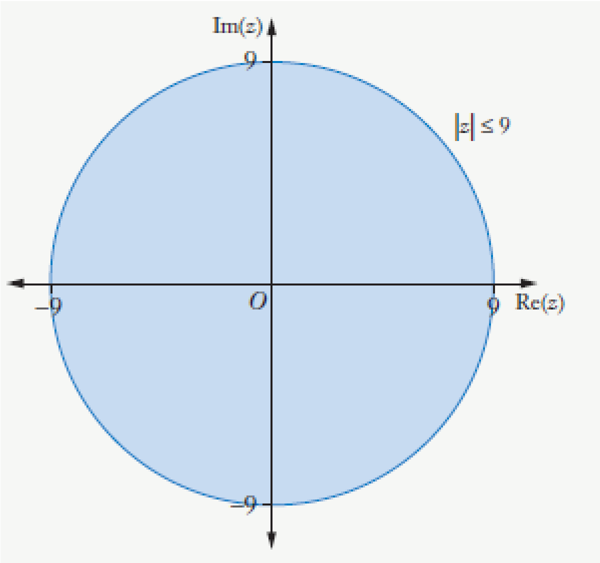
c $-\pi < \arg w < \frac{\pi}{2}$



Question 7

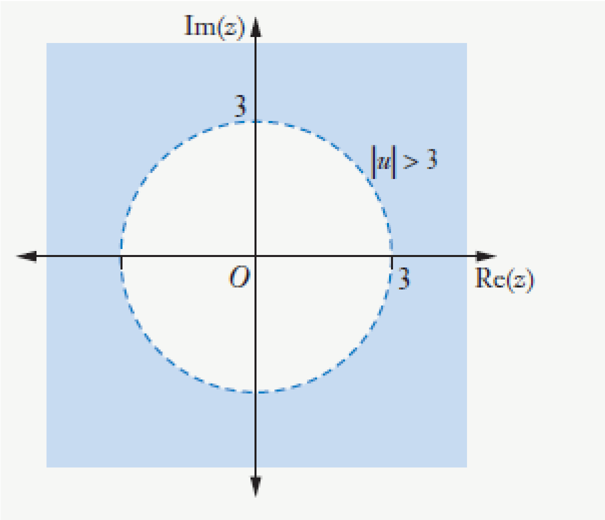
a $|z| \le 9$

Solid circle of radius 9 (including the boundary) centred on the origin.



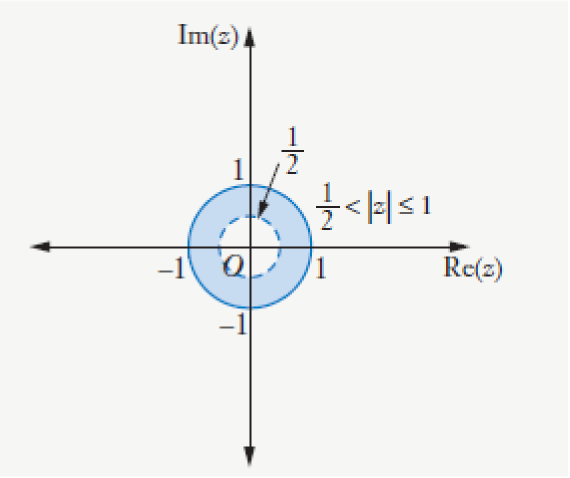
b $|u| > 3$

Outside of a circle of radius 3 (excluding the boundary) centred on the origin



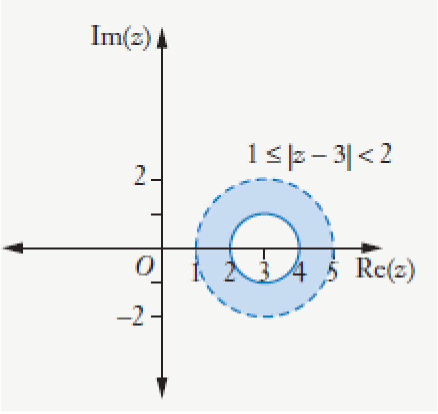
c $\frac{1}{2} < |z| \le 1$

A torus with inner radius $\frac{1}{2}$ (excluding the boundary) and outer radius 1 (including the boundary) centred on the origin.



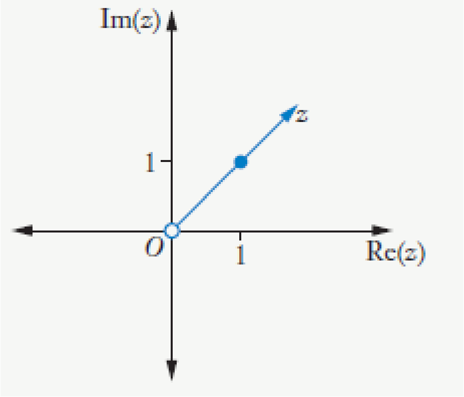
d $1 \le |z-3| < 2$

A torus with inner radius 1 and outer radius 2 (including the inner boundary excluding the outer boundary) centred on (3, 0).

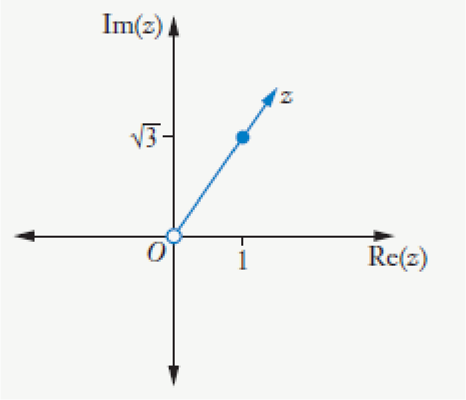


Question 8

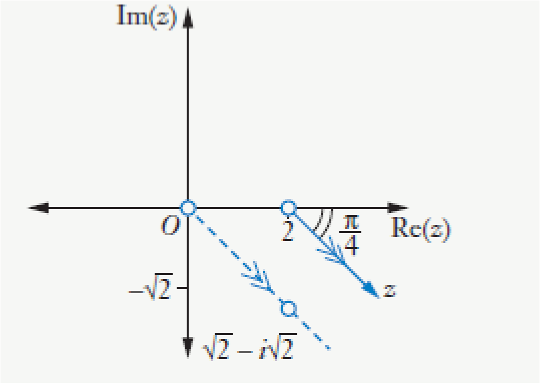
a $\arg(z) = \arg(1 + i)$



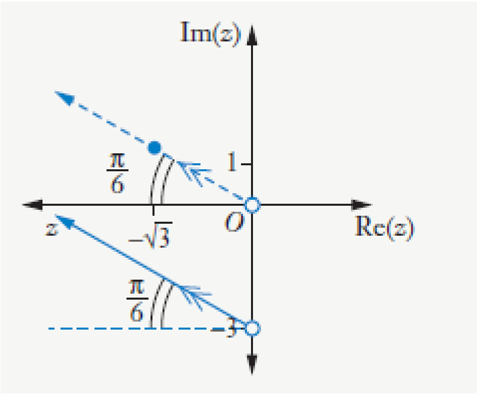
b $\arg(z) = \arg\left(1 + i\sqrt{3}\right)$



c $\arg(z-2) = \arg\left(\sqrt{2} - i\sqrt{2}\right)$



d $\arg(z+3i)=\arg\left(-\sqrt{3}+i\right)$

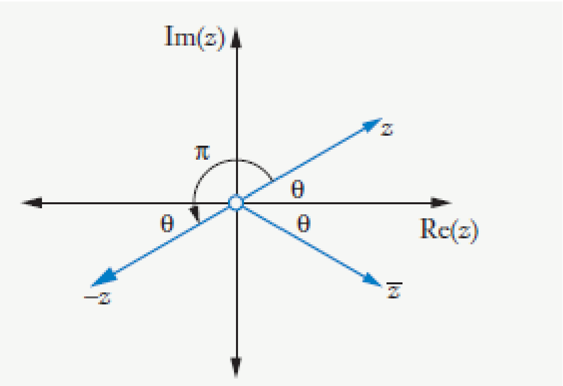


Question 9

$\arg(z)=\theta$

$\arg(-z)=\theta+180°$

$-\arg(z)=-\theta$

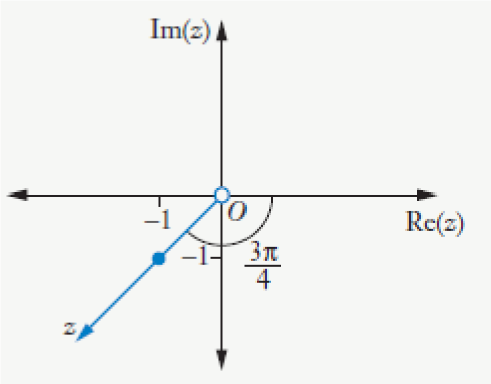


Arg $(-z)$ = arg $(-1 \times z)$ = arg (-1) + arg $z = \pi + \theta$

Question 10

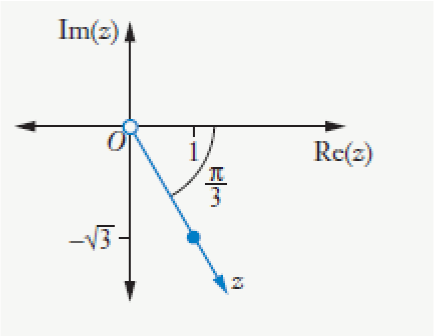
a

$$\arg(z) - \arg(-1-i) = 0$$
$$\arg(z) = \arg(-1-i)$$



b

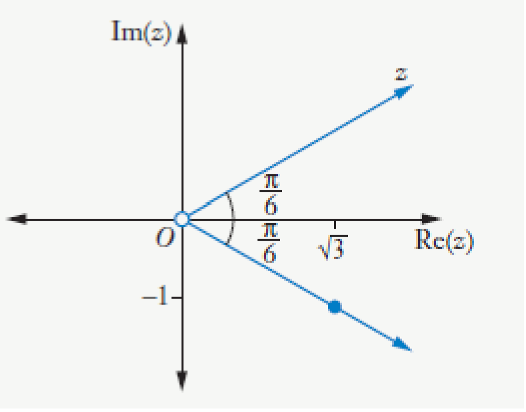
$$\arg(z) - \arg\left(1-i\sqrt{3}\right) = 0$$
$$\arg(z) = \arg\left(1-i\sqrt{3}\right)$$



c

$$\arg(z)+\arg\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)=0$$

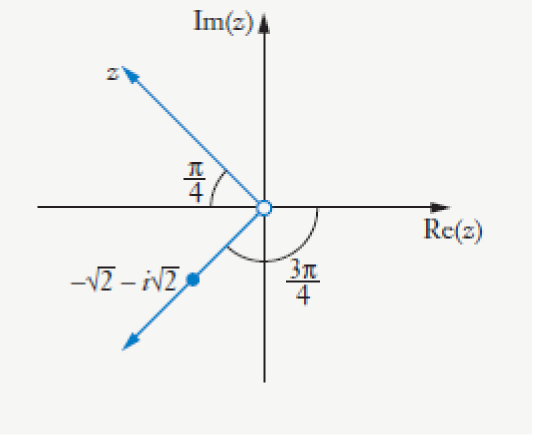
$$\arg(z)=-\arg\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)$$



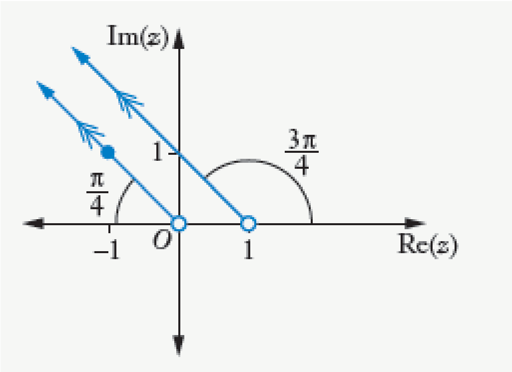
d

$$\arg(z)+\arg\left(-\sqrt{2}-i\sqrt{2}\right)=0$$

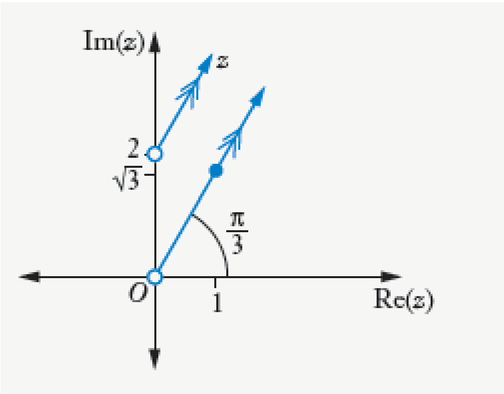
$$\arg(z)=-\arg\left(-\sqrt{2}-i\sqrt{2}\right)$$



e $\arg(z-1)=\arg(-1+i)$

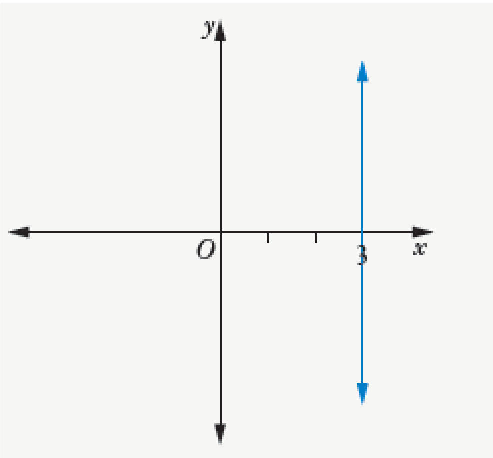


f $\arg(z-2i)=\arg\left(1+i\sqrt{3}\right)$

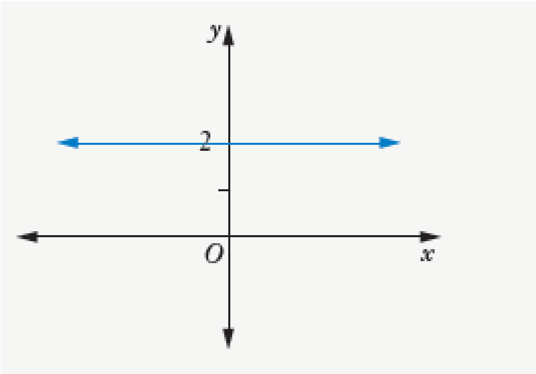


Question 11

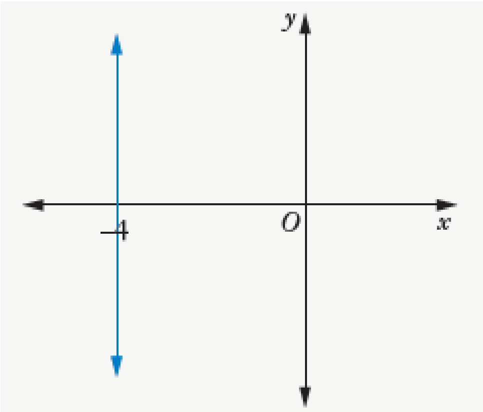
a $\text{Re}(z)=3$



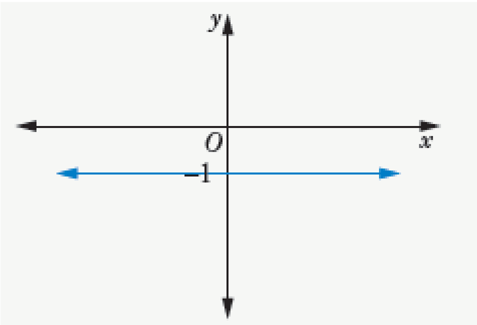
b $\text{Im}(z) = 2$



c $\text{Re}(z) = -4$



d $\text{Im}(z) = -1$

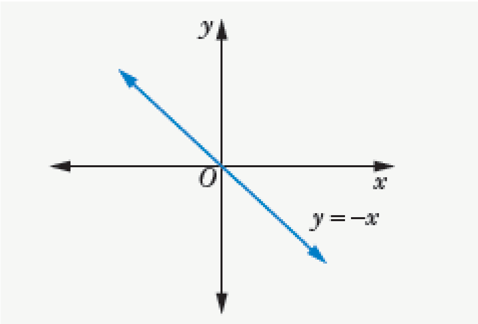


Question 12

a

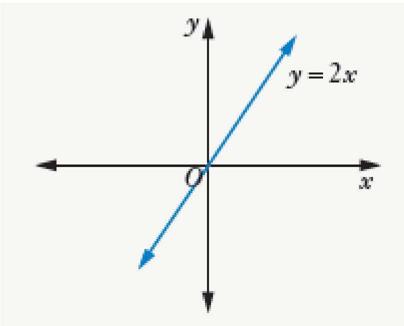
$$\text{Re}(z) + \text{Im}(z) = 0$$
$$\text{Re}(z) = -\text{Im}(z)$$

$x + y = 0$



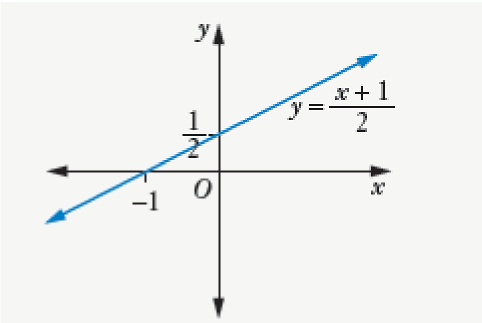
b

$$\text{Im}(z) = 2\,\text{Re}(z)$$
$$y = 2x$$



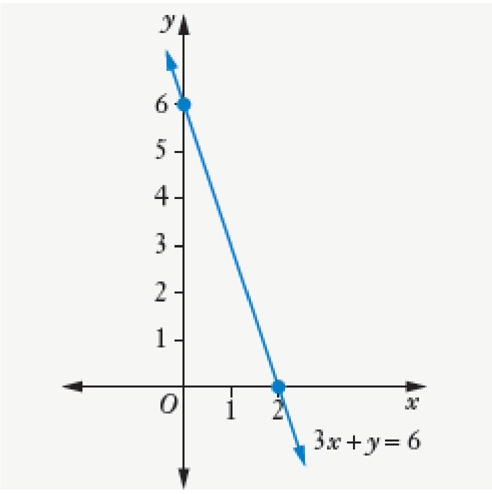
c

$$\text{Re}(z) = 2\,\text{Im}(z) - 1$$
$$x = 2y - 1$$
$$2y = x + 1$$
$$y = \frac{x+1}{2}$$



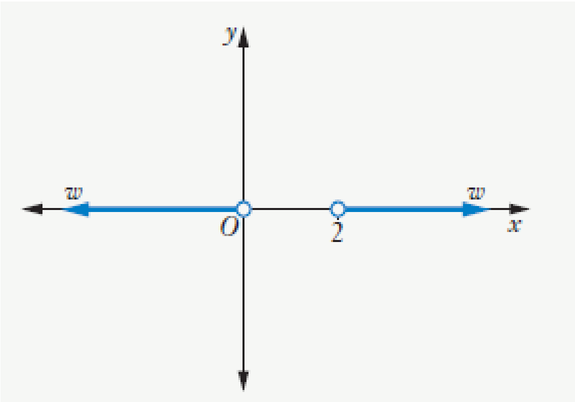
d

$$\text{Im}(z) + 3\,\text{Re}(z) = 6$$
$$y + 3x = 6$$
$$y = -3x + 6$$

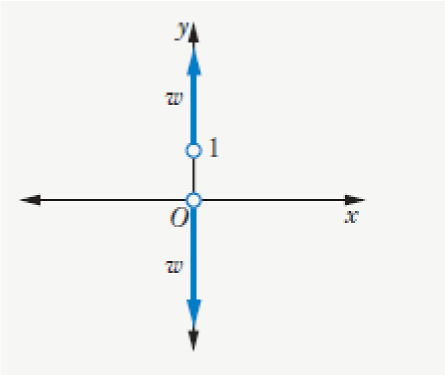


Question 13

a $\arg(w) = \arg(w-2)$

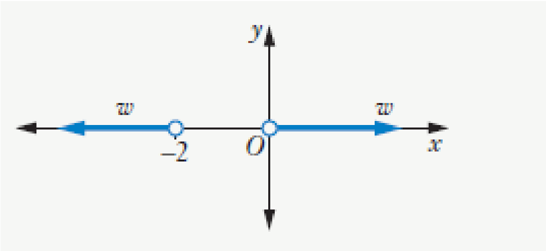


b $\arg(w) = \arg(w-i)$



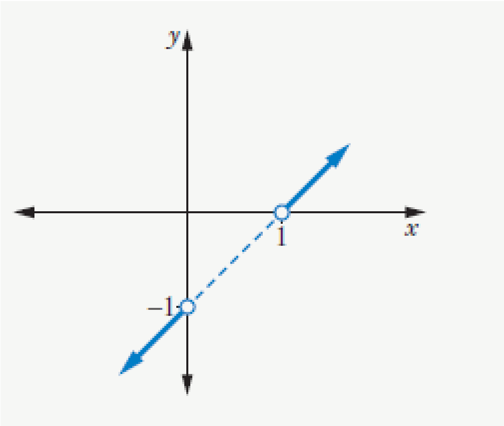
c

$$\arg(w+2) - \arg(w) = 0$$
$$\arg(w) = \arg(w+2)$$



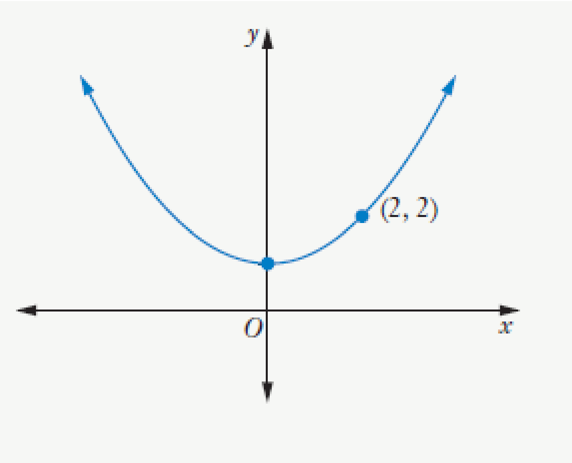
d

$$\arg(w+i)-\arg(w-1)=0$$
$$\arg(w+i)=\arg(w-1)$$



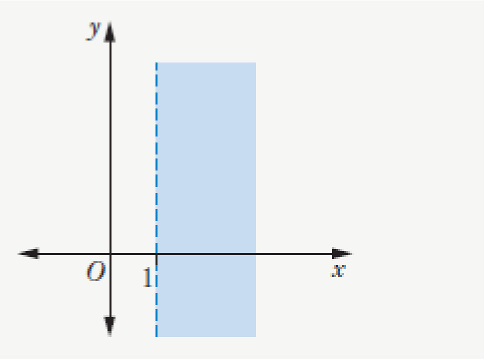
Question 14

$$\text{Im}(z)=|z-2i|$$

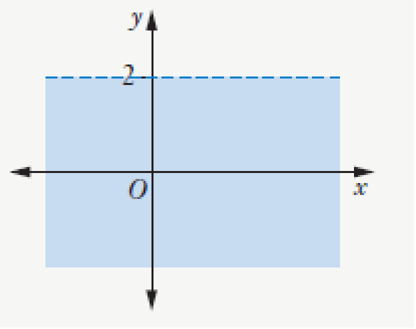


Question 15

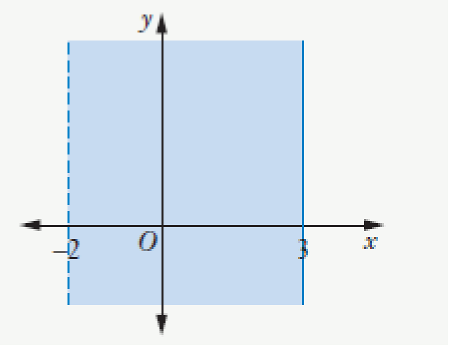
a $\text{Re}(z) > 1$



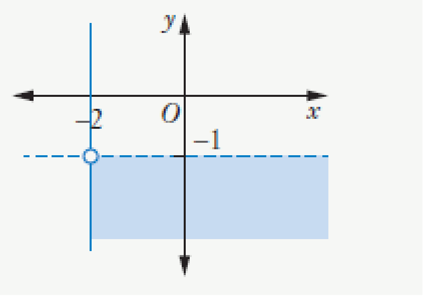
b $\text{Im}(z) < 2$



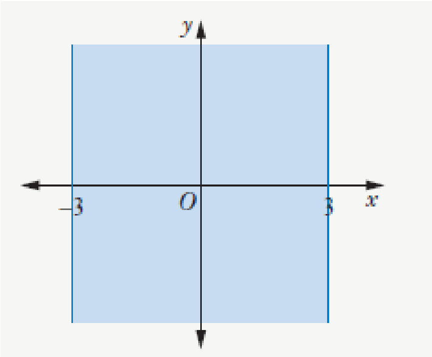
c $-2 < \text{Re}(z) \le 3$



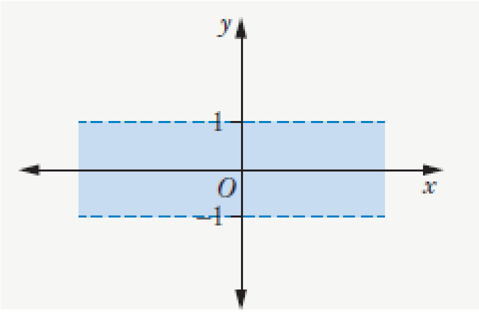
d $\text{Im}(z) < -1$ and $\text{Re}(z) \ge -2$



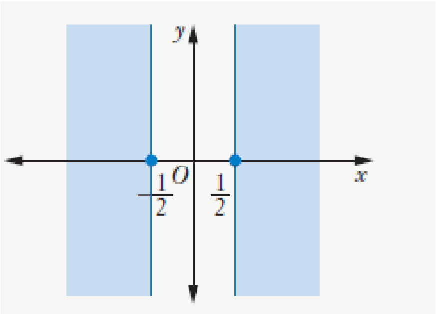
e $|\text{Re}(z)| \le 3$



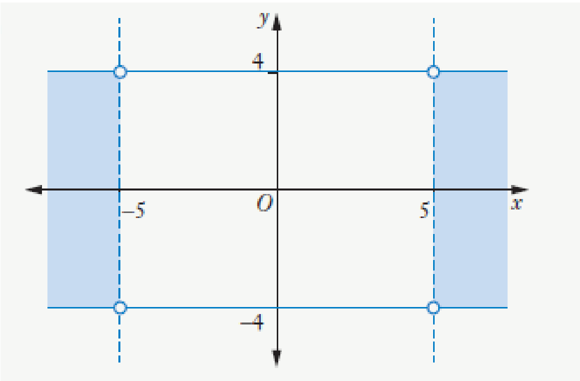
f $|\text{Im}(z)| < 1$



g $|\text{Re}(z)| \ge \frac{1}{2}$

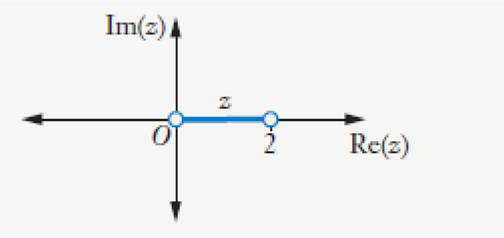


h $|\text{Re}(z)| > 5$ and $|\text{Im}(z)| \le 4$

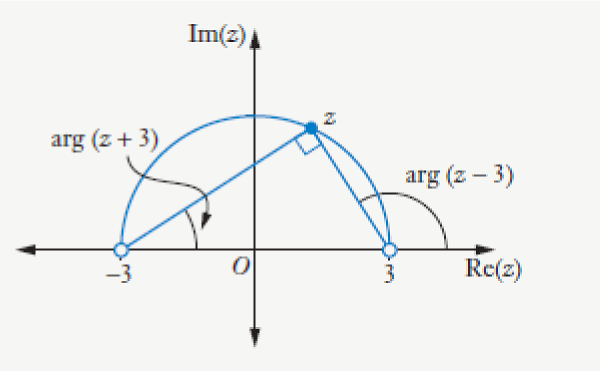


Question 16

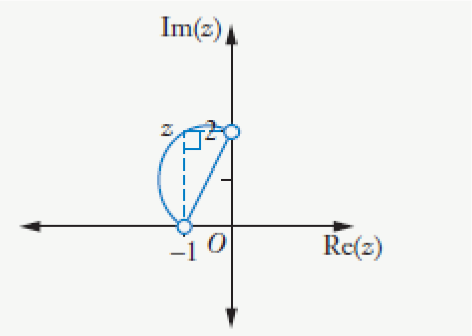
a $\arg(z-2)-\arg(z)=\pm\pi$



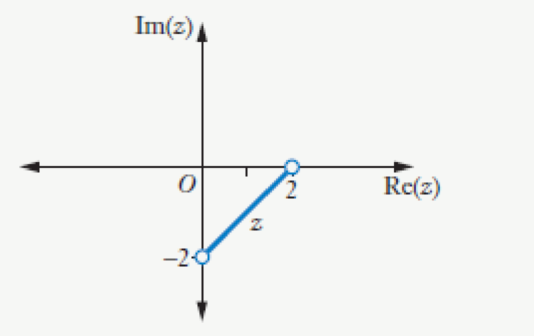
b $\arg(z-3)-\arg(z+3)=\frac{\pi}{2}$



c $\arg(z-2i)-\arg(z+i)=\frac{\pi}{2}$



d $\arg(z+2i)-\arg(z-2)=\pm\pi$



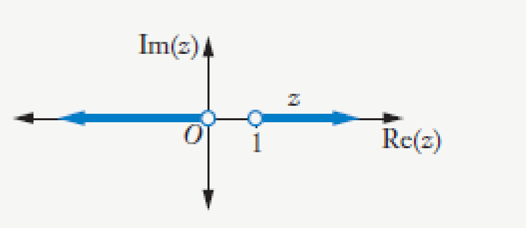
Question 17

a

$$\arg\left(\frac{z}{z-1}\right)=0$$

$$\arg(z)-\arg(z-1)=0$$

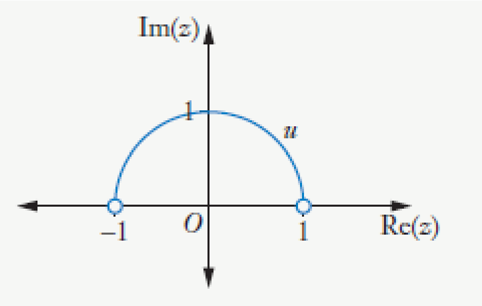
$$\arg(z)=\arg(z-1)$$



b

$$\arg\left(\frac{u-1}{u+1}\right)=\frac{\pi}{2}$$

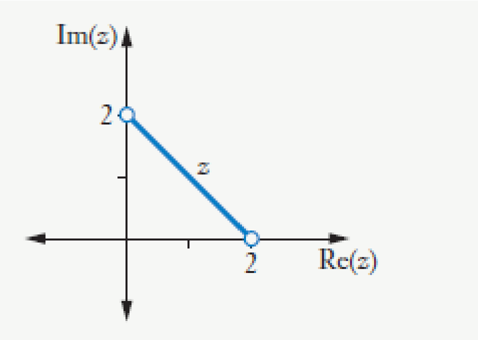
$$\arg(u-1)-\arg(u+1)=\frac{\pi}{2}$$



c

$$\arg\left(\frac{z-2}{z-2i}\right)=\pm\pi$$

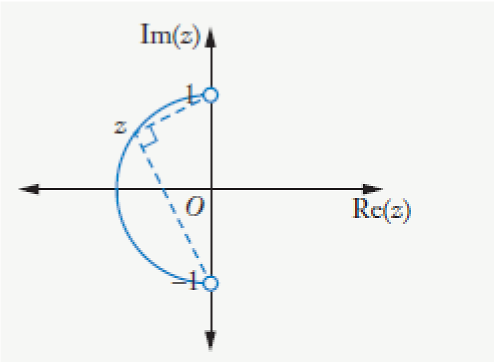
$$\arg(z-2)-\arg(z-2i)=\pm\pi$$



d

$$\arg\left(\frac{z-i}{z+i}\right)=\frac{\pi}{2}$$

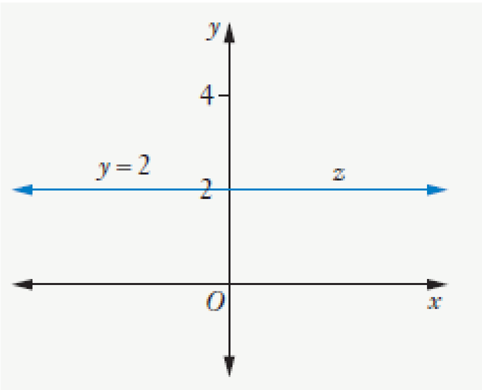
$$\arg(z-i)-\arg(z+i)=\frac{\pi}{2}$$



Question 18

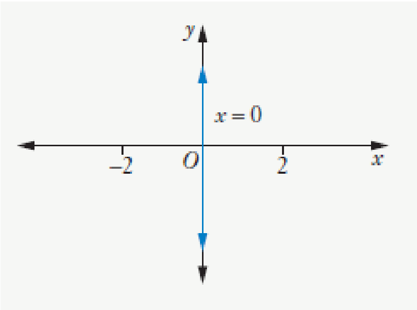
a $|z|=|z-4i|$

This is a perpendicular line bisecting (0, 0) and (0, 4*i*).



b $|z-2|=|z+2|$

This is a perpendicular line bisecting (–2, 0) and (2, 0).

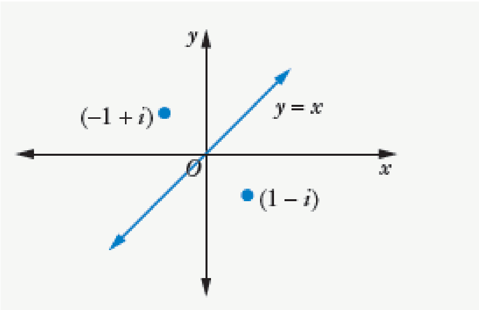


c

$|z-1+i|=|z+1-i|$

$|z-(1-i)|=|z-(-1+i)|$

This is a perpendicular line bisecting (1, –*i*) and (–1, *i*).

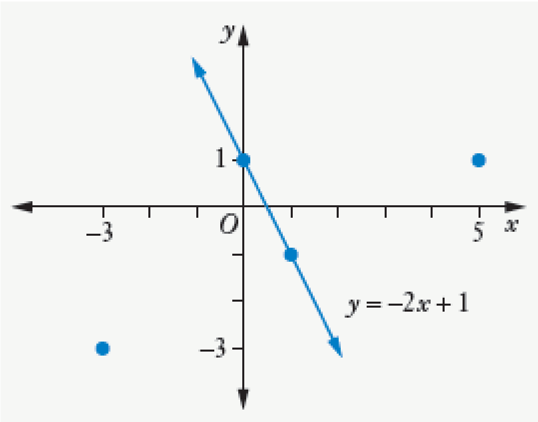


d

$|z-5-i|=|z+3+3i|$

$|z-(5+i)|=|z-(-3-3i)|$

This is a perpendicular line bisecting (5, *i*) and (–3, –3*i*).



Question 19

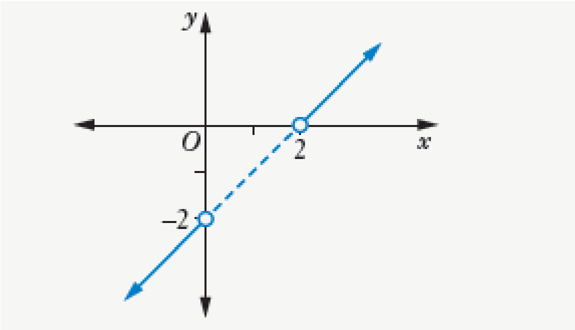
a

$$\arg\left(\frac{w-2}{w+2i}\right)=0$$

$$\arg(w-2)-\arg(w+2i)=0$$

$$\arg(w-2)=\arg(w+2i)$$

$y = x + 2$ for $x > 2$ or $x < 0$.

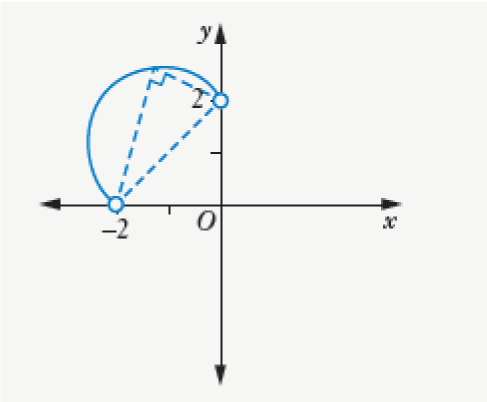


b

$$\arg\left(\frac{u-2i}{u+2}\right)=\frac{\pi}{2}$$

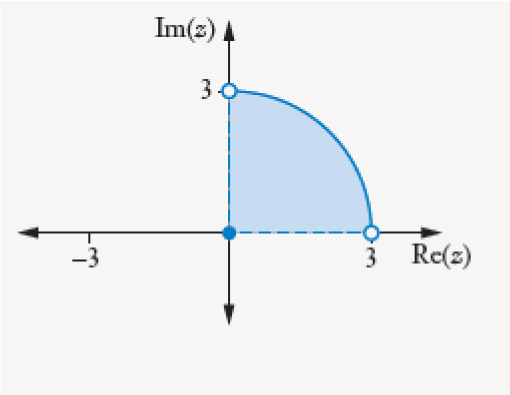
$$\arg(u-2i)-\arg(u+2)=\frac{\pi}{2}$$

Semicircle, centre (–1, 1), radius $\sqrt{2}$.

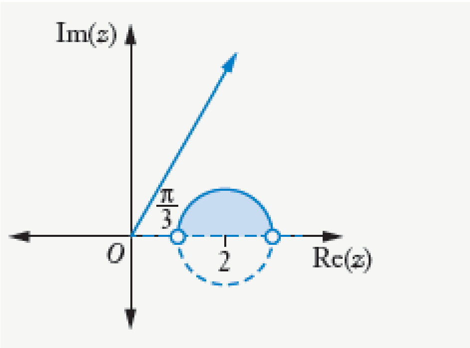


Question 20

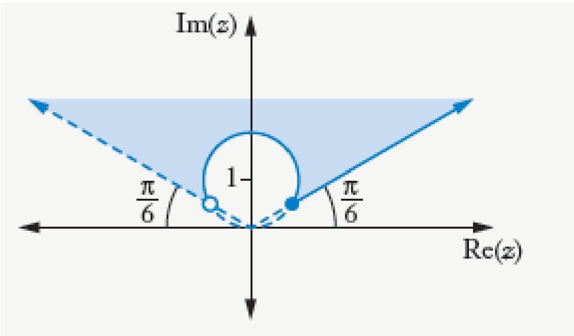
a $|z| \le 3$ and $0 < \arg(z) < \frac{\pi}{2}$



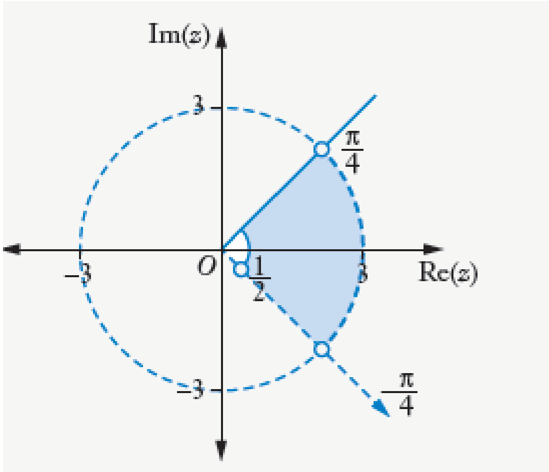
b $|z-2| \le 1$ and $0 < \arg(z) < \frac{\pi}{3}$



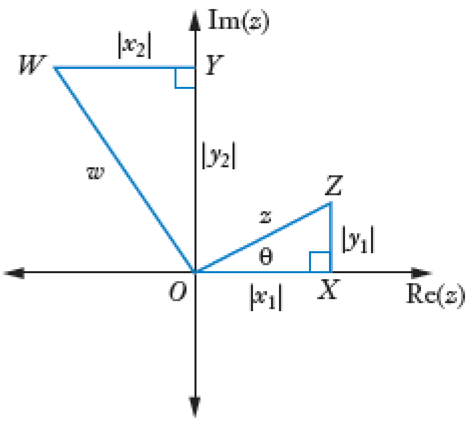
c $|z-i| \ge 1$ and $\frac{\pi}{6} < \arg(z) < \frac{5\pi}{6}$



d $\quad \frac{1}{2} \le |z| < 3$ and $-\frac{\pi}{4} < \arg(z) < \frac{\pi}{4}$



Question 21



As ΔOXZ and ΔOWY are similar

$\frac{|x|}{|y_2|} = \frac{|z|}{|w|}$

$|z||y_2| = |x_1||w|$

As ΔOXZ and ΔOWY are similar

$\angle WOY = \theta$

$\angle YOZ = 90° - \theta$

$\angle ZOW = \angle YOZ + \angle WOY = 90° - \theta + \theta = 90°$

$w|x_1| = iz|y_2|$

Question 22

a

$|p-q| = \text{length } PQ$

$|r-q| = \text{length } QR$

$\triangle PQR$ is isosceles

$|PQ| = |QR|$

$|p-q| = |r-q|$

b

$\arg\left(\dfrac{p-q}{r-q}\right) = \dfrac{\pi}{2}$

$\arg\left(\dfrac{p-q}{r-q}\right) = \arg(p-q) - \arg(r-q)$

As $\triangle PQR$ is a right-angled triangle

$\arg(p-q) - \arg(r-q) = \dfrac{\pi}{2}$

c follows directly from **b**

d Using **c**

$p-q = i(r-q)$

$(p-q)^2 = (i(r-q))^2$

$(p-q)^2 = -(r-q)^2$

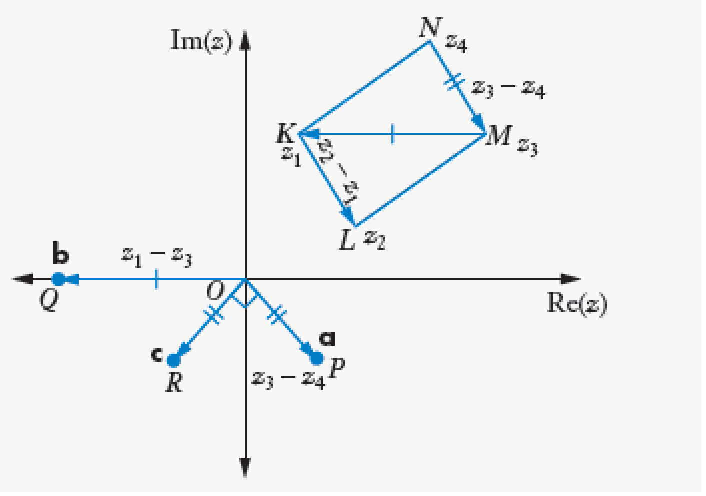
$(p-q)^2 + (r-q)^2 = 0$

e Using Pythagoras' theorem

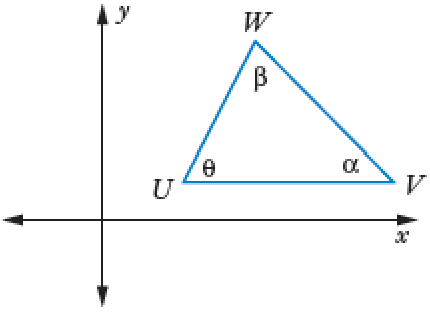
$(PQ)^2 + (QR)^2 = (PR)^2$

$(p-q)^2 + (r-q)^2 = (r-p)^2$

Question 23



Question 24



a

$\arg(w-u)$

$=$ angle UW makes with the x-axis

As UV is parallel to the x-axis

$\angle$VUW is corresponding and equal

$\therefore \arg(w-u) = \theta$

b

$\arg(w-v)$

$=$ angle VW makes with the x-axis measured counter clockwise

$\therefore \arg(w-v)=\pi-\alpha$

c

$\arg\left(\dfrac{w-v}{w-u}\right)$

$=\arg(w-v)-\arg(w-u)$

$=(\pi-\alpha)-\theta$

$=\pi-(\alpha+\theta)$

From ΔUVW

$=\beta$

Question 25

$a=5\operatorname{cis}\dfrac{\pi}{6}$

$b=5\operatorname{cis}\left(-\dfrac{\pi}{6}\right)=\overline{a}$

$c=5\operatorname{cis}\dfrac{5\pi}{12}$

$=5\operatorname{cis}\left(\dfrac{\pi}{6}+\dfrac{\pi}{4}\right)$

$=5\operatorname{cis}\dfrac{\pi}{6}\operatorname{cis}\dfrac{\pi}{4}$

$=a\operatorname{cis}\dfrac{\pi}{4}$

$d=2\times(-1)c$

$=-2c$

$=-2a\operatorname{cis}\dfrac{\pi}{4}$

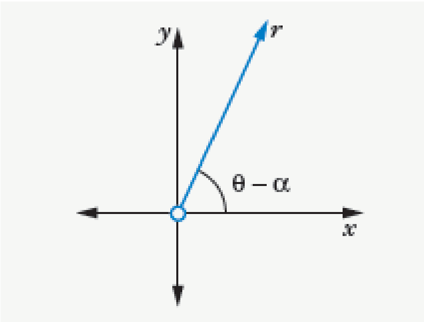
Question 26

$|p| = |q| = 1$
$\arg(p) = \theta$
$\arg(q) = \alpha$

a

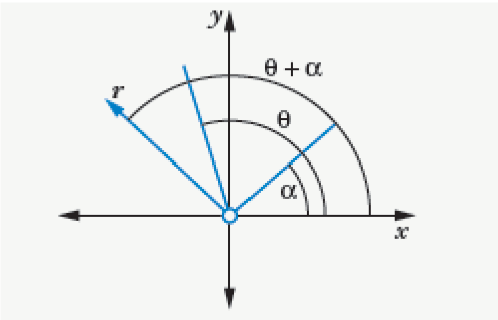
$\arg(r) = \theta - \alpha$

$r = k\dfrac{p}{q}$



b

$\arg(r) = \theta + \alpha$
$r = kpq$



Question 27

a $\alpha = \frac{2\pi}{5}$

b

$$z_1 = \text{cis}\frac{2\pi}{5}$$
$$z_2 = \text{cis}\frac{4\pi}{5}$$
$$z_3 = \text{cis}\left(-\frac{4\pi}{5}\right)$$
$$z_4 = \text{cis}\left(-\frac{2\pi}{5}\right)$$
$$z_5 = 1$$

c

$$(z_1)^2 = \left(\text{cis}\frac{2\pi}{5}\right)^2$$
$$= \text{cis}\frac{4\pi}{5}$$
$$= z_2$$

d

$$(z_2)^2 = \left(\text{cis}\frac{4\pi}{5}\right)^2$$
$$= \text{cis}\frac{8\pi}{5}$$
$$= \text{cis}\left(-\frac{2\pi}{5}\right)$$
$$= z_4$$

e

$$(z_3)^2 = \left(\text{cis}\left[-\frac{4\pi}{5}\right]\right)^2$$
$$= \text{cis}\left(-\frac{8\pi}{5}\right)$$
$$= \text{cis}\frac{2\pi}{5}$$
$$= z_1$$

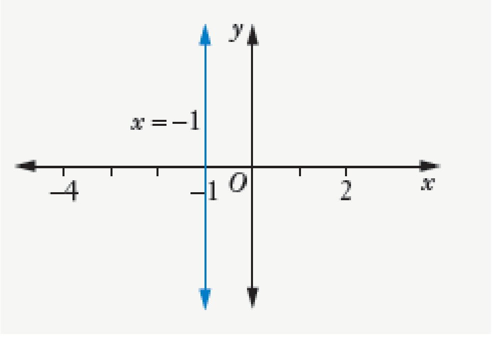
f

$$(z_4)^2 = \left(\text{cis}\left[-\frac{2\pi}{5}\right]\right)^2$$
$$= \text{cis}\left(-\frac{4\pi}{5}\right)$$
$$= z_3$$
$$(z_1)^2 + (z_2)^2 + (z_3)^2 + (z_4)^2 + 1$$
$$= z_2 + z_4 + z_1 + z_3 + 1$$
$$= 0$$

Question 28

a

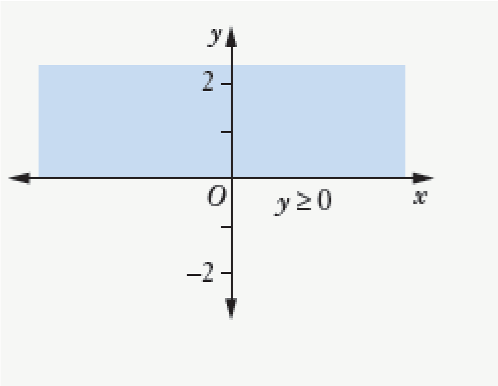
$$\frac{|z+4|}{|z-2|} = 1$$
$$|z+4| = |z-2|$$



b

$$\frac{|z-2i|}{|z+2i|} \le 1$$

$$|z-2i| \le |z+2i|$$



c

$$\frac{|z-3|}{|2z|} = 1$$

$$|z-3| = |2z|$$

$$|z-3| = 2|z|$$

for $z = x + iy$

$$|x-3+iy| = 2|x+iy|$$

$$\sqrt{(x-3)^2 + y^2} = 2\sqrt{x^2+y^2}$$

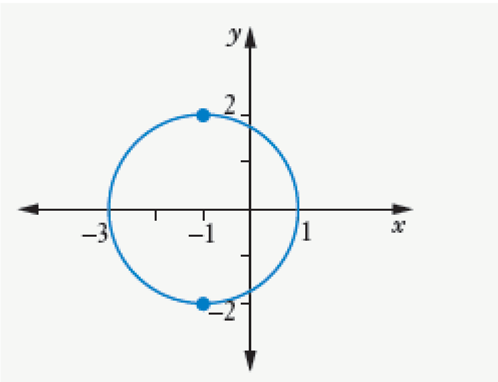
$$(x-3)^2 + y^2 = 4(x^2+y^2)$$

$$3x^2 + 6x - 9 + 3y^2 = 0$$

$$x^2 + 2x - 3 + y^2 = 0$$

$$x^2 + 2x + 1 + y^2 = 4$$

$$(x+1)^2 + y^2 = 4$$



Question 29

a

$$|z+3| = 2|z-1|$$

for $z = x + iy$

$$|x+3+iy| = 2|x-1+iy|$$

$$\sqrt{(x+3)^2 + y^2} = 2\sqrt{(x-1)^2 + y^2}$$

$$(x+3)^2 + y^2 = 4\left(x^2 - 2x + 1 + y^2\right)$$

$$3x^2 - 14x - 5 + 3y^2 = 0$$

$$x^2 - \frac{14}{3}x + \frac{49}{9} + y^2 = \frac{64}{9}$$

$$\left(x - \frac{7}{3}\right)^2 + y^2 = \frac{64}{9}$$

Which is a circle of the form $(x - a)^2 + (y - b)^2 = r^2$.

b from a, centre $\left(\frac{7}{3}, 0\right)$, radius $\frac{8}{3}$.

Question 30

From $|z - 1| = 1$, for $z = x + iy$,

$|x - 1 + iy| = 1$

$$\sqrt{(x-1)^2 + y^2} = 1$$
$$(x-1)^2 + y^2 = 1$$
$$y^2 = 1-(x-1)^2$$
$$y^2 = 1-(x^2-2x+1)$$
$$y^2 = 2x - x^2$$
$$y = \sqrt{2x-x^2}$$

$\arg(z + 1) = \arg(x + 1 + iy)$

$$\tan^{-1}\left(\frac{y}{x+1}\right) = \tan^{-1}\left(\frac{\sqrt{2x-x^2}}{x+1}\right)$$

Using calculus, maximum argument of $\tan^{-1}$ is $x = \dfrac{1}{2}$.

$$\tan^{-1}\left(\frac{\sqrt{2x-x^2}}{x+1}\right) = \tan^{-1}\left(\frac{\sqrt{2\left(\frac{1}{2}\right)-\left(\frac{1}{2}\right)^2}}{\frac{1}{2}+1}\right)$$
$$= \tan^{-1}\left(\frac{\sqrt{1-\frac{1}{4}}}{\frac{3}{2}}\right) = \tan^{-1}\left(\frac{\sqrt{\frac{3}{4}}}{\frac{3}{2}}\right)$$
$$= \tan^{-1}\left(\frac{\sqrt{3}}{2}\times\frac{2}{3}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$$
$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{\pi}{6}$$

Question 1

a

$$\left(\sqrt{3}\operatorname{cis}\delta\right)^4$$
$$=\left(\sqrt{3}\right)^4\operatorname{cis}\left(4\times\delta\right)$$
$$=9\operatorname{cis}4\delta$$

b

$$\left(\operatorname{cis}\left[-\frac{5\pi}{6}\right]\right)^7$$
$$=\operatorname{cis}\left(7\times-\frac{5\pi}{6}\right)$$
$$=\operatorname{cis}\left(-\frac{35\pi}{6}\right)$$
$$=\operatorname{cis}\left(\frac{\pi}{6}\right)$$

c

$$\left(9\operatorname{cis}72°\right)^{\frac{1}{2}}$$
$$=9^{\frac{1}{2}}\operatorname{cis}\left(\frac{1}{2}\times72°\right)$$
$$=3\operatorname{cis}36°$$

Question 2

$\left(\text{cis}\,\theta\right)^5 = \text{cis}\,5\theta$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

a Equating Imaginary parts

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$= 5\sin\theta\left(1-\sin^2\theta\right)\left(1-\sin^2\theta\right) - 10\left(1-\sin^2\theta\right)\sin^3\theta + \sin^5\theta$$
$$= 5\sin\theta\left(1-2\sin^2\theta+\sin^4\theta\right) - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$
$$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$
$$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$$

b Equating real parts

$$\left(cis\theta\right)^5 = cis5\theta$$
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$= \cos^5\theta - 10\cos^3\theta\left(1-\cos^2\theta\right) + 5\cos\theta\left(1-\cos^2\theta\right)\left(1-\cos^2\theta\right)$$
$$= \cos^5\theta - 10\cos^3\theta\left(1-\cos^2\theta\right) + 5\cos\theta\left(1-2\cos^2\theta+\cos^4\theta\right)$$
$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$$
$$= 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta$$

Question 3

a

$$
\begin{aligned}
\tan 5\theta &= \frac{\sin 5\theta}{\cos 5\theta} \\
&= \frac{5\sin\theta - 20\sin^3\theta + 16\sin^5\theta}{5\cos\theta - 20\cos^3\theta + 16\cos^5\theta} \\
&= \frac{\left(\dfrac{5\sin\theta - 20\sin^3\theta + 16\sin^5\theta}{\cos^5\theta}\right)}{\left(\dfrac{5\cos\theta - 20\cos^3\theta + 16\cos^5\theta}{\cos^5\theta}\right)} \\
&= \frac{5\tan\theta\sec^4\theta - 20\tan^3\theta\sec^2\theta + 16\tan^5\theta}{5\sec^4\theta - 20\sec^2\theta + 16} \\
&= \frac{5\tan\theta\left(1+\tan^2\theta\right)\left(1+\tan^2\theta\right) - 20\tan^3\theta\left(1+\tan^2\theta\right) + 16\tan^5\theta}{5\left(1+\tan^2\theta\right)\left(1+\tan^2\theta\right) - 20\left(1+\tan^2\theta\right) + 16} \\
&= \frac{5\tan\theta\left(1+2\tan^2\theta+\tan^4\theta\right) - 20\tan^3\theta\left(1+\tan^2\theta\right) + 16\tan^5\theta}{5\left(1+2\tan^2\theta+\tan^4\theta\right) - 20\left(1+\tan^2\theta\right) + 16} \\
&= \frac{5\tan\theta + 10\tan^3\theta + 5\tan^5\theta - 20\tan^3\theta - 20\tan^5\theta + 16\tan^5\theta}{5 + 10\tan^2\theta + 5\tan^4\theta - 20 - 20\tan^2\theta + 16} \\
&= \frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{1 - 10\tan^2\theta + 5\tan^4\theta} \\
&= \frac{\tan\theta\left(\tan^4\theta - 10\tan^2\theta + 5\right)}{1 - 10\tan^2\theta + 5\tan^4\theta}
\end{aligned}
$$

b

$$\tan 5\theta = \frac{\tan\theta\left(\tan^4\theta - 10\tan^2\theta + 5\right)}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

$\tan 5\theta = 0$ when $\tan\theta\left(\tan^4\theta - 10\tan^2\theta + 5\right) = 0$

then $5\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$\theta = 0, \pm\frac{\pi}{5}, \pm\frac{2\pi}{5}, \pm\frac{3\pi}{5}, \ldots$

For $\tan\theta\left(\tan^4\theta - 10\tan^2\theta + 5\right) = 0$

$\tan\theta = 0$ or $\tan^4\theta - 10\tan^2\theta + 5 = 0$

For $\tan\theta = 0$, $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

So other solutions to θ above must yield 4 unique solutions to $x^4 - 10x^2 + 5 = 0$ where $x = \tan\theta$

Test for $\tan\left(\pm\frac{\pi}{5}\right), \tan\left(\pm\frac{2\pi}{5}\right), \tan\left(\pm\frac{3\pi}{5}\right), \tan\left(\pm\frac{4\pi}{5}\right)$ etc

So unique solutions are:

$x = \tan\left(\pm\frac{\pi}{5}\right), \tan\left(\pm\frac{2\pi}{5}\right)$

$= \mp\tan\left(\frac{\pi}{5}\right), \mp\tan\left(\frac{2\pi}{5}\right)$ tan is an odd function

$= \pm\tan\left(\frac{\pi}{5}\right), \pm\tan\left(\frac{2\pi}{5}\right)$

Question 4

$$z-\frac{1}{z}=2i\sin\theta$$

$$\left(z-\frac{1}{z}\right)^7=\left(2i\sin\theta\right)^7$$

$$=-128i\sin^7\theta$$

$$=z^7-\binom{7}{1}z^6z^{-1}+\binom{7}{2}z^5z^{-2}-\binom{7}{3}z^4z^{-3}+\binom{7}{4}z^3z^{-4}-\binom{7}{5}z^2z^{-5}+\binom{7}{6}z^1z^{-6}-\binom{7}{7}z^{-7}$$

$$=z^7-7z^5+21z^3-35z^1+35z^{-1}-21z^{-3}+7z^{-5}-z^{-7}$$

$$=\left(z^7-z^{-7}\right)-7\left(z^5-z^{-5}\right)+21\left(z^3-z^{-3}\right)-35\left(z^1-z^{-1}\right)$$

$$=2i\sin7\theta-7\times2i\sin5\theta+21\times2i\sin3\theta-35\times2i\sin\theta$$

$$=2i\left(\sin7\theta-7\sin5\theta+21\sin3\theta-35\sin\theta\right)$$

$$\therefore -128i\sin^7\theta=2i\left(\sin7\theta-7\sin5\theta+21\sin3\theta-35\sin\theta\right)$$

$$64\sin^7\theta=35\sin\theta-21\sin3\theta+7\sin5\theta-\sin7\theta$$

$$35\sin\theta-64\sin^7\theta=21\sin3\theta-7\sin5\theta+\sin7\theta$$

$$\int\left(35\sin\theta-64\sin^7\theta\right)d\theta$$

$$=\int\left(21\sin3\theta-7\sin5\theta+\sin7\theta\right)d\theta$$

$$-7\cos3\theta+\frac{7}{5}\cos5\theta-\frac{1}{7}\cos7\theta+c$$

Question 5

a

$$z^2+2iz+3=0$$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-2i\pm\sqrt{\left(2i\right)^2-4\times1\times3}}{2}$$

$$=\frac{-2i\pm\sqrt{-4-12}}{2}$$

$$=\frac{-2i\pm\sqrt{-16}}{2}$$

$$=\frac{-2i\pm4i}{2}$$

$$=-3i,i$$

b

$$w^2-(2-3i)w-1-3i=0$$

$$w=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{2-3i\pm\sqrt{(2-3i)^2-4\times1\times(-1-3i)}}{2}$$

$$=\frac{2-3i\pm\sqrt{-5-12i+4+12i}}{2}$$

$$=\frac{2-3i\pm\sqrt{-1}}{2}$$

$$=\frac{2-3i\pm i}{2}$$

$$=1-2i,1-i$$

c

$$ix^2-9=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{0\pm\sqrt{-4\times i\times(-9)}}{2i}$$

$$=\frac{\pm\sqrt{36i}}{2i}$$

$$=\frac{\pm6\sqrt{i}}{2i}$$

$$=\pm3\frac{\sqrt{i}}{i}$$

$$=\pm3i^{-\frac{1}{2}}$$

$$=\pm3\left(\operatorname{cis}\frac{\pi}{2}\right)^{-\frac{1}{2}}$$

$$=\pm3\operatorname{cis}\left(-\frac{\pi}{4}\right)$$

$$=\pm3\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)$$

Question 6

a

$$z^2 = 1 - i\sqrt{3}$$

$$\left|z^2\right| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$$

$$\arg\left(z^2\right) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$$

$$z^2 = 2\text{cis}\left(-\frac{\pi}{3}\right)$$

$$z = \sqrt{2\text{cis}\left(-\frac{\pi}{3}\right)}$$

$$= \sqrt{2}\text{cis}\left(-\frac{\pi}{3} \times \frac{1}{2}\right)$$

$$= \sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right)$$

$$z = \sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right), \sqrt{2}\text{cis}\left(\frac{5\pi}{6}\right)$$

b

$$z^2 = -\sqrt{2} - i\sqrt{2}$$

$$\left|z^2\right| = \sqrt{\left(-\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2} = \sqrt{4} = 2$$

$$\arg\left(z^2\right) = \tan^{-1}\left(\frac{-\sqrt{2}}{-\sqrt{2}}\right) = -\frac{3\pi}{4}$$

$$z^2 = 2\text{cis}\left(-\frac{3\pi}{4}\right)$$

$$z = \sqrt{2\text{cis}\left(-\frac{3\pi}{4}\right)}$$

$$= \sqrt{2}\text{cis}\left(-\frac{3\pi}{4} \times \frac{1}{2}\right)$$

$$= \sqrt{2}\text{cis}\left(-\frac{3\pi}{8}\right)$$

$$z = \sqrt{2}\text{cis}\left(-\frac{3\pi}{8}\right), \sqrt{2}\text{cis}\left(\frac{5\pi}{8}\right)$$

Question 7

$P(z) = z^4 - 4z^3 + 11z^2 - 14z + 12$

a

$$P\left(1-i\sqrt{2}\right) = \left(1-i\sqrt{2}\right)^4 - 4\left(1-i\sqrt{2}\right)^3 + 11\left(1-i\sqrt{2}\right)^2 - 14\left(1-i\sqrt{2}\right) + 12$$

$$z = 1-i\sqrt{2}$$

$$z^2 = \left(1-i\sqrt{2}\right)^2$$

$$= 1 - 2i\sqrt{2} - 2$$

$$= -1 - 2i\sqrt{2}$$

$$z^3 = \left(-1-2i\sqrt{2}\right)\left(1-i\sqrt{2}\right)$$

$$= -1 + i\sqrt{2} - 2i\sqrt{2} - 4$$

$$= -5 - i\sqrt{2}$$

$$z^4 = \left(-5-i\sqrt{2}\right)\left(1-i\sqrt{2}\right)$$

$$= -5 + 5i\sqrt{2} - i\sqrt{2} - 2$$

$$= -7 + 4i\sqrt{2}$$

$$\left(1-i\sqrt{2}\right)^4 - 4\left(1-i\sqrt{2}\right)^3 + 11\left(1-i\sqrt{2}\right)^2 - 14\left(1-i\sqrt{2}\right) + 12$$

$$= -7 + 4i\sqrt{2} - 4\left(-5-i\sqrt{2}\right) + 11\left(-1-2i\sqrt{2}\right) - 14\left(1-i\sqrt{2}\right) + 12$$

$$= -7 + 20 - 11 - 14 + 12 + 4i\sqrt{2} + 4i\sqrt{2} - 22i\sqrt{2} + 14i\sqrt{2}$$

$$= 0$$

$\therefore\ 1-i\sqrt{2}$ is a root

b As $P(z)$ has only real coefficients $1+i\sqrt{2}$ is also a root

$$\left(z-\left(1+i\sqrt{2}\right)\right)\left(z-\left(1-i\sqrt{2}\right)\right)$$

$$= z^2 - z\left(1+i\sqrt{2}\right) - z\left(1-i\sqrt{2}\right) + \left(1+i\sqrt{2}\right)\left(1-i\sqrt{2}\right)$$

$$= z^2 - 2z + 3$$

$$
\begin{array}{r}
z^2-2z+4 \\
z^2-2z+3\overline{\big)z^4-4z^3+11z^2-14z+12} \\
\underline{z^4-2z^3+3z^2} \\
-2z^3+8z^2-14z \\
\underline{-2z^3+4z^2-6z} \\
11z^2-14z+12 \\
\underline{11z^2-14z+12} \\
0
\end{array}
$$

$P(z)=z^4-4z^3+11z^2-14z+12$

$=(z^2-2z+3)(z^2-2z+4)$

$z^2-2z+4=0$

$z=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\dfrac{2\pm\sqrt{(-2)^2-4\times1\times4}}{2}$

$=\dfrac{2\pm\sqrt{-12}}{2}$

$=\dfrac{2\pm2\sqrt{-3}}{2}$

$=1\pm i\sqrt{3}$

Solutions

$x=1\pm i\sqrt{3},1\pm i\sqrt{2}$

c

$P(x)=x^4-4x^3+11x^2-14x+12$

$=(x^2-2x+3)(x^2-2x+4)$

Question 8

$P(z) = 2z^3 + bz^2 + cz + 13$

As all coefficients are real if $\alpha = -2 + 3i$ is a root then $\overline{\alpha} = -2 - 3i$ must also be a root.

$\left(z - (-2+3i)\right)\left(z - (-2-3i)\right)$

$= z^2 - z(-2+3i) - z(-2-3i) + (-2+3i)(-2-3i)$

$= z^2 + 4z + 4 + 9$

$= z^2 + 4z + 13$

Let the third root = β

$\alpha\overline{\alpha}\beta = -\dfrac{c}{a}$

$13\beta = -\dfrac{13}{2}$

$\beta = -\dfrac{1}{2}$

$\therefore 2z^3 + bz^2 + cz + 13 = \left(z^2 + 4z + 13\right)(2z + 1)$

$= 2z^3 + z^2 + 8z^2 + 4z + 26z + 13$

$= 2z^3 + 9z^2 + 30z + 13$

$P(x) = 2x^3 + 9z^2 + 30x + 13 = (2x+1)\left(x^2 + 4x + 13\right)$

$x = -\dfrac{1}{2}, -2 \pm 3i$

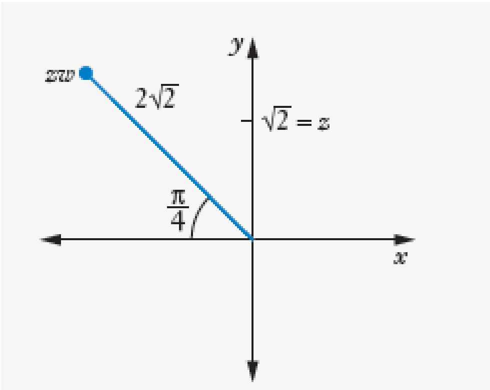
Question 9

a

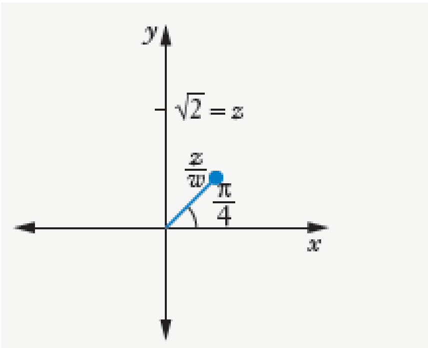
$$z = \sqrt{2}\text{cis}\left(\frac{\pi}{2}\right)$$

$$w = 2\text{cis}\left(\frac{\pi}{4}\right)$$

i $$zw = 2\sqrt{2}\text{cis}\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = 2\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$$



ii $$\frac{z}{w} = \frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\text{cis}\left(\frac{\pi}{4}\right)$$

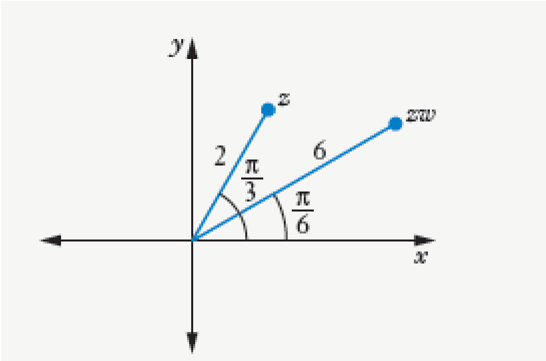


b

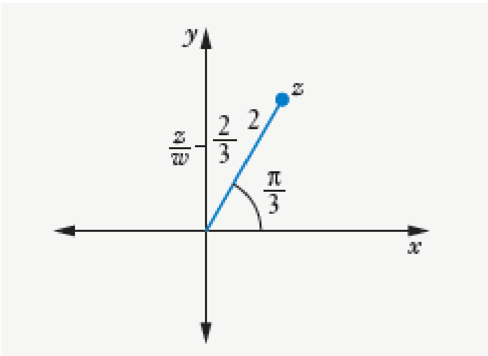
$$z = 2\text{cis}\left(\frac{\pi}{3}\right)$$

$$w = 3\text{cis}\left(-\frac{\pi}{6}\right)$$

i $$zw = 2 \times 3\text{cis}\left(\frac{\pi}{3} + \left[-\frac{\pi}{6}\right]\right) = 6\text{cis}\left(\frac{\pi}{6}\right)$$



ii $$\frac{z}{w} = \frac{2}{3}\text{cis}\left(\frac{\pi}{3} - \left[-\frac{\pi}{6}\right]\right) = \frac{2}{3}\text{cis}\left(\frac{\pi}{2}\right)$$

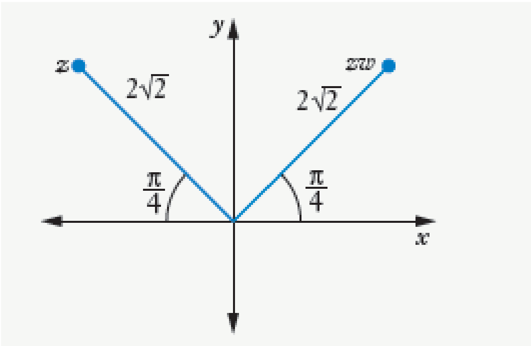


c

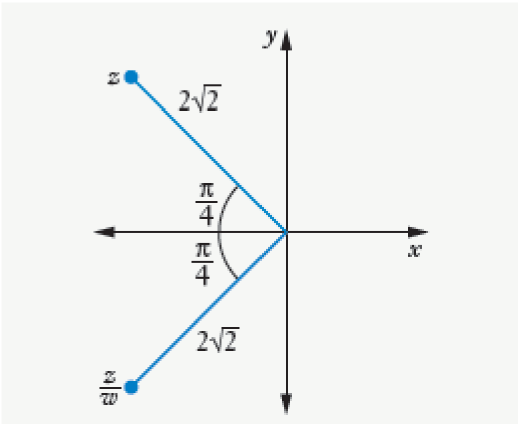
$$z=(1+i)^3=\left(\sqrt{2}\text{cis}\left[\frac{\pi}{4}\right]\right)^3=2\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$$

$$w=\left(\frac{\sqrt{3}+i}{2}\right)^9=\left(\text{cis}\left[\frac{\pi}{6}\right]\right)^9=\text{cis}\left(\frac{3\pi}{2}\right)$$

i $$zw=2\sqrt{2}\text{cis}\left(\frac{3\pi}{4}+\frac{3\pi}{2}\right)=2\sqrt{2}\left(\frac{9\pi}{4}\right)=2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$$



ii $$\frac{z}{w}=\frac{2\sqrt{2}}{1}\text{cis}\left(\frac{3\pi}{4}-\frac{3\pi}{2}\right)=2\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$$



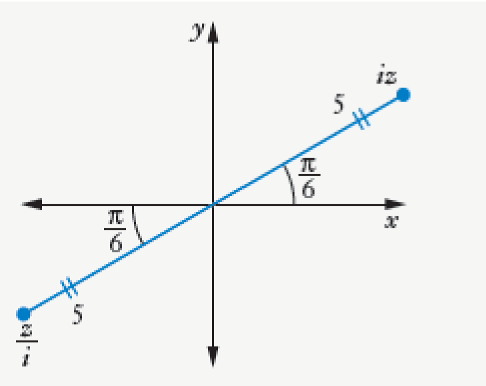
Question 10

$$z = 5\text{cis}\left(-\frac{\pi}{3}\right)$$

$$iz = 5\text{cis}\left(\frac{\pi}{6}\right)$$

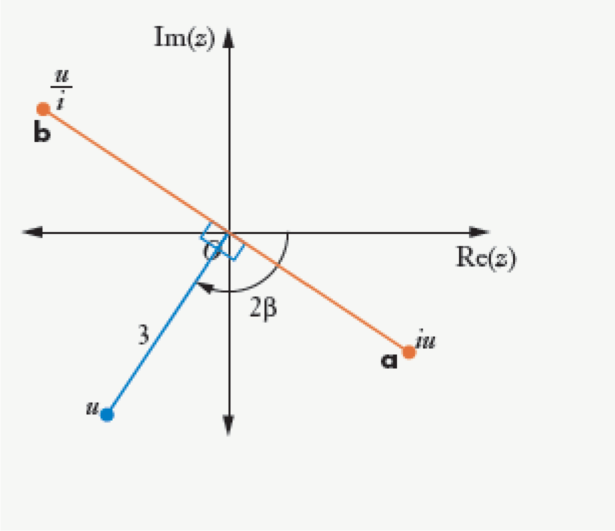
$$\frac{z}{i} = \frac{iz}{-1} = -iz$$

$$= 5\text{cis}\left(-\frac{5\pi}{6}\right)$$

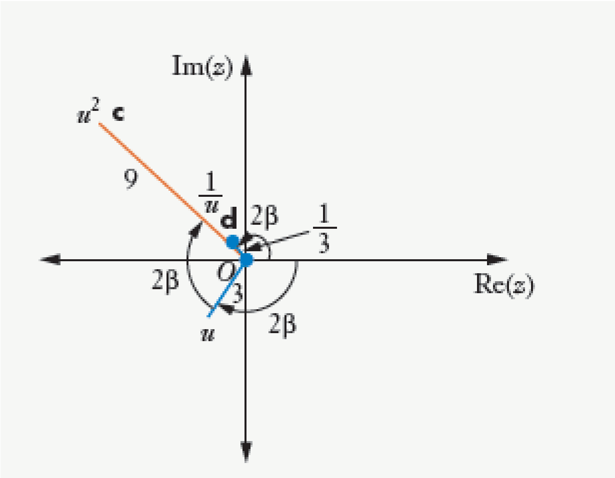


Question 11

a, b



c, d



Question 12

a $w = u + v$

b

As $|v| = |u|$ and v is $\perp$ to u clockwise

$v = iu$

$\therefore\ w = u + iu$

c As *OVWU* is a square, all adjacent lines are perpendicular.

$u - w$ represents the vector $\overrightarrow{WU}$.

$v - w$ represents the vector $\overrightarrow{WV}$.

$\overrightarrow{WU}$ is the vector $\overrightarrow{WV}$ rotated anticlockwise $\frac{\pi}{2}$.

Hence $u - w = i(v - w)$.

d

$$
\begin{aligned}
&u^2 + v^2 \\
&= u^2 + (iu)^2 \\
&= u^2 - u^2 \\
&= 0
\end{aligned}
$$

e

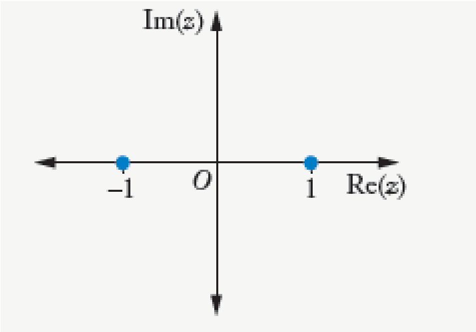
$$
\begin{aligned}
m &= \frac{w}{2} \\
&= \frac{u+v}{2} \\
&= \frac{u+iu}{2}
\end{aligned}
$$

Question 13

a

$$z^2 = 1$$

$$z_1 = 1, z_2 = -1$$



b

$$z^3 = 1$$

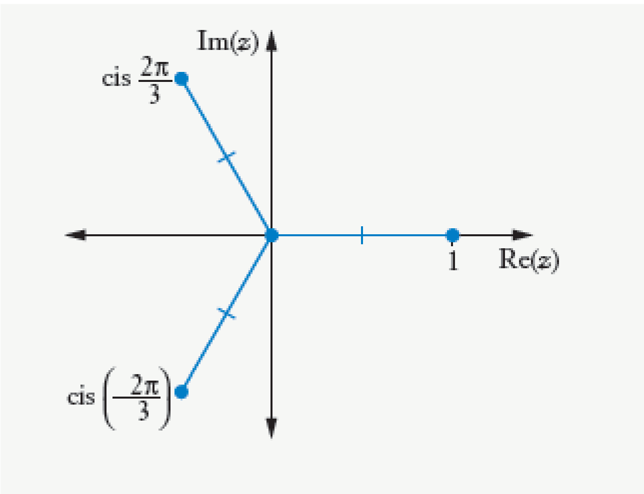
3 roots of unity must be equally spaced around the unit circle

$$z_1 = 1$$

$$z_2 = \text{cis}\frac{2\pi}{3}$$

$$z_3 = \bar{z}_2 = \text{cis}\left(-\frac{2\pi}{3}\right)$$

$$\text{conjuates: cis } \frac{2\pi}{3}, \text{cis } \left(-\frac{2\pi}{3}\right)$$



c

$z^6 = 1$

6 roots of unity must be equally spaced around the unit circle

$z_1 = 1$

$z_4 = -1$

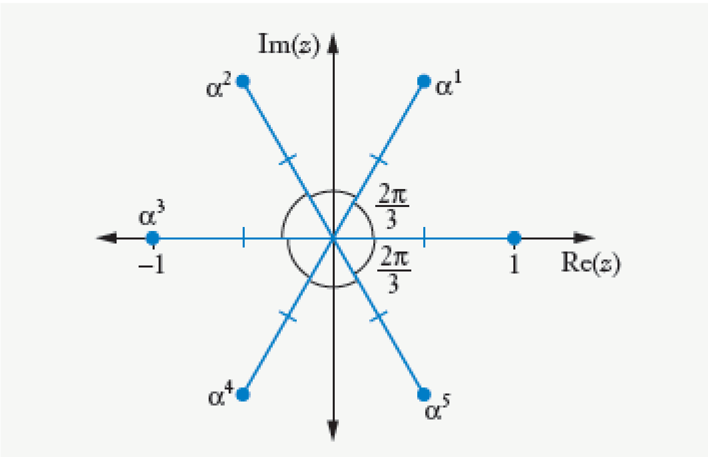
$z_2 = \text{cis}\dfrac{\pi}{3}$

$z_6 = \bar{z}_2 = \text{cis}\left(-\dfrac{\pi}{3}\right)$

$z_3 = \text{cis}\dfrac{2\pi}{3}$

$z_5 = \bar{z}_4 = \text{cis}\left(-\dfrac{2\pi}{3}\right)$

conjugates: $\text{cis}\dfrac{\pi}{3}, \text{cis}\left(-\dfrac{\pi}{3}\right)$ and $\text{cis}\dfrac{2\pi}{3}, \text{cis}\left(-\dfrac{2\pi}{3}\right)$



d

$z^8 = 1$

8 roots of unity must be equally spaced around the unit circle

$z_1 = 1, z_5 = -1,$

$z_3 = i \;\; z_7 = \bar{z}_3 = -i$

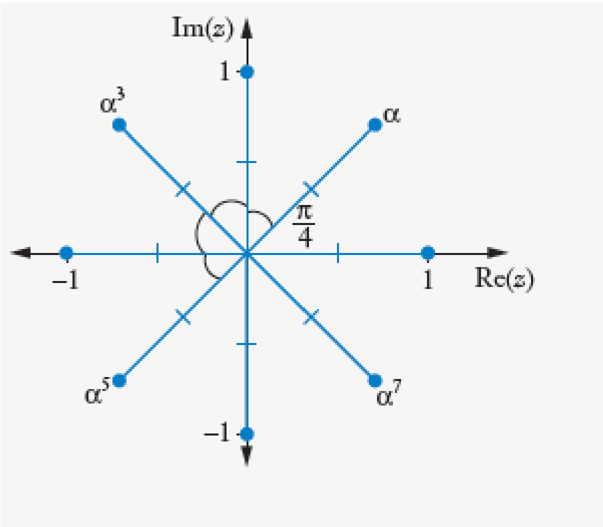
$z_2 = \text{cis}\frac{\pi}{4}$

$z_8 = \bar{z}_2 = \text{cis}\left(-\frac{\pi}{4}\right)$

$z_4 = \text{cis}\frac{3\pi}{4}$

$z_6 = \bar{z}_4 = \text{cis}\left(-\frac{3\pi}{4}\right)$

$\text{conjugates}: \text{cis}\frac{\pi}{4}, \text{cis}\left(-\frac{\pi}{4}\right); \text{cis}\frac{3\pi}{4}, \text{cis}\left(-\frac{3\pi}{4}\right)$ and $i, -i$



Question 14

a

$z^7 = 1$

Seven roots of unity must be equally spaced around the unit circle

$z_1 = 1$

$z_2 = \text{cis}\dfrac{2\pi}{7}$

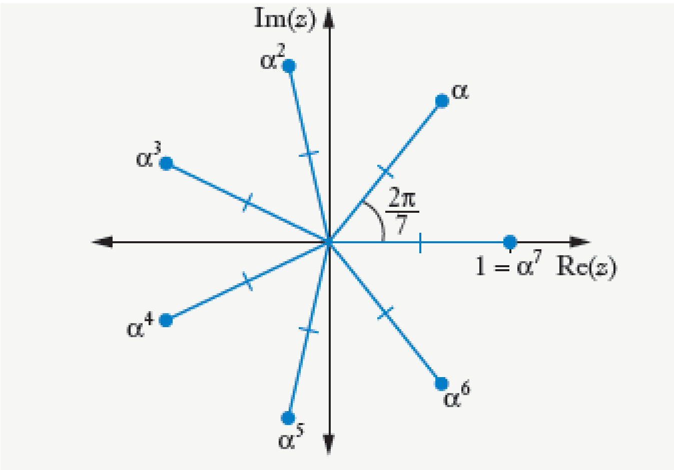
$z_3 = \overline{z_2} = \text{cis}\left(-\dfrac{2\pi}{7}\right)$

$z_4 = \text{cis}\dfrac{4\pi}{7}$

$z_5 = \overline{z_4} = \text{cis}\left(-\dfrac{4\pi}{7}\right)$

$z_6 = \text{cis}\dfrac{6\pi}{7}$

$z_7 = \overline{z_6} = \text{cis}\left(-\dfrac{6\pi}{7}\right)$



b

$z^7 = 1$

$z^7 - 1 = 0$

$(z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$

As α is a complex root, $\alpha - 1 \neq 0$

$\therefore \alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$

As the roots are equally spaced around a unit circle they form a regular heptagon, adding these vectors returns you to the origin.

c

$$(z-z_2)(z-\bar{z}_2)=\left(z-\text{cis}\left[\frac{2\pi}{7}\right]\right)\left(z-\text{cis}\left[-\frac{2\pi}{7}\right]\right)$$

$$=z^2-2z\cos\left(\frac{2\pi}{7}\right)+1$$

$$(z-z_4)(z-\bar{z}_4)=\left(z-\text{cis}\left[\frac{4\pi}{7}\right]\right)\left(z-\text{cis}\left[-\frac{4\pi}{7}\right]\right)$$

$$=z^2-2z\cos\left(\frac{4\pi}{7}\right)+1$$

$$(z-z_6)(z-\bar{z}_6)=\left(z-\text{cis}\left[\frac{6\pi}{7}\right]\right)\left(z-\text{cis}\left[-\frac{6\pi}{7}\right]\right)$$

$$=z^2-2z\cos\left(\frac{6\pi}{7}\right)+1$$

$$z^7-1=(z-1)\left(z^2-2z\cos\left(\frac{2\pi}{7}\right)+1\right)\left(z^2-2z\cos\left(\frac{4\pi}{7}\right)+1\right)\left(z^2-2z\cos\left(\frac{6\pi}{7}\right)+1\right)$$

d

$$z^7-1=(z-1)\left(z^2-2z\cos\left(\frac{2\pi}{7}\right)+1\right)\left(z^2-2z\cos\left(\frac{4\pi}{7}\right)+1\right)\left(z^2-2z\cos\left(\frac{6\pi}{7}\right)+1\right)$$

$$(z-1)\left(z^6+z^5+z^4+z^3+z^2+z+1\right)=(z-1)\left(z^2-2z\cos\left(\frac{2\pi}{7}\right)+1\right)$$

$$\left(z^2-2z\cos\left(\frac{4\pi}{7}\right)+1\right)\left(z^2-2z\cos\left(\frac{6\pi}{7}\right)+1\right)$$

$$z^6+z^5+z^4+z^3+z^2+z+1=\left(z^2-2z\cos\left(\frac{2\pi}{7}\right)+1\right)\left(z^2-2z\cos\left(\frac{4\pi}{7}\right)+1\right)$$

$$\left(z^2-2z\cos\left(\frac{6\pi}{7}\right)+1\right)$$

$$=\left(z^2-2z\left[-\cos\left(\pi-\frac{2\pi}{7}\right)\right]+1\right)\left(z^2-2z\left[-\cos\left(\pi-\frac{4\pi}{7}\right)\right]+1\right)\left(z^2-2z\left[-\cos\left(\pi-\frac{6\pi}{7}\right)\right]+1\right)$$

$$=\left(z^2+2z\cos\left(\frac{5\pi}{7}\right)+1\right)\left(z^2+2z\cos\left(\frac{3\pi}{7}\right)+1\right)\left(z^2+2z\cos\left(\frac{\pi}{7}\right)+1\right)$$

e

$$z^6+z^5+z^4+z^3+z^2+z+1=0$$
$$\text{cis}\left(\frac{2\pi}{7}\right)+\text{cis}\left(-\frac{2\pi}{7}\right)+\text{cis}\left(\frac{4\pi}{7}\right)+\text{cis}\left(-\frac{4\pi}{7}\right)+\text{cis}\left(\frac{6\pi}{7}\right)+\text{cis}\left(-\frac{-6\pi}{7}\right)=0$$
$$1+2\cos\left(\frac{2\pi}{7}\right)+2\cos\left(\frac{4\pi}{7}\right)+2\cos\left(\frac{6\pi}{7}\right)=0$$
$$2\left(\cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)\right)=-1$$
$$\cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)=-\frac{1}{2}$$

Question 15

a

$(\text{cis}\,\theta)^3=\cos 3\theta+i\sin\theta$ b De Moivre's theorem

$(\text{cis}\,\theta)^3=\cos^3\theta+3i\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta$ by expanding

Equating real parts:

$$\begin{aligned}\cos 3\theta&=\cos^3\theta-3\cos\theta\sin^2\theta\\&=\cos^3\theta-3\cos\theta\left(1-\cos^2\theta\right)\\&=4\cos^3\theta-3\cos\theta\end{aligned}$$

b

Let $x=\cos\theta$

Then $\cos 3\theta = 4\cos^3\theta - 3\cos\theta = 4x^3 - 3x$

If $8x^3-6x-1=0$, then:

$8x^3-6x=1$

$4x^3-3x=\frac{1}{2}$

$\cos 3\theta=\frac{1}{2}$

$3\theta=\frac{\pi}{3},\frac{5\pi}{3},\frac{7\pi}{3},\frac{11\pi}{3},...$

$\theta=\frac{\pi}{9},\frac{5\pi}{9},\frac{7\pi}{9},\frac{11\pi}{9},...$

$x=\cos\theta=\cos\frac{\pi}{9},\cos\frac{5\pi}{9},\cos\frac{7\pi}{9},\cos\frac{11\pi}{9},...$

but $8x^3-6x-1=0$ has 3 roots and $\cos\frac{11\pi}{9}=\cos\left(-\frac{7\pi}{9}\right)=\cos\frac{7\pi}{9}$, so

$x=\cos\frac{\pi}{9},\cos\frac{5\pi}{9},\cos\frac{7\pi}{9}$

c

$8x^3-6x-1=0$ the sum of the roots = 0

$\cos\frac{\pi}{9}+\cos\frac{5\pi}{9}+\cos\frac{7\pi}{9}=0$

$\cos\frac{\pi}{9}+\left[-\cos\left(\pi-\frac{5\pi}{9}\right)\right]+\left[-\cos\left(\pi-\frac{7\pi}{9}\right)\right]=0$

$\cos\frac{\pi}{9}-\cos\frac{4\pi}{9}-\cos\frac{2\pi}{9}=0$

$-\cos\frac{\pi}{9}+\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}=0$

$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}-\cos\frac{\pi}{9}=0$

Question 16

Let $z^3 = 1$

$z^3 - 1 = 0$

$(z-1)(z^2 + z + 1) = 0$

As ω is a complex cube root of unity $\omega \neq 1$

$\therefore \omega^2 + \omega + 1 = 0$

a

$$\begin{aligned}&(\omega^2 + 1)^3\\&= (-\omega)^3\\&= -\omega^3\\&= -1\end{aligned}$$

b

$$\begin{aligned}&1 + \frac{1}{\omega} + \frac{1}{\omega^2}\\&= 1 + \frac{\omega + \omega^2}{\omega^3}\\&= 1 + \frac{-1}{1}\\&= 0\end{aligned}$$

c

$$\begin{aligned}&(1 - \omega - \omega^2)(1 - \omega + \omega^2)(1 + \omega - \omega^2)\\&= (1 + 1 - 1 - \omega - \omega^2)(1 - 2\omega + \omega^2)(1 + \omega + \omega^2 - 2\omega^2)\\&= (2 - (1 + \omega + \omega^2))(-2\varpi + (1 + \omega + \omega^2))((1 + \omega + \omega^2) - 2\omega^2)\\&= (2)(-2\omega)(-2\omega^2)\\&= 8\omega^3\\&= 8\end{aligned}$$

Question 17

a

$$\frac{1}{1+\omega}+\frac{1}{1+\omega^2}$$
$$=\frac{1+\omega+1+\omega^2}{(1+\omega)(1+\omega^2)}$$
$$=\frac{1+1+\omega+\omega^2}{1+\omega+\omega^2+\omega^3}$$
$$=\frac{1}{\omega^3}$$
$$=1$$

b

$$\frac{k+l\omega+m\omega^2}{l+m\omega+k\omega^2}$$
$$=\frac{k+l\omega+m\omega^2}{l+m\omega+k\omega^2}\times\frac{\omega^2}{\omega^2}$$
$$=\frac{k\omega^2+l\omega\omega^2+m\omega^2\omega^2}{\omega^2\left(l+m\omega+k\omega^2\right)}$$
$$=\frac{k\omega^2+l\omega^3+m\omega^4}{\omega^2\left(l+m\omega+k\omega^2\right)}$$
$$=\frac{k\omega^2+l+m\omega}{\omega^2\left(l+m\omega+k\omega^2\right)}$$
$$=\frac{1}{\omega^2}$$
$$=\frac{\omega}{\omega^3}$$
$$=\omega$$

Question 18

a

Let $z^2 = 15 - 8i$

$(x + iy)^2 = 15 - 8i$

$x^2 - y^2 + 2ixy = 15 - 8i$

Equating real and imaginary coefficients

$x^2 - y^2 = 15$

$2xy = -8$

$y = \frac{-4}{x}$

$x^2 - \left(\frac{-4}{x}\right)^2 = 15$

$x^4 - 16 = 15x^2$

$x^4 - 15x^2 - 16 = 0$

$(x^2 - 16)(x^2 + 1) = 0$

$x^2 - 16 = 0$

$x = \pm 4$

$y = \mp 1$

$z = 4 - i, -4 + i$

b

$z^2 = e^{i\frac{\pi}{4}}$

$z = \left(e^{\frac{\pi}{4}}\right)^{\frac{1}{2}}$

$= \pm e^{i\frac{\pi}{8}}$

c

$z^2 = 4\text{cis}\left(\frac{\pi}{6}\right)$

$z = \sqrt{4\text{cis}\left(\frac{\pi}{6}\right)}$

$= \pm 2\text{cis}\left(\frac{\pi}{12}\right)$

Question 19

a

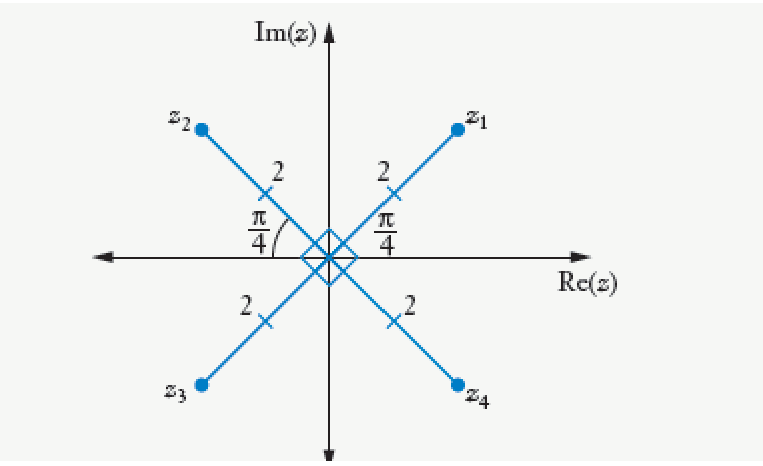
$z^4 = -16$

$z^4 = 16\text{cis}(\pi)$

$z = 16^{\frac{1}{4}}\,\text{cis}\left(\frac{1}{4}\times\pi\right) = 2\text{cis}\frac{\pi}{4}$

The four roots of 16 are equally spaced $\frac{\pi}{2}$ apart fro $z = 2\text{cis}\frac{\pi}{4}$

$z = 2\text{cis}\frac{\pi}{4}, 2\text{cis}\frac{3\pi}{4}, 2\text{cis}\left(-\frac{\pi}{4}\right), 2\text{cis}\left(-\frac{3\pi}{4}\right)$



b

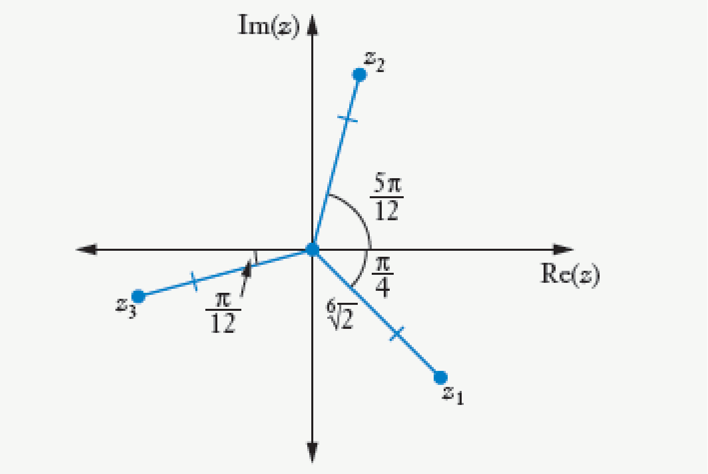
$z^3 = -1 - i$

$z^3 = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

$z = \sqrt[6]{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

The three roots of $-1-i$ are equally spaced $\frac{2\pi}{3}$ apart from $z = \sqrt[6]{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$.

$z = \sqrt[6]{2}\,\text{cis}\left(-\frac{\pi}{4}\right), \sqrt[6]{2}\,\text{cis}\left(\frac{5\pi}{12}\right), \sqrt[6]{2}\,\text{cis}\left(-\frac{11\pi}{12}\right)$



c

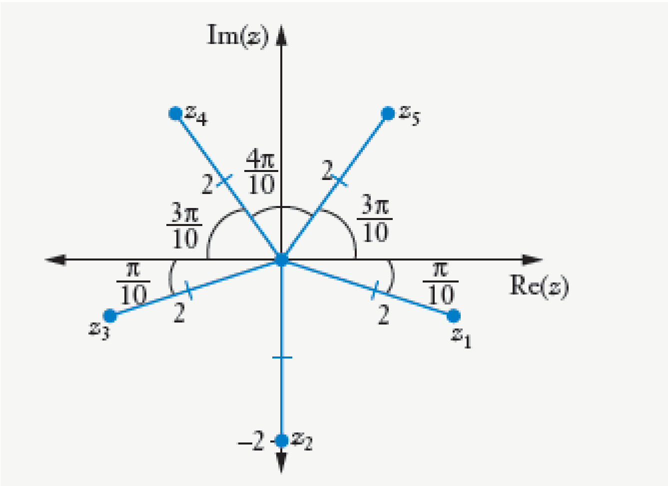
$z^5 = 32e^{-i\frac{\pi}{2}}$

$z = \left(32e^{-\frac{\pi}{2}}\right)^{\frac{1}{5}}$

$z = 2e^{-i\frac{\pi}{10}}$

The five roots of $32e^{-i\frac{\pi}{2}}$ are equally spaced $\frac{2\pi}{5}$ apart from $z = 2e^{-i\frac{\pi}{0}}$.

$z = 2e^{-i\frac{\pi}{0}}, 2e^{-\frac{\pi}{2}}, 2e^{-i\frac{9\pi}{10}}, 2e^{i\frac{7\pi}{10}}, 2e^{i\frac{3\pi}{10}}$



Question 20

$$z^5 = \frac{1+i\sqrt{3}}{2}$$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\theta = \tan^{-}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

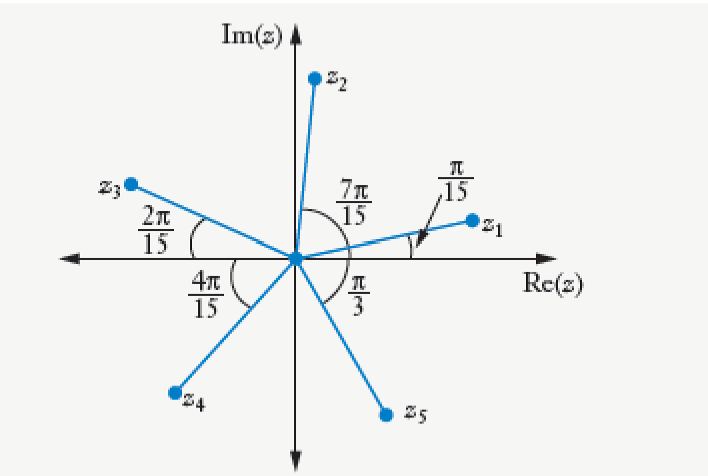
$$z^5 = e^{i\frac{\pi}{3}}$$

$$z = \left(e^{\frac{\pi}{3}}\right)^{\frac{1}{5}}$$

$$z = e^{i\frac{\pi}{15}}$$

The five roots of $e^{i\frac{\pi}{3}}$ are equally spaced $\frac{2\pi}{5}$ apart from $z = e^{i\frac{\pi}{15}}$.

$$z = e^{i\frac{\pi}{15}}, e^{i\frac{7\pi}{15}}, e^{i\frac{13\pi}{15}}, e^{-i\frac{11\pi}{15}}, e^{-i\frac{\pi}{3}}$$



Question 21

a

$z^9 + 1 = 0$

$z^9 = -1$

The ninth roots of 1 are equally spaced $\frac{2\pi}{9}$ apart from $z = -1$.

$z = -1, \text{cis}\frac{7\pi}{9}, \text{cis}\frac{5\pi}{9}, \text{cis}\frac{\pi}{3}, \text{cis}\frac{\pi}{9}, \text{cis}\left(-\frac{\pi}{9}\right), \text{cis}\left(-\frac{\pi}{3}\right), \text{cis}\left(-\frac{5\pi}{9}\right), \text{cis}\left(-\frac{7\pi}{9}\right)$

b

$$\begin{aligned} z^9 + 1 &= \left(z^3\right)^3 + 1 \\ &= \left(z^3 + 1\right)\left(\left[z^3\right]^2 - z^3 + 1\right) \\ &= \left(z^3 + 1\right)\left(z^6 - z^3 + 1\right) \\ &= \left(z + 1\right)\left(z^2 - z + 1\right)\left(z^6 - z^3 + 1\right) \end{aligned}$$

c

$z^9 + 1 = \left(z^3 + 1\right)\left(z^6 - z^3 + 1\right)$

The solutions of $\left(z^6 - z^3 + 1\right)$ are the 6 solutions of $z^9 + 1$

that do not include the solutions of $\left(z^3 + 1\right)$.

The solutions not required are -1, $\text{cis}\frac{\pi}{3}, \text{cis}\left(-\frac{\pi}{3}\right)$, leaving

$\text{cis}\frac{\pi}{9}, \text{cis}\frac{5\pi}{9}, \text{cis}\frac{7\pi}{9}, \text{cis}\left(-\frac{7\pi}{9}\right), \text{cis}\left(-\frac{5\pi}{9}\right), \text{cis}\left(-\frac{\pi}{9}\right)$

d

The real quadratic actors of $z^6 - z^3 + 1$ are:

$\left(z - cis\frac{\pi}{9}\right)\left(z - cis\left[-\frac{\pi}{9}\right]\right), \left(z - cis\frac{5\pi}{9}\right)\left(z - cis\left[-\frac{5\pi}{9}\right]\right), \left(z - cis\frac{7\pi}{9}\right)\left(z - cis\left[-\frac{7\pi}{9}\right]\right)$

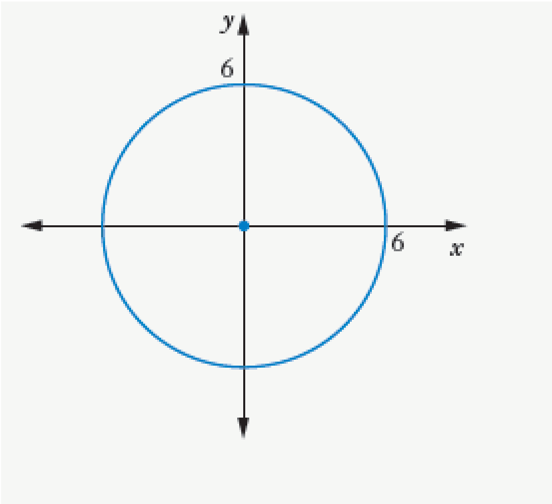
$= \left(z^2 - 2z\cos\frac{\pi}{9} + 1\right), \left(z^2 - 2z\cos\frac{5\pi}{9} + 1\right), \left(z^2 - 2z\cos\frac{7\pi}{9} + 1\right).$

Question 22

a **i**

$$|z| = 6$$
$$x^2 + y^2 = 6^2$$
$$x^2 + y^2 = 36$$

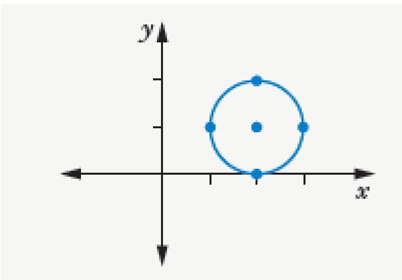
ii Set of points equidistant 6 units from *O*.



b **i**

$$|z - 2 - i| = 1$$
$$|x + iy + (-2 - i)| = 1$$
$$\sqrt{(x-2)^2 + (y-1)^2} = 1$$
$$(x-2)^2 + (y-1)^2 = 1^2$$

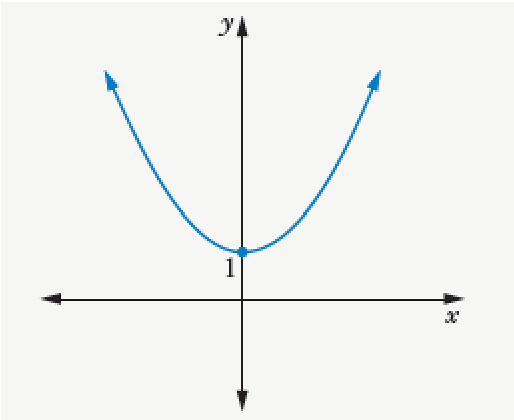
ii Set of points equidistant 1 unit from (2, 1).



c **i**

$$|z-2i| = \mathrm{Im}(z)$$
$$|x+iy-2i| = y$$
$$\sqrt{(x)^2+(y-2)^2} = y$$
$$x^2+(y-2)^2 = y^2$$
$$x^2-4y+4=0$$
$$y=\frac{1}{4}(x^2+4)$$

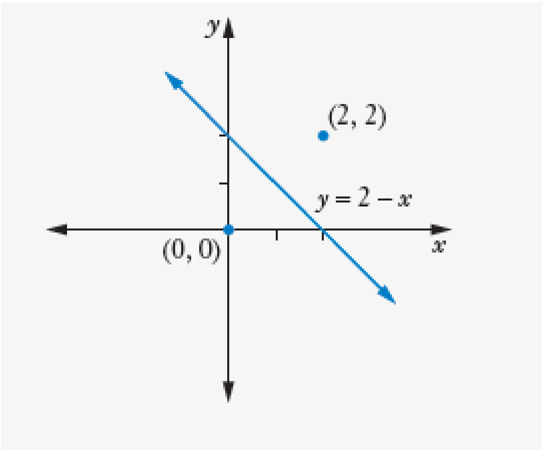
ii Set of points equidistant from (0, 2) and the *x*-axis.



d **i**

$$|z| = |z - 2 - 2i|$$
$$|x + iy| = |x + iy - (2 + 2i)|$$
$$\sqrt{(x)^2 + (y)^2} = \sqrt{(x-2)^2 + (y-2)^2}$$
$$x^2 + y^2 = x^2 - 4x + 4 + y^2 - 4y + 4$$
$$4x + 4y - 8 = 0$$
$$y = 2 - x$$

ii It is the perpendicular bisector of the line joining (0, 0) and (2, 2).



e **i**

$$\frac{|z+6|}{|z-4i|}=1$$

$$|z+6|=|z-4i|$$

$$|x+iy-(-6)|=|x+iy-4i|$$

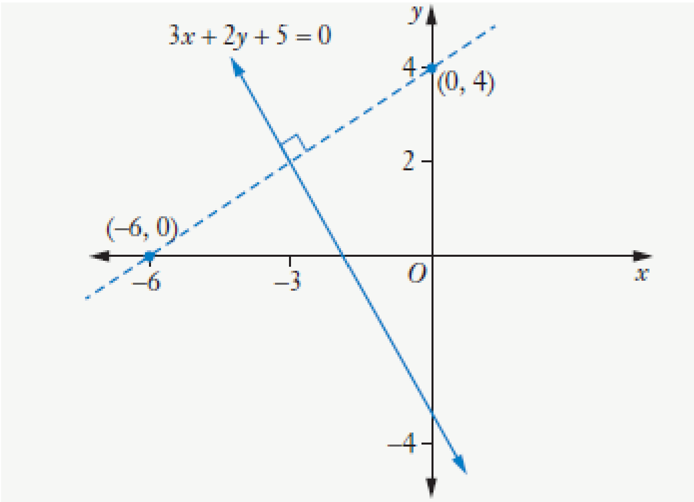
$$\sqrt{(x+6)^2+(y)^2}=\sqrt{(x)^2+(y-4)^2}$$

$$x^2+12x+36+y^2=x^2+y^2-8y+16$$

$$12x+8y+20=0$$

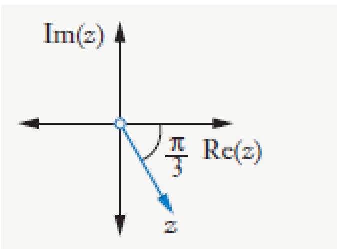
$$3x+2y+5=0$$

ii It is the perpendicular bisector of the line joining (–6, 0) and (0, 4).

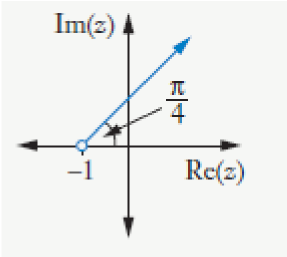


Question 23

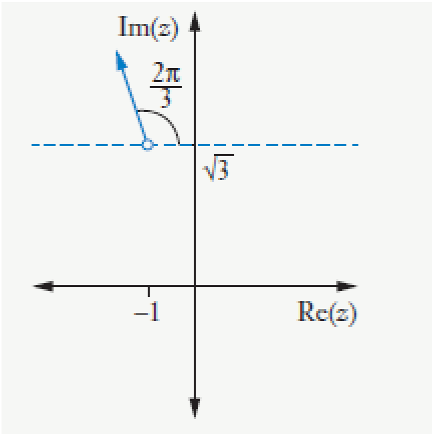
a $\arg(z)=-\frac{\pi}{3}$



b $\arg(z+1)=\dfrac{\pi}{4}$

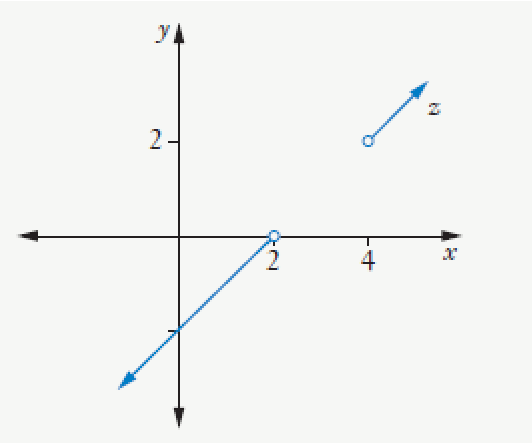


c $\arg\left(z+1-i\sqrt{3}\right)=\dfrac{2\pi}{3}$

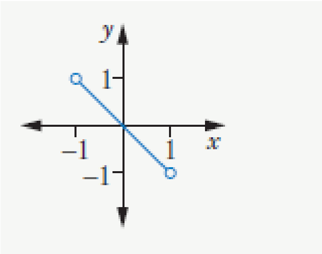


Question 24.

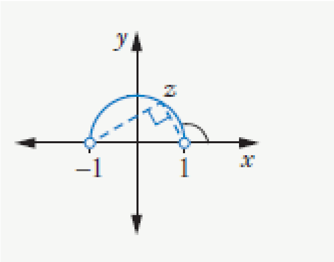
a $\arg(z-2)=\arg(z-4-2i)$



b $\arg(z+1-i)-\arg(z-1+i)=\pi$

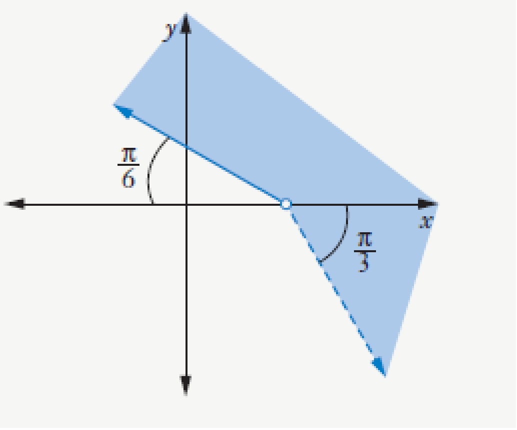


c $\arg(z-1)-\arg(z+1)=\dfrac{\pi}{2}$

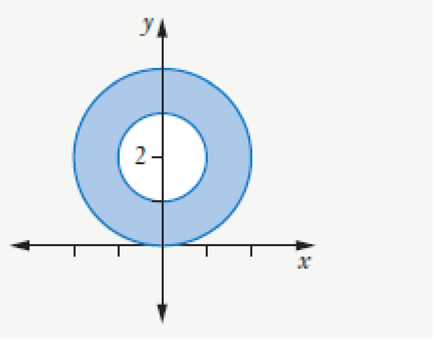


Question 25

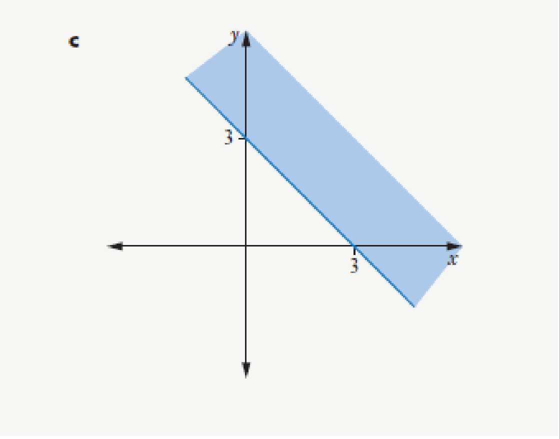
a $-\frac{\pi}{3} < \arg(z-1) \le \frac{5\pi}{6}$



b $1 \le \arg(z-2i) \le 2$

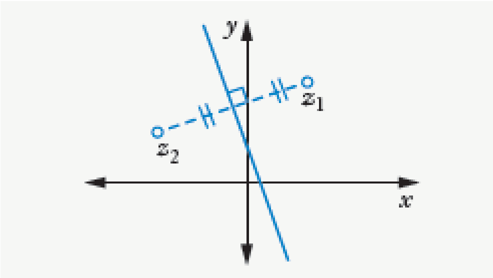


c $\text{Re}(z)+\text{Im}(z) \ge 3$

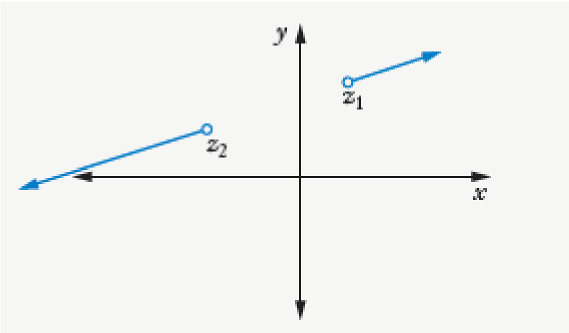


Question 26

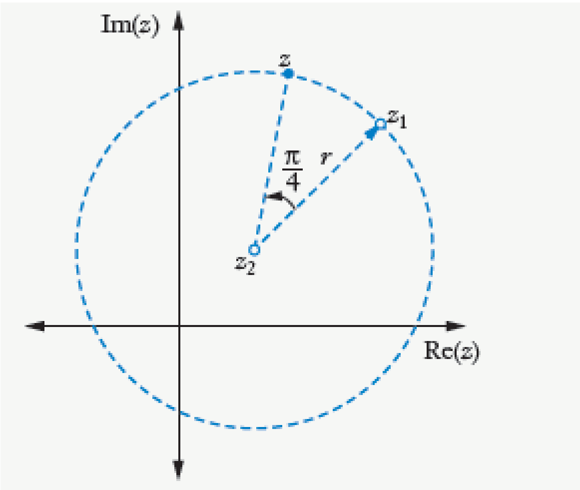
a perpendicular bisector



b



c *z* is intersection of ray and circle



Question 27

$$\frac{w_2 - w_1}{w_3 - w_1} = \frac{w_3 - w_2}{w_1 - w_2}$$

$\therefore$ Moduli and arguments are equal

$$\therefore \left|\frac{w_2 - w_1}{w_3 - w_1}\right| = \left|\frac{w_3 - w_2}{w_1 - w_2}\right|$$

$$\arg\left(\frac{w_2 - w_1}{w_3 - w_1}\right) = \arg\left(\frac{w_3 - w_2}{w_1 - w_2}\right)$$

$$\arg(w_2 - w_1) - \arg(w_3 - w_1) = \arg(w_3 - w_2) - \arg(w_1 - w_2)$$

$$\therefore \angle w_3 w_2 w_1 = \angle w_3 w_1 w_2$$

$$\therefore |w_3 - w_2| = |w_3 - w_1|$$

$$\therefore |w_2 - w_1|^2 = |w_3 - w_1|^2$$

$$|w_3 - w_2| = |w_3 - w_1| = |w_2 - w_1|$$

All 3 sides are equal, hence it is an equilateral triangle.

MATHS IN FOCUS 12
MATHEMATICS EXTENSION 2

WORKED SOLUTIONS

Chapter 5: Further mathematical induction

Exercise 5.01 Review of mathematical induction

Question 1

a

$P(n): 1+2+3+4+....+n = \frac{n}{2}(n+1)$

$P(1)$ LHS $=1$

RHS $= \frac{1}{2}(1+1) = \frac{1}{2}\times 2 = 1 =$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $1+2+3+4+....+k = \frac{k}{2}(k+1)$

$P(k+1)$ LHS

$1+2+3+4+....+k+(k+1)$

$= \frac{k}{2}(k+1)+(k+1)$

$= \left(\frac{k}{2}+1\right)(k+1)$

$= (k+1)\left(\frac{k}{2}+\frac{2}{2}\right)$

$= (k+1)\frac{1}{2}(k+2)$

$= \frac{k+1}{2}((k+1)+1)$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$P(n): 1+3+5+....+(2n-1)=n^2$

$P(1)$ LHS$=1$

RHS $=1^2 = 1 =$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $1+3+5+....+(2k-1)=k^2$

$P(k+1)$ LHS

$1+2+3+4+....+(2k-1)+(2(k+1)-1)$

$= k^2+(2(k+1)-1)$

$= k^2+(2k+1)$

$=(k+1)^2$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

c

$P(n): 1^2+2^2+3^2+4^2+....+n^2=\frac{n}{6}(n+1)(2n+1)$

$P(1)\quad \text{LHS}=1^2=1$

$\text{RHS}=\frac{1}{6}(1+1)(2+1)=\frac{1}{6}\times 2\times 3=1=\text{LHS}$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $1^2+2^2+3^2+4^2+....+k^2=\frac{k}{6}(k+1)(2k+1)$

Required to prove

$P(k+1): 1^2+2^2+3^2+4^2+....+k^2+(k+1)^2=\frac{(k+1)}{6}((k+1)+1)(2(k+1)+1)$

LHS $1^2+2^2+3^2+4^2+....+k^2+(k+1)^2$

$=\frac{k}{6}(k+1)(2k+1)+(k+1)^2$

$=(k+1)\left(\frac{k}{6}(2k+1)+(k+1)\right)$

$=(k+1)\left(\frac{k(2k+1)}{6}+\frac{6(k+1)}{6}\right)$

$=\frac{(k+1)}{6}(k(2k+1)+6(k+1))$

$=\frac{(k+1)}{6}(2k^2+k+6k+6)$

$=\frac{(k+1)}{6}(2k^2+7k+6)$

$=\frac{(k+1)}{6}(k+2)(2k+3)$

$=\frac{(k+1)}{6}((k+1)+1)(2(k+1)+1)$

$=\text{RHS}$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 2

a

$$P(n): 1+3+3^2+3^3+3^4+....+3^{n-1}=\frac{3^n-1}{2}$$

$P(1)$ LHS $=1$

$$\text{R.S}=\frac{3\ \ -1}{2}=\frac{2}{2}=1=\text{LHS}$$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $1+3+3^2+3^3+3^4+....+3^{k-1}=\dfrac{3^k-1}{2}$

Required to prove

$$P(k+1): 1+3+3^2+3^3+3^4+....+3^{k-1}+3^k=\frac{3^{k+}\ -1}{2}$$

LHS $1+3+3^2+3^3+3^4+....+3^{k-1}+3^k$

$$=\frac{3^k-1}{2}+3^k$$

$$=\frac{3^k-1}{2}+\frac{2\times 3^k}{2}$$

$$=\frac{3^k-1+2\times 3^k}{2}$$

$$=\frac{3^k(1+2)-1}{2}$$

$$=\frac{3\times 3^k-1}{2}$$

$$=\frac{3^{k+}\ -1}{2}$$

= RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$$P(n): 1+4+4^2+4^3+4^4+....+4^{n-1}=\frac{4^n-1}{3}$$

$P(1)$ LHS $=1$

$$\text{RHS}=\frac{4\ -1}{3}=\frac{3}{3}=1=\text{LHS}$$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $1+4+4^2+4^3+4^4+....+4^{k-1}=\frac{4^k-1}{3}$

Required to prove

$$P(k+1): 1+4+4^2+4^3+4^4+....+4^{k-1}+4^k=\frac{4^{k+}\ -1}{3}$$

LHS $1+4+4^2+4^3+4^4+....+4^{k-1}+4^k$

$$=\frac{4^k-1}{3}+4^k$$

$$=\frac{4^k-1}{3}+\frac{3\times4^k}{3}$$

$$=\frac{4^k-1+3\times4^k}{3}$$

$$=\frac{4^k(1+3)-1}{3}$$

$$=\frac{4\times4^k-1}{3}$$

$$=\frac{4^{k+}\ -1}{3}$$

= RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

c

$P(n): 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+....+\frac{1}{2^{n-1}}=2-\frac{1}{2^{n-1}}$

$P(1)$ LHS $=1$

RHS $= 2-\frac{1}{2^0}=2-1=1=$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+....+\frac{1}{2^{k-1}}=2-\frac{1}{2^{k-1}}$

Required to prove

$P(k+1): 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\quad+\frac{1}{2^{k-1}}+\frac{1}{2^k}=2-\frac{1}{2^k}$

LHS $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\quad+\frac{1}{2^{k-1}}+\frac{1}{2^k}$

$=2-\frac{1}{2^{k-}}+\frac{1}{2^k}$

$=2-\frac{2}{2^k}+\frac{1}{2^k}$

$=2-\frac{2-1}{2^k}$

$=2-\frac{1}{2^k}$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 3

a

$P(n): 4^n - 1$ is divisible by 3 $\forall n \geq 1$

i.e. $4^n - 1 = 3m \quad m \in \mathbb{N}$

$P(1)$ LHS $= 4 \ - 1 = 3$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $4^k - 1 = 3m$

$P(k+1): 4^{k+} \ - 1$

$= 4 \times 4^k - 1$

$= 3 \times 4^k + 4^k - 1$

$= 3 \times 4^k + 3m$

$= 3\left(4^k + m\right)$

As $k, m, 4^k + m \in \mathbb{N}$

$3\left(4^k + m\right) = 3q$

$\therefore 4^{k+} \ - 1$ is divisible by 3

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$P(n): 7^n - 1$ is divisible by 6 $\forall n \geq 1$

i.e. $7^n - 1 = 6m \quad m \in \mathbb{N}$

$P(1)$ LHS $= 7 \ - 1 = 6$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $7^k - 1 = 6m$

$P(k+1): 7^{k+} \ - 1$

$= 7 \times 7^k - 1$

$= 6 \times 7^k + 7^k - 1$

$= 6 \times 7^k + 6m$

$= 6\left(7^k + m\right)$

As $k, m, 7^k + m \in \mathbb{N}$

$6\left(7^k + m\right) = 6q$

$\therefore 7^{k+} \ - 1$ is divisible by 6

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

c

$P(n): 3^{2n}-1$ is divisible by 8 $\forall n \geq 1$

i.e. $3^{2n}-1=8m \quad m \in \mathbb{N}$

$P(1)$ LHS $=3^2-1=8$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $3^{2k}-1=8m$

$P(k+1): 3^{2\;k+1}-1$

$=3^2 \times 3^{2k}-1$

$=9 \times 3^{2k}-1$

$=8 \times 3^{2k}+3^{2k}-1$

$=8 \times 3^{2k}+8m$

$=8\left(3^{2k}+m\right)$

As $k, m, 3^{2k}+m \in \mathbb{N}$

$8\left(3^{2k}+m\right)=8q$

$\therefore 3^{2\;k+1}-1$ is divisible by 8

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 4

a

$P(n): 1^3 + 2^3 + 3^3 + 4^3 + ... + n^3 = \frac{n^2}{4}(n+1)^2$

$P(1)$ LHS $= 1^3 = 1$

RHS $= \frac{1^2}{4}(1+1)^2 = \frac{1}{4} \times 2^2 = 1 =$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $1^3 + 2^3 + 3^3 + 4^3 + ... + k^3 = \frac{k^2}{4}(k+1)^2$

Required to prove

$P(k+1): 1^3 + 2^3 + 3^3 + 4^3 + ... + k^3 + (k+1)^3 = \frac{(k+1)^2}{4}((k+1)+1)^2$

LHS $1^3 + 2^3 + 3^3 + 4^3 + ... + k^3 + (k+1)^3$

$= \frac{k^2}{4}(k+1)^2 + (k+1)^3$

$= (k+1)^2\left(\frac{k^2}{4} + (k+1)\right)$

$= (k+1)^2\left(\frac{k^2}{4} + \frac{4(k+1)}{4}\right)$

$= \frac{(k+1)^2}{4}\left(k^2 + 4(k+1)\right)$

$= \frac{(k+1)^2}{4}\left(k^2 + 4k + 4\right)$

$= \frac{(k+1)^2}{4}(k+2)^2$

$= \frac{(k+1)^2}{4}((k+1)+1)^2$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$P(n): 3 + 3^2 + 3^3 + 3^4 + ... + 3^n = \dfrac{3(3^n - 1)}{2}$

$P(1)$ LHS $= 3$

RHS $= \dfrac{3(3 \ -1)}{2} = \dfrac{3 \times 2}{2} = 3 =$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $3 + 3^2 + 3^3 + 3^4 + ... + 3^k = \dfrac{3(3^k - 1)}{2}$

Required to prove

$P(k+1): 3 + 3^2 + 3^3 + 3^4 + ... + 3^{k-1} + 3^k + 3^{k+1} = \dfrac{3(3^{k+} \ -1)}{2}$

LHS $1 + 3 + 3^2 + 3^3 + 3^4 + ... + 3^{k-1} + 3^k + 3^{k+1}$

$= \dfrac{3(3^k - 1)}{2} + 3^{k+1}$

$= \dfrac{3(3^k - 1)}{2} + \dfrac{2 \times 3^{k+1}}{2}$

$= \dfrac{3^{k+1} - 3 + 2 \times 3^{k+1}}{2}$

$= \dfrac{3^{k+} \ (1+2) - 3}{2}$

$= \dfrac{3(3^{k+} \ -1)}{2}$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

c

$$P(n): \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\quad +\frac{1}{n\times(n+1)}=\frac{n}{n+1}$$

$P(1)$ LHS $=\frac{1}{2}$

RHS $=\frac{1}{2}=$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\quad +\frac{1}{k\times(k+1)}=\frac{k}{k+1}$

Required to prove

$$P(k+1): \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\quad +\frac{1}{k\times(k+1)}+\frac{1}{(k+1)\times((k+1)+1)}=\frac{(k+1)}{((k+1)+1)}$$

LHS $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+....+\frac{1}{k\times(k+1)}+\frac{1}{(k+1)\times((k+1)+1)}$

$=\frac{k}{k+1}+\frac{1}{(k+1)\times(k+2)}$

$=\frac{k(k+2)+1}{(k+1)(k+2)}$

$=\frac{k^2+2k+1}{(k+1)(k+2)}$

$=\frac{(k+1)^2}{(k+1)(k+2)}$

$=\frac{(k+1)}{(k+2)}$

$=\frac{(k+1)}{((k+1)+1)}$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 5

a

$P(n): 9^{n+2} - 4^n$ is divisible by 5 $\forall n \geq 1$

i.e. $9^{n+2} - 4^n = 5m \quad m \in \mathbb{N}$

$P(1)$ LHS $= 9^3 - 4^1 = 725 = 5 \times 145$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $9^{k+2} - 4^k = 5m$

$P(k+1)$: LHS $9^{k+1+2} - 4^{k+1}$

$= 9^{k+3} - 4^{k+1}$

$= 9 \times 9^{k+2} - 4 \times 4^k$

$= 5 \times 9^{k+2} + 4 \times 9^{k+2} - 4 \times 4^k$

$= 5 \times 9^{k+2} + 4\left(9^{k+2} - 4^k\right)$

$= 5 \times 9^{k+2} + 4(5m)$

$= 5\left(9^{k+2} + 4m\right)$

As $k, m, 9^{k+2} + 4m \in \mathbb{N}$

$5\left(9^{k+2} + 4m\right) = 5q$

$\therefore 9^{k+1+2} - 4^{k+1}$ is divisible by 5

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$P(n): n(n+1)$ is divisible by 2 $\forall n \geq 1$

i.e. $n(n+1) = 2m \quad m \in \mathbb{N}$

$P(1)$ LHS $= 1(1+1) = 2$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $k(k+1) = 2m$

$P(k+1)$: LHS $(k+1)((k+1)+1)$

$= (k+1)(k+2)$

$= k(k+1) + 2(k+1)$

$= 2m + 2(k+1)$

$= 2(m+k+1)$

As $k, m, m+k+1 \in \mathbb{N}$

$2(m+k+1) = 2q$

$\therefore (k+1)((k+1)+1)$ is divisible by 2

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

c This is actually a repeat of question **5 a** as $3^2 = 9$.

d

$P(n): n(n+1)(n+2)$ is divisible by 6 $\forall n \geq 1$

i.e. $n(n+1)(n+2) = 6m \quad m \in \mathbb{N}$

$P(1)$ LHS $= 1(1+1)(1+2) = 6$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $k(k+1)(k+2) = 6m$

$P(k+1)$: $(k+1)((k+1)+1)((k+1)+2)$

$(k+1)(k+2)(k+3)$

$= k(k+1)(k+2) + 3(k+1)(k+2)$

$= 6m + 3(k+1)(k+2)$

from **5b** $(k+1)(k+2)$ must be an even integer so we can write

$= 6m + 3 \times 2a$

$= 6(m+a)$

As $a, m, a+m \in \mathbb{N}$

$6(m+a) = 6q$

$\therefore (k+1)((k+1)+1)((k+1)+2)$ is divisible by 6

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

e

$P(n): n^3 + 2n$ is divisible by 3 $\forall n \geq 1$

i.e. $n^3 + 2n = 3m \quad m \in \mathbb{N}$

$P(1)$ LHS $= 1^3 + 2 \times 1 = 3$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $k^3 + 2k = 3m$

$P(k+1)$: LHS $(k+1)^3 + 2(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= k^3 + 2k + 3k^2 + 3k + 3$

$= k^3 + 2k + 3(k^2 + k + 1)$

$= 3m + 3(k^2 + k + 1)$

$= 3(k^2 + k + 1 + m)$

As $k, m, k^2 + k + 1 + m \in \mathbb{N}$

$(k+1)^3 + 2(k+1) = 3q$

$\therefore (k+1)^3 + 2(k+1)$ is divisible by 3

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 6

a

$P(n)\ \ a+(a+d)+(a+2d)+(a+3d)+....+(a+(n-1)d)=\frac{n}{2}(2a+(n-1)d)$

$P(1)$ LHS $= a$

RHS $=\frac{1}{2}(2a+(1-1)d)=\frac{1}{2}2a=a=$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $a+(a+d)+(a+2d)+(a+3d)+....+(a+(k-1)d)=\frac{k}{2}(2a+(k-1)d)$

Required to prove

$$P(k+1): a+(a+d)+(a+2d)+(a+3d)+....+(a+(k-1)d)+(a+((k+1)-1)d)$$
$$=\frac{(k+1)}{2}(2a+((k+1)-1)d)$$

LHS $a+(a+d)+(a+2d)+(a+3d)+....+(a+(k-1)d)+(a+((k+1)-1)d)$

$=\frac{k}{2}(2a+(k-1)d)+(a+kd)$

$=ka+\frac{k(k-1)d}{2}+a+kd$

$=a(k+1)+\frac{k(k-1)d}{2}+\frac{2kd}{2}$

$=a(k+1)+\frac{k^2-k+2k}{2}d$

$=\frac{2a(k+1)}{2}+\frac{k(k+1)}{2}d$

$=\frac{k+1}{2}(2a+kd)$

$=\frac{(k+1)}{2}(2a+((k+1)-1)d)$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$$P(n)\ \ a+ar+ar^2+ar^3+\quad +ar^{n-1}=\frac{a\left(r^n-1\right)}{r-1}$$

$P(1)$ LHS $=a$

$$\text{RHS}=\frac{a\left(r-1\right)}{r-1}=a=\text{LHS}$$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $a+ar+ar^2+ar^3+\quad +ar^{k-1}=\dfrac{a\left(r^k-1\right)}{r-1}$

Required to prove

$$P(k+1): a+ar+ar^2+ar^3+\quad +ar^{k-1}+ar^{\ k+1\ -1}=\frac{a\left(r^{k+1}-1\right)}{r-1}$$

LHS $\ a+ar+ar^2+ar^3+....+ar^{k-}\ +ar^{(k+1\ -1}$

$$=\frac{a\left(r^k-1\right)}{r-1}+ar^k$$

$$=\frac{a\left(r^k-1\right)}{r-1}+\frac{ar^k\left(r-1\right)}{\left(r-1\right)}$$

$$=\frac{ar^k-a+ar^kr-ar^k}{r-1}$$

$$=\frac{ar^{k+1}-a}{r-1}$$

$$=\frac{a\left(r^{k+1}-1\right)}{r-1}$$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Exercise 5.02 Further mathematical induction

Question 1

a

$P(n): 5^n - 1$ is divisible by 8 $\forall$ even $n \geq 2$

i.e $5^n - 1 = 8m \quad m \in \mathbb{N}$

$P(2)$ LHS $= 5^2 - 1 = 24$

$\therefore P(2)$ is true

Let $P(k)$ be true

i.e. $5^k - 1 = 8m$

$P(k+2)$: $5^{k+2} - 1$

$= 5^2 \times 5^k - 1$

$= 24 \times 5^k + 5^k - 1$

$= 8 \times 3 \times 5^k + 8m$

$= 8\left(3 \times 5^k + m\right)$

As $k, m, 3 \times 5^k + m \in \mathbb{N}$

$8\left(3 \times 5^k + m\right) = 8q$

$\therefore 5^{k+2} - 1$ is divisible by 8

$\therefore P(k+2)$ is true

$P(2)$ is true and the truth of $P(k)$ implies the truth of $P(k+2)$

by mathematical induction $P(n)$ is true.

b

$P(n): 3^n - 2^n$ is divisible by 5 $\forall$ even $n \geq 2$

i.e $3^n - 2^n = 5m \quad m \in \mathbb{N}$

$P(2)$ LHS $= 3^2 - 2^2 = 9 - 4 = 5$

$\therefore P(2)$ is true

Let $P(k)$ be true

i.e. $3^k - 2^k = 5m$

$P(k+2):$ LHS $3^{k+2} - 2^{k+2}$

$= 3^2 \times 3^k - 2^2 \times 2^k$

$= 9 \times 3^k - 4 \times 2^k$

$= 5 \times 3^k + 4 \times 3^k - 4 \times 2^k$

$= 5 \times 3^k + 4\left(3^k - 2^k\right)$

$= 5 \times 3^k + 4(5m)$

$= 5\left(3^k + 4m\right)$

As $k, m, 3^k + 4m \in \mathbb{N}$

$5\left(3^k + 4m\right) = 5q$

$\therefore 3^{k+2} - 2^{k+2}$ is divisible by 5

$\therefore P(k+2)$ is true

$P(2)$ is true and the truth of $P(k)$ implies the truth of $P(k+2)$

by mathematical induction $P(n)$ is true.

c

$P(n): x^n - 1$ is divisible by $x^2 - 1$ $\forall$ even $n \geq 2$

i.e $x^n - 1 = (x^2 - 1)Q(x) \quad m \in \mathbb{N}$

$P(2)$ LHS $= x^2 - 1 = (x^2 - 1) \times 1$

$\therefore P(2)$ is true

Let $P(k)$ be true

i.e. $x^k - 1 = (x^2 - 1)Q(x)$

$P(k+2)$: LHS $x^{k+2} - 1$

$= (x^2 + 1 - 1)x^k - 1$

$= (x^2 - 1)x^k + x^k - 1$

$= (x^2 - 1)x^k + (x^2 - 1)Q(x)$

$= (x^2 - 1)(Q(x) + x^k)$

$x^{k+2} - 1 = (x^2 - 1)(Q(x) + x^k)$

$\therefore x^{k+2} - 1$ is divisible by $x^2 - 1$

$\therefore P(k+2)$ is true

$P(2)$ is true and the truth of $P(k)$ implies the truth of $P(k+2)$

by mathematical induction $P(n)$ is true.

Question 2

a

$P(n): 5^n + 2^n$ is divisible by 7 $\forall$ odd $n \geq 1$

i.e $5^n + 2^n = 7m \quad m \in \mathbb{N}$

$P(1)$ LHS $= 5^1 + 2^1 = 7$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $5^k + 2^k = 7m$

$P(k+2)$: LHS $5^{k+2} + 2^{k+2}$

$= 5^2 \times 5^k + 2^2 \times 2^k$

$= 25 \times 5^k + 4 \times 2^k$

$= 21 \times 5^k + 4 \times 5^k + 4 \times 2^k$

$= 7 \times 3 \times 5^k + 4\left(5^k + 2^k\right)$

$= 7 \times 3 \times 5^k + 4(7m)$

$= 7\left(3 \times 5^k + 4m\right)$

As $k, m, 3 \times 5^k + 4m \in \mathbb{N}$

$7\left(3 \times 5^k + 4m\right) = 7q$

$\therefore 5^{k+2} + 2^{k+2}$ is divisible by 7

$\therefore P(k+2)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+2)$

by mathematical induction $P(n)$ is true.

b

$P(n): 6^n + 3^n$ is divisible by 9 $\forall$ odd $n \geq 1$

i.e $6^n + 3^n = 9m \quad m \in \mathbb{N}$

$P(1)$ LHS $= 6^1 + 3^1 = 9$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $6^k + 3^k = 9m$

$P(k+2)$: LHS $6^{k+2} + 3^{k+2}$

$= 6^2 \times 6^k + 3 \times 3^k$

$= 36 \times 6^k + 9 \times 3^k$

$= 27 \times 6^k + 9 \times 6^k + 9 \times 3^k$

$= 9 \times 3 \times 6^k + 9\left(6^k + 3^k\right)$

$= 9\left(3 \times 6^k + 6^k + 3^k\right)$

$= 9\left(3 \times 6^k + 9m\right)$

As $k, m, 3 \times 6^k + 9m \in \mathbb{N}$

$9\left(3 \times 6^k + 9m\right) = 9q$

$\therefore 6^{k+2} + 3^{k+2}$ is divisible by 9

$\therefore P(k+2)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+2)$

by mathematical induction $P(n)$ is true.

c

$P(n): 4^{n-2} + 7^{n-2}$ is divisible by 11 $\forall$ odd $n \geq 3$

i.e. $4^{n-2} + 7^{n-2} = 11m \quad m \in \mathbb{N}$

$P(3)$ LHS$= 4^{3-2} + 7^{3-2} = 11$

$\therefore P(3)$ is true

Let $P(k)$ be true

i.e. $4^{k-2} + 7^{k-2} = 11m$

$P(k+2)$: LHS $4^{k+2-2} + 7^{k+2-2}$

$= 4^2 \times 4^{k-2} + 7^2 \times 7^{k-2}$

$= 16 \times 4^{k-2} + 49 \times 7^{k-2}$

$= 16 \times 4^{k-2} + 16 \times 7^{k-2} + 33 \times 7^{k-2}$

$= 16\left(4^{k-2} + 7^{k-2}\right) + 11 \times 3 \times 7^{k-2}$

$= 16(11m) + 11 \times 3 \times 7^{k-2}$

$= 11\left(16m + 3 \times 7^{k-2}\right)$

As $k, m, 16m + 3 \times 7^{k-2} \in \mathbb{N}$

$11\left(16m + 3 \times 7^{k-2}\right) = 11q$

$\therefore 4^{k+2-2} + 7^{k+2-2}$ is divisible by 11

$\therefore P(k+2)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+2)$

by mathematical induction $P(n)$ is true.

Question 3

a

$P(n): 9^n - 8(n-1) - 9$ is divisible by 64 $\forall n \geq 2$

i.e. $9^n - 8(n-1) - 9 = 64m \quad m \in \mathbb{N}$

$P(2)$ LHS $= 9^2 - 8(2-1) - 9 = 81 - 8 - 9 = 64$

$\therefore P(2)$ is true

Let $P(k)$ be true

i.e. $9^k - 8(k-1) - 9 = 64m$

$P(k+1): 9^{k+} \quad -8((k+1)-1) - 9$

$= 9 \times 9^k - 8(k+1) + 8 - 9$

$= 8 \times 9^k + 9^k - 8k - 8 + 8 - 9$

$= 8 \times 9^k + 9^k - 8(k-1) - 9 - 8$

$= 8 \times 9^k - 8 + 9^k - 8(k-1) - 9$

$= 8 \times 9^k - 8 + 64m$

$= 8(9^k - 1) + 64m$

$= 8\left((9-1)\left(9^{k-1} + 9^{k-2} + 9^{k-3} + ... + 1\right)\right) + 64m$

$= 8\left(8\left(9^{k-1} + 9^{k-2} + 9^{k-3} + ... + 1\right)\right) + 64m$

$= 64\left(9^{k-1} + 9^{k-2} + 9^{k-3} + ... + 1\right) + 64m$

$= 64\left(9^{k-1} + 9^{k-2} + 9^{k-3} + ... + 1 + m\right)$

As $k, m, 9^{k-1} + 9^{k-2} + 9^{k-3} + ... + 1 + m \in \mathbb{N}$

$9^{k+} \quad -8((k+1)-1) - 9 = 64q$

$\therefore 9^{k+} \quad -8((k+1)-1) - 9$ is divisible by 64

$\therefore P(k+1)$ is true

$P(2)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$P(n): 13^{n+} -12n-13$ is divisible by 144 $\forall n \geq 1$

i.e $13^{n+} -12n-13=144m \quad m \in \mathbb{N}$

$P(1)$ LHS $=13^{1+1}-12-13=169-12-13=144$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $13^{k+} -12k-13=144m$

$P(k+1): 13^{k+1+1}-12(k+1)-13$

$=13\times 13^{k+} -12k-12-13$

$=12\times 13^{k+1}+13^{k+1}-12k-12-13$

$=12\times 13^{k+1}+13^{k+1}-12k-13-12$

$=12\times 13^{k+1}-12+13^{k+1}-12k-13$

$=12\times 13^{k+} -12+144m$

$=12(13^{k+} -1)+144m$

$=12((13-1)(13^{k}+13^{k-1}+13^{k-2}+....+1))+144m$

$=12(12(13^{k}+13^{k-1}+13^{k-2}+....+1))+144m$

$=144(13^{k}+13^{k-1}+13^{k-2}+....+1+m)$

As $k, m, 13^{k}+13^{k-} +13^{k-2}+....+1+m \in \mathbb{N}$

$13^{k+1+1}-12(k+1)-13=144q$

$\therefore 13^{k+1+1}-12(k+1)-13$ is divisible by 144

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Exercise 5.03 Series and sigma notation

Question 1

a $$\sum_{r=1}^{10} r^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2$$

b $$\sum_{k=1}^{n} (2k+3) = (2\times1+3) + (2\times2+3) + (2\times3+3) + ... + (2\times n+3)$$

c $$\sum_{n=1}^{M+1} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \quad + \frac{1}{M+1}$$

d

$$\sum_{r=2}^{9} (-1)^{r-1} r = (-1)^{2-1}\times2 + (-1)^{3-1}\times3 + (-1)^{4-1}\times4 + (-1)^{5-1}\times5 + (-1)^{6-1}\times6$$
$$+ (-1)^{7-1}\times7 + (-1)^{8-1}\times8 + (-1)^{9-1}\times9$$

e $$\sum_{r=1}^{\infty} \frac{1}{2^{r-1}} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + ...$$

Question 2

a

$$\sum_{k=1}^{4}(k+2)$$
$$=(1+2)+(2+2)+(3+2)+(4+2)$$
$$=18$$

b

$$\sum_{r=1}^{5}3^{r-1}$$
$$=3^0+3^1+3^2+3^3+3^4$$
$$=1+3+9+27+81$$
$$=121$$

c

$$\sum_{j=1}^{3}j(j+1)$$
$$=1\times(1+1)+2\times(2+1)+3\times(3+1)$$
$$=2+6+12$$
$$=20$$

d

$$\sum_{k=3}^{8}\frac{(-1)^{k-1}}{k^2}$$
$$=\frac{(-1)^{3-1}}{3^2}+\frac{(-1)^{3-1}}{3^2}+\frac{(-1)^{3-1}}{3^2}+\frac{(-1)^{3-1}}{3^2}+\frac{(-1)^{3-1}}{3^2}+\frac{(-1)^{3-1}}{3^2}$$
$$=\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\frac{1}{49}-\frac{1}{64}$$
$$=\frac{15433}{235200}$$
$$\approx 0.065616$$

Question 3

a

$$-1^2 + 2^2 - 3^2 + 4^2 - 5^2 + ... - 77^2$$

$$\sum_{r=1}^{77} (-1)^r r^2 =$$

b

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \quad + \frac{1}{n}$$

$$= \sum_{r=2}^{n} \frac{1}{r}$$

c

$$3^1 + 3^2 + 3^3 + 3^4 + ... + 3^{99}$$

$$= \sum_{r=1}^{99} 3^r$$

d

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + ...$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{2^r}$$

Question 4

a

$P(n): \sum_{r=1}^{n} 3r-2 = \frac{n}{2}(3n-1)$

$P(1)$ LHS $3-2=1$

RHS $= \frac{1}{2}(3-1) = \frac{1}{2}\times 2 = 1 =$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\sum_{r=1}^{k} 3r-2 = \frac{k}{2}(3k-1)$

Required to prove

$P(k+1): \sum_{r=1}^{k+1} 3r-2 = \frac{(k+1)}{2}(3(k+1)-1)$

$P(k+1)$ LHS $\sum_{r=1}^{k+1} 3r-2$

$= \sum_{r=1}^{k} 3r-2+(3(k+1)-2)$

$= \frac{k}{2}(3k-1)+3(k+1)-2$

$= \frac{3k^2-k}{2}+\frac{6k+6}{2}-\frac{4}{2}$

$= \frac{3k^2-k+6k+6-4}{2}$

$= \frac{3k^2+5k+2}{2}$

$= \frac{3k^2+3k+2k+2}{2}$

$= \frac{3k(k+1)+2(k+1)}{2}$

$= \frac{(k+1)(3k+2)}{2}$

$= \frac{(k+1)}{2}(3(k+1)-1)$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$P(n): \sum_{r=1}^{n} 6^{r-1} = \dfrac{6^n - 1}{5}$

$P(1)$ LHS $6^{1-1} = 1$

$\text{RHS} = \dfrac{6-1}{5} = \dfrac{5}{5} = 1 = \text{LHS}$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\sum_{r=1}^{k} 6^{r-1} = \dfrac{6^k - 1}{5}$

Required to prove

$P(k+1): \sum_{r=1}^{k+1} 6^{r-} = \dfrac{6^{k+} - 1}{5}$

$P(k+1)$ LHS $\sum_{r=1}^{k+1} 6^{r-1}$

$= \sum_{r=1}^{k} 6^{r-1} + 6^{k+1-1}$

$= \dfrac{6^k - 1}{5} + 6^k$

$= \dfrac{6^k - 1}{5} + \dfrac{5 \times 6^k}{5}$

$= \dfrac{6^k + 5 \times 6^k - 1}{5}$

$= \dfrac{6^k (1+5) - 1}{5}$

$= \dfrac{6^{k+} - 1}{5}$

$= \text{RHS}$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

c

$P(n): \sum_{r=1}^{n} 2r-1=n^2$

$P(1)$ LHS $2-1=1$

RHS $=1^2=1=$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\sum_{r=1}^{k} 2r-1=k^2$

Required to prove

$P(k+1): \sum_{r=1}^{k+1} 2r-1=(k+1)^2$

$P(k+1)$ LHS $\sum_{r=1}^{k+1} 2r-1$

$=\sum_{r=1}^{k} 2r-1+2(k+1)-1$

$=k^2+2k+1$

$=(k+1)^2$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

d

$P(n): \sum_{n=1}^{N} \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{N-1}}$

$P(1)$ LHS $\frac{1}{2^{1-1}} = 1$

RHS $= 2 - \frac{1}{2^{1-1}} = 2 - 1 = 1 = \text{LHS}$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\sum_{n=1}^{k} \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{k-1}}$

Required to prove

$P(k+1): \sum_{n=1}^{k+1} \frac{1}{2^{n-}} = 2 - \frac{1}{2^{(k+1\ -1}}$

$P(k+1)$ LHS $\sum_{n=1}^{k+1} \frac{1}{2^{n-1}}$

$= \sum_{n=1}^{k} \frac{1}{2^{n-}} + \frac{1}{2^{(k+1\ -1}}$

$= 2 - \frac{1}{2^{k-}} + \frac{1}{2^{k}}$

$= 2 - \frac{2}{2^{k}} + \frac{1}{2^{k}}$

$= 2 - \frac{1}{2^{k}}$

$= 2 - \frac{1}{2^{(k+1\ -1}}$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

e

$$P(n): \sum_{k=1}^{n}(2k-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

$P(1)$ LHS $(2-1)^2 = 1$

RHS = $\frac{1(2-1)(2+1)}{3} = \frac{3}{3} = 1 = \text{LHS}$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\sum_{k=1}^{k}(2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$

Required to prove

$$P(k+1): \sum_{k=1}^{k+1}(2k-1)^2 = \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

$P(k+1)$ LHS $\sum_{k=1}^{k+1}(2k-1)^2$

$= \sum_{k=1}^{k}(2k-1)^2 + (2(k+1)-1)^2$

$= \frac{k(2k-1)(2k+1)}{3} + \frac{3(2k+1)^2}{3}$

$= \frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}$

$= \frac{(2k+1)(k(2k-1)+3(2k+1))}{3}$

$= \frac{(2(k+1)-1)(2k^2-k+6k+3)}{3}$

$= \frac{(2(k+1)-1)(2k^2+5k+3)}{3}$

$= \frac{(2(k+1)-1)(2k+3)(k+1)}{3}$

$= \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$

= RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 5

$P(n): \sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$

$P(1)$ LHS $\frac{1}{(1+1)!} = \frac{1}{2}$

$\text{RHS} = 1 - \frac{1}{(1+1)!} = 1 - \frac{1}{2} = \frac{1}{2} = \text{LHS}$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\sum_{r=1}^{k} \frac{r}{(r+1)!} = 1 - \frac{1}{(k+1)!}$

Required to prove

$P(k+1): \sum_{r=1}^{k+1} \frac{r}{(r+1)!} = 1 - \frac{1}{((k+1)+1)!}$

$P(k+1)$ LHS $\sum_{r=1}^{k+1} \frac{r}{(r+1)!}$

$= \sum_{r=1}^{k} \frac{r}{(r+1)!} + \frac{k+1}{(k+1+1)!}$

$= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$

$= 1 - \frac{k+2}{(k+2)!} + \frac{k+1}{(k+2)!}$

$= 1 - \frac{1}{(k+2)!}$

$= 1 - \frac{1}{((k+1)+1)!}$

$= \text{RHS}$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 6

$P(n): \sum_{r=1}^{n} \log\left(\frac{r+1}{r}\right) = \log(n+1)$

$P(1)$ LHS $\log\left(\frac{1+1}{1}\right) = \log 2$

$\text{RHS} = \log(1+1) = \log 2 = \text{LHS}$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\sum_{r=1}^{k} \log\left(\frac{r+1}{r}\right) = \log(k+1)$

Required to prove

$P(k+1): \sum_{r=1}^{k+1} \log\left(\frac{r+1}{r}\right) = \log((k+1)+1)$

$P(k+1)$ LHS $\sum_{r=1}^{k+1} \log\left(\frac{r+1}{r}\right)$

$= \sum_{r=1}^{k} \log\left(\frac{r+1}{r}\right) + \log\left(\frac{k+1+1}{k+1}\right)$

$= \log(k+1) + \log\left(\frac{k+2}{k+1}\right)$

$= \log\left((k+1)\frac{k+2}{k+1}\right)$

$= \log(k+2)$

$= \log((k+1)+1)$

$= \text{RHS}$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Exercise 5.04 Applications of mathematical induction

Question 1

a

$P(n): 3^n \ge 1+2n \quad n \in \mathbb{N}$

$P(1) \;\; 3 \;\; = 3 \ge 1+2\times 1 = 3$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\;\; 3^k \ge 1+2k$

Required to prove

$P(k+1): 3^{k+} \;\; \ge 1+2(k+1)$

$$
\begin{aligned}
\text{LHS} &= 3^{k+1} \\
&= 3\times 3^k \\
&\ge 3\times(1+2k) \\
&= 3+6k \\
&= 2k+2+1+4k \\
&= 2(k+1)+1+4k \\
&> 2(k+1)+1
\end{aligned}
$$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

By mathematical indution $P(n)$ is true.

b

$P(n): 2^n \geq 1+n \quad n \in \mathbb{N}$

$P(1)\ \ 2\ = 2 \geq 1+1 = 2$

$\therefore P(2)$ is true

Assume $P(k)$ is true

i.e. $2^k \geq 1+k$

Required to prove

$P(k+1): 2^{k+}\ \geq 1+k+1 = k+2$

$\text{LHS} = 2^{k+1}$

$= 2 \times 2^k$

$\geq 2 \times (1+k)$

$= 2+2k$

$= 2k+2$

$> k+2$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

By mathematical induction $P(n)$ is true.

c

$P(n)\ 3^{n} > n^3 \qquad n \in \mathbb{N}, n \geq 4$

$P(4)\ 3^4 = 81 > 3^3 = 27$

$\therefore P(4)$ is true

Assume $P(k)$ is true

i.e. $3^k > k^3$

Required to prove

$P(k+1): 3^{k+} > (k+1)^3$

$\text{LHS} = 3^{k+1}$

$= 3 \times 3^k$

$> 3 \times k^3$

$= k^3 + k^3 + k^3$

As $k \geq 4, \quad k^3 > 3k^2,\ k^3 > 3k+1$

$\therefore k^3 + k^3 + k^3 > k^3 + 3k^2 + 3k + 1$

$= (k+1)^3$

$\therefore P(k+1)$ is true

$P(4)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

By mathematical indution $P(n)$ is true.

d

$P(n):(1+y)^n \geq 1+ny \quad n \in \mathbb{N}, y \in \mathbb{R}, y > -1$

$P(1) \ (1+y) \ =1+y$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $(1+y)^k \geq 1+ky$

Requiuired to prove

$P(k+1):(1+y)^{k+} \ \geq 1+(k+1)y$

LHS $(1+y)^{k+1}$

$=(1+y)(1+y)^k$

$\geq (1+y)(1+ky)$

$=1+y+ky+ky^2$

As $k, y^2, ky^2 \geq 0$

$1+y+ky+ky^2$

$\geq 1+(k+1)y$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

e

$P(n): n! > 2^n \quad n \in \mathbb{N}, n \geq 4$

$P(4) = 4! = 24 \geq 2^4 = 16$

$\therefore P(4)$ is true

Assume $P(k)$ is true

i.e. $k! > 2^k$

Required to prove:

$P(k+1): (k+1)! > 2^{k+1}$

$\text{LHS} = (k+1)!$

$= k! \times (k+1)$

$> 2^k \times (k+1)$

$> (1+1) \times 2^k \quad$ as $k \geq 4$

$= 2 \times 2^k$

$= 2^{k+1}$

$\therefore P(k+1)$ is true

$P(4)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

By mathematical indution $P(n)$ is true.

Question 2

a

Let $y = x^M$

$P(n): \dfrac{d^n y}{dx^n} = \dfrac{M \;\; x^{M-n}}{(M-n)!} \quad n, M \in \mathbb{N}, M \geq n$

$P(1) \;\; \dfrac{d\left(x^M\right)}{dx} = Mx^{M-1}$

$= \dfrac{M \;\; x^{M-1}}{(M-1)!}$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\dfrac{d^k y}{dx^k} = \dfrac{M \;\; x^{M-k}}{(M-k)!}$

Required to prove

$P(k+1): \dfrac{d^{k+1} y}{dx^{k+}} = \dfrac{M \;\; x^{M-k+1}}{(M-k+1)!}$

LHS $\dfrac{d^{k+} y}{dx^{k+}} = \dfrac{d\left(\dfrac{M \;\; x^{M-k}}{(M-k)}\right)}{dx}$

$= \dfrac{M \;\; (M-k) x^{M-k-1}}{(M-k)!}$

$= \dfrac{M \;\; (M-k) x^{M- \;\; k+1)}}{(M-k)(M-k-1)!}$

$= \dfrac{M \;\; x^{M- \;\; k+1)}}{(M-(k+1))!}$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

Let $y = \dfrac{1}{x}$

$P(n): \dfrac{d^n y}{dx^n} = \dfrac{(-1)^n n!}{x^{n+1}} \quad n \in \mathbb{N}$

$P(1) \ \dfrac{d\left(\dfrac{1}{x}\right)}{dx} = -\dfrac{1}{x^2}$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\dfrac{d^k y}{dx^k} = \dfrac{(-1)^k k!}{x^{k+1}}$

Required to prove

$P(k+1): \dfrac{d^{k+} y}{dx^{k+}} = \dfrac{(-1)^{k+} (k+1)!}{x^{k+1+1}}$

LHS $\dfrac{d^{k+} y}{dx^{k+}} = \dfrac{d\left(\dfrac{(-1)^k k!}{x^{k+}}\right)}{dx}$

$= -(k+1)\dfrac{(-1)^k k!}{x^{k+1+1}}$

$= \dfrac{(-1)^{k+} (k+1)k!}{x^{k+1+1}}$

$= \dfrac{(-1)^{k+} (k+1)!}{x^{k+1+1}}$

= RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 3

a

$$P(n) \quad (x+a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r$$

$P(1)$ LHS $= x+a$

$$\text{RHS} = \sum_{r=0}^{1} {}^1C_r x^{1-r} a^r \;= x+a = \text{LHS}$$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $(x+a)^k = \sum_{r=0}^{k} {}^kC_r x^{k-r} a^r$

Required to prove

$$P(k+1):(x+a)^{k+} \;= \sum_{r=0}^{k+1} {}^{k+1}C_r x^{k+1-r} a^r$$

$P(k+1)$ LHS $(x+a)^{k+1}$

$$= (x+a)(x+a)^k$$

$$= x(x+a)^k + a(x+a)^k$$

$$= x\sum_{r=0}^{k} {}^kC_r x^{k-r} a^r + a\sum_{r=0}^{k} {}^kC_r x^{k-r} a^r$$

$$= \sum_{r=0}^{k} {}^kC_r x^{k+1-r} a^r + \sum_{r=0}^{k} {}^kC_r x^{k-r} a^{r+1}$$

Collecting powers of x

$$= {}^kC_0 x^{k+1-0} a^0 + \left({}^kC_1 + {}^kC_0\right) x^{k+1-1} a^1 + \left({}^kC_2 + {}^kC_1\right) x^{k+1-2} a^2 + \ldots + \left(0 + {}^kC_k\right) x^{k+1-k-1} a^{k+1}$$

$$= {}^{k+1}C_0 x^{k+1-0} a^0 + {}^{k+1}C_1 x^{k+1-1} a^1 + {}^{k+1}C_2 x^{k+1-2} a^2 + \quad + {}^{k+}C_{k+1} x^{k+\;-(k+1)} a^{k+1}$$

$$\sum_{r=0}^{k+1} {}^{k+1}C_r x^{k+1-r} a^r$$

$= \text{RHS}$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$$P(n)\ \ n(x+a)^{n-}\ \ =\sum_{r=0}^{n} r\times {}^{n}C_r x^{r-}\ a^{n-r}$$

$$P(1):$$

$$\text{LHS}=1(x+a)^{0}=1$$

$$\text{RHS}=\sum_{r=0}^{1} r\times {}^{1}C_r x^{r-1}a^{1-r}$$

$$=0+1x^{1-1}a^{1-1}$$

$$=1$$

$$=\text{LHS}$$

$\therefore P(1)$ is true

Assume $P(k)$ is true:

$$k(x+a)^{k-}\ \ =\sum_{r=0}^{k} r\, {}^{k}C_r x^{r-}\ a^{k-r}$$

Required to prove true for $P(k+1)$:

$$(k+1)(x+a)^{k}=\sum_{r=0}^{k+1} r\, {}^{k+1}C_r x^{r-1}a^{k+1-r}$$

$$=0+{}^{k+1}C_1x^{1-1}a^{k+1-1}+2\,{}^{k+1}C_2x^{2-1}a^{k+1-2}+3\,{}^{k+1}C_3x^{3-1}a^{k+1-3}+\quad k\,{}^{k+1}C_kx^{k-1}a^{k+1-k}$$

$$+(k+1)\,{}^{k+1}C_{k+}\ x^{k+1-1}a^{k+1-\ k+1)}$$

$$={}^{k+1}C_1a^{k}+2\,{}^{k+1}C_2xa^{k-1}+3\,{}^{k+1}C_3x^{2}a^{k-2}+\quad +k\,{}^{k+1}C_kx^{k-1}a+(k+1)x^{k}$$

$$
\begin{aligned}
\text{LHS} &= (k+1)(x+a)^k \\
&= k(x+a)^k + (x+a)^k \\
&= (x+a)\left[k(x+a)^{k-}\quad\right] + (x+a)^k \\
&= (x+a)\sum_{r=0}^{k} r\,{}^kC_r x^{r-}\, a^{k-r} + (x+a)^k \text{ from assumption} \\
&= (x+a)\left[0 + {}^kC_1 x^{1-1}a^{k-1} + 2\,{}^kC_2 x^{2-1}a^{k-2} + 3\,{}^kC_3 x^{3-1}a^{k-3} + \quad + k\,{}^kC_k x^{k-}\, a^{k-k}\right] + (x+a)^k \\
&= (x+a)\left[{}^kC_1 a^{k-} \; + 2\,{}^kC_2 xa^{k-2} + 3\,{}^kC_3 x^2a^{k-3} + \ldots + k\,{}^kC_k x^{k-1}\right] + (x+a)^k \\
&= x\left[{}^kC_1 a^{k-} \; + 2\,{}^kC_2 xa^{k-2} + 3\,{}^kC_3 x^2a^{k-3} + \ldots + k\,{}^kC_k x^{k-}\quad\right] \\
&\qquad + a\left[{}^kC_1 a^{k-} \; + 2\,{}^kC_2 xa^{k-2} + 3\,{}^kC_3 x^2a^{k-3} + \ldots + k\,{}^kC_k x^{k-}\quad\right] + (x+a)^k \\
&= \left[{}^kC_1 xa^{k-1} + 2\,{}^kC_2 x^2a^{k-2} + 3\,{}^kC_3 x^3a^{k-3} + \ldots + k\,{}^kC_k x^k\right] \\
&\qquad + \left[{}^kC_1 a^k + 2\,{}^kC_2 xa^{k-1} + 3\,{}^kC_3 x^2a^{k-2} + \ldots + k\,{}^kC_k x^{k-}\, a\right] \\
&\qquad + {}^kC_0 x^0a^k + {}^kC_1 x^1a^{k-1} + {}^kC_2 x^2a^{k-2} + \quad + {}^kC_{k-}\, x^{k-1}a^1 + {}^kC_k x^ka^0 \\
&= \left({}^kC_0 + {}^kC_1\right)a^k + \left(2\,{}^kC_1 + 2\,{}^kC_2\right)xa^{k-} \; + \left(3\,{}^kC_2 + 3\,{}^kC_3\right)x^2a^{k-2} + \quad + \left(k\,{}^kC_{k-} \; + k\,{}^kC_k\right)x^{k-}\, a + \left([k+1]\,{}^kC_k\right)x^k \\
&= \left({}^kC_0 + {}^kC_1\right)a^k + 2\left({}^kC_1 + {}^kC_2\right)xa^{k-} \; + 3\left({}^kC_2 + {}^kC_3\right)x^2a^{k-2} + \quad + k\left({}^kC_{k-} \; + {}^kC_k\right)x^{k-}\, a + (k+1)\,{}^kC_k x^k \\
&= \left({}^{k+}\,C\;\right)a^k + \left(2\,{}^{k+1}C_2\right)xa^{k-1} + \left(3\,{}^{k+1}C_3\right)x^2a^{k-2} + \quad + \left(k\,{}^{k+1}C_k\right)x^{k-1}a + (k+1)x^k \text{ by Pacal's triangle identity} \\
&= \text{RHS}
\end{aligned}
$$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true

Question 4

a

$P(n)\ \sin\ (n\pi+\theta)=(-1)^n\sin\theta \quad n\in\mathbb{N}$

$P(1)\ \sin(\pi+\theta)=-\sin\theta$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\sin(k\pi+\theta)=(-1)^k\sin\theta$

Required to prove

$P(k+1):\sin((k+1)\pi+\theta)=(-1)^{k+}\ \sin\theta$

LHS $\sin((k+1)\pi+\theta)$

$=\sin(\pi+(k\pi+\theta))$

$=\sin\pi\cos(k\pi+\theta)+\cos\pi\sin(k\pi+\theta)$

$=0+(-1)\sin(k\pi+\theta)$

$=0+(-1)(-1)^k\sin\theta$

$=(-1)^{k+}\ \sin\theta$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$P(n)\ \cos\ (n\pi-\theta)=(-1)^n\cos\theta \quad n\in\mathbb{N}$

$P(1)$

$\text{LHS}=\cos(\pi-\theta)=-\cos\theta$

$\text{RHS}=(-1)\ \cos\theta=-\cos\theta$

$\therefore P(1)$ is true

Assume $P(k)$ is true

i.e. $\cos(k\pi+\theta)=(-1)^k\cos\theta$

Required to prove

$P(k+1):\cos((k+1)\pi+\theta)=(-1)^{k+}\ \ \cos\theta$

$$\begin{aligned}
\text{LHS} &= \cos((k+1)\pi+\theta) \\
&= \cos(k\pi+(\pi+\theta)) \\
&= \cos k\pi\cos(\pi+\theta)-\sin k\pi\sin(\pi+\theta) \\
&= \cos k\pi[-\cos\theta]-0[-\sin\theta] \\
&= -\cos k\pi\cos\theta \\
&= -(-1)^k\cos\theta \\
&= (-1)^{k+}\ \cos\theta
\end{aligned}$$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

By mathematical indution $P(n)$ is true.

Question 5

$P(n): S_n = (n-2)\times 180° \quad n \in \mathbb{N}, n \geq 3$

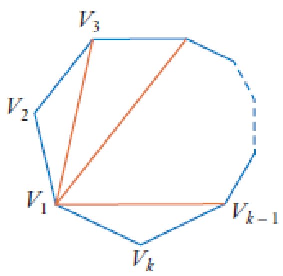
$P(3)$ LHS $S_3 = 180°$

RHS $(3-2)\times 180° = 180° = \text{LHS}$

$\therefore P(3)$ is true

Let $P(k)$ be true

i.e. $S_k = (k-2)\times 180°$



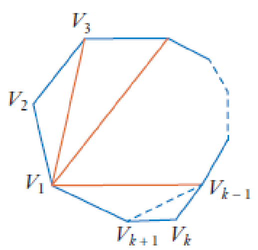
We see from the diagram that there are k vertices labelled V_1, V_2, V_3, … V_{k-1}, V_k.

Joining the diagonals from V_1 we create $k - 2$ triangles, with a total interior angle sum of $(k - 2) \times 180°$.

Required to prove

$P(k + 1)$: $S_{k+1} = ((k + 1) - 2) \times 180°$

Consider the diagram with $(k + 1)$ vertices.



By adding an extra vertex it has created an extra triangle

$\therefore \text{LHS} = S_{k+1}$

$= S_k + 180$

$= (k-2)\times 180 + 180 \qquad$ using S_k

$= ((k+1)-2)\times 180 = \text{RHS}$

$\therefore P(k+1)$ is true

$P(3)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 6

Formula for maximum number of slices for a pizza

$$S_n = \frac{n(n+1)}{2} + 1 \quad n \in \mathbb{N}$$

$$P(n): S_n = \frac{n(n+1)}{2} + 1 \quad n \in \mathbb{N}$$

$P(1)$ LHS $S = 2$

RHS $\frac{1(1+1)}{2} + 1 = 2 = \text{LHS}$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $S_k = \frac{k(k+1)}{2} + 1$

Re*quired to prove*

$$P(k+1): S_{k+} = \frac{(k+1)((k+1)+1)}{2} + 1$$

The *n*th cut will create *n* extra pieces

$$S_{k+1)} = \frac{(k+1)((k+1)+1)}{2} + 1$$

$$= \frac{k(k+1)}{2} + 1 + k + 1$$

$$= \frac{k(k+1)}{2} + \frac{2+2k}{2} + 1$$

$$= \frac{k(k+1) + 2(k+1)}{2} + 1$$

$$= \frac{(k+1)(k+2)}{2} + 1$$

$$= \frac{(k+1)((k+1)+1)}{2} + 1$$

$= \text{RHS}$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 7

$P(n)$ $\left|z_1 + z_2 + z_3 + z_4 + \quad + z_n\right| \le \left|z_1\right| + \left|z_2\right| + \left|z_3\right| + \left|z_4\right| + \quad + \left|z_n\right| \qquad n \in \mathbb{N}$

$P(2)$ $\left|z_1 + z_2\right| \le \left|z_1\right| + \left|z_2\right|$, given in question

$\therefore P(2)$ is true

Assume $P(k)$ is true

i.e. $\left|z_1 + z_2 + z_3 + z_4 + \quad + z_k\right| \le \left|z_1\right| + \left|z_2\right| + \left|z_3\right| + \left|z_4\right| + \quad + \left|z_k\right|$

Required to prove

$P(k+1)$

i.e. $\left|z_1 + z_2 + z_3 + z_4 + \quad + z_{k+1}\right| \le \left|z_1\right| + \left|z_2\right| + \left|z_3\right| + \left|z_4\right| + ... + \left|z_{k+1}\right|$

$$\begin{aligned}
\text{LHS} &= \left|z_1 + z_2 + z_3 + z_4 + \quad + z_k + z_{k+1}\right| \\
&\le \left|z_1 + z_2 + z_3 + z_4 + \quad + z_k\right| + \left|z_{k+1}\right| \\
&\le \left|z_1\right| + \left|z_2\right| + \left|z_3\right| + \left|z_4\right| + \quad + \left|z_{k+1}\right|
\end{aligned}$$

$\therefore P(k+1)$ is true

$P(2)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

By mathematical indution $P(n)$ is true.

Exercise 5.05 Recursive formula proofs

Question 1

a

$T_1 = 2, T_n = T_{n-1} + 2$

$T_n = 2n, \quad n \in \mathbb{N}$

$P(n): T_n = 2n$

$P(1)$ LHS $= 2$

RHS $= 2 \times 1 = 2 =$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $T_k = 2k$

Required to prove

$P(k+1): T_{k+} = 2(k+1)$

LHS T_{k+1}

$= T_k + 2$

$= 2k + 2$

$= 2(k+1)$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$T_1 = 2, T_n = 2T_{n-1}$

$T_n = 2^n, \quad n \in \mathbb{N}$

$P(n): T_n = 2^n$

$P(1)$ LHS $= 2$

RHS $= 2 \quad = 2 =$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $T_k = 2^k$

Required to prove

$P(k+1): T_{k+} \ = 2^{(k+1)}$

LHS T_{k+1}

$= 2T_k$

$= 2 \times 2^k$

$= 2^{(k+1)}$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

c

$T_1 = 1, T_n = T_{n-1} + 5$

$T_n = 5n - 4, \quad n \in \mathbb{N}$

$P(n): T_n = 5n - 4$

$P(1)$ LHS $= 1$

RHS $= 5 \times 1 - 4 = 1 =$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $T_k = 5k - 4$

Required to prove

$P(k+1): T_{k+} = 5(k+1) - 4$

LHS T_{k+1}

$= T_k + 5$

$= 5k - 4 + 5$

$= 5k + 5 - 4$

$= 5(k+1) - 4$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

d

$T_1 = 7, T_n = 3T_{n-1}$

$T_n = 7 \times 3^{n-}$, $n \in \mathbb{N}$

$P(n): T_n = 7 \times 3^{n-1}$

$P(1)$ LHS $= 7$

RHS $= 7 \times 3^{1-1} = 7 =$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $T_k = 7 \times 3^{k-1}$

Required to prove

$P(k+1): T_{k+} = 7 \times 3^{(k+1-1}$

LHS T_{k+1}

$= 3T_k$

$= 3 \times 7 \times 3^{k-1}$

$= 7 \times 3^{k-1+1}$

$= 7 \times 3^{(k+1-1}$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 2

a

$T_1 = 5, T_2 = 7, T_n = 3T_{n-1} - 2T_{n-2}$

$T_n = 2^n + 3, \quad n \in \mathbb{N}, n \geq 3$

$P(n): T_n = 2^n + 3$

$P(3)$ $\text{LHS} = 3T_{3-1} - 2T_{3-2} = 3 \times T_2 - 2 \times T_1 = 3 \times 7 - 2 \times 5 = 21 - 10 = 11$

$\text{RHS} = 2^3 + 3 = 8 + 3 = 11 = \text{LHS}$

$\therefore P(3)$ is true

Let $P(k)$ be true

i.e. $T_k = 2^k + 3$

This also implies $P(k-1)$ is true

i.e. $T_{k-} = 2^{k-} + 3$

Required to prove

$P(k+1): T_{k+} = 2^{k+} + 3$

LHS T_{k+1}

$= 3T_{k+1-1} - 2T_{k+1-2}$

$= 3T_k - 2T_{k-1}$

$= 3\left(2^k + 3\right) - 2\left(2^{k-} + 3\right)$

$= 3 \times 2^k + 9 - 2 \times 2^{k-} - 6$

$= 3 \times 2^k + 3 - 2^k$

$= 2 \times 2^k + 3$

$= 2^{k+} + 3$

$= \text{RHS}$

$\therefore P(k+1)$ is true

$P(3)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$T_1 = 2, T_2 = 16, T_n = 8T_{n-1} - 15T_{n-2}$

$T_n = 5^n - 3^n,\ \ n \in \mathbb{N}, n \geq 3$

$P(n): T_n = 5^n - 3^n$

$P(3)$ LHS $= 8T_{3-1} - 15T_{3-2} = 8 \times T_2 - 15 \times T_1 = 8 \times 16 - 15 \times 2 = 128 - 30 = 98$

RHS $= 5^3 - 3^3 = 125 - 27 = 98 =$ LHS

$\therefore P(3)$ is true

Let $P(k)$ be true

i.e. $T_k = 5^k - 3^k$

This also implies $P(k-1)$ is true

i.e. $T_{k-1} = 5^{k-1} - 3^{k-1}$

Required to prove

$P(k+1): T_{k+} = 5^{k+1} - 3^{k+1}$

LHS T_{k+1}

$= 8T_{k+1-1} - 15T_{k+1-2}$

$= 8T_k - 15T_{k-1}$

$= 8(5^k - 3^k) - 15(5^{k-1} - 3^{k-1})$

$= 8 \times 5^k - 8 \times 3^k - 15 \times 5^{k-1} - 15 \times 3^{k-1}$

$= 8 \times 5^k - 3 \times 5 \times 5^{k-1} - 8 \times 3^k - 3 \times 5 \times 3^{k-1}$

$= 8 \times 5^k - 3 \times 5^k - 8 \times 3^k - 5 \times 3^k$

$= 5 \times 5^k - 3 \times 3^k$

$= 5^{k+1} - 3^{k+1}$

$=$ RHS

$\therefore P(k+1)$ is true

$P(3)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

c

$T_1 = 1, T_n = T_{n-1} + 2n - 1$

$T_n = n^2, \quad n \in \mathbb{N}, n \geq 2$

$P(n) \;\; T_n = n^2$

$P(2)$ LHS $= T_{2-1} + 2 \times 2 - 1 = T + 4 - 1 = 1 + 3 = 4$

RHS = $2^2 = 4 =$ LHS

$\therefore P(2)$ is true

Let $P(k)$ be true

i.e. $T_k = k^2$

Required to prove

$P(k+1) : T_{k+} = (k+1)^2$

LHS $\;T_{k+1}$

$= T_k + 2(k+1) - 1$

$= k^2 + 2k + 2 - 1$

$= k^2 + 2k + 1$

$= (k+1)^2$

= RHS

$\therefore P(k+1)$ is true

$P(2)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

d

$T_1 = 1$ $T_n = T_{n-1} + (n-1)(n-1)!$

$T_n = n!, \quad n \in \mathbb{N}, n \geq 2$

$P(n): T_n = n!$

$P(2)$ $\text{LHS} = T_{2-1} + (2-1)(2-1)! = T \;\; + (1)(1) = 1 + 1 = 2$

$\text{RHS} = 2! = 2 = \text{LHS}$

$\therefore P(2)$ is true

Let $P(k)$ be true

i.e. $T_k = k!$

Required to prove

$P(k+1): T_{k+} \;\; = (k+1)!$

LHS T_{k+1}

$= T_k + ((k+1)-1)((k+1)-1)!$

$= k! + (k)(k)!$

$= k!(k+1)$

$= (k+1)!$

$=$ RHS

$\therefore P(k+1)$ is true

$P(2)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 3

$T_1 = 1, T_2 = 1, T_n = T_{n-1} + T_{n-2}$

$$T_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}, \quad n \in \mathbb{N}, n \geq 3$$

$P(3)$ LHS $= T_{3-1} + T_{3-2} = T_2 + T_1 = 1 + 1 = 2$

$$\text{RHS} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^3}{\sqrt{5}}$$

$$= \frac{\frac{1+3\sqrt{5}+15+5\sqrt{5}}{8} - \frac{1-3\sqrt{5}+15-5\sqrt{5}}{8}}{\sqrt{5}}$$

$$= \frac{\frac{6\sqrt{5}+10\sqrt{5}}{8}}{\sqrt{5}} = \frac{2\sqrt{5}}{\sqrt{5}} = 2 = \text{LHS}$$

$\therefore P(3)$ is true

Let $P(k)$ be true

i.e. $$T_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}}$$

This also implies $P(k-1)$ is true

i.e $$T_{k-1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

Required to prove

$$P(k+1):T_{k+} \;=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}$$

LHS $T_{k+} \;=T_{k+1\;-1}+T_{k+1\;-2}$

$=T_k+T_{k-1}$

$$=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k}-\left(\frac{1-\sqrt{5}}{2}\right)^{k}}{\sqrt{5}}+\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$=\frac{\left(1+\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}-\left(1+\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$=\frac{\left(\frac{2+1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}-\left(\frac{2+1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$=\frac{\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}-\left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$=\frac{\left(\frac{6+2\sqrt{5}}{4}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}-\left(\frac{6-2\sqrt{5}}{4}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2}\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{2}\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}$$

= RHS

$\therefore P(k+1)$ is true

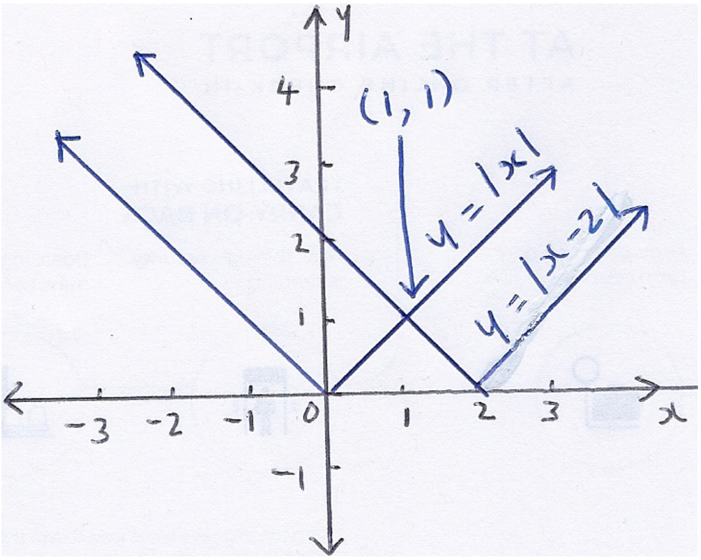
$P(3)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 1

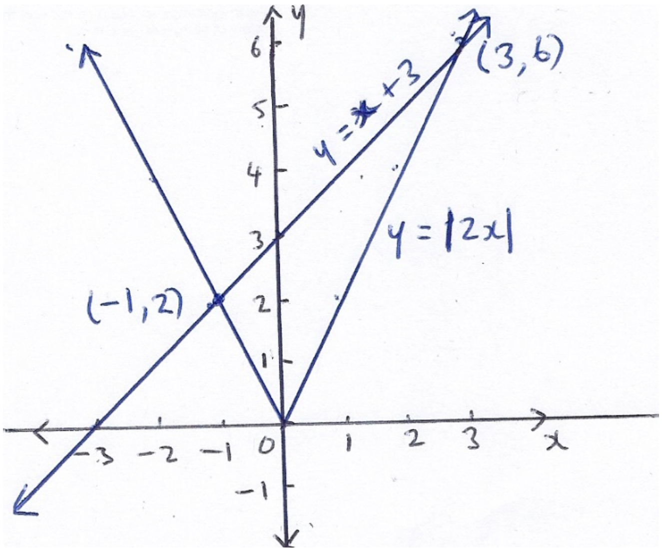
a

$$|x-2|>|x|$$
$$x<1$$



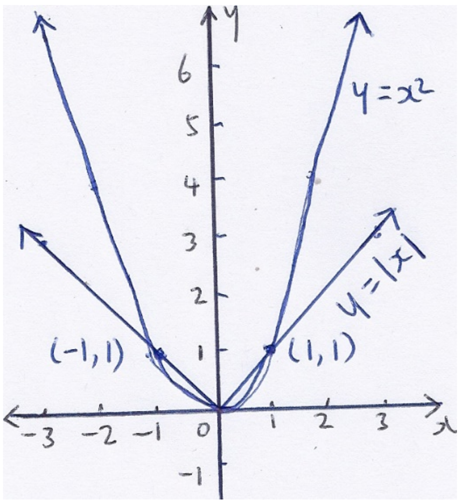
b

$$|2x|\le x+3$$
$$-1\le x\le 3$$



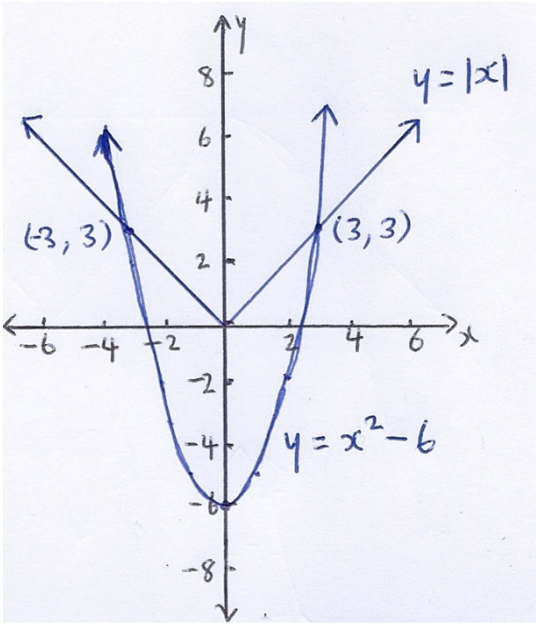
c

$$x^2<|x|$$
$$-1<x<1, x\ne 0$$



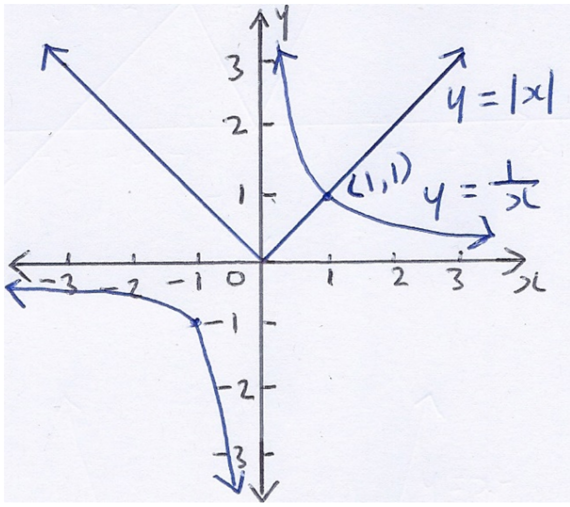
d

$$x^2 - 6 \geq |x|$$
$$x \leq -3, x \geq 3$$



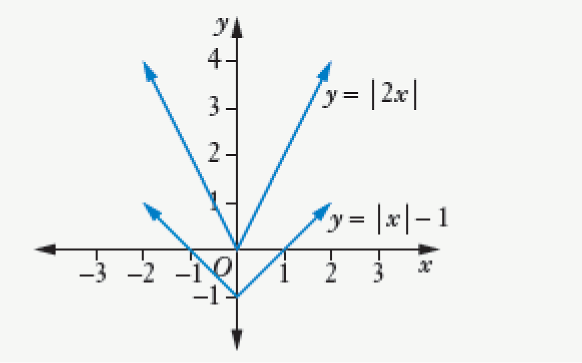
e

$$|x| > \frac{1}{x}$$
$$x < 0, x > 1$$



Question 2

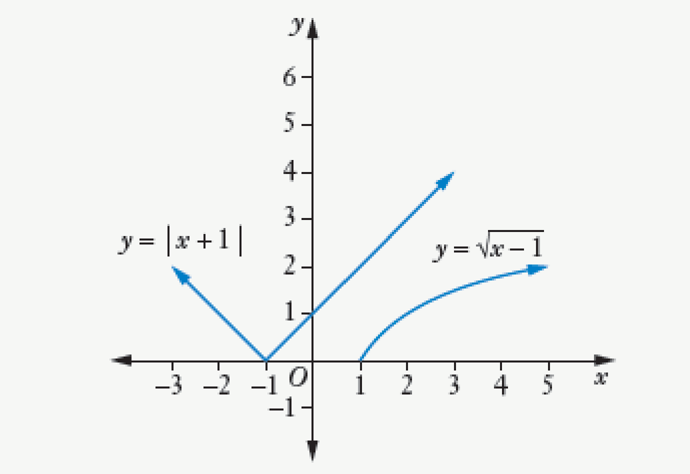
a



Clearly from the graph

$|2x| > |x| - 1$

b



From the graph $|x+1|$ is always above $\sqrt{x-1}$ for $x \geq 1$

$|x+1| \not< \sqrt{x-1}$

Question 3

$3x^2 - 2x - 2 > |3x|$

$3x^2 - 2x - 2 = 3x$

$3x^2 - 5x - 2 = 0$

$(3x+1)(x-2) = 0$

take the positive solution

$x = 2$

$3x^2 - 2x - 2 = -3x$

$3x^2 + x - 2 = 0$

$(3x-2)(x+1) = 0$

take the negative solution

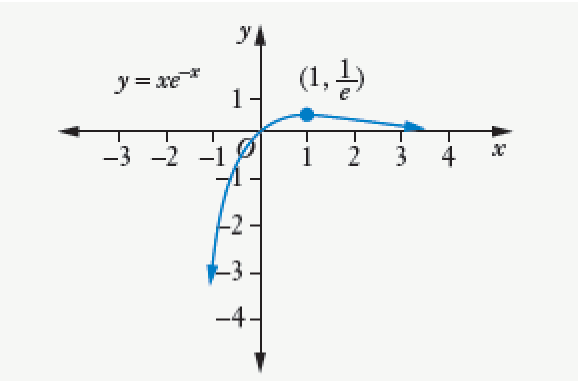
$x = -1$

$x < -1, x > 2$

Question 4

a

$$y = xe^{-x}$$
$$y' = -xe^{-x} + e^{-x}$$
$$y' = e^{-x}(1-x)$$
$$\text{Let } y' = 0$$
$$0 = e^{-x}(1-x)$$
$$1-x=0$$
$$x=1$$
$$y = 1\times e^{-1}$$
$$y = \frac{1}{e}$$
$$\text{Turning point } \left(1,\frac{1}{e}\right),\text{maximum}$$



b As $\frac{1}{e}$ is the maximum value of xe^{-x}

$$xe^{-x} \le \frac{1}{e}$$
$$x \le \frac{e^x}{e}$$
$$x \le e^{x-1}$$

Question 5

a

$$m=\frac{\ln\left(1+\frac{1}{n}\right)-0}{1+\frac{1}{n}-1}$$

$$m=\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}$$

$$m=n\ln\left(1+\frac{1}{n}\right)$$

$$m=n\ln\left(\frac{n+1}{n}\right)$$

b

$$m=n\ln\left(1+\frac{1}{n}\right)$$

As $n\to\infty, m\to$ gradient of tangent at $x=1$

$$\frac{d}{dx}(\ln x)=\frac{1}{x}=1 \text{at } x=1$$

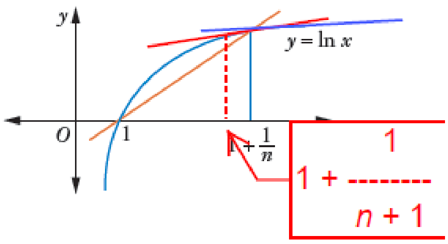
As $n\to\infty, m\to 1$

$$\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)=1$$

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n=1$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n=e \;\; =e$$

c Consider the point if $y = \ln x$ where $x = 1 + \dfrac{1}{n+1}$, slightly left of $x = 1 + \dfrac{1}{n}$ as shown on the graph.



The tangent at $x = 1 + \dfrac{1}{n+1}$ is more steep than at $x = 1 + \dfrac{1}{n}$, so its gradient will be greater. This can be seen on the graph or by noting that $\dfrac{dy}{dx} = \dfrac{1}{x}$ is a decreasing gradient function for $x >$ 1.

$$m = \frac{\ln\left(1 + \frac{1}{n+1}\right) - 0}{1 + \frac{1}{n+1} - 1}$$

$$m = \frac{\ln\left(1 + \frac{1}{n+1}\right)}{\frac{1}{n+1}}$$

$$m = (n+1)\ln\left(1 + \frac{1}{n+1}\right)$$

Comparing gradients:

$$(n+1)\ln\left(1 + \frac{1}{n+1}\right) > n\ln\left(1 + \frac{1}{n}\right)$$

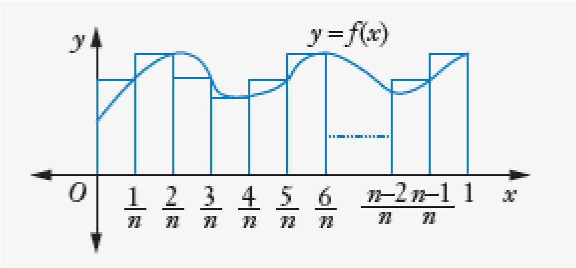
$$\ln\left(1 + \frac{1}{n+1}\right)^{n+} > \ln\left(1 + \frac{1}{n}\right)^{n}$$

$$\left(1 + \frac{1}{n+1}\right)^{n+} > \left(1 + \frac{1}{n}\right)^{n}$$

d The compound interest formula is $A = P(1 + r)^n$ where r is the interest rate and n is the number of compounding periods. For a fixed term, for example, one year, if the value of n increases, the interest is compounded more often and the value of r is smaller as it is the annual interest divided by the number of compounding periods per year. What part **c**'s result shows is that the higher the number of compounding periods in $A = P\left(1 + \dfrac{r}{n+1}\right)^{n+1}$, the faster the investment grows and the higher the interest earned on the investment.

Question 6.

a



Consider rectangles with width $\frac{1}{n}$ above the curve.

$f\left(\frac{a}{n}\right)$ is the height of the rectangle $1 \le a \le n$.

Area of the rectangles will be

$$A = \frac{1}{n} \times f\left(\frac{1}{n}\right) + \frac{1}{n} \times f\left(\frac{2}{n}\right) + \frac{1}{n} \times f\left(\frac{3}{n}\right) + \quad + \frac{1}{n} \times f\left(\frac{n}{n}\right)$$

Area under the curve will approximately equal area of the rectangles.

$$\int_0^1 f(x)\, dx \approx \lim_{n \to \infty} \text{Area of the rectangles}$$

$$= \lim_{n \to \infty} \frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \quad + f\left(\frac{n}{n}\right)\right)$$

b

$$= \lim_{n \to \infty} \frac{1}{n}\left(\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{3\pi}{n}\right) + \ldots + \sin\left(\frac{n\pi}{n}\right)\right)$$

$$\approx \int_0^1 \sin(\pi x)\, dx$$

$$= \left[-\frac{1}{\pi}\cos(\pi x)\right]_0^1$$

$$= \left(-\frac{1}{\pi}\cos(\pi)\right) - \left(-\frac{1}{\pi}\cos(0)\right)$$

$$= \frac{1}{\pi} + \frac{1}{\pi}$$

$$= \frac{2}{\pi}$$

Question 7

a

$$P(n)\ \ a+ar+ar^2+ar^3+\quad +ar^{n-1}=\frac{a\left(r^n-1\right)}{r-1}$$

$P(1)$ LHS $=a$

$$\text{RHS}=\frac{a\left(r^1-1\right)}{r-1}=a=\text{LHS}$$

$\therefore P(1)$ is true

Let $P(k)$ be true

$$\text{i.e.}\ \ a+ar+ar^2+ar^3+\quad +ar^{k-1}=\frac{a\left(r^k-1\right)}{r-1}$$

Required to prove

$$P(k+1): a+ar+ar^2+ar^3+\quad +ar^{\ k+1\ -1}=\frac{a\left(r^{(k+1)}-1\right)}{r-1}$$

$$\text{LHS}\ \ a+ar+ar^2+ar^3+\quad +ar^{k-1}+ar^{(k+1\ -1}$$

$$=\frac{a\left(r^k-1\right)}{r-1}+ar^k$$

$$=\frac{a\left(r^k-1\right)}{r-1}+\frac{ar^k\left(r-1\right)}{r-1}$$

$$=\frac{a\left(r^k-1\right)+ar^k\left(r-1\right)}{r-1}$$

$$=\frac{ar^k-a+ar^{k+}\ -ar^k}{r-1}$$

$$=\frac{ar^{k+1}-a}{r-1}$$

$$=\frac{a\left(r^{k+1}-1\right)}{r-1}$$

= RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b **i**

$$A_0A_1 = a\sin 30 = \frac{a}{2}$$

$$A_1A_2 = \frac{a}{2}\cos 30 = \frac{a\sqrt{3}}{4}$$

$$A_2A_3 = \frac{a\sqrt{3}}{4}\cos 30 = \frac{3a}{8}$$

$$\frac{A_2A_3}{A_1A_2} = \frac{\frac{3a}{8}}{\frac{a\sqrt{3}}{4}} = \frac{\sqrt{3}}{2}$$

$$\frac{A_1A_2}{A_0A_1} = \frac{\frac{a\sqrt{3}}{4}}{\frac{a}{2}} = \frac{\sqrt{3}}{2}$$

$$\therefore r = \frac{\sqrt{3}}{2}$$

ii

$$A_{n-}A_n = ar^{n-1}$$

$$A_{n-}A_n = \frac{a}{2}\left(\frac{\sqrt{3}}{2}\right)^{n-1} = \frac{a\sqrt{3}^{\,n-1}}{2^n}$$

iii

$$A_0A_1 + A_1A_2 + A_2A_3 + \ldots + A_{n-}A_n$$

$$= a + ar + ar^2 + \quad + ar^{n-1}$$

$$a = \frac{a}{2} \;\; r = \frac{\sqrt{3}}{2}$$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{\frac{a}{2}}{1-\frac{\sqrt{3}}{2}} = \frac{\frac{a}{2}}{\frac{2}{2}-\frac{\sqrt{3}}{2}}$$

$$S_\infty = \frac{a}{2-\sqrt{3}}$$

$$S_\infty = \frac{a\left(2+\sqrt{3}\right)}{4-3}$$

$$S_\infty = a\left(2+\sqrt{3}\right)$$

Question 8

a

Rectangle has height = 1

width = $\sqrt{p}-1$

Area = $\sqrt{p}-1$

Area under the curve is less than the rectangle

$$\therefore \int^{\sqrt{p}} \frac{dx}{x} < \sqrt{p}-1$$

b

As $p>1$

$0<\ln p$

$0<\int^{\sqrt{p}} \frac{dx}{x} < \sqrt{p}-1$

$0<[\ln x]^{\sqrt{p}} < \sqrt{p}-1$

$0<\ln\sqrt{p}-\ln 1<\sqrt{p}-1$

$0<\ln\sqrt{p}<\sqrt{p}-1$

$0<\frac{1}{2}\ln p<\sqrt{p}-1$

$0<\ln p<2\sqrt{p}-2$

c

From **b:**

$0<\ln\sqrt{p}<\sqrt{p}-1$

Let $x=\sqrt{p}$:

$0<\ln x<x-1$

$0<\ln x<x$

$0<\frac{\ln x}{x}<1$

As x approaches infinity, x grows faster than ln x, so $\frac{\ln x}{x}$ approaches 0.

$$\lim_{x\to\infty}\frac{\ln x}{x}=0$$

Question 9

a

$$y = x\ln x$$
$$Let\ u = x,\quad v = \ln x$$
$$u' = 1 \qquad v' = \frac{1}{x}$$
$$y' = uv' + u'v$$
$$= x\frac{1}{x} + \ln x$$
$$= 1 + \ln x$$

b

$$\int^{n} \ln x \ dx$$
$$= \int^{n} (1 + \ln x - 1)\ dx$$
$$= \int_{1}^{n} (1 + \ln x)\ dx - \int_{1}^{n} 1\ dx$$
$$= \left[x\ln x - x\right]_{1}^{n}$$
$$= (n\ln n - n) - (1\ln 1 - 1)$$
$$= n\ln n - n + 1$$

c **i**

$$S_b = \ln 2 + \ln 3 + \ln 4 + ... + \ln(n-1)$$
$$= \ln\left[2\times 3\times 4\times 5......\times(n-1)\right]$$
$$= \ln\left[(n-1)!\right]$$

ii

$$S_a = \ln 2 + \ln 3 + \ln 4 + ... + \ln n$$
$$= \ln(2\times 3\times 4\times 5......\times n)$$
$$= \ln(n!)$$

iii

The area under the curve $y = \ln x$ from $1 \to n$ must be between S_a and S_b

$$\ln\left[(n-1)!\right] < \int^{n} \ln\ x\ dx < \ln(n!)$$
$$\Rightarrow \ln\left[(n-1)!\right] < n\ln n - n + 1 < \ln(n!)$$

iv

$$\ln\left[(n-1)!\right] < n\ln n - n + 1 < \ln(n!)$$

$$e^{\ln[\ n-1\ !]} < e^{n\ln n - n+1} < e^{\ln!\,n\ )}$$

$$(n-1)! < e^{n\ln n - n+1} < n!$$

$$(n-1)! < e^{1-n}e^{\ln n^n} < n!$$

$$(n-1)! < e^{\ -n}n^n < n!$$

Question 1

a

$P(n): 5+11+17+23+....+(6n-1)=3n^2+2n$

$P(1)$ LHS $=5$

RHS $=3+2=5=$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $5+11+17+23+....+(6k-1)=3k^2+2k$

Required to prove

$P(k+1): 5+11+17+23+....+(6(k+1)-1)=3(k+1)^2+2(k+1)$

$P(k+1)$ LHS $5+11+17+23+....+(6(k+1)-1)$

$=3k^2+2k+(6(k+1)-1)$

$=3k^2+2k+6k+5$

$=3k^2+6k+3+2k+2$

$=3(k+1)^2+2(k+1)$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$$P(n): \frac{1}{1\times4}+\frac{1}{4\times7}+\frac{1}{7\times10}+\quad+\frac{1}{(3n-2)\times(3n+1)}=\frac{n}{3n+1}$$

$P(1)$ LHS $=\frac{1}{4}$

RHS $=\frac{1}{3+1}=\frac{1}{4}=$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\frac{1}{1\times4}+\frac{1}{4\times7}+\frac{1}{7\times10}+\quad+\frac{1}{(3k-2)\times(3k+1)}=\frac{k}{3k+1}$

Required to prove

$$P(k+1): \frac{1}{1\times4}+\frac{1}{4\times7}+\frac{1}{7\times10}+\quad+\frac{1}{(3(k+1)-2)\times(3(k+1)+1)}=\frac{(k+1)}{3(k+1)+1}$$

LHS $\frac{1}{1\times4}+\frac{1}{4\times7}+\frac{1}{7\times10}+\quad+\frac{1}{(3(k+1)-2)\times(3(k+1)+1)}$

$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$

$=\frac{k(3k+4)}{(3k+1)(3k+4)}+\frac{1}{(3k+1)(3k+4)}$

$=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$

$=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$

$=\frac{(k+1)}{(3k+4)}$

$=\frac{(k+1)}{(3(k+1)+1)}$

= RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical inducion $P(n)$ is true.

c

$$P(n)\quad a+\frac{a}{p}+\frac{a}{p^2}+\frac{a}{p^3}+\quad +\frac{a}{p^{n-1}}=\frac{a\left(1-p^n\right)}{p^{n-1}(1-p)}$$

$P(1)$ LHS $= a$

$$\text{RHS}=\frac{a\left(1-p^1\right)}{p^{1-1}(1-p)}=a=\text{LHS}$$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $a+\frac{a}{p}+\frac{a}{p^2}+\frac{a}{p^3}+\quad +\frac{a}{p^{k-1}}=\frac{a\left(1-p^k\right)}{p^{k-1}(1-p)}$

Required to prove

$$P(k+1): a+\frac{a}{p}+\frac{a}{p^2}+\frac{a}{p^3}+\quad +\frac{a}{p^{\ k+1\ -1}}=\frac{a\left(1-p^{(k+1)}\right)}{p^{\ k+1\ -1}(1-p)}$$

LHS $a+\frac{a}{p}+\frac{a}{p^2}+\frac{a}{p^3}+\quad +\frac{a}{p^{(k+1\ -1}}$

$$=\frac{a\left(1-p^k\right)}{p^{k-}\ (1-p)}+\frac{a}{p^{(k+1\ -1}}$$

$$=\frac{ap\left(1-p^k\right)}{p^{k+1-1}(1-p)}+\frac{a(1-p)}{p^{\ k+1\ -1}(1-p)}$$

$$=\frac{ap\left(1-p^k\right)+a(1-p)}{p^{k+1-1}(1-p)}$$

$$=\frac{ap-ap^{k+}\ +a-ap}{p^{k+1-1}(1-p)}$$

$$=\frac{a-ap^{k+1}}{p^{k+1-1}(1-p)}$$

$$=\frac{a\left(1-p^{k+1}\right)}{p^{\ k+1\ -1}(1-p)}$$

= RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 2

a

$P(n): 9^n - 1$ is divisible by 8 $\forall n \geq 1$

i.e $9^n - 1 = 8m \quad m \in \mathbb{N}$

$P(1)$ LHS $= 9 - 1 = 8$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $9^k - 1 = 8m$

$P(k+1): 9^{k+} \;\; -1$

$= 9 \times 9^k - 1$

$= 9 \times 9^k - 1$

$= 8 \times 9^k + 9^k - 1$

$= 8 \times 9^k + 8m$

$= 8\left(9^k + m\right)$

As $k, m, 9^k + m \in \mathbb{N}$

$8\left(9^k + m\right) = 8q$

$\therefore 9^{k+} \;\; -1$ is divisible by 8

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$P(n): 2^{n+2}+3^{2n+1}$ is divisible by 7 $\forall$ odd $n \geq 1$

i.e $2^{n+2}+3^{2n+1}=7m \quad m \in \mathbb{N}$

$P(1)$ LHS $=2^{1+2}+3^{2+1}=8+27=35$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $2^{k+2}+3^{2k+1}=7m$

$P(k+1)$: LHS $2^{\ k+1\ +2}+3^{2\ k+1\ +1}$

$=2\times 2^{k+2}+9\times 3^{2k+1}$

$=2\times 2^{k+2}+2\times 3^{2k+1}+7\times 3^{2k+1}$

$=2\times\left(2^{k+2}+3^{2k+1}\right)+7\times 3^{2k+1}$

$=2\times 7m+7\left(3^{2k+1}\right)$

$=7\left(2m+3^{2k+1}\right)$

As $k, m, 2m+3^{2k+1} \in \mathbb{N}$

$7\left(2m+3^{2k+1}\right)=7q$

$\therefore 2^{\ k+1\ +2}+3^{2\ k+1\ +1}$ is divisible by 7

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 3

a

$P(n): \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

$P(1)$ LHS $1^3 = 1$

$\text{RHS} = \frac{1}{4}\times 1^2(1+1)^2 = \frac{1}{4}\times 4 = 1 = \text{LHS}$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$

Required to prove

$P(k+1): \sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2((k+1)+1)^2$

$P(k+1)$ LHS $\sum_{r=1}^{k+1} r^3$

$= \sum_{r=1}^{k} r^3 + (k+1)^3$

$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$

$= \frac{1}{4}(k+1)^2(k^2 + 4(k+1))$

$= \frac{1}{4}(k+1)^2(k^2 + 4k + 1)$

$= \frac{1}{4}(k+1)^2(k+2)^2$

$= \frac{1}{4}(k+1)^2((k+1)+1)^2$

$= \text{RHS}$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$P(n): \sum_{k=1}^{n} k(k)! = (n+1)! - 1$

$P(1)$ LHS $1 \times 1 = 1$

$\text{RHS} = (1+1)! - 1 = 2 - 1 = 1 = \text{LHS}$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\sum_{k=1}^{k} k(k)! = (k+1)! - 1$

Required to prove

$P(k+1): \sum_{k=1}^{k+1} (k+1)(k+1)! = ((k+1)+1)! - 1$

$P(k+1)$ LHS $\sum_{k=1}^{k+1} (k+1)(k+1)!$

$= \sum_{k=1}^{k} k(k)! + (k+1)(k+1)!$

$= (k+1)! - 1 + (k+1)(k+1)!$

$= (k+1+1)(k+1)! - 1$

$= (k+2)(k+1)! - 1$

$= (k+2)! - 1$

$= ((k+1)+1)! - 1$

$= \text{RHS}$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

c

$$P(n): \sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$$

$$P(1) \text{ LHS } \frac{1}{1(1+1)} = \frac{1}{2}$$

$$\text{RHS} = \frac{1}{1+1} = \frac{1}{2} = \text{LHS}$$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\displaystyle\sum_{j=1}^{k} \frac{1}{j(j+1)} = \frac{k}{k+1}$

Required to prove

$$P(k+1): \sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \frac{k+1}{(k+1)+1}$$

$$P(k+1) \text{ LHS } \sum_{j=1}^{k+1} \frac{1}{j(j+1)}$$

$$= \sum_{j=1}^{k} \frac{1}{j(j+1)} + \frac{1}{(k+1)((k+1)+1)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)((k+1)+1)}$$

$$= \frac{k((k+1)+1)}{(k+1)((k+1)+1)} + \frac{1}{(k+1)((k+1)+1)}$$

$$= \frac{k(k+2)+1}{(k+1)((k+1)+1)}$$

$$= \frac{k^2+2k+1}{(k+1)((k+1)+1)}$$

$$= \frac{(k+1)^2}{(k+1)((k+1)+1)}$$

$$= \frac{(k+1)}{((k+1)+1)}$$

$= \text{RHS}$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 4

a

$7^n + 13^n + 19^n$ is divisible by 13 $\forall$ odd $n \geq 1$

Since 13^n is already divisible by 13 we only need to show

$P(n): 7^n + 19^n$ is divisible by 13 $\forall$ odd n ≥ 1

i.e $7^n + 19^n = 13m \quad m \in \mathbb{N}$

$P(1)$ LHS $= 7^1 + 19^1 = 26$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $7^k + 19^k = 13m$

$P(k+2)$: LHS $7^{k+2} + 19^{k+2}$

$= 7^2 \times 7^k + 19^2 \times 19^k$

$= 49 \times 7^k + 361 \times 19^k$

$= 49 \times 7^k + 49 \times 19^k + 13 \times 24 \times 19^k$

$= 49 \times \left(7^k + 19^k\right) + 13\left(24 \times 19^k\right)$

$= 49 \times (13m) + 13\left(24 \times 19^k\right)$

$= 13\left(24 \times 19^k + 49m\right)$

As $k, m, 24 \times 19^k + 49m \in \mathbb{N}$

$13\left(24 \times 19^k + 49m\right) = 13q$

$\therefore 7^{k+2} + 19^{k+2}$ is divisible by 13

$\therefore P(k+2)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+2)$

by mathematical induction $P(n)$ is true.

$\therefore 7^n + 19^n$ is divisible by 13 $\forall$ odd $n \geq 1$

$\therefore 7^n + 13^n + 19^n$ is divisible by 13 $\forall$ odd $n \geq 1$

b

$P(n): n^4+4n^2+11$ is divisible by 16 $\forall$ odd $n \geq 1$

i.e $n^4+4n^2+11=16m \quad m \in \mathbb{N}$

$P(1)$ LHS $=1+4+11=16$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $k^4+4k^2+11=16m$

$P(k+2)$: LHS $(k+2)^4+4(k+2)^2+11$

$=k^4+8k^3+24k^2+32k+16+4k^2+16k+16+11$

$=k^4+4k^2+11+8k^3+24k^2+48k+16$

$=16m+8(k^3+3k^2+6k+2)$

As k is odd k^3+3k^2+6k+2 must be even

so we can write $k^3+3k^2+6k+2=2b$ for some $b \in \mathbb{N}$

$=16m+8(2b)$

$=16m+16b$

$=16(m+b)$

As $b, m \in \mathbb{N}$

$16(m+b)=16q$

$\therefore (k+2)^4+4(k+2)^2+11$ is divisible by 16

$\therefore P(k+2)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+2)$

by mathematical induction $P(n)$ is true.

Question 5

a

$P(n): 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{5^2}+....+\frac{1}{n^2}\le 2-\frac{1}{n}$

$P(1)$ LHS $=1$

RHS $=2-\frac{1}{1}=2-1=1=$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{5^2}+....+\frac{1}{k^2}\le 2-\frac{1}{k}$

Required to prove

$P(k+1): 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{5^2}+....+\frac{1}{(k+1)^2}\le 2-\frac{1}{k+1}$

LHS $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{5^2}+\quad+\frac{1}{(k+1)^2}$

$\le 2-\frac{1}{k}+\frac{1}{(k+1)^2}$

$=2-\frac{(k+1)^2-k}{k(k+1)^2}$

$=2+\frac{-k^2-2k-1+k}{k(k+1)^2}$

$=2+\frac{-k^2-k-1}{k(k+1)^2}$

$=2+\frac{-k(k+1)-1}{k(k+1)^2}$

$=2-\frac{1}{k+1}+\frac{-1}{k(k+1)^2}$

$\le 2-\frac{1}{k+1}$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$$P(n): \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{x^4} - .... - \frac{1}{x^n} = \frac{1}{x^n(x-1)}$$

$$P(1) \text{ LHS} = \frac{1}{x-1} - \frac{1}{x} = \frac{x-x+1}{x(x-1)} = \frac{1}{x(x-1)}$$

$$\text{RHS} = \frac{1}{x^1(x-1)} = \text{LHS}$$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{x^4} - .... - \frac{1}{x^k} = \frac{1}{x^k(x-1)}$

Required to prove

$$P(k+1): \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{x^4} - \quad - \frac{1}{x^{k+1}} = \frac{1}{x^{k+1}(x-1)}$$

$$\text{LHS } \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{x^4} - .... - \frac{1}{x^{k+1}}$$

$$= \frac{1}{x^k(x-1)} - \frac{1}{x^{k+1}}$$

$$= \frac{x^{k+} \ - x^k(x-1)}{x^{k+} \ x^k(x-1)}$$

$$= \frac{x^{k+1} - x^{k+1} + x^k}{x^{k+} \ x^k(x-1)}$$

$$= \frac{x^k}{x^{k+} \ x^k(x-1)}$$

$$= \frac{1}{x^{k+1}(x-1)}$$

$= \text{RHS}$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 6

$P(n)$ $(ab)^n = a^n b^n$

$P(1)$ LHS $= (ab) = ab$

RHS $= a^1 b^1 = ab =$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $(ab)^k = a^k b^k$

$P(k+1)$ LHS $(ab)^{k+1}$

$= (ab)(ab)^{k+1}$

$= aba^k b^k$

$= aa^k bb^k$

$= a^{k+1} b^{k+1}$

= RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical inductio $P(n)$ is true.

Question 7

$P(n)\ \ \frac{d}{dx}\left(x^{-n}\right)=-nx^{-n-1}$

$P(1)\ \ \text{LHS}=\frac{d}{dx}\left(x^{-1}\right)=\frac{-1}{x^2}=-1x^{-2}=\text{RHS}$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\frac{d}{dx}\left(x^{-k}\right)=-nx^{-k-1}$

$P(k+1)\ \ \text{LHS}\ \ \frac{d}{dx}\left(x^{-\ k+1)}\right)$

$=\frac{d}{dx}\left(x^{-}\ x^{-k}\right)$

$=x^{-k}\frac{d}{dx}\left(x^{-1}\right)+x^{-1}\frac{d}{dx}\left(x^{-k}\right)$

$=x^{-k}\left(-1x^{-2}\right)+x^{-1}\left(-kx^{-k-1}\right)$

$=-x^{-k-2}-kx^{-k-1-1}$

$=-(k+1)x^{-\ k+1\ -1}$

$=\text{RHS}$

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 8

$P(n): \frac{d^n}{d\theta^n}(\sin p\theta) = p^n \sin\left(p\theta + \frac{n\pi}{2}\right)$

$P(1)$ LHS $= \frac{d}{d\theta}(\sin p\theta)$

$= p\cos p\theta$

$= p\sin\left(p\theta + \frac{\pi}{2}\right)$

$=$ RHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\frac{d^k}{d\theta^k}(\sin p\theta) = p^k \sin\left(p\theta + \frac{k\pi}{2}\right)$

$P(k+1)$ LHS $\frac{d^{k+1}}{d\theta^{k+}}(\sin p\theta)$

$= \frac{d}{dx}\left(p^k \sin\left(p\theta + \frac{k\pi}{2}\right)\right)$

$= p \times p^k \cos\left(p\theta + \frac{k\pi}{2}\right)$

$= p^{k+1} \sin\left(p\theta + \frac{k\pi}{2} + \frac{\pi}{2}\right)$

$= p^{k+} \sin\left(p\theta + \frac{(k+1)\pi}{2}\right)$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 9.

$x^{2n} - y^{2n} = (x^n - y^n)(x^n + y^n)$

It must be divisible by $x^n + y^n$.

Question 10

$P(n):(x+1)^n - nx - 1$ is divisible by $x^2 \ \forall \ n \geq 1$

i.e $(x+1)^n - nx - 1 = mx^2 \quad m \in \mathbb{N}$

$P(1)$ LHS $= (x+1) - x - 1 = 0$

$\therefore P(1)$ is true

$P(2)$ LHS $= (x+1)^2 - 2x - 1 = x^2 + 2x + 1 - 2x - 1 = x^2$

$\therefore P(2)$ is true

Let $P(k)$ be true

i.e. $(x+1)^k - kx - 1 = mx^2$

$P(k+1)$: LHS $(x+1)^{k+} - (k+1)x - 1$

$= (x+1)(x+1)^k - kx - x - 1$

$= (x+1)^k - kx - 1 + x(x+1)^k - x$

$= mx^2 + x\left((x+1)^k - 1\right)$

$= mx^2 + x\left((x+1)^k - kx - 1 + kx\right)$

$= mx^2 + x\left(mx^2 + kx\right)$

$= mx^2 + xmx^2 + kx^2$

$= x^2(m + xm + k)$

$\therefore (x+1)^{k+} - (k+1)x - 1$ is divisible by x^2

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 11

a

$P(m): m(m+3)$ is divisible by 2 $\forall m \geq 1$

i.e $m(n+m) = 2n \quad n \in \mathbb{N}$

$P(1)\,\text{LHS} = 1(1+3) = 4$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $k(k+3) = 2n$

$P(k+1): \text{LHS } (k+1)((k+1)+3)$

$= (k+1)(k+4)$

$= k^2 + 5k + 4$

$= k^2 + 3k + 2k + 4$

$= k(k+3) + 2(k+2)$

$= 2n + 2(k+2)$

$= 2(n+k+2)$

As $k, n, n+k+2 \in \mathbb{N}$

$2(n+k+2) = 2q$

$\therefore (k+1)((k+1)+3)$ is divisible by 2

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

b

$P(n): n(n+1)(n-1)$ is divisible by 6 $\forall n \geq 2$

i.e $n(n+1)(n-1) = 6m \quad m \in \mathbb{N}$

$P(2)\text{LHS} = 2(2+1)(2-1) = 6$

$\therefore P(2)$ is true

Assume $P(k)$ is true

i.e. $k(k+1)(k-1) = 6m$

$P(k+1): (k+1)((k+1)+1)((k+1)-1)$

$= (k+1)(k-1+3)k$

$= k(k+1)(k-1) + 3k(k+1)$

$= 6m + 3k(k+1)$

from **5.01, Q5 b** $k(k+1)$ must be an even integer so we can write

$= 6m + 3\times 2a$

$= 6(m+a)$

As $a, m, a+m \in \mathbb{N}$

$6(m+a) = 6q$

$\therefore (k+1)((k+1)+1)((k+1)-1)$ is divisible by 6

$\therefore P(k+1)$ is true

$P(2)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

By mathematical indution $P(n)$ is true.

c

$P(n): n(n+2)+(n+2)(n+4)$ is divisible by 2 $\forall n \geq 1$ and n is odd

this is actually true $\forall n \in \mathbb{N}$

$P(n): n(n+2)+(n+2)(n+4)$ is divisible by 2 $\forall n \geq 1$

i.e $n(n+2)+(n+2)(n+4)=2m \quad m \in \mathbb{N}$

$P(1)$LHS $=1(1+2)+(1+2)(1+4)=3+15=18$

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $k(k+2)+(k+2)(k+4)=2m$

$P(k+1)$: LHS $(k+1)((k+1)+2)+((k+1)+2)((k+1)+4)$

$=(k+1)(k+3)+(k+3)(k+5)$

$=k^2+4k+3+k^2+8k+15$

$=k^2+2k+2k+3+k^2+6k+2k+8+7$

$=k^2+2k+k^2+6k+8+2k+2k+3+7$

$=k(k+2)+(k+2)(k+4)+2(2k+5)$

$=2m+2(2k+5)$

$=2(m+2k+5)$

As $k, m, m+2k+5 \in \mathbb{N}$

$2(m+2k+5)=2q$

$\therefore (k+1)((k+1)+2)+((k+1)+2)((k+1)+4)$ is divisible by 2

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 12

a

$$\frac{K^{n+1}-K^nL+L^{n+1}-KL^n}{K-L}$$
$$=\frac{K^n(K-L)+L^n(L-K)}{K-L}$$
$$=\frac{K^n(K-L)-L^n(K-L)}{K-L}$$
$$=\frac{(K^n-L^n)(K-L)}{K-L}$$
$$=K^n-L^n$$

b

If $K>L \Rightarrow K^n>L^n$

$\Rightarrow K^n-L^n>0$

$\Rightarrow \frac{K^{n+1}-K^nL+L^{n+1}-KL^n}{K-L}>0$

$\Rightarrow K^{n+1}-K^nL+L^{n+1}-KL^n>0$

$\Rightarrow K^{n+1}+L^{n+1}\geq K^nL+KL^n$

c

$P(n):\left(\frac{K+L}{2}\right)^n \le \frac{K^n+L^n}{2}$

$P(1)$ LHS $=\frac{K+L}{2}$

RHS $=\frac{K+L}{2}=$ LHS

$\therefore P(1)$ is true

Let $P(k)$ be true

i.e. $\left(\frac{K+L}{2}\right)^k \le \frac{K^k+L^k}{2}$

Required to prove

$P(k+1):\left(\frac{K+L}{2}\right)^{k+} \le \frac{K^{k+1}+L^{k+1}}{2}$

LHS $\left(\frac{K+L}{2}\right)^{k+1}$

$=\left(\frac{K+L}{2}\right)^k\left(\frac{K+L}{2}\right)$

$\le \frac{K^k+L^k}{2}\left(\frac{K+L}{2}\right)$

$=\frac{\left(K^k+L^k\right)\left(K+L\right)}{4}$

$=\frac{K^{k+1}+L^{k+1}+KL^k+LK^k}{4}$

$\le \frac{K^{k+1}+L^{k+1}+K^{k+1}+L^{k+1}}{4}$

$=\frac{2\left(K^{k+1}+L^{k+1}\right)}{4}$

$=\frac{K^{k+1}+L^{k+1}}{2}$

= RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 13

a

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\begin{aligned}\cos(2P) &= \cos(P+P)\\ &= \cos P\cos P - \sin P\sin P\\ &= \cos^2 P - \sin^2 P\\ &= 1-\sin^2 P - \sin^2 P\\ &= 1-2\sin^2 P\end{aligned}$$

b

$$\begin{aligned}\frac{\cos Q - \cos(Q+2P)}{2\sin P} &= \frac{\cos Q - (\cos Q\cos 2P - \sin Q\sin 2P)}{2\sin P}\\ &= \frac{\cos Q(1-\cos 2P) + \sin Q\sin 2P}{2\sin P}\\ &= \frac{\cos Q\left(1-\left(1-2\sin^2 P\right)\right) + \sin Q(2\sin P\cos P)}{2\sin P}\\ &= \frac{\cos Q\left(2\sin^2 P\right) + 2\sin Q\sin P\cos P}{2\sin P}\\ &= \cos Q\sin P + \sin Q\cos P\\ &= \sin(Q+P)\end{aligned}$$

c

$$P(n): \sin P + \sin 3P + \sin 5P + \ldots + \sin\left[(2n-1)P\right] = \frac{1-\cos 2nP}{2\sin P}$$

$P(1)$ LHS $= \sin P$

$$\text{RHS} = \frac{1-\cos 2P}{2\sin P}$$

$$= \frac{1-\left(1-2\sin^2 P\right)}{2\sin P}$$

$$= \frac{2\sin^2 P}{2\sin P}$$

$$= \sin P$$

$= \text{LHS}$

$\therefore P(1)$ is true

Assume $P(k)$ is true.

$$\sin P + \sin 3P + \sin 5P + \ldots + \sin\left[(2k-1)P\right] = \frac{1-\cos 2kP}{2\sin P}$$

Required to prove

$$P(k+1): \sin P + \sin 3P + \sin 5P + \ldots + \sin\left((2k-1)P\right) + \sin\left(\left(2(k+1)-1\right)P\right) = \frac{1-\cos\left(2(k+1)P\right)}{2\sin P}$$

$$\text{LHS} = \sin P + \sin 3P + \sin 5P + \ldots + \sin\left((2k-1)P\right) + \sin\left(\left(2(k+1)-1\right)P\right)$$

$$= \frac{1-\cos 2kP}{2\sin P} + \sin\left((2k+1)P\right)$$

$$= \frac{1-\cos 2kP}{2\sin P} + \sin\left(2kP + P\right)$$

$$= \frac{1-\cos 2kP}{2\sin P} + \frac{\cos 2kP - \cos\left(2kP + 2P\right)}{2\sin P}$$ from part **b** where $Q = 2kP$

$$= \frac{1-\cos 2kP + \cos 2kP - \cos\left((2k+2)P\right)}{2\sin P}$$

$$= \frac{1-\cos\left(2(k+1)P\right)}{2\sin P}$$

$= RHS$

$\therefore P(k+1)$ is true

As $P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

By mathematical indution $P(n)$ is true.

Question 14

a

$T_1 = 5, T_n = 2T_{n-1} + 1, n \geq 2$

$T_n = 6\left(2^{n-}\ \right) - 1, \quad n \in \mathbb{N}$

$P(n): T_n = 6\left(2^{n-}\ \right) - 1$

$P(2)$ LHS $= 2T_{2-1} + 1 = 2T_1 + 1 = 2 \times 5 + 1 = 11$

RHS $= 6\left(2^{2-1}\right) - 1 = 6(2) - 1 = 12 - 1 = 11 =$ LHS

$\therefore P(2)$ is true

Assume $P(k)$ is true

i.e. $T_k = 6\left(2^{k-}\ \right) - 1$

Required to prove

$P(k+1): T_{k+}\ = 6\left(2^{\ k+1\ -1}\right) - 1 = 6\left(2^k\right) - 1$

LHS $= 2T_k + 1$

$= 2\left(6\left(2^{k-1}\right) - 1\right) + 1$

$= 6 \times 2\left(2^{k-}\ \right) - 2 + 1$

$= 6\left(2^k\right) - 1$

$=$ RHS

$\therefore P(k+1)$ is true

$P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

By mathematical indution $P(n)$ is true.

b **i**

$T\ = 1\ T_n = \dfrac{2T_{n-}\ - 1}{3}$

$T_2 = \dfrac{2T\ - 1}{3} = \dfrac{2 \times 1 - 1}{3} = \dfrac{1}{3}$

$T_3 = \dfrac{2T_2 - 1}{3} = \dfrac{2 \times \frac{1}{3} - 1}{3} = \dfrac{-\frac{1}{3}}{3} = -\dfrac{1}{9}$

$T_4 = \dfrac{2T_3 - 1}{3} = \dfrac{2 \times \left(-\frac{1}{9}\right) - 1}{3} = \dfrac{-\frac{11}{9}}{3} = -\dfrac{11}{27}$

ii

$$T \ = 1 \ T_n = \frac{2T_{n-} \ -1}{3}$$

$$T_n = 3\left(\frac{2}{3}\right)^n - 1, \quad n \in \mathbb{N}, n \geq 2$$

$$P(n): T_n = 3\left(\frac{2}{3}\right)^n - 1$$

$$P(2) \text{ LHS} = \frac{2T_{2-1} - 1}{3} = \frac{2T_1 - 1}{3} = \frac{2 \times 1 - 1}{3} = \frac{1}{3}$$

$$\text{RHS} = 3\left(\frac{2}{3}\right)^2 - 1 = 3 \times \frac{4}{9} - 1 = \frac{4}{3} - 1 = \frac{1}{3} = \text{LHS}$$

$\therefore P(2)$ is true

Let $P(k)$ be true

i.e. $T_k = 3\left(\frac{2}{3}\right)^k - 1$

Required to prove

$$P(k+1): T_{k+} \ = 3\left(\frac{2}{3}\right)^{k+1} - 1$$

$$\text{LHS } \ T_{k+1} = \frac{2T_k - 1}{3}$$

$$= \frac{2\left(3\left(\frac{2}{3}\right)^k - 1\right) - 1}{3}$$

$$= \frac{3\left(\frac{2^{k+}}{3^k}\right) - 2 - 1}{3}$$

$$= \frac{3\left(\frac{2^{k+}}{3^k}\right)}{3} - \frac{3}{3}$$

$$= 3\left(\frac{2^{k+}}{3^{k+}}\right) - 1$$

$$= 3\left(\frac{2}{3}\right)^{k+1} - 1$$

$= \text{RHS}$

$\therefore P(k+1)$ is true

$P(2)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

c

$T_1 = 5, T_2 = 11, T_n = 4T_{n-1} - 3T_{n-2}$

$T_n = 3^n + 2, \quad n \in \mathbb{N}, n \geq 3$

$P(n): T_n = 3^n + 2$

$P(3)$ LHS $= 4T_{3-1} - 3T_{3-2} = 4T_2 - 3T_1 = 4\times 11 - 3\times 5 = 44 - 15 = 29$

RHS $= 3^3 + 2 = 27 + 2 = 29 =$ LHS

$\therefore P(3)$ is true

Let $P(k)$ be true

i.e. $T_k = 3^k + 2$

This also implies $P(k-1)$ is true

i.e. $T_{k-} = 3^{k-} + 2$

Required to prove

$P(k+1): T_{k+} = 3^{k+} + 2$

LHS $T_{k+1} = 4T_{k+1\ -1} - 3T_{k+1\ -2} = 4T_k - 3T_{k-1}$

$= 4\left(3^k + 2\right) - 3\left(3^{k-} + 2\right)$

$= 4\times 3^k + 8 - 3\times 3^{k-} - 6$

$= 4\times 3^k - 3^k + 2$

$= 3^k(4-1) + 2$

$= 3\times 3^k + 2$

$= 3^{k+} + 2$

$=$ RHS

$\therefore P(k+1)$ is true

$P(3)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true.

Question 15

The sum S_n of the exterior angles of an n-sided convex polygon is given by

$P(n): S_n = 360 \quad n \in \mathbb{N}, n \geq 3$

$P(3)$

The sum of the exterior angles of a triangle is360 °.

Sum of interior and exterior angles of a triangle $= 3 \times 180° = 540°$(3 straight angles)

Sum of interior angles of a triangle $= 180°$

$\therefore$ Sum of exterior angles of a triangle $= 540° - 180° = 360°$

$\therefore P(3)$ is true

Assume $P(k)$ is true

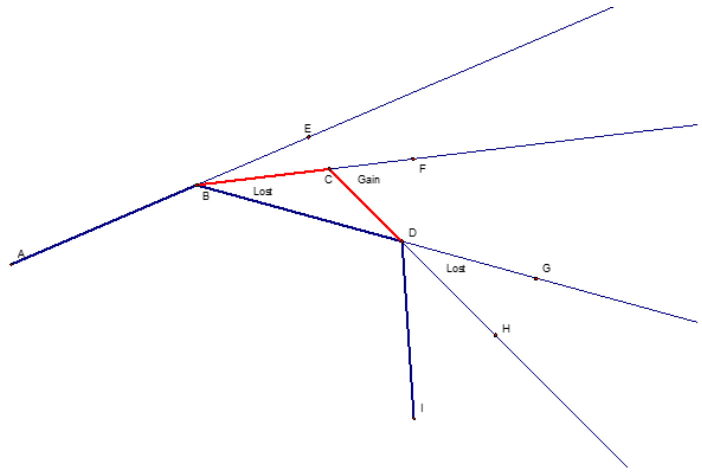
i.e. $S_k = 360$

Required to prove

$P(k+1): S_{k+} = 360$

We see from the diagram a convex polygon with k sides.

Consider a convex polygon with $k+1$ sides by replacing side BD with the 2 red sides BC and CD.

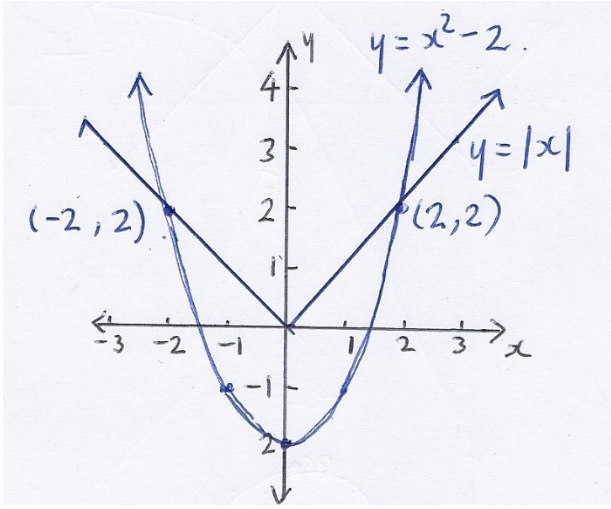


Adding an extra side has created an extra triangle.
From the original polygon's exterior angles,
we lose the angle sizes $\angle CBD$ and $\angle GDH$
and gain $\angle FCD$ which is the exterior angle of a triangle and equal to the
2 opposite interior angles $\angle CBD$ and $\angle CDB$.
But $\angle GDH = \angle CDB$ (vertically opposite).
So there is no change in the exteropr angle sum of the polygon because the angle size gained is equal to the angle lost from the original polygon.
Hence $S_{k+} = 360$
$\therefore P(k+1)$ is true.
$P(3)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$.
By mathematical indution $P(n)$ is true.

Question 16

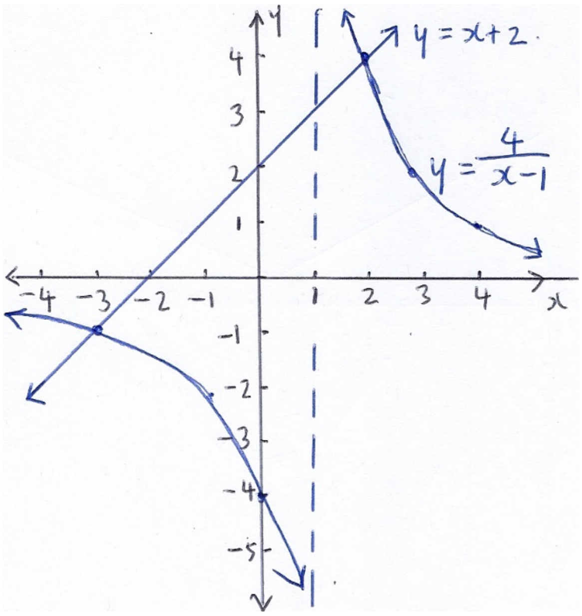
a

$$|x| \geq x^2 - 2$$
$$-2 \leq x \leq 2$$



b

$$\frac{4}{x-1} \leq x + 2$$
$$-3 \leq x < 1,\ x \geq 2$$



Question 17

a

From the diagram

Area of the triangle < Area of the semicircle < Area of the rectangle

$$\frac{1}{2}\times 2r\times r<\frac{1}{2}\pi r^2<2r\times r$$

$$\Rightarrow r^2<\frac{1}{2}\pi r^2<2r^2$$

$$\Rightarrow 2r^2<\pi r^2<4r^2$$

$$\Rightarrow 2<\pi<4$$

b A better approximation for π could be found by using polygons with a greater number of sides for inscribing and escribing the semicircle in a similar process to Archimedes.

Question 18

a From the diagram the chord AB is always below the curve $y = \ln x$.

Hence the area under the curve must be greater than the area of the trapezium.

b

$$\int_a^b \ln x \, dx > \frac{b-a}{2}(\ln a + \ln b)$$

$$\Rightarrow [x\ln x - x]_a^b > \frac{b-a}{2}(\ln a + \ln b)$$

$$\Rightarrow b\ln b - b - a\ln a + a > \frac{b}{2}\ln b + \frac{b}{2}\ln a - \frac{a}{2}\ln b - \frac{a}{2}\ln a$$

$$\Rightarrow \frac{b}{2}\ln b - \frac{a}{2}\ln a + (a-b) > \frac{b}{2}\ln a - \frac{a}{2}\ln b$$

$$\Rightarrow \frac{b}{2}\ln b - \frac{b}{2}\ln a - \frac{a}{2}\ln a + \frac{a}{2}\ln b > b - a$$

$$\Rightarrow \frac{b}{2}\ln\left(\frac{b}{a}\right) + \frac{a}{2}\ln\left(\frac{b}{a}\right) > b - a$$

$$\Rightarrow \left(\frac{b}{2} + \frac{a}{2}\right)\ln\left(\frac{b}{a}\right) > b - a$$

$$\Rightarrow \frac{1}{2}(b+a)\ln\left(\frac{b}{a}\right) > b - a$$

$$\Rightarrow (b+a)\ln\left(\frac{b}{a}\right) > 2(b-a)$$

$$\Rightarrow \ln\left(\frac{b}{a}\right) > 2\frac{(b-a)}{(b+a)}$$

$$\Rightarrow \frac{b}{a} > e^{2\frac{(b-a)}{(b+a)}}$$

$$\Rightarrow e^{2\frac{(b-a)}{b+a}} < \frac{b}{a}$$

MATHS IN FOCUS 12
MATHEMATICS EXTENSION 2

WORKED SOLUTIONS

Chapter 6: Further integration

Exercise 6.01 Integration by substitution

Question 1

a $\int \frac{e^x}{e^x+1}\,dx$

Let $u = e^x + 1$, $\frac{du}{dx} = e^x$, so $du = e^x\,dx$.

$\int \frac{e^x}{e^x+1}\,dx = \int \frac{du}{u}$

$= \ln u + C$

$= \ln(e^x + 1) + C$

b $\int \frac{e^{\frac{x}{2}}}{e^x+1}\,dx$

Let $u = e^{\frac{x}{2}}$, $\frac{du}{dx} = \frac{1}{2}e^{\frac{x}{2}}$, so $du = \frac{1}{2}e^{\frac{x}{2}}\,dx$.

$\int \frac{e^{\frac{x}{2}}}{e^x+1}\,dx = \int \frac{e^{\frac{x}{2}}}{\left(e^{\frac{x}{2}}\right)^2+1}\,dx = 2\int \frac{\frac{1}{2}e^{\frac{x}{2}}\,dx}{\left(e^{\frac{x}{2}}\right)^2+1} = 2\int \frac{du}{u^2+1}$

$= 2\tan^{-1} u + C$

$= 2\tan^{-1}\left(e^{\frac{x}{2}}\right) + C$

c $\int x\left(1+x^2\right)^4 dx$

Let $u = 1 + x^2$, $\frac{du}{dx} = 2x$, so $du = 2x\,dx$

$\int x\left(1+x^2\right)^4 dx = \frac{1}{2}\int\left(1+x^2\right)^4 \times 2\ dx = \frac{1}{2}\int u^4\,du$

$= \frac{1}{2}\times\frac{u^5}{5}+C$

$= \frac{1}{10}\left(1+x^2\right)^5+C$

d $\int \frac{x}{\sqrt{1-x}}\,dx$

Let $u = 1 - x$, $\frac{du}{dx} = -1$, and $x = 1 - u$, so $du = -1\,dx$ and $dx = -du$.

$\int \frac{x}{\sqrt{1-x}}\,dx = \int \frac{1-u}{\sqrt{u}} - du = \int \frac{u-1}{\sqrt{u}}\,du$

$= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du$

$= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C$

$= \frac{2}{3}u^{\frac{1}{2}}(u-3)+C$

$= \frac{2}{3}\sqrt{1-x}\,(1-x-3)+C$

$= \frac{2}{3}\sqrt{1-x}\,(-2-x)+C$

$= -\frac{2}{3}\sqrt{1-x}\,(x+2)+C$

$= -\frac{2}{3}(x+2)\sqrt{1-x}+C$

e $\int \frac{e^x\ dx}{\sqrt{e^x - 1}}$, $x > 0$

Let $u = e^x - 1$, $\frac{du}{dx} = e^x$, so $du = e^x\ dx$.

$$\int \frac{e^x\ dx}{\sqrt{e^x-1}} = \int \frac{du}{\sqrt{u}} = \int u^{-\frac{1}{2}}\ du$$
$$= 2u^{\frac{1}{2}} + C$$
$$= 2\sqrt{e^x - 1} + C$$

f $\int \frac{dx}{a^2 + x^2}$

Let $x = a \tan \theta$, $\frac{dx}{d\theta} = a \sec^2 \theta$, so $dx = a \sec^2 \theta\ d\theta$.

$$\int \frac{dx}{a^2+x^2} = \int \frac{a \sec^2 \theta\ d\theta}{a^2 + a^2 \tan^2 \theta} = \int \frac{a \sec^2 \theta\ d\theta}{a^2\left(1 + \tan^2 \theta\right)}$$
$$= \int \frac{a \sec^2 \theta\ d\theta}{a^2 \sec^2 d\theta} = \int \frac{1}{a}\ d\theta$$
$$= \frac{1}{a}\theta + C$$
$$= \frac{1}{a}\tan^{-}\left(\frac{x}{a}\right) + C$$

g $\int x\sqrt{x-3}\ dx$

Let $u = x - 3$, $\frac{du}{dx} = 1$ and $x = u + 3$, so $du = dx$.

$$\int x\sqrt{x-3}\ dx = \int (u+3)\sqrt{u}\ du$$
$$= \int u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\ du$$
$$= \frac{2}{5}u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + C$$
$$= \frac{2}{5}(x-3)^2\sqrt{x-3} + 2(x-3)\sqrt{x-3} + C$$

h $\int \frac{x}{\sqrt{x+1}}\,dx$

Let $x = u^2 - 1$, $\frac{dx}{du} = 2u$ and $x + 1 = u^2$, so $dx = 2u\,du$.

$$\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{u^2-1}{\sqrt{u^2}} \times 2u \times du = \int \frac{u^2-1}{\cancel{u}} \times 2\cancel{u}\,du$$
$$= 2\int u^2 - 1\,du$$
$$= \frac{2}{3}u^3 - 2u + C$$
$$= \frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C$$
$$= (x+1)^{\frac{1}{2}}\left[\frac{2}{3}(x+1) - 2\right] + C$$
$$= \frac{2}{3}(x+1)^{\frac{1}{2}}\left[(x+1) - 3\right] + C$$
$$= \frac{2}{3}(x-2)\sqrt{x+1} + C$$

i $\int \frac{x}{\sqrt{x-1}}\,dx$

Let $x = u^2 + 1$, $\frac{dx}{du} = 2u$ and $x - 1 = u^2$, so $dx = 2u\,du$.

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u^2+1}{\sqrt{u^2}} \times 2u \times du = \int \frac{u^2+1}{\cancel{u}} \times 2\cancel{u}\,du$$
$$= 2\int u^2 + 1\,du$$
$$= \frac{2}{3}u^3 + 2u + C$$
$$= \frac{2}{3}(x-1)^{\frac{3}{2}} - 2(x-1)^{\frac{1}{2}} + C$$
$$= (x-1)^{\frac{1}{2}}\left[\frac{2}{3}(x-1) - 2\right] + C$$
$$= \frac{2}{3}(x-1)^{\frac{1}{2}}\left[(x-1) - 3\right] + C$$
$$= \frac{2}{3}(x+2)\sqrt{x-1} + C$$

Question 2

a $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Let $u = \sqrt{x}$, so $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ and du = $\frac{1}{2\sqrt{x}} dx$.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2\int e^{\sqrt{x}} \times \frac{1}{2\sqrt{x}}\ dx$$
$$= 2\int e^{u}\ du$$
$$= 2e^{u} + C$$
$$= 2e^{\sqrt{x}} + C$$

b $\int \frac{x}{e^{x^2}} dx$

Let $u = x^2$, $\frac{du}{dx} = 2x$, so $du = 2x\ dx$.

$$\int \frac{x}{e^{x^2}} dx = \frac{1}{2}\int \frac{1}{e^{x^2}} \times 2x\ dx$$
$$= \frac{1}{2}\int e^{-u}\ du$$
$$= -\frac{1}{2}e^{-u} + C$$
$$= -\frac{1}{2e^{x^2}} + C$$

c $\int \frac{x}{\sqrt{x^2+1}}\ dx$

Let $u = x^2 + 1$, $\frac{du}{dx} = 2x$, so $du = 2x\ dx$.

$$\int \frac{x}{\sqrt{x^2+1}}\ dx = \frac{1}{2}\int \frac{1}{\sqrt{x^2+1}} \times 2x\ dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{u}} du = \frac{1}{2}\int u^{-\frac{1}{2}}\ du$$
$$= u^{\frac{1}{2}} + C$$
$$= \sqrt{x^2+1} + C$$

d $\displaystyle\int \frac{(1+\ln x)^2}{x}\,dx$

Let $u = 1 + \ln x$, $\dfrac{du}{dx} = \dfrac{1}{x}$, so $du = \dfrac{1}{x}\,dx$.

$$\int \frac{(1+\ln x)^2}{x}\,dx = \int (1+\ln x)^2 \times \frac{1}{x}\,dx$$
$$= \int u^2\,du$$
$$= \frac{u^3}{3} + C$$
$$= \frac{(1+\ln x)^3}{3} + C$$

e $\displaystyle\int \frac{\left(\sin^{-}[x]+1\right)^2}{\sqrt{1-x^2}}\,dx$

Let $u = \sin^{-1}[x] + 1$, $\dfrac{du}{dx} = \dfrac{1}{\sqrt{1-x^2}}$.

$$\int \frac{\left(\sin^{-}[x]+1\right)^2}{\sqrt{1-x^2}}\,dx = \int \left(\sin^{-}[x]+1\right)^2 \times \frac{1}{\sqrt{1-x^2}}\,dx$$
$$= \int u^2\,du$$
$$= \frac{u^3}{3} + C$$
$$= \frac{1}{3}\left(\sin^{-}[x]+1\right)^3 + C$$
$$= \frac{1}{3}\left(1+\sin^{-}x\right)^3 + C$$

f $\int \frac{\tan^{-}\left(x+1\right)}{\left(x+1\right)^2+1}\,dx$

Let $u = \tan^{-1}(x + 1)$, $\frac{du}{dx}=\frac{1}{\left(x+1\right)^2+1}$.

$$\int \frac{\tan^{-}\left(x+1\right)}{\left(x+1\right)^2+1}\,dx = \int \tan^{-}\left(x+1\right)\times\frac{1}{\left(x+1\right)^2+1}\,dx$$
$$=\int u\ du$$
$$=\frac{1}{2}u^2+C$$
$$=\frac{1}{2}\left[\tan^{-}\left(x+1\right)\right]^2+C$$

g $\int \frac{dx}{x\left(\ln x\right)^2}$

Let $u = \ln x$, $\frac{du}{dx}=\frac{1}{x}$, so $du = \frac{1}{x}\ dx$.

$$\int \frac{dx}{x\left(\ln x\right)^2}=\int \frac{1}{\left(\ln x\right)^2}\times\frac{1}{x}dx$$
$$=\int \frac{1}{u^2}\ du=\int u^{-2}\ du$$
$$=-\frac{1}{u}+C$$
$$=-\frac{1}{\ln\ x}+C$$

h $\int \frac{dx}{x\ln x}$

Let $u = \ln x$, $\frac{du}{dx}=\frac{1}{x}$, so $du = \frac{1}{x}\ dx$.

$$\int \frac{dx}{x\ln x}=\int \frac{1}{\ln x}\times\frac{1}{x}dx$$
$$=\int \frac{1}{u}du$$
$$=\ln u+C$$
$$=\ln\left|\ln x\right|+C$$

Question 3

a $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x} dx$

Let $u = 1 + \sin x$, $\frac{du}{dx} = \cos x$.

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} \times \cos x \; dx$$

When $x = 0$, $u = 1$, and when $x = \frac{\pi}{2}$, $u = 2$.

$$= \int^2 \frac{1}{u} du$$
$$= [\ln u]_1^2$$
$$= \ln 2 - \ln 1$$
$$= \ln 2$$

b $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)^2} dx$

Let $u = 1 + \sin x$, $\frac{du}{dx} = \cos x$.

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)^2} dx = \int_0^{\frac{\pi}{2}} \frac{1}{(1+\sin x)^2} \times \cos x \; dx$$

When $x = 0$, $u = 1$, and when $x = \frac{\pi}{2}$, $u = 2$.

$$= \int^2 \frac{1}{u^2} du$$
$$= \left[-\frac{1}{u} \right]_1^2$$
$$= -\frac{1}{2} - (-1)$$
$$= \frac{1}{2}$$

c $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)^3}\,dx$

Let $u = 1 + \sin x$, $\frac{du}{dx} = \cos x$.

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)^3}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{(1+\sin x)^3} \times \cos x\ dx$$

When $x = 0$, $u = 1$, and when $x = \frac{\pi}{2}$, $u = 2$.

$$= \int^2 \frac{1}{u^3}\,du$$

$$= \left[-\frac{1}{2u^2}\right]_1^2$$

$$= -\frac{1}{8} - \left(-\frac{1}{2}\right)$$

$$= \frac{3}{8}$$

d $\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{1+\sin x} dx$

Let $u = 1 + \sin x$, so $\sin x = u - 1$, and $\frac{du}{dx} = \cos x$.

$$\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{1+\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin x} \times \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1-\sin^2 x}{1+\sin x} \times \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(1-\sin x)\cancel{(1+\sin x)}}{\cancel{1+\sin x}} \times \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (1-\sin x) \times \cos x \, dx$$

When $x = 0$, $u = 1$, and when $x = \frac{\pi}{2}$, $u = 2$.

$$= \int_1^2 \left[1-(u-1)\right] du = \int_1^2 (2-u) \, du$$

$$= \left[2u - \frac{u^2}{2}\right]_1^2$$

$$= \left(2[2] - \frac{[2]^2}{2}\right) - \left(2[1] - \frac{[1]^2}{2}\right)$$

$$= 4 - 2 - 2 + \frac{1}{2}$$

$$= \frac{1}{2}$$

e $\int \sin^2 x \cos^3 x \, dx$

Let $u = \sin x$, so $\frac{du}{dx} = \cos x$.

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \times \cos x \, dx$$
$$= \int \sin^2 x \left(1 - \sin^2 x\right) \times \cos x \, dx$$
$$= \int u^2 \left(1 - u^2\right) du = \int \left(u^2 - u^4\right) du$$
$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$
$$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$
$$= -\frac{1}{5}\sin^5 x + \frac{1}{3}\sin^3 x + C$$

f $\int \sin^3 x \cos^2 x \, dx$

Let $u = \cos x$, so $\frac{du}{dx} = -\sin x$.

$$\int \sin^3 x \cos^2 x \, dx = -\int \sin^2 x \cos^2 x \times \sin x \, dx$$
$$= -\int \left(1 - \cos^2 x\right) \cos^2 x \times \sin x \, dx$$
$$= -\int \left(1 - u^2\right) u^2 \, du = \int \left(u^4 - u^2\right) du$$
$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$
$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

g $\int \sin^4 x \cos^5 x \, dx$

Let $u = \sin x$, so $\dfrac{du}{dx} = \cos x$.

$$\int \sin^4 x \cos^5 x \, dx = \int \sin^4 x \cos^4 x \times \cos x \, dx$$
$$= \int \sin^4 x \left(\cos^2 x\right)^2 \times \cos x \, dx$$
$$= \int \sin^4 x \left(1 - \sin^2 x\right)^2 \times \cos x \, dx$$
$$= \int u^4 \left(1 - u^2\right)^2 \, du = \int u^4 \left(1 - 2u^2 + u^4\right) du$$
$$= \int \left(u^4 - 2u^6 + u^8\right) du$$
$$= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C$$
$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 + C$$
$$= \frac{1}{9}\sin^9 x - \frac{2}{7}\sin^7 x + \frac{1}{5}\sin^5 x + C$$

h $\int \sin^5 x \cos^4 x \, dx$

Let $u = \cos x$, so $\dfrac{du}{dx} = -\sin x$.

$$\int \sin^5 x \cos^4 x \, dx = -\int \sin^4 x \cos^4 x \times -\sin x \, dx$$
$$= -\int \left(\sin^2 x\right)^2 \cos^4 x \times -\sin x \, dx$$
$$= -\int \left(1 - \cos^2 x\right)^2 \cos^4 x \times -\sin x \, dx$$
$$= -\int \left(1 - u^2\right) u^4 \, du = -\int \left(1 - 2u^2 + u^4\right) u^4 \, du$$
$$= -\int \left(u^4 - 2u^6 + u^8\right) du$$
$$= -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C$$
$$= -\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C$$
$$= -\frac{1}{9}\cos^9 x + \frac{2}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$$

i $\int \frac{\sin\theta}{\cos^2\theta}\,d\theta$

Let $u = \cos\theta$, so $\frac{du}{dx} = -\sin\theta$.

$$\int \frac{\sin\theta}{\cos^2\theta}\,d\theta = -\int \frac{1}{\cos^2\theta} \times -\sin\theta\,d\theta$$
$$= -\int \frac{1}{u^2}\,du$$
$$= -\left(-\frac{1}{u}\right) + C$$
$$= \frac{1}{u} + C$$
$$= \frac{1}{\cos\theta} + C$$
$$= \sec\theta + C$$

j $\int \frac{\sin\theta}{\cos^3\theta}\,d\theta$

Let $u = \cos\theta$, so $\frac{du}{dx} = -\sin\theta$.

$$\int \frac{\sin\theta}{\cos^3\theta}\,d\theta = -\int \frac{1}{\cos^3\theta} \times -\sin\theta\,d\theta$$
$$= -\int \frac{1}{u^3}\,du$$
$$= -\left(-\frac{1}{2u^2}\right) + C$$
$$= \frac{1}{2u^2} + C$$
$$= \frac{1}{2\cos^2\theta} + C$$
$$= \frac{1}{2}\sec^2\theta + C$$

k $\int \frac{\sin\theta}{\cos^4\theta}\,d\theta$

Let $u = \cos\theta$, so $\frac{du}{dx} = -\sin\theta$.

$$\int \frac{\sin\theta}{\cos^4\theta}\,d\theta = -\int \frac{1}{\cos^4\theta} \times -\sin\theta\,d\theta$$
$$= -\int \frac{1}{u^4}\,du$$
$$= -\left(-\frac{1}{3u^3}\right) + C$$
$$= \frac{1}{3u^3} + C$$
$$= \frac{1}{3\cos^3\theta} + C$$
$$= \frac{1}{3}\sec^3\theta + C$$

l $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

Let $u = \sin\theta$, so $\frac{du}{dx} = \cos\theta$.

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{1}{\sin^2\theta} \times \cos\theta\,d\theta$$
$$= \int \frac{1}{u^2}\,du$$
$$= -\frac{1}{u} + C$$
$$= -\frac{1}{\sin\theta} + C$$
$$= -\operatorname{cosec}\theta + C$$

m $\int \frac{\cos\theta}{\sin^3\theta}\, d\theta$

Let $u = \sin\theta$, so $\frac{du}{dx} = \cos\theta$.

$$\int \frac{\cos\theta}{\sin^3\theta}\, d\theta = \int \frac{1}{\sin^3\theta} \times \cos\theta\, d\theta$$
$$= \int \frac{1}{u^3}\, du$$
$$= -\frac{1}{2u^2} + C$$
$$= -\frac{1}{2\sin^2\theta} + C$$

n $\int \frac{\cos\theta}{\sin^4\theta}\, d\theta$

Let $u = \sin\theta$, so $\frac{du}{dx} = \cos\theta$.

$$\int \frac{\cos\theta}{\sin^4\theta}\, d\theta = \int \frac{1}{\sin^4\theta} \times \cos\theta\, d\theta$$
$$= \int \frac{1}{u^4}\, du$$
$$= -\frac{1}{3u^3} + C$$
$$= -\frac{1}{3\sin^3\theta} + C$$

o $\int \frac{\sin^3 \theta}{\cos^4 \theta}\, d\theta$

Let $u = \cos\theta$, so $\frac{du}{dx} = -\sin\theta$.

$$\int \frac{\sin^3 \theta}{\cos^4 \theta}\, d\theta = -\int \frac{\sin^2 \theta}{\cos^4 \theta} \times -\sin\theta\, d\theta$$
$$= -\int \frac{\left(1-\cos^2\theta\right)}{\cos^4\theta} \times -\sin\theta\, d\theta$$
$$= -\int \frac{\left(1-u^2\right)}{u^4}\, du$$
$$= \int -\frac{1}{u^4} + \frac{1}{u^2}\, du$$
$$= \frac{1}{3u^3} - \frac{1}{u} + C$$
$$= \frac{1}{3\cos^3\theta} - \frac{1}{\cos\theta} + C$$

p $\int \frac{\sin^3 \theta}{\cos^2 \theta}\, d\theta$

Let $u = \cos\theta$, so $\frac{du}{dx} = -\sin\theta$.

$$\int \frac{\sin^3 \theta}{\cos^2 \theta}\, d\theta = -\int \frac{\sin^2 \theta}{\cos^2 \theta} \times -\sin\theta\, d\theta$$
$$= -\int \frac{\left(1-\cos^2\theta\right)}{\cos^2\theta} \times -\sin\theta\, d\theta$$
$$= -\int \frac{\left(1-u^2\right)}{u^2}\, du$$
$$= \int -\frac{1}{u^2} + 1\, du$$
$$= \frac{1}{u} + u + C$$
$$= \frac{1}{\cos\theta} + \cos\theta + C$$
$$= \sec\theta + \cos\theta + C$$

Question 4

a $\int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x\, dx$

Let $u = \tan x$, $\frac{du}{dx} = \sec^2 x$.

When $x = 0$, $u = 0$, and when $x = \frac{\pi}{4}$, $u = 1$.

$$\int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x\, dx = \int_0 u^4\, du$$
$$= \left[\frac{u^5}{5}\right]_0^1$$
$$= \frac{(1)^5}{5} - \frac{(0)^5}{5}$$
$$= \frac{1}{5}$$

b $\int_0^{\frac{\pi}{4}} \frac{e^{\tan\theta}}{\cos^2\theta}\, d\theta$

Let $u = \tan\theta$, so $\frac{du}{d\theta} = \sec^2\theta = \frac{1}{\cos^2\theta}$

When $x = 0$, $u = 0$, and when $x = \frac{\pi}{4}$, $u = 1$.

$$\int_0^{\frac{\pi}{4}} \frac{e^{\text{an}\,\theta}}{\cos^2\theta}\, d\theta = \int_0^{\frac{\pi}{4}} e^{\text{an}\,\theta} \times \frac{1}{\cos^2\theta}\, d\theta$$
$$= \int_0^1 e^u\, du$$
$$= \left[e^u\right]_0^1$$
$$= e^1 - e^0$$
$$= e - 1$$

c $\int_{2}^{\sqrt{5}} \frac{x}{\sqrt{x^2-4}}\,dx$

Let $u = x^2 - 4$, so $\frac{du}{dx} = 2x$, $du = 2x\,dx$.

When $x = 2$, $u = 0$, and when $x = \sqrt{5}$, $u = 1$.

$$\int_{2}^{\sqrt{5}} \frac{x}{\sqrt{x^2-4}}\,dx = \frac{1}{2}\int_{2}^{\sqrt{5}} \frac{1}{\sqrt{x^2-4}} \times 2x\,dx$$
$$= \frac{1}{2}\int_{0}^{1} u^{-\frac{1}{2}}\,du$$
$$= \frac{1}{2}\left[2\sqrt{u}\right]_{0}^{1}$$
$$= \frac{1}{2}\left(2\sqrt{1} - 2\sqrt{0}\right)$$
$$= \frac{1}{2}(2)$$
$$= 1$$

d $\int_{\ln\frac{\pi}{6}}^{\ln\frac{\pi}{4}} e^x \sin e^x\,dx$

Let $u = e^x$, $\frac{du}{dx} = e^x$, so $du = e^x\,dx$.

When $x = \ln\frac{\pi}{6}$, $u = \frac{\pi}{6}$, and when $x = \ln\frac{\pi}{4}$, $u = \frac{\pi}{4}$

$$\int_{\ln\frac{\pi}{6}}^{\ln\frac{\pi}{4}} e^x \sin e^x\,dx = \int_{\ln\frac{\pi}{6}}^{\ln\frac{\pi}{4}} \left(\sin e^x\right) \times e^x\,dx$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin u\,du$$
$$= \left[-\cos u\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$
$$= -\cos\left(\frac{\pi}{4}\right) - \left[-\cos\left(\frac{\pi}{6}\right)\right]$$
$$= -\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{3}-\sqrt{2}}{2}$$

e

$$\int_0^{\frac{\pi}{4}} \cos^2 x\, dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1+\cos 2x)\ dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos 2x)\ dx$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x \right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left\{\left[\frac{\pi}{4} + \frac{1}{2}\sin\left(2\times\frac{\pi}{4} \right)\right] - \left[0 + \frac{1}{2}\sin\left(2\times 0\right)\right]\right\}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\sin\left[\frac{\pi}{2}\right]\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right)$$

$$= \frac{1}{4}\left(\frac{\pi}{2} + 1\right)$$

f

$$\int_0^{\frac{\pi}{4}} \cos^2 4x\, dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1+\cos 8x)\ dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos 8x)\ dx$$

$$= \frac{1}{2}\left[x + \frac{1}{8}\sin 8x \right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left\{\left[\frac{\pi}{4} + \frac{1}{8}\sin\left(8\times\frac{\pi}{4} \right)\right] - \left[0 + \frac{1}{8}\sin\left(8\times 0\right)\right]\right\}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{8}\sin\left[2\pi\right]\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} \right)$$

$$= \frac{\pi}{8}$$

g

$$\int_0^{\frac{\pi}{4}} \sin^2 2x\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1-\cos 4x)\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1-\cos 4x)\,dx$$

$$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left\{\left[\frac{\pi}{4} - \frac{1}{4}\sin\left(4\times\frac{\pi}{4}\right)\right] - \left[0 - \frac{1}{4}\sin(4\times 0)\right]\right\}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{4}\sin[\pi]\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{8}$$

Question 5

a $\int_3^8 \frac{x}{(x+1)\sqrt{x+1}}\ dx$

Let $u = x + 1$, $\frac{du}{dx} = 1$, so $du = dx$, and $x = u - 1$.

When $x = 3$, $u = 4$, and when $x = 8$, $u = 9$.

$$\int_3^8 \frac{x}{(x+1)\sqrt{x+1}}\ dx = \int_4^9 \frac{u-1}{u\sqrt{u}} du$$

$$= \int_4^9 \frac{u-1}{u^{\frac{3}{2}}} du = \int_4^9 u^{-\frac{1}{2}} - u^{-\frac{3}{2}}\ du$$

$$= \left[2u^{\frac{}{2}} + \frac{2}{u^{\frac{}{2}}} \right]_4^9$$

$$= \left(2\sqrt{9} + \frac{2}{\sqrt{9}} \right) - \left(2\sqrt{4} + \frac{2}{\sqrt{4}} \right)$$

$$= 6 + \frac{2}{3} - 4 - 1$$

$$= 1\frac{2}{3}$$

b $\int_0^3 \sqrt{9-x^2}\, dx$

Let $x = 3 \sin \theta$, so $\dfrac{dx}{d\theta} = 3 \cos \theta$, $dx = 3 \cos \theta \, d\theta$.

When $x = 0$, $\theta = 0$, and when $x = 3$, $\theta = \dfrac{\pi}{2}$.

$$\int_0^3 \sqrt{9-x^2}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{9-(3\sin\theta)^2} \times 3\cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{9(1-\sin^2\theta)} \times 3\cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 3\sqrt{\cos^2\theta} \times 3\cos\theta \, d\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \frac{1}{2}[1+\cos 2\theta] \, d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{9}{2}\left\{\left[\frac{\pi}{2} + \frac{1}{2}\sin\left(2\times\frac{\pi}{2}\right)\right] - \left[0 + \frac{1}{2}\sin(2\times 0)\right]\right\}$$

$$= \frac{9}{2}\left[\frac{\pi}{2} + \frac{1}{2}\sin(\pi)\right]$$

$$= \frac{9\pi}{4}$$

c $\int_0^3 \frac{6}{9+x^2}\,dx$

Let $x = 3\tan\theta$, $\frac{dx}{d\theta} = 3\sec^2\theta$, $dx = 3\sec^2\theta\, d\theta$.

When $x = 0$, $\theta = 0$, and when $x = 3$, $\theta = \frac{\pi}{4}$.

$$\int_0^3 \frac{6}{9+x^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{6}{9+(3\tan\theta)^2}\times 3\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{18\sec^2\theta}{9+9\tan^2\theta}\,d\theta$$

$$= \frac{18}{9}\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^2\theta}\,d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} d\theta$$

$$= 2[\theta]_0^{\frac{\pi}{4}}$$

$$= 2\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{2}$$

d $\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$

Let $u = 1 + e^x$, $\frac{du}{dx} = e^x$, $du = e^x\,dx$.

When $x = 0$, $u = 2$, and when $x = 1$, $u = 1 + e$.

$$\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx = \int_0^{1+e} \frac{1}{u^2}\,du$$

$$= \left[-\frac{1}{u}\right]_2^{1+e}$$

$$= -\frac{1}{1+e} - \left(-\frac{1}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{1+e}$$

e $\int_0^1 \frac{\sqrt{x}}{1+x}\,dx$

Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$, so $dx = 2\sqrt{x}\,du = 2u\,du$.

When $x = 0$, $u = 0$, and when $x = 1$, $u = 1$.

$$\int_0^1 \frac{\sqrt{x}}{1+x}\,dx = \int_0^1 \frac{u}{1+u^2} \times 2u\,du$$

$$= 2\int_0^1 \frac{u^2+1-1}{u^2+1}\,du$$

$$= 2\int_0^1 1 - \frac{1}{u^2+1}\,du$$

$$= 2\left[u - \tan^{-} u\right]_0^1$$

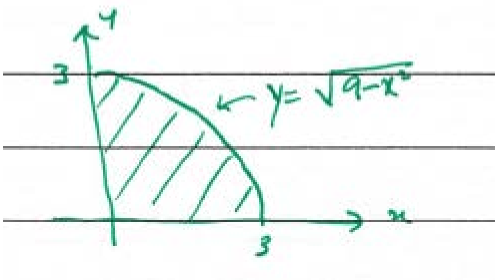
$$= 2\left\{\left[1-\tan^{-1}(1)\right] - \left[0 - \tan^{-1}(0)\right]\right\}$$

$$= 2\left(1 - \frac{\pi}{4}\right)$$

$$= 2 - \frac{\pi}{2}$$

Question 6

$\int_0^3 \sqrt{9-x^2}\,dx$



Area = $\frac{1}{4} \times \pi \times r^2$

$$= \frac{1}{4} \times \pi \times 3^2$$

$$= \frac{9\pi}{4}$$

Question 7

a $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$

Let $t = \tan \frac{x}{2}$, $dx = \frac{2}{1+t^2}\,dt$, $\sin x = \frac{2t}{1+t^2}$.

When $x = 0$, $t = 0$, when $x = \frac{\pi}{2}$, $t = \tan \frac{\pi}{4} = 1$.

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx = \int_0 \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{\frac{2}{\cancel{1+t^2}}}{\frac{1+t^2+2t}{\cancel{1+t^2}}}\,dt$$

$$= \int_0^1 \frac{2}{t^2+2t+1}\,dt = 2\int_0^1 \frac{1}{(t+1)^2}\,dt$$

$$= 2\int_0^1 (t+1)^{-2}\,dt$$

$$= 2\left[-\frac{1}{t+1}\right]_0^1$$

$$= 2\left[-\frac{1}{(1)+1} - \left(-\frac{1}{(0)+1}\right)\right]$$

$$= 2\left(-\frac{1}{2} - [-1]\right)$$

$$= 2\left(\frac{1}{2}\right)$$

$$= 1$$

b $\displaystyle\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{dx}{\sin x}$

Let $t = \tan\dfrac{x}{2}$, $dx = \dfrac{2}{1+t^2}\,dt$, $\sin x = \dfrac{2t}{1+t^2}$.

When $x = \dfrac{\pi}{2}$, $t = \tan\dfrac{\pi}{4} = 1$, when $x = \dfrac{2\pi}{3}$, $t = \tan\dfrac{\pi}{3} = \sqrt{3}$.

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{dx}{\sin x} = \int^{\sqrt{3}} \frac{1}{\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int_1^{\sqrt{3}} \frac{2}{2t}\,dt = \int_1^{\sqrt{3}} \frac{1}{t}\,dt$$

$$= \left[\ln t\right]_1^{\sqrt{3}}$$

$$= \ln\sqrt{3} - \ln 1$$

$$= \ln\sqrt{3} = \ln\left(3^{\frac{1}{2}}\right) = \frac{1}{2}\ln 3$$

c $\displaystyle\int \frac{1}{1-\cos x}\,dx$

Let $t = \tan\dfrac{x}{2}$, $dx = \dfrac{2}{1+t^2}\,dt$, $\cos x = \dfrac{1-t^2}{1+t^2}$.

$$\int \frac{1}{1-\cos x}\,dx = \int \frac{1}{1-\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{1+t^2-\left(1-t^2\right)}\,dt$$

$$= \int \frac{2}{2t^2}\,dt = \int \frac{1}{t^2}\,dt = \int t^{-2}\,dt$$

$$= -\frac{1}{t} + C$$

$$= -\frac{1}{\tan\left(\frac{x}{2}\right)} + C$$

$$= -\cot\left(\frac{x}{2}\right) + C$$

d $\int_0^{\frac{\pi}{3}} \frac{1}{1+\sin\theta} d\theta$

Let $t = \tan\frac{\theta}{2}$, $d\theta = \frac{2}{1+t^2}\, dt$, $\sin\theta = \frac{2t}{1+t^2}$.

When $\theta = 0$, $t = 0$, when $\theta = \frac{\pi}{3}$, $t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

$$\int_0^{\frac{\pi}{3}} \frac{1}{1+\sin\theta}\, d\theta = \int_0^{\sqrt{3}} \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\, dt$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{\frac{2}{\cancel{1+t^2}}}{\frac{1+t^2+2t}{\cancel{1+t^2}}}\, dt$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{t^2+2t+1}\, dt = 2\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{(t+1)^2}\, dt$$

$$= 2\int_0^{\frac{1}{\sqrt{3}}} (t+1)^{-2}\, dt$$

$$= 2\left[-\frac{1}{t+1}\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= 2\left[-\frac{1}{\left(\frac{1}{\sqrt{3}}\right)+1} - \left(-\frac{1}{(0)+1}\right)\right]$$

$$= 2\left(-\frac{\sqrt{3}}{1+\sqrt{3}} - [-1]\right)$$

$$= 2 - \frac{2\sqrt{3}}{1+\sqrt{3}}$$

$$= 2 - \frac{2\sqrt{3}}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}}$$

$$= 2 - \left[\frac{2\sqrt{3}-6}{1-3}\right]$$

$$= 2 - \left(\sqrt{3}-3\right)$$

$$= -1 + \sqrt{3}$$

As required.

e $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta + 2}$

Let $t = \tan\frac{\theta}{2}$, $d\theta = \frac{2}{1+t^2}\,dt$, $\sin\theta = \frac{2t}{1+t^2}$.

When $\theta = 0$, $t = 0$, when $\theta = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4} = 1$.

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta + 2} = \int_0^1 \frac{1}{\frac{2t}{1+t^2}+2} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{2t+2\left(1+t^2\right)}\,dt = \frac{2}{2}\int_0^1 \frac{1}{t+\left(1+t^2\right)}\,dt$$

$$= \int_0^1 \frac{1}{t^2+t+1}\,dt = \int_0^1 \frac{1}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}\,dt$$

$$= \left[\frac{2}{\sqrt{3}}\tan^{-}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left\{\tan^{-1}\left(\frac{[1]+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) - \tan^{-1}\left(\frac{[0]+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right\}$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}}\right) - \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{3}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{2\pi}{6\sqrt{3}} = \frac{\pi}{3\sqrt{3}}$$

As required.

f $\int_0^{\frac{\pi}{3}} \frac{d\theta}{1+\cos 2\theta}$

Let $t = \tan\theta$, $d\theta = \frac{1}{1+t^2}\,dt$, $\cos 2\theta = \frac{1-t^2}{1+t^2}$.

When θ = 0, t = 0, and when $\theta = \frac{\pi}{3}$, t = tan $\frac{\pi}{3} = \sqrt{3}$.

$$\int_0^{\frac{\pi}{3}} \frac{d\theta}{1+\cos 2\theta} = \int_0^{\sqrt{3}} \frac{1}{1+\frac{1-t^2}{1+t^2}} \times \frac{1}{1+t^2}\,dt$$

$$= \int_0^{\sqrt{3}} \frac{1}{1+t^2+\left(1-t^2\right)}\,dt = \int_0^{\sqrt{3}} \frac{1}{2}\,dt$$

$$= \frac{1}{2}\left[t\right]_0^{\sqrt{3}}$$

$$= \frac{1}{2}\left(\sqrt{3}\right) = \frac{\sqrt{3}}{2}$$

Question 8

a $\displaystyle\int \frac{dx}{x^2+4}$

Let $x = 2\tan\theta$, $\dfrac{dx}{d\theta} = 2\sec^2\theta$, so $dx = 2\sec^2\theta\, d\theta$, $\theta = \tan^{-1}\left(\dfrac{x}{2}\right)$

$$\int \frac{dx}{x^2+4} = \int \frac{2\sec^2\theta}{4\tan^2\theta+4}\,d\theta$$

$$= \frac{2}{4}\int \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta = \frac{1}{2}\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{2}\int d\theta$$

$$= \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\tan^{-}\left(\frac{x}{2}\right) + C$$

b $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$

Let $x = 3\sin\theta$, $\dfrac{dx}{d\theta} = 3\cos\theta$, so $dx = 3\cos\theta\, d\theta$, $\theta = \sin^{-1}\left(\dfrac{x}{3}\right)$.

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3\cos\theta}{\sqrt{9-(3\sin\theta)^2}}\,d\theta$$

$$= \int \frac{3\cos\theta}{\sqrt{9-9\sin^2\theta}}\,d\theta = \int \frac{3\cos\theta}{\sqrt{9\left(1-\sin^2\theta\right)}}\,d\theta$$

$$= \frac{3}{3}\int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta = \int \frac{\cos\theta}{\cos\theta}\,d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \sin^{-}\left(\frac{x}{3}\right) + C$$

c $\int \frac{dx}{4x^2+9}$

Let $x = \frac{3}{2}\tan\theta$, $\frac{dx}{d\theta} = \frac{3}{2}\sec^2\theta$, so $dx = \frac{3}{2}\sec^2\theta\, d\theta$, $\theta = \tan^{-1}\left(\frac{2x}{3}\right)$

$$\int \frac{dx}{4x^2+9} = \int \frac{\frac{3}{2}\sec^2\theta}{4\left(\frac{3}{2}\tan\theta\right)^2+9}\, d\theta$$

$$= \frac{3}{2}\int \frac{\sec^2\theta}{9\left(\tan^2\theta+1\right)}\, d\theta$$

$$= \frac{3}{2}\times\frac{1}{9}\int \frac{\sec^2\theta}{\sec^2\theta} d\theta$$

$$= \frac{1}{6}\int d\theta$$

$$= \frac{\theta}{6} + C$$

$$= \frac{\tan^{-1}\left(\frac{2x}{3}\right)}{6} + C$$

Question 9

a $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

Let $x = 2\sin\theta$, $\frac{dx}{d\theta} = 2\cos\theta$, so $dx = 2\cos\theta\, d\theta$.

When $x = 0$, $\theta = 0$, and when $x = 1$, $\theta = \frac{\pi}{6}$.

$$\int_0 \frac{dx}{\sqrt{4-x^2}} = \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{\sqrt{4-(2\sin\theta)^2}}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}}\, d\theta = \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{\sqrt{4(1-\sin^2\theta)}}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{\sqrt{4\cos^2\theta}}\, d\theta = \int_0^{\frac{\pi}{6}} \frac{2\cos\theta}{2\cos\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} d\theta$$

$$= [\theta]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{6}$$

b $\int_0^2 \frac{2\,dx}{\sqrt{16-x^2}}$

Let $x = 4\sin\theta$, $\frac{dx}{d\theta} = 4\cos\theta$, so $dx = 4\cos\theta\, d\theta$.

When $x = 0$, $\theta = 0$, and when $x = 2$, $\theta = \frac{\pi}{6}$.

$$\int_0^2 \frac{2\,dx}{\sqrt{16-x^2}} = \int_0^{\frac{\pi}{6}} \frac{2\times 4\cos\theta}{\sqrt{16-(4\sin\theta)^2}}\,d\theta$$

$$= 2\int_0^{\frac{\pi}{6}} \frac{4\cos\theta}{\sqrt{16-16\sin^2\theta}}\,d\theta = 2\int_0^{\frac{\pi}{6}} \frac{4\cos\theta}{\sqrt{16(1-\sin^2\theta)}}\,d\theta$$

$$= 2\int_0^{\frac{\pi}{6}} \frac{4\cos\theta}{\sqrt{16\cos^2\theta}}\,d\theta = 2\int_0^{\frac{\pi}{6}} \frac{4\cos\theta}{4\cos\theta}\,d\theta$$

$$= 2\int_0^{\frac{\pi}{6}} d\theta$$

$$= 2[\theta]_0^{\frac{\pi}{6}}$$

$$= 2\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3}$$

c $\displaystyle\int_0^{\frac{1}{3}} \frac{3\,dx}{1+9x^2}$

Let $x = \dfrac{1}{3}\tan\theta$, $\dfrac{dx}{d\theta} = \dfrac{1}{3}\sec^2\theta$, so $dx = \dfrac{1}{3}\sec^2\theta\,d\theta$.

When $x = 0$, $\theta = 0$, and when $x = \dfrac{1}{3}$, $\theta = \dfrac{\pi}{4}$.

$$\int_0^{\frac{1}{3}} \frac{3\,dx}{1+9x^2} = \int_0^{\frac{\pi}{4}} \frac{3\times\frac{1}{3}\sec^2\theta}{1+9\left(\frac{1}{3}\tan\theta\right)^2}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{1+9\left(\frac{1}{9}\tan^2\theta\right)}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \int_0^{\frac{\pi}{4}} d\theta$$

$$= [\theta]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4}$$

d $\displaystyle\int_3^4 \frac{2x\,dx}{\sqrt{25-x^2}}$

Let $u = 25 - x^2$, $\dfrac{du}{dx} = -2x$, so $-du = 2x\,dx$.

When $x = 3$, $u = 16$, and when $x = 4$, $u = 9$.

$$\int_3^4 \frac{2x\,dx}{\sqrt{25-x^2}} = \int_6^9 \frac{-du}{\sqrt{u}}$$

$$= -\int_6^9 u^{-\frac{1}{2}}\,du$$

$$= -\left[2u^{\frac{\,}{2}}\right]_{16}^{9}$$

$$= -2(3-4)$$

$$= 2$$

Question 10

a $\int_0^3 x^3\sqrt{9-x^2}\,dx$

Let $u^2 = 9 - x^2$, so $x^2 = 9 - u^2$, and $u = \sqrt{9-x^2}$.

So $\frac{du}{dx}=\frac{-x}{\sqrt{9-x^2}}, du=\frac{-x}{\sqrt{9-x^2}}dx, dx=\frac{\sqrt{9-x^2}}{-x}du$

When $x = 0$, $u^2 = 9 - 0 = 9$, so $u = 3$, and when $x = 3$, $u^2 = 9 - 9 = 0$, so $u = 0$.

$$\int_0^3 x^3\sqrt{9-x^2}\,dx=\int_3^0 x^3\sqrt{9-x^2}\ \frac{\sqrt{9-x^2}}{-x}\,du$$

$$=\int_3^0 -x^2\times u^2\,du$$

$$=-\int_3^0\left(9-u^2\right)u^2\,du$$

$$=-\int_3^0 9u^2-u^4\,du$$

$$=-\left[3u^3-\frac{u^5}{5}\right]_3^0$$

$$=-\left\{\left[3(0)^3-\frac{(0)^5}{5}\right]-\left[3(3)^3-\frac{(3)^5}{5}\right]\right\}$$

$$=-\left(-81+\frac{243}{5}\right)$$

$$=-\left(-\frac{162}{5}\right)=\frac{162}{5}=32\frac{2}{5}$$

b $\displaystyle\int \frac{dx}{1+\cos x+\sin x}$

Let $t = \tan\frac{x}{2}$, $\frac{dx}{dt}=\frac{2}{1+t^2}$ $dx=\frac{2}{1+t^2}dt$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$.

$$\int \frac{1}{1+\cos x+\sin x}dx=\int \frac{\frac{2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}\,dt$$

$$=\int \frac{\frac{2}{1+t^2}}{\frac{1+t^2}{1+t^2}+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}\,dt$$

$$=\int \frac{2}{1+t^2+1-t^2+2t}\,dt=\int \frac{2}{2+2t}\,dt$$

$$=\int \frac{dt}{1+t}$$

$$=\ln(1+t)+C$$

$$=\ln\left|1+\tan\left(\frac{x}{2}\right)\right|+C$$

c $\displaystyle\int \sec^3 x\,\tan x\,dx$

Let $u = \sec x$, $\frac{du}{dx}$ = $\sin x\sec^2 x = \tan x\sec x$, so $du = \tan x\sec x\,dx$.

$$\int \sec^3 x\,\tan x\,dx=\int u^2\,du$$

$$=\frac{u^3}{3}+C$$

$$=\frac{\sec^3 x}{3}+C$$

d $\displaystyle\int_0^{\sqrt{2}} \frac{dx}{\sqrt{\left(4-x^2\right)^3}}$

Let $x = 2\sin\theta$, $\dfrac{dx}{d\theta} = 2\cos\theta$, so $dx = 2\cos\theta\, d\theta$.

When $x = 0$, $\theta = 0$, and when $x = \sqrt{2}$, $\theta = \dfrac{\pi}{4}$.

$$\int_0^{\sqrt{2}} \frac{dx}{\sqrt{\left(4-x^2\right)^3}} = \int_0^{\frac{\pi}{4}} \frac{2\cos\theta}{\sqrt{\left(4-[2\sin\theta]^2\right)^3}}\, d\theta$$

$$=\int_0^{\frac{\pi}{4}} \frac{2\cos\theta}{\sqrt{\left(4-4\sin^2\theta\right)^3}}\, d\theta = \int_0^{\frac{\pi}{4}} \frac{2\cos\theta}{\sqrt{\left[4\left(1-\sin^2\theta\right)\right]^3}}\, d\theta$$

$$=2\times\frac{1}{\left(4^3\right)^{\frac{}{2}}}\int_0^{\frac{\pi}{4}} \frac{\cos\theta}{\sqrt{\left(\cos^2\theta\right)^3}}\, d\theta$$

$$=\frac{2}{8}\int_0^{\frac{\pi}{4}} \frac{\cos\theta}{\cos^3\theta}\, d\theta = \frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2\theta}\, d\theta$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{4}} \sec^2\theta\, d\theta$$

$$=\frac{1}{4}\left[\tan\theta\right]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{4}\left\{\tan\left(\frac{\pi}{4}\right)-\tan(0)\right\}$$

$$=\frac{1}{4}$$

e $\int \frac{e^x + e^{2x}}{e^{2x}+1}\ dx$

Let $u = e^x$, so $\frac{du}{dx} = e^x$, so $du = e^x\ dx$.

$$\int \frac{e^x + e^{2x}}{e^{2x}+1}\ dx = \int \frac{e^x\left(1+e^x\right)}{e^{2x}+1} dx$$
$$= \int \frac{1+e^x}{e^{2x}+1} \times e^x\ dx = \int \frac{1+u}{u^2+1}\ du$$
$$= \int \frac{1}{u^2+1}\ du + \frac{1}{2}\int \frac{2u}{u^2+1}\ du$$
$$= \tan^{-1} u + \frac{1}{2}\ln\left|u^2+1\right| + C$$
$$= \tan^{-1}\left(e^x\right) + \frac{1}{2}\ln\left|e^{2x}+1\right| + C$$

f

$$\int_0^2 \sqrt{x\left(4-x\right)}\ dx = \int_0^2 \sqrt{4x-x^2}\ dx$$
$$= \int_0^2 \sqrt{-4+4x-x^2+4}\ dx$$
$$= \int_0^2 \sqrt{4-\left(x-2\right)^2}\ dx$$

Let $u = x - 2$, so $\frac{du}{dx} = 1$, so $dx = du$.

When $x = 0$, $u = -2$, and when $x = 2$, $u = 0$.

$$\int_0^2 \sqrt{x\left(4-x\right)}\ dx = \int_{-2}^0 \sqrt{4-u^2}\ du$$

The graph of $y = \sqrt{4-u^2}$ is a semicircle with radius 2 and centred at (0, 0), so the integral is the area of a quadrant with radius 2.

$$\int_0^2 \sqrt{x\left(4-x\right)}\ dx = \frac{1}{4}\pi\left(2^2\right)$$
$$= \frac{1}{4}\pi\left(4\right)$$
$$= \pi$$

Alternatively, let $u = 2\sin\theta$, so $\theta = \sin^{-1}\left(\frac{u}{2}\right)$, $\frac{du}{d\theta} = 2\cos\theta$, $du = 2\cos\theta\, d\theta$.

When $u = -2$, $\theta = \sin^{-1}(-1) = -\frac{\pi}{2}$, and when $u = 0$, $\theta = \sin^{-1}(0) = 0$.

$$\begin{aligned}
\int_0^2 \sqrt{x(4-x)}\ dx &= \int_{-2}^0 \sqrt{4-u^2}\ du \\
&= \int_{-\frac{\pi}{2}}^0 \sqrt{4-4\sin^2\theta}\ 2\cos\theta\ d\theta \\
&= \int_{-\frac{\pi}{2}}^0 \sqrt{4\left(1-\sin^2\theta\right)}\ 2\cos\theta\ d\theta \\
&= \int_{-\frac{\pi}{2}}^0 \sqrt{4\cos^2\theta}\ 2\cos\theta\ d\theta \\
&= \int_{-\frac{\pi}{2}}^0 2\cos\theta\ 2\cos\theta\ d\theta * \\
&= \int_{-\frac{\pi}{2}}^0 4\cos^2\theta\ d\theta \\
&= \int_{-\frac{\pi}{2}}^0 4\left[\frac{1}{2}\left(1+\cos 2\theta\right)\right] d\theta \\
&= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^0 \\
&= 2\left[\left(0+\frac{1}{2}\sin 0\right) - \left(-\frac{\pi}{2}+\frac{1}{2}\sin\{-\pi\}\right)\right] \\
&= 2\left[(0+0) - \left(-\frac{\pi}{2}+0\right)\right] \\
&= 2\left(\frac{\pi}{2}\right) \\
&= \pi
\end{aligned}$$

*$\cos\theta \geq 0$ for $-\frac{\pi}{2} \leq \theta \leq 0$.

g

$$\int_0^{\frac{2}{3}} \sqrt{4-9x^2}$$

Let $x=\frac{2}{3}\sin\theta, dx=\frac{2}{3}\cos\theta\, d\theta$

When $x=0, \theta=0$; when $x=\frac{2}{3}, \theta=\sin^{-}(1)=\frac{\pi}{2}$

$$\int_0^{\frac{2}{3}} \sqrt{4-9x^2}=\int_0^{\frac{\pi}{2}} \sqrt{4-9\left(\frac{4}{9}\sin^2\theta\right)}\times\frac{2}{3}\cos\theta\, d\theta$$

$$=\int_0^{\frac{\pi}{2}} 2\cos\theta\times\frac{2}{3}\cos\theta\, d\theta$$

$$=\frac{4}{3}\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$=\frac{2}{3}\int_0^{\frac{\pi}{2}}\left(1+\cos 2\theta\right) d\theta$$

$$=\frac{2}{3}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$=\frac{2}{3}\times\frac{\pi}{2}$$

$$=\frac{\pi}{3}$$

As required.

Exercise 6.02 Rational functions with quadratic denominators

Question 1

a $\int \frac{1}{(x+3)^2}dx = \int (x+3)^{-2}\,dx = \frac{(x+3)^{-1}}{-1}+C = \frac{-1}{x+3}+C$

b $\int \frac{1}{(x-4)^2}dx = \int (x-4)^{-2}\,dx = \frac{(x-4)^{-1}}{-1}+C = \frac{-1}{x-4}+C$

c $\int \frac{-2}{x^2+4x+4}dx = -2\int \frac{1}{(x+2)^2}dx = -2\int (x+2)^{-2}\,dx = -2\times\frac{(x+2)^{-1}}{-1}+C = \frac{2}{x+2}+C$

d $\int \frac{1}{x^2+1}dx = \tan^{-}(x)+C$

e $\int \frac{1}{x^2+9}dx = \frac{1}{3}\tan^{-}\left(\frac{x}{3}\right)+C$

f $\int \frac{1}{x^2+3}dx = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+C$

g $\int \frac{1}{x^2+5}dx = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$

h $\int \frac{1}{x^2-4x+4}dx = \int \frac{1}{(x-2)^2}dx = \int (x-2)^{-2}\,dx = \frac{(x-2)^{-1}}{-1}+C = \frac{-1}{x-2}+C$

Question 2

a $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-}\left(x\right)+C$

b $\int \frac{1}{\sqrt{9-x^2}}\,dx = \sin^{-}\left(\frac{x}{3}\right)+C$

c $\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-}\left(\frac{x}{2}\right)+C$

d $\int \frac{1}{\sqrt{9-4x^2}}\,dx = \frac{1}{2}\sin^{-}\left(\frac{2x}{3}\right)+C$

e

$$\begin{aligned}\int \frac{1}{\sqrt{4x-x^2}}\,dx &= \int \frac{1}{\sqrt{-4+4x-x^2+4}}\,dx\\ &= \int \frac{1}{\sqrt{-\left(4-4x+x^2\right)+4}}\,dx\\ &= \int \frac{1}{\sqrt{4-\left(x-2\right)^2}}\,dx\\ &= \sin^{-1}\left(\frac{x-2}{2}\right)+C\end{aligned}$$

f

$$\begin{aligned}\int \frac{1}{\sqrt{-9x^2+12x}}\,dx &= \int \frac{1}{\sqrt{-4-9x^2+12x+4}}\,dx\\ &= \int \frac{1}{\sqrt{-\left(4+9x^2-12x\right)+4}}\,dx\\ &= \int \frac{1}{\sqrt{-\left(3x-2\right)^2+4}}\,dx\\ &= \int \frac{1}{\sqrt{4-\left(3x-2\right)^2}}\,dx\\ &= \frac{1}{3}\sin^{-}\left(\frac{3x-2}{2}\right)+C\end{aligned}$$

Question 3

a

$$x^2+2x+2=x^2+2x+1+1$$
$$=(x+1)^2+1$$

$$\int \frac{1}{x^2+2x+2}dx=\int \frac{1}{(x+1)^2+1}dx$$
$$=\tan^{-}(x+1)+C$$

b

$$2x-x^2=1-1+2x-x^2$$
$$=1-(x-1)^2$$

$$\int \frac{1}{\sqrt{2x-x^2}}dx=\int \frac{1}{\sqrt{1-(x-1)^2}}dx$$
$$=\sin^{-}(x-1)+C$$

c

$$\frac{x^2+2}{x^2+1}=\frac{x^2+1+1}{x^2+1}=\frac{x^2+1}{x^2+1}+\frac{1}{x^2+1}$$
$$=1+\frac{1}{x^2+1}$$

$$\int \frac{x^2+2}{x^2+1}dx=\int \left(1+\frac{1}{x^2+1}\right)dx$$
$$=x+\tan^{-}(x)+C$$

d

$$5-2x+x^2=4+1-2x+x^2$$
$$=4+(x-1)^2$$

$$\int \frac{1}{5-2x+x^2}dx=\int \frac{1}{4+(x-1)^2}dx$$
$$=\frac{1}{2}\tan^{-}\left(\frac{x-1}{2}\right)+C$$

Question 4

a

$$\int \frac{x^2}{x^2+1}dx = \int \frac{x^2+1-1}{x^2+1}dx$$
$$= \int\left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right)dx$$
$$= \int\left(1 - \frac{1}{x^2+1}\right)dx$$
$$= x - \tan^{-}(x) + C$$

b

$$\int \frac{x^2-1}{x^2+1}dx = \int \frac{x^2+1-2}{x^2+1}dx$$
$$= \int\left(\frac{x^2+1}{x^2+1} - \frac{2}{x^2+1}\right)dx$$
$$= \int\left(1 - \frac{2}{x^2+1}\right)dx$$
$$= x - 2\tan^{-}(x) + C$$

c

$$\int \frac{x^2+4}{x^2+2}dx = \int \frac{x^2+2+2}{x^2+2}dx$$
$$= \int\left(\frac{x^2+2}{x^2+2} + \frac{2}{x^2+2}\right)dx$$
$$= \int\left(1 + \frac{2}{x^2+2}\right)dx$$
$$= x + 2\frac{1}{\sqrt{2}}\tan^{-}\left(\frac{x}{\sqrt{2}}\right) + C$$
$$= x + \sqrt{2}\tan^{-}\left(\frac{x}{\sqrt{2}}\right) + C$$

d

$$\int \frac{x^2}{x^2+9}dx = \int \frac{x^2+9-9}{x^2+9}dx$$
$$= \int\left(\frac{x^2+9}{x^2+9} - \frac{9}{x^2+9}\right)dx$$
$$= \int\left(1 - \frac{1}{x^2+9}\right)dx$$
$$= x - \frac{9}{3}\tan^{-}\left(\frac{x}{3}\right) + C$$
$$= x - 3\tan^{-}\left(\frac{x}{3}\right) + C$$

e

$$\int \frac{(x+1)^2}{x^2+1}dx = \int \frac{x^2+2x+1}{x^2+1}dx$$
$$= \int\left(\frac{x^2+1}{x^2+1} + \frac{2x}{x^2+1}\right)dx$$
$$= \int\left(1 + \frac{2x}{x^2+1}\right)dx$$
$$= x + \ln\left(x^2+1\right) + C$$

f

$$\int \frac{(x-1)^2}{x^2+2}dx$$
$$= \int \frac{x^2-2x+1}{x^2+2}dx$$
$$= \int\left(\frac{x^2+2}{x^2+2} - \frac{2x-1}{x^2+2}\right)dx$$
$$= \int\left(1 - \frac{2x}{x^2+2} - \frac{1}{x^2+2}\right)dx$$
$$= x - \ln\left(x^2+2\right) - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

Question 5

a

$$\int \frac{2x+1}{x^2+4x+5}dx = \int \frac{2x+4-3}{x^2+4x+5}dx$$
$$= \int \left(\frac{2x+4}{x^2+4x+5} - \frac{3}{x^2+4x+4+1} \right) dx$$
$$= \int \left(\frac{2x+4}{x^2+4x+5} - \frac{3}{(x+2)^2+1} \right) dx$$
$$= \ln\left(x^2+4x+5\right) - 3\tan^{-1}(x+2) + C$$

b

$$\int \frac{4x+3}{x^2+1}dx = \int \left(2\frac{2x}{x^2+1} + \frac{3}{x^2+1} \right) dx$$
$$= 2\ln\left(x^2+1\right) + 3\tan^{-1}(x) + C$$

c

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{2x}{x^2+1}dx = \frac{1}{2}\ln\left(x^2+1\right) + C$$

d

$$\int \frac{x-1}{x^2+1}dx = \int \left(\frac{1}{2}\frac{2x}{x^2+1} - \frac{1}{x^2+1} \right) dx$$
$$= \frac{1}{2}\ln\left(x^2+1\right) - \tan^{-1}(x) + C$$

e

$$\int \frac{x}{x^2-2x+2}dx = \int \frac{1}{2}\left(\frac{2x-2+2}{x^2-2x+2} \right) dx$$
$$= \int \frac{1}{2}\left(\frac{2x-2}{x^2-2x+2} + \frac{2}{x^2-2x+2} \right) dx$$
$$= \frac{1}{2}\int \left(\frac{2x-2}{x^2-2x+2} + \frac{2}{(x-1)^2+1} \right) dx$$
$$= \frac{1}{2}\ln\left(x^2-2x+2\right) + \tan^{-1}(x-1) + C$$

f

$$\int \frac{x+1}{x^2-1}dx = \int \left(\frac{x+1}{(x+1)(x-1)} \right) dx$$
$$= \int \frac{1}{(x-1)}dx$$
$$= \ln|x-1| + C$$

Question 6

a $\int \frac{-1}{\sqrt{1-x^2}}\,dx = -\sin^{-1} x + C \text{ or } \cos^{-1} x + C$

b $\int \frac{1}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\sin^{-}(2x) + C$

c $\int \frac{dx}{1+4x^2} = \frac{1}{2}\tan^{-}(2x) + C$

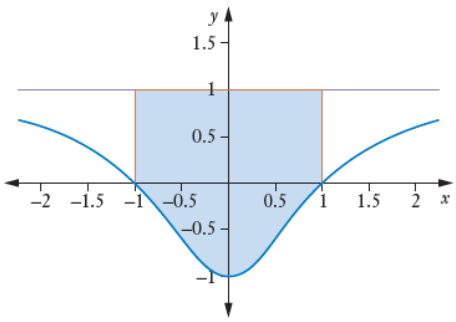
d
$$\begin{aligned}\int \frac{dx}{x^2+4x+8} &= \int \frac{dx}{x^2+4x+4+4}\\ &= \int \frac{1}{(x+2)^2+4}dx\\ &= \frac{1}{2}\tan^{-}\left(\frac{x+2}{2}\right)+C\end{aligned}$$

e
$$\begin{aligned}\int \frac{dx}{4x^2+4x+10} &= \int \frac{dx}{4x^2+4x+1+9}\\ &= \int \frac{1}{(2x+1)^2+9}dx\\ &= \frac{1}{6}\tan^{-}\left(\frac{2x+1}{3}\right)+C\end{aligned}$$

f
$$\begin{aligned}\int \frac{2}{x^2-6x+13}dx &= 2\int \frac{1}{x^2-6x+9+4}dx\\ &= 2\int \frac{1}{(x-3)^2+4}dx\\ &= 2\times\frac{1}{2}\tan^{-}\left(\frac{x-3}{2}\right)+C\\ &= \tan^{-1}\left(\frac{x-3}{2}\right)+C\end{aligned}$$

Question 7

a Top rectangle



A = 2 × 1 = 2

For the bottom bell

$$A=\int_{-1}^{1}\frac{x^2-1}{x^2+1}dx=\int_{-1}^{1}\frac{x^2+1-2}{x^2+1}dx$$

$$=\int_{-1}^{1}\left(\frac{x^2+1}{x^2+1}-\frac{2}{x^2+1}\right)dx=2\int_{0}^{1}\left(1-\frac{2}{x^2+1}\right)dx$$

$$=2\left(x-2\tan^{-1}(x)\right)_0^1$$

$$=2\left|1-2\tan^{-}(1)-0\right|=2\left|1-2\times\frac{\pi}{4}\right|$$

$$=\pi-2$$

Total area = 2 + π – 2 = π units^2

b Volume of top rectangle

r = 1, h = 1

V = $\pi r^2 h$

V = π units^3

Volume of bottom bell (rotated)

$$V=\pi\int_{-}^{0}x^2dy$$

$$y=\frac{x^2-1}{x^2+1}=\frac{x^2+1}{x^2+1}-\frac{2}{x^2+1}=1-\frac{2}{x^2+1}$$

$$y-1=-\frac{2}{x^2+1}$$

$$x^2+1=-\frac{2}{y-1}$$

$$x^2=-1-\frac{2}{y-1}$$

$$\therefore\ V=\pi\int_{-1}^{0}\left(-1-\frac{2}{y-1}\right)dy=\pi\left|\left(-x-2\ln|y-1|\right)\right|_{-1}^{0}$$

$$=\pi\left|\left(-0-2\ln|0-1|\right)-\left(-(-1)-2\ln|-2|\right)\right|$$

$$=\pi\left|-1+2\ln(2)\right|$$

$$=\pi(2\ln 2-1)$$

Total volume

π(2 ln 2 – 1) + π = 2π ln2 units^3

c

$$V = \pi\int_{-1}^{2}\left(\frac{1}{\sqrt{x^2+4x+8}}\right)^2 dx$$

$$= \pi\int_{-1}^{2}\frac{1}{x^2+4x+8}dx$$

$$= \pi\int_{-1}^{2}\frac{1}{x^2+4x+4+4}dx$$

$$= \pi\int_{-1}^{2}\frac{1}{(x+2)^2+4}dx$$

$$= \pi\left[\frac{1}{2}\tan^{-}\left(\frac{x+2}{2}\right)\right]_{-1}^{2}$$

$$= \frac{\pi}{2}\left[\tan^{-1}\left(\frac{2+2}{2}\right)-\tan^{-1}\left(\frac{-1+2}{2}\right)\right]$$

$$= \frac{\pi}{2}\left[\tan^{-1}2-\tan^{-1}\left(\frac{1}{2}\right)\right]$$

We can simplify $\tan^{-1} 2 - \tan^{-1}\left(\frac{1}{2}\right)$ by letting $A = \tan^{-1} 2$ and $B = \tan^{-1}\left(\frac{1}{2}\right)$

So $A - B = \tan^{-1} 2 - \tan^{-1}\left(\frac{1}{2}\right)$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$

$$= \frac{2-\frac{1}{2}}{1+2\times\frac{1}{2}}$$

$$= \frac{1\frac{1}{2}}{2}$$

$$= \frac{3}{4}$$

So $A - B = \tan^{-1}\left(\frac{3}{4}\right)$

So $V = \frac{\pi}{2}\left[\tan^{-1}2-\tan^{-1}\left(\frac{1}{2}\right)\right]$

$$= \frac{\pi}{2}\tan^{-1}\left(\frac{3}{4}\right)$$

Exercise 6.03 Partial fractions

Question 1

a

$$\int \frac{3x+1}{(x-3)(x+2)}dx$$

$$\frac{3x+1}{(x-3)(x+2)} = \frac{A}{(x-3)} + \frac{B}{(x+2)}$$

$$\Rightarrow 3x+1 = A(x+2) + B(x-3)$$

Let $x = -2$

$$-6+1 = 0A + -5B$$

$$-5 = -5B$$

$$B = 1$$

Let $x = 3$

$$10 = 5A + 0B$$

$$A = 2$$

$$\Rightarrow \frac{3x+1}{(x-3)(x+2)} = \frac{2}{(x-3)} + \frac{1}{(x+2)}$$

$$\int \frac{3x+1}{(x-3)(x+2)}dx$$

$$= \int \left(\frac{2}{(x-3)} + \frac{1}{(x+2)} \right) dx$$

$$= 2\ln|x-3| + \ln|x+2| + C$$

b

$$\int \frac{5x+8}{(x+3)(2x-1)}dx$$

$$\frac{5x+8}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1}$$

$$\Rightarrow 5x+8 = A(2x-1) + B(x+3)$$

Let $x = -3$

$$-15+8 = -7A + 0B$$

$$-7 = -7A$$

$$A = 1$$

Let $x = 0$

$$8 = -1 + 3B$$

$$9 = 3B$$

$$B = 3$$

$$\Rightarrow \frac{5x+8}{(x+3)(2x-1)} = \frac{1}{x+3} + \frac{3}{2x-1}$$

$$\int \frac{5x+8}{(x+3)(2x-1)}dx$$

$$= \int \left(\frac{1}{x+3} + \frac{3}{2x-1} \right) dx$$

$$= \ln|x+3| + \frac{3}{2}\ln|2x-1| + C$$

c

$$\int \frac{3x+1}{(x+3)(x+2)}\,dx$$

$$\frac{3x+1}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2}$$

$$\Rightarrow 3x+1 = A(x+2) + B(x+3)$$

Let $x = -2$

$$-6+1 = 0A + B$$

$$-5 = B$$

Let $x = -3$

$$-9+1 = -A + 0B$$

$$-8 = -8A$$

$$A = 8$$

$$\Rightarrow \frac{3x+1}{(x+3)(x+2)} = \frac{8}{x+3} - \frac{5}{x+2}$$

$$\int \frac{3x+1}{(x+3)(x+2)}\,dx$$

$$= \int \left(\frac{8}{x+3} - \frac{5}{x+2} \right) dx$$

$$= 8\ln|x+3| - 5\ln|x+2| + C$$

d

$$\int \frac{3x+7}{(x-3)(x+5)}dx$$

$$\frac{3x+7}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$$

$$\Rightarrow 3x+7 = A(x+5)+B(x-3)$$

Let $x = -5$

$$-15+7 = 0A-8B$$

$$-8 = -8B$$

$$B = 1$$

Let $x = 3$

$$9+7 = 8A+0B$$

$$16 = 8A$$

$$A = 2$$

$$\Rightarrow \frac{3x+7}{(x-3)(x+5)} = \frac{2}{x-3} + \frac{1}{x+5}$$

$$\int \frac{3x+7}{(x-3)(x+5)}dx$$

$$= \int \left(\frac{2}{x-3} + \frac{1}{x+5} \right) dx$$

$$= 2\ln|x-3| + \ln|x+5| + C$$

e

$$\int \frac{3+x}{(1+2x)(1-3x)}dx$$

$$\frac{3+x}{(1+2x)(1-3x)} = \frac{A}{1+2x} + \frac{B}{1-3x}$$

$$\Rightarrow 3+x = A(1-3x) + B(1+2x)$$

$$3+x = A - 3xA + B + 2xB$$

Equating coefficients

$$3 = A+B$$

$$9 = 3A+3B$$

$$1 = -3A+2B$$

$$10 = 5B$$

$$B = 2$$

$$3 = A+B$$

$$3 = A+2$$

$$1 = A$$

$$\Rightarrow \frac{3+x}{(1+2x)(1-3x)} = \frac{1}{1+2x} + \frac{2}{1-3x}$$

$$\int \frac{3+x}{(1+2x)(1-3x)}dx$$

$$= \int \left(\frac{1}{1+2x} + \frac{2}{1-3x}\right)dx$$

$$= \frac{1}{2}\ln|1+2x| - \frac{2}{3}\ln|1-3x| + C$$

f

$$\int \frac{(a-b)x}{(x-a)(x-b)}dx$$

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

$$\Rightarrow (a-b)x = A(x-b) + B(x-a)$$

Let $x = a$

$$(a-b)a = A(a-b) + B(a-a)$$

$$(a-b)a = A(a-b)$$

$$A = a$$

Let $x = b$

$$(a-b)b = A(b-b) + B(b-a)$$

$$(a-b)b = -B(a-b)$$

$$B = -b$$

$$\Rightarrow \frac{(a-b)x}{(x-a)(x-b)} = \frac{a}{x-a} - \frac{b}{x-b}$$

$$\int \frac{(a-b)x}{(x-a)(x-b)}dx$$

$$= \int \left(\frac{a}{x-a} - \frac{b}{x-b} \right) dx$$

$$= a\ln|x-a| - b\ln|x-b| + C$$

Question 2

a

$$\int \frac{2x}{(x+2)(x-2)}dx$$

$$= \int \frac{2x}{x^2-4}dx$$

$$\ln|x^2-4| + C$$

b

$$\int \frac{11}{6x^2+5x-4}\,dx$$
$$=\int \frac{11}{(3x+4)(2x-1)}\,dx$$
$$\frac{11}{(3x+4)(2x-1)}=\frac{A}{3x+4}+\frac{B}{2x-1}$$
$$\Rightarrow 11=A(2x-1)+B(3x+4)$$
$$11=2xA-A+3xB+4B$$

Equating coefficients

$$0=2A+3B$$
$$11=-A+4B$$
$$22=-2A+8B$$
$$22=11B$$
$$B=2$$
$$0=2A+6$$
$$-6=2A$$
$$-3=A$$
$$\Rightarrow \frac{11}{(3x+4)(2x-1)}=\frac{-3}{3x+4}+\frac{2}{2x-1}$$
$$\int \frac{11}{(3x+4)(2x-1)}\,dx$$
$$=\int\left(\frac{-3}{3x+4}+\frac{2}{2x-1}\right)dx$$
$$=\ln|2x-1|-\ln|3x+4|+C$$
$$=\ln\left|\frac{2x-1}{3x+4}\right|+C$$

c

$$\int \frac{x-7}{2x^2-3x-2}dx$$

$$=\int \frac{x-7}{(2x+1)(x-2)}dx$$

$$\frac{x-7}{(2x+1)(x-2)}=\frac{A}{2x+1}+\frac{B}{x-2}$$

$$\Rightarrow x-7=A(x-2)+B(2x+1)$$

$$x-7=xA-2A+2xB+B$$

Equating coefficients

$$1=A+2B$$

$$2=2A+4B$$

$$-7=-2A+B$$

$$-5=5B$$

$$B=-1$$

$$1=A-2$$

$$3=A$$

$$\Rightarrow \frac{x-7}{(2x+1)(x-2)}=\frac{3}{2x+1}-\frac{1}{x-2}$$

$$\int \frac{x-7}{(2x+1)(x-2)}dx$$

$$=\int\left(\frac{3}{2x+1}-\frac{1}{x-2}\right)dx$$

$$=\frac{3}{2}\ln|2x+1|-\ln|x-2|+C$$

d

$$\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx$$

$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

$$\Rightarrow 3x^2-12x+11=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$$

Let $x=1$

$$3-12+11=A(-1)(-2)+0B+0C$$

$$2=2A$$

$$A=1$$

Let $x=2$

$$12-24+11=0A+B(1)(-1)+0C$$

$$-1=-B$$

$$B=1$$

Let $x=3$

$$27-36+11=0A+0B+C(2)(1)$$

$$2=2C$$

$$C=1$$

$$\Rightarrow \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}$$

$$\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx$$

$$=\int\left(\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}\right)dx$$

$$=\ln|x-1|+\ln|x-2|+\ln|x-3|+C$$

e

$$\int \frac{5x^2+9x+6}{\left(x^2-1\right)\left(2x+3\right)}dx$$

$$\frac{5x^2+9x+6}{(x+1)(x-1)(2x+3)}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{2x+3}$$

$$\Rightarrow 5x^2+9x+6=A(x-1)(2x+3)+B(x+1)(2x+3)+C(x+1)(x-1)$$

Let $x=1$

$$5+9+6=0A+B(2)(5)+0C$$

$$20=10B$$

$$B=2$$

Let $x=-1$

$$5-9+6=A(-2)(1)+0B+0C$$

$$2=-2A$$

$$A=-1$$

Let $x=0$

$$6=-1(-1)(3)+2(1)(3)+C(1)(-1)$$

$$6=3+6-C$$

$$C=3$$

$$\Rightarrow \frac{5x^2+9x+6}{(x+1)(x-1)(2x+3)}=\frac{-1}{x+1}+\frac{2}{x-1}+\frac{3}{2x+3}$$

$$\int \frac{5x^2+9x+6}{(x+1)(x-1)(2x+3)}dx$$

$$=\int\left(\frac{-1}{x+1}+\frac{2}{x-1}+\frac{3}{2x+3}\right)dx$$

$$=-\ln|x+1|+2\ln|x-1|+\frac{3}{2}\ln|2x+3|+C$$

f

$$\int \frac{x-1}{x^2-7x+6}dx$$

$$=\int\left(\frac{x-1}{(x-1)(x-6)}\right)dx$$

$$=\int \frac{1}{(x-6)}dx$$

$$=\ln|x-6|+C$$

Question 3

a

$$\int \frac{4+7x}{(x+1)^2(2+3x)}dx$$

$$\frac{4+7x}{(x+1)^2(2+3x)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{2+3x}$$

multiply by $(x+1)^2(2+3x)$

$$4+7x = A(2+3x) + B(x+1)(2+3x) + C(x+1)^2$$

$$4+7x = A(2+3x) + B(3x^2+5x+2) + C(x^2+2x+1)$$

Equating coefficients

$3B + C = 0$

$3A + 5B + 2C = 7$

$2A + 2B + C = 4$

$6A + 10B + 4C = 14$

$6A + 6B + 3C = 12$

$4B + C = 2$

$3B + C = 0$

$B = 2$

$3 \times 2 + C = 0$

$C = -6$

$2A + 2 \times 2 - 6 = 4$

$2A = 6$

$A = 3$

$$\Rightarrow \frac{4+7x}{(x+1)^2(2+3x)} = \frac{3}{(x+1)^2} + \frac{2}{x+1} + \frac{-6}{2+3x}$$

$$\int \frac{4+7x}{(x+1)^2(2+3x)}dx$$

$$= \int \left(\frac{3}{(x+1)^2} + \frac{2}{x+1} - \frac{6}{2+3x} \right) dx$$

$$= \frac{-3}{x+1} + 2\ln|x+1| - 2\ln|2+3x| + C$$

$$= 2\ln\left|\frac{x+1}{2+3x}\right| - \frac{3}{x+1} + C$$

b

$$\int \frac{1}{(x+1)^2(x-1)}dx$$

$$\frac{1}{(x+1)^2(x-1)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x-1}$$

multiply by $(x+1)^2(x-1)$

$$1 = A(x-1) + B(x+1)(x-1) + C(x+1)^2$$

$$1 = A(x-1) + B(x^2-1) + C(x^2+2x+1)$$

Equating coefficients

$B + C = 0$

$A + 2C = 0$

$-A - B + C = 1$

$-B + 3C = 1$

$B + C = 0$

$4C = 1$

$C = \frac{1}{4}$

$B = -\frac{1}{4}$

$A + 2 \times \frac{1}{4} = 0$

$A = -\frac{1}{2}$

$$\Rightarrow \frac{1}{(x+1)^2(x-1)} = \frac{-1}{2(x+1)^2} - \frac{1}{4(x+1)} + \frac{1}{4(x-1)}$$

$$\int \frac{1}{(x+1)^2(x-1)}dx$$

$$= \int \left(\frac{-1}{2(x+1)^2} - \frac{1}{4(x+1)} + \frac{1}{4(x-1)} \right) dx$$

$$= \frac{1}{2(x+1)} - \frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + C$$

$$= \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2(x+1)} + C$$

c

$$\int \frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}\,dx$$

$$\frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}=\frac{A}{(x+1)^2}+\frac{B}{x+1}+\frac{C}{(x-2)^2}+\frac{D}{x-2}$$

multiply by $(x+1)^2(x-2)^2$

$$x^3-6x^2+25=A(x-2)^2+B(x+1)(x-2)^2+C(x+1)^2+D(x-2)(x+1)^2$$

$$x^3-6x^2+25=A\left(x^2-4x+4\right)+B(x+1)\left(x^2-4x+4\right)+C\left(x^2+2x+1\right)+D(x-2)\left(x^2+2x+1\right)$$

$$x^3-6x^2+25=A\left(x^2-4x+4\right)+B\left(x^3-3x^2+4\right)+C\left(x^2+2x+1\right)+D\left(x^3-3x-2\right)$$

Equating coefficients

$B+D=1$

$A-3B+C=-6$

$-4A+2C-3D=0$

$4A+4B+C-2D=25$

Solve system of linear equations with 4 unknowns using TI Nspire

$A=2$

$B=3$

$C=1$

$D=-2$

$$\Rightarrow \frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}=\frac{2}{(x+1)^2}+\frac{3}{x+1}+\frac{1}{(x-2)^2}-\frac{2}{x-2}$$

$$\int \frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}\,dx$$

$$=\int\left(\frac{2}{(x+1)^2}+\frac{3}{x+1}+\frac{1}{(x-2)^2}-\frac{2}{x-2}\right)dx$$

$$=\frac{-2}{x+1}+3\ln|x+1|-\frac{1}{x-2}-2\ln|x-2|+C$$

Question 4

a

$$\int \frac{x-2}{\left(x^2+1\right)\left(x+1\right)}dx$$

$$\frac{x-2}{\left(x^2+1\right)\left(x+1\right)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1}$$

multiply by $\left(x^2+1\right)\left(x+1\right)$

$$x-2=\left(Ax+B\right)\left(x+1\right)+C\left(x^2+1\right)$$

$$x-2=Ax^2+\left(A+B\right)x+B+Cx^2+C$$

Equating coefficients

$A+C=0$ [1]

$A+B=1$ [2]

$B+C=-2$ [3]

$A+B=1\Rightarrow A=1-B$ rearranging[2]

$1-B+C=0$ substituting for A in[1]

$-B+C=-1$ [4]

$2C=-3$ [3]+[4]

$$C=-\frac{3}{2}$$

$$A=\frac{3}{2}$$

$$B=1-A$$

$$B=1-\frac{3}{2}$$

$$B=-\frac{1}{2}$$

$$\Rightarrow \frac{x-2}{\left(x^2+1\right)\left(x+1\right)}=\frac{3x-1}{2\left(x^2+1\right)}-\frac{3}{2\left(x+1\right)}$$

$$\int \frac{x-2}{\left(x^2+1\right)\left(x+1\right)}dx$$

$$=\int\left(\frac{3x-1}{2\left(x^2+1\right)}-\frac{3}{2\left(x+1\right)}\right)dx$$

$$=\int\left(\frac{3x}{2\left(x^2+1\right)}-\frac{1}{2\left(x^2+1\right)}-\frac{3}{2\left(x+1\right)}\right)dx$$

$$=\frac{3}{4}\ln\left|x^2+1\right|-\frac{1}{2}\tan^{-1}\left(x\right)-\frac{3}{2}\ln\left|x+1\right|+C$$

b

$$\int \frac{x}{\left(x^2+1\right)\left(x-1\right)}dx$$

$$\frac{x}{\left(x^2+1\right)\left(x-1\right)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}$$

multiply by $\left(x^2+1\right)\left(x-1\right)$

$$x=\left(Ax+B\right)\left(x-1\right)+C\left(x^2+1\right)$$

$$=Ax^2-Ax+Bx-B+Cx^2+C$$

$$x=\left(A+C\right)x^2+\left(-A+B\right)x-B+C$$

Equating coefficients

$A+C=0$ [1]

$-A+B=1$ [2]

$-B+C=0$ [3]

$C+B=1$ [4]: Add [1] and [2]:

$2C=1$ Add [3] and [4]:

$$C=\frac{1}{2}$$

$A+\frac{1}{2}=0$ Sub into [1]:

$$A=-\frac{1}{2}$$

$--\frac{1}{2}+B=1$ Sub into [2]:

$$B=\frac{1}{2}$$

So $\frac{x}{\left(x^2+1\right)\left(x-1\right)}=\frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}+\frac{\frac{1}{2}}{x-1}$

$$=\frac{1}{2}\left(\frac{-x+1}{x^2+1}+\frac{1}{x-1}\right)$$

$$\int \frac{x}{\left(x^2+1\right)\left(x-1\right)}dx=\int \frac{1}{2}\left(\frac{-x+1}{x^2+1}+\frac{1}{x-1}\right)dx$$

$$=\frac{1}{2}\int \left(\frac{-x}{x^2+1}+\frac{1}{x^2+1}+\frac{1}{x-1}\right)dx$$

$$=\frac{1}{2}\left(-\frac{1}{2}\ln\left(x^2+1\right)+\tan^{-1}x+\ln\left|x-1\right|\right)+C$$

$$=-\frac{1}{4}\ln\left(x^2+1\right)+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\ln\left|x-1\right|+C$$

c

$$\int \frac{x+3}{x^3+3x^2+x+3}\,dx$$
$$=\int\left(\frac{x+3}{\left(x^2+1\right)\left(x+3\right)}\right)dx$$
$$=\int \frac{1}{x^2+1}\,dx$$
$$=\tan^{-}\ \left(x\right)+C$$

Question 5

a

$$\int \frac{12}{\left(x^3+8\right)}dx = \int \frac{12}{(x+2)\left(x^2-2x+4\right)}dx$$

$$\frac{12}{(x+2)\left(x^2-2x+4\right)} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4}$$

multiply by $(x+2)\left(x^2-2x+4\right)$

$$12 = A\left(x^2-2x+4\right) + (Bx+C)(x+2)$$

$$12 = Ax^2 - 2xA + 4A + Bx^2 + (2B+C)x + 2C$$

Equating coefficients

$A + B = 0 \Rightarrow A = -B$

$-2A + 2B + C = 0$ [1]

$4A + 2C = 12 \Rightarrow 2A + C = 6$ [2]

$2B + 2C = 6$ [1]+[2]

$B + C = 3$

$C = 3 - B$

$2B + 2B + 3 - B = 0$ substituting for A and C in [1]

$3B = -3$

$B = -1$

$A = 1$

$C = 3 - (-1)$

$C = 4$

$$\Rightarrow \frac{12}{(x+2)\left(x^2-2x+4\right)} = \frac{1}{x+2} + \frac{-x+4}{x^2-2x+4}$$

$$\int \frac{12}{(x+2)\left(x^2-2x+4\right)}dx$$

$$= \int \left(\frac{1}{x+2} - \frac{x-4}{x^2-2x+4}\right)dx$$

$$= \int \left(\frac{1}{x+2} - \frac{1}{2}\frac{2x-8}{x^2-2x+4}\right)dx$$

$$= \int \left(\frac{1}{x+2} - \frac{1}{2}\left(\frac{2x-2}{x^2-2x+4} - \frac{6}{x^2-2x+1+3}\right)\right)dx$$

$$= \int \left(\frac{1}{x+2} - \frac{1}{2}\frac{2x-8}{x^2-2x+4} + \frac{3}{(x-1)^2+3}\right)dx$$

$$= \ln|x+2| - \frac{1}{2}\ln\left|x^2-2x+4\right| + \sqrt{3}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + C$$

b

$$\int \frac{9x+6}{\left(x^3-8\right)}dx$$

$$= \int \frac{9x+6}{(x-2)\left(x^2+2x+4\right)}dx$$

$$\frac{9x+6}{(x-2)\left(x^2+2x+4\right)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$$

multiply by $(x-2)\left(x^2+2x+4\right)$

$$9x+6 = A\left(x^2+2x+4\right) + (Bx+C)(x-2)$$

$$9x+6 = Ax^2+2xA+4A+Bx^2+(-2B+C)x-2C$$

Equating coefficients

$A+B=0 \ \Rightarrow A=-B$

$2A-2B+C=9$

$-4B+C=9$

$4A-2C=6$

$-4B-2C=6$

$3C=3$

$C=1$

$-4B+C=9$

$-4B+1=9$

$-4B=8$

$B=-2$

$A=2$

$$\Rightarrow \frac{9x+6}{(x-2)\left(x^2+2x+4\right)} = \frac{2}{x-2} + \frac{-2x+1}{x^2+2x+4}$$

$$\int \frac{9x+6}{(x-2)\left(x^2+2x+4\right)}dx$$

$$= \int \left(\frac{2}{x-2} - \frac{2x-1}{x^2+2x+4}\right)dx$$

$$= \int \left(\frac{2}{x-2} - \frac{2x+2}{x^2+2x+4} + \frac{3}{x^2+2x+1+3}\right)dx$$

$$= \int \left(\frac{2}{x-2} - \frac{2x+2}{x^2+2x+4} + \frac{3}{(x+1)^2+3}\right)dx$$

$$= 2\ln|x-2| - \ln\left|x^2+2x+4\right| + \sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

Question 6

a

$$\frac{2}{(1+x)(1+x^2)}=\frac{A}{1+x}+\frac{Bx+C}{1+x^2}$$

multiply by $(1+x)(1+x^2)$

$2=A(1+x^2)+(Bx+C)(1+x)$

$2=A+Ax^2+Bx^2+(B+C)x+C$

Equating coefficients

$A+B=0$

$\Rightarrow A=-B$

$B+C=0$

$A+C=2$

$-A+C=0$

$2C=2$

$C=1$

$B+C=0$

$B=-1$

$A=1$

b

$$\frac{2}{(1+x)(1+x^2)}=\frac{1}{1+x}+\frac{-x+1}{1+x^2}$$

$$\int\frac{2}{(1+x)(1+x^2)}dx$$

$$=\int\left(\frac{1}{1+x}-\frac{x-1}{1+x^2}\right)dx$$

$$=\int\left(\frac{1}{1+x}-\frac{1}{2}\frac{2x-2}{1+x^2}\right)dx$$

$$=\int\left(\frac{1}{1+x}-\frac{1}{2}\frac{2x}{1+x^2}+\frac{1}{1+x^2}\right)dx$$

$$=\ln|1+x|-\frac{1}{2}\ln\left|1+x^2\right|+\tan^{-1}(x)+C$$

Question 7

a

$$\frac{1}{(x+3)(x+1)}=\frac{A}{x+3}+\frac{B}{x+1}$$

$$\Rightarrow 1=A(x+1)+B(x+3)$$

Let $x=-1$

$$1=0A+2B$$

$$B=\frac{1}{2}$$

Let $x=-3$

$$1=-2A+0B$$

$$A=-\frac{1}{2}$$

b

$$\frac{1}{(x+3)(x+1)}=\frac{-1}{2(x+3)}+\frac{1}{2(x+1)}$$

$$\int_0^1\frac{1}{(x+3)(x+1)}\,dx=\int_0^1\left(\frac{-1}{2(x+3)}+\frac{1}{2(x+1)}\right)dx$$

$$=\left[-\frac{1}{2}\ln|x+3|+\frac{1}{2}\ln|x+1|\right]_0^1$$

$$=-\frac{1}{2}\ln|1+3|+\frac{1}{2}\ln|1+1|-\left(-\frac{1}{2}\ln|0+3|+\frac{1}{2}\ln|0+1|\right)$$

$$=-\frac{1}{2}\ln 4+\frac{1}{2}\ln 2+\frac{1}{2}\ln 3-\frac{1}{2}\ln 1$$

$$=\frac{1}{2}(\ln 2+\ln 3-\ln 4)$$

$$=\frac{1}{2}\ln\left(\frac{2\times 3}{4}\right)$$

$$=\frac{1}{2}\ln\frac{3}{2}$$

Question 8

Using partial fractions,

$$\frac{2x^2+5x+3}{(x-1)^2(x^2+1)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{Cx+D}{(x^2+1)}$$

On multiplying both sides by $(x-1)^2(x^2+1)$;

$$2x^2+5x+3=A(x^2+1)(x-1)+B(x^2+1)+(Cx+D)(x-1)^2$$

Substituting $x = 1$, we get $10 = 2B$, so $B = 5$.

Substituting $x = –1$, we get $0 = –4A + 2B – 4C + 4D$.

Substituting $x = 0$, we get $3 = –A + B + D$.

Substituting $x = 2$, we get $21 = 5A + 5B + 2C + D$.

Solving simultaneously, $A=-\frac{1}{2}$ $B=5, C=\frac{1}{2}$ and $D=-\frac{5}{2}$.

Hence,

$$\int\frac{2x^2+5x+3}{(x-1)^2(x^2+1)}dx=\int\frac{-\frac{1}{2}}{(x-1)}+\frac{5}{(x-1)^2}+\frac{\frac{1}{2}x-\frac{5}{2}}{(x^2+1)}dx$$

Integrating,

$$\int\frac{-\frac{1}{2}}{(x-1)}+\frac{5}{(x-1)^2}+\frac{\frac{1}{2}x-\frac{5}{2}}{(x^2+1)}\,dx=-\frac{1}{2}\ln|x-1|-\frac{5}{(x-1)}+\frac{1}{4}\ln(x^2+1)-\frac{5}{2}\tan^{-1}(x)+C$$

Question 9

$$\frac{2x^2 - x - 7}{(x+2)^2(x^2+x+1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+x+1}$$

multiply by $(x+2)^2(x^2+x+1)$

$2x^2 - x - 7 = A(x+2)(x^2+x+1) + B(x^2+x+1) + (Cx+D)(x+2)^2$

Let $x = -2$, so $8 + 2 - 7 = A(0) + 3B + (Cx+D)(0)$

$\Rightarrow 3 = 3B \Rightarrow B = 1$

$2x^2 - x - 7 = A(x+2)(x^2+x+1) + (x^2+x+1) + (Cx+D)(x+2)^2$

$x^2 - 2x - 8 = A(x+2)(x^2+x+1) + (Cx+D)(x^2+4x+4)$

$= A(x^3 + x^2 + x + 2x^2 + 2x + 2) + Cx^3 + 4Cx^2 + 4Cx + Dx^2 + 4Dx + 4D$

$= Ax^3 + Ax^2 + Ax + 2Ax^2 + 2Ax + 2A + Cx^3 + 4Cx^2 + 4Cx + Dx^2 + 4Dx + 4D$

$= (A+C)x^3 + (3A+4C+D)x^2 + (3A+4C+4D)x + 2A + 4D$

Equating coefficients

$A + C = 0$ [1]

$3A + 4C + D = 1$ [2]

$3A + 4C + 4D = -2$ [3]

$2A + 4D = -8$ [4]

$[3]-[2]$:

$3D = -3$

$D = -1$

Sub into [4]:

$2A - 4 = -8$

$2A = -4$

$A = -2$

Sub into [1]:

$-2 + C = 0$

$C = 2$

$$\frac{2x^2 - x - 7}{(x+2)^2(x^2+x+1)} = \frac{-2}{x+2} + \frac{1}{(x+2)^2} + \frac{2x-1}{x^2+x+1}$$

$$
\begin{aligned}
\int_0^1 \frac{2x^2 - x - 7}{(x+2)^2(x^2+x+1)}\,dx &= \int_0^1 \left(\frac{-2}{x+2} + \frac{1}{(x+2)^2} + \frac{2x-1}{x^2+x+1} \right) dx \\
&= \int_0^1 \left(\frac{-2}{x+2} + \frac{1}{(x+2)^2} + \frac{2x+1}{x^2+x+1} - \frac{2}{x^2+x+\frac{1}{4}+\frac{3}{4}} \right) dx \\
&= \int_0^1 \left(\frac{-2}{x+2} + (x+2)^{-2} + \frac{2x+1}{x^2+x+1} - \frac{2}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \right) dx \\
&= \left[-2\ln|x+2| - (x+2)^{-} + \ln|x^2+x+1| - 2 \times \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) \right]_0^1 \\
&= \left[-2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}} \right) \right]_0^1 \\
&= \left[-2\ln 3 - \frac{1}{3} + \ln 3 - \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{3}{\sqrt{3}} \right) \right] - \left[-2\ln 2 - \frac{1}{2} + \ln 1 - \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}} \right) \right] \\
&= -\ln 3 - \frac{1}{3} - \frac{4}{\sqrt{3}} \tan^{-} \sqrt{3} + 2\ln 2 + \frac{1}{2} + \frac{4}{\sqrt{3}} \left(\frac{\pi}{6} \right) \\
&= -\ln 3 + \frac{1}{6} - \frac{4}{\sqrt{3}} \left(\frac{\pi}{3} \right) + \ln 2^2 + \frac{4}{\sqrt{3}} \left(\frac{\pi}{6} \right) \\
&= -\ln 3 + \frac{1}{6} - \frac{4}{\sqrt{3}} \left(\frac{\pi}{6} \right) + \ln 4 \\
&= \ln\left(\frac{4}{3} \right) - \frac{2\pi}{3\sqrt{3}} + \frac{1}{6}
\end{aligned}
$$

Question 10

a

$$\frac{(x+1)^3}{x^2}$$
$$=\frac{x^3+3x^2+3x+1}{x^2}$$
$$=\frac{x^3}{x^2}+\frac{3x^2}{x^2}+\frac{3x}{x^2}+\frac{1}{x^2}$$
$$=x+3+\frac{3}{x}+\frac{1}{x^2}$$

So A = 1, B = 3, C = 3, D = 1.

b

$$\int^3 \frac{(x+1)^3}{x^2}dx$$
$$=\int^3\left(x+3+\frac{3}{x}+\frac{1}{x^2}\right)dx$$
$$=\left(\frac{x^2}{2}+3x+3\ln|x|-\frac{1}{x}\right)_1^3$$
$$=\left(\frac{3^2}{2}+3\times 3+3\ln|3|-\frac{1}{3}\right)-\left(\frac{1}{2}+3+3\ln|1|-\frac{1}{1}\right)$$
$$=\frac{32}{3}+3\ln 3$$

Question 11

a

$$\frac{x}{x^2+2x+3}$$

$$=\frac{1}{2}\left[\frac{2x}{x^2+2x+3}\right]$$

$$=\frac{1}{2}\left(\frac{2x+2-2}{x^2+2x+3}\right)$$

$$=\frac{1}{2}\left(\frac{2x+2}{x^2+2x+3}\right)-\frac{1}{2}\left(\frac{2}{x^2+2x+3}\right)$$

$$=\frac{1}{2}\left(\frac{2x+2}{x^2+2x+3}\right)-\frac{1}{x^2+2x+3}$$

(or prove RHS = LHS)

b

$$\int\frac{x}{x^2+2x+3}dx$$

$$=\int\left(\frac{1}{2}\left(\frac{2x+2}{x^2+2x+3}\right)-\frac{1}{x^2+2x+3}\right)dx$$

$$=\int\left(\frac{1}{2}\left(\frac{2x+2}{x^2+2x+3}\right)-\frac{1}{x^2+2x+1+2}\right)dx$$

$$=\int\left(\frac{1}{2}\left(\frac{2x+2}{x^2+2x+3}\right)-\frac{1}{(x+1)^2+\left(\sqrt{2}\right)^2}\right)dx$$

$$=\frac{1}{2}\ln\left|x^2+2x+3\right|-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+C$$

Question 12

a

$$\frac{1}{\left(x^2+a^2\right)\left(x^2+b^2\right)}=\frac{Ax+B}{x^2+a^2}+\frac{Cx+D}{x^2+b^2}$$

$$\Rightarrow 1=(Ax+B)\left(x^2+b^2\right)+(Cx+D)\left(x^2+a^2\right)$$

$$1=Ax^3+Bx^2+Ab^2x+Bb^2+Cx^3+Dx^2+Ca^2x+Da^2$$

Equating coefficients

$A+C=0$

$\Rightarrow A=-C$

$B+D=0$

$\Rightarrow B=-D$

$Ab^2+Ca^2=0$

$Bb^2+Da^2=1$

$-Db^2+Da^2=1$

$D\left(a^2-b^2\right)=1$

$$D=\frac{1}{\left(a^2-b^2\right)}$$

$$B=\frac{-1}{\left(a^2-b^2\right)}$$

$Ab^2+Ca^2=0$

$-Cb^2+Ca^2=0$

$C\left(a^2-b^2\right)=0$

As $a^2-b^2\neq 0$

$\Rightarrow C=0$

$A=0$

$$\Rightarrow \frac{1}{\left(x^2+a^2\right)\left(x^2+b^2\right)}=\frac{\frac{-1}{\left(a^2-b^2\right)}}{x^2+a^2}+\frac{\frac{1}{\left(a^2-b^2\right)}}{x^2+b^2}$$

$$=\frac{1}{\left(a^2-b^2\right)}\left(\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}\right)$$

b

$$\int_0^{\infty} \frac{1}{\left(x^2+a^2\right)\left(x^2+b^2\right)}\,dx$$

$$=\frac{1}{\left(a^2-b^2\right)}\int_0^{\infty}\left(\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}\right)dx$$

$$=\frac{1}{\left(a^2-b^2\right)}\left[\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)-\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]_0^{\infty}$$

$$=\frac{1}{\left(a^2-b^2\right)}\left[\left(\frac{1}{b}\tan^{-1}\left(\frac{\infty}{b}\right)-\frac{1}{a}\tan^{-1}\left(\frac{\infty}{a}\right)\right)-\left(\frac{1}{b}\tan^{-1}\left(\frac{0}{b}\right)-\frac{1}{a}\tan^{-1}\left(\frac{0}{a}\right)\right)\right]$$

$$=\frac{1}{\left(a^2-b^2\right)}\left[\left(\frac{1}{b}\frac{\pi}{2}-\frac{1}{a}\frac{\pi}{2}\right)-0\right]$$

$$=\frac{\pi}{2(a-b)(a+b)}\left(\frac{1}{b}-\frac{1}{a}\right)$$

$$=\frac{\pi}{2(a-b)(a+b)}\left(\frac{a-b}{ab}\right)$$

$$=\frac{\pi}{2ab(a+b)}$$

Exercise 6.04 Integration by parts

Question 1

a

$$\int \ln(x+1)\,dx$$

$$\text{Let } u'(x)=1 \qquad v(x)=\ln(x+1)$$

$$u(x)=x \qquad v'(x)=\frac{1}{x+1}$$

$$\int u'(x)v(x)dx = u(x)v(x)-\int u(x)v'(x)dx$$

$$\int 1\times\ln(x+1)\,dx = x\ln(x+1)-\int x\times\frac{1}{x+1}dx$$

$$= x\ln(x+1)-\int\frac{x}{x+1}dx$$

$$= x\ln(x+1)-\int\frac{x+1-1}{x+1}dx$$

$$= x\ln(x+1)-\int 1-\frac{1}{x+1}dx$$

$$= x\ln(x+1)-x+\ln(x+1)+C$$

$$=(x+1)\ln(x+1)-x+C$$

b

$$\int \ln(x^2)\,dx$$

$$=\int 2\ln(x)\,dx$$

$$=2\int \ln(x)\,dx$$

$$\text{Let } u'(x)=1 \qquad v(x)=\ln(x)$$

$$u(x)=x \qquad v'(x)=\frac{1}{x}$$

$$\int u'(x)v(x)dx = u(x)v(x)-\int u(x)v'(x)dx$$

$$2\int 1\times\ln(x)\,dx = 2\left(x\ln(x)-\int x\times\frac{1}{x}dx\right)$$

$$=2\left(x\ln(x)-\int 1dx\right)$$

$$=2\left(x\ln(x)-x\right)+C$$

c

$$\int x\cos\left(x\right)dx$$

Let $u'(x)=\cos\left(x\right)$ $\quad v(x)=x$

$u(x)=\sin\left(x\right)$ $\quad v'(x)=1$

$$\int u'(x)v(x)dx=u(x)v(x)-\int u(x)v'(x)dx$$

$$\int \cos\left(x\right)\times xdx=x\sin\left(x\right)-\int \sin\left(x\right)\times 1dx$$

$$=x\sin\left(x\right)-\left(-\cos\left(x\right)\right)+C$$

$$=x\sin\left(x\right)+\cos\left(x\right)+C$$

d

$$\int xe^{-x}dx$$

Let $u'(x)=e^{-x}$ $\quad v(x)=x$

$u(x)=-e^{-x}$ $\quad v'(x)=1$

$$\int u'(x)v(x)dx=u(x)v(x)-\int u(x)v'(x)dx$$

$$\int e^{-x}\times xdx=x-e^{-x}-\int -e^{-x}\times 1dx$$

$$=x\left(-e^{-x}\right)-\left(e^{-x}\right)+C$$

$$=-e^{-x}\left(x+1\right)+C$$

e

$$\int x\sin\left(2x\right)dx$$

Let $u'(x)=\sin\left(2x\right)$ $\quad v(x)=x$

$u(x)=-\frac{1}{2}\cos\left(2x\right)$ $\quad v'(x)=1$

$$\int u'(x)v(x)dx=u(x)v(x)-\int u(x)v'(x)dx$$

$$\int \sin\left(2x\right)\times xdx=x\left(-\frac{1}{2}\cos\left(2x\right)\right)-\int -\frac{1}{2}\cos\left(2x\right)\times 1dx$$

$$=-\frac{x}{2}\cos\left(2x\right)+\frac{1}{4}\sin\left(2x\right)+C$$

f

$$\int xe^{2x}dx$$

Let $u'(x)=e^{2x}$ $\quad v(x)=x$

$$u(x)=\frac{1}{2}e^{2x} \qquad v'(x)=1$$

$$\int u'(x)v(x)dx=u(x)v(x)-\int u(x)v'(x)dx$$

$$\int e^{2x}\times xdx=x\frac{1}{2}e^{2x}-\int\frac{1}{2}e^{2x}\times 1dx$$

$$=\frac{xe^{2x}}{2}-\frac{1}{4}e^{2x}+C$$

$$=\frac{1}{4}e^{2x}\left(2x-1\right)+C$$

g

$$\int x\ln xdx$$

Let $u'(x)=x$ $\quad v(x)=\ln x$

$$u(x)=\frac{1}{2}x^2 \qquad v'(x)=\frac{1}{x}$$

$$\int u'(x)v(x)dx=u(x)v(x)-\int u(x)v'(x)dx$$

$$\int x\times\ln xdx=\frac{1}{2}x^2\ln x-\int\frac{1}{2}x^2\times\frac{1}{x}dx$$

$$=\frac{1}{2}x^2\ln x-\int\frac{1}{2}xdx$$

$$=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$$

Question 2

a

$$\int e^x \sin(x)\,dx$$

Let $u'(x) = e^x \qquad v(x) = \sin(x)$

$\qquad u(x) = e^x \qquad v'(x) = \cos(x)$

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

$$\int e^x \times \sin(x)\,dx = e^x \sin(x) - \int e^x \times \cos(x)\,dx$$

$$\int e^x \times \cos(x)\,dx$$

Let $u'(x) = e^x \qquad v(x) = \cos(x)$

$\qquad u(x) = e^x \qquad v'(x) = -\sin(x)$

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

$$\int e^x \times \cos(x)\,dx = e^x \cos(x) - \int e^x \times \left(-\sin(x)\right)dx$$

$$= e^x \cos(x) + \int e^x \times \sin(x)\,dx$$

$$\int e^x \sin(x)\,dx = e^x \sin(x) - \left(e^x \cos(x) + \int e^x \times \sin(x)\,dx\right)$$

$$2\int e^x \sin(x)\,dx = e^x \sin(x) - e^x \cos(x) + C$$

$$= \frac{e^x}{2}\left(\sin(x) - \cos(x)\right) + C$$

b

$$\int e^x \cos(x)\,dx$$

Let $u'(x) = e^x \qquad v(x) = \cos(x)$

$u(x) = e^x \qquad v'(x) = -\sin(x)$

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

$$\int e^x \times \cos(x)\,dx = e^x \cos(x) + \int e^x \times \sin(x)\,dx$$

$$\int e^x \times \sin(x)\,dx$$

Let $u'(x) = e^x \qquad v(x) = \sin(x)$

$u(x) = e^x \qquad v'(x) = \cos(x)$

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

$$\int e^x \times \sin(x)\,dx = e^x \sin(x) - \int e^x \times \cos(x)\,dx$$

$$= e^x \sin(x) - \int e^x \times \cos(x)\,dx$$

$$\int e^x \cos(x)\,dx = e^x \cos(x) + \left(e^x \sin(x) - \int e^x \times \cos(x)\,dx\right)$$

$$2\int e^x \cos(x)\,dx = e^x \cos(x) + e^x \sin(x) + C$$

$$= \frac{e^x}{2}\left(\sin(x) + \cos(x)\right) + C$$

c

$$\int xe^{x^2}\,dx$$

$$= \frac{1}{2}\int 2xe^{x^2}\,dx$$

$$= \frac{1}{2}e^{x^2} + C$$

d

$$\int x^2 \ln x dx$$

$$\text{Let } u'(x) = x^2 \qquad v(x) = \ln x$$

$$u(x) = \frac{1}{3}x^3 \qquad v'(x) = \frac{1}{x}$$

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

$$\int x^2 \ln x dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \times \frac{1}{x} dx$$

$$\int x^2 \ln x dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \times \frac{1}{x} dx$$

$$= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2 dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

e

$$\int x^2 \sin(x) dx$$

$$\text{Let } u'(x) = \sin(x) \qquad v(x) = x^2$$

$$u(x) = -\cos(x) \qquad v'(x) = 2x$$

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

$$\int \sin(x) \times x^2 dx = x^2(-\cos(x)) - \int -\cos(x) \times 2x dx$$

$$= x^2(-\cos(x)) + 2\int x\cos(x) dx$$

$$\int x\cos(x) dx$$

$$\text{Let } u'(x) = \cos(x) \qquad v(x) = x$$

$$u(x) = \sin(x) \qquad v'(x) = 1$$

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

$$\int \cos(x) \times x dx = x\sin(x) - \int \sin(x) \times 1 dx$$

$$= x\sin(x) - (-\cos(x)) + C$$

$$= x\sin(x) + \cos(x) + C$$

$$\int \sin(x) \times x^2 dx = -x^2\cos(x) + 2(x\sin(x) + \cos(x)) + C$$

f

$$\int x\tan^{-}(x)dx$$

Let $u'(x)=x \qquad v(x)=\tan^{-1}(x)$

$u(x)=\frac{1}{2}x^2 \qquad v'(x)=\frac{1}{x^2+1}$

$$\int u'(x)v(x)dx=u(x)v(x)-\int u(x)v'(x)dx$$

$$\int x\times\tan^{-}(x)dx=\frac{1}{2}x^2\tan^{-1}(x)-\int\frac{1}{2}x^2\times\frac{1}{x^2+1}dx$$

$$=\frac{1}{2}x^2\tan^{-1}(x)-\frac{1}{2}\int\frac{x^2+1-1}{x^2+1}dx$$

$$=\frac{1}{2}x^2\tan^{-1}(x)-\frac{1}{2}\int\left(\frac{x^2+1}{x^2+1}-\frac{1}{x^2+1}\right)dx$$

$$=\frac{1}{2}x^2\tan^{-1}(x)-\frac{1}{2}\int\left(1-\frac{1}{x^2+1}\right)dx$$

$$=\frac{1}{2}x^2\tan^{-1}(x)-\frac{1}{2}\left(x-\tan^{-}(x)\right)+C$$

$$=\frac{1}{2}\left(x^2\tan^{-1}(x)+\tan^{-1}(x)-x\right)+C$$

g

$$\int x^2 e^{4x} dx$$

Let $u'(x) = e^{4x} \qquad v(x) = x^2$

$u(x) = \frac{1}{4}e^{4x} \qquad v'(x) = 2x$

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

$$\int e^{4x} \times x^2 dx = x^2 \frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x} \times 2x dx$$

$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\int xe^{4x} dx$$

$$\int xe^{4x} dx$$

Let $u'(x) = e^{4x} \qquad v(x) = x$

$u(x) = \frac{1}{4}e^{4x} \qquad v'(x) = 1$

$$\int u\ (x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

$$\int e^{4x} \times x dx = x\frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x} \times 1 dx$$

$$= \frac{xe^{4x}}{4} - \frac{1}{16}e^{4x}$$

$$\int e^{4x} \times x^2 dx = \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\left(\frac{xe^{4x}}{4} - \frac{1}{16}e^{4x}\right) + C$$

$$= \frac{e^{4x}}{32}\left(8x^2 - 4x + 1\right) + C$$

Question 3

a

$$\int^{2}\ln\left(x\right)dx$$

$$\text{Let } u'(x)=1 \qquad v(x)=\ln\left(x\right)$$

$$u(x)=x \qquad v'(x)=\frac{1}{x}$$

$$\int_{1}^{2}u'(x)v(x)d=\left(u(x)v(x)\right)^{2}-\int_{1}^{2}u(x)v'(x)dx$$

$$\int_{1}^{2}1\times\ln\left(x\right)dx=\left(x\ln\left(x\right)\right)^{2}-\int_{1}^{2}x\times\frac{1}{x}dx$$

$$=\left(x\ln\left(x\right)\right)_{1}^{2}-\int^{2}1dx$$

$$=\left(x\ln\left(x\right)\right)_{1}^{2}-\left(x\right)_{1}^{2}$$

$$=2\ln 2-2-(1\ln 1-1)$$

$$=2\ln 2-1$$

b

$$\int_{0}^{1}e^{\sqrt{x}}dx$$

$$a=\sqrt{x}\ \Rightarrow a^{2}=x$$

$$\frac{dx}{da}=2a$$

$$dx=2ada$$

$$\int_{0}^{1}e^{\sqrt{x}}dx=\int_{0}^{1}e^{a}2ada$$

$$=2\int_{0}^{1}e^{a}ada$$

$$\text{Let } u'(a)=e^{a} \qquad v(x)=a$$

$$u(a)=e^{a} \qquad v'(x)=1$$

$$\int_{0}^{1}u'(x)v(x)d=\left(u(x)v(x)\right)_{0}-\int_{0}^{1}u(x)v'(x)dx$$

$$2\int_{0}^{1}e^{a}ada=2\left(\left(ae^{a}\right)_{0}-\int_{0}^{1}e^{a}da\right)$$

$$=2\left(\left(ae^{a}\right)_{0}^{1}-\left(e^{a}\right)_{0}^{1}\right)$$

$$=2\left(\left(e^{1}-e^{1}\right)-\left(0-e^{0}\right)\right)$$

$$=2$$

c

$$\int^{e}\ln\left(x^{2}\right)dx$$

$$=2\int^{e}\ln\left(x\right)dx$$

Let $u'(x)=1 \qquad v(x)=\ln\left(x\right)$

$$u(x)=x \qquad v'(x)=\frac{1}{x}$$

$$\int_{1}^{e}u'(x)v(x)d=\left(u(x)v(x)\right)^{e}-\int_{1}^{e}u(x)v'(x)dx$$

$$2\int_{1}^{e}1\times\ln\left(x\right)dx=2\left(\left(x\ln\left(x\right)\right)^{e}-\int_{1}^{e}x\times\frac{1}{x}dx\right)$$

$$=2\left(\left(x\ln\left(x\right)\right)_{1}^{e}-\int^{e}1dx\right)$$

$$=2\left(\left(x\ln\left(x\right)\right)_{1}^{e}-\left(x\right)_{1}^{e}\right)$$

$$=2\left(e-e-(0-1)\right)$$

$$=2$$

d

$$\int_0^{0.5} \cos^{-}\left(2x\right)dx$$

Let $u'(x) = 1 \qquad v(x) = \cos^{-}\left(2x\right)$

$$u(x) = x \qquad v'(x) = \frac{-2}{\sqrt{1-\left(2x\right)^2}} = \frac{-2}{\sqrt{1-4x^2}}$$

$$\int_0^{0.5} \cos^{-1}\left(2x\right)dx = \left[x\cos^{-1}\left(2x\right)\right]_0^{0.5} - \int_0^{0.5} \frac{-2x}{\sqrt{1-4x^2}}\,dx$$

$$= \left[0.5\cos^{-1}\left(1\right) - 0\cos^{-1}\left(0\right)\right]_0^{0.5} - \frac{1}{4}\int_0^{0.5} -8x\left(1-4x^2\right)^{-\frac{1}{2}}dx$$

$$= \left[0-0\right] - \frac{1}{4}\left[2\left(1-4x^2\right)^{\frac{}{2}}\right]_0^{0.5}$$

$$= -\frac{1}{2}\left[\sqrt{1-4x^2}\right]_0^{0.5}$$

$$= -\frac{1}{2}\left[\sqrt{0}-\sqrt{1}\right]$$

$$= -\frac{1}{2}\left(-1\right)$$

$$= \frac{1}{2}$$

e

$$\int x\cos\left(x\right)dx$$

From **1c**

$$\int x\cos\left(x\right)dx = x\sin\left(x\right) + \cos\left(x\right) + C$$

$$\int_0^{\pi} x\cos\left(x\right)dx = \left(x\sin\left(x\right) + \cos\left(x\right)\right)_0^{\pi}$$

$$= \left(\pi\sin\left(\pi\right) + \cos\left(\pi\right)\right) - \left(0\sin\left(0\right) + \cos\left(0\right)\right)$$

$$= 0 + \left(-1\right) - 1$$

$$= -2$$

f

$$\int_0^2 \ln\left(x^2+1\right)dx$$

Let $u'(x)=1 \qquad v(x)=\ln\left(x^2+1\right)$

$u(x)=x \qquad v'(x)=\dfrac{2x}{x^2+1}$

$$\int_0^2 u'(x)v(x)d=\left(u(x)v(x)\right)_0^2-\int_0^2 u(x)v'(x)dx$$

$$\int_0^2 1\times\ln\left(x^2+1\right)dx=\left(x\ln\left(x^2+1\right)\right)_0^2-\int_0^2\frac{2x^2}{x^2+1}dx$$

$$=\left(x\ln\left(x^2+1\right)\right)_0^2-2\int_0^2\frac{x^2+1-1}{x^2+1}dx$$

$$=\left(x\ln\left(x^2+1\right)\right)_0^2-2\int_0^2\left(\frac{x^2+1}{x^2+1}-\frac{1}{x^2+1}\right)dx$$

$$=\left(x\ln\left(x^2+1\right)\right)_0^2-2\int_0^2\left(1-\frac{1}{x^2+1}\right)dx$$

$$=\left(x\ln\left(x^2+1\right)-2x+2\tan^{-1}(x)\right)_0^2$$

$$=\left(2\ln 5-4+2\tan^{-}\ (2)\right)-(0)$$

$$=2\ln 5-4+2\tan^{-}\ (2)$$

g

$$\int_{-\pi}^{\pi} x\sin(x)dx$$

$Let\ u'(x)=\sin(x) \qquad v(x)=x$

$u(x)=-\cos(x) \quad v'(x)=1$

$$\int_{-\pi}^{\pi} u'(x)v(x)d=\left(u(x)v(x)\right)_{-\pi}^{\pi}-\int_{-\pi}^{\pi}u(x)v'(x)dx$$

$$\int_{-\pi}^{\pi} x\sin(x)dx=\left(-x\cos(x)\right)_{-\pi}^{\pi}-\int_{-\pi}^{\pi}-\cos(x)dx$$

$$=\left(-x\cos(x)+\sin(x)\right)_{-\pi}^{\pi}$$

$$=\left(-\pi\cos(\pi)+\sin(\pi)\right)-\left(-x\cos(-\pi)+\sin(-\pi)\right)$$

$$=2\pi$$

Question 1

a

$$\int x^n \cos(x)\,dx$$

$$\text{Let } u'(x) = \cos(x) \qquad v(x) = x^n$$

$$u(x) = \sin(x) \qquad v'(x) = nx^{n-1}$$

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

$$\int \cos(x) \times x^n dx = x^n \sin(x) - \int \sin(x) \times nx^{n-}\ dx$$

$$\int \cos(x) \times x^n dx = x^n \sin(x) - n\int x^{n-}\ \sin(x)dx$$

b

$$\int \left(\ln(x)\right)^n dx$$

$$\text{Let } u'(x) = 1 \qquad v(x) = \left(\ln(x)\right)^n$$

$$u(x) = x \qquad v'(x) = \frac{n}{x}\left(\ln(x)\right)^{n-1}$$

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

$$\int 1 \times \left(\ln(x)\right)^n dx = x\left(\ln(x)\right)^n - \int x \times \frac{n}{x}\left(\ln(x)\right)^{n-}\ dx$$

$$= x\left(\ln(x)\right)^n - n\int \left(\ln(x)\right)^{n-}\ dx$$

c

$$\int x^n e^{2x} dx$$

$$\text{Let } u'(x) = e^{2x} \qquad v(x) = x^n$$

$$u(x) = \frac{1}{2}e^{2x} \qquad v'(x) = nx^{n-1}$$

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

$$\int e^{2x} \times x^n dx = \frac{e^{2x}}{2}x^n - \int \frac{1}{2}e^{2x} \times nx^{n-1}dx$$

$$= \frac{1}{2}x^n e^{2x} - \frac{n}{2}\int x^{n-1}e^{2x}dx$$

d

$$\int (\tan(x))^n \, dx$$
$$= \int (\tan(x))^{n-2} (\tan(x))^2 \, dx$$
$$= \int (\tan(x))^{n-2} \left((\sec(x))^2 - 1\right) dx$$
$$= \int \left((\tan(x))^{n-2} (\sec(x))^2 - (\tan(x))^{n-2}\right) dx$$
$$= \frac{1}{n-1}(\tan(x))^{n-1} - \int (\tan(x))^{n-2} \, dx$$

Question 2

a

$$I_n = \int_0^1 \left(1-x^2\right)^{\frac{n}{2}} dx$$

$$\text{Let } u'(x) = 1 \qquad v(x) = \left(1-x^2\right)^{\frac{n}{2}}$$

$$u(x) = x \qquad v'(x) = -xn\left(1-x^2\right)^{\frac{n-2}{2}}$$

$$\int_0^1 u'(x)v(x)d = \left(u(x)v(x)\right)_0 - \int_0^1 u(x)v'(x)dx$$

$$\int_0^1 1\times\left(1-x^2\right)^{\frac{n}{2}} dx = \left[x\left(1-x^2\right)^{\frac{n}{2}}\right]_0 - \int_0^1 x\times\left(-xn\left(1-x^2\right)^{\frac{n-2}{2}}\right)dx$$

$$= 0 + n\int_0^1 \left(x^2-1+1\right)\left(1-x^2\right)^{\frac{n-2}{2}} dx$$

$$= -n\int_0^1 \left(1-x^2\right)\left(1-x^2\right)^{\frac{n-2}{2}} dx + n\int_0^1 \left(1-x^2\right)^{\frac{n-2}{2}} dx$$

$$= -n\int_0^1 \left(1-x^2\right)^{\frac{n}{2}} dx + n\int_0^1 \left(1-x^2\right)^{\frac{n-2}{2}} dx$$

$$= -nI_n + nI_{n-2}$$

$$I_n = -nI_n + nI_{n-2}$$

$$I_n + nI_n = nI_{n-2}$$

$$I_n\left(1+n\right) = nI_{n-2}$$

$$I_n = \frac{n}{n+1} I_{n-2}$$

b

$$I_5 = \frac{5}{6} I_3$$

$$= \frac{5}{6}\times\frac{3}{4} I$$

$$I = \int_0^1 \left(1-x^2\right)^{\frac{}{2}} dx = \int_0^1 \sqrt{1-x^2}\,dx$$

This is the area of a quarter of a circle, radius $=1$.

$$I = \frac{1}{4}\times\pi\times1^2 = \frac{\pi}{4}$$

OR let $x = \cos\theta, dx = -\sin\theta d\theta$

$\theta = \cos^{-1} x$

When $x = 0,\ \theta = \cos^{-}\ 0 = -\frac{\pi}{2}.$

When $x = 1,\ \theta = \cos^{-}\ 1 = 0.$

$$
\begin{aligned}
I &= \int_0^1 \sqrt{1-x^2}\,dx \\
&= \int_{-\frac{\pi}{2}}^{0} \sqrt{1-\cos^2\theta}\left(-\sin\theta\right)d\theta \\
&= \int_{-\frac{\pi}{2}}^{0} \sqrt{\sin^2\theta}\left(-\sin\theta\right)d\theta \\
&= \int_{-\frac{\pi}{2}}^{0} \left(-\sin\theta\right)\left(-\sin\theta\right)d\theta *
\end{aligned}
$$

$*\sin\theta \le 0$ for $-\dfrac{\pi}{2} \le \theta \le 0$

$$
\begin{aligned}
&= \int_{-\frac{\pi}{2}}^{0} \sin^2\theta\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{0} \frac{1}{2}\left(1-\cos 2\theta\right)d\theta \\
&= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{0} \\
&= \frac{1}{2}\left[\left(0-\frac{1}{2}\sin\left\{-\pi\right\}\right)-\left(-\frac{\pi}{2}-\frac{1}{2}\sin 0\right)\right] \\
&= \frac{1}{2}\left[\left(0-0\right)-\left(-\frac{\pi}{2}-0\right)\right] \\
&= \frac{1}{2}\left(\frac{\pi}{2}\right) \\
&= \frac{\pi}{4}
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= \frac{5}{6}\times\frac{3}{4}\times\frac{\pi}{4} \\
&= \frac{5\pi}{32}
\end{aligned}
$$

Question 3

$$I_n = \int_0^1 x^n \left(x^2 - 1\right)^5 dx$$

Let $u'(x) = 12x\left(x^2 - 1\right)^5 \qquad v(x) = \frac{1}{12}x^{n-1}$

$u(x) = \left(x^2 - 1\right)^6 \qquad v'(x) = \frac{n-1}{12}x^{n-2}$

$$\int_0^1 u'(x)v(x)d = \left(u(x)v(x)\right)_0 - \int_0^1 u(x)v'(x)dx$$

$$\int_0^1 \frac{1}{12}x^{n-1} \times 12x\left(x^2 - 1\right)^5 dx = \left(\left(x^2 - 1\right)^6 \frac{1}{12}x^{n-1}\right)_0 - \int_0^1 \left(x^2 - 1\right)^6 \times \left(\frac{n-1}{12}x^{n-2}\right)dx$$

$$= 0 - \frac{n-1}{12}\int_0^1 \left(x^2 - 1\right)\left(x^2 - 1\right)^5 \times \left(x^{n-2}\right)dx$$

$$= -\frac{n-1}{12}\left(\int_0^1 x^2\left(x^2 - 1\right)^5 \times \left(x^{n-2}\right) - \left(x^2 - 1\right)^5 \times \left(x^{n-2}\right)\right)dx$$

$$= -\frac{n-1}{12}\int_0^1 \left(x^2 - 1\right)^5 \times \left(x^n\right)dx + \frac{n-1}{12}\int_0^1 \left(x^2 - 1\right)^5 \times \left(x^{n-2}\right)dx$$

$$= -\frac{n-1}{12}I_n + \frac{n-1}{12}I_{n-2}$$

$$I_n = -\frac{n-1}{12}I_n + \frac{n-1}{12}I_{n-2}$$

$$I_n + \frac{n-1}{12}I_n = \frac{n-1}{12}I_{n-2}$$

$$I_n\left(1 + \frac{n-1}{12}\right) = \frac{n-1}{12}I_{n-2}$$

$$I_n\left(\frac{11+n}{12}\right) = \frac{n-1}{12}I_{n-2}$$

$$I_n = \frac{n-1}{11+n}I_{n-2}$$

Question 4

a

$$y = \sin^{n-} \theta \cos\theta$$

$$u = \sin^{n-} \theta \qquad v = \cos\theta$$

$$u' = (n-1)\cos\theta \sin^{n-2}\theta \qquad v' = -\sin\theta$$

$$\begin{aligned} y' &= u'v + uv' \\ &= (n-1)\cos\theta \sin^{n-2}\theta \cos\theta - \sin^{n-1}\theta \sin\theta \\ &= (n-1)\cos^2\theta \sin^{n-2}\theta - \sin^n\theta \\ &= (n-1)\left(1-\sin^2\theta\right)\sin^{n-2}\theta - \sin^n\theta \\ &= (n-1)\sin^{n-2}\theta - (n-1)\sin^n\theta - \sin^n\theta \\ &= (n-1)\sin^{n-2}\theta - (n-1+1)\sin^n\theta \\ &= (n-1)\sin^{n-2}\theta - n\sin^n\theta \end{aligned}$$

b From part **a**

$$\frac{d}{d\theta}\sin^{n-1}\theta\cos\theta = (n-1)\sin^{n-2}\theta - n\sin^n\theta$$

So $\int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}\theta - n\sin^n\theta d\theta = \left[\sin^{n-1}\theta\cos\theta\right]_0^{\frac{\pi}{2}}$

$$\begin{aligned} &= \left[\left(\sin^{n-1}\frac{\pi}{2}\cos\frac{\pi}{2}\right) - \left(\sin^{n-1}0\cos 0\right)\right] \\ &= \left[(1\times 0) - (0\times 1)\right] \\ &= 0 \end{aligned}$$

So $\int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}\theta d\theta - \int_0^{\frac{\pi}{2}} n\sin^n\theta d\theta = 0$

$$(n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta d\theta = n\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta d\theta$$

c

$$\text{So } \int_0^{\frac{\pi}{2}} \sin^4\theta d\theta = \frac{4-1}{4}\int_0^{\frac{\pi}{2}} \sin^{4-2}\theta d\theta$$
$$= \frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta$$

$$I_4 = \frac{3}{4} I_2$$
$$I_2 = \int_0^{\frac{\pi}{2}} \sin^2\theta d\theta$$
$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2\theta) d\theta$$
$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$$
$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - 0\right) - (0-0)\right]$$
$$= \frac{1}{2}\left(\frac{\pi}{2}\right)$$
$$= \frac{\pi}{4}$$
$$\text{So } \int_0^{\frac{\pi}{2}} \sin^4\theta d\theta = \frac{3}{4}\times\frac{\pi}{4}$$
$$= \frac{3\pi}{16}$$

Question 5

$$I_n = \int^{e^2} \left(\ln\left(x\right)\right)^n dx$$

$$\int \left(\ln\left(x\right)\right)^n dx$$

Let $u'(x) = 1 \qquad v(x) = \left(\ln\left(x\right)\right)^n$

$$u(x) = x \qquad v'(x) = \frac{n}{x}\left(\ln\left(x\right)\right)^{n-1}$$

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

$$\int 1 \times \left(\ln\left(x\right)\right)^n dx = x\left(\ln\left(x\right)\right)^n - \int x \times \frac{n}{x}\left(\ln\left(x\right)\right)^{n-} dx$$

$$= x\left(\ln\left(x\right)\right)^n - n\int \left(\ln\left(x\right)\right)^{n-} dx$$

$$\int_1^{e^2} \left(\ln\left(x\right)\right)^n dx = \left(x\left(\ln\left(x\right)\right)^n\right)^{e^2} - n\int_1^{e^2} \left(\ln\left(x\right)\right)^{n-} dx$$

$$= \left(e^2\left(\ln\left(e^2\right)\right)^n\right) - \left(1\left(\ln\left(1\right)\right)^n\right) - n\int^{e^2} \left(\ln\left(x\right)\right)^{n-} dx$$

$$= e^2 2^n - 0 - nI_{n-1}$$

$$I_n = e^2 2^n - nI_{n-1}$$

Question 6

$$I_n = \int_0^1 \frac{x^{2n}}{x^2+1} dx$$

a

$$I_0 = \int_0^1 \frac{x^0}{x^2+1} dx$$

$$= \int_0^1 \frac{1}{x^2+1} dx$$

$$= \left[\tan^{-}\left(x\right)\right]_0^1$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}$$

b

$$I_n = \int_0^1 \frac{x^{2n}}{x^2+1}dx$$

$$I_{n-1} = \int_0^1 \frac{x^{2\ n-1)}}{x^2+1}dx$$

$$I_n + I_{n-} \ = \int_0^1 \frac{x^{2n}}{x^2+1}dx + \int_0^1 \frac{x^{2n-2}}{x^2+1}dx$$

$$= \int_0^1 \left(\frac{x^{2n}}{x^2+1} + \frac{x^{2n-2}}{x^2+1} \right) dx$$

$$= \int_0^1 \left(\frac{x^{2n} + x^{2n-2}}{x^2+1} \right) dx$$

$$= \int_0^1 \left(\frac{x^{2n}\left(1+\frac{1}{x^2}\right)}{x^2+1} \right) dx$$

$$= \int_0^1 \left(x^{2n} \left(\frac{x^2+1}{x^2} \right) \div \left(x^2+1\right) \right) dx$$

$$= \int_0^1 \left(\frac{x^{2n}}{x^2} \right) dx$$

$$= \int_0^1 x^{2n-2} dx$$

$$= \left[\frac{1}{2n-1} x^{2n-1} \right]_0^1$$

$$= \frac{1}{2n-1}$$

c The required integral = I_2.

$$I_2 = \int_0^1 \frac{x^4}{x^2+1}dx$$

$$I_2 + I_1 = \frac{1}{2(2)-1} = \frac{1}{3}$$ using part **b**

$$I_1 + I_0 = \frac{1}{2(1)-1} = 1$$ using part **b**

$$I_1 + \frac{\pi}{4} = 1$$ substituting in for I_0 from **a**

$$I \ = 1 - \frac{\pi}{4}$$

$$I_2 + 1 - \frac{\pi}{4} = \frac{1}{3}$$

$$I_2 = \frac{1}{3} - 1 + \frac{\pi}{4}$$

$$I_2 = -\frac{2}{3} + \frac{\pi}{4}$$

$$I_2 = \frac{3\pi - 8}{12}$$

Question 7

$I_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx$

a

$$I_n = \int_0^{\frac{\pi}{4}} \sec^n(x)dx = \int_0^{\frac{\pi}{4}} \sec^2(x)\sec^{n-2}(x)dx$$

Let $u'(x) = \sec^2(x) \qquad v(x) = \sec^{n-2}(x)$

$\qquad u(x) = \tan(x) \qquad v'(x) = (n-2)\sec(x)\tan(x)\sec^{n-3}(x)$

$$\int_0^{\frac{\pi}{4}} u'(x)v(x)d = \left(u(x)v(x)\right)_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} u(x)v'(x)dx$$

$$\int_0^{\frac{\pi}{4}} \sec^2(x)\sec^{n-2}(x)dx = \left(\tan(x)\sec^{n-2}(x)\right)_0^{\frac{\pi}{4}}$$

$$- \int_0^{\frac{\pi}{4}} \tan(x)(n-2)\sec(x)\tan(x)\sec^{n-3}(x)dx$$

$$= \left(\tan\left(\frac{\pi}{4}\right)\sec^{n-2}\left(\frac{\pi}{4}\right)\right) - \left(\tan(0)\sec^{n-2}(0)\right) - (n-2)\int_0^{\frac{\pi}{4}} \tan^2(x)\sec^{n-2}(x)dx$$

$$= \left(\sqrt{2}^{n-2}\right) - (0) - (n-2)\int_0^{\frac{\pi}{4}} \left(\sec^2(x) - 1\right)\sec^{n-2}(x)dx$$

$$= \sqrt{2}^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} \left(\sec^n(x) - \sec^{n-2}(x)\right)dx$$

$$I_n = \sqrt{2}^{n-2} - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n + (n-2)I_n = \sqrt{2}^{n-2} + (n-2)I_{n-2}$$

$$(n-1)I_n = \sqrt{2}^{n-2} + (n-2)I_{n-2}$$

$$I_n = \frac{1}{n-1}\left(\sqrt{2}^{n-2} + (n-2)I_{n-2}\right)$$

b

$$I_4 = \frac{1}{3}\left(\sqrt{2}^2 + 2I_2\right)$$

$$I_2 = 1\left(\sqrt{2}^0 + 0I_0\right)$$

$$= 1$$

$$I_4 = \frac{1}{3}\left(\sqrt{2}^2 + 2 \times 1\right)$$

$$= \frac{4}{3}$$

Test yourself 6

Question 1

$$\int x^3 e^{6x^4+1}\ dx$$

Let $u = 6x^4 + 1, \frac{du}{dx} = 24x^3, dx = \frac{du}{24x^3}$.

$$\int x^3 e^{6x^4+1}\ dx = \int e^u\ x^3 \times \frac{du}{24x^3}$$

$$= \frac{1}{24}\int e^u\ du$$

$$= \frac{1}{24}e^u + C$$

$$= \frac{1}{24}e^{6x^4+1} + C$$

Question 2

$$\int_{\sqrt{2}}^{2} \frac{dx}{x\sqrt{x^2-1}}$$

Let $x = \sec\theta = \frac{1}{\cos\theta} = (\cos\theta)^{-1}$

$\cos\theta = \frac{1}{x}$

$\theta = \cos^{-1}\left(\frac{1}{x}\right)$

$\frac{dx}{d\theta} = -(\cos\theta)^{-2}(-\sin\theta) = \frac{\sin\theta}{\cos^2\theta} = \tan\theta\sec\theta, dx = \tan\theta\sec\theta\ d\theta$

When $x = \sqrt{2}, \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$, when $x = 2, \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$$\int_{\sqrt{2}}^{2} \frac{dx}{x\sqrt{x^2-1}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan\theta\sec\theta}{\sec\theta\sqrt{\sec^2\theta-1}}\ d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan\theta}{\sqrt{\tan^2\theta}}\ d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan\theta}{\tan\theta}\ d\theta *$$

*$\tan\theta \geq 0$ for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{3}$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1\, d\theta$$

$$= [\theta]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Question 3

$$\int \sin\theta \sec^3\theta \, d\theta = \int \sin\theta \times \frac{1}{\cos^3\theta} \, d\theta$$

Let $u = \cos\theta, du = -\sin\theta \, d\theta \Rightarrow d\theta = -\frac{du}{\sin\theta}$

$$\begin{aligned}
\int \sin\theta \times \frac{1}{\cos^3\theta} \, d\theta &= \int \sin\theta \times \frac{1}{u^3} \times \left(-\frac{du}{\sin\theta}\right) \\
&= -\int \frac{1}{u^3} du \\
&= -\int u^{-3} \, du \\
&= -\frac{1}{-2} u^{-2} + C \\
&= \frac{1}{2u^2} + C \\
&= \frac{1}{2\cos^2\theta} + C \\
&= \frac{1}{2}\sec^2\theta + C
\end{aligned}$$

Question 4

Show that $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta} = \frac{\pi\sqrt{3}}{9}$

Let $t = \tan\frac{\theta}{2}, d\theta = \frac{2\,dt}{1+t^2}, \cos\theta = \frac{1-t^2}{1+t^2}$.

When $x=0, t=\tan 0 = 0$, when $x=\frac{\pi}{2}, t=\tan\frac{\pi}{4}=1$.

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta} = \int_0^1 \frac{\frac{2\,dt}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}$$

$$= \int_0^1 \frac{2}{2\left(1+t^2\right)+1-t^2}\,dt$$

$$= \int_0^1 \frac{2}{3+t^2}\,dt$$

$$= 2\left[\frac{1}{\sqrt{3}}\tan^{-}\left(\frac{x}{\sqrt{3}}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-}(0)\right]$$

$$= \frac{2}{\sqrt{3}}\left[\frac{\pi}{6}-0\right]$$

$$= \frac{2\pi}{6\sqrt{3}}$$

$$= \frac{\pi}{3\sqrt{3}}$$

$$= \frac{\pi\sqrt{3}}{3(3)}$$

$$= \frac{\pi\sqrt{3}}{9}$$

Question 5

$$\int_0^3 \frac{dx}{(3x+1)^2} = \left(-\frac{1}{3(3x+1)}\right)_0^3$$

$$= -\left(\frac{1}{10} - \frac{1}{3}\right)$$

$$= \frac{3}{10}$$

Question 6

$$\int \frac{1}{\sqrt{6x - x^2}}\,dx = \int \frac{1}{\sqrt{-9 + 6x - x^2 + 9}}\,dx$$

$$= \int \frac{1}{\sqrt{-\left(9 - 6x + x^2\right) + 9}}\,dx$$

$$= \int \frac{1}{\sqrt{9 - (x-3)^2}}\,dx$$

$$= \sin^{-1}\left(\frac{x-3}{3}\right) + C$$

Question 7

$$\int_{-\frac{\pi}{2}}^{0} \frac{2\,dx}{x^2 + 4} = \tan^{-}\left(\frac{x}{2}\right)_{-\frac{\pi}{2}}^{0}$$

$$= \tan^{-1}(0) - \tan^{-1}\left(-\frac{\pi}{4}\right)$$

$$= \tan^{-1}\left(\frac{\pi}{4}\right)$$

Question 8

$$\int \frac{1}{x^2 - 4x + 7}dx = \int \frac{1}{(x-2)^2 + 3}dx$$

$$= \frac{1}{\sqrt{3}} \tan^{-}\left(\frac{x-2}{\sqrt{3}}\right) + C$$

Question 9

$$\int_0^{0.5} \frac{x^2}{x^2-1}dx = \int_0^{0.5} \frac{x^2-1+1}{x^2-1}dx$$

$$= \int_0^{0.5} \frac{x^2-1}{x^2-1} + \frac{1}{x^2-1}dx$$

$$= \int_0^{0.5} 1 + \frac{1}{x^2-1}dx$$

$$= \int_0^{0.5} 1 + \frac{1}{(x-1)(x+1)}dx$$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$\Rightarrow 1 = A(x+1) + B(x-1)$$

Let $x = 1$

$$1 = 2A + 0B$$

$$A = \frac{1}{2}$$

Let $x = -1$

$$1 = 0A - 2B$$

$$B = -\frac{1}{2}$$

$$\Rightarrow \frac{1}{(x-1)(x+1)} = \frac{1}{2x-2} - \frac{1}{2x+2}$$

$$= \int_0^{0.5} \left(1 + \frac{1}{(x-1)(x+1)}\right)dx$$

$$= \int_0^{0.5} \left(1 + \frac{1}{2x-2} - \frac{1}{2x+2}\right)dx$$

$$= \left(x + \frac{1}{2}\ln|2x-2| - \frac{1}{2}\ln|2x+2|\right)_0^{0.5}$$

$$= \left(x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right)_0^{0.5}$$

$$= \left(0.5 + \frac{1}{2}\ln\left|\frac{-0.5}{1.5}\right|\right) - \left(0 + \frac{1}{2}\ln|-1|\right)$$

$$= 0.5 + \frac{1}{2}\ln\left(\frac{1}{3}\right)$$

Question 10

$$\frac{1}{x(x-1)}=\frac{A}{x}+\frac{B}{x-1}$$

$\Rightarrow 1=A(x-1)+Bx$

Let $x=1$

$1=0A+B$

$B=1$

Let $x=0$

$1=-A+0B$

$A=-1$

$$\Rightarrow \frac{1}{x(x-1)}=\frac{-1}{x}+\frac{1}{x-1}$$

$$\int\frac{1}{x(x-1)}dx=\int\left(\frac{1}{x-1}-\frac{1}{x}\right)dx$$

$=\ln|x-1|-\ln|x|+C$

$$=\ln\left|\frac{x-1}{x}\right|+C$$

Question 11

$$\frac{1}{(x+1)(x-1)^2}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

multiply by $(x+1)(x-1)^2$

$1=A(x-1)^2+B(x+1)(x-1)+C(x+1)$

$1=A\left(x^2-2x+1\right)+B\left(x^2-1\right)+C(x+1)$

$1=Ax^2-2Ax+A+Bx^2-B+Cx+C$

$1=(A+B)x^2(-2A+C)x+A-B+C$

Equating coefficients

$A+B=0$ [1]

$-2A+C=0$ [2]

$A-B+C=1$ [3]

[1] + [3]:

$2A+C=1$ [4]

[2] + [4]:

$2C=1 \Rightarrow C=\frac{1}{2}$

Sub into [2]:

$-2A+\frac{1}{2}=0$

$\frac{1}{2}=2A$

$A=\frac{1}{4}$

Sub into [1]:

$\frac{1}{4}+B=0$

$B=-\frac{1}{4}$

So $\frac{1}{(x+1)(x-1)^2}=\frac{1}{4(x+1)}-\frac{1}{4(x-1)}+\frac{1}{2(x-1)^2}$

$$\begin{aligned}\int\frac{1}{(x+1)(x-1)^2}dx&=\int\left(\frac{1}{4(x+1)}-\frac{1}{4(x-1)}+\frac{1}{2(x-1)^2}\right)dx\\&=\frac{1}{4}\ln|x+1|-\frac{1}{4}\ln|x-1|-\frac{1}{2(x-1)}+C\\&=\frac{1}{4}\ln\left|\frac{x+1}{x-1}\right|-\frac{1}{2(x-1)}+C\end{aligned}$$

Question 12

$$\frac{2x-3}{x^2+3x+2}=\frac{2x-3}{(x+2)(x+1)}$$

$$\frac{2x-3}{(x+2)(x+1)}=\frac{A}{x+2}+\frac{B}{x+1}$$

$$\Rightarrow 2x-3=A(x+1)+B(x+2)$$

Let $x=-1$

$$-5=0A+B$$

$$B=-5$$

Let $x=-2$

$$-7=-A+0B$$

$$A=7$$

$$\Rightarrow \frac{2x-3}{(x+2)(x+1)}=\frac{7}{x+2}-\frac{5}{x+1}$$

$$\int \frac{2x-3}{(x+2)(x+1)}dx$$

$$=\int\left(\frac{7}{x+2}-\frac{5}{x+1}\right)dx$$

$$=7\ln|x+2|-5\ln|x+1|+C$$

Question 13

$$\frac{2x^2+3x-1}{x^3-x^2+x-1}=\frac{2x^2+3x-1}{x^2(x-1)+x-1}=\frac{2x^2+3x-1}{(x^2+1)(x-1)}$$

$$\frac{2x^2+3x-1}{(x^2+1)(x-1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}$$

Multiply by $(x^2+1)(x-1)$

$$2x^2+3x-1=(Ax+B)(x-1)+C(x^2+1)$$

Let $x = 1$

$$2+3-1=0(Ax+B)+2C$$

$$4=2C$$

$$C=2$$

$$2x^2+3x-1=Ax^2+(B-A)x-B+2x^2+2$$

$$3x-3=Ax^2+(B-A)x-B$$

Equating coefficients

$$A=0$$

$$-B=-3$$

$$B=3$$

$$\Rightarrow \frac{2x^2+3x-1}{(x^2+1)(x-1)}=\frac{3}{x^2+1}+\frac{2}{x-1}$$

$$\int\frac{2x^2+3x-1}{(x^2+1)(x-1)}dx=\int\left(\frac{3}{x^2+1}+\frac{2}{x-1}\right)dx$$

$$=3\tan^{-}(x)+2\ln|x-1|+C$$

Question 14

$$\int_0^{\frac{1}{2}} \frac{x^2+3}{\left(x^2+1\right)\left(x^2+2\right)}dx$$

$$\frac{x^2+3}{\left(x^2+1\right)\left(x^2+2\right)} = \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2}$$

Multiply by $\left(x^2+1\right)\left(x^2+2\right)$.

$$\begin{aligned} x^2+3 &= (Ax+B)\left(x^2+2\right)+(Cx+D)\left(x^2+1\right) \\ &= Ax^3+Bx^2+2Ax+2B+Cx^3+Dx^2+Cx+D \\ &= (A+C)x^3+(B+D)x^2+(2A+C)x+2B+D \end{aligned}$$

Equating coefficients

$$A+C=0 \Rightarrow A=-C$$
$$2A+C=0 \Rightarrow 2A-A=0 \Rightarrow A=0, C=0$$
$$B+D=1$$
$$2B+D=3 \Rightarrow B=2,\ D=-1$$

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{x^2+3}{\left(x^2+1\right)\left(x^2+2\right)}dx &= \int_0^{\frac{1}{2}} \frac{2}{x^2+1}-\frac{1}{x^2+2}dx \\ &= \left[2\tan^{-1}x-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^{\frac{1}{2}} \\ &= 2\tan^{-1}\left(\frac{1}{2}\right)-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{2\sqrt{2}}\right) \\ &\approx 0.69 \end{aligned}$$

Question 15

$$\int^{e} \frac{\ln(x)}{x^2} dx$$

Let $u'(x) = \frac{1}{x^2}$ $\qquad v(x) = \ln(x)$

$u(x) = -\frac{1}{x}$ $\qquad v'(x) = \frac{1}{x}$

$$\int_1^{e} u'(x)v(x)dx = \left(u(x)v(x)\right)^{e} - \int_1^{e} u(x)v'(x)dx$$

$$\int_1^{e} \frac{\ln(x)}{x^2} dx = \left(-\frac{1}{x}\ln(x)\right)^{e} - \int_1^{e} -\frac{1}{x} \times \frac{1}{x} dx$$

$$= \left(-\frac{1}{x}\ln(x)\right)^{e} + \int^{e} \frac{1}{x^2} dx$$

$$= \left(-\frac{1}{x}\ln(x)\right)_1^{e} - \left(\frac{1}{x}\right)_1^{e}$$

$$= \left(-\frac{1}{x}\ln(x) - \frac{1}{x}\right)_1^{e}$$

$$= \left(-\frac{1}{e}\ln(e) - \frac{1}{e}\right) - \left(-\frac{1}{1}\ln(1) - \frac{1}{1}\right)$$

$$= -\frac{2}{e} + 1$$

Question 16

$\int \left(\ln(x)\right)^2 dx$

Let $u'(x) = 1 \qquad v(x) = \left(\ln(x)\right)^2$

$u(x) = x \qquad v'(x) = \frac{2}{x}\ln(x)$

$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$

$\int 1 \times \left(\ln(x)\right)^2 dx = x\left(\ln(x)\right)^2 - \int x \times \frac{2}{x}\ln(x)dx$

$= x\left(\ln(x)\right)^2 - 2\int \ln(x)dx$

$\int \ln(x)dx$

Let $u'(x) = 1 \qquad v(x) = \ln(x)$

$u(x) = x \qquad v'(x) = \frac{1}{x}$

$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$

$\int 1 \times \ln(x)dx = x\ln(x) - \int x \times \frac{1}{x}dx$

$= x\ln(x) - \int 1dx$

$= x\ln(x) - x$

$\int \left(\ln(x)\right)^2 dx = x\left(\ln(x)\right)^2 - 2\int \ln(x)dx$

$= x\left(\ln(x)\right)^2 - 2\left(x\ln(x) - x\right) + C$

$= x\left(\left(\ln(x)\right)^2 - 2\ln(x) + 2\right) + C$

Question 17

$$\int^{e} \frac{\ln(x)}{\sqrt{x}} dx$$

Let $u'(x) = \frac{1}{\sqrt{x}}$ $\qquad v(x) = \ln(x)$

$\qquad u(x) = 2\sqrt{x}$ $\qquad v'(x) = \frac{1}{x}$

$$\int_1^e u'(x)v(x)d = \left(u(x)v(x)\right)^e - \int_1^e u(x)v'(x)dx$$

$$\int_1^e \frac{\ln(x)}{\sqrt{x}} dx = \left(2\sqrt{x}\ln(x)\right)^e - \int_1^e 2\sqrt{x} \times \frac{1}{x} dx$$

$$= \left(2\sqrt{x}\ln(x)\right)_1^e - 2\int^e \frac{1}{\sqrt{x}} dx$$

$$= \left(2\sqrt{x}\ln(x) - 4\sqrt{x}\right)_1^e$$

$$= \left(2\sqrt{e}\ln(e) - 4\sqrt{e}\right) - \left(2\sqrt{1}\ln(1) - 4\sqrt{1}\right)$$

$$= -2\sqrt{e} + 4$$

Question 18

$$\int_0^{\ln 2} xe^x dx$$

Let $u'(x) = e^x$ $\qquad v(x) = x$

$\qquad u(x) = e^x$ $\qquad v'(x) = 1$

$$\int_0^{\text{n}\,2} u'(x)v(x)d = \left(u(x)v(x)\right)_0^{\text{n}\,2} - \int_0^{\ln 2} u(x)v'(x)dx$$

$$\int_0^{\text{n}\,2} xe^x dx = \left(xe^x\right)_0^{\text{n}\,2} - \int_0^{\ln 2} e^x dx$$

$$= \left(xe^x - e^x\right)_0^{\ln 2}$$

$$= \left(\ln 2e^{\ln 2} - e^{\ln 2}\right) - \left(0e^0 - e^0\right)$$

$$= 2\ln 2 - 2 + 1$$

$$= 2\ln 2 - 1$$

Question 19

$I_n = \int_0^1 \left(x^2 - 1\right)^n dx$

Let $u'(x) = 1 \qquad v(x) = \left(x^2 - 1\right)^n$

$u(x) = x \qquad v'(x) = 2nx\left(x^2 - 1\right)^{n-1}$

$\int_0^1 u'(x)v(x)d = \left(u(x)v(x)\right)_0 - \int_0^1 u(x)v'(x)dx$

$\int_0^1 1 \times \left(x^2 - 1\right)^n dx = \left(x\left(x^2 - 1\right)^n\right)_0 - \int_0^1 x \times \left(2nx\left(x^2 - 1\right)^{n-1}\right)dx$

$= 0 - 2n\int_0^1 x^2\left(x^2 - 1\right)^{n-1} dx$

$= -2n\int_0^1 \left(x^2 - 1 + 1\right)\left(x^2 - 1\right)^{n-1} dx$

$= -2n\int_0^1 \left(\left(x^2 - 1\right)\left(x^2 - 1\right)^{n-1} + \left(x^2 - 1\right)^{n-1}\right)dx$

$= -2n\int_0^1 \left(\left(x^2 - 1\right)^n + \left(x^2 - 1\right)^{n-1}\right)dx$

$= -2n\int_0^1 \left(x^2 - 1\right)^n dx - 2n\int_0^1 \left(x^2 - 1\right)^{n-1} dx$

$I_n = -2nI_n - 2nI_{n-1}$

$I_n + 2nI_n = -2nI_{n-1}$

$I_n\left(1 + 2n\right) = -2nI_{n-1}$

$I_n = \frac{-2n}{2n+1} I_{n-1}$

MATHS IN FOCUS 12 MATHEMATICS EXTENSION 2

WORKED SOLUTIONS

Chapter 7: Mechanics

Exercise 7.01 Velocity and acceleration in terms of *x*

Question 1

$$v = \sqrt{x^2 + 2}$$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}\left(\sqrt{x^2+2}\right)^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}\left(x^2+2\right)\right)$$

$$= \frac{1}{2} \times 2x$$

$$= x$$

Question 2

$$\ddot{x} = -3x$$

a When $t = 0$, $\ddot{x} = -3 \times 5 = -15 \text{ ms}^{-2}$

As it was at rest (v = 0) and the acceleration is negative, the velocity will be negative as well.

It will move to in the negative direction (to the left, towards the origin).

b

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -3x$$

$$d\left(\frac{1}{2}v^2\right) = -3x\,dx$$

$$\frac{1}{2}v^2 = \frac{-3x^2}{2} + c$$

When $v = 0$, $x = 5$

$$0 = \frac{-3 \times 5^2}{2} + c$$

$$c = \frac{75}{2}$$

$$v^2 = -3x^2 + 75$$

c Particle comes to rest when $v = 0$.

$$0 = -3x^2 + 75$$

$$3x^2 = 75$$

$$x^2 = 25$$

$$x = -5 \text{ (as } x = 5 \text{ at the start)}$$

d When $x = -5$, $\ddot{x} = -3x = -3(-5) = 15$ ms^{-2}.

As it was at rest ($v = 0$) and the acceleration is positive, the velocity will be positive as well.

It will move to in the positive direction (to the right, towards the origin).

e The particle oscillates between –5 and 5, greatest speed occurs when $a = 0$.

$$0 = -3x$$

$$x = 0$$

When $x = 0$

$$v^2 = 0 + 75$$

$$v = \sqrt{75}$$

$$v = 5\sqrt{3} \text{ m s}^{-1}$$

Question 3

a

$$\ddot{x} = -10$$
$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -10$$
$$\frac{1}{2}v^2 = -10x + c$$
$$v^2 = -20x + c$$
$$x = 0, v = 60$$
$$60^2 = c$$
$$c = 3600$$
$$v^2 = -20x + 3600$$

b

$$\frac{dv}{dt} = -10$$
$$v = -10t + c$$
$$v = 60, t = 0$$
$$60 = 0 + c$$
$$c = 60$$
$$v = -10t + 60$$

c The greatest height occurs when $v = 0$.

$$v^2 = -20x + 3600$$
$$0 = -20x + 3600$$
$$20x = 3600$$
$$x = 180 \text{ m}$$

d

$$v = -10t + 60$$
$$0 = -10t + 60$$
$$10t = 60$$
$$t = 6 \text{ s}$$

e

$$v = -10t + 60$$
$$v = -10 \times 3 + 60$$
$$v = 30$$
$$v^2 = -20x + 3600$$
$$30^2 = -20x + 3600$$
$$20x = 3600 - 900$$
$$20x = 2700$$
$$x = 135 \text{ m}$$

f

$$v^2 = -20x + 3600$$
$$v^2 = -20 \times 105 + 3600$$
$$v^2 = 1500$$
$$v = \pm 10\sqrt{15}$$
$$v = -10t + 60$$
$$10\sqrt{15} = -10t + 60$$
$$10t = 60 - 10\sqrt{15}$$
$$t = 2.13 \text{ s}$$

By symmetry of the flight:

t = 12 – 2.13 = 9.87 on the downward path.

Question 4

a

$$v^2 = 4\left(7 + 6x - x^2\right)$$
$$\ddot{x} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$
$$= \frac{d\left(\frac{1}{2}\left(4\left(7 + 6x - x^2\right)\right)\right)}{dx}$$
$$= 2\left(6 - 2x\right)$$
$$= 4\left(3 - x\right) \text{ m s}^{-2}$$

b Particle is at rest when v = 0.

$$0 = 4\left(7 + 6x - x^2\right)$$
$$0 = 7 + 6x - x^2$$
$$0 = (x - 7)(x + 1)$$
$$x = -1, 7$$
$$x = -1 \text{ m}$$
$$\ddot{x} = -4(-1 - 3) = 16 \text{ m s}^{-2}$$
$$x = 7 \text{ m}$$
$$\ddot{x} = -4(7 - 3) = -16 \text{ m s}^{-2}$$

c From part **b**: x = –1, x = 7. [–1, 7]

d Greatest speed of the particle when a = 0.

$$\ddot{x} = -4(x - 3)$$
$$0 = -4(x - 3)$$
$$0 = x - 3$$
$$x = 3$$
$$x = -1$$
$$v^2 = 4\left(7 + 6x - x^2\right)$$
$$= 4(7 + 18 - 9)$$
$$v^2 = 64$$
$$v = 8 \text{ m s}^{-1}$$

Question 5

a

$$\ddot{x} = \frac{1}{\sqrt{4x+9}}$$

When $x = 0$, $\ddot{x} = \frac{1}{\sqrt{9}} > 0$

So the particle moves to the right.

b

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{1}{\sqrt{4x+9}}$$

$$d\left(\frac{1}{2}v^2\right) = \frac{1}{\sqrt{4x+9}}dx$$

$$\frac{1}{2}v^2 = \frac{\sqrt{4x+9}}{2} + c$$

$$v^2 = \sqrt{4x+9} + c$$

$v = 0$, $x = 0$

$$0 = \sqrt{9} + c$$

$$c = -3$$

$$v^2 = \sqrt{4x+9} - 3$$

c As the particle is initially at rest and then moves to the right, $x > 0$.

$$v^2 = \sqrt{4x+9} - 3$$

As $x \geq 0$,

$$\sqrt{4x+9} \geq 3$$

$$\therefore v^2 \geq 0$$

So the particle will never slow down and hence will keep moving to the right.

d $x = 4$

$$v^2 = \sqrt{4 \times 4 + 9} - 3$$

$$= \sqrt{25} - 3$$

$$= 2$$

$$v = \sqrt{2}\ \text{m s}^{-1}$$

Question 6

a

$$\ddot{x} = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = -50e^{-4x}$$

$$d\left(\frac{1}{2}v^2\right) = -50e^{-4x}dx$$

$$\frac{1}{2}v^2 = 125\,e^{-4x} + c$$

$$v^2 = 25e^{-4x} + c$$

$$x = 0, v = 5$$

$$5^2 = 25e^0 + c$$

$$c = 0$$

$$v^2 = 25e^{-4x}$$

$$v = \sqrt{25e^{-4x}}$$

$$v = 5e^{-2x}\ \mathrm{m\,s^{-1}}$$

b

$$\frac{dx}{dt} = 5e^{-2x}$$

$$\frac{dt}{dx} = \frac{1}{5}e^{2x}$$

$$t = \frac{1}{10}e^{2x} + c$$

$$t = 0, x = 0$$

$$0 = \frac{1}{10}e^0 + c$$

$$c = -\frac{1}{10}$$

$$t = \frac{1}{10}\left(e^{2x} - 1\right)$$

$$10t = e^{2x} - 1$$

$$e^{2x} = 10t + 1$$

$$2x = \ln\left(10t + 1\right)$$

$$x = \frac{1}{2}\ln\left(10t + 1\right)$$

Question 7

$$v = \frac{dx}{dt} = 3x$$

a

$$\frac{dt}{dx} = \frac{1}{3x}$$

$$dt = \frac{1}{3x}dx$$

$$t = \frac{1}{3}\ln|x| + c$$

$$t = 0, x = 1$$

$$0 = \frac{1}{3}\ln 1 + c$$

$$c = 0$$

$$t = \frac{1}{3}\ln|x|$$

$$\ln|x| = 3t$$

$$x = e^{3t}$$

$$t = 3$$

$$x = e^{9} \text{ cm}$$

b

$$x = e^{3t}$$

$$v = \frac{dx}{dt}$$

$$= 3e^{3t}$$

$$v(3) = 3e^{9} \text{ cm s}^{-1}$$

c Particle starts to the right of the origin.

$$t \geq 0$$

$$v = 3e^{3t}$$

$$v \geq 0$$

So it continues to accelerate to the right.

d

$$v = 3x$$
$$v^2 = 9x^2$$
$$\frac{v^2}{2} = \frac{9x^2}{2}$$
$$\frac{d\left(\frac{v^2}{2}\right)}{dx} = 9x$$

Question 8

a

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{16+x^2}$$
$$\frac{1}{2}v^2 = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + c$$
$$v = 0, x = 0$$
$$0 = \frac{1}{4}\tan^{-}\ (0) + c$$
$$c = 0$$
$$\frac{1}{2}v^2 = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)$$
$$v^2 = \frac{1}{2}\tan^{-1}\left(\frac{x}{4}\right)$$

b Limiting maximum speed occurs when $v^2 = \frac{1}{2}\tan^{-1}\left(\frac{x}{4}\right)$ is at its maximum,

which is at its horizontal asymptote $v^2 = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$.

$$v^2 = \frac{\pi}{4}$$
$$v = \pm\sqrt{\frac{\pi}{4}}\ \text{m s}^{-1}$$

Limiting speed = $\sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}\ \text{m s}^{-1}$

Question 9

$$v = \sqrt{4x+6}$$

$$\frac{dx}{dt} = \sqrt{4x+6}$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{4x+6}}$$

$$t = \frac{\sqrt{4x+6}}{2} + c$$

$$t = 0, x = 0$$

$$0 = \frac{\sqrt{6}}{2} + c$$

$$c = -\frac{\sqrt{6}}{2}$$

$$t = \frac{\sqrt{4x+6} - \sqrt{6}}{2}$$

$$2t = \sqrt{4x+6} - \sqrt{6}$$

$$2t + \sqrt{6} = \sqrt{4x+6}$$

$$4x + 6 = \left(2t + \sqrt{6}\right)^2$$

$$4x = \left(2t + \sqrt{6}\right)^2 - 6$$

$$4x = 4t^2 + 4t\sqrt{6} + 6 - 6$$

$$x = t^2 + t\sqrt{6}$$

Question 10

$a = -8e^{-x}$

$\dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = -8e^{-x}$

$d\left(\frac{1}{2}v^2\right) = -8e^{-x}dx$

$\frac{1}{2}v^2 = 8e^{-x} + c$

$v^2 = 16e^{-x} + c$

$x = 0, v = 4$

$4^2 = 16e^0 + c$

$c = 0$

$v^2 = 16e^{-x}$

$v = 4e^{-\frac{x}{2}}$

$\frac{dx}{dt} = 4e^{-\frac{x}{2}}$

$\frac{dt}{dx} = \frac{1}{4}e^{\frac{x}{2}}$

$t = \frac{1}{2}e^{\frac{x}{2}} + c$

$t = 0, x = 0$

$0 = \frac{1}{2}e^0 + c$

$c = -\frac{1}{2}$

$t = \frac{1}{2}\left(e^{\frac{x}{2}} - 1\right)$

$e^{\frac{x}{2}} - 1 = 2t$

$e^{\frac{x}{2}} = 2t + 1$

$\frac{x}{2} = \ln(2t+1)$

$x = 2\ \ln(2t+1)$

Question 11

a

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx}=8$$

$$d\left(\frac{1}{2}v^2\right)=8\,dx$$

$$\frac{1}{2}v^2=8x+c$$

$$v^2=16x+c$$

$$x=0, v=4$$

$$4^2=c$$

$$c=16$$

$$v^2=16(x+1)$$

$$v=4\sqrt{x+1}\ \text{m s}^{-1}$$

b

$$x=3$$

$$v=4\sqrt{(3+1)}=8\ \text{m s}^{-1}$$

c

$$\frac{dx}{dt} = 4\sqrt{x+1}$$

$$\frac{dt}{dx} = \frac{1}{4\sqrt{x+1}}$$

$$t = \frac{\sqrt{x+1}}{2} + c$$

$$t = 0, x = 0$$

$$0 = \frac{\sqrt{1}}{2} + c$$

$$c = -\frac{1}{2}$$

$$t = \frac{\sqrt{x+1} - 1}{2}$$

$$2t = \sqrt{x+1} - 1$$

$$\sqrt{x+1} = 2t + 1$$

$$x + 1 = (2t+1)^2$$

$$x + 1 = 4t^2 + 4t + 1$$

$$x = 4t^2 + 4t \text{ m}$$

Question 12

$$\ddot{x} = -\frac{160\ 000}{x^2}$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -\frac{160\ 000}{x^2}$$

$$d\left(\frac{1}{2}v^2\right) = -\frac{160\ 000}{x^2}dx$$

$$\frac{1}{2}v^2 = \frac{160\ 000}{x} + c$$

$$v^2 = \frac{320\ 000}{x} + c$$

$$x = 6400, v = 4$$

$$4^2 = \frac{320\ 000}{6400} + c$$

$$16 = 50 + c$$

$$c = -34$$

$$v^2 = \frac{320\ 000}{x} - 34$$

Let $v = 0$

$$0 = \frac{320\ 000}{x} - 34$$

$$34 = \frac{320\ 000}{x}$$

$$x = \frac{320\ 000}{34}$$

$$x = 9411.76...$$

Distance above Earth 1.76...   6400

$= 3011.76...$

≈ 3012 km

Question 13

$$a = v^2 - 8$$

$$v\frac{dv}{dx} = v^2 - 8$$

$$\frac{dv}{dx} = \frac{v^2 - 8}{v}$$

$$\frac{dx}{dv} = \frac{v}{v^2 - 8}$$

$$= \frac{1}{2}\frac{2v}{v^2 - 8}$$

$$x = \frac{1}{2}\ln\left|v^2 - 8\right| + c$$

$$x = 1, v = -3$$

$$1 = \frac{1}{2}\ln 1 + c$$

$$c = 1$$

$$x = \frac{1}{2}\ln\left|v^2 - 8\right| + 1$$

$$\frac{1}{2}\ln\left|v^2 - 8\right| = x - 1$$

$$\ln\left|v^2 - 8\right| = 2(x - 1)$$

$$v^2 - 8 = e^{2\ x-1)}$$

$$v^2 = e^{2\ x-1} + 8$$

$$v = -\sqrt{e^{2\ x-1} + 8}\ \text{m s}^{-1}$$

Question 14

$$v = -2x^2$$

$$v^2 = \left(-2x^2\right)^2 = 4x^4$$

$$\frac{1}{2}v^2 = 2x^4$$

$$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

$$a = 8x^3$$

$$a(4) = 8 \times 4^3 = 512 \text{ m s}^{-2}$$

$$\frac{dx}{dt} = -2x^2$$

$$\frac{dt}{dx} = -\frac{1}{2x^2}$$

$$t = \frac{1}{2x} + c$$

$$t = 0, x = 4$$

$$0 = \frac{1}{8} + c$$

$$c = -\frac{1}{8}$$

$$t = \frac{1}{2x} - \frac{1}{8}$$

$$\frac{1}{2x} = t + \frac{1}{8}$$

$$\frac{1}{2x} = \frac{8t+1}{8}$$

$$2x = \frac{8}{8t+1}$$

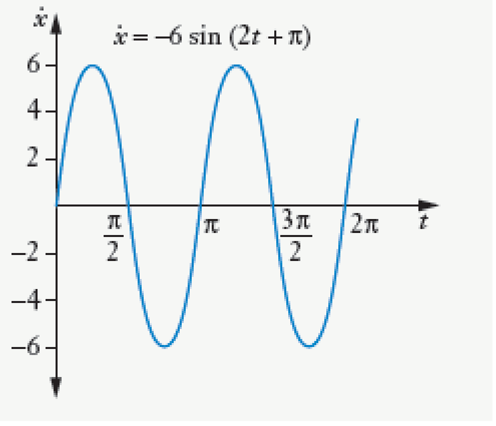
$$x = \frac{4}{8t+1}$$

Question 1

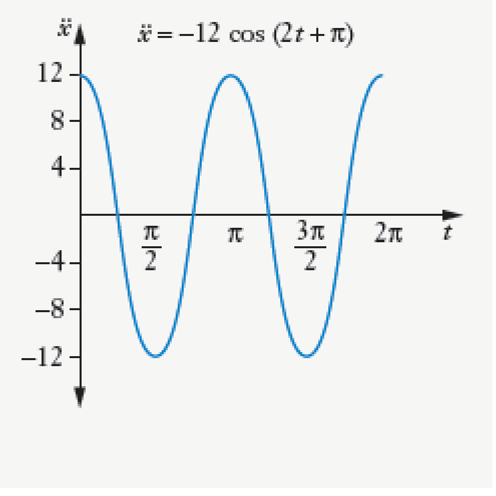
a

$$x = A\cos(nt + \alpha)$$
$$A = 3, n = 2, \alpha = \pi$$
$$x = 3\cos(2t + \pi)$$

b $\dot{x} = -6\sin(2t + \pi)$



c $\ddot{x} = -12\cos(2t + \pi)$

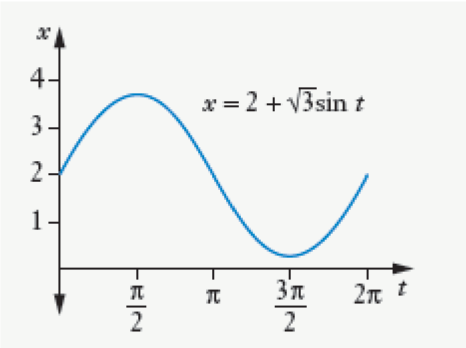


Question 2

a

$$x = 2 + \sqrt{3}\sin t$$
$$\dot{x} = \sqrt{3}\cos t$$
$$\ddot{x} = -\sqrt{3}\sin t$$
$$= 2 - 2 - \sqrt{3}\sin t$$
$$= 2 - x$$
$$= -(x-2) \ \text{m s}^{-2}$$

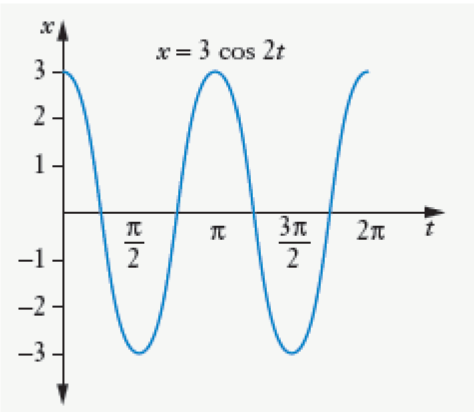
b



$A = \sqrt{3}$, $T = 2\pi$.

Question 3

a

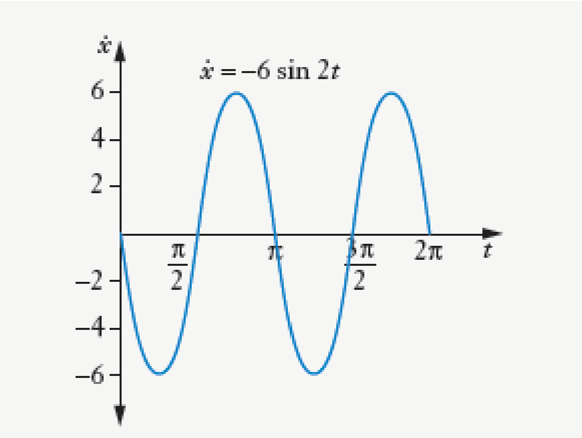


b Maximum at $n\pi$, so first 3 times 0, π and 2π.

Maximum displacement = 3

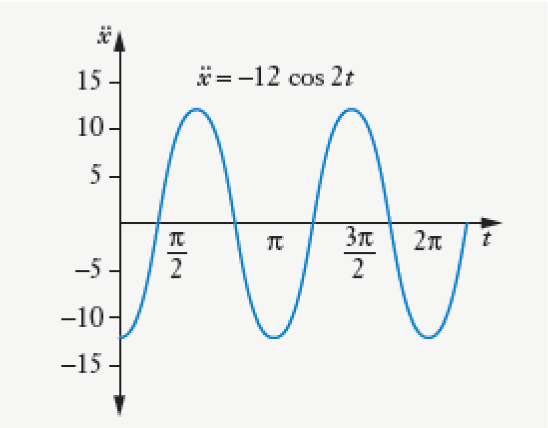
c

$$x = 3\cos(2t)$$
$$\dot{x} = -6\sin(2t)$$



d Velocity at maximum displacement = 0.

e $\ddot{x} = -12\cos(2t)$



f When the particle is at the origin $a = 0$.

Question 4

a

$$x = 5\cos 3t - 12\sin 3t$$
$$\dot{x} = -15\sin 3t - 36\cos 3t$$
$$\ddot{x} = -45\cos 3t + 108\cos 3t$$
$$= -9\left(5\cos 3t - 12\sin 3t\right)$$
$$= -9x$$

$\therefore$ It is simple Harmonic Motion

b

$$5\cos 3t - 12\sin 3t = R\sin\left(t + \alpha\right)$$
$$R = \sqrt{5^2 + 12^2} = 13$$
$$\alpha = \tan^{-1}\left(\frac{-12}{5}\right)$$
$$5\cos 3t - 12\sin 3t = 13\sin\left(3t + \tan^{-}\left(\frac{-12}{5}\right)\right)$$
$$\text{Period} = T = \frac{2\pi}{3}$$

c

$$x = 13\sin\left(3t + \tan^{-}\left(\frac{-12}{5}\right)\right)$$
$$\dot{x} = 39\cos\left(3t + \tan^{-}\left(\frac{-12}{5}\right)\right)$$

Maximum speed = 39 m s^{-1}

Question 5

$\ddot{x} = -n^2 x = -16x$

$n^2 = 16$

$n = 4$

$x = A\cos(4t + \alpha)$

When $t = 0,\ x = 0,\ v = 6.$

$0 = A\cos(0 + \alpha)$

$0 = \cos\alpha$

$\alpha = \frac{\pi}{2}$

$x = A\cos\left(4t + \frac{\pi}{2}\right)$

$v = -4A\sin\left(4t + \frac{\pi}{2}\right)$

When $t = 0,\ v = 6.$

$6 = -4A\sin\left(\frac{\pi}{2}\right)$

$6 = -4A$

$A = -\frac{3}{2}$

$x = -\frac{3}{2}\cos\left(4t + \frac{\pi}{2}\right)$

This can also be written as $\frac{3}{2}\cos\left(4t - \frac{\pi}{2}\right)$ or $\frac{3}{2}\sin 4t.$

OR

$$\ddot{x} = -16x$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -16x$$

$$\frac{1}{2}v^2 = -8x^2 + c$$

$$v^2 = -16x^2 + c$$

$$x = 0, v = 6$$

$$6^2 = c$$

$$v^2 = 36 - 16x^2$$

$$v = \sqrt{36 - 16x^2}$$

$$\frac{dx}{dt} = \sqrt{36 - 16x^2}$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{36 - 16x^2}}$$

$$\frac{dt}{dx} = \frac{1}{4}\frac{1}{\sqrt{\frac{36}{16} - x^2}}$$

$$t = \frac{1}{4}\sin^{-}\left(\frac{4x}{6}\right) + c$$

$$t = \frac{1}{4}\sin^{-}\left(\frac{2x}{3}\right) + c$$

$$t = 0, x = 0$$

$$c = 0$$

$$t = \frac{1}{4}\sin^{-}\left(\frac{2x}{3}\right)$$

$$\sin^{-1}\left(\frac{2x}{3}\right) = 4t$$

$$\frac{2x}{3} = \sin 4t$$

$$x = \frac{3}{2}\sin 4t$$

Question 6

a

$$v^2 = 225 - 625x^2$$

$$\frac{v^2}{2} = \frac{225 - 625x^2}{2}$$

$$\frac{d\left(\frac{v^2}{2}\right)}{dx} = a = -625x$$

b

$$v^2 = 225 - 625x^2$$

$$v^2 = 625\left(\frac{225}{625} - x^2\right)$$

$$v^2 = 25^2\left(\left(\frac{15}{25}\right)^2 - x^2\right)$$

$$n = 2, \quad A = \frac{15}{25}$$

amplitude = $A = \frac{15}{25} = \frac{3}{5}$

Period = $T = \frac{2\pi}{n} = \frac{2\pi}{25}$

Maximum speed occur when $x = 0$.

$$v^2 = 225 - 0$$

$$v = 15 \text{ m s}^{-1}$$

Question 7

$$\dot{x}^2 = 6x - x^2$$
$$= 9 - 9 + 6x - x^2$$
$$= 9 - \left(x^2 - 6x + 9\right)$$
$$= 9 - \left(x - 3\right)^2$$
$$n = 1, A = 3, c = 3$$

a As A = 3 and c = 3 the points must be 6 m, 0 m.

b Centre of motion is 3 m.

c Maximum speed occurs when x = 3.

$$\dot{x}^2 = 6x - x^2$$
$$\dot{x}^2\left(3\right) = 18 - 9 = 9$$
$$\dot{x} = 3 \text{ m s}^{-1}$$

d $\ddot{x} = -n^2\left(x - c\right) = -\left(x - 3\right) \text{ m s}^{-2}$

e $\text{Period} = T = \dfrac{2\pi}{n} = \dfrac{2\pi}{1} = 2\pi \text{ s}$

Question 8

$$\dot{x}^2 = 60 - 8x - 4x^2$$
$$= 4\left(15 - 2x - x^2\right)$$
$$= 4\left(16 - 1 - 2x - x^2\right)$$
$$= 4\left(16 - \left(1 + 2x + x^2\right)\right)$$
$$= 2^2\left(4^2 - \left(x+1\right)^2\right)$$
$$n = 2, A = 4, c = -1$$

a $\ddot{x} = -n^2\left(x - c\right) = -4\left(x+1\right)$

b Centre of motion $x = -1$.

c

$$\text{Period} = T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$$
$$\text{Frequency} = f = \frac{1}{T} = \frac{1}{\pi}$$

d

$$x = A\cos\left(nt + \alpha\right) + c$$
$$x = 4\cos\left(2t + \alpha\right) - 1$$
$$t = 0, x = -1$$
$$-1 = 4\cos\alpha - 1$$
$$0 = 4\cos\alpha$$
$$\alpha = \pm\frac{\pi}{2}$$

But $\dot{x} > 0$ when $t = 0$:

$$\dot{x} = -8\sin\left(2t + \alpha\right) > 0$$
$$-8\sin\alpha > 0$$
$$\sin\alpha < 0$$
$$\therefore \alpha = -\frac{\pi}{2}$$
$$x = 4\cos\left(2t - \frac{\pi}{2}\right) - 1$$

Question 9

$x = 3\cos(2t + \alpha)$

a Amplitude = 3

Period = $\frac{2\pi}{2} = \pi$

$\frac{dx}{dt} = -6\sin(2t + \alpha)$

Greatest speed = 6 m/s

b

$$
\begin{aligned}
x &= 3\cos(2t+\alpha) \\
\dot{x} &= -6\sin(2t+\alpha) \\
\dot{x}^2 &= 36\sin^2(2t+\alpha) \\
&= 36\left(1-\cos^2(2t+\alpha)\right) \\
&= 36 - 36\cos^2(2t+\alpha) \\
&= 4\left(9 - 9\cos^2(2t+\alpha)\right) \\
&= 4\left(9 - \left(3\cos(2t+\alpha)\right)^2\right) \\
&= 4\left(9 - x^2\right)
\end{aligned}
$$

c

$$
\begin{aligned}
\dot{x} &= -6\sin(2t+\alpha) \\
\ddot{x} &= -12\cos(2t+\alpha) \\
&= -4\left[3\cos(2t+\alpha)\right] \\
&= -4x \text{ ms}^{-2}
\end{aligned}
$$

d

$$x = 3\cos(2t + \alpha)$$

$$t = 0\,\text{s}, x = 1.5\,\text{m},\ \dot{x} = 3\sqrt{3}\ \text{m s}^{-1}$$

$$1.5 = 3\cos(0 + \alpha)$$

$$\cos\alpha = \frac{1}{2}$$

$$\alpha = \pm\frac{\pi}{3}$$

$$\text{But } \dot{x} = -6\sin(2t + \alpha) = 3\sqrt{3} \text{ when } t = 0$$

$$-6\sin\alpha = 3\sqrt{3}$$

$$\sin\alpha = -\frac{\sqrt{3}}{2}$$

$$\alpha = -\frac{\pi}{3}$$

$$\dot{x} = -6\sin\left(2t - \frac{\pi}{3}\right)$$

Question 10

$$\dot{x}^2 = 28 - 24x - 4x^2$$
$$= 4\left(7 - 6x - x^2\right)$$
$$= 4\left(16 - 9 - 6x - x^2\right)$$
$$= 4\left(16 - \left(9 + 6x + x^2\right)\right)$$
$$= 2^2\left(4^2 - \left(x+3\right)^2\right)$$
$$n = 2, A = 4, c = -3$$

a

$$\dot{x}^2 = 2^2\left(4^2 - \left(x+3\right)^2\right)$$
$$\frac{\dot{x}^2}{2} = 2\left(4^2 - \left(x+3\right)^2\right)$$
$$\frac{d\left(\frac{\dot{x}^2}{2}\right)}{dx} = a = 2\left(-2x - 6\right) = -4\left(x+3\right) \text{ m s}^{-2}$$

b

$$\ddot{x} = -nx$$
$$\dot{x}^2 = n^2\left(A^2 - x^2\right)$$

These 2 equations satisfy the conditions for simple harmonic motion.

Period = $T = \dfrac{2\pi}{n} = \dfrac{2\pi}{2} = \pi$.

c A = 4, c = –3.

The particle is at rest at –3 ± 4 = = –7 m, 1 m

d Greatest speed occurs when

$$\left(x+3\right) = 0$$
$$\dot{x}^2 = 2^2\left(4^2 - 0\right) = 64$$
$$\dot{x} = 8 \text{ m s}^{-1}$$

e

$$\dot{x}^2 = 28 - 24x - 4x^2$$
$$= 4\left(7 - 6x - x^2\right)$$
$$= 4\left(16 - 9 - 6x - x^2\right)$$
$$= 4\left(16 - \left(9 + 6x + x^2\right)\right)$$
$$= 2^2\left(4^2 - \left(x + 3\right)^2\right)$$
$$n = 2, A = 4, c = -3$$

Question 11

a

$$x = \cos 5t - \sqrt{3}\sin 5t = R\sin(t+\alpha)$$

$$R = \sqrt{1^2 + \sqrt{3}^2} = 2$$

$$\alpha = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6}$$

$$x = 2\sin\left(5t - \frac{\pi}{6}\right)$$

It is simple harmonic motion with

$$\text{Period} = T = \frac{2\pi}{5}$$

$$\text{Amplitude} = A = 2$$

b

$$x = 2\sin\left(5t - \frac{\pi}{6}\right)$$

$$\dot{x} = 10\cos\left(5t - \frac{\pi}{6}\right)$$

$$\dot{x} = 5$$

$$5 = 10\cos\left(5t - \frac{\pi}{6}\right)$$

$$\cos\left(5t - \frac{\pi}{6}\right) = \frac{1}{2}$$

$$5t - \frac{\pi}{6} = \frac{\pi}{3}$$

$$5t = \frac{\pi}{2}$$

$$t = \frac{\pi}{10}\text{ s}$$

Question 12

$x = 3 \sin (4t)$

a

$$x = 3\sin\left(4t\right)$$
$$\dot{x} = 12\cos\left(4t\right)$$
$$\ddot{x} = -48\sin\left(4t\right)$$
$$= -12x$$

which satisfies the condition for simple harmonic motion.

$$\text{Period} = T = \frac{2\pi}{4} = \frac{\pi}{2}$$

b

$$x = 3\sin\left(4t\right)$$
$$1.5 = 3\sin\left(4t\right)$$
$$\sin\left(4t\right) = \frac{1}{2}$$
$$4t = \frac{\pi}{6}$$
$$t = \frac{\pi}{24}\text{ s}$$
$$\dot{x} = 12\cos\left(4t\right)$$
$$\dot{x}\left(\frac{\pi}{6}\right) = 12\cos\left(4 \times \frac{\pi}{24}\right)$$
$$= 12 \times \left(\frac{\sqrt{3}}{2}\right)$$
$$= 6\sqrt{3}\ \text{cm s}^{-1}$$

c Greatest speed

$$\dot{x} = 12\cos\left(4t\right)$$

Maximum occurs when $4t = 0$, so $\dot{x} = 12 \times 1 = 12$ cm s^{-}.

The particle is centred on the origin and its amplitude is 3.

Hence the interval [–3, 3].

Question 13

a

$$\ddot{x} = -4x$$

$$\frac{d\left(\frac{1}{2}\dot{x}^2\right)}{dx} = -4x$$

$$\frac{1}{2}\dot{x}^2 = -2x^2 + c$$

$$\dot{x}^2 = -4x^2 + c$$

$$x = 0, v = 12$$

$$144 = c$$

$$\dot{x}^2 = -4x^2 + 144$$

$$\dot{x} = \pm\sqrt{-4x^2 + 144}$$

$$\dot{x} = \pm 2\sqrt{36 - x^2}$$

b

$$\frac{dx}{dt} = \sqrt{-4x^2 + 144}$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{-4x^2 + 144}}$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{4\left(36 - x^2\right)}}$$

$$\frac{dt}{dx} = \frac{1}{2}\frac{1}{\sqrt{36 - x^2}}$$

$$t = \frac{1}{2}\sin^{-}\left(\frac{x}{6}\right) + c$$

$$t = 0, x = 0, c = 0$$

$$\sin^{-}\left(\frac{x}{6}\right) = 2t$$

$$\frac{x}{6} = \sin\left(2t\right)$$

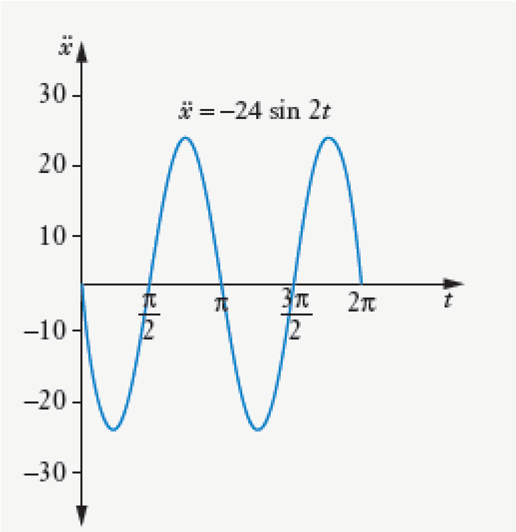
$$x = 6\sin\left(2t\right)$$

c Period = $T = \frac{2\pi}{2} = \pi$

Frequency = $f = \frac{1}{T} = \frac{1}{\pi}$

d

$$x = 6\sin(2t)$$
$$\dot{x} = 12\cos(2t)$$
$$\ddot{x} = -24\sin(2t)$$



Question 14

$$\dot{x}^2 = \pi^2\left(4 - x^2\right)$$
$$n = \pi, A = 2$$

a Amplitude = A = 2

$$\text{Period} = T = \frac{2\pi}{\pi} = 2$$

b

$$x = A\cos(nt + \alpha)$$
$$x = 2\cos(\pi t + \alpha)$$
$$t = 0, x = 2$$
$$2 = 2\cos\alpha$$
$$\cos\alpha = 1$$
$$\alpha = 0$$
$$x = 2\cos\pi t$$

c

$$\pm\sqrt{2} = 2\cos\pi t$$
$$\cos\pi t = \pm\frac{1}{\sqrt{2}}$$
$$\pi t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$
$$t = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$$

It is within $\sqrt{2}$ for t values between $\frac{1}{4}, \frac{3}{4}$ and $\frac{5}{4}, \frac{7}{4}$.

This gives a total of $\frac{3}{4} - \frac{1}{4} + \frac{7}{4} - \frac{5}{4} = 1$ for a period of 2 hence it is 50%.

Question 15

a Amplitude = A = 14.5 – 8.5 = 6 m

Period = T = 16 hours (2 × 8 hours [midnight to 8 a.m., both 8.5 m])

b

3:00 p.m. = 15 hours after midnight, t = 15.

$$x = 8.5 + 6\sin\left(\frac{\pi}{8}t\right)$$

$$x(15) = 8.5 + 6\sin\left(\frac{\pi}{8}\times 15\right) = 6.2 \text{ m}$$

c

$$10 = 8.5 + 6\sin\left(\frac{\pi}{8}t\right)$$

$$1.5 = 6\sin\left(\frac{\pi}{8}t\right)$$

$$\sin\left(\frac{\pi}{8}t\right) = \frac{1}{4}$$

$$\frac{\pi}{8}t = \sin^{-1}\left(\frac{1}{4}\right)$$

$$\frac{\pi}{8}t = 0.2526, 2.8889$$

$$t = 0.64, 7.36$$

$$7.36 - 0.64 = 6.72 \text{ hours} \approx 6 \text{ hours } 43 \text{ minutes}$$

Exercise 7.03 Projectile motion

Question 1

$v = 35 \text{ m s}^{-1}$

$y_{\text{max}} = \dfrac{v^2}{2g}$

$= \dfrac{35^2}{20}$

$= 61.25 \text{ m}$

Question 2

$y = 980 \text{ m}, g = 9.8 \text{ m s}^{-2}$

$s = ut + \dfrac{1}{2}at^2$

$980 = \dfrac{1}{2} \times 9.8 \times t^2$

$t^2 = 200$

$t = 10\sqrt{2} \text{ s}$

Question 3

v = 45 m s^{-1}, g = 9.8 m s^{-2}

a

$\ddot{y} = -9.8$

$\dot{y} = -98\ t + c$

When $t = 0, \dot{y} = 45$

$45 = 0 + c$

$\dot{y} = -9.8t + 45$

When $\dot{y} = 0$:

$0 = -9.8t + 45$

$9.8t = 45$

$t = \frac{45}{9.8}$

$= 4.5919...$

≈ 4.6 s

b

$\dot{y} = -9.8t + 45$

$y = -4.9t^2 + 45t + d$

When $t = 0,\ y = 0$:

$0 = 0 + 0 + d$

$y = -4.9t^2 + 45t$

When $t = 4.6$,

$y = -4.9(4.6)^2 + 45(4.6)$

$= 103.316.31$

≈ 103.3 m

c

$y = 0$:

$0 = -4.9t^2 + 45t$

$0 = t(-4.9t + 45)$

$0 = -4.9t + 45 \qquad t \neq 0$

$4.9t = 45$

$t = \frac{45}{4.9}$

$= 9.1836...$

≈ 9.2s

d

$$\dot{y} = -9.8t + 45$$

When $t = 8$,

$$y = -9.8(8) + 45$$

$$= 33.4 \text{ ms}^{-1}$$

Question 4

a

$$\ddot{x} = 0$$
$$\dot{x} = c$$
When $t = 0, \dot{x} = 20\cos 45° = \frac{20}{\sqrt{2}} = \frac{20\sqrt{2}}{2} = 10\sqrt{2}$
$$c = 10\sqrt{2}$$
$$\dot{x} = 10\sqrt{2}$$
$$x = 10t\sqrt{2} + d$$
When $t = 0,\ x = 0$:
$$0 = 0 + d$$
$$d = 0$$
$$x = 10t\sqrt{2}$$
$$\ddot{y} = -10$$
$$\dot{y} = -10t + e$$
When $t = 0, \dot{y} = 20\sin 45° = \frac{20}{\sqrt{2}} = \frac{20\sqrt{2}}{2} = 10\sqrt{2}$
$$e = 10\sqrt{2}$$
$$\dot{y} = -10t + 10\sqrt{2}$$
$$y = -5t^2 + 10t\sqrt{2} + f$$
When $t = 0,\ y = 0$
$$0 = 0 + 0 + f$$
$$f = 0$$
$$y = -5t^2 + 10t\sqrt{2}$$

b

$$y = -5t^2 + 10t\sqrt{2}$$
$$y = 0$$
$$0 = t\left(-5t + 10\sqrt{2}\right)$$
$$0 = -5t + 10\sqrt{2} \qquad t \neq 0$$
$$5t = 10\sqrt{2}$$
$$t = 2\sqrt{2} \text{ s}$$

c

$$x = 10t\sqrt{2} \qquad [1]$$

$$y = -5t^2 + 10t\sqrt{2} \qquad [2]$$

From [1]:

$$t = \frac{x}{10\sqrt{2}}$$

Sub into [2]:

$$y = -5\left(\frac{x}{10\sqrt{2}}\right)^2 + 10\sqrt{2}\left(\frac{x}{10\sqrt{2}}\right)$$

$$y = -\frac{5x^2}{200} + x$$

$$y = -\frac{x^2}{40} + x$$

Question 5

$$\text{Range} = \frac{V^2 \sin 2\theta}{g}$$

$$\text{Range} = 60\,\text{m}, \theta = 30°$$

$$60 = \frac{V^2 \sin 60°}{10}$$

$$V^2 = \frac{600}{\frac{\sqrt{3}}{2}}$$

$$V^2 = \frac{1200}{\sqrt{3}} = 692.8203...$$

$$V = \sqrt{692.8203...}$$

$$= 26.3214...$$

$$\approx 26.3\,\text{m}\,\text{s}^{-1}$$

Question 6

$\ddot{x} = 0$

$\dot{x} = c$

When $t = 0, \dot{x} = 100\cos\alpha$:

$c = 100\cos\alpha$

$\dot{x} = 100\cos\alpha$

$x = 100t\cos\alpha + d$

When $t = 0, x = 0$:

$d = 0$

$x = 100t\cos\alpha$ [1]

$\ddot{y} = -g = -10$

$\dot{y} = -10t + e$

When $t = 0, \dot{y} = 100\sin\alpha$:

$100\sin\alpha = 0 + e$

$\dot{y} = -10t + 100\sin\alpha$

$y = -5t^2 + 100t\sin\alpha + f$

When $t = 0, y = 50$:

$50 = 0 + 0 + f$

$f = 50$

$y = -5t^2 + 100t\sin\alpha + 50$

$t = \dfrac{x}{100\cos\alpha}$ from[1]

$$y = -5\left(\frac{x}{100\cos\alpha}\right)^2 + 100\sin\alpha\left(\frac{x}{100\cos\alpha}\right) + 50$$

$$= -\frac{x^2}{2000\cos^2\alpha} + x\tan\alpha + 50$$

$$= -\frac{x^2}{2000}\sec^2\alpha + x\tan\alpha + 50$$

$$= -\frac{x^2}{2000}\left(1 + \tan^2\alpha\right) + x\tan\alpha + 50$$

When $x = 500,\ y = 0$:

$$0 = -\frac{500^2}{2000}\left(1+\tan^2\alpha\right)+500\tan\alpha+50$$

$$0 = -125\left(1+\tan^2\alpha\right)+500\tan\alpha+50$$

$$125\tan^2\alpha-500\tan\alpha+75=0$$

$$5\tan^2\alpha-20\tan\alpha+3=0$$

$$\tan\alpha = \frac{-(-20)\pm\sqrt{(-20)^2-4\times5\times3}}{10}$$

$$= \frac{20\pm\sqrt{340}}{10}$$

$$= 3.8439...,0.1560...$$

$$\alpha = 75.417...^\circ \approx 75^\circ \quad \text{or} \quad \alpha = 8.971...^\circ \approx 9^\circ$$

Question 7

$$v = 50$$

$$x = \frac{v^2\sin 2\theta}{g}$$

$$250 = \frac{50^2\sin 2\theta}{10}$$

$$\sin 2\theta = \frac{2500}{2500}$$

$$\sin 2\theta = 1$$

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

Question 8

$\ddot{x} = 0$

$\dot{x} = c$

When $t = 0, \dot{x} = V\cos\alpha$:

$c = V\cos\alpha$

$\dot{x} = V\cos\alpha$

$x = Vt\cos\alpha + d$

When $t = 0, x = 0$:

$d = 0$

$x = Vt\cos\alpha$ [1]

$\ddot{y} = -g = -10$

$\dot{y} = -10t + e$

When $t = 0, \dot{y} = V\sin\alpha$:

$V\sin\alpha = 0 + e$

$\dot{y} = -10t + V\sin\alpha$

$y = -5t^2 + Vt\sin\alpha + f$

When $t = 0, y = 0$:

$0 = 0 + 0 + f$

$f = 0$

$y = -5t^2 + Vt\sin\alpha$ [2]

$t = \dfrac{x}{V\cos\alpha}$ from[1]

$$y = -5\left(\frac{x}{V\cos\alpha}\right)^2 + V\sin\alpha\left(\frac{x}{V\cos\alpha}\right)$$

$$= -\frac{5x^2}{V^2\cos^2\alpha} + x\tan\alpha$$

Let R be the horizotal range.

When $x = R,\ y = 0$:

$$0 = -\frac{5R^2}{V^2 \cos^2 \alpha} + R \tan \alpha$$

$$0 = -\frac{5R^2}{V^2} \sec^2 \alpha + R \tan \alpha$$

$$0 = -\frac{5R^2}{V^2}\left(1 + \tan^2 \alpha\right) + R \tan \alpha$$

$$\frac{5R^2}{V^2}\left(1 + \tan^2 \alpha\right) - R \tan \alpha = 0$$

$$5R^2 + 5R^2 \tan^2 \alpha - RV^2 \tan \alpha = 0$$

$$5R^2 \tan^2 \alpha - RV^2 \tan \alpha + 5R^2 = 0$$

$$\tan \alpha = \frac{-\left(-RV^2\right) \pm \sqrt{\left(-RV^2\right)^2 - 4 \times 5R^2 \times 5R^2}}{2\left(5R^2\right)}$$

$$\tan \alpha = \frac{RV^2 \pm \sqrt{R^2V^4 - 100R^4}}{10R^2}$$

$$\tan \alpha = \frac{RV^2 \pm R\sqrt{V^4 - 100R^2}}{10R^2}$$

$$\tan \alpha = \frac{V^2 \pm \sqrt{V^4 - 100R^2}}{10R}$$

2 solutionsfor α

Note: $\sqrt{V^4 - 100R^2} < V^2$ so RHS > 0, so both solutions are less than $\frac{\pi}{2}$.

OR, continuing from [2]

Finding the range R when $y = 0$:

$0 = -5t^2 + Vt\sin\alpha$

$5t^2 = Vt\sin\alpha$

$5t = V\sin\alpha \quad (t \neq 0)$

$t = \dfrac{V\sin\alpha}{5}$

Sub into [1]:

$R = Vt\cos\alpha$

$R = V\left(\dfrac{V\sin\alpha}{5}\right)\cos\alpha$

$R = \dfrac{V^2\sin\alpha\cos\alpha}{5}$

$R = \dfrac{V^2\sin 2\alpha}{10}$

$\sin 2\alpha = \dfrac{10R}{V^2}$

$\alpha = \dfrac{1}{2}\sin^{-1}\left(\dfrac{10R}{V^2}\right)$ or $\dfrac{1}{2}\left[\pi - \sin^{-1}\left(\dfrac{10R}{V^2}\right)\right]$

Note: Both solutionsless than $\dfrac{\pi}{2}$.

Question 9

$\ddot{x} = 0$

$\dot{x} = c$

When $t = 0,\ \dot{x} = V\cos 0° = V$

$\dot{x} = V$

$x = Vt + d$

When $t = 0,\ x = 0$

$0 = 0 + d$

$d = 0$

$x = Vt \qquad [1]$

$\ddot{y} = -g = -10$

$\dot{y} = -10t + e$

When $t = 0,\ \dot{y} = V\sin 0° = 0$

$\dot{y} = -10t$

$y = -5t^2 + f$

When $t = 0,\ y = 90$

$90 = 0 + f$

$f = 90$

$y = -5t^2 + 90 \quad [2]$

Landing at $x = 180$:

$180 = Vt \qquad [\text{from } 1]$

$$t = \frac{180}{V}$$

Landing at $y = 0$:

$0 = -5t^2 + 90 \qquad [\text{from } 2]$

$$0 = -5\left(\frac{180}{V}\right)^2 + 90$$

$$5\left(\frac{180^2}{V^2}\right) = 90$$

$$\frac{180^2}{V^2} = 18$$

$$V^2 = \frac{180^2}{18} = 1800$$

$$V = \sqrt{1800}$$

$$= 30\sqrt{2}\ \text{m s}^{-1}$$

Question 10

a

$\ddot{y} = -g$

$\dot{y} = -gt + c$

When $t = 0, \dot{y} = V\sin 0° = 0$

$0 = 0 + c$

$c = 0$

$\dot{y} = -gt$

$y = -\frac{gt^2}{2} + d$

When $t = 0, y = h$

$h = 0 + d$

$d = h$

$y = -\frac{gt^2}{2} + h$

At the ground $y = 0$

$0 = -\frac{gt^2}{2} + h$

$h = \frac{gt^2}{2}$

$t^2 = \frac{2h}{g}$

$t = \sqrt{2hg^{-1}}$

As required.

b

$\dot{y} = -gt = -g\sqrt{2hg^{-1}}$ [substituting in using answer to part **a**]

$\ddot{x} = 0$

$\dot{x} = k$

When $x = 0, \dot{x} = V\cos 0° = V$

$\dot{x} = V$

$\tan 60° = \frac{|\dot{y}|}{\dot{x}} = \frac{g\sqrt{2hg^{-1}}}{V}$

$V\sqrt{3} = g\sqrt{2hg^{-1}}$

$3V^2 = g^2 2hg^{-1}$

$3V^2 = 2gh$

Question 11

$\ddot{y} = -g = -9.81$

$\dot{y} = -981\,t + c$

When $t = 0, \dot{y} = 16\sin 22°$

$16\sin 22° = 0 + c$

$\dot{y} = -9.81t + 16\sin 22°$

$y = -4.905gt^2 + 16t\sin 22° + d$

When $t = 0$, let $y = h$ (height of building)

$h = 0 + 0 + d$

$y = -4.905t^2 + 16t\sin 22° + h$

When $t = 3$, $y = 0$ (ground)

$0 = -4.905(3^2) + 16(3)\sin 22° + h$

$h = 26.1639...$

≈ 26.16 m

Question 12

$$x = \frac{v^2 \sin 2\theta}{g}$$

$$10 = \frac{v^2 \sin 24}{9.81}$$

$$v^2 = \frac{98.1}{\sin 24}$$

$v^2 = 241.188$

$v = 15.5302 \approx 15.53 \text{ m s}^{-1}$

Question 13

$$y = \frac{-gx^2}{2V^2}\left(1+\tan^2\theta\right) + x\tan\theta$$

$$\theta = 45°, \tan\theta = 1, y = 1\,\text{m}, x = 10\,\text{m}$$

$$1 = \frac{-10\times10^2}{2V^2}(1+1)+10(1)$$

$$1 = \frac{-500}{V^2}(2)+10$$

$$\frac{1000}{V^2} = 9$$

$$\frac{1000}{9} = V^2$$

$$V = \sqrt{\frac{1000}{9}}$$

$$= \frac{10\sqrt{10}}{3}\,\text{m}\,\text{s}^{-1}$$

Question 14

$$y = \frac{-gx^2}{2V^2}\left(1+\tan^2\alpha\right) + x\tan\alpha$$

$$y = 4\text{ m}, x = 15\,\text{m}, v = 15\text{ m}\,\text{s}^{-1}$$

$$4 = \frac{-10\times15^2}{2\times15^2}\left(1+\tan^2\alpha\right)+15\tan\alpha$$

$$4 = -5\left(1+\tan^2\alpha\right)+15\tan\alpha$$

$$4 = -5-5\tan^2\alpha+15\tan\alpha$$

$$5\tan^2\alpha-15\tan\alpha+9 = 0$$

As required.

Question 15

a

$$y=\frac{-gx^2}{2V^2}\left(1+\tan^2\alpha\right)+x\tan\alpha$$

For wall 1

$y=6, x=6$

$$6=\frac{-10\times6^2}{2V^2}\left(1+\tan^2\alpha\right)+6\tan\alpha$$

$$6=\frac{-180}{V^2}\left(1+\tan^2\alpha\right)+6\tan\alpha$$

For wall 2

$y=6, x=12$

$$6=\frac{-10\times12^2}{2V^2}\left(1+\tan^2\alpha\right)+12\tan\alpha$$

$$6=\frac{-720}{V^2}\left(1+\tan^2\alpha\right)+12\tan\alpha$$

Range

$$x=\frac{V^2\sin2\alpha}{g}$$

$$18=\frac{V^2\sin2\alpha}{10}$$

$$V^2=\frac{180}{\sin2\alpha}$$

Using wall 1

$$6=\frac{-180}{V^2}\left(1+\tan^2\alpha\right)+6\tan\alpha$$

$$6=\frac{-180\sin2\alpha}{180}\left(1+\tan^2\alpha\right)+6\tan\alpha$$

$$6=-\sin2\alpha\left(1+\tan^2\alpha\right)+6\tan\alpha$$

$$6-6\tan\alpha=-\sin2\alpha\left(1+\tan^2\alpha\right)$$

$$6-6\tan\alpha=\frac{-2\tan\alpha}{1+\tan^2\alpha}\left(1+\tan^2\alpha\right)$$

$$6-6\tan\alpha=-2\tan\alpha$$

$$6=4\tan\alpha$$

$$\tan\alpha=\frac{3}{2}$$

b

For walls x_1 and x_2 distance

Range = $x_1 + x_2$

For height h

$$y = \frac{-gx^2}{2V^2}\left(1+\tan^2\alpha\right)+x\tan\alpha$$

$$h = \frac{-10x^2}{2V^2}\left(1+\tan^2\alpha\right)+x\tan\alpha$$

$$h = \frac{-5x^2}{V^2}\left(1+\tan^2\alpha\right)+x\tan\alpha$$

Using range

$$x = \frac{V^2\sin 2\alpha}{g}$$

$$x_1 + x_2 = \frac{V^2\sin 2\alpha}{10}$$

$$V^2 = \frac{10\left(x_1+x_2\right)}{\sin 2\alpha}$$

sub V^2 into wall 1

$$h = \frac{-5x_1^2}{\frac{10\left(x_1+x_2\right)}{\sin 2\alpha}}\left(1+\tan^2\alpha\right)+x\ \tan\alpha$$

$$h = \frac{-5x^2\sin 2\alpha}{10\left(x_1+x_2\right)}\left(1+\tan^2\alpha\right)+x\ \tan\alpha$$

$$h = \frac{-x^2}{2\left(x_1+x_2\right)}\frac{2\tan\alpha}{1+\tan^2\alpha}\left(1+\tan^2\alpha\right)+x\ \tan\alpha$$

$$h = \frac{-x^2\tan\alpha}{\left(x_1+x_2\right)}+x\ \tan\alpha$$

$$h\left(x_1+x_2\right) = -x_1^2\tan\alpha + x_1\tan\alpha\left(x_1+x_2\right)$$

$$h\left(x_1+x_2\right) = \tan\alpha\left(-x_1^2+x_1^2+x_1x_2\right)$$

$$h\left(x_1+x_2\right) = \left(x_1x_2\right)\tan\alpha$$

$$\tan\alpha = \frac{h\left(x_1+x_2\right)}{x_1x_2}$$

Question 16

a

$$\ddot{y} = -g$$
$$\dot{y} = -gt + c$$
$$t = 0, y = V\sin\alpha$$
$$\dot{y} = -gt + V\sin\alpha$$
$$y = -\frac{gt^2}{2} + Vt\sin\alpha + c$$
$$t = 0, y = 0, c = 0$$
$$y = -\frac{gt^2}{2} + Vt\sin\alpha$$
$$\dot{y}^2 = \left(-gt + V\sin\alpha\right)^2$$
$$= g^2t^2 - 2gtV\sin\alpha + V^2\sin^2\alpha$$
$$= V^2\sin^2\alpha + 2g\left(\frac{gt^2}{2} - tV\sin\alpha\right)$$
$$= V^2\sin^2\alpha - 2gy$$

b

$$\dot{x} = V\cos\alpha$$
$$S^2 = V^2\cos^2\alpha + V^2\sin^2\alpha - 2gy$$
$$= V^2\left(\cos^2\alpha + \sin^2\alpha\right) - 2gy$$
$$= V^2 - 2gy$$

Question 1

a

$$F_H = 16\cos 60° = 8 \text{ N}$$
$$F_V = 16\sin 60° = 8\sqrt{3} \text{ N}$$

b

$$F_H = 20\cos 30° = 10\sqrt{3} \text{ N}$$
$$F_V = 20\sin 30° = 10 \text{ N}$$

c

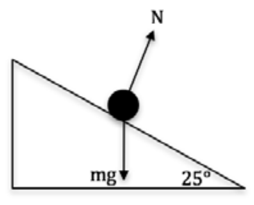
$$F_H = 24\sin 20° = 8.2 \text{ N}$$
$$F_V = 24\cos 20° = 22.6 \text{ N}$$

d

$$F_H = 18\sin 40° = 11.6 \text{ N}$$
$$F_V = 18\cos 40° = 13.8 \text{ N}$$

Question 2

a



N

$N_{||} = 0 \text{ N along plane}$

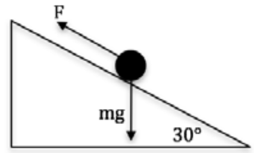
$N_{\perp} = N \text{ N perpendicular to plane}$

mg

$mg_{||} = mg\sin 25° \text{ N along plane}$

$mg_{\perp} = mg\cos 25° \text{ N perpendicular to plane}$

b



F

$F_{\parallel} = F$ N along plane

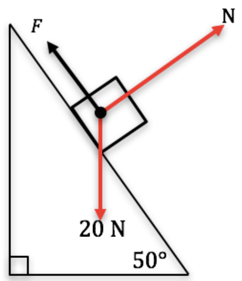
$F_{\perp} = 0$ N perpendicular to plane

mg

$mg_{\parallel} = mg \sin 30°$ N along plane

$mg_{\perp} = mg \cos 30°$ N perpendicular to plane

c



F

$F_{\parallel} = F$ N along plane

$F_{\perp} = 0$ N perpendicular to plane

N

$N_{\parallel} = 0$ N along plane

$N_{\perp} = N$ N perpendicular to plane

20 N force

$20\ \text{N}_{\parallel} = 20 \sin 50°$ N along plane

$20\ \text{N}_{\perp} = 20 \cos 50°$ N perpendicular to plane

Question 3

a

$$mg = 5\,\text{N}$$
$$N + F\sin\theta = mg\cos 40° \qquad [1]$$
$$\mu N + mg\sin 40° = F\cos 40° \qquad [2]$$
$$\frac{1}{\sqrt{3}}N + 5\sin 40° = F\cos 40° \qquad [3:\text{from } 2]$$
$$N + F\sin\theta = 5\cos 40° \qquad [4:\text{from } 1]$$
$$N = 5\cos 40° - F\sin\theta$$
$$(\text{substitute into } 3)$$
$$\frac{1}{\sqrt{3}}\left(5\cos 40° - F\sin\theta\right) + 5\sin 40° = F\cos 40°$$
$$5\cos 40° - F\sin\theta + 5\sqrt{3}\sin 40° = F\sqrt{3}\cos\theta$$
$$F\left(\sin\theta + \sqrt{3}\cos\theta\right) = 5\cos 40° + 5\sqrt{3}\sin 40°$$
$$F = \frac{5\left(\cos 40° + \sqrt{3}\sin 40°\right)}{\sin\theta + \sqrt{3}\cos\theta}$$

b

$$\theta = 45°$$
$$F = \frac{5\left(\cos 40° + \sqrt{3}\sin 40°\right)}{\sin 45° + \sqrt{3}\cos 45°}$$
$$= 4.86421...$$
$$\approx 4.9\ \text{N}$$

Question 4

$N = mg = 0.5 \times 9.8 = 4.9$

$Fr = \mu N = 49\ \mu$

$|Fr| = |F|$

$4.9\mu = 2$

$\mu = \frac{2}{49} = 0408163 \approx 4$

Question 5

$N = mg\cos 20°$

$N = 45\ g\cos 20\ °$

$T = \mu N + mg\sin 20$

$8g = \mu \times 4.5g\cos 20 + 4.5g\sin 20$

$\mu \times 4.5g\cos 20 = 8g - 4.5g\sin 20$

$\mu = \frac{8g - 4.5g\sin 20}{45\ g\cos 20} = 1.53$

Question 6

a

$N = 170g$

For no acceleration

$P = \mu N = 0.85 \times 170 \times g$

$P = 14.59 \times 8$

$P = 1416.1 \approx 1420\ \text{N}$

b

$F = ma = 0.5 \times 170 = 85$

$\therefore P - \mu N = 85$

$P = 85 + 1416.1 = 1501.1 \approx 1500\ \text{N}$

Question 7

a

$$(20-x)^2 = 10^2 + x^2$$
$$400 - 40x + x^2 = 100 + x^2$$
$$40x = 300$$
$$x = 7.5$$
$$\tan\theta = \frac{7.5}{10}$$
$$\theta = 36°52'$$

Resolving forces vertically

$$mg = T + T\sin\theta$$
$$T\left(1+\sin 36°52'\right) = 0.005g$$
$$T = \frac{0005 \times 8}{1+\sin 36°52'}$$
$$T = 0030626 \approx .031 \text{ N}$$

b Resolving forces horizontally

$$T\cos\theta = F$$
$$F = 0031\cos 36°52'$$
$$F = 0.025 \text{ N}$$

Question 8

$\cos\theta = \frac{6}{10} = \frac{3}{5}$

$\sin\theta = \frac{4}{5}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$= 6^2 + 5^2 - 2\times5\times6\cos\frac{3}{5}$

$= 25$

$c = 5$

Resolving vertical forces

$15g = T_1\cos\theta + T_2\sin\theta$

$15g = \frac{3}{5}T_1 + \frac{4}{5}T_2$

$75g = 3T_1 + 4T_2$

Resolving horizontal forces

$0 = T_1\sin\theta + T_2\sin(90-\theta)$

$T_1\sin\theta = T_2\cos\theta$

$\frac{4}{5}T_1 = \frac{3}{5}T_2$

$4T_1 = 3T_2$

$T = \frac{3T_2}{4}$

$75g = 3T_1 + 4T_2$

$75g = 3\frac{3T_2}{4} + 4T_2$

$75g = \frac{9T_2}{4} + 4T_2$

$75g = \frac{25T_2}{4}$

$T_2 = 12g$ N

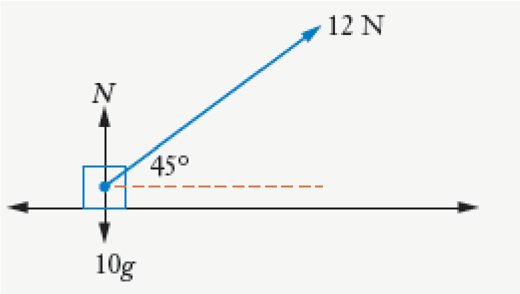
$T = \frac{3T_2}{4}$

$T = \frac{3\times12g}{4}$

$T = 9g$ N

Question 9

a



b Unresolved force

$$T\cos 45° = 12 \times \frac{\sqrt{2}}{2} = 6\sqrt{2}\ \text{N}$$

$$F = ma$$

$$6\sqrt{2} = 10a$$

$$a = \frac{3\sqrt{2}}{5}\ \text{m s}^{-2}$$

c

$$N + T\sin 45° = mg$$

$$N + 6\sqrt{2} = 10g$$

$$N = 10g - 6\sqrt{2}\ \ \text{N}$$

Question 10

a For body *M*: $N = Mg$

Resolving horizontal forces:

$$F = T - \mu N = mg - \mu Mg$$
$$F = ma$$
$$(M + m)a = mg - \mu Mg$$
$$a = \frac{mg - \mu Mg}{M + m}$$
$$= \frac{(m - \mu M)g}{M + m}$$

b For body *m*:

$$a = \frac{(m - \mu M)g}{M + m}$$
$$F = ma$$
$$= \frac{(m - \mu M)mg}{M + m}$$
$$F = mg - T$$
$$T = mg - F$$
$$= mg - \frac{(m - \mu M)mg}{M + m}$$
$$= mg\left[1 - \frac{m - \mu M}{M + m}\right]$$
$$= mg\left[\frac{M + m - m + \mu M}{M + m}\right]$$
$$= \frac{mgM(1 + \mu)}{M + m}$$

c

System won't move if the resultant force = 0

$0 = mg - \mu Mg$

$\mu Mg = mg$

$\mu = \dfrac{m}{M}$

This is the minimum possible value of the friction coefficient μ for the system to remain stationary.

If $\mu \geq \dfrac{m}{M}$, the system will not move.

Question 11

Horizontal forces

$\mu N + Mg\sin\alpha = F\cos\theta$ [1]

Vertical forces

$N + F\sin\theta = Mg\cos\alpha$

$N = Mg\cos\alpha - F\sin\theta$ [2]

Sub [2] into [1]:

$\mu\left(Mg\cos\alpha - F\sin\theta\right) + Mg\sin\alpha = F\cos\theta$

$\mu Mg\cos\alpha - \mu F\sin\theta + Mg\sin\alpha = F\cos\theta$

$\mu Mg\cos\alpha + Mg\sin\alpha = F\cos\theta + \mu F\sin\theta$

$Mg\left(\mu\cos\alpha + \sin\alpha\right) = F\left(\cos\theta + \mu\sin\theta\right)$

$F = \dfrac{Mg\left(\mu\cos\alpha + \sin\alpha\right)}{\cos\theta + \mu\sin\theta}$

$F = \dfrac{\left(\sin\alpha + \mu\cos\alpha\right)}{\cos\theta + \mu\sin\theta}Mg$

Exercise 7.05 Resisted horizontal motion

Question 1

$$F \alpha -v$$
$$F = -kv$$
$$ma = -kv$$
$$a = -\frac{kv}{m}$$
$$v\frac{dv}{dx} = -\frac{kv}{m}$$
$$\frac{dv}{dx} = -\frac{k}{m}$$
$$v = -\frac{k}{m}x + c$$
$$x = 0 \quad v = u$$
$$u = c$$
$$v = -\frac{k}{m}x + u$$
$$\frac{k}{m}x = u - v$$
$$x = \frac{m(u-v)}{k}$$

Question 2

$F = -kv^2$

$\ddot{x} = -\frac{k}{m}v^2$

$v\frac{dv}{dx} = -\frac{k}{m}v^2$

$\frac{dv}{dx} = -\frac{kv}{m}$

$\frac{dx}{dc} = -\frac{m}{k}\frac{1}{v}$

$x = -\frac{m}{k}\ln v + c$

$x = 0 \; v = u$

$0 = -\frac{m}{k}\ln u + c$

$c = \frac{m}{k}\ln u$

$x = \frac{m}{k}\ln\frac{u}{v}$

$\ln\frac{u}{v} = \frac{xk}{m}$

$\frac{u}{v} = e^{\frac{xk}{m}}$

$\frac{v}{u} = e^{-\frac{xk}{m}}$

$v = ue^{-\frac{xk}{m}}$

Question 3

$\ddot{x} = -\left(v + v^3\right)$

$v\dfrac{dv}{dx} = -v\left(1 + v^2\right)$

$\dfrac{dv}{dx} = -\left(1 + v^2\right)$

$\dfrac{dx}{dc} = -\dfrac{1}{\left(1 + v^2\right)}$

$x = -\tan^{-}\ \left(v\right) + c$

$x = 0\ v = V$

$0 = -\tan^{-}\ \left(V\right) + c$

$c = \tan^{-}\ \left(V\right)$

$x = -\tan^{-1}\left(v\right) + \tan^{-1}\left(V\right)$

$\tan x = \tan\left(-\tan^{-1}\left(v\right) + \tan^{-1}\left(V\right)\right)$

$\tan x = \dfrac{V - v}{1 + Vv}$

$x = \tan^{-1}\left(\dfrac{V - v}{1 + Vv}\right)$

Question 4

$M\ \ddot{x} = F - kv^2$

a

$M\ddot{x} = 0, v = 430$

$0 = F - k\left(430^2\right)$

$F = k\left(430^2\right)$

$k = \dfrac{F}{430^2}$

$M\ \ddot{x} = F - kv^2$

$M\ \ddot{x} = F - \dfrac{F}{430^2}v^2$

$M\ \ddot{x} = F\left[1 - \left(\dfrac{v}{430}\right)^2\right]$

b

$$M\ddot{x} = F\left[1-\left(\frac{v}{430}\right)^2\right]$$

$$\ddot{x} = \frac{dv}{dt} = \frac{F}{M}\left[1-\left(\frac{v}{430}\right)^2\right]$$

$$\frac{dt}{dv} = \frac{M}{F}\left[\frac{1}{1-\left(\frac{v}{430}\right)^2}\right]$$

$$= \frac{M}{F}\frac{1}{2}\left(\frac{1}{1-\frac{v}{430}} + \frac{1}{1+\frac{v}{430}}\right)$$

$$t = \frac{M}{2F}\left[-430\ln\left(1-\frac{v}{430}\right) + 430\left(1+\frac{v}{430}\right)\right] + c$$

$$t = \frac{430M}{2F}\ln\left(\frac{1+\frac{v}{430}}{1-\frac{v}{430}}\right) + c$$

$$t = \frac{215M}{F}\ln\left(\frac{1+\frac{v}{430}}{1-\frac{v}{430}}\right) + c$$

When $t = 0, v = 0$

$$0 = \frac{215M}{F}\ln\left(\frac{1}{1}\right) + c; \quad c = 0$$

$$t = \frac{215M}{F}\ln\left(\frac{1+\frac{v}{430}}{1-\frac{v}{430}}\right)$$

When $v = 400$

$$t = \frac{215M}{F}\ln\left(\frac{1+\frac{400}{430}}{1-\frac{400}{430}}\right)$$

$$= \frac{215M}{F}\ln\left(\frac{830}{30}\right)$$

$$= \frac{215M}{F}\ln\left(\frac{83}{3}\right)$$

Question 5

a

$$F = m\ddot{x} = -kv^3$$

$$\ddot{x} = -kv^3 \qquad m = 1$$

$$v\frac{dv}{dx} = -kv^3$$

$$\frac{dv}{dx} = -kv^2$$

$$\frac{dx}{dv} = -\frac{1}{kv^2}$$

$$x = \frac{1}{kv} + c$$

When $x = 0,\ v = v_0$

$$0 = \frac{1}{kv_0} + c$$

$$c = -\frac{1}{kv_0}$$

$$x = \frac{1}{kv} - \frac{1}{kv_0}$$

$$\frac{1}{v} - \frac{1}{v_0} = kx$$

$$\frac{1}{v} = kx + \frac{1}{v_0}$$

$$\frac{1}{v} = \frac{kxv_0 + 1}{v_0}$$

$$v = \frac{v_0}{kv_0x + 1}$$

b

$$v = \frac{dx}{dt} = \frac{v_0}{kv_0x + 1}$$

$$\frac{dt}{dx} = \frac{kv_0x + 1}{v_0}$$

$$t = \frac{1}{v_0}\left(\frac{kv_0x^2}{2} + x\right) + c$$

When $t = 0, x = 0$

$$0 = \frac{1}{v_0}(0 + 0) + c$$

$$c = 0$$

$$t = \frac{1}{v_0}\left(\frac{kv_0x^2}{2} + x\right)$$

$$= \frac{1}{2}kx^2 + \frac{x}{v_0}$$

c

$$t=\frac{1}{2}kx^2+\frac{x}{v_0}$$

When $t=0.8, x=800$

$$0.8=\frac{k\times 800^2}{2}+\frac{800}{v_0}$$

$$0.8=320\,000k+\frac{800}{v_0} \quad [1]$$

When $t=1.8, x=1600$

$$1.8=\frac{k\times 1600^2}{2}+\frac{1600}{v_0}$$

$$1.8=1280\,000k+\frac{1600}{v_0}$$

$$0.9=640\,000k+\frac{800}{v_0} \quad [2]$$

$[2]-[1]$:

$$0.1=320\,000k$$

$$k=\frac{1}{3200\,000}$$

Sub into [1]:

$$0.8=\frac{320\,000}{3200\,000}+\frac{800}{v_0}$$

$$0.8=0.1+\frac{800}{v_0}$$

$$0.7=\frac{800}{v_0}$$

$$0.7=\frac{800}{v_0}$$

$$v_0=\frac{800}{0.7}=1142.8571...\text{ m s}^{-1}$$

Time taken to travel 2400 m

$$t=\frac{1}{2}kx^2+\frac{x}{v_0}$$

$$=\frac{1}{2}\frac{1}{3200\,000}\left(2400^2\right)+\frac{2400}{1142.8571...}$$

$$=3\text{ s}$$

Time taken to travel ast 800 m = $3-1.8$

$$=1.2\text{ s}$$

Question 6

a

$$a = k\left(1-v^2\right)$$

$$\frac{dv}{dt} = k\left(1-v^2\right)$$

$$\frac{dt}{dv} = \frac{1}{k\left(1-v^2\right)}$$

$$\frac{dt}{dv} = \frac{1}{k(1+v)(1-v)}$$

$$\frac{dt}{dv} = \frac{1}{2k}\left(\frac{1}{1+v}+\frac{1}{1-v}\right)$$

$$2kt = \ln(1+v)-\ln(1-v)+c$$

$$2kt = \ln\left(\frac{1+v}{1-v}\right)+c$$

$$t = 0, v = 0, c = 0$$

$$2kt = \ln\left(\frac{1+v}{1-v}\right)$$

$$\ln\left(\frac{1+v}{1-v}\right) = 2kt$$

$$\frac{1+v}{1-v} = e^{2kt}$$

$$1+v = (1-v)e^{2kt}$$

$$v+ve^{2kt} = e^{2kt}-1$$

$$v\left(1+e^{2kt}\right) = e^{2kt}-1$$

$$v = \frac{e^{2kt}-1}{1+e^{2kt}}$$

$$v = \frac{1-e^{-2kt}}{1+e^{-2kt}}$$

Terminal velocity occurs when $a = 0$

$$0 = k\left(1-v^2\right)$$

$$1-v^2 = 0$$

$$v^2 = 1$$

$$v = 1 \text{ ms}^{-1}$$

b

$$a = k\left(1-v^2\right)$$

$$v\frac{dv}{dx} = k\left(1-v^2\right)$$

$$\frac{dv}{dx} = \frac{k\left(1-v^2\right)}{v}$$

$$\frac{dx}{dv} = \frac{v}{k\left(1-v^2\right)}$$

$$-2k\frac{dx}{dv} = \frac{-2v}{1-v^2}$$

$$-2kx = \ln\left(1-v^2\right) + c$$

$$x = 0, v = 0, c = 0$$

$$x = -\frac{\ln\left(1-v^2\right)}{2k}$$

Question 7

a

$$F = ma = -\left(mk + mv^2\right)$$

$$a = -k - v^2$$

$$v\frac{dv}{dx} = -k - v^2$$

$$\frac{dv}{dx} = -\frac{k}{v} - v$$

$$\frac{dv}{dx} = -\frac{k+v^2}{v}$$

$$\frac{dx}{dv} = -\frac{v}{k+v^2}$$

$$\frac{dx}{dv} = -\frac{1}{2}\frac{2v}{k+v^2}$$

$$x = -\frac{1}{2}\ln\left(k+v^2\right) + c$$

$$x = 0 \quad v = u$$

$$0 = -\frac{1}{2}\ln\left(k+u^2\right) + c$$

$$c = \frac{1}{2}\ln\left(k+u^2\right)$$

$$x = -\frac{1}{2}\ln\left(k+v^2\right) + \frac{1}{2}\ln\left(k+u^2\right)$$

$$x = \frac{1}{2}\ln\left(\frac{k+u^2}{k+v^2}\right)$$

b

$$a = -k - v^2$$

$$\frac{dv}{dt} = -k - v^2$$

$$\frac{dt}{dv} = \frac{-1}{k + v^2}$$

$$t = -\frac{1}{\sqrt{k}}\tan^{-}\left(\frac{v}{\sqrt{k}}\right) + c$$

$$t = 0 \;\; v = u$$

$$0 = -\frac{1}{\sqrt{k}}\tan^{-}\left(\frac{u}{\sqrt{k}}\right) + c$$

$$c = \frac{1}{\sqrt{k}}\tan^{-1}\left(\frac{u}{\sqrt{k}}\right)$$

$$t = -\frac{1}{\sqrt{k}}\tan^{-1}\left(\frac{v}{\sqrt{k}}\right) + \frac{1}{\sqrt{k}}\tan^{-1}\left(\frac{u}{\sqrt{k}}\right)$$

$$t = \frac{1}{\sqrt{k}}\left(\tan^{-1}\left(\frac{u}{\sqrt{k}}\right) - \tan^{-1}\left(\frac{v}{\sqrt{k}}\right)\right)$$

$$v = 0$$

$$t = \frac{1}{\sqrt{k}}\tan^{-1}\left(\frac{u}{\sqrt{k}}\right)$$

Question 8

a

$$F=-mk\left(c+v\right)$$
$$ma=-mk\left(c+v\right)$$
$$a=-k\left(c+v\right)$$

b

$$a=-k\left(c+v\right)$$
$$\frac{dv}{dt}=-k\left(c+v\right)$$
$$\frac{dt}{dv}=\frac{1}{-k\left(c+v\right)}$$
$$-kt=\ln\left(c+v\right)+b$$
$$t=0\ v=U$$
$$0=\ln\left(c+U\right)+b$$
$$b=-\ln\left(c+U\right)$$
$$-kt=\ln\left(c+v\right)-\ln\left(c+U\right)$$
$$-kt=\ln\left(\frac{c+v}{c+U}\right)$$
$$t=T,v=0$$
$$-kT=\ln\left(\frac{c}{c+U}\right)\qquad[1]$$
$$t=\frac{T}{2}\ \ v=\frac{1}{8}U$$
$$\frac{-kT}{2}=\ln\left(\frac{c+\frac{U}{8}}{c+U}\right)$$
$$-kT=2\ln\left(\frac{c+\frac{U}{8}}{c+U}\right)$$
$$-kT=\ln\left(\frac{c+\frac{U}{8}}{c+U}\right)^2\qquad[2]$$

equating[1] and [2]

$$\ln\left(\frac{c}{c+U}\right)=\ln\left(\frac{c+\frac{U}{8}}{c+U}\right)^2$$

$$\frac{c}{c+U}=\left(\frac{c+\frac{U}{8}}{c+U}\right)^2$$

$$c(c+U)^2=(c+U)\left(c+\frac{U}{8}\right)^2$$

$$c(c+U)=\left(c+\frac{U}{8}\right)^2$$

$$c^2+cU=c^2+\frac{cU}{4}+\frac{U^2}{64}$$

$$cU-\frac{cU}{4}=\frac{U^2}{64}$$

$$\frac{3cU}{4}=\frac{U^2}{64}$$

$$3c=\frac{U}{16}$$

$$c=\frac{U}{48}$$

c

$$-kt = \ln\left(\frac{c+v}{c+U}\right)$$

$$e^{-kt} = \frac{c+v}{c+U}$$

$$e^{-kt} = \frac{\frac{U}{48}+v}{\frac{U}{48}+U}$$

$$e^{-kt} = \frac{\frac{U}{48}+\frac{48v}{48}}{\frac{U}{48}+\frac{48U}{48}}$$

$$e^{-kt} = \frac{U+48v}{49U}$$

$$49e^{-kt} = \frac{U+48v}{U}$$

$$49e^{-kt} = 1+\frac{48v}{U}$$

$$\frac{48v}{U} = 49e^{-kt} - 1$$

Exercise 7.06 Resisted vertical motion

Question 1

a

$$F = mg - k\,v^2$$

$$ma = mg - k\,v^2$$

$$a = g - kv^2$$

$$k = \frac{k}{m}$$

b

$$a = v\frac{dv}{dx} = g - kv^2$$

$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$x = -\frac{1}{2k}\ln\left|g - kv^2\right| + c$$

When $x = 0, v = 0$:

$$0 = -\frac{1}{2k}\ln g + c$$

$$c = \frac{1}{2k}\ln g$$

$$x = -\frac{1}{2k}\ln\left|g - kv^2\right| + \frac{1}{2k}\ln g$$

$$x = \frac{1}{2k}\left(\ln g - \ln\left|g - kv^2\right|\right)$$

$$x = \frac{1}{2k}\ln\left(\frac{g}{\left|g - kv^2\right|}\right)$$

$$2kx = \ln\left(\frac{g}{\left|g - kv^2\right|}\right)$$

$$e^{2kx} = \frac{g}{g - kv^2}$$

$$e^{-2kx} = \frac{g - kv^2}{g}$$

$$ge^{-2kx} = g - kv^2$$

$$kv^2 = g - ge^{-2kx}$$

$$v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$$

c When it hits the ground, $x = h$.

$$v^2 = \frac{g}{k}\left(1 - e^{-2kh}\right)$$

$$v = \sqrt{\frac{g}{k}\left(1 - e^{-2kh}\right)}$$

Question 2

a

$$a = \frac{dv}{dt} = -g - kv^2$$

$$\frac{dt}{dv} = \frac{-1}{g + kv^2}$$

$$= -\frac{1}{k}\frac{1}{\frac{g}{k} + v^2}$$

$$t = -\frac{1}{k}\frac{1}{\sqrt{\frac{g}{k}}}\tan^{-}\left(\frac{v}{\sqrt{\frac{g}{k}}}\right) + c$$

$$t = -\frac{1}{\sqrt{kg}}\tan^{-}\left(v\sqrt{\frac{k}{g}}\right) + c$$

When $t = 0, v = U$:

$$0 = -\frac{1}{\sqrt{kg}}\tan^{-}\left(U\sqrt{\frac{k}{g}}\right) + c$$

$$c = \frac{1}{\sqrt{kg}}\tan^{-}\left(U\sqrt{\frac{k}{g}}\right)$$

$$t = -\frac{1}{\sqrt{kg}}\tan^{-1}\left(v\sqrt{\frac{k}{g}}\right) + \frac{1}{\sqrt{kg}}\tan^{-1}\left(U\sqrt{\frac{k}{g}}\right)$$

$$t = \frac{1}{\sqrt{kg}}\left[\tan^{-}\left(U\sqrt{\frac{k}{g}}\right) - \tan^{-}\left(v\sqrt{\frac{k}{g}}\right)\right]$$

$v = 0$ for maximum height:

$$t = \frac{1}{\sqrt{kg}}\left[\tan^{-}\left(U\sqrt{\frac{k}{g}}\right) - 0\right]$$

$$t = \frac{1}{\sqrt{kg}}\tan^{-}\left(U\sqrt{\frac{k}{g}}\right)$$

b

$$a = v\frac{dv}{dx} = -g - kv^2$$

$$\frac{dv}{dx} = \frac{-g - kv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{-g - kv^2} = -\frac{v}{g + kv^2}$$

$$x = -\frac{1}{2k}\ln\left(g + kv^2\right) + c$$

When $x = 0, v = U$

$$0 = -\frac{1}{2k}\ln\left(g + kU^2\right) + c$$

$$c = \frac{1}{2k}\ln\left(g + kU^2\right)$$

$$x = -\frac{1}{2k}\ln\left(g + kv^2\right) + \frac{1}{2k}\ln\left(g + kU^2\right)$$

$$x = \frac{1}{2k}\ln\left(\frac{g + kU^2}{g + kv^2}\right)$$

Maximum height occurs when $v = 0$

$$x = \frac{1}{2k}\ln\left(\frac{g + kU^2}{g}\right)$$

$$x = \frac{1}{2k}\ln\left(1 + \frac{kU^2}{g}\right)$$

Question 3

a

$$F = -g - \frac{v}{4}$$

as $m = 1$

$$a = -g - \frac{v}{4}$$

$$\ddot{x} = -10 - \frac{v}{4}$$

$$\ddot{x} = \frac{-40 - v}{4}$$

b

$$v\frac{dv}{dx} = \frac{-40 - v}{4}$$

$$\frac{dv}{dx} = \frac{-40 - v}{4v}$$

$$\frac{dx}{dv} = -\frac{4v}{40 + v}$$

$$\frac{dx}{dv} = -\frac{4(v + 40 - 40)}{40 + v}$$

$$\frac{dx}{dv} = -\frac{4(v + 40) - 160}{40 + v}$$

$$\frac{dx}{dv} = -4 + \frac{160}{40 + v}$$

$$x = -4v + 160\ln(40 + v) + c$$

$$x = 0, v = 40$$

$$0 = -160 + 160\ln(40 + 40) + c$$

$$c = 160 - 160\ln(80)$$

$$x = -4v + 160\ln(40 + v) + 160 - 160\ln(80)$$

$$x = -4v + 160\ln\left(\frac{40 + v}{80}\right) + 160$$

Maximum height when $v = 0$

$$x = 160\ln\left(\frac{1}{2}\right) + 160$$

$$x = 160 - 160\ln 2$$

$$x = 160(1 - \ln 2) \text{ m}$$

c

$$\ddot{x} = \frac{-40 - v}{4}$$

$$\frac{dv}{dt} = \frac{-40 - v}{4}$$

$$\frac{dt}{dv} = \frac{-4}{40 + v}$$

$$t = -4\ln(40 + v) + c$$

$$t = 0, v = 40$$

$$0 = -4\ln(40 + 40) + c$$

$$c = 4\ln(80)$$

$$t = -4\ln(40 + v) + 4\ln(80)$$

$$t = 4\ln\left(\frac{80}{40 + v}\right)$$

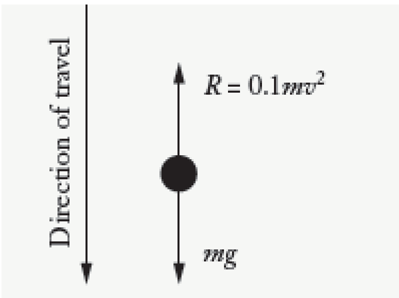
Maximum height occurs when v = 0

$$t = 4\ln\left(\frac{80}{40}\right)$$

$$t = 4\ln 2 \text{ s}$$

Question 4

a



b

$$F = mg - 01\,mv^2$$
$$ma = mg - 01\,mv^2$$
$$a = g - 01\,v^2$$
$$= 10 - 0.1v^2$$
$$= 01\,(100 - v^2)$$

c

$$\frac{dv}{dt} = 01\left(100 - v^2\right)$$

$$\frac{dt}{dv} = \frac{10}{100 - v^2}$$

$$\frac{dt}{dv} = \frac{10}{(10+v)(10-v)}$$

$$\frac{dt}{dv} = 10\left(\frac{\frac{1}{20}}{10+v} + \frac{\frac{1}{20}}{10-v}\right)$$

$$\frac{dt}{dv} = \frac{1}{2}\left(\frac{1}{10+v} + \frac{1}{10-v}\right)$$

$$t = \frac{1}{2}\ln|10+v| - \frac{1}{2}\ln|10-v| + c$$

$$t = \frac{1}{2}\ln\left|\frac{10+v}{10-v}\right| + c$$

When $t = 0, v = 0$

$$0 = \frac{1}{2}\ln\left(\frac{10}{10}\right) + c$$

$$c = 0$$

$$t = \frac{1}{2}\ln\left|\frac{10+v}{10-v}\right|$$

$$\ln\left|\frac{10+v}{10-v}\right| = 2t$$

$$\frac{10+v}{10-v} = e^{2t}$$

$$10 + v = (10 - v)e^{2t} = 10e^{2t} - ve^{2t}$$

$$v + ve^{2t} = 10e^{2t} - 10$$

$$v\left(1 + e^{2t}\right) = 10\left(e^{2t} - 1\right)$$

$$v = \frac{10\left(e^{2t} - 1\right)}{e^{2t} + 1}$$

d

$$v = \frac{10\left(e^{2t} - 1\right)}{e^{2t} + 1}$$

$t = \ln\left(1 + \sqrt{2}\right)$ when it hits the ground

$$v = \frac{10\left(e^{2\ln\left(1+\sqrt{2}\right)} - 1\right)}{1 + e^{2\ln\left(1+\sqrt{2}\right)}}$$

$$= \frac{10\left(e^{\ln\left(1+\sqrt{2}\right)^2} - 1\right)}{1 + e^{\ln\left(1+\sqrt{2}\right)^2}}$$

$$= \frac{10\left(\left[1+\sqrt{2}\right]^2 - 1\right)}{1 + \left[1+\sqrt{2}\right]^2}$$

$$= \frac{10\left(1 + 2\sqrt{2} + 2 - 1\right)}{1 + 1 + 2\sqrt{2} + 2}$$

$$= \frac{10\left(2 + 2\sqrt{2}\right)}{4 + 2\sqrt{2}}$$

$$= \frac{10\left(1 + \sqrt{2}\right)}{2 + \sqrt{2}}$$

$$= \frac{10 + 10\sqrt{2}}{2 + \sqrt{2}} \times \frac{2 - \sqrt{2}}{2 - \sqrt{2}}$$

$$= \frac{20 - 10\sqrt{2} + 20\sqrt{2} - 20}{4 - 2}$$

$$= \frac{10\sqrt{2}}{2}$$

$$= 5\sqrt{2}\ \text{m s}^{-1}$$

e

$$v\frac{dv}{dx} = 01\left(100 - v^2\right)$$

$$\frac{dv}{dx} = \frac{0.1\left(100-v^2\right)}{v}$$

$$\frac{dx}{dv} = \frac{v}{0.1\left(100-v^2\right)}$$

$$x = -\frac{1}{2}\frac{1}{0.1}\ln\left|100-v^2\right| + c$$

$$x = -5\ln\left|100-v^2\right| + c$$

When $t = 0, v = 0$

$$x = -5\ln 100 + c$$

$$c = 5\ln 100$$

$$x = -5\ln\left|100-v^2\right| + 5\ln 100$$

$$= 5\left(\ln 100 - \ln\left|100-v^2\right|\right)$$

$$= 5\ln\left|\frac{100}{100-v^2}\right|$$

When $v = 5\sqrt{2}$:

$$x = 5\ln\left(\frac{100}{100-50}\right)$$

$$= 5\ln\left(\frac{100}{50}\right)$$

$$= 5\ln 2 \text{ m}$$

Question 5

a When the stone is dropped, it falls under gravity with force *mg* and air resistance *mkv*.

So Total force = *ma* = *mg* – *mkv*

Dividing both sides by *m* gives: *a* = *g* – *kv*

b

$$a=\frac{dv}{dt}=g-kv$$

$$\frac{dt}{dv}=\frac{1}{g-kv}$$

$$t=-\frac{1}{k}\ln\left|g-kv\right|+c$$

When $t=0, v=0.$

$$0=-\frac{1}{k}\ln\left|g-0\right|+c$$

$$c=\frac{1}{k}\ln g$$

$$t=-\frac{1}{k}\ln\left|g-kv\right|+\frac{1}{k}\ln g$$

$$=\frac{1}{k}\left(\ln g-\ln\left|g-kv\right|\right)$$

$$kt=\ln\left|\frac{g}{g-kv}\right|$$

$$e^{kt}=\frac{g}{g-kv}$$

$$e^{-kt}=\frac{g-kv}{g}$$

$$ge^{-k}=g-kv$$

$$kv=g-ge^{-kt}$$

$$v=\frac{g}{k}\left(1-e^{-kt}\right)$$

c Terminal velocity occurs when $t \to \infty$ or $a = 0$.

As $t \to \infty$,

$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$
$$\to \frac{g}{k}(1-0)$$
$$= \frac{g}{k}$$

OR for $a = 0$

$$g - kv = 0$$
$$kv = g$$
$$v = \frac{g}{k}$$

d

$$a = v\frac{dv}{dx} = g - kv$$
$$\frac{dv}{dx} = \frac{g-kv}{v}$$
$$\frac{dx}{dv} = \frac{v}{g-kv}$$
$$= -\frac{1}{k}\left(\frac{g-kv-g}{g-kv}\right)$$
$$= -\frac{1}{k}\left(1 - \frac{g}{g-kv}\right)$$
$$x = -\frac{1}{k}\left(v + \frac{g}{k}\ln\left(g-kv\right)\right) + c$$

When $x = 0, v = 0$.

$$0 = -\frac{1}{k}\left(0 + \frac{g}{k}\ln g\right) + c$$
$$c = \frac{g}{k^2}\ln g$$
$$x = -\frac{1}{k}\left(v + \frac{g}{k}\ln\left(g-kv\right)\right) + \frac{g}{k^2}\ln g$$
$$= \frac{g}{k^2}\left(\ln g - \ln\left(g-kv\right)\right) - \frac{v}{k}$$
$$= \frac{g}{k^2}\ln\left(\frac{g}{g-kv}\right) - \frac{v}{k}$$

e

$$x = \frac{g}{k^2}\ln\left(\frac{g}{g-kv}\right) - \frac{v}{k}$$

$$x = \frac{10}{0.2^2}\ln\left(\frac{10}{10-0.2v}\right) - \frac{v}{0.2}$$

$$x = 250\ln\left(\frac{10}{10-0.2v}\right) - \frac{v}{0.2}$$

Sub $t = 3$ into (b):

$$v = \frac{g}{k}\left(1-e^{-kt}\right)$$

$$v = \frac{10}{0.2}\left(1-e^{-0.2\times 3}\right)$$

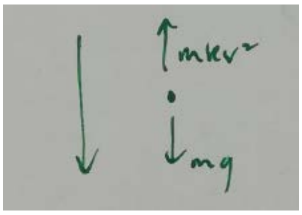
$$= 22.5594...\text{ms}^{-1}$$

$$x = 250\ln\left(\frac{10}{10-0.2\times 22.5594...}\right) - \frac{22.5594...}{0.2}$$

$$= 37.2029...$$

$$\approx 37.2 \text{ m}$$

Question 6



For downward motion:

$$m\ddot{x} = mg - mkv^2$$
$$\ddot{x} = g - kv^2$$

For terminal velocity, $\ddot{x} = 0$

$$\ddot{x} = g - kV^2 = 0$$
$$kV^2 = g$$
$$V^2 = \frac{g}{k}$$
$$V = \sqrt{\frac{g}{k}}$$

$$\ddot{x} = v\frac{dv}{dx} = g - kv^2$$

$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$x = -\frac{1}{2k}\ln\left(g - kv^2\right) + c$$

When $x = 0, v = 0$:

$$0 = -\frac{1}{2k}\ln\left(g - 0\right) + c$$

$$c = \frac{1}{2k}\ln g$$

$$x = -\frac{1}{2k}\ln\left(g - kv^2\right) + \frac{1}{2k}\ln g$$

$$= \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$$

When $v = \frac{1}{2}V = \frac{1}{2}\sqrt{\frac{g}{k}}$:

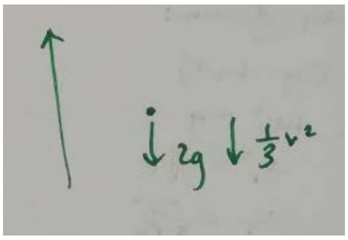
$$x = \frac{1}{2k}\ln\left(\frac{g}{g - k\left[\frac{g}{4k}\right]}\right)$$

$$= \frac{1}{2k}\ln\left(\frac{g}{\frac{3g}{4}}\right)$$

$$= \frac{1}{2k}\ln\left(\frac{4}{3}\right)$$

Question 7

a



$$m\ddot{x} = -2g - \frac{1}{3}v^2$$

$$2\ddot{x} = -2(10) - \frac{1}{3}v^2$$

$$\ddot{x} = \frac{dv}{dt} = -10 - \frac{1}{6}v^2$$

$$\frac{dv}{dt} = -\frac{60 + v^2}{6}$$

$$\frac{dt}{dv} = -\frac{6}{60 + v^2}$$

$$t = -\frac{6}{\sqrt{60}}\tan^{-}\left(\frac{v}{\sqrt{60}}\right) + c$$

$$t = -\frac{6}{2\sqrt{15}}\tan^{-}\left(\frac{v}{2\sqrt{15}}\right) + c$$

$$t = -\frac{3\sqrt{15}}{15}\tan^{-}\left(\frac{v\sqrt{15}}{30}\right) + c$$

$$t = -\frac{\sqrt{15}}{5}\tan^{-}\left(\frac{v\sqrt{15}}{30}\right) + c$$

When $t = 0, v = 15$:

$$0 = -\frac{\sqrt{15}}{5}\tan^{-}\left(\frac{15\sqrt{15}}{30}\right) + c$$

$$c = \frac{\sqrt{15}}{5}\tan^{-}\left(\frac{\sqrt{15}}{2}\right)$$

$$t = -\frac{\sqrt{15}}{5}\tan^{-1}\left(\frac{v\sqrt{15}}{30}\right) + \frac{\sqrt{15}}{5}\tan^{-1}\left(\frac{\sqrt{15}}{2}\right)$$

$$t = \frac{\sqrt{15}}{5}\left[\tan^{-1}\left(\frac{\sqrt{15}}{2}\right) - \tan^{-1}\left(\frac{v\sqrt{15}}{30}\right)\right]$$

When $v = 0$:

$$t = \frac{\sqrt{15}}{5}\left[\tan^{-1}\left(\frac{\sqrt{15}}{2}\right) - \tan^{-1}(0)\right]$$

$$t = \frac{\sqrt{15}}{5}\tan^{-}\left(\frac{\sqrt{15}}{2}\right)\ s$$

b

$$\ddot{x} = v\frac{dv}{dx} = -\frac{60+v^2}{6}$$

$$\frac{dv}{dx} = -\frac{60+v^2}{6v}$$

$$\frac{dx}{dv} = -\frac{6v}{60+v^2}$$

$$x = -3\ln\left(60+v^2\right)+c$$

When $x = 0, v = 15$:

$$0 = -3\ln\left(60+225\right)+c$$

$$c = 3\ln 285$$

$$x = -3\ln\left(60+v^2\right)+3\ln 285$$

$$x = 3\ln\left(\frac{285}{60+v^2}\right)$$

$$\frac{x}{3} = \ln\left(\frac{285}{60+v^2}\right)$$

$$e^{\frac{x}{3}} = \frac{285}{60+v^2}$$

$$e^{-\frac{x}{3}} = \frac{60+v^2}{285}$$

$$60+v^2 = 285e^{-\frac{x}{3}}$$

$$v^2 = 285e^{-\frac{x}{3}}-60$$

c

$$\ddot{x} = v\frac{dv}{dx} = -\frac{60+v^2}{6}$$

$$\frac{dv}{dx} = -\frac{60+v^2}{6v}$$

$$\frac{dx}{dv} = -\frac{6v}{60+v^2}$$

$$x = -3\ln\left(60+v^2\right)+c$$

When $x = 0, v = 15$:

$$0 = -3\ln\left(60+225\right)+c$$

$$c = 3\ln 285$$

When $x = H, v = 0$:

$$0^2 = 285e^{-\frac{H}{3}} - 60$$

$$60 = 285e^{-\frac{H}{3}}$$

$$\frac{4}{19} = e^{-\frac{H}{3}}$$

$$-\frac{H}{3} = \ln\frac{4}{19}$$

$$H = -3\ln\frac{4}{19}$$

This can also be written as $H = 3\ln\left(\frac{4}{19}\right)^{-} = 3\ln\left(\frac{19}{4}\right)$

Question 8

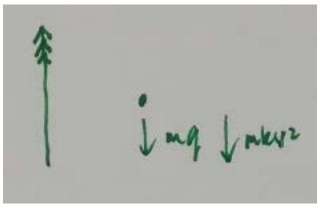
Going downwards:

$$m\ddot{x} = mg - mkv^2$$
$$\ddot{x} = g - kv^2$$

For terminal speed = initial speed = V, let $\ddot{x} = 0$:

$$0 = g - kV^2$$
$$kV^2 = g$$
$$V^2 = \frac{g}{k}$$
$$V = \sqrt{\frac{g}{k}}$$

Going upwards:



$$m\ddot{x} = -mg - mkv^2$$
$$\ddot{x} = v\frac{dv}{dx} = -g - kv^2$$
$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$
$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$
$$x = -\frac{1}{2k}\ln\left(g + kv^2\right) + c$$

When $x = 0, v = V = \sqrt{\frac{g}{k}}$:

$$0 = -\frac{1}{2k}\ln\left(g + k\frac{g}{k}\right) + c$$
$$c = \frac{1}{2k}\ln 2g$$
$$x = -\frac{1}{2k}\ln\left(g + kv^2\right) + \frac{1}{2k}\ln 2g$$
$$= \frac{1}{2k}\ln\left(\frac{2g}{g + kv^2}\right)$$

Going downwards:

$$\ddot{x} = v\frac{dv}{dx} = g - kv^2$$

$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$x = -\frac{1}{2k}\ln\left(g - kv^2\right) + c$$

When $x = 0, v = 0$:

$$0 = -\frac{1}{2k}\ln\left(g - 0\right) + c$$

$$c = \frac{1}{2k}\ln g$$

$$x = -\frac{1}{2k}\ln\left(g - kv^2\right) + \frac{1}{2k}\ln g$$

$$= \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$$

For greatest height, sub $v = 0$ into 'going upwards' equation:

$$x = \frac{1}{2k}\ln\left(\frac{2g}{g + 0}\right)$$

$$= \frac{1}{2k}\ln 2$$

Distance travelled upwards = distance travelled downwards:

Sub this *x*-value into 'going downwards' equation to find final speed:

$$\frac{1}{2k}\ln 2 = \frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right)$$

$$2 = \frac{g}{g-kv^2}$$

$$g-kv^2 = \frac{g}{2}$$

$$kv^2 = \frac{g}{2}$$

$$v^2 = \frac{g}{2k}$$

$$v = \sqrt{\frac{g}{2k}}$$

$$= V\sqrt{\frac{1}{2}} \qquad \left(V = \sqrt{\frac{g}{k}}\right)$$

$$= \frac{V\sqrt{2}}{2}\,\text{ms}^{-1}$$

Question 9

a

$$m\ddot{x} = mg - mkv^2$$

$$\ddot{x} = v\frac{dv}{dx} = g - kv^2$$

$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$x = -\frac{1}{2k}\ln\left(g - kv^2\right) + c$$

When $x = 0, v = 0$:

$$0 = -\frac{1}{2k}\ln\left(g - 0\right) + c$$

$$c = \frac{1}{2k}\ln g$$

$$x = -\frac{1}{2k}\ln\left(g - kv^2\right) + \frac{1}{2k}\ln g$$

$$= \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$$

$$2kx = \ln\left(\frac{g}{g - kv^2}\right)$$

$$e^{2kx} = \frac{g}{g - kv^2}$$

$$e^{-2kx} = \frac{g - kv^2}{g}$$

$$ge^{-2kx} = g - kv^2$$

$$kv^2 = g - ge^{-2kx}$$

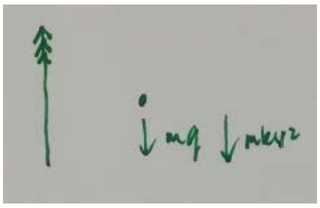
$$v^2 = \frac{g\left(1 - e^{-2kx}\right)}{k}$$

b $k = 0.003$, $x = 1200$:

$$v^2 = \frac{10\left(1 - e^{-2\ 0.003\ 1200}\right)}{0.003}$$
$$= 3330.8447...$$
$$v = \sqrt{3330.8447...}$$
$$= 57.7134...$$
$$\approx 57.7\,\text{ms}^{-1}$$

Question 10

a Going upwards:



$$m\ddot{x} = -mg - mkv^2$$
$$\ddot{x} = v\frac{dv}{dx} = -g - kv^2$$
$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$
$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$
$$x = -\frac{1}{2k}\ln\left(g + kv^2\right) + c$$

When $x = 0, v = v_0$:

$$0 = -\frac{1}{2k}\ln\left(g + k{v_0}^2\right) + c$$
$$c = \frac{1}{2k}\ln\left(g + k{v_0}^2\right)$$
$$x = -\frac{1}{2k}\ln\left(g + kv^2\right) + \frac{1}{2k}\ln\left(g + k{v_0}^2\right)$$
$$= \frac{1}{2k}\ln\left(\frac{g + k{v_0}^2}{g + kv^2}\right)$$

$v = 0$ for greatest height reached.

$$x = \frac{1}{2k}\ln\left(\frac{g + k{v_0}^2}{g + 0}\right)$$
$$= \frac{1}{2k}\ln\left(\frac{g + k{v_0}^2}{g}\right)$$

b Going downwards:

$m\ddot{x} = mg - mkv^2$

$\ddot{x} = g - kv^2$

Terminal velocity when $\ddot{x} = 0$

$0 = g - kv^2$

$kv^2 = g$

$v^2 = \dfrac{g}{k}$

$v = \sqrt{\dfrac{g}{k}}$

c

$m\ddot{x} = mg - mkv^2$

$\ddot{x} = v\dfrac{dv}{dx} = g - kv^2$

$\dfrac{dv}{dx} = \dfrac{g - kv^2}{v}$

$\dfrac{dx}{dv} = \dfrac{v}{g - kv^2}$

$x = -\dfrac{1}{2k}\ln\left(g - kv^2\right) + c$

When $x = 0, v = 0$:

$0 = -\dfrac{1}{2k}\ln\left(g - 0\right) + c$

$c = \dfrac{1}{2k}\ln g$

$x = -\dfrac{1}{2k}\ln\left(g - kv^2\right) + \dfrac{1}{2k}\ln g$

$\quad = \dfrac{1}{2k}\ln\left(\dfrac{g}{g - kv^2}\right)$

Distance travelled upwards = distance travelled downwards

Sub greatest height and $v = V$ into 'going downwards' equation above:

$\dfrac{1}{2k}\ln\left(\dfrac{g + kv_0{}^2}{g}\right) = \dfrac{1}{2k}\ln\left(\dfrac{g}{g - kV^2}\right)$

$\dfrac{g + kv_0{}^2}{g} = \dfrac{g}{g - kV^2}$

$\left(g + kv_0{}^2\right)\left(g - kV^2\right) = g^2$

Exercise 7.07 Resisted projectile motion

Question 1

a

$$m\ddot{y} = -mg - kv$$
$$5\ddot{y} = -5\times10 - kv$$
$$\ddot{y} = -10 - \frac{k}{5}v$$

b Terminal velocity occurs when $\ddot{y} = 0$

$$\ddot{y} = -10 - \frac{k}{5}v$$
$\ddot{y} = 0$ for terminal elocity, $k = 2.5$
$$0 = -10 - \frac{2.5}{5}v$$
$$0.5v = -10$$
$$v = -20\ \text{m s}^{-1}$$

c

$$\ddot{y} = -10 - \frac{k}{5}v$$

$$v\frac{dv}{dy} = -10 - \frac{2.5}{5}v$$

$$= -\left(10 + \frac{v}{2}\right)$$

$$= -\frac{20+v}{2}$$

$$\frac{dv}{dy} = -\frac{20+v}{2v}$$

$$\frac{dy}{dv} = -\frac{2v}{20+v}$$

$$= -\frac{2(20+v) - 40}{20+v}$$

$$= -2 + \frac{40}{20+v}$$

$$y = -2v + 40\ln(20+v) + c$$

When $y = 0$, $v = 100 \sin 15°$:

$$0 = -2(100\sin 15°) + 40\ln(20 + 100\sin 15°) + c$$

$$0 = 101.2790... + c$$

$$c = -101.2790...$$

$$y = -2v + 40\ln(20+v) - 101.2790...$$

Maximum height when $v = 0$:

$$y = -2(0) + 40\ln(20+0) - 101.2790...$$

$$= 40\ln 20 - 101.0...$$

$$= 18.5502...$$

$$\approx 18.55\,\text{m}$$

Question 2

a

$$m\ddot{x} = -mkv_x$$
$$\ddot{x} = -kv_x$$
$$\frac{dv_x}{dt} = -0.3v_x$$
$$\frac{dt}{dv_x} = -\frac{1}{03\ v_x}$$
$$t = -\frac{1}{0.3}\ln v_x + c$$
When $t = 0, v_x = 380\cos 5°$
$$0 = -\frac{1}{0.3}\ln\left(380\cos 5°\right) + c$$
$$c = \frac{1}{0.3}\ln\left(380\cos 5°\right)$$
$$t = -\frac{1}{0.3}\ln v_x + \frac{1}{0.3}\ln\left(380\cos 5°\right)$$
$$t = \frac{1}{0.3}\ln\left(\frac{380\cos 5°}{v_x}\right)$$
$$0.3t = \ln\left(\frac{380\cos 5°}{v_x}\right)$$
$$e^{0.3t} = \frac{380\cos 5°}{v_x}$$
$$e^{-0.3t} = \frac{v_x}{380\cos 5°}$$
$$v_x = 380\cos 5° e^{-0.3t}$$
$$x = \frac{380\cos 5° e^{-0.3t}}{-0.3} + d$$

When $t=0, x=0$:

$$0=\frac{380\cos 5°e^{0}}{-0.3}+d$$

$$d=\frac{380\cos 5°}{0.3}$$

$$x=-\frac{380\cos 5°e^{-0.3}}{0.3}+\frac{380\cos 5°}{0.3}$$

$$x=\frac{380\cos 5°}{0.3}\left(1-e^{-0.3t}\right)$$

When $t=2$:

$$x=\frac{380\cos 5°}{0.3}\left(1-e^{-0.6}\right)$$

$$=569.3305...$$

$$\approx 570 \text{ m}$$

b

$$c=\frac{1}{0.3}\ln\left(380\cos 5°\right)$$

$$t=-\frac{1}{0.3}\ln v_x+\frac{1}{0.3}\ln\left(380\cos 5°\right)$$

$$x=\frac{380\cos 5°}{0.3}\left(1-e^{-0.3t}\right)$$

Make t thesubject:

$$\frac{0.3x}{380\cos 5°}=1-e^{-0.3t}$$

$$e^{-0.3t}=1-\frac{0.3x}{380\cos 5°}$$

$$-0.3t=\ln\left(1-\frac{03\ x}{380\cos 5°}\right)$$

$$t=-\frac{1}{0.3}\ln\left(1-\frac{0.3x}{380\cos 5°}\right)$$

When $x=2\times 570=1140$:

$$t=-\frac{1}{0.3}\ln\left(1-\frac{0.3\times 1140}{380\cos 5°}\right)$$

$$=7.7918...$$

$$\approx 7.8 \text{ s}$$

c

$$\text{When } x = \frac{1}{2} \times 570 = 285:$$
$$t = -\frac{1}{0.3}\ln\left(1 - \frac{0.3 \times 285}{380\cos 5°}\right)$$
$$= 0.8533...$$
$$\approx 0.85 \text{ s}$$

Question 3

$$m\ddot{x} = -kv$$
$$0.145\ddot{x} = -kv$$
$$\ddot{x} = \frac{dv}{dt} = -\frac{kv}{0.145}$$
$$\frac{dt}{dv} = -\frac{0.145}{kv}$$
$$t = -\frac{0.145}{k}\ln v + c$$
$$\text{When } t = 0, v = 30\cos 10°:$$
$$0 = -\frac{0145}{k}\ln\left(30\cos 10°\right) + c$$
$$c = \frac{0.145}{k}\ln\left(30\cos 10°\right)$$
$$t = -\frac{0.145}{k}\ln v + \frac{0.145}{k}\ln\left(30\cos 10°\right)$$
$$t = \frac{0.145}{k}\ln\left(\frac{30\cos 10°}{v}\right)$$
$$\frac{kt}{0.145} = \ln\left(\frac{30\cos 10°}{v}\right)$$
$$e^{\frac{kt}{0.145}} = \frac{30\cos 10°}{v}$$
$$e^{-\frac{kt}{0.145}} = \frac{v}{30\cos 10°}$$
$$v = 30\cos 10° e^{-\frac{kt}{0.145}}$$

$$x = 30\cos 10°\left(-\frac{0145}{k}\right)e^{-\frac{kt}{0.145}} + d$$

$$x = \frac{-435\cos 10\ °}{k}e^{-\frac{kt}{0.145}} + d$$

When $t = 0, x = 0$:

$$0 = \frac{-435\cos 10\ °}{k}e^{0} + d$$

$$d = \frac{4.35\cos 10°}{k}$$

$$x = \frac{-4.35\cos 10°}{k}e^{-\frac{k}{0.145}} + \frac{4.35\cos 10°}{k}$$

$$x = \frac{4.35\cos 10°}{k}\left(1 - e^{-\frac{k}{0.145}}\right)$$

When $t = 1, x = 18$:

$$18 = \frac{4.35\cos 10°}{k}\left(1 - e^{-\frac{k}{0.145}}\right)$$

Test $k = 0.158$:

$$\text{RHS} = \frac{4.35\cos 10°}{0.158}\left(1 - e^{-\frac{0.158}{0.145}}\right)$$

$$= 18.2718...$$

$$\approx 18 \text{ (LHS)}$$

So $k \approx 0.158$.

Question 4

a

$$m = 0.16 \text{ kg}, g = 9.8, k = 0.09$$

$$m\ddot{y} = -mg - kv_y$$

$$\ddot{y} = -g - \frac{k}{m}v_y$$

$$\ddot{y} = \frac{dv_y}{dt} = -98 \; - \frac{0.09}{016}v_y$$

$$\frac{dv_y}{dt} = -98 \; -.5625 \quad v_y$$

$$\frac{dt}{dv_y} = -\frac{1}{98 \; +.5625 \quad v_y}$$

$$t = -\frac{1}{0.5625}\ln\left(9.8 + 0.5625v_y\right) + c$$

When $t = 0, v_y = 50\sin 5°$:

$$0 = -\frac{1}{0.5625}\ln\left(9.8 + 0.5625 \times 50\sin 5°\right) + c$$

$$c = \frac{1}{0.5625}\ln\left(9.8 + 0.5625 \times 50\sin 5°\right)$$

$$= 4.4544...$$

$$t = -\frac{1}{0.5625}\ln\left(9.8 + 0.5625v_y\right) + 4.4544...$$

Maximum height reahed when $v_y = 0$:

$$t = -\frac{1}{0.5625}\ln\left(9.8 + 0\right) + 4.4544...$$

$$= 0.3968...$$

$$\approx 0.4 \text{ s}$$

b

$$m\ddot{x} = -kv_x$$

$$\ddot{x} = -\frac{k}{m}v_x$$

$$\ddot{x} = -\frac{0.09}{0.16}v_x$$

$$\frac{dv}{dt} = -0.5625v_x$$

$$\frac{dt}{dv} = -\frac{1}{05625\ v_x}$$

$$t = -\frac{1}{0.5625}\ln v_x + c$$

When $t = 0, v_x = 50\cos 5°$:

$$0 = -\frac{1}{0.5625}\ln\left(50\cos 5°\right) + c$$

$$c = \frac{1}{0.5625}\ln\left(50\cos 5°\right)$$

$$t = -\frac{1}{0.5625}\ln v_x + \frac{1}{0.5625}\ln\left(50\cos 5°\right)$$

$$t = \frac{1}{0.5625}\ln\left(\frac{50\cos 5°}{v_x}\right)$$

$$0.5625t = \ln\left(\frac{50\cos 5°}{v_x}\right)$$

$$e^{0.5625t} = \frac{50\cos 5°}{v_x}$$

$$e^{-0.5625t} = \frac{v_x}{50\cos 5°}$$

$$v_x = 50\cos 5° e^{-0.5625t}$$

$$x = \frac{50\cos 5°}{-0.5625}e^{-0.5625} + d$$

When $t = 0,\ x = 0$:

$$0 = \frac{50\cos 5°}{-0.5625}e^{0} + d$$

$$d = \frac{50\cos 5°}{0.5625}$$

$$x = \frac{50\cos 5°}{-0.5625}e^{-0.5625t} + \frac{50\cos 5°}{0.5625}$$

$$= \frac{50\cos 5°}{0.5625}\left(1 - e^{-0.5625t}\right)$$

When $t = 0.3...$ at greatest height:

$$x = \frac{50\cos 5°}{0.5625}\left(1 - e^{-0.5625 \times 0.3...}\right)$$

$$= 17.7141...$$

$$\approx 17.7 \text{ m}$$

c

$v_x = 50\cos 5° e^{-0.5625t}$ (from b)

When $t = 1$:

$$v_x = 50\cos 5° e^{-0.5625}$$

$$= 28.3807...$$

$t = -\frac{1}{0.5625}\ln\left(9.8 + 0.5625v_y\right) + 4.4544...$ (from a)

When $t = 1$:

$$1 = -\frac{1}{0.5625}\ln\left(9.8 + 0.5625v_y\right) + 4.4544...$$

$$-3.4544... = -\frac{1}{0.5625}\ln\left(9.8 + 0.5625v_y\right)$$

$$1.9431... = \ln\left(9.8 + 0.5625v_y\right)$$

$$e^{1943..} = 9.8 + 0.5625v_y$$

$$e^{1943..} - 9.8 = 0.5625v_y$$

$$\frac{e^{1.9431...} - 9.8}{0.5625} = v_y$$

$$v_y = -5.0126...$$

$$v^2 = {v_x}^2 + {v_y}^2$$

$$v^2 = 28.3807...^2 + \left(-5.0126...\right)^2$$

$$v^2 = 830.5920...$$

$$v = 28.8199...$$

$$= 28.8 \text{ ms}^{-1}$$

Question 5

a

$$m\ddot{y} = mg - kv^2$$

$$v = 44 \text{ ms}^{-} \quad \text{when } \ddot{y} = 0,$$

$$\therefore mg = kv^2$$

$$k = \frac{mg}{v^2}$$

$$k = \frac{0046 \times 8}{44^2}$$

$$k = 0.000233$$

b

$$x = \frac{m}{k}\ln\left(1 + \frac{kt}{m}v\cos\theta\right)$$

$$e^{\frac{kx}{m}} = 1 + \frac{kt}{m}v\cos\theta$$

$$t = \frac{m}{k}\frac{\left(e^{\frac{kx}{m}} - 1\right)}{v\cos\theta}$$

$$4.53 = \frac{0.046}{k}\frac{\left(e^{\frac{k\times155}{0.046}} - 1\right)}{60\cos\theta}$$

$$\cos\theta = \frac{0.046}{k}\frac{\left(e^{\frac{k\times155}{0.046}} - 1\right)}{60\times.53}$$

$$\cos\theta = 0.86629$$

$$\theta = 29.9696\ldots \approx 30°$$

Test yourself 7

Question 1

$$v = 2 - e^{-2x}$$

$$v^2 = \left(2 - e^{-2x}\right)^2$$

$$\frac{1}{2}v^2 = \frac{1}{2}\left(2 - e^{-2x}\right)^2$$

$$\frac{1}{2}v^2 = \frac{1}{2}\left(4 - 4e^{-2x} + e^{-4x}\right)$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{1}{2}\left(8e^{-2x} - 4e^{-4x}\right)$$

$$a = \frac{1}{2}\left(8e^{-2x} - 4e^{-4x}\right)$$

$$x = 0$$

$$a = \frac{1}{2}\left(8e^{0} - 4e^{0}\right)$$

$$a = 2 \text{ cm s}^{-2}$$

Question 2

$$a = -40e^{-8x}$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -40e^{-8x}$$

$$\frac{1}{2}v^2 = 5e^{-8x} + c$$

$$x = 0, v = 10$$

$$\frac{1}{2} \times 10^2 = 5e^{0} + c$$

$$50 = 5 + c$$

$$c = 45$$

$$\frac{1}{2}v^2 = 5e^{-8x} + 45$$

$$v^2 = 10e^{-8x} + 90$$

$$v = \sqrt{10e^{-8x} + 90} \text{ m s}^{-1}$$

Question 3

$\ddot{x} = -36x$

$n^2 = 36$

$n = 6$

$x = a\cos(nt + \alpha)$

$x = a\cos(6t + \alpha)$

When $t = 0, x = 0, v = -5$

$0 = a\cos(0 + \alpha)$

$\cos\alpha = 0$

$\alpha = \frac{\pi}{2}$

$x = a\cos\left(6t + \frac{\pi}{2}\right)$

$v = \dot{x} = -6a\sin\left(6t + \frac{\pi}{2}\right)$

When $t = 0, v = -5$

$-5 = -6a\sin\left(\frac{\pi}{2}\right)$

$-5 = -6a$

$a = \frac{5}{6}$

$x = \frac{5}{6}\cos\left(6t + \frac{\pi}{2}\right)$

Question 4

a

$$v^2 = 40 - 8x - 4x^2$$
$$\frac{1}{2}v^2 = 20 - 4x - 2x^2$$
$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -4 - 4x^2$$
$$a = -4(1 + x)$$

b Particle is at rest when $v = 0$.

$$0 = 40 - 8x - 4x^2$$
$$4x^2 + 8x + 40 = 0$$
$$x^2 + 2x - 10 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times (-10)}}{2}$$
$$x = \frac{-2 \pm \sqrt{44}}{2}$$
$$x = \frac{-2 - 2\sqrt{11}}{2} \quad x = \frac{-2 + 2\sqrt{11}}{2}$$
$$x = -1 - \sqrt{11} \text{ m}, x = -1 + \sqrt{11} \text{ m}$$

c Greatest speed when $a = 0$.

$$0 = -4(x + 1)$$
$$x + 1 = 0$$
$$x = -1$$
$$v^2 = 40 - 8x - 4x^2$$
$$v^2 = 40 - 8(-1) - 4(-1)^2$$
$$v^2 = 44$$
$$v = \pm 2\sqrt{11} \text{ m s}^{-1}$$
$$\text{speed} = |v| = 2\sqrt{11} \text{ m s}^{-1}$$

d

$$v^2 = 40 - 8x - 4x^2$$
$$v^2 = 4\left(10 - 2x - x^2\right)$$
$$v^2 = 4\left(11 - 1 - 2x - x^2\right)$$
$$v^2 = 4\left(11 - \left(1 + 2x + x^2\right)\right)$$
$$v^2 = 2^2\left(11 - \left(1 + x\right)^2\right)$$

Question 5

$v = 25$

$a - -9.8$

$\ddot{y} = -9.8$

$\dot{y} = -98\ t + c$

$t = 0, \dot{y} = 25$

$25 = 0 + c$

$\dot{y} = -9.8t + 25$

$y = -4.9t^2 + 25t + c$

$t = 0, y = 0, c = 0$

$y = -4.9t^2 + 25t$

Maximimum height when $\dot{y}$ = 0

$\dot{y} = -9.8t + 25$

$0 = -9.8t + 25$

$9.8t = 25$

$t = 2.55$

Total flight time 2 $\times$2.55 = 5.1 s

Maximum height

$y = -4.9t^2 + 25t$

$y = -4.9 \times \left(2.55\right)^2 + 25 \times 2.55$

$y = 31.9$ m

Question 6

Using the trajectory formula

$$y = \frac{-gx^2}{2v^2}\left(1+\tan^2\theta\right)+x\tan\theta$$

$$y = 1, x = 104.2, v = 75.4, g = 9.8$$

$$1 = \frac{-9.8\times(104.2)^2}{2\times(75.4)^2}\left(1+\tan^2\theta\right)+104.2\tan\theta$$

$$1 = -9.35812\left(1+\tan^2\theta\right)+104.2\tan\theta$$

$$9.35812\tan^2\theta - 104.2\tan\theta + 10.35812 = 0$$

$$\tan\theta = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\tan\theta = \frac{104.2\pm\sqrt{(-104.2)^2-4(9.358)10.358}}{2\times 9.358}$$

$$\tan\theta = \frac{104.2\pm\sqrt{10\,469.9}}{18.716}$$

$$\tan\theta = 0.1003, 11.0345$$

$$\theta = 57^\circ, 4.8^\circ$$

Question 7

Resolving forces

Vertical

$$N\cos\theta + \mu N\sin\theta = mg$$

$$N(\cos\theta + \mu\sin\theta) = 20$$

$$N\left(\cos\theta + \frac{1}{\sqrt{3}}\sin\theta\right) = 20$$

Horizontal

$$N\sin\theta = \mu N\cos\theta$$

$$N\sin\theta = \frac{1}{\sqrt{3}}N\cos\theta$$

$$\sin\theta = \frac{1}{\sqrt{3}}\cos\theta$$

$$\frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{3}}$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6} = 30^\circ$$

Question 8

Resolving forces

Vertical

$$N + 50\sin 30 = mg$$

$$N = 10g - 25$$

Horizontal

$$F = 50\cos 0 \ -0.3 \ N$$

$$F = 50\cos 30 - 0.3(10g - 25)$$

$$F = 25\sqrt{3} - \frac{3(10g - 25)}{10}$$

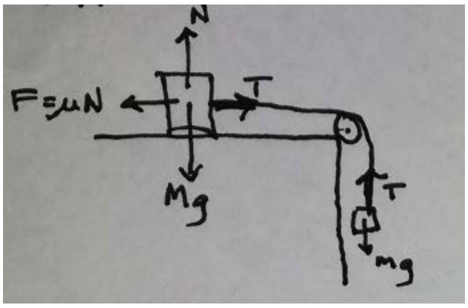
$$F = \frac{250\sqrt{3} - 3(10g - 25)}{10}$$

$$a = \frac{F}{m}$$

$$a = \frac{250\sqrt{3} - 3(10g - 25)}{100}\ \text{m s}^{-2}$$

Question 9

a



$$M = 10, m = 5, g = 9.8, \mu = 0.6$$

$$\begin{aligned}(M+m)\ddot{x} &= T - F \\ &= mg - \mu N \\ &= mg - \mu Mg\end{aligned}$$

$$(10+5)\ddot{x} = 5\times 9.8 - 0.6\times 10\times 9.8$$

$$15\ddot{x} = -9.8$$

$$\begin{aligned}\ddot{x} &= \frac{-9.8}{15} \\ &= -0.65333.... \\ &\approx -0.7 \text{ ms}^{-2}\end{aligned}$$

b

$M\ddot{x} = T - F$ on the table

$$M\ddot{x} = T - \mu Mg$$

$$10\times(-0.65333...) = T - 0.6\times 10\times 9.8$$

$$-6.5333... = T - 58.8$$

$$\begin{aligned}T &= -6.5333... + 58.8 \\ &= 52.2666... \\ &= 52.3 \text{ N}\end{aligned}$$

Question 10

$$F = -kv^3$$

$$a = -\frac{kv^3}{m}$$

$$v\frac{dv}{dx} = -\frac{kv^3}{m}$$

$$\frac{dv}{dx} = -\frac{kv^2}{m}$$

$$\frac{dx}{dv} = -\frac{m}{kv^2}$$

$$kdx = -mv^{-2}dv$$

$$kx = mv^{-} + c$$

$$x = 0 \;\; v = v_0$$

$$0 = mv_0^{\,-} + c$$

$$c = -mv_0^{\,-1}$$

$$kx = mv^{-1} - mv_0^{\,-1}$$

$$kx = m\left(\frac{1}{v} - \frac{1}{v_0}\right)$$

$$x = \frac{m}{k}\left(\frac{1}{v} - \frac{1}{v_0}\right)$$

$$\frac{1}{v} - \frac{1}{v_0} = \frac{xk}{m}$$

$$\frac{1}{v} = \frac{xk}{m} + \frac{1}{v_0}$$

$$\frac{1}{v} = \frac{xkv_0 + m}{mv_0}$$

$$v = \frac{mv_0}{xkv_0 + m}$$

Question 11

$F = a = -\frac{v}{40}$

$\frac{dv}{dt} = -\frac{v}{40}$

$\frac{dt}{dv} = -\frac{40}{v}$

$t = -40\ln v + c$

$t = 0, v = 30$

$0 = -40\ln 30 + c$

$c = 40\ln 30$

$t = 40\ln\left(\frac{30}{v}\right)$

$\frac{t}{40} = \ln\left(\frac{30}{v}\right)$

$\frac{30}{v} = e^{\frac{t}{40}}$

$\frac{v}{30} = e^{-\frac{t}{40}}$

$v = 30e^{-\frac{t}{40}}$

When $t = 40$

$v = 30e^{-\frac{40}{40}}$

$v_{40} = \frac{30}{e}\ \mathrm{m\,s^{-1}}$

$$a = -\frac{v}{40}$$

$$v\frac{dv}{dx} = -\frac{v}{40}$$

$$\frac{dv}{dx} = -\frac{1}{40}$$

$$v = -\frac{x}{40} + c$$

$$x = 0, v = 30$$

$$30 = c$$

$$v = -\frac{x}{40} + 30$$

$$30e^{-\frac{t}{40}} = -\frac{x}{40} + 30$$

$$\frac{x}{40} = 30 - 30e^{-\frac{t}{40}}$$

$$x = 1200\left(1 - e^{-\frac{}{40}}\right)$$

$$x(40) = 1200\left(1 - e^{-\frac{40}{40}}\right) = 1200\left(1 - \frac{1}{e}\right) \text{ m}$$

Question 12

a Terminal velocity occurs when $a = 0$.

$$F = ma = mg - \frac{v}{6}$$

$$0 = mg - \frac{v}{6}$$

$$\frac{v}{6} = mg$$

$$v_T = 6mg$$

b

$$F = ma = mg - 2v$$

If she hits the ground at terminal velocity with the chute

$$0 = mg - 2v$$

$$2v = mg$$

$$v = \frac{mg}{2}$$

Maximum velocity = $6mg$

Minimum velocity $= \frac{mg}{2}$

Question 13

$$F = m\ddot{x} = -mg - \frac{mv}{10}$$

$$\ddot{x} = -98 \ -\frac{v}{10}$$

$$\frac{dv}{dt} = -\frac{98+v}{10}$$

$$\frac{dt}{dv} = -\frac{10}{98+v}$$

$$t = -10\ln(98+v) + c$$

When $t = 0, v = 13$

$$0 = -10\ln(98+13) + c$$

$$c = 10\ln 111$$

$$t = -10\ln(98+v) + 10\ln 111$$

$$t = 10\ln\left(\frac{111}{98+v}\right)$$

Maximum height occur when $v = 0$.

$$t = 10\ln\left(\frac{111}{98}\right)$$

$$= 1.2456...$$

$$\approx 1.2\,\text{s}$$

$$t = 10\ln\left(\frac{111}{98+v}\right)$$

$$\frac{t}{10} = \ln\left(\frac{111}{98+v}\right)$$

$$\frac{111}{98+v} = e^{\frac{t}{10}}$$

$$\frac{98+v}{111} = e^{-\frac{t}{10}}$$

$$98 + v = 111e^{-\frac{t}{10}}$$

$$v = 111e^{-\frac{t}{0}} - 98$$

$$x = \frac{111e^{-\frac{t}{10}}}{-\frac{1}{\frac{1}{10}}} - 98t + d$$

$$x = -1110e^{-\frac{t}{0}} - 98t + d$$

When $t = 0, x = 0$:

$$0 = -1110e^{0} - 0 + d$$

$$d = 1110$$

$$x = -1110e^{-\frac{t}{0}} - 98t + 1110$$

When $t = 1.2456...$

$$x = -1110e^{-\frac{1.2456...}{0}} - 98(1.2456...) + 1110$$

$$= 7.9312... \quad \approx 7.9 \text{ m}$$

Question 14

$$y = \left(\frac{mg}{kv\cos\theta} + \tan\theta\right)x + \frac{m^2 g}{k^2}\ln\left(1 - \frac{kx}{mv\cos\theta}\right)$$

$$y = \left(\frac{2g}{8\cos 30^\circ} + \tan 30^\circ\right)x + 4g\ln\left(1 - \frac{x}{16\cos 30^\circ}\right)$$

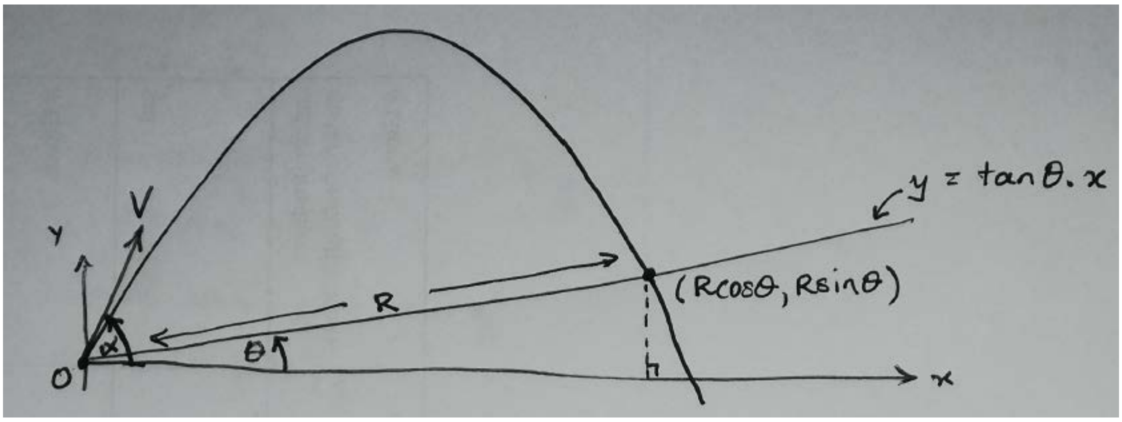
$$y = \left(\frac{2g}{4\sqrt{3}} + \frac{4}{4\sqrt{3}}\right)x + 4g\ln\left(1 - \frac{x}{8\sqrt{3}}\right)$$

$$y = \left(\frac{2g+4}{4\sqrt{3}}\right)x + 4g\ln\left(1 - \frac{x}{8\sqrt{3}}\right)$$

$$y = \left(\frac{g+2}{2\sqrt{3}}\right)x + 4g\ln\left(1 - \frac{x}{8\sqrt{3}}\right)$$

Question 15

a



Path of the projectile

$$y = \frac{-gx^2}{2V^2\cos^2\alpha} + x\tan\alpha$$

Let $P(R\cos\theta, R\sin\theta)$ be the point on the incline where the particle lands on the plane.

Substitute into x nd $y$above:

$$R\sin\theta = \frac{-g(R\cos\theta)^2}{2V^2\cos^2\alpha} + R\cos\theta\tan\alpha$$

$$2V^2R\sin\theta\cos^2\alpha = -gR^2\cos^2\theta + 2V^2R\cos\theta\tan\alpha\cos^2\alpha$$

$$2V^2R\sin\theta\cos^2\alpha = -gR^2\cos^2\theta + 2V^2R\cos\theta\sin\alpha\cos\alpha$$

$$gR^2\cos^2\theta = 2V^2R\cos\alpha(\cos\theta\sin\alpha - \sin\theta\cos\alpha)$$

$$gR^2\cos^2\theta = 2V^2R\cos\alpha\sin(\alpha-\theta)$$

$$gR\cos^2\theta = 2V^2\cos\alpha\sin(\alpha-\theta) \qquad (R \neq 0)$$

$$R = \frac{2V^2\cos\alpha\sin(\alpha-\theta)}{g\cos^2\theta}$$

Using $\cos A\sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$:

$$R = \frac{2V^2\frac{1}{2}\left[\sin(\alpha+\{\alpha-\theta\}) - \sin(\alpha-\{\alpha-\theta\})\right]}{g\cos^2\theta}$$

$$R = \frac{V^2\left[\sin(2\alpha-\theta) - \sin\theta\right]}{g\cos^2\theta}$$

As α is the variable, R has a maximum when $\sin(2\alpha-\theta)$ is a maximum of 1.

$$R_{\max} = \frac{V^2[1-\sin\theta]}{g\cos^2\theta}$$

$$= \frac{V^2(1-\sin\theta)}{g(1-\sin^2\theta)}$$

$$= \frac{V^2(1-\sin\theta)}{g(1+\sin\theta)(1-\sin\theta)}$$

$$= \frac{V^2}{g(1+\sin\theta)}$$

b

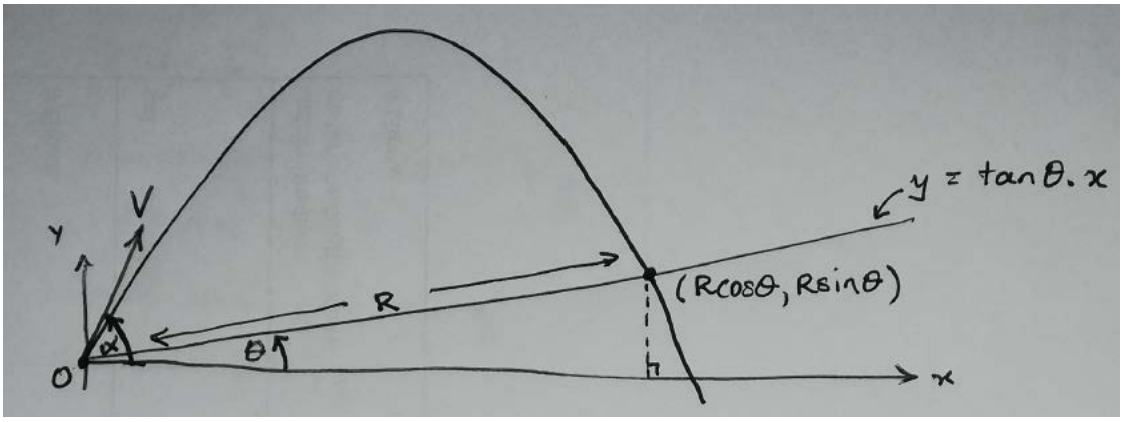
$$R_{max} = \frac{V^2}{g(1+\sin\theta)}$$

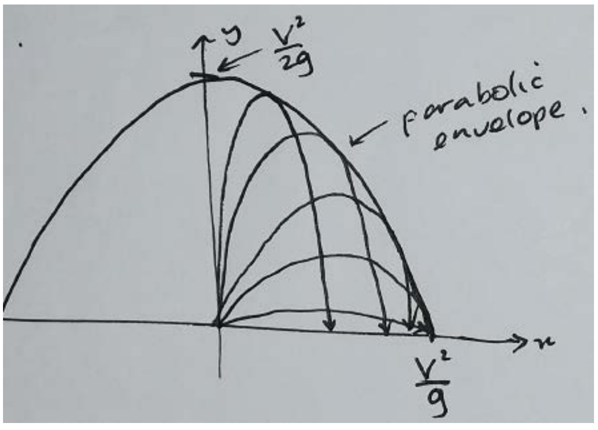
For $0° \leq \theta \leq 90°$, R_{max} lies between $\frac{V^2}{g(1+\sin 0°)}$ and $\frac{V^2}{g(1+\sin 90°)}$

between $\frac{V^2}{g(1+0)}$ and $\frac{V^2}{g(1+1)}$

between $\frac{V^2}{g}$ and $\frac{V^2}{2g}$

The graph of $y = R_{max}$ is a concave-down parabola of the form $y = -ax^2 + c$.





This is the graph of $y = R_{max}$

When $x = 0, \theta = 90°, y = \frac{V^2}{2g}$.

$$\frac{V^2}{2g} = 0 + c$$

$$c = \frac{V^2}{2g}$$

$$y = -ax^2 + \frac{V^2}{2g}$$

When $\theta = 0°, x = \frac{V^2}{g}, y = 0$.

$$0 = -a\left(\frac{V^2}{g}\right)^2 + \frac{V^2}{2g}$$

$$\frac{aV^4}{g^2} = \frac{V^2}{2g}$$

$$\frac{aV^2}{g} = \frac{1}{2}$$

$$a = \frac{g}{2V^2}$$

$$y = -\frac{g}{2V^2}x^2 + \frac{V^2}{2g}$$

$$-\frac{2V^2}{g}y = x^2 - \frac{2V^2}{g}\frac{V^2}{2g}$$

$$-\frac{2V^2}{g}y = x^2 - \frac{V^4}{g^2}$$

$$x^2 + \frac{2V^2}{g}y - \frac{V^4}{g^2} = 0$$

MATHS IN FOCUS 12 MATHEMATICS EXTENSION 2

WORKED SOLUTIONS

Practice set 1

Question 1

C

The correct rules are:

$$|z_1 z_2| = |z_1||z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$|z|^2 = z\overline{z}$$

$$\overline{z_1}\,\overline{z_2} = \overline{z_1 z_2}$$

Question 2

C

$$(\sin\theta - i\cos\theta)^n$$

$$= (-i(i\sin\theta + \cos\theta))^n$$

$$= (-i)^n (i\sin\theta + \cos\theta)^n$$

$$= (-i)^n (\cos n\theta + i\sin n\theta)$$

Question 3

$z^3 = 1$

$z^3 - 1 = 0$

$(z - 1)(z^2 + z + 1) = 0$

If ω is complex root

A

$\omega^2 + \omega + 1 = 0$

True

B

As the roots are equally spaced around the unit circle

$\omega^2 = \overline{\omega}$

True

C

$\omega^4 = \omega \times \omega^3$

$\omega^4 = \omega \times 1$

$\omega^4 = \omega$

True

D

Let $\omega^{-2} = \frac{1}{\omega}$

$\Rightarrow \omega^{-} \ = 1$

$\Rightarrow \omega = 1$

But ω is complex

False

Question 4

C

The negation is false.

Question 5

If $a > b$

A False

Counterexample

$$a = 1, b = -2$$
$$a > b$$
$$\frac{1}{a^2} = 1$$
$$\frac{1}{b^2} = \frac{1}{4}$$
$$\frac{1}{a^2} > \frac{1}{b^2}$$

B False

Counterexample

$$a = \frac{1}{10} \quad b = \frac{-1}{4}$$
$$a > b$$
$$\frac{1}{a^2} = 100$$
$$\frac{1}{b^2} = 16$$
$$\frac{1}{a^2} > \frac{1}{b^2}$$

C False

Counterexample

$$a = 1, b = -10$$
$$a > b$$
$$a^2 = 1$$
$$b^2 = 100$$
$$a^2 < b^2$$

D True

D

Question 6

$$\underset{\sim}{u} \times -3 = -3\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -9 \\ 3 \end{pmatrix} = \underset{\sim}{w}$$

So $\underset{\sim}{u}$ and $\underset{\sim}{w}$ are parallel (though opposite in direction)

B

Question 7

D

Question 8

D

Question 9

$$v = (1 - 2\cos\alpha) - 2i\sin\alpha$$

$$v^{-1} = \frac{1}{(1-2\cos\alpha) - 2i\sin\alpha}$$

$$= \frac{1}{(1-2\cos\alpha) - 2i\sin\alpha} \times \frac{(1-2\cos\alpha) + 2i\sin\alpha}{(1-2\cos\alpha) + 2i\sin\alpha}$$

$$= \frac{(1-2\cos\alpha) + 2i\sin\alpha}{(1-2\cos\alpha)^2 - (2i\sin\alpha)^2}$$

$$= \frac{1-2\cos\alpha + 2i\sin\alpha}{(1-2\cos\alpha)^2 + 4\sin^2\alpha}$$

Real part

$$\frac{1-2\cos\alpha}{1-4\cos\alpha + 4\cos^2\alpha + 4\sin^2\alpha}$$

$$= \frac{1-2\cos\alpha}{1-4\cos\alpha + 4(\cos^2\alpha + \sin^2\alpha)}$$

$$= \frac{1-2\cos\alpha}{5-4\cos\alpha}$$

A

Question 10

D

Question 11

a $\sqrt{-16} = 4i$

b

$$\sqrt{\frac{-7}{4}}$$
$$= \frac{i\sqrt{7}}{2}$$

c

$$\frac{6 \pm \sqrt{-12}}{2}$$
$$= \frac{6 \pm 2\sqrt{-3}}{2}$$
$$= 3 \pm \sqrt{-3}$$
$$= 3 \pm i\sqrt{3}$$

Question 12

a $i^8 = 1$

b

$$i^{22} + i^{23} + i^{24} + i^{25} + i^{26} + \quad + i^{98} + i^{99}$$
$$= i^2 + i^3 + i^0 + i^1 + i^2 + i^3 + \quad + i^2 + i^3$$
$$i^0 + i^1 + i^2 + i^3 = 0$$
$$\therefore i^{22} + i^{23} + i^{24} + i^{25} + i^{26} + \quad + i^{98} + i^{99} = i^2 + i^3$$
$$= -1 - i$$

Question 13

a

$$x^2 + 64 = 0$$
$$x^2 = -64$$
$$x = \pm 8i$$

b

$$x^2 + 2x + 7 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 7}}{2}$$
$$x = \frac{-2 \pm \sqrt{-24}}{2}$$
$$x = \frac{-2 \pm 2\sqrt{-6}}{2}$$
$$x = -1 \pm i\sqrt{6}$$

c

$$(x-3)^2 + 9 = 0$$
$$(x-3)^2 = -9$$
$$x - 3 = \pm 3i$$
$$x = 3 \pm 3i$$

Question 14

a

$$z = \frac{5-2i}{3}$$
$$\operatorname{Re}(z) = \frac{5}{3}$$
$$\operatorname{Im}(z) = -\frac{2}{3}$$

b

$$z = \frac{x+2i-iy+7}{x^2+y^2}$$
$$\operatorname{Re}(z) = \frac{x+7}{x^2+y^2}$$
$$\operatorname{Im}(z) = \frac{2-y}{x^2+y^2}$$

Question 15

a

$$z = 5x - 3iy$$
$$\overline{z} = 5x + 3iy$$

b

$$z = \frac{ai+6b-2a-ib}{4}$$
$$\overline{z} = \frac{6b-2a+ib-ai}{4}$$

Question 16

$$\omega = \frac{m+in}{m+n}$$

$$\overline{\omega} = \frac{m-in}{m+n}$$

$$\omega\overline{\omega} = \frac{m+in}{m+n}\frac{m-in}{m+n}$$

$$= \frac{m^2-(in)^2}{(m+n)^2}$$

$$= \frac{m^2+n^2}{(m+n)^2} \in \mathbb{R}$$

Question 17

a

$3x+2iy-18+6i=0$

Equating real terms

$3x-18=0$

$3x=18$

$x=6$

Equating imaginary terms

$2y+6=0$

$2y=-6$

$y=-3$

b

$x+y-i(x-y)=6-2i$

Equating real and imaginary terms

$x+y=6$

$x-y=2$

$2x=8$

$x=4$

$4+y=6$

$y=2$

Question 18

a

$$3-4i\left(5+2i\right)+i$$
$$=3-20i+8+i$$
$$=11-19i$$

b

$$\left(2-i\sqrt{3}\right)\left(2+i\sqrt{3}\right)$$
$$=4+3$$
$$=7$$

c

$$\left(1+5i\right)^2-\left(1-5i\right)^2$$
$$=\left(1+5i+1-5i\right)\left(1+5i-1+5i\right)$$
$$=2\times10i$$
$$=20i$$

Question 19

a

$$\left(x-1-2i\right)\left(x-1+2i\right)=0$$
$$x^2+x\left(-1-2i\right)+x\left(-1+2i\right)+\left(-1-2i\right)\left(-1+2i\right)=0$$
$$x^2-2x+1+4=0$$
$$x^2-2x+5=0$$

b

$$\left(x-\frac{-1-i\sqrt{2}}{6}\right)\left(x-\frac{-1+i\sqrt{2}}{6}\right)=0$$
$$x^2+x\left(-\frac{1-i\sqrt{2}}{6}\right)+x\left(-\frac{1+i\sqrt{2}}{6}\right)+\left(-\frac{1-i\sqrt{2}}{6}\right)\left(-\frac{1+i\sqrt{2}}{6}\right)=0$$
$$x^2+\frac{x}{6}+\frac{x}{6}+\frac{1+2}{36}=0$$
$$x^2+\frac{x}{3}+\frac{1}{12}=0$$
$$12x^2+4x+1=0$$

Question 20

a

$$\frac{2}{1-i\sqrt{3}}$$

$$=\frac{2}{1-i\sqrt{3}}\times\frac{1+i\sqrt{3}}{1+i\sqrt{3}}$$

$$=\frac{2\left(1+i\sqrt{3}\right)}{1+3}$$

$$=\frac{2+2i\sqrt{3}}{4}$$

$$=\frac{1+i\sqrt{3}}{2}$$

b

$$\frac{\sqrt{5}+2i}{\sqrt{5}-2i}+\frac{\sqrt{5}-2i}{\sqrt{5}+2i}$$

$$=\frac{\left(\sqrt{5}+2i\right)^2+\left(\sqrt{5}-2i\right)^2}{\left(\sqrt{5}+2i\right)\left(\sqrt{5}-2i\right)}$$

$$=\frac{5+4i\sqrt{5}-4+5-4i\sqrt{5}-4}{5+4}$$

$$=\frac{2}{9}$$

c

$$\frac{1}{(1-i)^2}$$

$$=\frac{1}{1-2i-1}$$

$$=\frac{1}{-2i}$$

$$=\frac{i}{2}$$

Question 21

$z^2 = 24 - 10i$

$(x + iy)^2 = 24 - 10i$

$x^2 - y^2 + 2ixy = 24 - 10i$

Equating real and imaginary parts

$x^2 - y^2 = 24$

$2xy = -10$

$y = \frac{-5}{x}$

$x^2 - \left(\frac{-5}{x}\right)^2 = 24$

$x^2 - \frac{25}{x^2} = 24$

$x^4 - 25 = 24x^2$

$x^4 - 24x^2 - 25 = 0$

$(x^2 + 1)(x^2 - 25) = 0$

Only take the real solutions as $x \in \mathbb{R}$

$x^2 = 25$

$x = \pm 5$

$y = \frac{-5}{\pm 5}$

$y = \mp 1$

$z = 5 - i, -5 + i$

Question 22

$z^2 = -48 + 14i$

$(x + iy)^2 = -48 + 14i$

$x^2 - y^2 + 2ixy = -48 + 14i$

Equating real and imaginary parts

$x^2 - y^2 = -48$

$2xy = 14$

$y = \frac{7}{x}$

$x^2 - \left(\frac{7}{x}\right)^2 = -48$

$x^2 - \frac{49}{x^2} = -48$

$x^4 - 49 = -48x^2$

$x^4 + 48x^2 - 49 = 0$

$(x^2 - 1)(x^2 + 49) = 0$

Only take the real solutions as $x \in \mathbb{R}$

$x^2 = 1$

$x = \pm 1$

$y = \frac{7}{\pm 1}$

$y = \pm 7$

$z = 1 + 7i, -1 - 7i$

Question 23

$x^2 - (1 + 2i)x + 1 + 7i$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{1 + 2i \pm \sqrt{(1 + 2i)^2 - 4 \times 1 \times (1 + 7i)}}{2}$

$x = \frac{1 + 2i \pm \sqrt{1 + 4i - 4 - 4 - 28i}}{2}$

$x = \frac{1 + 2i \pm \sqrt{-7 - 24i}}{2}$

$z^2 = -7 - 24i$

$\left(x + iy\right)^2 = -7 - 24i$

$x^2 - y^2 + 2ixy = -7 - 24i$

Equating real and imaginary parts

$x^2 - y^2 = -7$

$2xy = 24$

$y = \frac{-12}{x}$

$x^2 - \left(\frac{-12}{x}\right)^2 = -7$

$x^2 - \frac{144}{x^2} = -7$

$x^4 - 144 = -7x^2$

$x^4 + 7x^2 - 144 = 0$

$\left(x^2 + 16\right)\left(x^2 - 9\right) = 0$

Only take the real solutions as $x \in \mathbb{R}$

$x^2 = 9$

$x = \pm 3$

$y = \frac{-12}{\pm 3} = \pm 4$

$z = \pm\left(3 - 4i\right)$

$x = \frac{1 + 2i \pm \sqrt{-7 - 24i}}{2}$

$x = \frac{1 + 2i \pm \left(3 - 4i\right)}{2}$

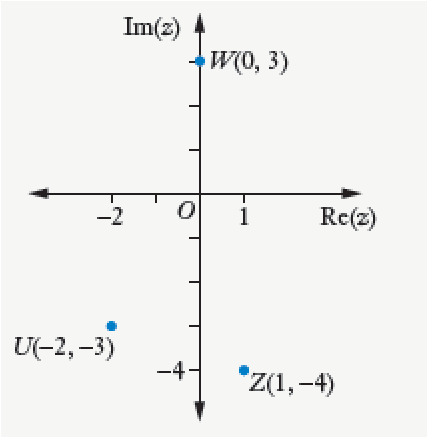
$x = \frac{1 + 2i + 3 - 4i}{2}\ \ \frac{1 + 2i - 3 + 4i}{2}$

$x = \frac{4 - 2i}{2}\ \ \frac{-2 + 6i}{2}$

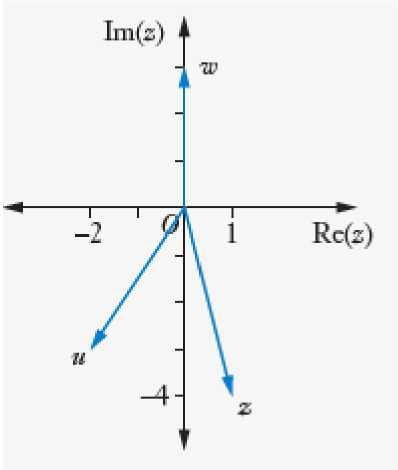
$x = 2 - i, -1 + 3i$

Question 24

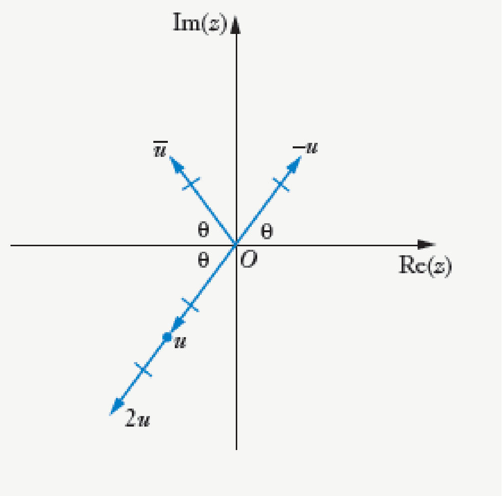
i



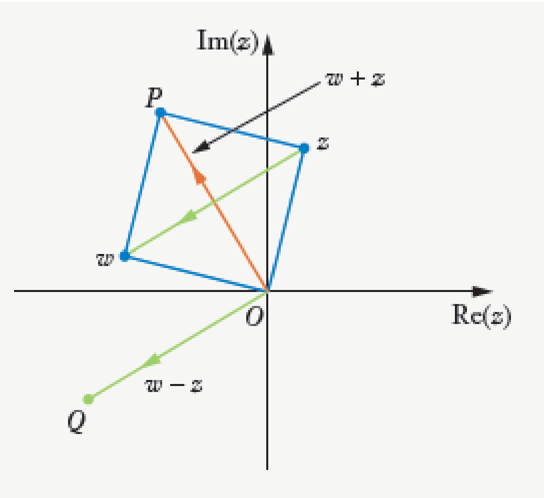
ii



Question 25



Question 26



Question 27

a $z = -1 + i\sqrt{3}$

i

$$|z| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2}$$
$$= \sqrt{4} = 2$$

ii

$$\arg(z) = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$$
$$= \frac{2\pi}{3}$$

Which is in the 2nd quadrant as required.

b $z = 2 - 2i$

i

$$|z| = \sqrt{2^2 + (-2)^2}$$
$$= \sqrt{8}$$
$$= 2\sqrt{2}$$

ii

$$\arg(z) = \tan^{-1}\left(\frac{-2}{2}\right)$$

$$= \frac{3\pi}{4}$$

As z is in the 4th quadrant

$$\arg(z) = -\frac{\pi}{4}$$

c

$$z = \frac{-\sqrt{6} - i\sqrt{2}}{2}$$

i

$$|z| = \sqrt{\left(\frac{-\sqrt{6}}{2}\right)^2 + \left(\frac{-\sqrt{2}}{2}\right)^2}$$

$$= \sqrt{\frac{6}{4} + \frac{2}{4}}$$

$$= \sqrt{2}$$

ii

$$\arg(z) = \tan^{-1}\left(\frac{\frac{-\sqrt{2}}{2}}{\frac{-\sqrt{6}}{2}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6}$$

As z is in the third quadrant

$$\arg(z) = \pi + \frac{\pi}{6}$$

$$= \frac{7\pi}{6}$$

$$= -\frac{5\pi}{6}$$

Question 28

a $3\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)=3\text{cis}\left(-\frac{\pi}{3}\right)$

b

$$\sqrt{2}\left(\sin\frac{3\pi}{4}+i\cos\frac{3\pi}{4}\right)$$
$$=\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)=\sqrt{2}\ \text{cis}\left(-\frac{\pi}{4}\right)$$

c $5i=5\ \text{cis}\left(\frac{\pi}{2}\right)$

d

$$z=-2+2i\sqrt{3}$$
$$|z|=\sqrt{(-2)^2+\left(2\sqrt{3}\right)^2}$$
$$=\sqrt{4+12}=\sqrt{16}=4$$
$$\arg(z)=\tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right)=\frac{2\pi}{3}$$
$$z=4\ \text{cis}\left(\frac{2\pi}{3}\right)$$

e

$$z=\frac{1+i}{3}$$
$$|z|=\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}=\sqrt{\frac{2}{9}}=\frac{\sqrt{2}}{3}$$
$$\arg(z)=\tan^{-1}\left(\frac{\frac{1}{3}}{\frac{1}{3}}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$
$$z=\frac{\sqrt{2}}{3}\text{cis}\left(\frac{\pi}{4}\right)$$

Question 29

a

$$u = \sqrt{8}\ \text{cis}\left(\frac{3\pi}{4}\right)$$

$$v = 2\ \text{cis}\left(-\frac{\pi}{3}\right)$$

b

$$\begin{aligned} u &= \sqrt{8}\ \text{cis}\left(\frac{3\pi}{4}\right) \\ &= \sqrt{8}\cos\left(\frac{3\pi}{4}\right) + i\sqrt{8}\sin\left(\frac{3\pi}{4}\right) \\ &= \sqrt{8}\left(-\frac{1}{\sqrt{2}}\right) + i\sqrt{8}\left(\frac{1}{\sqrt{2}}\right) \\ &= -2 + 2i \end{aligned}$$

$$\begin{aligned} v &= 2\ \text{cis}\left(-\frac{\pi}{3}\right) \\ &= 2\cos\left(-\frac{\pi}{3}\right) + 2i\sin\left(-\frac{\pi}{3}\right) \\ &= 2\times\frac{1}{2} + 2i\times\frac{\sqrt{3}}{2} \\ &= 1 - i\sqrt{3} \end{aligned}$$

Question 30

a $r_1 \text{cis}\,\alpha_1 \times r_2 \text{cis}\,\alpha_2 = r_1 r_2 \text{cis}(\alpha_1 + \alpha_2)$

b

$$\frac{r_1 \text{cis}\,\alpha_1}{r_2 \text{cis}\,\alpha_2} = \frac{r_1}{r_2}\text{cis}(\alpha_1 - \alpha_2)$$

Question 31

a $\arg\left(\text{cis}\,\theta\right)^n = n\theta$

b

$$\arg\left(\cos\theta - i\sin\theta\right)^n$$
$$= \arg\left(\text{cis}\left(-\theta\right)\right)^n$$
$$= -n\theta$$

c $\arg\left(\text{cis}\,\theta\right)^{-n} = -n\theta$

d

$$\arg\left(\cos\theta - i\sin\theta\right)^{-n}$$
$$= \arg\left(\text{cis}\left(-\theta\right)\right)^{-n}$$
$$= n\theta$$

Question 32

$$z_1 = r_1\,\text{cis}\,\alpha_1$$
$$z_2 = r_2\,\text{cis}\,\alpha_2$$
$$\left|z_1\right| = r_1$$
$$\left|z_2\right| = r_2$$
$$\frac{\left|z_1\right|}{\left|z_2\right|} = \frac{r}{r_2}$$
$$\frac{z_1}{z_2} = \frac{r_1\,\text{cis}\,\alpha_1}{r_2\,\text{cis}\,\alpha_2}$$
$$= \frac{r}{r_2}\text{cis}\left(\alpha_1 - \alpha_2\right)$$
$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{\left|z_1\right|}{\left|z_2\right|}$$

Question 33

$z_1 = 3\,\text{cis}\frac{5\pi}{6}$

$z_2 = 2\,\text{cis} - \frac{\pi}{3}$

a

$$z_1 z_2 = \left(3\,\text{cis}\frac{5\pi}{6}\right) \times \left(2\,\text{cis} - \frac{\pi}{3}\right)$$
$$= 6\,\text{cis}\left(\frac{5\pi}{6} + \frac{-\pi}{3}\right)$$
$$= 6\,\text{cis}\left(\frac{3\pi}{6}\right)$$
$$= 6\,\text{cis}\left(\frac{\pi}{2}\right) = 6i$$

b

$$\frac{z_1}{z_2} = \frac{3\,\text{cis}\frac{5\pi}{6}}{2\,\text{cis} - \frac{\pi}{3}}$$
$$= \frac{3}{2}\,\text{cis}\left(\frac{5\pi}{6} - \frac{-\pi}{3}\right)$$
$$= \frac{3}{2}\,\text{cis}\left(\frac{7\pi}{6}\right)$$
$$= \frac{3}{2}\,\text{cis}\left(\frac{-5\pi}{6}\right)$$

c

$$(z_2)^3 = \left(2\,\text{cis}\frac{-\pi}{3}\right)^3$$
$$= 2^3\,\text{cis}\left(\frac{-\pi}{3} \times 3\right)$$
$$= 8\,\text{cis}(-\pi)$$
$$= -8$$

d

$$\left(z_1\right)^{-4} = \left(3\operatorname{cis}\frac{5\pi}{6}\right)^{-4}$$
$$= (3)^{-4}\operatorname{cis}\left(\frac{5\pi}{6}\times(-4)\right)$$
$$= \frac{1}{81}\operatorname{cis}\left(\frac{-10\pi}{3}\right)$$
$$= \frac{1}{81}\operatorname{cis}\left(\frac{-4\pi}{3}\right)$$
$$= \frac{1}{81}\operatorname{cis}\left(\frac{2\pi}{3}\right)$$

Question 34

a

$$z = (1-i)^8$$
$$= \left(\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)\right)^8$$
$$= \left(\sqrt{2}\right)^8\operatorname{cis}\left(-\frac{\pi}{4}\times 8\right)$$
$$= 16\operatorname{cis}(-2\pi)$$
$$= 16$$

b

$$\frac{\left(1+i\sqrt{3}\right)^2}{\sqrt{2}-i\sqrt{2}}$$

$$z\ =1+i\sqrt{3}$$

$$=2\operatorname{cis}\left(\frac{\pi}{3}\right)$$

$$z_2=\sqrt{2}-i\sqrt{2}$$

$$=2\operatorname{cis}\left(-\frac{\pi}{4}\right)$$

$$\frac{{z_1}^2}{z_2}=\frac{\left(2\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^2}{2\operatorname{cis}\left(-\frac{\pi}{4}\right)}$$

$$=\frac{4}{2}\left(\operatorname{cis}\left(\frac{2\pi}{3}-\frac{-\pi}{4}\right)\right)$$

$$=2\operatorname{cis}\left(\frac{11\pi}{12}\right)$$

c

$$\frac{1}{\left(\sqrt{3}+i\right)^4}$$

$$=\left(\sqrt{3}+i\right)^{-4}$$

$$=\left(2\operatorname{cis}\left(\frac{\pi}{6}\right)\right)^{-4}$$

$$=\frac{1}{16}\operatorname{cis}\left(-\frac{4\pi}{6}\right)$$

$$=\frac{1}{16}\operatorname{cis}\left(-\frac{2\pi}{3}\right)$$

Question 35

$(1+i)(\sqrt{3}+i)=\sqrt{3}-1+i(1+\sqrt{3})$

$1+i=\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

$\sqrt{3}+i=2\,\text{cis}\left(\frac{\pi}{6}\right)$

$(1+i)(\sqrt{3}+i)=\left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right)\left(2\,\text{cis}\left(\frac{\pi}{6}\right)\right)$

$=2\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$

$=2\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)$

Equating imaginary coefficients

$2\sqrt{2}\sin\left(\frac{5\pi}{12}\right)=1+\sqrt{3}$

$\sin\left(\frac{5\pi}{12}\right)=\frac{1+\sqrt{3}}{2\sqrt{2}}$

Question 36

a $\text{cis}\,3=e^{3i}$

b $4\left(\cos\left(\frac{\pi}{5}\right)-i\sin\left(\frac{\pi}{5}\right)\right)=4\,\text{cis}\left(-\frac{\pi}{5}\right)=4e^{\frac{-\pi}{5}}$

c

$z=-\sqrt{3}-i$

$r=2$

$\theta=\tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right)=\frac{\pi}{6}$

As θ is in the third quadrant

$\theta=\pi+\frac{\pi}{6}=\frac{7\pi}{6}=-\frac{5\pi}{6}$

$z=2\,\text{cis}\left(-\frac{5\pi}{6}\right)$

$=2e^{\frac{-5\pi}{6}}$

Question 37

a $2e^{3i\alpha} = 2\,\text{cis}(3\alpha)$

b $e^{\frac{-\pi}{7}} = \text{cis}\left(-\frac{\pi}{7}\right)$

c $-\frac{1}{2}e^{\frac{i\pi}{3}} = -\frac{1}{2}\text{cis}\left(\frac{\pi}{3}\right) = \frac{1}{2}\text{cis}\left(-\frac{2\pi}{3}\right)$

Question 38

a

$$2e^{\frac{i}{2}} \times 3e^{2i}$$
$$= 6e^{\frac{i}{2}+2i}$$
$$= 6e^{\frac{5i}{2}}$$

b

$$\frac{e^{i\pi} \times (-1)}{e^{\frac{i\pi}{5}}}$$
$$= \frac{(-1)\times(-1)}{e^{\frac{i\pi}{5}}}$$
$$= e^{\frac{-\pi}{5}}$$

Question 39

$$z = e^{i\theta}$$
$$w = e^{i\alpha}$$
$$\arg(z) = \theta$$
$$\arg(w) = \alpha$$
$$\arg(z) + \arg(w) = \theta + \alpha$$
$$zw = e^{i\theta} \times e^{i\alpha} = e^{i\theta + i\alpha}$$
$$\arg(zw) = \theta + \alpha = \arg(z) + \arg(w)$$

Question 40

If it rains $\Rightarrow$ The dam is full

Question 41

If the people are starving then there is not enough food

Question 42

a If n is even $\Rightarrow$ n is divisible by 2

Converse

If n is divisible by 2 $\Rightarrow$ n is even

True

It is an equivalence.

b If n is positive $\Rightarrow \frac{1}{n}$ is positive

Converse

If $\frac{1}{n}$ is positive $\Rightarrow$ n is positive

True

It is an equivalence.

c If a quadrilateral has 4 equal angles $\Rightarrow$ it is a rectangle

Converse

If a quadrilateral is a rectangle $\Rightarrow$ it has 4 equal angles

True

It is an equivalence.

d If an animal is a kangaroo $\Rightarrow$ it eats grass

Converse

If an animal eats grass $\Rightarrow$ it is a kangaroo

False

Question 43

a The dam is full.

Negative

The dam is not full.

b The teacher is good.

Negative

The teacher is not good.

c All cats are fluffy.

Negative

There is at least one cat that is not fluffy.

d There is at least one smart politician.

Negative

There are no smart politicians.

e No wine is sweet.

Negative

There is at least one wine that is sweet.

f Some sheep are black

Negative

No sheep are black.

Question 44

a If you get a speeding ticket, then you speed.

Contrapositive

If you do not speed you then you do not get a speeding ticket.

True

b If you get the old-age pension, then you are over 65.

Contrapositive

If you are not over 65 you do not get the old-age pension.

True

c If a triangle is equilateral, then it has 3 equal sides.

Contrapositive

If a triangle does not have 3 equal sides then it is not equilateral.

True

d If you go swimming, then you get wet.

Contrapositive

If you do not get wet then you do not go swimming.

True

Question 45

$\forall x \in \mathbb{N}, \exists y \in \mathbb{N} : y = 2x$

For every natural number x, there exists a natural number y such that $y = 2x$.

Question 46

For every natural number x such that x is a multiple of 4,

there exists a natural number y such that $\sqrt{x} = 2\sqrt{y}$

$(\forall x \in \mathbb{N} : x \text{ is divisible by } 4), \exists y \in \mathbb{N} : \sqrt{x} = 2\sqrt{y}$

Question 47

C

Question 48

Suppose $a, b \in \mathbb{N}$ with a and b having no common factors

$\frac{a}{b} = \sqrt{10}$

$\Rightarrow \left(\frac{a}{b}\right)^2 = 10$

$\Rightarrow \frac{a^2}{b^2} = 10$

$\Rightarrow a^2 = 10b^2$

$\Rightarrow a^2$ is even

$\Rightarrow a$ is even

$\therefore$ we can write $a = 2m$

$\frac{a^2}{b^2} = 10$

$\Rightarrow \frac{(2m)^2}{b^2} = 10$

$\Rightarrow \frac{4m^2}{b^2} = 10$

$\Rightarrow 4m^2 = 10b^2$

$\Rightarrow 2m^2 = 5b^2$

$\Rightarrow b^2$ is even

$\Rightarrow b$ is even

$\therefore$ we can write $b = 2n$

This is a contradiction as both a and b have no common factors

$\therefore \sqrt{10} \notin \mathbb{Q}$

Question 49

Let $y = f(x)$ and $f''(p) = 0$

$f(x) = (x-p)^6$

$f'(x) = 6(x-p)^5$

$f''(x) = 30(x-p)^4$

But p is not a point of inflection but a turning point.

Question 50

a

$M, N \in \mathbb{N}, M > N > 0$

If M and N are even we can write

$M = 2m$

$N = 2n$

$M^2 - N^2$

$= (2m)^2 - (2n)^2$

$= 4m^2 - 4n^2$

$= 4(m^2 - n^2)$

$= 2 \times 2(m^2 - n^2)$

As $m, n, m^2 - n^2 \in \mathbb{N}$

$2(m^2 - n^2)$ is even

$\therefore M^2 - N^2$ is even

b

$M, N \in \mathbb{N}, M > N > 0$
If M and N are odd we can write
$M = 2m+1$
$N = 2n+1$
$M^2 - N^2$
$= (2m+1)^2 - (2n+1)^2$
$= 4m^2 + 4m + 1 - (4n^2 + 4n + 1)$
$= 4(m^2 + m - n^2 - n)$
$= 2 \times 2(m^2 + m - n^2 - n)$
As $m, n, m^2 + m - n^2 - n \in \mathbb{N}$
$2(m^2 + m - n^2 - n)$ is even
$\therefore M^2 - N^2$ is even

c

$M, N \in \mathbb{N}, M > N > 0$
If M is even and N are odd we can write
$M = 2m$
$N = 2n+1$
$M^2 - N^2$
$= (2m)^2 - (2n+1)^2$
$= 4m^2 - (4n^2 + 4n + 1)$
$= 4(m^2 + m - n^2 - n) + 1$
$= 2 \times 2(m^2 + m - n^2 - n) + 1$
As $m, n, m^2 + m - n^2 - n \in \mathbb{N}$
$2(m^2 + m - n^2 - n)$ is even
So $2 \times 2(m^2 + m - n^2 - n) + 1$ is odd
$\therefore M^2 - N^2$ is odd

Question 51

a

$$\forall x, y \in \mathbb{R} : x, y > 0$$

Required to prove

$$\frac{x^2 + y^2}{2} > xy$$

$$(x - y)^2 > 0$$

$$\Rightarrow x^2 - 2xy + y^2 > 0$$

$$\Rightarrow x^2 + y^2 > 2xy$$

$$\Rightarrow \frac{x^2 + y^2}{2} > xy$$

b

$$\forall x \in \mathbb{R} : x > 0$$

Required to prove

$$x + \frac{1}{x} > 2$$

$$\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 > 0$$

$$\Rightarrow x - 2 + \frac{1}{x} > 0$$

$$\Rightarrow x + \frac{1}{x} > 2$$

c

$$\frac{\sqrt{ab} + \sqrt{cd}}{2} \geq \sqrt[4]{abcd}$$

$$\Rightarrow \frac{\sqrt{ab}}{2} + \frac{\sqrt{cd}}{2} \geq \sqrt[4]{abcd}$$

From AM-GM inequality

$$\frac{a+b}{2} \geq \sqrt{ab} \text{ and } \frac{c+d}{2} \geq \sqrt{cd}$$

$$\therefore \frac{\sqrt{ab}}{2} + \frac{\sqrt{cd}}{2} \geq \sqrt[4]{abcd}$$

$$\Rightarrow \frac{\frac{a+b}{2}}{2} + \frac{\frac{c+d}{2}}{2} \geq \sqrt[4]{abcd}$$

$$\Rightarrow \frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$$

Question 52

$\underset{\sim}{u} = 3i - 2j$

$\underset{\sim}{v} = 4i + 6j$

$\underset{\sim}{u}.\underset{\sim}{v} = 3 \times 4 + (-2) \times 6 = 0$

The vectors are perpendicular.

Question 53

a

$A = (1, -2, 4)$

$B = (3, 1, 2)$

$\overrightarrow{AB} = -(1, -2, 4) + (3, 1, 2) = (2, 3, -2)$

b $\left|\overrightarrow{AB}\right| = \sqrt{2^2 + 3^2 + (-2)^2} = \sqrt{17}$

c $\hat{u} = \frac{1}{\sqrt{17}}(2, 3, -2)$

Question 54

$\underset{\sim}{p} = 2i + 5j + k$

$\underset{\sim}{q} = -7i + j + nk$

$\underset{\sim}{p}$ and $\underset{\sim}{q}$ are orthoganal if

$\underset{\sim}{p}.\underset{\sim}{q} = 0$

$2 \times (-7) + 5 \times 1 + 1 \times n = 0$

$-14 + 5 + n = 0$

$-9 + n = 0$

$n = 9$

Question 55

a

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (2,5,8) - (-2,3,4)$$
$$= (4,2,4)$$
$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$$
$$= (1,-3,2) - (3,-2,4)$$
$$= (-2,-1,-2)$$
$$\overrightarrow{CD} = -2\overrightarrow{AB}$$

$\therefore \overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel

b

$$\overrightarrow{AB} = (4,2,4)$$
$$\left|\overrightarrow{AB}\right| = \sqrt{4^2 + 2^2 + 4^2}$$
$$= \sqrt{36}$$
$$= 6$$
$$\overrightarrow{CD} = (-2,-1,-2)$$
$$\left|\overrightarrow{CD}\right| = \sqrt{(-2)^2 + (-1)^2 + (-2)^2}$$
$$= \sqrt{9}$$
$$= 3$$

c *ABCD* is a trapezium.

Question 56

$$\underset{\sim}{u} = \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}$$

$$|\underset{\sim}{u}| = \sqrt{(-2)^2 + 4^2 + (-3)^2} = \sqrt{29}$$

$$\underset{\sim}{v} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$|\underset{\sim}{v}| = \sqrt{5^2 + (-1)^2 + 3^2} = \sqrt{29}$$

$$\underset{\sim}{u} \bullet \underset{\sim}{v} = 5 \times (-2) + (-1) \times 4 + 3 \times (-3) = -23$$

$$\cos\theta = \frac{\underset{\sim}{u} \bullet \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$$

$$\cos\theta = \frac{-23}{\sqrt{35}\sqrt{29}}$$

$$\theta = 180° - 43°47' = 136°13'$$

Question 57

$$F = (1, 3, -2)$$

$$G = (4, -2, 7)$$

$$\overrightarrow{FG} = \overrightarrow{OG} - \overrightarrow{OF}$$

$$= (4, -2, 7) - (1, 3, -2)$$

$$= (3, -5, 9)$$

The vector equation of the line is

$$(1, 3, -2) + \lambda(3, -5, 9)$$

Question 58

$P = (3, -1, 3)$

$Q = (4, 5, 1)$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\frac{x - 3}{4 - 3} = \frac{y + 1}{5 - (-1)} = \frac{z - 3}{1 - 3}$$

$$\frac{x - 3}{1} = \frac{y + 1}{6} = \frac{z - 3}{-2}$$

$$x - 3 = \frac{y + 1}{6} = \frac{-z + 3}{2}$$

Question 59

$K(-5, 18, 1)$ lies on

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 7 \\ 1 \end{pmatrix}$$

True if

$1 - 3\lambda = -5$

$4 + 7\lambda = 18$

$-2 + \lambda = 1$

$1 - 3\lambda = -5$

$6 = 3\lambda$

$\lambda = 2$

$4 + 7\lambda = 18$

$7\lambda = 14$

$\lambda = 2$

$-2 + \lambda = 1$

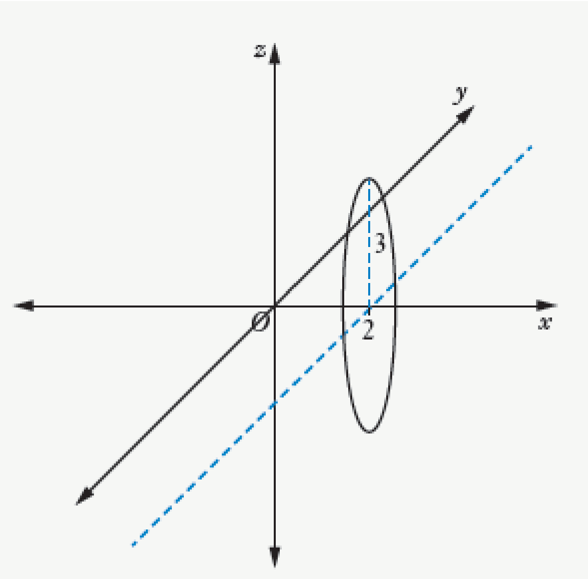
$\lambda = 3$

It is not on the line.

Question 60

$$\frac{x-2}{6}=\frac{y-1}{2}=\frac{z-4}{3}$$
$$=(2,1,4)+\lambda(6,2,3)$$

Question 61



Question 62

Requiredto prove: $(\text{cis }\theta)^n = \text{cis }(n\theta)$

For n = 1

$(\text{cis}\,\theta) = \text{cis}(\theta)$

Let the kth term be true

$(\text{cis}\,\theta)^k = \text{cis}(k\theta)$

The $(k+1)$th term

$(\text{cis}\,\theta)^{k+} = (\text{cis}\,\theta)(\text{cis}\,\theta)^k$

$= (\text{cis}\,\theta)(\text{cis }k\theta)$

$= \cos\theta\cos k\theta + \cos\theta i\sin k\theta + i\sin\theta\cos k\theta + i\sin\theta i\sin k\theta$

$= \cos\theta\cos k\theta - \sin\theta\sin k\theta + i(\cos\theta\sin k\theta + \sin\theta\cos k\theta)$

$= \cos(\theta + k\theta) + i\sin(\theta + k\theta)$

$= \cos\theta(1+k) + i\sin\theta(1+k)$

Therefore the $(k+1)$h term is true.

If the first term is true and given the kth term is true then the $(\text{k}+1)$th term is true, by mathematical inducton it is true $\forall n \in \mathbb{N}$

Question 63

a

$$\left(\sqrt{2}\,\text{cis}(3\beta)\right)^5 = \left(\sqrt{2}\right)^5 \text{cis}(5\times 3\beta)$$

$$= 4\sqrt{2}\ \text{cis}(15\beta)$$

b

$$\left(512\,\text{cis}(-144^\circ)\right)^{\frac{1}{9}} = (512)^{\frac{1}{9}}\,\text{cis}\left(\frac{-144^\circ}{9}\right)$$

$$= 2\,\text{cis}(16^\circ)$$

c

$$\left(\frac{1}{2}\text{cis}\left(\frac{-2\pi}{3}\right)\right)^8 = \left(\frac{1}{2}\right)^8 \text{cis}\left(8\times\frac{-2\pi}{3}\right)$$

$$= \frac{1}{256}\text{cis}\left(-\frac{16\pi}{3}\right)$$

$$= \frac{1}{256}\text{cis}\left(\frac{-4\pi}{3}\right)$$

$$= \frac{1}{256}\text{cis}\left(\frac{2\pi}{3}\right)$$

Question 64

a

$$(\operatorname{cis}\theta)^6 = \operatorname{cis}(6\theta)$$

$$(\cos\theta + i\sin\theta)^6$$

$$= \cos^6\theta + 6\cos^5\theta i\sin\theta - 15\cos^4\theta\sin^2\theta - 20\cos^3\theta i\sin^3\theta$$
$$+15\cos^2\theta\sin^4\theta + 6\cos\theta i\sin^5\theta - \sin^6\theta$$

Equating real coefficients

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$
$$= \cos^6\theta - 15\cos^4\theta\left(1-\cos^2\theta\right) + 15\cos^2\theta\left(1-\cos^2\theta\right)\left(1-\cos^2\theta\right)$$
$$-\left(1-\cos^2\theta\right)\left(1-\cos^2\theta\right)\left(1-\cos^2\theta\right)$$
$$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta\left(1-2\cos^2\theta+\cos^4\theta\right)$$
$$-\left(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta\right)$$
$$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

b

$$\cos 6\theta = 0$$

$$6\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \ldots$$

$$\theta = \frac{\pi}{12}, \frac{3\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}, \frac{11\pi}{12}, \ldots \text{ or } \frac{(2k-1)\pi}{12} \text{ for integer } k$$

c

Let $32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = 0$

Roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$

$$32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$
$$= (\cos\theta - \alpha_1)(\cos\theta - \alpha_2)(\cos\theta - \alpha_3)(\cos\theta - \alpha_4)(\cos\theta - \alpha_5)(\cos\theta - \alpha_6)$$

6 unique roots are $\cos\left(\frac{\pi}{12}\right), \cos\left(\frac{3\pi}{12}\right), \cos\left(\frac{5\pi}{12}\right), \cos\left(\frac{7\pi}{12}\right), \cos\left(\frac{9\pi}{12}\right), \cos\left(\frac{11\pi}{12}\right)$

$-\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{11\pi}{12}\right), -\cos\left(\frac{3\pi}{12}\right) = \cos\left(\frac{9\pi}{12}\right), -\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{7\pi}{12}\right)$

$$\left(\cos\theta - \cos\left(\frac{\pi}{12}\right)\right)\left(\cos\theta - \cos\left(\frac{3\pi}{12}\right)\right)\left(\cos\theta - \cos\left(\frac{5\pi}{12}\right)\right)$$
$$\left(\cos\theta - \cos\left(\frac{7\pi}{12}\right)\right)\left(\cos\theta - \cos\left(\frac{9\pi}{12}\right)\right)\left(\cos\theta - \cos\left(\frac{11\pi}{12}\right)\right)$$
$$= \left(\cos\theta - \cos\left(\frac{\pi}{12}\right)\right)\left(\cos\theta + \cos\left(\frac{\pi}{12}\right)\right)\left(\cos\theta - \cos\left(\frac{3\pi}{12}\right)\right)\left(\cos\theta + \cos\left(\frac{3\pi}{12}\right)\right)$$
$$\left(\cos\theta - \cos\left(\frac{5\pi}{12}\right)\right)\left(\cos\theta + \cos\left(\frac{5\pi}{12}\right)\right)$$
$$= \left(\cos^2\theta - \cos^2\left(\frac{\pi}{12}\right)\right)\left(\cos^2\theta - \cos^2\left(\frac{3\pi}{12}\right)\right)\left(\cos^2\theta - \cos^2\left(\frac{5\pi}{12}\right)\right)$$
$$= \left(\cos^2\theta - \cos^2\left(\frac{\pi}{12}\right)\right)\left(\cos^2\theta - \frac{1}{2}\right)\left(\cos^2\theta - \cos^2\left(\frac{5\pi}{12}\right)\right)$$

Question 65

a

$$z = cis\theta$$

$$z - \frac{1}{z} = \text{cis}\,\theta - \text{cis}(-\theta)$$

$$= \cos\theta + i\sin\theta - (\cos(-\theta) + i\sin(-\theta))$$

$$= \cos\theta + i\sin\theta - \cos\theta + i\sin\theta$$

$$= 2i\sin\theta$$

b

$$\left(z - \frac{1}{z}\right)^5 = (2i\sin\theta)^5$$

$$= 32i\sin^5\theta$$

$$\left(z - \frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}$$

$$= z^5 - z^{-5} - 5(z^3 - z^{-3}) + 10(z - z^{-1})$$

$$= \left(\text{cis}(5\theta) - \frac{1}{\text{cis}(5\theta)}\right) - 5\left(\text{cis}(3\theta) - \frac{1}{\text{cis}(3\theta)}\right) + 10\left(\text{cis}(\theta) - \frac{1}{\text{cis}(\theta)}\right)$$

$$= 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$

$$\therefore 32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$

$$32\sin^5\theta = 2\sin 5\theta - 10\sin 3\theta + 20\sin\theta$$

$$\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$$

$$A = \frac{1}{16}, B = \frac{5}{16}, C = \frac{5}{8}$$

c

$$\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$$
$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{16}\sin 5x - \frac{5}{16}\sin 3x + \frac{5}{8}\sin x \right) dx$$
$$= \frac{1}{16}\int_0^{\frac{\pi}{2}} \left(\sin 5x - 5\sin 3x + 10\sin x \right) dx$$
$$= \frac{1}{16}\left[-\frac{1}{5}\cos 5x + \frac{5}{3}\cos 3x - 10\cos x \right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{16}\left[\left(-\frac{1}{5}\cos\frac{5\pi}{2} + \frac{5}{3}\cos\frac{3\pi}{2} - 10\cos\frac{\pi}{2} \right) - \left(-\frac{1}{5}\cos 0 + \frac{5}{3}\cos 0 - 10\cos 0 \right) \right]$$
$$= \frac{1}{16}\left[(0) - \left(-\frac{1}{5} + \frac{5}{3} + 10 \right) \right]$$
$$= \frac{1}{16} \times \frac{128}{15}$$
$$= \frac{8}{15}$$

Question 66

$$z^2 - 4iz - 12 = 0$$
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$z = \frac{4i \pm \sqrt{(4i)^2 - 4\times 1\times(-12)}}{2}$$
$$z = \frac{4i \pm \sqrt{32}}{2}$$
$$z = \frac{4i \pm 4\sqrt{2}}{2}$$
$$z = 2i \pm 2\sqrt{2}$$

Question 67

$z^2 = -1 + i\sqrt{3}$

$(x+iy)^2 = -1 + i\sqrt{3}$

$x^2 - y^2 + 2ixy = -1 + i\sqrt{3}$

Equating real and imaginary parts

$x^2 - y^2 = -1$

$2xy = \sqrt{3}$

$y = \dfrac{\sqrt{3}}{2x}$

$x^2 - \left(\dfrac{\sqrt{3}}{2x}\right)^2 = -1$

$x^2 - \dfrac{3}{4x^2} = -1$

$x^4 - \dfrac{3}{4} = -x^2$

$x^4 + x^2 - \dfrac{3}{4} = 0$

$4x^4 + 4x^2 - 3 = 0$

$(2x^2 - 1)(2x^2 + 3) = 0$

Only take the real solutions as $x \in \mathbb{R}$

$2x^2 = 1$

$x^2 = \dfrac{1}{2}$

$x = \pm\dfrac{1}{\sqrt{2}}$

$y = \dfrac{\sqrt{3}}{2\left(\pm\dfrac{1}{\sqrt{2}}\right)}$

$y = \pm\dfrac{\sqrt{3}}{\sqrt{2}}$

$z = \pm\left(\dfrac{1}{\sqrt{2}} + \dfrac{\sqrt{3}}{\sqrt{2}}i\right)$

$z = \dfrac{1}{\sqrt{2}} + \dfrac{\sqrt{3}}{\sqrt{2}}i = \sqrt{2}\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right) = \sqrt{2}\left[\cos\left(\dfrac{\pi}{3}\right) + i\sin\left(\dfrac{\pi}{3}\right)\right] = \sqrt{2}\,\mathrm{cis}\left(\dfrac{\pi}{3}\right)$

$z = -\dfrac{1}{\sqrt{2}} - \dfrac{\sqrt{3}}{\sqrt{2}}i = \sqrt{2}\left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right) = \sqrt{2}\left[\cos\left(-\dfrac{2\pi}{3}\right) + i\sin\left(-\dfrac{2\pi}{3}\right)\right] = \sqrt{2}\,\mathrm{cis}\left(-\dfrac{2\pi}{3}\right)$

Question 68

a Given a polynomial equation $P(z) = 0$, by the fundamental theorem of algebra $P(z)$ can be factorised into a product of linear and quadratic functions all with real coefficients. If the highest power is odd, then there must be at least one linear function and hence one real root.

b

$$P(x) = x^5 + 2x^3 - x^2 - 2$$

$$P(i\sqrt{2}) = (i\sqrt{2})^5 + 2(i\sqrt{2})^3 - (i\sqrt{2})^2 - 2$$

$$= i4\sqrt{2} - 2\times 2i\sqrt{2} - (-2) - 2$$

$$= 0$$

$\therefore i\sqrt{2}$ is a root

c

As $i\sqrt{2}$ is a root of a polynomial with real coefficients then the conjugate $-i\sqrt{2}$ is also a root

$$(x + i\sqrt{2})(x - i\sqrt{2}) = x^2 + 2$$

$$\begin{array}{r} x^3 - 1 \\ x^2 + 2 \overline{\smash{\big)} x^5 + 0x^4 + 2x^3 - x^2 + 0x - 2} \\ \underline{x^5 + 0x^4 + 2x^3} \qquad\qquad\quad \\ 0 - x^2 + 0x - 2 \\ \underline{-x^2 + 0x - 2} \\ 0 \end{array}$$

$$P(x) = x^5 + 2x^3 - x^2 - 2$$

$$= (x^2 + 2)(x^3 - 1)$$

$$= (x - 1)(x^2 + 2)(x^2 + x + 1)$$

$$x^2 + x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1 - 4}}{2}$$

$$x = \frac{-1 \pm i\sqrt{3}}{2}$$

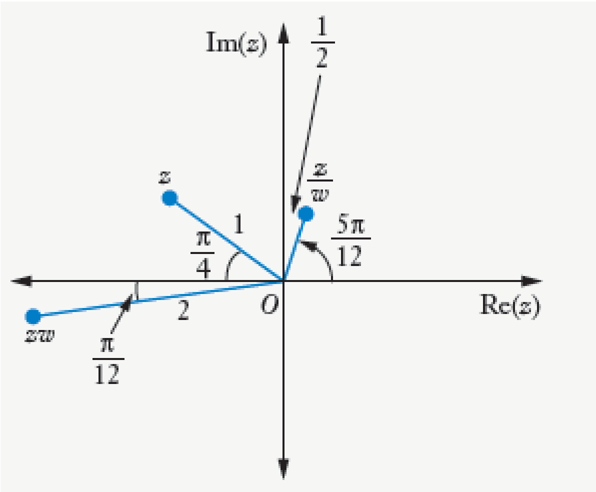
$\therefore \pm i\sqrt{2},\ \dfrac{-1 \pm i\sqrt{3}}{2}, 1$ are the roots

d

$$P(x) = x^5 + 2x^3 - x^2 - 2$$
$$= (x^2 + 2)(x^3 - 1)$$
$$= (x - 1)(x^2 + 2)(x^2 + x + 1)$$

Question 69

a



$$zw = \text{cis}\left(\frac{3\pi}{4}\right) \times 2\,\text{cis}\left(\frac{\pi}{3}\right)$$
$$= 2\,\text{cis}\left(\frac{3\pi}{4} + \frac{\pi}{3}\right)$$
$$= 2\,\text{cis}\left(\frac{13\pi}{12}\right)$$
$$= 2\,\text{cis}\left(-\frac{11\pi}{12}\right)$$

b

$$\frac{z}{w} = \frac{\text{cis}\left(\frac{3\pi}{4}\right)}{2\,\text{cis}\left(\frac{\pi}{3}\right)}$$
$$= \frac{1}{2}\,\text{cis}\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)$$
$$= \frac{1}{2}\,\text{cis}\left(\frac{5\pi}{12}\right)$$

Question 70

$u = iz$

$w = i^2 z = -z$

$v = -iz = \frac{z}{i}$

Question 71

a

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$$
$$d = a + \overrightarrow{BC}$$
$$= a + c - b$$

b

$\arg\left(\frac{a-b}{c-b}\right)$ is the angle between the lines CB and BA, and is given $\frac{\pi}{4}$.

As $ABCD$ is a parallelogram, the angle between BA and AD must be supplementary.

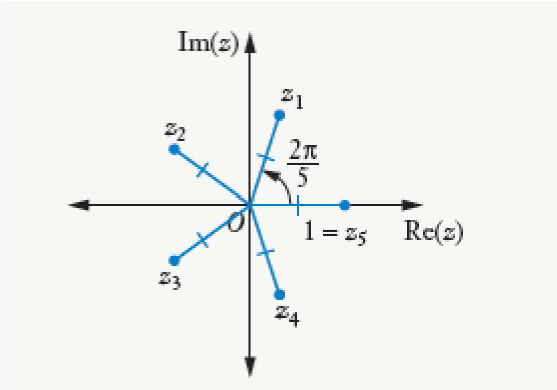
$\arg\left(\frac{d-a}{b-a}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

c

$$M = \frac{1}{2}\left(\overrightarrow{BA} + \overrightarrow{AD}\right)$$
$$= \frac{1}{2}(a - b + d - a)$$
$$= \frac{1}{2}(d - b)$$
$$= \frac{1}{2}(a + c - b - b)$$
$$= \frac{1}{2}(a + c - 2b)$$

Question 72

a



The fifth roots of unity must be equally spaced around the unit circle

$$z = 1, \text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right)$$

b

Let α be a root and $\alpha \neq 1$

$$z^5 - 1 = 0$$

$$(z-1)\left(z^4 + z^3 + z^2 + z + 1\right) = 0$$

$$\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$$

c

$$z^5 - 1$$

$$= (z-1)\left(z - \text{cis}\left(\frac{2\pi}{5}\right)\right)\left(z - \text{cis}\left(-\frac{2\pi}{5}\right)\right)\left(z - \text{cis}\left(\frac{4\pi}{5}\right)\right)\left(z - \text{cis}\left(-\frac{4\pi}{5}\right)\right)$$

$$= (z-1)\left(z^2 - 2z\cos\left(\frac{2\pi}{5}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{4\pi}{5}\right) + 1\right)$$

d

$$\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$$

$$\alpha^4 + \alpha^3 + \alpha^2 + \alpha = -1$$

$$\text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right) + \text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right) = -1$$

$$2\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right) = -1$$

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$$

Question 73

a

Let ω be a complex cube root of unity

$$\omega^3 - 1 = 0$$

$$(\omega - 1)(\omega^2 + \omega + 1) = 0$$

$$\text{As}\,\omega \neq 1$$

$$\omega^2 + \omega + 1 = 0$$

b

$$\omega^9 + \omega^8 + \omega^7 + \omega^6 + \omega^5 + \omega^4$$

$$= (\omega^3)^3 + (\omega^3)^2\omega^2 + (\omega^3)^2\omega + (\omega^3)^2 + \omega^3\omega^2 + \omega^3\omega$$

$$= 1 + \omega^2 + \omega + 1 + \omega^2 + \omega$$

$$= 2(1 + \omega + \omega^2)$$

$$= 0$$

c

$$(1 - \omega^{-1})(1 - \omega^{-2})$$

$$= 1 - \omega^{-1} - \omega^{-2} + \omega^{-3}$$

$$= 1 - \frac{1}{\omega} - \frac{1}{\omega^2} + \frac{1}{\omega^3}$$

$$= 1 - \frac{\omega^2}{\omega^3} - \frac{\omega}{\omega^3} + 1$$

$$= 3 - (1 + \omega + \omega^2)$$

$$= 3$$

Question 74

a

$$z^2 = e^{\frac{i\pi}{2}}$$

$$z = \left(e^{\frac{\pi}{2}}\right)^{\frac{1}{2}}$$

$$z = e^{\frac{i\pi}{4}} \quad e^{-\frac{3i\pi}{4}}$$

b

$$z^2 = 9\,\text{cis}\left(\frac{\pi}{3}\right)$$

$$z = 3\,\text{cis}\left(\frac{\pi}{6}\right), 3\,\text{cis}\left(-\frac{5\pi}{6}\right)$$

Question 75

a $z^3 = -8$

The 3 roots will be equally spaced around the origin.

Let $z_1 = r(\cos\theta + i\sin\theta)$

Converting –8 to polar form, we have $-8 = 8(\cos\pi + i\sin\pi)$

Then $z_1^3 = r^3(\cos 3\theta + i\sin 3\theta) = 8(\cos\pi + i\sin\pi)$

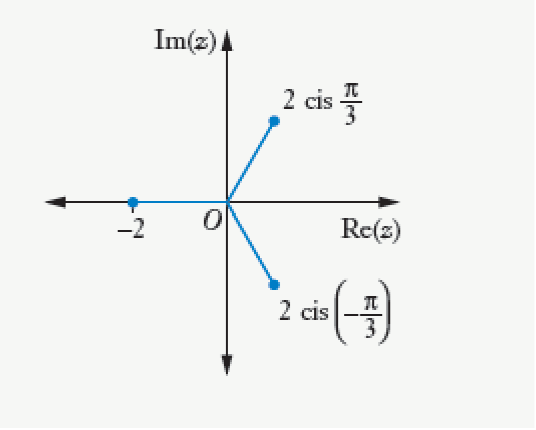
So $r = \sqrt[3]{8} = 2$ and $3\theta = \pi$, so $\theta = \frac{\pi}{3}$.

Therefore, $z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + i\sqrt{3}$.

The other 2 roots will have a spacing of $\frac{2\pi}{3}$. They are:

$z_2 = 2(\cos\pi + i\sin\pi) = 2(-1 + 0) = -2$

$z_3 = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - i\sqrt{3}$



Solutions are: $z = -2,\ 2\operatorname{cis}\left(\frac{\pi}{3}\right),\ 2\operatorname{cis}\left(-\frac{\pi}{3}\right)$

b $z^4 = -1 + i\sqrt{3}$

The 4 roots will be equally spaced around the origin.

Let $z_1 = r(\cos\theta + i\sin\theta)$

Converting $-1 + i\sqrt{3}$ to polar form, we have $-1 + i\sqrt{3} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

Then $z_1^4 = r^4(\cos 4\theta + i\sin 4\theta) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

So $r = \sqrt[4]{2}$ and $4\theta = \frac{2\pi}{3}$, so $\theta = \frac{\pi}{6}$.

Therefore, $z_1 = \sqrt[4]{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt[4]{2}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \frac{\sqrt[4]{2}}{2}\left(\sqrt{3} + i\right)$.

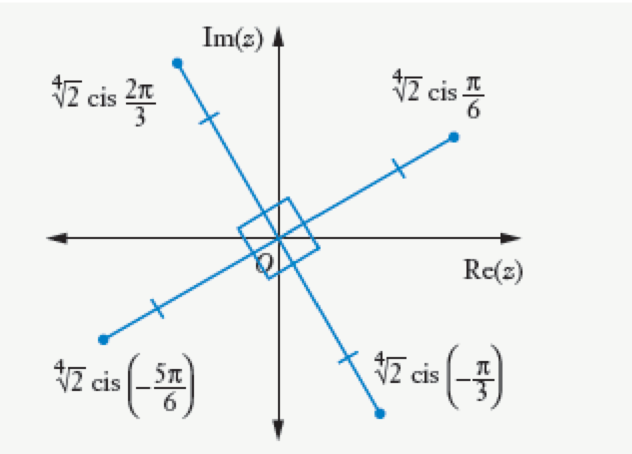
The other 2 roots will have a spacing of $\frac{2\pi}{4} = \frac{\pi}{2}$. They are:

$$z_2 = \sqrt[4]{2}\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] = \sqrt[4]{2}\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = \frac{\sqrt[4]{2}}{2}\left(-1 + i\sqrt{3}\right)$$

$$z_3 = \sqrt[4]{2}\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] = \sqrt[4]{2}\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = \frac{\sqrt[4]{2}}{2}\left(1 - i\sqrt{3}\right)$$

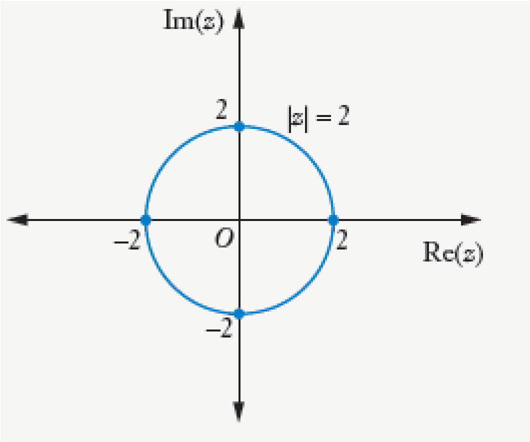
$$z_4 = \sqrt[4]{2}\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right] = \sqrt[4]{2}\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \frac{\sqrt[4]{2}}{2}\left(-\sqrt{3} + i\right)$$

So the solutions are: $\sqrt[4]{2}\,\text{cis}\left(\frac{2\pi}{3}\right), \sqrt[4]{2}\,\text{cis}\left(\frac{\pi}{6}\right), \sqrt[4]{2}\,\text{cis}\left(-\frac{\pi}{3}\right), \sqrt[4]{2}\,\text{cis}\left(-\frac{5\pi}{6}\right)$.

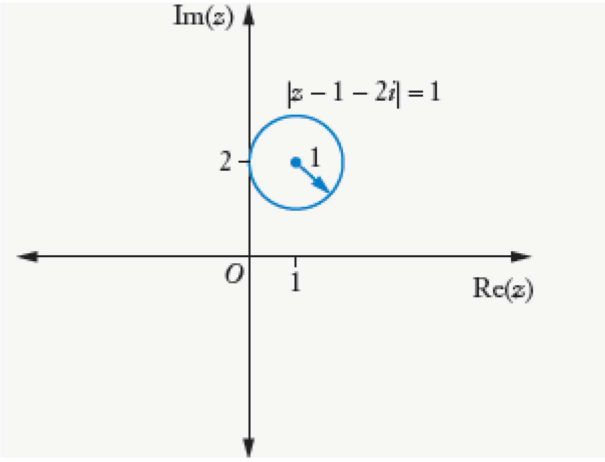


Question 76

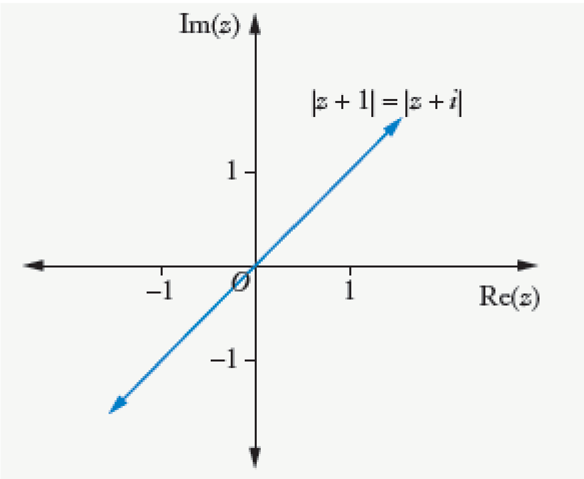
a



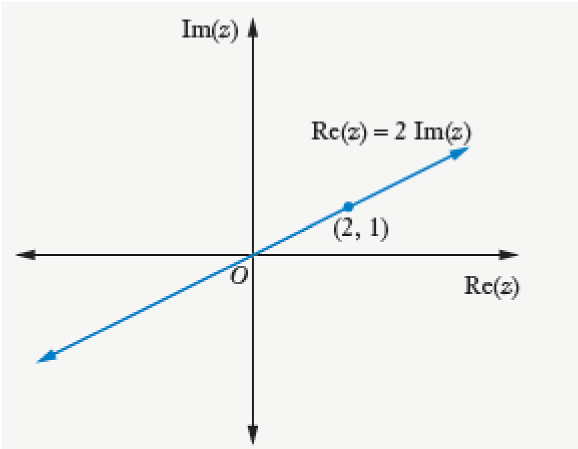
b



c

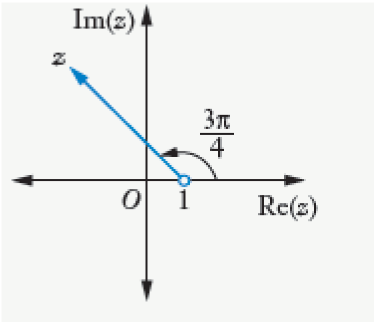


d

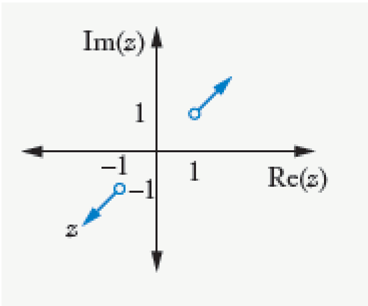


Question 77

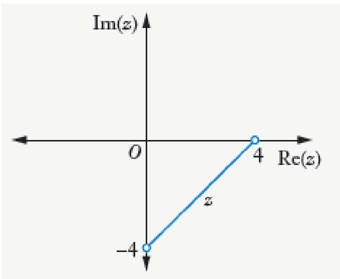
a



b

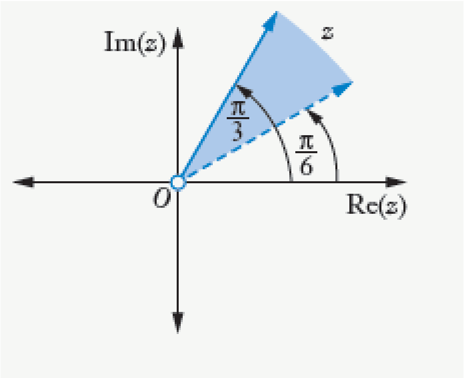


c

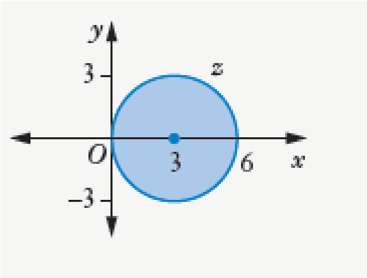


Question 78

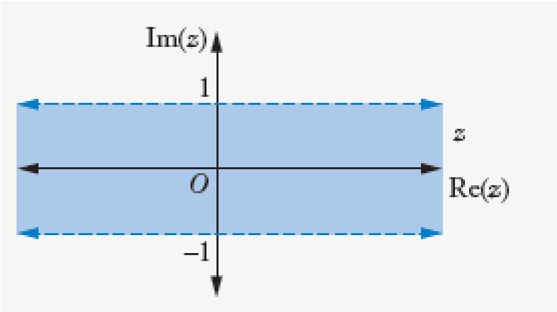
a



b



c



Question 79

$z_1 - z_2 = z_4 - z_3$

So opposite sides are parallel, so a parallelogram.

MATHS IN FOCUS 12 MATHEMATICS EXTENSION 2

WORKED SOLUTIONS

Practice set 2

Question 1

C

Question 2

A

Question 3

D

Question 4

C

Question 5

D

Question 6

$$\int \frac{2}{x^2+4x+13}dx$$
$$= \int \frac{2}{x^2+4x+4+9}dx$$
$$= \int \frac{2}{\left(x+2\right)^2+9}dx$$
$$= 2\int \frac{dx}{\left(x+2\right)^2+3^2}$$
$$= \frac{2}{3}\tan^{-}\left(\frac{x+2}{3}\right)+C$$

B

Question 7

$$\int x\ln\left(x\right)dx$$

Let $u'(x)=x$ $\quad v(x)=\ln\left(x\right)$

$u(x)=\frac{1}{2}x^2$ $\quad v'(x)=\frac{1}{x}$

$$\int u'(x)v(x)dx = u(x)v(x)-\int u(x)v'(x)dx$$
$$\int x\times\ln\left(x\right)dx = \frac{1}{2}x^2\ln\left(x\right)-\int\frac{1}{2}x^2\times\frac{1}{x}dx$$
$$= \frac{1}{2}x^2\ln\left(x\right)-\frac{1}{4}x^2+C$$
$$= \frac{1}{4}x^2\left(2\ln\left(x\right)-1\right)+C$$
$$= \frac{1}{2}x^2\ln\left(x\right)-\frac{x^2}{4}+C$$

A

Question 8

$$\frac{x}{(x-1)(x+4)}$$

$$= \frac{A}{x-1} + \frac{B}{x+4}$$

$$\Rightarrow A(x+4) + B(x-1) = x$$

Equating coefficients

$4A - B = 0$

$A + B = 1$

$B = 1 - A$

$4A - (1 - A) = 0$

$5A - 1 = 0$

$$A = \frac{1}{5}$$

$$B = 1 - \frac{1}{5}$$

$$B = \frac{4}{5}$$

$$\frac{x}{(x-1)(x+4)} = \frac{1}{5(x-1)} + \frac{4}{5(x+4)}$$

$$\int \frac{x}{(x-1)(x+4)}\,dx$$

$$= \frac{1}{5}\int \left(\frac{1}{x-1} + \frac{4}{x+4}\right) dx$$

$$= \frac{1}{5}\left(\ln|x-1| + 4\ln|x+4|\right) + C$$

$$= \frac{1}{5}\ln|x-1| + \frac{4}{5}\ln|x+4| + C$$

D

Question 9

$$F = \frac{m}{x^3}(8+10x)$$

$$a = \frac{8+10x}{x^3}$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{8}{x^3} + \frac{10}{x^2}$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 8x^{-3} + 10x^{-2}$$

$$\frac{1}{2}v^2 = -4x^{-2} - 10x^{-1} + C$$

$$v^2 = -8x^{-2} - 20x^{-1} + C$$

$$v = 0, x = 1$$

$$0 = -8 - 20 + C$$

$$C = 28$$

$$v^2 = -8x^{-2} - 20x^{-1} + 28$$

$$= 4\left(-2x^{-2} - 5x^{-1} + 7\right)$$

$$= \frac{4\left(-2-5x+7x^2\right)}{x^2}$$

$$v = \pm\frac{2}{x}\sqrt{7x^2 - 5x - 2}$$

C

Question 10

B

Question 11

$$F = g - kv^2$$

$$a = g - kv^2$$

$$\frac{v\,dv}{dx} = g - kv^2$$

$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$\frac{dx}{dv} = -\frac{1}{2k}\left(\frac{-2kv}{g - kv^2}\right)$$

$$x = -\frac{1}{2k}\ln\left(g - kv^2\right) + C$$

When $x = 0, v = 0.$

$$0 = -\frac{1}{2k}\ln g + C$$

$$C = \frac{1}{2k}\ln g$$

$$x = -\frac{1}{2k}\ln\left(g - kv^2\right) + \frac{1}{2k}\ln g$$

$$x = \frac{1}{2k}\left[\ln g - \ln\left(g - kv^2\right)\right]$$

$$= \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$$

$$\ln\left(\frac{g}{g - kv^2}\right) = 2kx$$

$$\frac{g}{g - kv^2} = e^{2kx}$$

$$\frac{g}{e^{2kx}} = g - kv^2$$

$$g\,e^{-2kx} = g - kv^2$$

$$kv^2 = g - g\,e^{-2kx}$$

$$v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$$

A

Question 12

For the journey up (assume $a = -10$ m s^{-2}).

$$v = u + at$$
$$0 = 21 - 10t$$
$$10t = 21$$
$$t = 2.1$$

Height

$$s = ut + \frac{1}{2}at^2$$
$$s = 21 \times 2.1 + \frac{1}{2}(-10)2.1^2$$
$$s = 22.05$$

Distance to water

22.05 + 20 = 42.05

$$s = ut + \frac{1}{2}at^2$$
$$42.05 = 0 + \frac{1}{2} \times 10 \times t^2$$
$$5t^2 = 42.05$$
$$t^2 = 8.41$$
$$t = 2.9$$

Total time = 2.9 + 2.1 = 5.1 ≈ 5

B

Question 13

$P(n):(n+1)(n+2)$ is even

$P(1):(1+1)(1+2)$

$=6$ which is even

$\therefore P(1)$ is true

Let $P(k)$ be true

$(k+1)(k+2)$ is even

$\Rightarrow (k+1)(k+2)=2m, \quad m\in\mathbb{N}$

$P(k+1)$ L.H.S

$((k+1)+1)((k+1)+2)$

$=(k+2)(k+3)$

$=(k+2)(k+1+2)$

$=(k+2)(k+1)+2(k+2)$

$=2m+2(k+2)$

$=2(m+(k+2))$

As $m,k,m+k+2\in\mathbb{N}$

$2(m+(k+2))=2q, \quad q\in\mathbb{N}$

$\therefore ((k+1)+1)((k+1)+2)$ is even

$\therefore P(k+1)$ is true

As $P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true

Question 14

$-2-4-6-...-2n$

$a=-2, d=-2, n=n$

$S_n=\frac{n}{2}(2a+(n-1)d)$

$S_n=\frac{n}{2}(-4-2(n-1))$

$S_n=\frac{n}{2}(-2-2n)$

$S_n=\frac{n}{2}(-2)(1+n)$

$S_n=-n(1+n)$

Question 15

$P(n)\quad f^{\,n}(x) = a^n \sin\left(ax + \frac{n\pi}{2}\right)\quad (\forall n \in \mathbb{N})$

If $f(x) = \sin(ax)$

$P(1)$ LHS

$f(x) = \sin(ax)$

$f'(x) = a\cos(ax)$

$= a\sin\left(ax + \frac{\pi}{2}\right)$

$\therefore P(1)$ is true

Let $P(k)$ be true

$f^{\,k}(x) = a^k \sin\left(ax + \frac{k\pi}{2}\right)$

$P(k+1)$ LHS

$\frac{d}{dx}\left(a^k \sin\left(ax + \frac{k\pi}{2}\right)\right)$

$= a \times a^k \cos\left(ax + \frac{k\pi}{2}\right)$

$= a^{k+} \sin\left(ax + \frac{k\pi}{2} + \frac{\pi}{2}\right)$

$= a^{k+} \sin\left(ax + \frac{(k+1)\pi}{2}\right)$

$\therefore P(k+1)$ is true

As $P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true

Question 16

$$\sum_{r=1}^{n} x^{r-1} = \frac{1-x^n}{1-x}$$

$a = x^{1-1}$

$a = x^0$

$a = 1$

$r = x$

$$S_n = \frac{a\left(r^n - 1\right)}{r-1}$$

$$S_n = \frac{1\left(x^n - 1\right)}{x-1}$$

$$S_n = \frac{x^n - 1}{x-1}$$

$$S_n = \frac{1-x^n}{1-x}$$

Question 17

$P(n): 2^{3n} - 3^{n}$ is divisible by 5

$P(1): 2^{3} - 3^{1}$

$= 8 - 3$

$= 5$ which is divisible by 5

$\therefore P(1)$ is true

Let $P(k)$ be true

$2^{3k} - 3^{k}$ is divisible by 5

$\Rightarrow 2^{3k} - 3^{k} = 5m, \quad m \in \mathbb{N}$

$P(k+1)$ L.H.S

$2^{3(k+1)} - 3^{k+1}$

$= 2^{3k+3} - 3^{k+1}$

$= 2^{3} \times 2^{3k} - 3 \times 3^{k}$

$= 8 \times 2^{3k} - 3 \times 3^{k}$

$= 5 \times 2^{3k} + 3 \times 2^{3k} - 3 \times 3^{k}$

$= 5 \times 2^{3k} + 3\left(2^{3k} - 3^{k}\right)$

$= 5 \times 2^{3k} + 3(5m)$

$= 5 \times 2^{3k} + 5(3m)$

$= 5\left(2^{3k} + 3m\right)$

As $m, k, 2^{3k} + 3m \in \mathbb{N}$

$5\left(2^{3k} + 3m\right) = 5q, \quad q \in \mathbb{N}$

$\therefore 5\left(2^{3k} + 3m\right)$ is divisible by 5

$\therefore P(k+1)$ is true

As $P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true

Question 18

$P(n): 4^n > 3n+7,\ n > 1$

$P(2)$ LHS

$4^2 = 16$

RHS $= 3\times 2 + 7 = 13$

$\therefore P(2)$ is true

Let $P(k)$ be true

$4^k > 3k+7$

$P(k+1)$ LHS

4^{k+1}

$= 4\times 4^k$

$> 4\times(3k+7)$

$= 12k + 4\times 7$

$= 4k + 4 + 8k + 24$

$= 4(k+1) + 7 + 8k + 17$

As $k > 0$ $8k > 0$

$> 4(k+1) + 7$

$\therefore P(k+1)$ is true

As $P(2)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true

Question 19

$P(n) \ \dfrac{d}{dx}x^n = nx^{n-1}$

$P(1)$ LHS

$f(x) = x^1$

$f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$

$f'(x) = \lim\limits_{h\to 0}\dfrac{(x+h) - x}{h}$

$f'(x) = \lim\limits_{h\to 0}\dfrac{h}{h}$

$f'(x) = \lim\limits_{h\to 0}1$

$f'(x) = 1$

$f'(x) = 1x^{1-1}$

$\therefore P(1)$ is true

Let $P(k)$ be true

$f'(x) = kx^{k-1}$

$P(k+1)$ LHS

$f(x) = x^k$

$f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0}\dfrac{(x+h)^{k+} - x^{k+1}}{h}$

$= \lim\limits_{h\to 0}\dfrac{(x+h)(x+h)^k - x \times x^k}{h}$

$= \lim\limits_{h\to 0}\dfrac{x(x+h)^k - x \times x^k + h(x+h)^k}{h}$

$= \lim\limits_{h\to 0}\dfrac{x\left[(x+h)^k - x^k\right]}{h} + \lim\limits_{h\to 0}\dfrac{h(x+h)^k}{h}$

$= x\lim\limits_{h\to 0}\dfrac{(x+h)^k - x^k}{h} + \lim\limits_{h\to 0}(x+h)^k$

$= x \times kx^{k-} + x^k$

$= kx^k + x^k$

$= (k+1)x^{(k+1 -1}$

$\therefore P(k+1)$ is true

As $P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true

Question 20

$a \ = 3, a_n = a_{n-} + 5 \quad n > 1$

Required to prove: $a_n = 5n - 2$

$P(n): a_n = 5n - 2$

$P(2):$ LHS

$a_2 = a_1 + 5$

$= 3 + 5$

$= 8$

RHS

$a_2 = 5 \times 2 - 2$

$= 8 \ =$ LHS

$\therefore P(2)$ is true

Let $P(k)$ be true

$a_k = 5k - 2$

$P(k+1)$ LHS

$a_{k+} \ = a_k + 5$

$= 5k - 2 + 5$

$= 5k + 5 - 2$

$= 5(k+1) - 2$

$=$ RHS

$\therefore P(k+1)$ is true

As $P(2)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true

Question 21

$u_1 = 8, u_2 = 20, u_n = 4u_{n-1} - 4u_{n-2} \quad n \geq 3$

Required to prove : $u_n = (n+3)2^n$

$P(n): u_n = (n+3)2^n$

$P(3)$: LHS

$u_3 = 4u_2 - 4u_1$

$= 4 \times 20 - 4 \times 8$

$= 48$

RHS

$u_3 = (3+3) \times 2^3$

$= 48 \ = \text{LHS}$

$\therefore P(3)$ is true

Let $P(k)$ be true

$u_k = (k+3)2^k$

This also assumes $P(k-1)$ be true

$u_{k\_} = (k+2)2^{k-1}$

$P(k+1)$ LHS

$u_{k+} = 4u_k - 4u_{k-1}$

$= 4(k+3)2^k - 4(k+2)2^{k-1}$

$= 2(k+3)2^{k+1} - (k+2)2^{k+1}$

$= 2^{k+} \ (2k+6-k-2)$

$= 2^{k+} \ (k+1+3)$

$= \text{RHS}$

$\therefore P(k+1)$ is true

As $P(3)$ is true and the truth of $P(k)$ and $P(k-1)$ implies the truth of $P(k+1)$ by mathematical induction $P(n)$ is true

Question 22

a $\cos(x-y) - \cos(x+y) = 2\sin x \sin y$

b

$P(n): \sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin((2n-1)\theta) = \dfrac{\sin^2(n\theta)}{\sin\theta}$

$P(1)$; LHS $\sin\theta$; RHS $= \dfrac{\sin^2(\theta)}{\sin\theta} = \sin\theta$

$\therefore P(1)$ is true

Let $P(k)$ be true: $\sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin((2k-1)\theta) = \dfrac{\sin^2(k\theta)}{\sin\theta}$

Required to prove:

$P(k+1); \sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin((2k-1)\theta) + \sin((2(k+1)-1)\theta) = \dfrac{\sin^2((k+1)\theta)}{\sin\theta}$

$P(k+1)$ LHS

$\sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin((2k-1)\theta) + \sin((2(k+1)-1)\theta)$

$= \dfrac{\sin^2(k\theta)}{\sin\theta} + \sin([2(k+1)-1]\theta)$

$= \dfrac{\sin^2(k\theta) + \sin(2k\theta+\theta)\sin\theta}{\sin\theta}$

$= \dfrac{\sin^2(k\theta) + \frac{1}{2}[\cos(2k\theta+\theta-\theta)\cos(2k\theta+\theta+\theta)]}{\sin\theta}$

$= \dfrac{\sin^2(k\theta) + \frac{1}{2}[\cos(2k\theta) - \cos(2[k\theta+\theta])]}{\sin\theta}$

$= \dfrac{\sin^2(k\theta) + \frac{1}{2}[\cos(2k\theta) - (1-2\sin^2(k\theta+\theta))]}{\sin\theta}$

$= \dfrac{2\sin^2(k\theta) + \cos(2k\theta) - (1-2\sin^2(k\theta+\theta))}{2\sin\theta}$

$= \dfrac{2\sin^2(k\theta) + \cos(2k\theta) - 1 + 2\sin^2(k\theta+\theta)}{2\sin\theta}$

$= \dfrac{2\sin^2(k\theta) + (1-2\sin^2(k\theta)) - 1 + 2\sin^2(\theta(k+1))}{2\sin\theta}$

$= \dfrac{\sin^2(\theta(k+1))}{\sin\theta}$

$\therefore P(k+1)$ is true

As $P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$

by mathematical induction $P(n)$ is true

Question 23

$$\int \frac{dx}{\sqrt{4+4x-x^2}}$$

$$= \int \frac{dx}{\sqrt{8-\left(x^2-4x+4\right)}}$$

$$= \int \frac{dx}{\sqrt{8-(x-2)^2}}$$

$$= -\sin^{-}\left(\frac{2-x}{2\sqrt{2}}\right)+C$$

Question 24

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos x+1}\,dx$$

$$t=\tan\left(\frac{x}{2}\right), \cos x=\frac{1-t^2}{1+t^2}, dx=\frac{2dt}{1+t^2}$$

$$x=\frac{\pi}{2} \Rightarrow t=1, x=0 \Rightarrow t=0$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos x+1}\,dx$$

$$= \int_0^1 \frac{1}{\frac{1-t^2}{1+t^2}+1}\frac{2\,dt}{1+t^2}$$

$$= \int_0^1 \frac{1}{\frac{1-t^2}{1+t^2}+\frac{1+t^2}{1+t^2}}\frac{2\,dt}{1+t^2}$$

$$= \int_0^1 \frac{1}{\frac{2}{1+t^2}}\frac{2\,dt}{1+t^2}$$

$$= \int_0^1 \frac{1+t^2}{2}\frac{2\,dt}{1+t^2}$$

$$= \int_0^1 1\,dt$$

$$= [x]_0^1$$

$$= 1$$

Question 25

$$\int \frac{dx}{x^2 - 2x + 10}$$
$$= \int \frac{dx}{x^2 - 2x + 1 + 9}$$
$$= \int \frac{dx}{(x-1)^2 + 3^2}$$
$$= \frac{1}{3}\tan^{-}\left(\frac{x-1}{3}\right) + c$$

Question 26

$$\int \frac{1}{x(x-2)}dx$$
$$\frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$
$$\Rightarrow A(x-2) + Bx = 1$$
Let $x = 0$
$$-2A = 1$$
$$A = -\frac{1}{2}$$
Let $x = 2$
$$2B = 1$$
$$B = \frac{1}{2}$$
$$\frac{1}{x(x-2)} = -\frac{1}{2x} + \frac{1}{2x-4}$$
$$\int \frac{1}{x(x-2)}dx = \int\left(-\frac{1}{2x} + \frac{1}{2x-4}\right)dx$$
$$= \int -\frac{1}{2x}\,dx + \int \frac{1}{2x-4}\,dx$$
$$= -\frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{x-2}\,dx$$
$$= -\frac{1}{2}\ln|x| + \frac{1}{2}\ln|x-2| + c$$
$$= \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + c$$

Question 27

$$\int \frac{2x-1}{x^2+3x+2}dx$$

$$= \int \frac{2x-1}{(x+2)(x+1)}dx$$

$$\frac{2x-1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

$$\Rightarrow A(x+1) + B(x+2) = 2x-1$$

Let $x = -1$

$$B(-1+2) = -2-1$$

$$B = -3$$

Let $x = -2$

$$A(-2+1) = -4-1$$

$$-A = -5$$

$$A = 5$$

$$\frac{2x-1}{(x+2)(x+1)} = \frac{5}{x+2} - \frac{3}{x+1}$$

$$\int \frac{2x-1}{(x+2)(x+1)}dx = \int \left(\frac{5}{x+2} - \frac{3}{x+1} \right) dx$$

$$= \int \frac{5}{x+2}dx - \int \frac{3}{x+1}dx$$

$$= 5\int \frac{1}{x+2}dx - 3\int \frac{1}{x+1}dx$$

$$= 5\ln|x+2| - 3\ln|x+1| + C$$

$$= \ln\left| \frac{(x+2)^5}{(x+1)^3} \right| + C$$

Question 28

$$\int \frac{x}{(x+2)(x+3)}\,dx$$

$$\frac{x}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$\Rightarrow A(x+3) + B(x+2) = x$$

Let $x = -3$

$$B(-3+2) = -3$$

$$-B = -3$$

$$B = 3$$

Let $x = -2$

$$A(-2+3) = -2$$

$$A = -2$$

$$\frac{x}{(x+2)(x+3)} = \frac{3}{x+3} - \frac{2}{x+2}$$

$$\int \frac{x}{(x+2)(x+3)}\,dx = \int \left(\frac{3}{x+3} - \frac{2}{x+2} \right) dx$$

$$= \int \frac{3}{x+3}\,dx - \int \frac{2}{x+2}\,dx$$

$$= 3\int \frac{1}{x+3}\,dx - 2\int \frac{1}{x+2}\,dx$$

$$= 3\ln|x+3| - 2\ln|x+2| + C$$

$$= \ln\left| \frac{(x+3)^3}{(x+2)^2} \right| + C$$

Question 29

$$\int^{e} \frac{\ln(x)}{x} dx$$

Let $u = \ln x,\ \frac{du}{dx} = \frac{1}{x},\ dx = x\ du$

$$\int_1^e \frac{\ln(x)}{x} dx = \int_1^e \frac{u}{x} x\ du = \int_1^e u\, du$$

$$= \left[\frac{1}{2}u^2\right]_1^e$$

$$= \left[\frac{1}{2}(\ln\ x)^2\right]_1^e$$

$$= \frac{1}{2}(\ln\ e)^2 - \frac{1}{2}(\ln 1)^2$$

$$= \frac{1}{2}$$

Question 30

$$\int_0^{\ln 3} x^2 e^x dx$$

Let $u'(x) = e^x \qquad v(x) = x^2$

$u(x) = e^x \qquad v'(x) = 2x$

$$\int_0^{n3} u\ (x)v(x)dx = u(x)v(x) - \int_0^{\ln 3} u(x)v'(x)dx$$

$$\int e^x \times x^2 dx = \left[e^x x^2\right]_0^{n3} - \int e^x \times 2x\ dx$$

$$= \left[e^x x^2\right]_0^{\ln 3} - 2\int xe^x dx$$

$$\int xe^x dx$$

Let $u'(x) = e^x \qquad v(x) = x$

$u(x) = e^x \qquad v'(x) = 1$

$$\int u'(x)v(x)\, dx = u(x)v(x) - \int u(x)v'(x)\, dx$$

$$\int e^x \times x\, dx = e^x\ x - \int e^x \times 1\ dx$$

$$= e^x\ x - e^x$$

$$\int_0^{\ln 3} x^2 e^x dx = \left[x^2 e^x\right]_0^{n3} - 2\left[xe^x - e^x\right]_0^{\ln 3}$$

$$= 3(\ln 3)^2 - 2(3\ln 3 - 2)$$

$$= 3(\ln 3)^2 - 6\ln 3 + 4$$

Question 31

$\int e^x \sin(x)dx$

Let $u'(x) = e^x \qquad v(x) = \sin(x)$

$\qquad u(x) = e^x \qquad v'(x) = \cos(x)$

$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$

$\int e^x \times \sin(x)dx = e^x \sin(x) - \int e^x \times \cos(x)dx$

$\int e^x \times \cos(x)dx$

Let $u'(x) = e^x \qquad v(x) = \cos(x)$

$\qquad u(x) = e^x \qquad v'(x) = -\sin(x)$

$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$

$\int e^x \times \cos(x)dx = e^x \cos(x) + \int e^x \times \sin(x)dx$

$= e^x \cos(x) + \int e^x \times \sin(x)dx$

$\int e^x \sin(x)dx = e^x \sin(x) - \left(e^x \cos(x) + \int e^x \times \sin(x)dx\right)$

$2\int e^x \sin(x)dx = e^x \sin(x) - e^x \cos(x) + C$

$= \frac{e^x}{2}\left(\sin(x) - \cos(x)\right) + C$

Question 32

$\int \sin^{-1} x \; dx$

Let $u'(x) = 1 \qquad v(x) = \sin^{-1} x$

$\qquad u(x) = x \qquad v'(x) = \frac{1}{\sqrt{1-x^2}}$

$\int u'(x)v(x)d = u(x)v(x) - \int u(x)v'(x)dx$

$\int 1 \times \sin^{-1} x dx = x\sin^{-1} x - \int x \frac{1}{\sqrt{1-x^2}} dx$

$= x\sin^{-1} x + \sqrt{1-x^2} + C$

Question 33

$$a=\frac{1}{(x+3)^2}$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx}=(x+3)^{-2}$$

$$\frac{1}{2}v^2=-(x+3)^{-1}+c$$

$$x=0, v=0$$

$$0=-(3)^{-1}+c$$

$$c=\frac{1}{3}$$

$$\frac{1}{2}v^2=\frac{1}{3}-\frac{1}{x+3}$$

$$v^2=\frac{2}{3}-\frac{2}{x+3}$$

$$v=\sqrt{\frac{2}{3}-\frac{2}{x+3}}$$

$$v=\sqrt{\frac{2x}{3x+9}}$$

Question 34

$$\text{amplitude} = a = \frac{18-12}{2} = 3 \text{ m}$$

$$\text{centre} = \frac{18+12}{2} = 15 \text{ m}$$

$$\text{period} = P = 6 \times 2 = 12 \text{ h}$$

$$\frac{2\pi}{n} = 12$$

$$\frac{2\pi}{12} = n$$

$$n = \frac{\pi}{6}$$

$$d = -3\cos\left(\frac{\pi t}{6}\right) + 15$$

$$16 = -3\cos\left(\frac{\pi t}{6}\right) + 15$$

$$-3\cos\left(\frac{\pi t}{6}\right) = 1$$

$$\cos\left(\frac{\pi t}{6}\right) = -\frac{1}{3}$$

$$\frac{\pi t}{6} = 1.91 \text{ for earliest time}$$

$$t = 365 \text{ h} \approx 3\text{ h } 39 \text{ minutes}$$

$$12{:}30 \text{ p.m.} + 3\text{h } 39 \text{ min} = 4{:}09 \text{ p.m.}$$

Question 35

Horizontal motion

$\ddot{x} = 0$
$\dot{x} = C$

When $t = 0$, $\dot{x} = V\cos\theta = 12\cos 15°$

$6\sqrt{3} = C$
$\dot{x} = 6\sqrt{3}$
$x = \left(6\sqrt{3}\right)t + D$

When $t = 0$, $x = 0$

$0 = 0 + D$

$D = 0$

$x = 12t\cos 15°$

Vertical motion

$\ddot{y} = -g = -9.8$
$\dot{y} = -98\,t + E$

When $t = 0$, $\dot{y} = V\sin\theta = 12\sin 15°$

$12t\sin 15° = 0 + E$
$E = 12t\sin 15°$
$\dot{y} = -98\,t + 12t\sin 15°$
$y = -49\,t^2 + 12t\sin 15° + F$

When $t = 0$, $y = 0$

$0 = 0 + 0 + F$

$F = 0$

$y = -49\,t^2 + 12t\sin 15°$

Long jumper lands when $y = 0$.

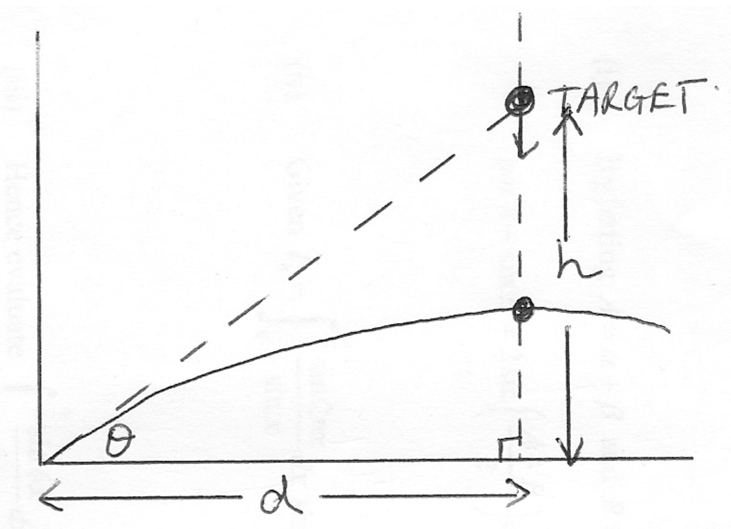
$$0 = -4.9t^2 + 12t\sin 15°$$
$$0 = t\left(-4.9t + 12\sin 15°\right)$$
$$4.9t = 12\sin 15° \qquad t > 0$$
$$t = \frac{12\sin 15°}{4.9}$$

Substitute into *x:*

$x = 12t \cos 15°$

$$x = 12\left(\frac{12\sin 15°}{49}\right)\cos 15°$$
$$= \frac{144\sin 15°\cos 15°}{4.9}$$
$$= \frac{72\left(2\sin 15°\cos 15°\right)}{4.9}$$
$$= \frac{72\left(\sin 30°\right)}{4.9}$$
$$= 7.346...$$
$$\approx 7.3 \text{ m}$$

Question 36



Let *h* be the height of the target and *d* be the horizontal distance between the projectile and the target at the start.

For the target:

$\ddot{y}_T = -g$

$\dot{y}_T = -gt + C$

When $t = 0, \dot{y}_T = 0$

$0 = 0 + C$

$C = 0$

$\dot{y}_T = -gt$

$y_T = -\frac{1}{2}gt^2 + D$

When $t = 0, y_T = h$

$h = 0 + D$

$D = h$

$y_T = -\frac{1}{2}gt^2 + h$

For the projectile:

$\ddot{y}_P = -g$

$\dot{y}_P = -gt + E$

When $t = 0, \dot{y}_P = V\sin\theta$

$V\sin\theta = 0 + E$

$E = V\sin\theta$

$\dot{y}_P = -gt + V\sin\theta$

$y_P = -\frac{1}{2}gt^2 + Vt\sin\theta + F$

When $t = 0, y_P = 0$

$0 = 0 + 0 + F$

$F = 0$

$y_P = -\frac{1}{2}gt^2 + Vt\sin\theta$

Need to find when and where the projectile will be at $x = d$.

For the projectile:

$\ddot{x}_P = 0$

$\dot{x}_P = G$

When $t = 0$, $\dot{x}_P = V\cos\theta$

$V\cos\theta = G$

$\dot{x}_P = V\cos\theta$

$x_P = Vt\cos\theta + H$

When $t = 0$, $x_P = 0$

$0 = 0 + H$

$H = 0$

$x_P = Vt\cos\theta$

$d = Vt\cos\theta$

$$t = \frac{d}{V\cos\theta}$$

Substituteinto y_P:

$$y_P = V\left(\frac{d}{V\cos\theta}\right)\sin\theta - \frac{1}{2}g\left(\frac{d}{V\cos\theta}\right)^2$$

$$y_P = d\tan\theta - \frac{gd^2}{2V^2\cos\theta}$$

Substituteinto y_T:

$$y_T = -\frac{1}{2}g\left(\frac{d}{V\cos\theta}\right)^2 + h$$

$$y_T = h - \frac{gd^2}{2V^2\cos\theta}$$

But $\tan\theta = \dfrac{h}{d}$ from the right-angled triangle.

$h = d\tan\theta$

$$y_P = d\tan\theta - \frac{gd^2}{2V^2\cos\theta}$$

which proves that $y_P = y_T$ at the same time $t = \dfrac{d}{V\cos\theta}$.

So the projectile will hit the target at that height $y_P = y_T$.

Question 37

Assume g = 9.8 m s^{-2}

$$v = u + at$$
$$0 = u + (-g) \times 1$$
$$u = g = 9.8 \text{ m s}^{-1}$$
$$s = ut + \frac{1}{2}at^2$$
$$s = g + \frac{1}{2}(-g)(1)^2$$
$$s = \frac{1}{2}g$$
$$s = 4.9 \text{ m}$$

Question 38

$$v^2 = 256 - 64x^2$$
$$\frac{1}{2}v^2 = 128 - 32x^2$$

a

$$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$
$$= -64x$$

b

Maximum speed occurs when x = 0

$$v^2 = 256$$
$$v = 16 \text{ m s}^{-1}$$

Question 39

a

Assume initial velocity is 0 m s^{-1}.

$$F = mg - \frac{mv}{10}$$

$$a = g - \frac{v}{10}$$

$$a = \frac{10g - v}{10} = \frac{100 - v}{10}$$

b

$$v\frac{dv}{dx} = \frac{10g - v}{10}$$

$$\frac{dv}{dx} = \frac{10g - v}{10v} = \frac{100 - v}{10v}$$

$$\frac{dx}{dv} = \frac{10v}{100 - v}$$

$$\frac{dx}{dv} = -10\left(1 - \frac{100}{100 - v}\right)$$

$$\frac{dx}{dv} = -10\left(1 + 100\frac{-1}{100 - v}\right)$$

$$\frac{dx}{dv} = -10 - 1000\frac{-1}{100 - v}$$

$$x = -10v - 1000\ln(100 - v) + c$$

When $x = 0$ m, $v = 0$ m s^{-1}

$$0 = -1000\ln(100) + C$$

$$C = 1000\ln(100)$$

$$x = -10v - 1000\ln(100 - v) + 1000\ln(100)$$

$$x = -10v + 1000\ln\left(\frac{100}{100 - v}\right)$$

When $x = 40(\text{water level})$

$$40 = -10v + 1000\ln\left(\frac{100}{100-v}\right)$$

$$40 + 10v - 1000\ln\left(\frac{100}{100-v}\right) = 0$$

$$\frac{40}{1000} + \frac{10}{1000}v - \ln\left(\frac{100}{100-v}\right) = 0$$

$$0.04 + \frac{v}{100} + \ln\left(\frac{100}{100-v}\right)^{-1} = 0$$

$$\frac{v}{100} + \ln\left(\frac{100-v}{100}\right) + 0.04 = 0$$

$$\frac{v}{100} + \ln\left(1 - \frac{v}{100}\right) + 0.04 = 0 \quad \text{as required}$$

c $F = ma = mg - \dfrac{mv^2}{10}$

$$a = 10 - \frac{v^2}{10}$$

$$= \frac{100 - v^2}{10}$$

d

$$a = \frac{100 - v^2}{10}$$

Terminal velocity occurs when $a = 0$

$$0 = \frac{100 - v_T^{\ 2}}{10}$$

$$0 = 100 - v_T^{\ 2}$$

$$v_T^{\ 2} = 100$$

$$v_T = 10 \text{ m s}^{-1}$$

Question 40

Equating horizontal forces

$$N\sin\theta = \mu N\cos\theta$$

$$\mu = \frac{\sin\theta}{\cos\theta}$$

$$\mu = \tan\theta$$

$$\frac{1}{\sqrt{3}} = \tan\theta$$

$$\theta = 30°$$

Question 41

Vertical forces

$$mg = N + 820\sin 10$$

$$N = mg - 820\sin 10$$

Horizontal forces

$$F = -\mu N + 820\cos 10°$$

$$F = 820\cos 10° - 0.04\left(mg - 820\sin 10°\right)$$

$$F = 13.24$$

$$F = ma$$

$$a = \frac{F}{m}$$

$$a = \frac{13.24}{2000}$$

$$a = 0.0066\ \text{m}\,\text{s}^{-2}$$

Question 42

$$\frac{dv_y}{dt} = -10 - kv_y$$

$$\frac{1}{k}\frac{kdv_y}{10+kv_y} = -dt$$

$$\frac{1}{k}\ln\left(10+kv_y\right) = -t + c$$

$$t = 0 \ v_y = \frac{15}{2}$$

$$\frac{1}{k}\ln\left(10+k\frac{15}{2}\right) = c$$

$$\frac{1}{k}\ln\left(10+kv_y\right) = -t + \frac{1}{k}\ln\left(10+k\frac{15}{2}\right)$$

For maximum height $v_y = 0$

$$\frac{1}{k}\ln\left(10\right) = -t + \frac{1}{k}\ln\left(10+\frac{15k}{2}\right)$$

$$t = \frac{1}{k}\ln\left(\frac{10+\frac{15k}{2}}{10}\right)$$

Given $v_{\mathrm{T}} = 11$

$$m\ddot{y} = mg - mkv$$

$$\ddot{y} = 0, v_{\mathrm{T}} = 11$$

$$g = kv$$

$$k = \frac{10}{11}$$

$$t = \frac{11}{10}\ln\left(\frac{10+\frac{15\times\frac{10}{11}}{2}}{10}\right)$$

$$t = 0.57 \text{ s}$$