Neap

Trial Examination 2023

Suggested Solutions

QCE Mathematical Methods Units 3&4

Paper 1 – Technology-free

SECTION 1 – MULTIPLE CHOICE QUESTIONS

	A	B	C	D
1.	○	○	●	○
2.	○	●	○	○
3.	○	○	●	○
4.	○	○	○	●
5.	○	○	●	○
6.	●	○	○	○
7.	●	○	○	○
8.	○	●	○	○
9.	●	○	○	○
10.	○	●	○	○

QUESTION 1 C

Let X be the continuous random variable of butterfly wingspan.

$$\begin{aligned} P(2.5 < X < 7.5) &= \frac{4+5}{15} \\ &= \frac{9}{15} \\ &= \frac{3}{5} \\ &= 0.6 \end{aligned}$$

QUESTION 2 B

$$y = 3e^{4x}$$

$$y' = 12e^{4x}$$

At $x = 0$, $y' = 12$.

$\therefore y = 12x + c$

At $x = 0$, $y = 3$.

$$\begin{aligned} \therefore 3 &= 12 \times 0 + c \\ c &= 3 \end{aligned}$$

Therefore, $y = 12x + 3$.

QUESTION 3 C

C is correct. Compared to the natural logarithmic function $y = \ln(x)$, $y = \ln(x - 3) - 1$ has been shifted 3 units to the right and 1 unit down, and has a vertical asymptote at $x = 3$. Only option **C** has an asymptote at $x = 3$.

A is incorrect. This is the graph of $y = \ln(x + 3) - 1$.

B is incorrect. This is the graph of $y = -\ln(x + 3) - 1$.

D is incorrect. This is the graph of $y = 0.5e^{x} - 2$.

QUESTION 4 D

Given that $\int 2x\cos\left(x^2\right) \times e^{\sin\left(x^2\right)} dx = e^{\sin\left(x^2\right)} + c$:

$$\begin{aligned} \int \frac{1}{9} x\cos\left(x^2\right) \times e^{\sin\left(x^2\right)} dx &= \frac{1}{18} \int 2x\cos\left(x^2\right) \times e^{\sin\left(x^2\right)} dx \\ &= \frac{1}{18} e^{\sin\left(x^2\right)} + d \end{aligned}$$

Note that the choice of constant does not matter, but d was chosen to distinguish it from c.

QUESTION 5 C

$$E(X)=\int_0^2 xf(x)dx$$
$$=\int_0^2 \frac{x^2}{2}dx$$
$$=\frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$$
$$=\frac{1}{6}\left(2^3-0^3\right)$$
$$=\frac{4}{3}$$

QUESTION 6 A

$$Var(X)=E\left(X^2\right)-\left[E(X)\right]^2$$
$$=\frac{1}{2}-\left(\frac{2}{3}\right)^2$$
$$=\frac{1}{18}$$

QUESTION 7 A

$$f(x)=-\frac{1}{3}\cos(3x)$$
$$\therefore f'(x)=-\frac{1}{3}\times\left(-\sin(3x)\right)\times 3$$
$$=\sin(3x)$$

Since it is negative, $3x$ is in quadrants 3 and 4.

$$\therefore 3x=\frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{19\pi}{6} \text{ or ...}$$
$$x=\frac{7\pi}{18} \text{ or } \frac{11\pi}{18} \text{ or } \frac{19\pi}{18} \text{ or ...}$$
$$x=\frac{7\pi}{18},\frac{11\pi}{18}$$

QUESTION 8 B

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \cos(2x)dx = \left[\frac{1}{2}\sin(2x)\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$$

$$= \frac{1}{2}\left[\sin(2x)\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$$

$$= \frac{1}{2}\left(\sin\left(2\times\frac{2\pi}{3}\right) - \sin\left(2\times\frac{\pi}{3}\right)\right)$$

$$= \frac{1}{2}\left(\sin\left(\frac{4\pi}{3}\right) - \sin\left(\frac{2\pi}{3}\right)\right)$$

$$= -\frac{\sqrt{3}}{2}$$

QUESTION 9 A

This is a Bernoulli distribution.

$$E(X) = np$$

$$= \frac{1}{10}\times 1$$

$$= \frac{1}{10}$$

$$\sigma(X) = \sqrt{np(1-p)}$$

$$= \sqrt{1\times\frac{1}{10}\times\left(1-\frac{1}{10}\right)}$$

$$= \sqrt{\frac{9}{100}}$$

$$= \frac{3}{10}$$

QUESTION 10 B

B is correct. This is the best option for representing the cumulative distribution function. The introduction of another variable, t, makes the expression clearer.

A is incorrect. The use of x as both the upper bound and the variable of integration is not the best mathematical notation to use.

C is incorrect. This option includes the derivative of the integrand, which is not required.

D is incorrect. This option is the derivative of the integrand.

SECTION 2

QUESTION 11 (4 marks)

a)
$$\ln(x)+\ln(x+1)=\ln 20$$
$$\ln\big(x(x+1)\big)=\ln 20$$
$$x(x+1)=20$$
$$x^2+x-20=0$$
$$(x+5)(x-4)=0$$

By the null factor law, $x = -5$ or $x = 4$.

However, only $x = 4$ can be substituted in the original equation. Therefore, $x = 4$.

[2 marks]
1 mark for using logarithmic laws to simplify to the quadratic equation.
1 mark for solving for the value of x.

b)
$$e^4\times\left(\left(e^{x-2}\right)^2-e^{12}\right)=0$$
$$e^4\times\left(e^{2x-4}-e^{12}\right)=0$$
$$e^{2x-4+4}-e^{12+4}=0$$
$$e^{2x}-e^{16}=0$$
$$e^{2x}=e^{16}$$
$$2x=16$$
$$x=8$$

[2 marks]
1 mark for using index laws to simplify to the simple linear equation.
Note: Allow for implied working.
1 mark for solving for the value of x.

QUESTION 12 (3 marks)

a) $y = \cos(x) \times \ln(x)$

Let $f = \cos(x)$ and $g = \ln(x)$.

$$y' = fg' + f'g$$
$$= \left(\cos(x)\times\frac{1}{x}\right)-\big(\sin(x)\times\ln(x)\big)$$
$$= \frac{\cos(x)}{x}-\big(\sin(x)\times\ln(x)\big)$$

[1 mark]
1 mark for determining the derivative using the product rule.

b) $y = \dfrac{\sin(x)}{e^{2x}}$

Let $f = \sin(x)$ and $g = e^{2x}$.

$$y' = \frac{f'g - fg'}{g^2}$$

$$= \frac{\left(\cos(x) \times e^{2x}\right) - \left(\sin(x) \times 2e^{2x}\right)}{\left(e^{2x}\right)^2}$$

$$= \frac{\cos(x) - 2\sin(x)}{e^{2x}}$$

[2 marks]
1 mark for determining the unsimplified derivative using either the product or quotient rule.
1 mark for determining the simplified derivative.

QUESTION 13 (6 marks)

a) $\text{area} = \int_{-3}^{3} 27 - 3x^2 dx$

[2 marks]
1 mark for showing correct bounds.
1 mark for writing an appropriate definite integral.

b) $\text{area} = \int_{-3}^{3} 27 - 3x^2 dx$

$$= \left[27x - x^3\right]_{-3}^{3}$$

$$= \left(27 \times 3 - 3^3\right) - \left(27 \times (-3) - (-3)^3\right)$$

$$= (27 \times 3 - 27) - \left(27 \times (-3) + 27\right)$$

$$= 108 \text{ units}^2$$

[2 marks]
1 mark for integrating both terms.
1 mark for determining the area.
*Note: Consequential on answer to **Question 13a)**.*

c) Let Δx be the width of a trapezoid.

$$\text{area} \approx \frac{\Delta x}{2}\left(f(-3) + 2f(-2) + 2f(-1) + 2f(0) + 2f(1) + 2f(2) + f(3)\right)$$

$$= \frac{0 + 2 \times 15 + 2 \times 24 + 2 \times 27 + 2 \times 24 + 2 \times 15 + 0}{2}$$

$$= 105 \text{ units}^2$$

[2 marks]
1 mark for using an appropriate formula and identifying that the function should be evaluated at x = –3, –2, –1, 0, 1, 2, 3.
1 mark for estimating the area.

QUESTION 14 (5 marks)

a) $E(X) = np$

$= 3 \times 0.5$

$= 1.5$ gold balls

[1 mark]

1 mark for determining the expected value.

b) $Var(X) = np(1-p)$

$= 3 \times \frac{1}{2} \times \frac{1}{2}$

$= 0.75$

$\therefore \sigma = \sqrt{0.75}$

[1 mark]

1 mark for determining the standard deviation using the rule for variance and taking the square root.

c) **Method 1:**

$P(\text{not winning a prize}) = P(X = 0)$

$= \binom{3}{0} p^0 (1-p)^3$

$= \left(\frac{1}{2}\right)^3$

$= \frac{1}{8}$

$P(\text{winning a prize}) = P(X \geq 1)$

$= 1 - P(X = 0)$

$= 1 - \frac{1}{8}$

$= \frac{7}{8}$

Method 2:

$P(\text{winning a prize}) = P(X = 1) + P(X = 2) + P(X = 3)$

$= \binom{3}{1} p^1 (1-p)^2 + \binom{3}{2} p^2 (1-p)^1 + \binom{3}{3} p^3 (1-p)^0$

$= \frac{7}{8}$

$\frac{7}{8} < \frac{9}{10}$

Thus, the vendor's claim is incorrect, as less than 90% of participants will win a prize.

[3 marks]

1 mark for using an appropriate strategy to determine the probability.

1 mark for determining the probability.

1 mark for making an appropriate conclusion regarding the vendor's claim.

QUESTION 15 (4 marks)

a) $z = \dfrac{X - \mu}{\sigma}$

For Mathematics:

$$z = \frac{70 - 45}{15}$$
$$= \frac{25}{15}$$
$$= \frac{5}{3}$$

For Chemistry:

$$x = \frac{76 - 51}{20}$$
$$= \frac{25}{20}$$
$$= \frac{5}{4}$$

Aleyna's z-score for Mathematics is higher, which means that she achieved a comparatively better result in the Mathematics competition.

[3 marks]
1 mark for determining the Mathematics z-score.
1 mark for determining the Chemistry z-score.
1 mark for determining that Aleyna achieved a better result in Mathematics.

b) $z = \dfrac{X - \mu}{\sigma}$

$$0.5 = \frac{X - 45}{15}$$
$$X = 52.5$$

Therefore, a student would need to score more than 52.5 marks to receive a distinction.

[1 mark]
1 mark for determining the minimum score.

QUESTION 16 (6 marks)

Graph 1: $y = a\cos x$

Graph 2: $y = a\sin x$

Points of intersection:

$$a\cos x = a\sin x$$
$$\cos x = \sin x$$

The intersections occur for x in quadrants 1 and 3.

Therefore, $x = \frac{\pi}{4}$ and $-\frac{3\pi}{4}$.

$$\begin{aligned}\text{shaded area} &= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} a\cos x - a\sin x \, dx \\ &= a\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \cos x - \sin x \, dx \\ &= a\left[\sin x + \cos x\right]_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \\ &= a\left(\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) - \sin\left(-\frac{3\pi}{4}\right) - \cos\left(-\frac{3\pi}{4}\right)\right) \\ &= a\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\ &= \frac{4a}{\sqrt{2}}\end{aligned}$$

Thus:

$$\frac{4a}{\sqrt{2}} = 6\sqrt{2}$$
$$a = 3$$

[6 marks]
1 mark for determining the functions for both graphs.
1 mark for determining the x-values for both points of intersection.
1 mark for setting up the definite integral.
1 mark for integrating the integrand.
1 mark for using trigonometric substitution to determine the area of the shaded region in terms of a.
1 mark for determining the value of a.
Note: Accept follow-through errors.

QUESTION 17 (6 marks)

$y = \ln\left(x^2 + 2\right)$

$y' = \dfrac{2x}{x^2 + 2}$

$y'' = \dfrac{4 - 2x^2}{\left(x^2 + 2\right)^2}$

Determining the critical points gives:

$$y' = \frac{2x}{x^2 + 2} = 0$$

Therefore, $x = 0$ is a critical point.

Categorising the critical point gives:

When $x = 0$, $y'' = \dfrac{4 - 2 \times 0^2}{\left(0^2 + 2\right)^2} > 0$.

Therefore, by the second derivative test, there is a local minimum when $x = 0$.

Determining the y-intercept gives:

Let $x = 0$.

$y = \ln(2)$

Therefore, both the y-intercept and local minimum occur at $(0, \ln(2))$.

Determining the x-intercept(s) gives:

Let $y = 0$.

$0 = \ln\left(x^2 + 2\right)$

$1 = x^2 + 2$

This has no solution; therefore, there are no x-intercepts.

Determining the x-coordinate of the points of inflection gives:

$$y'' = \frac{4 - 2x^2}{\left(x^2 + 2\right)^2} = 0$$

$$\therefore 4 - 2x^2 = 0$$

$$x = \pm\sqrt{2}$$

Determining the y-coordinate of the points of inflection gives:

$$\begin{aligned} y &= \ln\left(x^2 + 2\right) \\ &= \ln\left(\left(\sqrt{2}\right)^2 + 2\right) \\ &= \ln(2 + 2) \\ &= \ln(4) \end{aligned}$$

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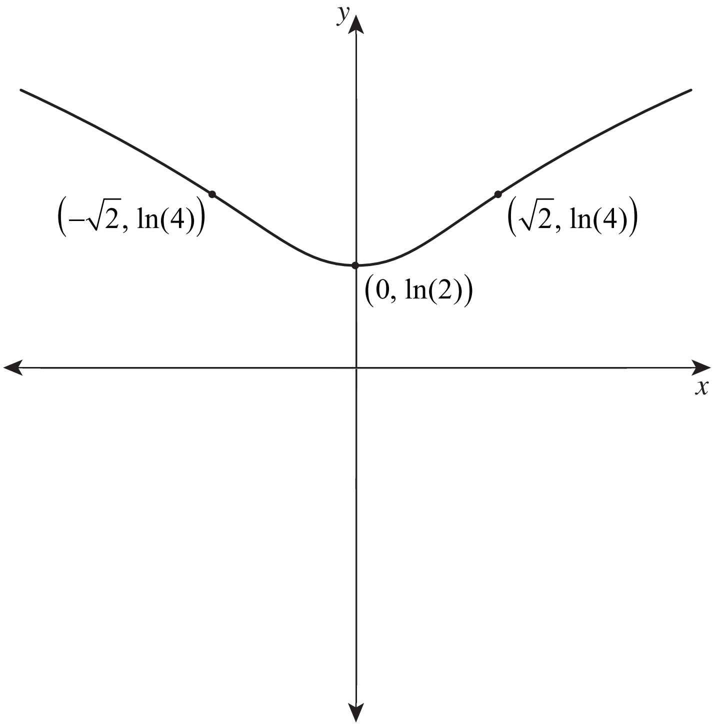
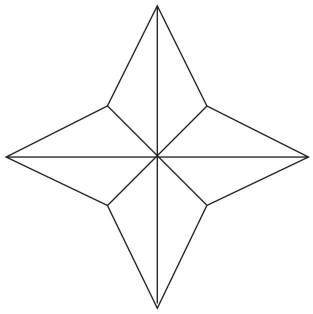


[6 marks]
1 mark for determining the first derivative.
1 mark for determining the second derivative.
1 mark for determining that there is a critical point at x = 0 and using the second derivative test to categorise the critical point as a local minimum.
1 mark for determining the y-coordinate of the local minimum.
1 mark for determining the coordinates of both points of inflection.
1 mark for sketching a graph with the correct shape and all relevant coordinates.
Note: The graph should be symmetrical and should not have any local maxima.

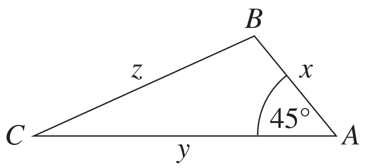
QUESTION 18 (6 marks)

For example:



The area of each triangle is 1 unit^2.

Let x and y be the lengths of the two interior sides and z be the length of a side of the octagon.



The angle at A is $\frac{360}{8} = 45°$.

$$\begin{aligned}
\text{area} = 1 &= \frac{1}{2}bc\sin(A) \\
1 &= \frac{1}{2}xy\sin(45) \\
1 &= \frac{1}{2}xy\frac{1}{\sqrt{2}} \\
1 &= \frac{1}{2\sqrt{2}}xy \\
xy &= 2\sqrt{2} \\
y &= \frac{2\sqrt{2}}{x}
\end{aligned}$$

Determining the length of z gives:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

$$z^2 = x^2 + y^2 - 2xy\cos(A)$$

$$\begin{aligned}
z^2 &= x^2 + \left(\frac{2\sqrt{2}}{x}\right)^2 - 2\left(2\sqrt{2}\right)\cos(45) \\
&= x^2 + \frac{8}{x^2} - 2\left(2\sqrt{2}\right)\times\frac{1}{\sqrt{2}} \\
&= x^2 + \frac{8}{x^2} - 4
\end{aligned}$$

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Minimising z^2 will also minimise z:

$$\frac{dz^2}{dx} = 2x - 2\times\frac{8}{x^3}$$
$$= 2x - \frac{16}{x^3}$$

Setting the derivative equal to 0 to determine the critical points gives:

$$2x - \frac{16}{x^3} = 0$$
$$2x = \frac{16}{x^3}$$
$$x^4 = 8$$
$$x = 8^{\frac{1}{4}}$$
$$= 2^{\frac{3}{4}} \text{ units}$$

Using the second derivative test to show that x minimises the value of z^2 gives:

$$\frac{d^2z^2}{dx^2} = 2 + \frac{48}{x^4} > 0$$

Since the second derivative (at the critical point) is greater than 0, by the second derivative test, it is a local minimum. Since z^2 is minimised, z is minimised and thus the perimeter of the octagon is minimised.

Showing that $x = y$ gives:

$$x = 2^{\frac{3}{4}} \text{ units}$$
$$y = \frac{2\sqrt{2}}{x}$$
$$= \frac{2^{\frac{3}{2}}}{2^{\frac{3}{4}}}$$
$$= 2^{\frac{3}{2}-\frac{3}{4}}$$
$$= 2^{\frac{3}{4}}$$
$$= x$$

Since $x = y$, the triangle ABC is isosceles.

[6 marks]

1 mark for setting up an accurate equation in terms of one variable for either z, z^2 or the entire perimeter.

1 mark for determining the derivative for the expression for z, z^2 or the perimeter.

1 mark for determining the critical value.

1 mark for using an appropriate method to categorise the critical point as a local minimum.

1 mark for using an appropriate method to show that the triangle is isosceles.

1 mark for showing clear and logical organisation of working.

Note: Accept follow-through errors.

QUESTION 19 (5 marks)

$$a(t)=\frac{2\sinh(\ln t)}{\ln\left(2\cosh t-e^{-t}\right)}$$

Using the definitions of the hyperbolic functions to simplify the numerator and denominator gives the following.

Numerator:

$$\begin{aligned}2\sinh(\ln t)&=2\times\frac{e^{\ln t}-e^{-\ln t}}{2}\\&=e^{\ln t}-e^{-\ln t}\\&=t-\frac{1}{t}\end{aligned}$$

Denominator:

$$\begin{aligned}\ln\left(2\cosh t-e^{-t}\right)&=\ln\left(2\times\frac{e^{t}+e^{-t}}{2}-e^{-t}\right)\\&=\ln\left(e^{t}\right)\\&=t\end{aligned}$$

$$\begin{aligned}\therefore a(t)&=\frac{t-\frac{1}{t}}{t}\\&=1-\frac{1}{t^2}\end{aligned}$$

Integrating to determine the velocity $v(t)$ and displacement $x(t)$ functions gives the following.

$$\begin{aligned}v(t)&=\int 1-\frac{1}{t^2}\,dt\\&=t+\frac{1}{t}+c\end{aligned}$$

Since $v(1)=2$, $v(1)=2=1+\frac{1}{1}+c$.

Therefore, $c=0$

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Thus:

$v(t)=t+\frac{1}{t}$, for $t>0$ seconds

$$\begin{aligned} x(t) &= \int v(t)\,dt \\ &= \int t+\frac{1}{t}\,dt \\ &= \frac{t^2}{2}+\ln t+d \end{aligned}$$

Since $x(1)=\frac{e^{-4}+1}{2}$, $x(1)=\frac{e^{-4}+1}{2}=\frac{1^2}{2}+\ln 1+d$.

Therefore, $d=\frac{e^{-4}}{2}$.

Thus, $x(t)=\frac{t^2}{2}+\ln t+\frac{e^{-4}}{2}$.

$$\begin{aligned} x\left(e^2\right) &= \frac{\left(e^2\right)^2}{2}+\ln\left(e^2\right)+\frac{e^{-4}}{2} \\ &= \frac{e^4}{2}+2+\frac{e^{-4}}{2} \\ &= \cosh(4)+2 \text{ m} \end{aligned}$$

[5 marks]

1 mark for simplifying the expression for the acceleration function. Note: This needs to be simplified so that it can be easily integrated.

1 mark for integrating for the velocity function and determining the constant of integration.

1 mark for integrating for the displacement function.

1 mark for determining the constant of integration for the displacement function.

1 mark for determining the displacement at e^2 seconds using appropriate units and an appropriate hyperbolic function.

Note: Accept follow-through errors.