Neap

Trial Examination 2021

Suggested solutions

QCE Specialist Mathematics Units 3&4

Paper 1 – Technology-free

SECTION 1 – MULTIPLE-CHOICE QUESTIONS

	A	B	C	D
1.	●	○	○	○
2.	○	○	○	●
3.	○	○	○	●
4.	○	●	○	○
5.	●	○	○	○
6.	●	○	○	○
7.	○	○	○	●
8.	○	○	●	○
9.	○	●	○	○
10.	○	●	○	○

QUESTION 1 A

Let $u = \cos(x)$ so $du = -\sin(x)dx$.

$$I = \int -\frac{1}{u^3}$$
$$= \frac{1}{2u^2}$$
$$= \frac{1}{2\cos^2(x)}$$
$$= \frac{1}{2}\sec^2(x)$$

QUESTION 2 D

D is correct. The derivative of $\ln(x)$ is $\frac{1}{x}$, so the slope decreases as x increases. **A** is incorrect. The slope field of the equation in **A** would show an exponential growth shape. **B** is incorrect. The slope field of the equation in **B** would show an exponential decay shape. **C** is incorrect. The slope field of the equation in **C** would show a trigonometric shape.

QUESTION 3 D

D is correct. As z is in the fourth quadrant, iz should be in the first quadrant. As a result, $-iz$ should be in the third quadrant. **A**, **B** and **C** are incorrect. These options do not show $-iz$ in the third quadrant.

QUESTION 4 B

The vector in the opposite direction is $-4\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$.

$$\frac{1}{\sqrt{(-4)^2 + 6^2 + (-3)^2}}(-4\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}) = \frac{1}{\sqrt{61}}(-4\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k})$$

QUESTION 5 A

$$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$
$$= 6x$$

$$\frac{1}{2}v^2 = 3x^2 + C$$

When $x = 0$, $v = -3$.

$$C = \frac{9}{2}$$

$$v^2 = 6x^2 + 9$$
$$v = \pm\sqrt{6x^2 + 9}$$

As $v = -3$ when $x = 0$, **A** is correct.

QUESTION 6 A

$f(x) = \tan^{-1}(x)$

$f'(x) = \dfrac{1}{x^2+1}$

$f''(x) = -\dfrac{2x}{\left(x^2+1\right)^2}$

$f'(x) = f''(x)$

$1 = -\dfrac{2x}{x^2+1}$

$x^2 + 2x + 1 = 0$

$(x+1)^2 = 0$

$x = -1$

QUESTION 7 D

$$\boldsymbol{a} + \lambda(\boldsymbol{b} - \boldsymbol{a}) = 2\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k} + \lambda\left((-3-2)\boldsymbol{i} + (1--3)\boldsymbol{j} + (-1-3)\boldsymbol{k}\right)$$
$$= 2\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k} + \lambda\left(-5\boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k}\right)$$

QUESTION 8 C

$\tan\theta = \dfrac{h}{40}$

$h = 40\tan\theta$

$\dfrac{dh}{dt} = \dfrac{dh}{d\theta} \times \dfrac{d\theta}{dt}$

$= 40\sec^2\theta \times \dfrac{d\theta}{dt}$

$\dfrac{d\theta}{dt} = \dfrac{1}{4\sec^2\theta}$

$= \dfrac{1}{4}\cos^2\theta$

When $h = 30$, $\tan\theta = \dfrac{3}{4}$, so $\cos\theta = \dfrac{4}{5}$.

$\dfrac{d\theta}{dt} = \dfrac{1}{4}\left(\dfrac{4}{5}\right)^2$

$= \dfrac{4}{25}$

QUESTION 9 B

$$\int \frac{1}{y^2}dy = -\frac{1}{10}\int e^x dx$$

$$-\frac{1}{y} = -\frac{1}{10}e^x + C$$

$$= \frac{C - e^x}{10}$$

$$y = \frac{10}{e^x - C}$$

$$\equiv \frac{10}{e^x + C}$$

QUESTION 10 B

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -2 & 1 \\ -2 & 3 & 1 \end{matrix}$$

$$= (-2\,.\,1 - 1\,.\,3)\,\boldsymbol{i} - \big(1\,.\,1 - (1\,.\,-2)\big)\,\boldsymbol{j} + \big(1\,.\,3 - (-2\,.\,-2)\big)\,\boldsymbol{k}$$

$$= -5\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$$

$$|\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{(-5)^2 + (-3)^2 + (-1)^2}$$

$$= \sqrt{35}$$

SECTION 2

QUESTION 11 (8 marks)

a) $\omega^3 = 1$, so $\omega^4 = \omega$

$$\omega^3 - 1 = 0$$

$$(\omega - 1)(\omega^2 + \omega + 1) = 0$$

$$\omega^2 + \omega + 1 = 0$$

[2 marks]

1 mark for correctly showing that $\omega^4 = \omega$.

1 mark for correctly showing that $1 + \omega + \omega^2 = 0$.

b) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5) = (1-\omega)^2(1-\omega^2)^2$

$\omega^4 = \omega$ and $\omega^5 = \omega^2$

$$\begin{aligned}(1+\omega^2-2\omega)(1+\omega^4-2\omega) &= (1+\omega^2+\omega-3\omega)(1+\omega+\omega^2-3\omega)\\ &= (-3\omega)(-3\omega^2)\\ &= 9\omega^3\\ &= 9\end{aligned}$$

[3 marks]

1 mark for using $\omega^4 = \omega$ and $\omega^5 = \omega^2$ to simplify brackets.

1 mark for expanding brackets and simplifying by using $1 + \omega + \omega^2 = 0$.

1 mark for final result.

c)

$$\begin{aligned}(1+\omega)(1+2\omega)(1+3\omega)(1+5\omega) &= (1+3\omega+2\omega^2)(1+8\omega+15\omega^2)\\ &= (2\omega+\omega^2)(7\omega+14\omega^2) \qquad \text{since } 1+\omega+\omega^2 = 0\\ &= 14\omega^2+7\omega^3+28\omega^3+14\omega^4\\ &= 14(1+\omega+\omega^2)+21\omega^3\\ &= 21\end{aligned}$$

[3 marks]

1 mark for expanding brackets in sets of 2.

1 mark for expanding and applying $1 + \omega + \omega^2 = 0$.

1 mark for final result.

QUESTION 12 (8 marks)

a) Let $u = x^n$ so $du = nx^{n-1}dx$, and let $\sqrt{1-x} = dv$ so $v = -\frac{2(1-x)^{\frac{3}{2}}}{3}$.

$$U_n = \left| \frac{-2(1-x)^{\frac{3}{2}}}{3} x^n \right|_0^1 + \frac{2n}{3}\int_0^1 x^{n-1}(1-x)^{\frac{3}{2}}dx$$

$$= \frac{2n}{3}\int_0^1 x^{n-1}\sqrt{1-x}(1-x)dx$$

$$= \frac{2n}{3}U_{n-1} - \frac{2n}{3}U_n$$

$$U_n + \frac{2n}{3}U_n = \frac{2n}{3}U_{n-1}$$

$$U_n = \frac{2n}{2n+3}U_{n-1}$$

[3 marks]

1 mark for assigning u and dv.

1 mark for finding $U_n = \frac{2n}{3}U_{n-1} - \frac{2n}{3}U_n$.

1 mark for rearranging for final result.

b) $U_n = \frac{2n}{2n+3}U_{n-1},\ U_{n-1} = \frac{2(n-1)}{2n+1}U_{n-2},\ \ldots$

$$U_0 = \int_0^1 \sqrt{1-x}\,dx$$

$$= \frac{2}{3}$$

$$U_n = \frac{2n}{2n+3} \times \frac{2(n-1)}{2n+1} \times \frac{2(n-2)}{2n-1}\ \ldots\ \frac{2}{3} = \frac{2^n n!}{(2n+3)(2n+1)\ldots 5} \times \frac{2}{3}$$

Multiplying numerator and denominator by $(2n+2)(2n)\ldots(4)(2)$:

$$U_n = \frac{2^n n!\, 2^{n+1}(n+1)(n)\ldots}{(2n+3)(2n+2)(2n+1)(2n)\ldots 5} \times \frac{2}{3(2)}$$

$$= \frac{2^n 2^{n+2} n!(n+1)!}{(2n+3)!}$$

$$= \frac{n!(n+1)!}{(2n+3)!}2^{2n+2}$$

$$= \frac{n!(n+1)!}{(2n+3)!}4^{n+1}$$

[5 marks]

1 mark for writing recursive results of U_n, U_{n-1}, …

1 mark for finding U_0.

1 mark for writing out U_n*, using previous steps.*

1 mark for multiplying numerator and denominator by (2n + 2)(2 n) … (4)(2).

1 mark for final result.

QUESTION 13 (6 marks)

a) $a = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx}$

$= x - 4$

$\int d\left(\frac{1}{2}v^2\right) = \int (x-4)\,dx$

$\frac{1}{2}v^2 = \frac{x^2}{2} - 4x + C$

$v = \sqrt{5}$ m s^{-1} at 7 m

$\frac{5}{2} = \frac{49}{2} - 28 + C$

$C = 6$

$\frac{1}{2}v^2 = \frac{x^2}{2} - 4x + 6$

$v^2 = x^2 - 8x + 12$

$= (x-6)(x-2)$

[4 marks]

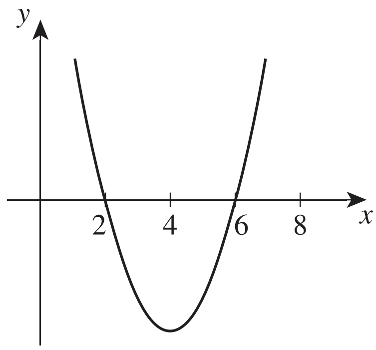
1 mark for finding $\frac{1}{2}v^2$.

1 mark for substituting to find C.

1 mark for simplifying v^2.

1 mark for correct v (since v > 0).

b) As $v^2 \geq 0$, so $(x-6)(x-2) \geq 0$.



Looking at the graph of the parabola, it follows that $x \leq 2$ or $x \geq 6$. As $x = 6$ initially, $x \geq 6$. As $v = +\sqrt{(x-6)(x-2)}$, the particle moves from $x = 6$ towards infinity with increasing velocity.

[2 marks]

1 mark for stating that x < 2 or x > 6 since $v^2 > 0$.

1 mark for stating that x > 6, with some description of motion.

Note: A graph may be drawn to find x and its restrictions, but is not required for full marks.

QUESTION 14 (5 marks)

Test for $n = 1$:

$$\begin{aligned}3^1 + 7^1 &= 10\\ &= 10 \times 1\end{aligned}$$

Therefore, the statement is true when $n = 1$.

Assume the statement is true for $n = k$, where k is an odd integer.

$$\begin{aligned}3^k + 7^k &= 10m \qquad \text{where } m \text{ is an integer}\\ 3^k &= 10m - 7^k\end{aligned}$$

Test for $n = k + 2$ (since we are only dealing with odd n values):

$$\begin{aligned}3^{k+2} + 7^{k+2} &= 9 \times 3^k + 49 \times 7^k\\ &= 9\left(10m - 7^k\right) + 49 \times 7^k \qquad \text{from assumption}\\ &= 90m + 40 \times 7^k\\ &= 10\left(9m + 4 \times 7^k\right)\end{aligned}$$

Since m and k are integers, $9m + 4 \times 7^k$ is an integer. Therefore, the statement is true for $n = \text{k} + 2$. Hence, by mathematical induction, the statement is true for all odd integers.

[5 marks]

1 mark for substituting n = 1.

1 mark for testing n = k + 2.

1 mark for substituting assumption.

1 mark for explaining why $10\left(9m + 4 \times 7^k\right)$ is an integer.

1 mark for conclusion statement.

QUESTION 15 (7 marks)

a)

$$\begin{aligned}\frac{1}{\sqrt{x^2+1}} &= \frac{\sqrt{2}}{\sqrt{x^2+4}}\\ \sqrt{x^2+4} &= \sqrt{2\left(x^2+1\right)}\\ x^2 + 4 &= 2\left(x^2+1\right)\\ x^2 + 4 &= 2x^2 + 2\\ x^2 &= 2\\ x &= \pm\sqrt{2}\end{aligned}$$

[2 marks]

1 mark for equating two expressions for x.

1 mark for final result.

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b) By considering one slice of the curve:

$V = \pi\left(y_1{}^2 - y_2{}^2\right)dx$, where $y_1 = \dfrac{1}{\sqrt{x^2+1}}$ and $y_2 = \dfrac{\sqrt{2}}{\sqrt{x^2+4}}$.

Therefore, $V = \pi\int_{-\sqrt{2}}^{\sqrt{2}}\left(\dfrac{1}{x^2+1} - \dfrac{2}{x^2+4}\right)dx$.

As the curves are symmetrical, $V = 2\pi\int_{0}^{\sqrt{2}}\left(\dfrac{1}{x^2+1} - \dfrac{2}{x^2+4}\right)dx$.

$$V = 2\pi\left[\arctan(x) - \arctan\left(\frac{x}{2}\right)\right]_0^{\sqrt{2}}$$

$$= 2\pi\left(\arctan\left(\sqrt{2}\right) - \arctan\left(\frac{\sqrt{2}}{2}\right)\right)$$

$$= 2\pi\arctan\left(\frac{\sqrt{2} - \frac{1}{\sqrt{2}}}{1 + \sqrt{2}\times\frac{1}{\sqrt{2}}}\right)$$

$$= 2\pi\arctan\left(\frac{\frac{2-1}{\sqrt{2}}}{2}\right)$$

$$= 2\pi\arctan\left(\frac{1}{2\sqrt{2}}\right)$$

[5 marks]
1 mark for correct volume of one slice.
1 mark for correct expression for overall volume.
1 mark for integrating individual expressions correctly.
1 mark for using double-angle formula correctly.
1 mark for final response.

QUESTION 16 (7 marks)

a) $$\int\frac{1}{\sqrt{4x+3}}dx = \frac{2\sqrt{4x+3}}{4} + C$$

$$= \frac{1}{2}\sqrt{4x+3} + C$$

[2 marks]
1 mark for correct substitution OR writing the expression as $(4x+3)^{-\frac{1}{2}}$ and then integrating by the power rule.

1 mark for final response.

b) $$\frac{dy}{dx}=\frac{\sqrt{4y+3}}{x^2}$$

$$\frac{1}{\sqrt{4y+3}}\,dy=\frac{1}{x^2}\,dx$$

Integrating both sides:

$$\frac{1}{2}\sqrt{4y+3}=-\frac{1}{x}+C$$

When $y = 1.5$, $x = -2$.

$$\frac{1}{2}\sqrt{9}=-\frac{1}{-2}+C$$

$$\frac{3}{2}=\frac{1}{2}+C$$

$$C=1$$

Therefore:

$$\frac{1}{2}\sqrt{4y+3}=-\frac{1}{x}+1$$

$$=\frac{(x-1)}{x}$$

$$\sqrt{4y+3}=\frac{2(x-1)}{x}$$

$$4y+3=\frac{4(x-1)^2}{x^2}$$

$$4y=\frac{4(x-1)^2}{x^2}-3$$

$$y=\frac{(x-1)^2}{x^2}-\frac{3}{4}$$

[5 marks]
1 mark for correctly integrating both sides.
1 mark for finding C.
1 mark for substituting in C and rearranging.
1 mark for squaring both sides to eliminate square root.
1 mark for final answer.

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QUESTION 17 (7 marks)

a) Equating P: $2 + 4n = 4 + q$

Equating H: $2p = 16 + 3q$

Equating O: $p = 4q$

Thus:

$4n - q = 2$

$-3q + 2p = 16$

$-4q + p = 0$

[2 marks]
1 mark for equating P, H and O.
1 mark for rearranging equations to move constants to one side, and n, p and q to the other side.

b) Assembling into a matrix (where $n = x$, $p = y$, $q = z$):

$$\left|\begin{array}{ccc|c} 4 & 0 & -1 & 2 \\ 0 & 2 & -3 & 16 \\ 0 & 1 & -4 & 0 \end{array}\right|$$

$R_3{}' \to R_2 - 2R_3$:

$$\left|\begin{array}{ccc|c} 4 & 0 & -1 & 2 \\ 0 & 2 & -3 & 16 \\ 0 & 0 & 5 & 16 \end{array}\right|$$

$5q = 16$

$q = \frac{16}{5}$

$2p - 3q = 16$

$2p - \frac{48}{5} = 16$

$2p = \frac{128}{5}$

$p = \frac{64}{5}$

$4n - q = 2$

$4n - \frac{16}{5} = 2$

$4n = \frac{26}{5}$

$n = \frac{13}{10}$

[4 marks]
1 mark for constructing matrix.
1 mark for $R_3{}' \to R_2 - 2R_3$ (or similar step to resolve matrix), thus finding q.
1 mark for substituting q back in to find p.
1 mark for substituting p back in to find n.
Note: Consequential on answer to Question 17a).

c) Verifying the equation:

$$P_2I_4 + \frac{13}{10}P_4 + \frac{64}{5}H_2O \rightarrow 4PH_4I + \frac{16}{5}H_3PO_4$$

Multiplying all terms by 10:

$$10P_2I_4 + 13P_4 + 128H_2O \rightarrow 40PH_4I + 32H_3PO_4$$

The equation is balanced for P, I, H, and O.

[1 mark]

1 mark for multiplying all terms by 10 and verifying that the equation is balanced.

Note: Consequential on answer to Question 17b).

QUESTION 18 (7 marks)

a) Cross product of the normals:

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$$

As the third plane passes through (2, 1, 0):

$$\begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = -1$$

Hence, the equation of the plane is $x - 3y - 5z = -1$.

[3 marks]

1 mark for cross product.

1 mark for substituting (2, 1, 0) into dot product.

1 mark for final response.

b) Equations of all planes:

$x + 2y - z = -3$ [1]

$2x - y + z = 10$ [2]

$x - 3y - 5z = -1$ [3]

$2 \times [1] - [2] = 5y - 3z = -16$ [4]

$[1] - [3] = 5y + 4z = -2$ [5]

$[5] - [4] \Rightarrow 7z = 14$

$z = 2$

Substituting into [5]:

$5y + 8 = -2$

$y = -2$

Substituting into [1]:

$x - 4 - 2 = -3$

$x = 3$

Therefore, the planes intersect at (3, –2, 2).

[4 marks]

1 mark for manipulating equations to solve for one of x, y or z.

2 marks for substituting obtained value to find values for the other two pronumerals.

1 mark for final point of intersection.