Neap

Trial Examination 2023

Question and Response Booklet

QCE Specialist Mathematics Units 1&2

Paper 2 – Technology-active

Student's Name: ______________________

Teacher's Name: ______________________

Time allowed

- Perusal time – 5 minutes
- Working time – 90 minutes

General instructions

- Answer all questions in this question and response booklet.
- QCAA-approved calculator **permitted**.
- Formula booklet provided.
- Planning paper will not be marked.

Section 1 (10 marks)

- 10 multiple choice questions

Section 2 (50 marks)

- 10 short response questions

SECTION 1

Instructions

- Choose the best answer for Questions 1–10.
- This section has 10 questions and is worth 10 marks.
- Use a 2B pencil to fill in the A, B, C or D answer bubble completely.
- If you change your mind or make a mistake, use an eraser to remove your response and fill in the new answer bubble completely.

	A	B	C	D
Example:	●	○	○	○

	A	B	C	D
1.	○	○	○	○
2.	○	○	○	○
3.	○	○	○	○
4.	○	○	○	○
5.	○	○	○	○
6.	○	○	○	○
7.	○	○	○	○
8.	○	○	○	○
9.	○	○	○	○
10.	○	○	○	○

SECTION 2

Instructions

- Write using black or blue pen.
- Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.
- If you need more space for a response, use the additional pages at the back of this booklet.
  - On the additional pages, write the question number you are responding to.
  - Cancel any incorrect response by ruling a single diagonal line through your work.
  - Write the page number of your alternative/additional response, i.e. See page …
  - If you do not do this, your original response will be marked.
- This section has 10 questions and is worth 50 marks.

DO NOT WRITE ON THIS PAGE

THIS PAGE WILL NOT BE MARKED

QUESTION 11 (4 marks)

Two complex numbers, z and w, are given as $z = -2 + bi$ and $w = a + (a + b)i$, where a and b are real numbers. Given that $\overline{z} = w$, determine the values of a and b.

QUESTION 12 (5 marks)

Consider the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, where $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = 11\boldsymbol{i} - \boldsymbol{j}$.

a) If $\boldsymbol{c} = \boldsymbol{a} + 2\boldsymbol{b}$, determine $\boldsymbol{c}$. *[2 marks]*

b) Calculate the scalar product, $\boldsymbol{a} \cdot \boldsymbol{b}$. *[1 mark]*

c) Determine the unit vector, $\hat{\boldsymbol{a}}$. *[2 marks]*

QUESTION 13 (4 marks)

A hula hoop with a diameter of 70 cm has a small price tag attached to its outer surface. When the hoop rolls along the ground, the height of the price tag above ground will change. An observer notes that, as it rolls, the time it takes for the hoop to complete one full revolution is 0.5 seconds.

To model the height of the price tag above the ground, an equation with the form $h(t) = A\sin(Bt) + A$ is used, where $h(t)$ represents the height, in cm, of the price tag at time, t seconds, since the hoop began rolling.

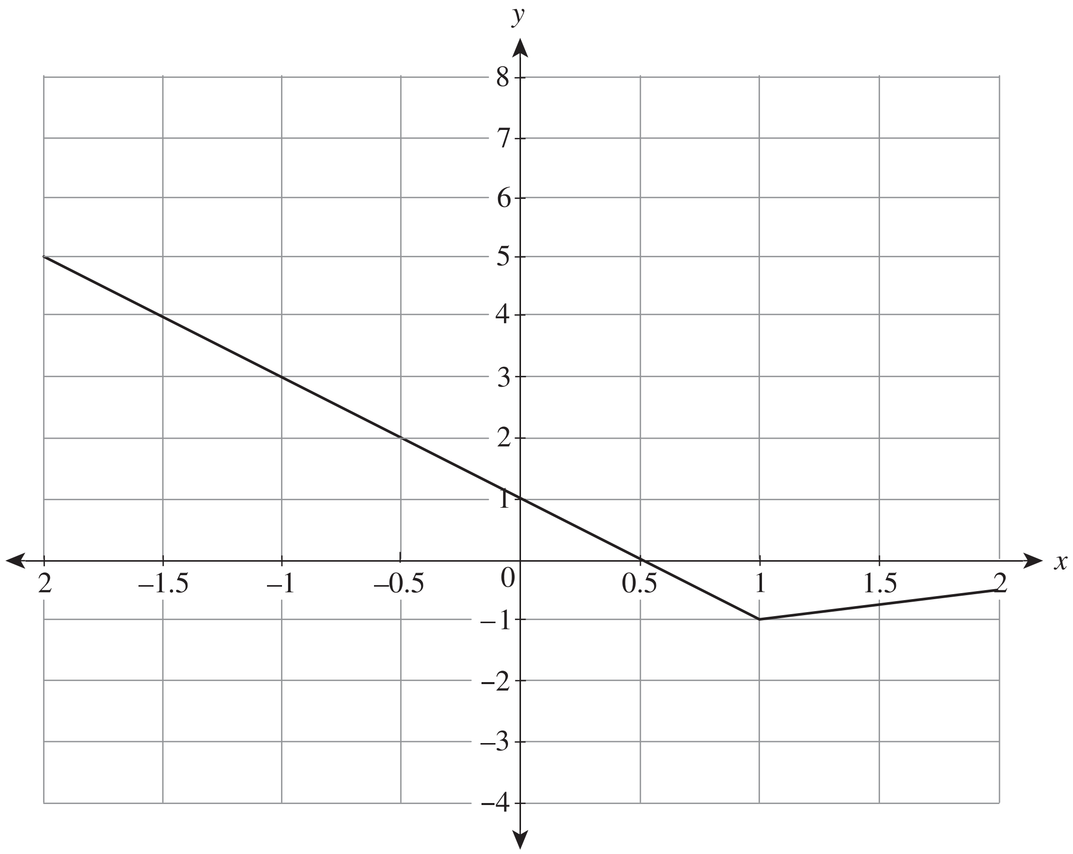
a) State a suitable value for A. *[1 mark]*

b) By performing a calculation, determine a suitable value for B. *[2 marks]*

c) Using the results of Questions 13a) and 13b), determine the initial height of the price tag, $h(0)$. *[1 mark]*

QUESTION 14 (4 marks)

The function $y = g(x)$ is graphed below over the domain $-2 \le x \le 2$.



Sketch the function $y = \dfrac{1}{g(x)}$ on the same set of axes, identifying any asymptotes that may exist within this domain.

QUESTION 15 (5 marks)

There are seven batteries in a drawer. A child needs two batteries for his new calculator.

a) Using the idea of combinations, write an expression for the number of selections the child can make when taking two batteries from the drawer containing seven batteries. *[1 mark]*

b) Evaluate the expression written in Question 15a). *[1 mark]*

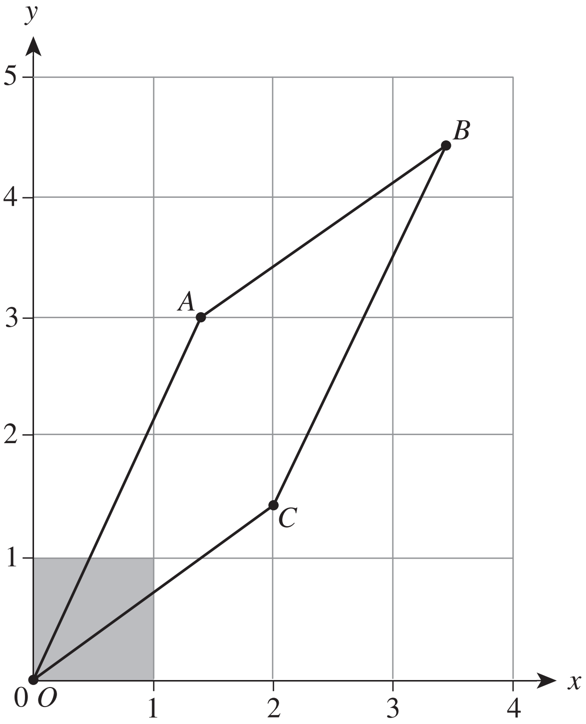
c) One of the seven batteries in the drawer is faulty.

Using the idea of combinations, determine how many selections of two batteries from the drawer of seven batteries will **not** contain the faulty battery. *[2 marks]*

d) Hence, or otherwise, state the probability that the calculator will operate when two batteries selected at random from the drawer are placed in it. Assume that the calculator will work, provided that neither of the batteries placed in it are faulty. *[1 mark]*

QUESTION 16 (7 marks)

A unit square with vertices at (0, 0), (0, 1), (1, 1) and (1, 0) undergoes a linear transformation represented by the matrix $\mathbf{T} = \begin{bmatrix} 2 & k \\ k & 3 \end{bmatrix}$. The original unit square and the transformed unit square are shown in the diagram.



The original unit square has been shaded. The quadrilateral formed by the application of **T**, *OABC*, has an area of four units^2.

a) Show that the value of k is $\sqrt{2}$. Briefly explain why it is not $-\sqrt{2}$. *[3 marks]*

b) Prove that quadrilateral *OABC* is a parallelogram. *[4 marks]*

QUESTION 17 (8 marks)

A plane undergoes a composition of linear transformations. Firstly, the plane is rotated by 45° clockwise; secondly, the plane is reflected through the x-axis; and thirdly, the plane is rotated by 45° anticlockwise. The application of these three transformations maps the point $\begin{pmatrix} x \\ y \end{pmatrix}$ to its image $\begin{pmatrix} x' \\ y' \end{pmatrix}$, according to the rule $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix}$.

a) Use matrix multiplication to determine the matrix **F**. *[4 marks]*

b) Determine the matrix **T** that corresponds to reflection through the line $y = x$. *[2 marks]*

c) Consider the following series of transformations.

- rotation of 45° clockwise
- reflection through the x-axis
- rotation of 45° anticlockwise

Evaluate the reasonableness of the claim that this series of transformations is equivalent to a reflection through the line $y = x$. *[2 marks]*

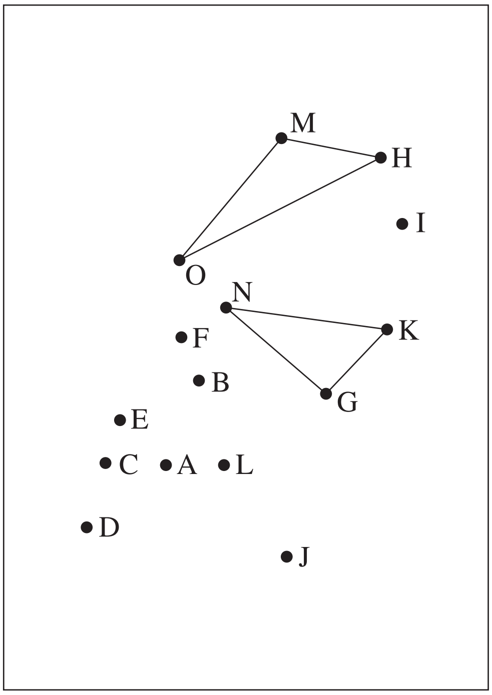
QUESTION 18 (4 marks)

The function $f(x) = 2\cos^2(2x) + \sin^2(2x)$ can be written in the form $f(x) = A\cos(B(x - C)) + D$.

Determine the values of A, B, C and D such that $2\cos^2(2x) + \sin^2(2x) = A\cos(B(x - C)) + D$.

QUESTION 19 (5 marks)

A sheet of paper is marked randomly with 15 different points, such that no three points are colinear. The claim has been made that no two distinct triangles constructed using these points as vertices will have the same area. To exhaustively test this claim, every distinct triangle that can be made is to be constructed and have its area calculated. The areas are then to be compared in pairs to check for equality.



If it takes 20 seconds to construct and calculate the area of each triangle and five seconds to compare the area of two triangles, determine the fraction of a week that it will take to exhaustively test the claim.

QUESTION 20 (4 marks)

It is asserted that if the 3×3 matrix $\mathbf{M}$ is non-singular, then it must be true that there is only one possible matrix that can serve as $\mathbf{M}^{-1}$.

A proof of this assertion can be completed by first assuming that the assertion is false. In addition, it can be assumed that both $\mathbf{K}$ and $\mathbf{L}$, where $\mathbf{K} \neq \mathbf{L}$, are inverse matrices for $\mathbf{M}$.

Use this structure of proof by contradiction to prove the assertion.

END OF PAPER

ADDITIONAL PAGE FOR STUDENT RESPONSES

Write the question number you are responding to.

ADDITIONAL PAGE FOR STUDENT RESPONSES

Write the question number you are responding to.

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Formula Booklet

QCE Specialist Mathematics Units 1&2

Mensuration			
circumference of a circle	$C = 2\pi r$	**area of a circle**	$A = \pi r^2$
area of a parallelogram	$A = bh$	**area of a trapezium**	$A = \frac{1}{2}(a+b)h$
area of a triangle	$A = \frac{1}{2}bh$	**total surface area of a cone**	$S = \pi rs + \pi r^2$
total surface area of a cylinder	$S = 2\pi rh + 2\pi r^2$	**surface area of a sphere**	$S = 4\pi r^2$
volume of a cone	$V = \frac{1}{3}\pi r^2 h$	**volume of a cylinder**	$V = \pi r^2 h$
volume of a prism	$V = Ah$	**volume of a pyramid**	$V = \frac{1}{3}Ah$
volume of a sphere	$V = \frac{4}{3}\pi r^3$		

Calculus	
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$
$\frac{d}{dx}e^x = e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}\ln(x) = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln\lvert x\rvert + c$
$\frac{d}{dx}\sin(x) = \cos(x)$	$\int \sin(x)dx = -\cos(x) + c$
$\frac{d}{dx}\cos(x) = -\sin(x)$	$\int \cos(x)dx = \sin(x) + c$
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\int \sec^2(x)dx = \tan(x) + c$
$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$	$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$
$\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2 - x^2}}$	$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c$
$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2}$	$\int \frac{a}{a^2 + x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$

Calculus		
chain rule	If $h(x)=f\left(g(x)\right)$ then $h'(x)=f'\left(g(x)\right)g'(x)$	If $y=f(u)$ and $u=g(x)$ then $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$
product rule	If $h(x)=f(x)g(x)$ then $h'(x)=f(x)g'(x)+f'(x)g(x)$	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$
quotient rule	If $h(x)=\frac{f(x)}{g(x)}$ then $h'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{\left(g(x)\right)^2}$	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
integration by parts	$\int f(x)g'(x)dx=f(x)g(x)-\int f'(x)g(x)dx$	$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$
volume of a solid of revolution	about the x-axis	$V=\pi\int_a^b\left[f(x)\right]^2dx$
	about the y-axis	$V=\pi\int_a^b\left[f(y)\right]^2dy$
Simpson's rule	$\int_a^b f(x)dx\approx\frac{w}{3}\left[f\left(x_0\right)+4\left[f\left(x_1\right)+f\left(x_3\right)+\ldots\right]+2\left[f\left(x_2\right)+f\left(x_4\right)+\ldots\right]+f\left(x_n\right)\right]$	
simple harmonic motion	If $\frac{d^2x}{dt^2}=-\omega^2x$ then $x=A\sin\left(\omega t+\alpha\right)$ or $x=A\cos\left(\omega t+\beta\right)$	
	$v^2=\omega^2\left(A^2-x^2\right)$	$T=\frac{2\pi}{\omega}$ $f=\frac{1}{T}$
acceleration	$a=\frac{dv}{dt}=\frac{d^2x}{dt^2}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$	

Real and complex numbers	
complex number forms	$z=x+yi=r\left(\cos(\theta)+i\sin(\theta)\right)=r\,\mathrm{cis}(\theta)$
modulus	$\lvert z\rvert=r=\sqrt{x^2+y^2}$
argument	$\arg(z)=\theta,\ \tan(\theta)=\frac{y}{x},-\pi<\theta\le\pi$
product	$z_1z_2=r_1r_2\,\mathrm{cis}\left(\theta_1+\theta_2\right)$
quotient	$\frac{z_1}{z_2}=\frac{r_1}{r_2}\mathrm{cis}\left(\theta_1-\theta_2\right)$
De Moivre's theorem	$\ddot{u}^n=\ {}^n\mathrm{cis}(\ \ \theta)$

Statistics		
binomial theorem	$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + y^n$	
permutation	${}^nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1)$	
combination	${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$	
sample means	mean	μ
	standard deviation	$\frac{\sigma}{\sqrt{n}}$
approximate confidence interval for $\boldsymbol{\mu}$	$\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$	

Trigonometry	
Pythagorean identities	$\sin^2(A) + \cos^2(A) = 1$ $\tan^2(A) + 1 = \sec^2(A)$ $\cot^2(A) + 1 = \operatorname{cosec}^2(A)$
angle sum and difference identities	$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ $\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$ $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$
double-angle identities	$\sin(2A) = 2\sin(A)\cos(A)$ $\cos(2A) = \cos^2(A) - \sin^2(A)$ $= 1 - 2\sin^2(A)$ $= 2\cos^2(A) - 1$
product identities	$\sin(A)\sin(B) = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$ $\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$ $\sin(A)\cos(B) = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$ $\cos(A)\sin(B) = \frac{1}{2}\left(\sin(A+B) - \sin(A-B)\right)$

Vectors and matrices		
magnitude	$\lvert \boldsymbol{a} \rvert = \left\lvert \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right\rvert = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}$	
scalar (dot) product	$\boldsymbol{a} \cdot \boldsymbol{b} = \lvert \boldsymbol{a} \rvert \lvert \boldsymbol{b} \rvert \cos(\theta)$	
	$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$	
vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + k\boldsymbol{d}$	
Cartesian equation of a line	$\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$	
vector (cross) product	$\boldsymbol{a} \times \boldsymbol{b} = \lvert \boldsymbol{a} \rvert \lvert \boldsymbol{b} \rvert \sin(\theta) \hat{\boldsymbol{n}}$	
	$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$	
vector projection	$\boldsymbol{a}$ on $\boldsymbol{b} = \lvert \boldsymbol{a} \rvert \cos(\quad) \hat{\boldsymbol{b}} = \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}} \right) \hat{\boldsymbol{b}}$	
vector equation of a plane	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$	
Cartesian equation of a plane	$ax + by + cz + d = 0$	
determinant	If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det(\mathbf{A}) = ad - bc$	
multiplicative inverse matrix	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, $\det(\mathbf{A}) \neq 0$	
linear transformations	dilation	$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$
	rotation	$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
	reflection (in the line $y = x\tan(\theta)$)	$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

Physical constant	
magnitude of mean acceleration due to gravity on Earth	$g = 9.8 \text{ m s}^{-2}$