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MATHEMATICAL METHODS

UNITS 3&4

CAMBRIDGE SENIOR MATHEMATICS
FOR QUEENSLAND

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INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

ISBN 978-1-108-45164-2

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108451642

First published 2019
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

Cover designed by Sardine Design
Text designed by Jane Pitkethly
Typeset by Jane Pitkethly & diacriTech
Printed in China by C & C Offset Printing Co. Ltd.

A catalogue record for this book is available from the National Library of Australia at www.nla.gov.au

ISBN 978-1-108-45164-2 Paperback

Additional resources for this publication at www.cambridge.edu.au/GO

Unit 3

ISBN 978-1-108-45164-2

ISBN 978-1-108-45164-2

Online assessment practice in the Interactive Textbook and Online Teaching Suite

IA1: Practice problem-solving and modelling task

IA2: Practice internal examination on Unit 3

Unit 4

ISBN 978-1-108-45164-2

ISBN 978-1-108-45164-2

Note: *A printable copy of the* **QCAA Formula sheet** *is available in the Interactive Textbook*

ISBN 978-1-108-45164-2

About the lead author and consultants

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He has also been active in the Australian Mathematics Trust, being involved with the writing of enrichment material and competition questions.

He has many years' experience as a Chief Examiner and Chairperson of examination panels.

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Introduction and overview

Cambridge Senior Mathematics for Queensland Mathematical Methods Units 3 & 4 provides complete and close coverage of the QCAA syllabus to be implemented in Year 12 from 2020. Its four components — the print book, downloadable PDF textbook, online Interactive Textbook (ITB) and Online Teaching Resource (OTS), both powered by the HOTmaths platform — contain a huge range of resources, including worked solutions available to schools in a single package at one convenient price (the OTS is included with class adoptions, conditions apply). There are no extra subscriptions or per-student charges to pay.

Preliminary topics (review of Units 1&2): The first four chapters can be considered as a review of Units 1&2: *Chapter 1 Functions and relations*, *Chapter 2 Coordinate geometry and transformations*, *Chapter3 Polynomial functions* and *Chapter 4 Trigonometric functions*. The topics covered in these chapters are important for Units 3&4 and of course may be examined at year 12. You may choose to complete these chapters prior to the beginning of Year 12.

In addition, two 'refresher' chapters are provided: *Chapter 7 Refresher on differentiation* and *Chapter 14 Refresher on probability and discrete random variables*. It is recommended that these be done just before the chapters for which they are preparation.

To help decide whether any students can be exempted from doing the preliminary topics and refresher chapters, the multiple-choice question sections from their chapter reviews are set up in the Online Teaching Suite to provide the option of being used as diagnostic tests for this purpose.

Degree of difficulty classification of questions: in the exercises, questions are classified as simple familiar **SF**, complex familiar **CF**, and complex unfamiliar **CU** questions. The revision chapters described below also contain model questions for each of these categories, and tests are also provided in the teacher resources, made up of such categorised model questions.

Three revision chapters of material covered in Units 3 and 4: These chapters contain sections on *Technology-free questions*, *Multiple choice questions*, *Extended-response questions*, and *Degree of difficulty classification of questions*. The first revision chapter occurs at the end of Unit 3, the second at the end of Unit 4 and there is a final revision chapter that will help with revision for the external examination

Calculator guidance: Throughout the book there is guidance for the use of the TI-Nspire CX non-CAS and the Casio fxCG20AU and fxCG50AU graphics calculators for the solution of problems. Guidance on the TI-84Plus CE is included in the interactive textbook, accessed via icons next to the TI-Nspire boxes. There are also online guides for the general use of each of these calculators.

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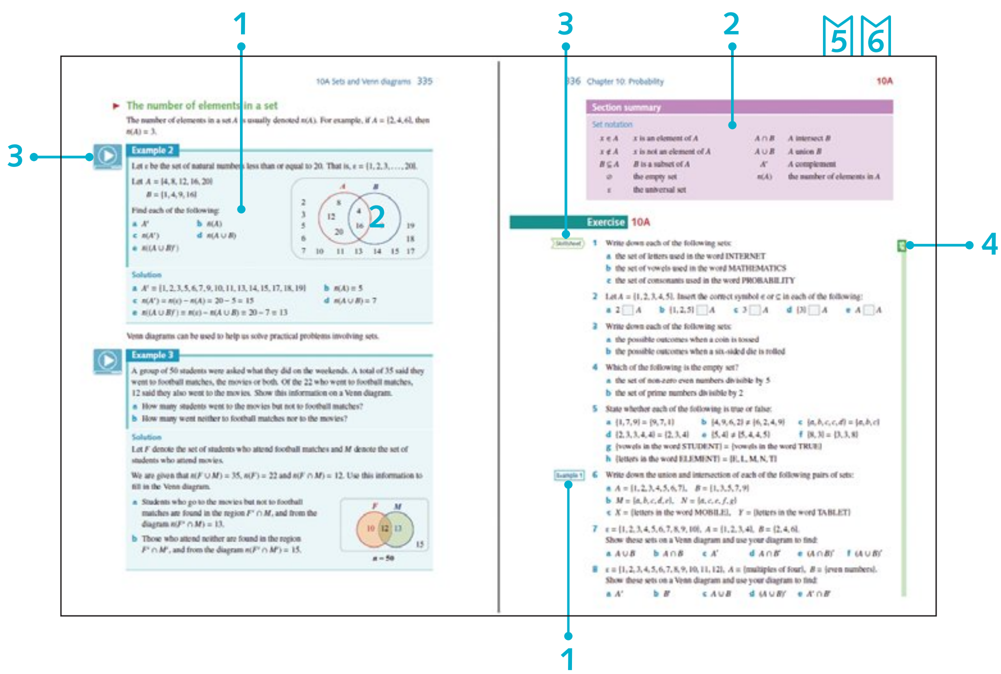
The online graphing calculator from Desmos.com is also embedded in the interactive textbook, as blank screens that students and teachers can use for their own calculations, or as widgets which have been set up for a variety of activities. The new Desmos geometry tool is also embedded in the Interactive Textbook, and activities and widgets using the tool will be added as they are developed.

Assessment practice: two sets of problem-solving and modelling tasks and internal and external examinations are provided, one in the Interactive Textbook which students can access, and a different set in the Online Teaching Suite for teacher-only access.

▶ Overview of the print book (shown below)

1. Graded step-by-step worked examples with precise explanations (and video versions) encourage independent learning, and are linked to exercises.
2. Section summaries provide important concepts in boxes for easy reference.
3. Additional linked resources in the Interactive Textbook are indicated by icons, such as skillsheets and video versions of examples.
4. Degree of difficulty categories are indicated in exercises (similar familiar, complex familiar and complex unfamiliar).
5. Chapter reviews contain a chapter summary and technology-free, multiple-choice, and extended-response questions. The latter are classified by degree of difficulty.
6. The glossary includes page numbers of the main explanation of each term.

Numbers refer to descriptions above. Content shown from Units 1 & 2.



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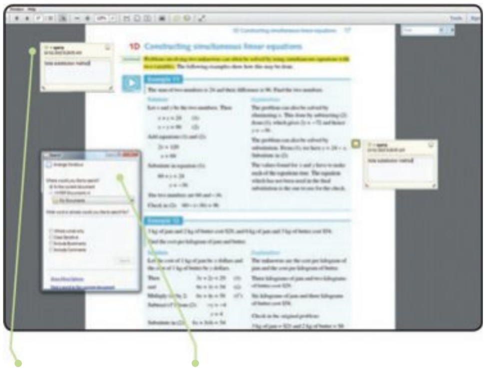
Overview of the downloadable PDF textbook

7 The convenience of a downloadable PDF textbook has been retained for times when users cannot go online.

8 PDF annotation and search features are enabled.



Overview of the Interactive Textbook (shown on the page opposite)

The **Interactive Textbook (ITB)** is an online HTML version of the print textbook powered by the HOTmaths platform, included with the print book or available as a separate purchase.

9 The material is formatted for on screen use with a convenient and easy-to-use navigation system and links to all resources.

10 The new **Workspaces** enable students to enter working and answers online and to save them. Input is by typing, with the help of a symbol palette, handwriting and drawing on tablets, or by uploading images of writing or drawing.

11 The new **self-assessment tools** enable students to check answers, mark their own work, and rate their confidence level in their work. This helps develop responsibility for learning and communicates progress and performance to the teacher. Student accounts can be linked to the learning management system used by the teacher in the Online Teaching Suite, so that teachers can review student self-assessment and provide feedback or adjust marks.

12 Examples have **video versions** to encourage independent learning.

13 **Worked solutions** are included and can be enabled or disabled in the student ITB accounts by the teacher.

14 Interactive **Desmos widgets** and activities based on embedded graphics calculator and geometry tool windows demonstrate key concepts and enable students to visualise the mathematics.

15 The **Desmos graphics calculator, scientific calculator,** and **geometry tool** are also embedded for students to use for their own calculations and exploration.

16 **Quick quizzes** containing automarked multiple choice questions enable students to check their understanding.

17 **Definitions** pop up for key terms in the text, and are also provided in a **dictionary**.

18 Messages from the teacher assign tasks and tests.

19 **Assessment practice** items for student access are provided in downloadable PDF files.

20 Online guides to technology are provided for three calculator models and Desmos.

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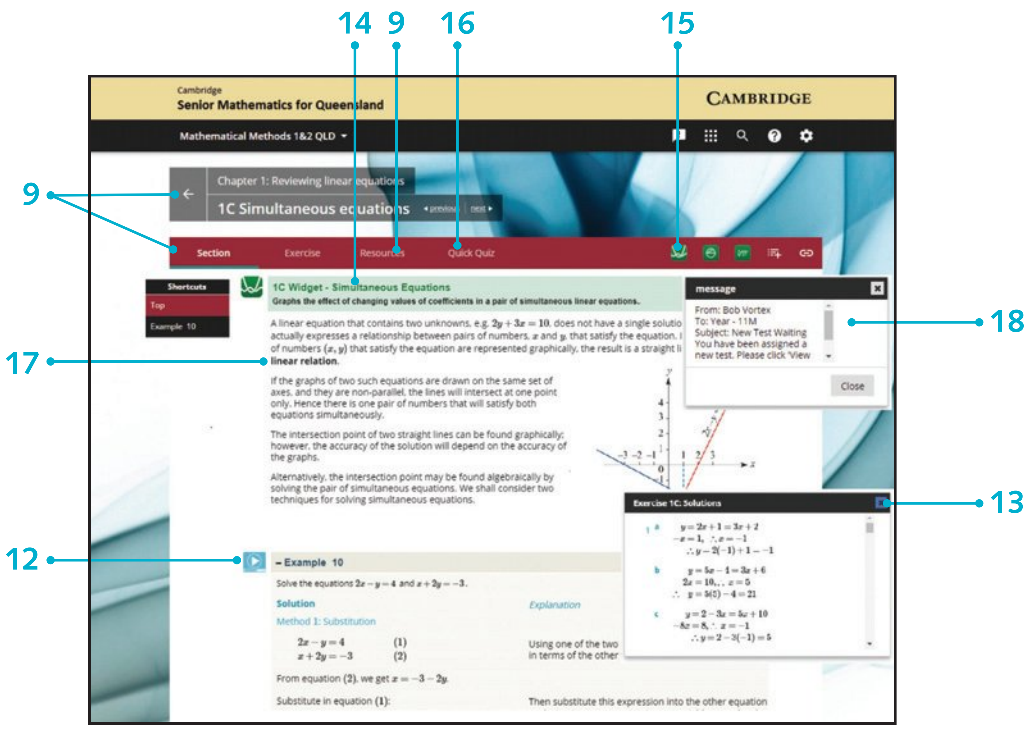
INTERACTIVE TEXTBOOK POWERED BY THE HOTmaths PLATFORM

A selection of features is shown. Numbers refer to the descriptions on the opposite page. HOTmaths platform features are updated regularly. Content shown from Units 1 & 2.



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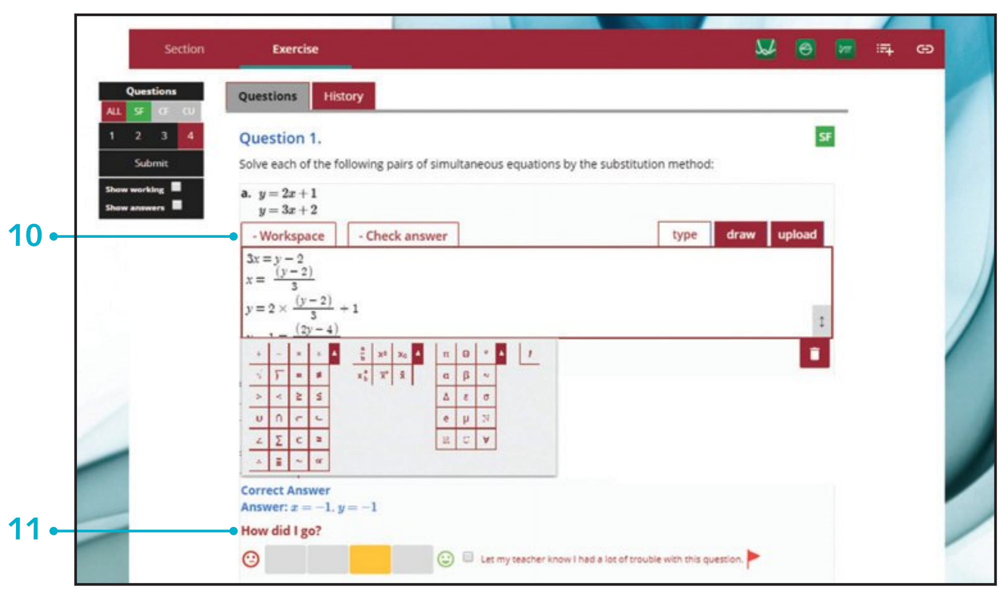

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Overview of the Online Teaching Suite powered by the HOTmaths platform (shown below)

The Online Teaching Suite is automatically enabled with a teacher account and is integrated with the teacher's copy of the Interactive Textbook. All the teacher resources are in one place for easy access. The features include:

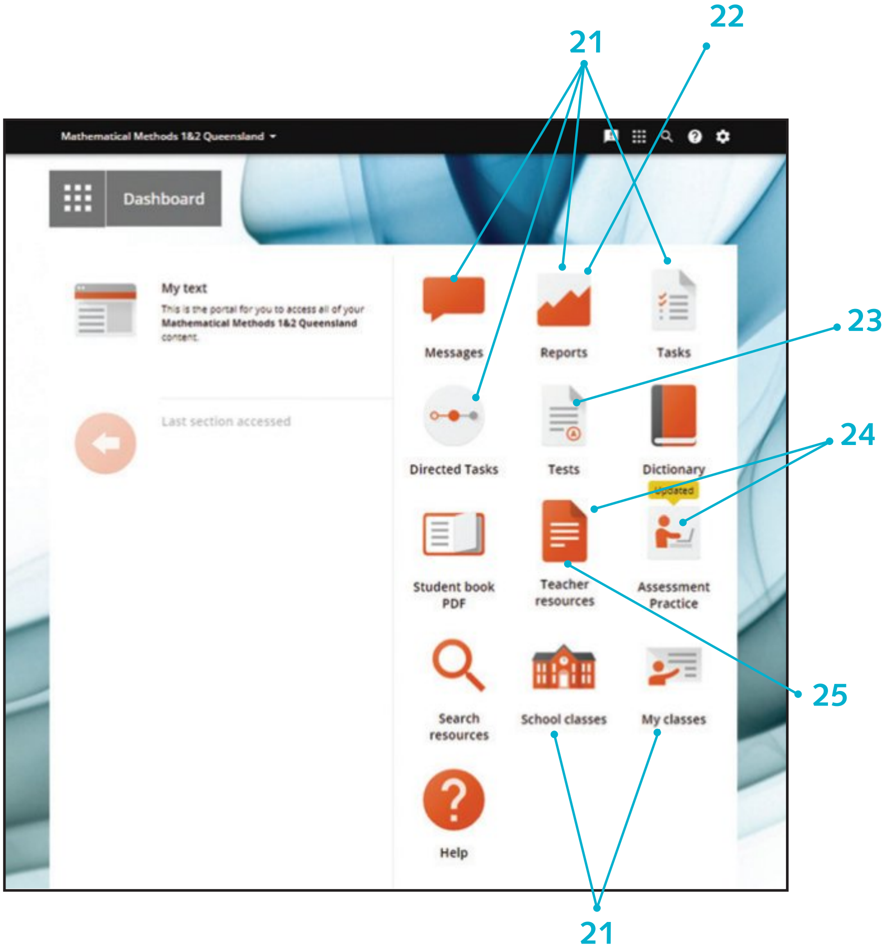
21 The HOTmaths learning management system with class and student analytics and reports, and communication tools.

22 Teacher's view of a student's working and self-assessment which enables them to modify the student's self-assessed marks, and respond where students flag that they had difficulty.

23 A HOTmaths-style test generator.

24 Chapter test worksheets and teachers' set of assessment practice items (these are listed in the table of contents of this textbook).

25 Editable curriculum grids and teaching programs.



ISBN 978-1-108-45164-2

Acknowledgements

The author and publisher wish to thank the following sources for permission to reproduce material:

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Chapter 1

1 Functions and relations

Objectives

- To revise **set notation**, including the notation for **sets of numbers**.
- To understand the concepts of **relation** and **function**.
- To find the **domain** and **range** of a given relation.
- To find the **implied (maximal) domain** of a function.
- To work with **restrictions of a function**, **piecewise-defined functions**, **odd functions** and **even functions**.
- To combine functions using **sums**, **products**, **quotients** and **compositions**.
- To understand the concepts of **strictly increasing** and **strictly decreasing**.
- To work with **power functions** and their graphs.
- To apply a knowledge of functions to **solving problems**.

The first five chapters of this book revise and extend some important concepts and techniques from Mathematical Methods Units 1 & 2 that will be built on in Units 3 & 4.

In this chapter we introduce the notation that will be used throughout the rest of the book. You will have met much of it before and this will serve as revision. The language introduced in this chapter helps to express important mathematical ideas precisely. Initially they may seem unnecessarily abstract, but later in the book you will find them used more and more in practical situations.

In Chapters 2 to 4 we study different families of functions and their graphs. We revise transformations of the plane in Chapter 2, and then study polynomial functions in Chapter 3 and trigonometric functions in Chapter 4.

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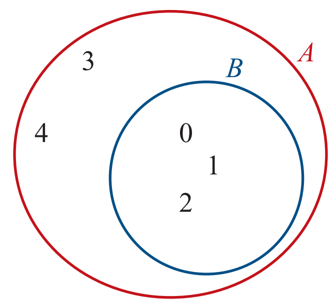
1A Set notation and sets of numbers

▶ Set notation

Set notation is used widely in mathematics and in this book where appropriate. This section summarises all of the set notation you will need.

- A **set** is a collection of objects.
- The objects that are in the set are known as **elements** or members of the set.
- If x is an element of a set A, we write $x \in A$. This can also be read as 'x is a member of the set A' or 'x belongs to A' or 'x is in A'.
- If x is **not an element** of A, we write $x \notin A$.
- A set B is called a **subset** of a set A if every element of B is also an element of A. We write $B \subseteq A$. This expression can also be read as 'B is contained in A' or 'A contains B'.

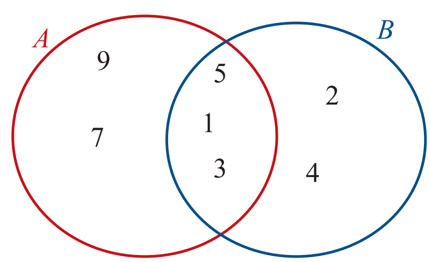


For example, let $B = \{0, 1, 2\}$ and $A = \{0, 1, 2, 3, 4\}$. Then

$$3 \in A, \quad 3 \notin B \quad \text{and} \quad B \subseteq A$$

as illustrated in the Venn diagram opposite.

- The set of elements common to two sets A and B is called the **intersection** of A and B, and is denoted by $A \cap B$. Thus $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- If the sets A and B have no elements in common, we say A and B are **disjoint**, and write $A \cap B = \varnothing$. The set $\varnothing$ is called the **empty set**.
- The set of elements that are in A or in B (or in both) is called the **union** of sets A and B, and is denoted by $A \cup B$.



For example, let $A = \{1, 3, 5, 7, 9\}$ and $B = \{1, 2, 3, 4, 5\}$. The intersection and union are illustrated by the Venn diagram shown opposite:

$$A \cap B = \{1, 3, 5\}$$

$$A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$$

Example 1

For $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 6, 7\}$, find:

a $A \cap B$ **b** $A \cup B$

Solution

a $A \cap B = \{3, 7\}$

b $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

Explanation

The elements 3 and 7 are common to sets A and B.

The set $A \cup B$ contains all elements that belong to A or B (or both).

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Sets of numbers

We begin by recalling that the elements of $\{1, 2, 3, 4, \dots\}$ are called **natural numbers**, and the elements of $\{\dots, -2, -1, 0, 1, 2, \dots\}$ are called **integers**.

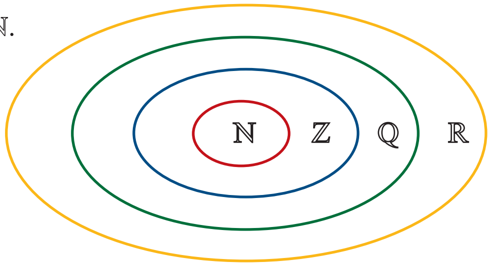
The numbers of the form $\frac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rational are called **irrational** (e.g. π and $\sqrt{2}$).

The rationals may be characterised as being those real numbers that can be written as a terminating or recurring decimal.

- The set of real numbers will be denoted by $\mathbb{R}$.
- The set of rational numbers will be denoted by $\mathbb{Q}$.
- The set of integers will be denoted by $\mathbb{Z}$.
- The set of natural numbers will be denoted by $\mathbb{N}$.

It is clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, and this may be represented by the diagram on the right.



Describing a set

It is not always possible to list the elements of a set. There is an alternative way of describing sets that is especially useful for infinite sets.

The set of all x such that ___ is denoted by $\{x : \_\_\_\}$.

For example:

- $\{x : 0 < x < 1\}$ is the set of all real numbers strictly between 0 and 1
- $\{x : x \geq 3\}$ is the set of all real numbers greater than or equal to 3
- $\{x : x \neq 0\}$ is the set of all real numbers other than 0.

The following are subsets of the real numbers for which we have special notation:

Positive real numbers $\mathbb{R}^+ = \{x : x > 0\}$

Negative real numbers $\mathbb{R}^- = \{x : x < 0\}$

Set difference

Sometimes we want to describe a set of real numbers by specifying which numbers are left out. We can do this using **set difference**.

The set $A \setminus B$ contains the elements of A that are not elements of B.

For example:

- $\mathbb{R} \setminus \{0\}$ is the set of all real numbers excluding 0
- $\mathbb{R} \setminus \{1\}$ is the set of all real numbers excluding 1
- $\mathbb{N} \setminus \{5, 7\}$ is the set of all natural numbers excluding 5 and 7.

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Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers with $a < b$.

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Intervals may be represented by diagrams as shown in Example 2.

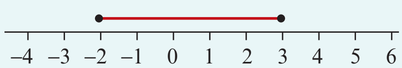
Example 2

Illustrate each of the following intervals of real numbers:

a $[-2, 3]$ **b** $(-3, 4]$ **c** $(-\infty, 5]$ **d** $(-2, 4)$ **e** $(-3, \infty)$

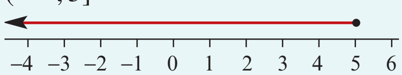
Solution

a $[-2, 3]$

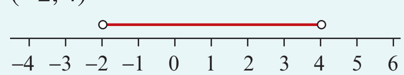


b $(-3, 4]$

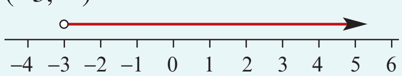
c $(-\infty, 5]$



d $(-2, 4)$



e $(-3, \infty)$



Note: The 'closed' circle (●) indicates that the number is included.
The 'open' circle (○) indicates that the number is not included.

Section summary

- If x is an element of a set A, we write $x \in A$.
- If x is not an element of a set A, we write $x \notin A$.
- If every element of B is an element of A, we say B is a **subset** of A and write $B \subseteq A$.
- Intersection The set $A \cap B$ contains the elements in common to A and B.
- Union The set $A \cup B$ contains the elements that are in A or in B (or in both).
- Set difference The set $A \setminus B$ contains the elements of A that are not in B.
- If the sets A and B have no elements in common, we say A and B are **disjoint** and write $A \cap B = \emptyset$. The set $\emptyset$ is called the **empty set**.

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- Sets of numbers:
  - Real numbers: $\mathbb{R}$
  - Rational numbers: $\mathbb{Q}$
  - Integers: $\mathbb{Z}$
  - Natural numbers: $\mathbb{N}$
- For real numbers a and b with $a < b$, we can consider the following **intervals**:

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Exercise 1A

SF

Example 1

1 For $A = \{3, 8, 11, 18, 22, 23, 24\}$, $B = \{8, 11, 25, 30, 32\}$ and $C = \{1, 8, 11, 25, 30\}$, find:

a $A \cap B$ **b** $A \cap B \cap C$ **c** $A \cup C$
d $A \cup B$ **e** $A \cup B \cup C$ **f** $(A \cap B) \cup C$

Example 2

2 Illustrate each of the following intervals on a number line:

a $[-2, 3)$ **b** $(-\infty, 4]$ **c** $[-3, -1]$
d $(-3, \infty)$ **e** $(-4, 3)$ **f** $(-1, 4]$

3 For $X = \{2, 3, 5, 7, 9, 11\}$, $Y = \{7, 9, 15, 19, 23\}$ and $Z = \{2, 7, 9, 15, 19\}$, find:

a $X \cap Y$ **b** $X \cap Y \cap Z$ **c** $X \cup Y$ **d** $X \setminus Y$
e $Z \setminus Y$ **f** $X \cap Z$ **g** $[-2, 8] \cap X$ **h** $(-3, 8] \cap Y$
i $(2, \infty) \cap Y$ **j** $(3, \infty) \cup Y$

4 For $X = \{a, b, c, d, e\}$ and $Y = \{a, e, i, o, u\}$, find:

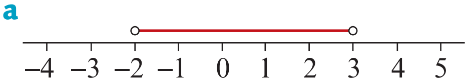
a $X \cap Y$ **b** $X \cup Y$ **c** $X \setminus Y$ **d** $Y \setminus X$

5 Use the appropriate interval notation (i.e. $[a, b]$, (a, b), etc.) to describe each of the following sets:

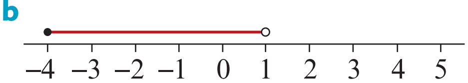
a $\{ x : -3 \le x < 1 \}$ **b** $\{ x : -4 < x \le 5 \}$ **c** $\{ y : -\sqrt{2} < y < 0 \}$
d $\left\{ x : -\dfrac{1}{\sqrt{2}} < x < \sqrt{3} \right\}$ **e** $\{ x : x < -3 \}$ **f** $\mathbb{R}^+$
g $\mathbb{R}^-$ **h** $\{ x : x \ge -2 \}$

6 Describe each of the following subsets of the real number line using the interval notation $[a, b)$, (a, b), etc.:

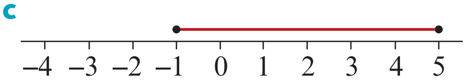
a



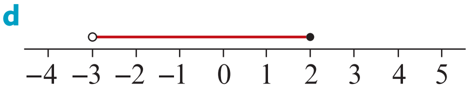
b



c



d



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SF

7 Illustrate each of the following intervals on a number line:

a $(-3, 2]$ **b** $(-4, 3)$ **c** $(-\infty, 3)$ **d** $[-4, -1]$ **e** $[-4, \infty)$ **f** $[-2, 5)$

CF

8 For each of the following, use one number line on which to represent the sets:

a $[-3, 6]$, $[2, 4]$, $[-3, 6] \cap [2, 4]$
b $[-3, 6]$, $\mathbb{R} \setminus [-3, 6]$
c $[-2, \infty)$, $(-\infty, 6]$, $[-2, \infty) \cap (-\infty, 6]$
d $(-8, -2)$, $\mathbb{R}^- \setminus (-8, -2)$

1B Identifying and describing relations and functions

▶ Relations, domain and range

An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.

A **relation** is a set of ordered pairs. The following are examples of relations:

a $S = \{(1, 1), (1, 2), (3, 4), (5, 6)\}$

b $T = \{(-3, 5), (4, 12), (5, 12), (7, -6)\}$

Every relation determines two sets:

- The set of all the first coordinates of the ordered pairs is called the **domain**.
- The set of all the second coordinates of the ordered pairs is called the **range**.

For the above examples:

a domain of $S = \{1, 3, 5\}$, range of $S = \{1, 2, 4, 6\}$

b domain of $T = \{-3, 4, 5, 7\}$, range of $T = \{5, 12, -6\}$

Some relations may be defined by a **rule** relating the elements in the domain to their corresponding elements in the range. In order to define the relation fully, we need to specify both the rule and the domain. For example, the set

$$\{ (x, y) : y = x + 1,\ x \in \{1, 2, 3, 4\} \}$$

is the relation

$$\{(1, 2), (2, 3), (3, 4), (4, 5)\}$$

The domain is the set $\{1, 2, 3, 4\}$ and the range is the set $\{2, 3, 4, 5\}$.

Describing relations

Often set notation is not used when describing a relation. For example:

- $\{ (x, y) : y = x^2 \}$ is written as $y = x^2$
- $\{ (x, y) : y = \sqrt{x},\ x \geq 0 \}$ is written as $y = \sqrt{x},\ x \geq 0$.

When the domain of a relation is not explicitly stated, it is understood to consist of all real numbers for which the defining rule has meaning. For example:

- $y = x^2$ is assumed to have domain $\mathbb{R}$
- $y = \sqrt{x}$ is assumed to have domain $[0, \infty)$.

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Example 3

Sketch the graph of each of the following relations and state the domain and range of each:

a $y = x^2$

b $\{(x, y) : y \leq x + 1\}$

c $\{(-2, -1), (-1, -1), (-1, 1), (0, 1), (1, -1)\}$

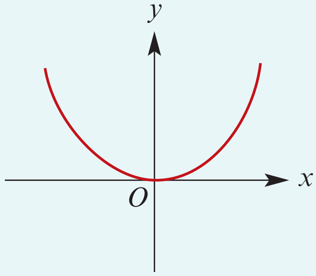
d $x^2 + y^2 = 1$

e $2x + 3y = 6, \ x \geq 0$

f $y = 2x - 1, \ x \in [-1, 2]$

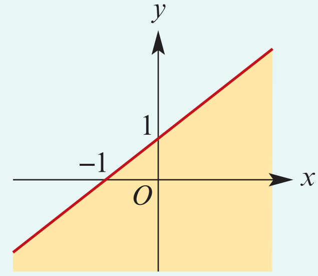
Solution

a



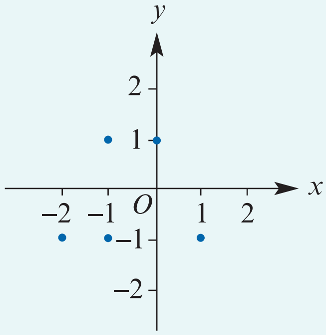
Domain = $\mathbb{R}$; Range = $[0, \infty)$

b



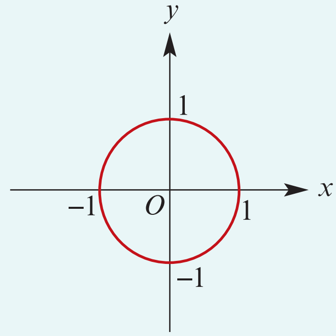
Domain = $\mathbb{R}$; Range = $\mathbb{R}$

c



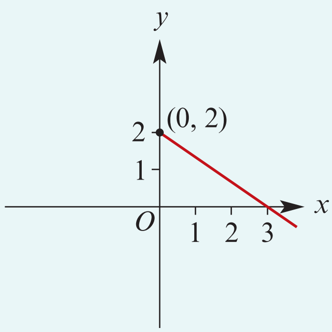
Domain = $\{-2, -1, 0, 1\}$
Range = $\{-1, 1\}$

d



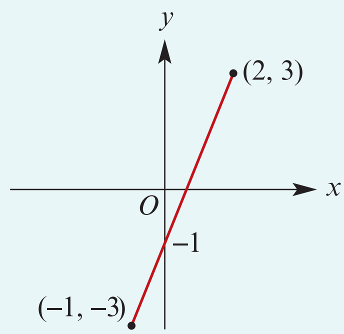
Domain = $[-1, 1]$; Range = $[-1, 1]$

e



Domain = $[0, \infty)$; Range = $(-\infty, 2]$

f



Domain = $[-1, 2]$; Range = $[-3, 3]$

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Functions

A **function** is a relation such that for each x-value there is only one corresponding y-value. This means that, if (a, b) and (a, c) are ordered pairs of a function, then $b = c$. In other words, a function cannot contain two different ordered pairs with the same first coordinate.

Example 4

Which of the following sets of ordered pairs defines a function?

a $S = \{(-3, -4), (-1, -1), (-6, 7), (1, 5)\}$ **b** $T = \{(-4, 1), (-4, -1), (-6, 7), (-6, 8)\}$

Solution

a S is a function, because for each x-value there is only one y-value.

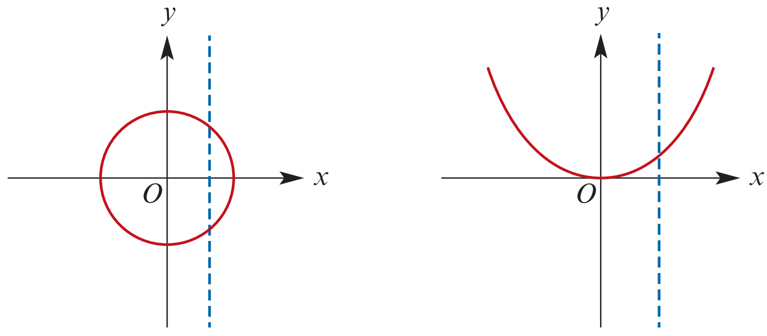
b T is not a function, because there is an x-value with two different y-values: the two ordered pairs $(-4, 1)$ and $(-4, -1)$ in T have the same first coordinate.

One way to identify whether a relation is a function is to draw a graph of the relation and then apply the following test.

Vertical-line test

If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a **function**.

For example:



$x^2 + y^2 = 1$ is not a function $\quad$ $y = x^2$ is a function

Functions are usually denoted by lowercase letters such as f, g, h.

If f is a function, then for each x in the domain of f there is a unique element y in the range such that $(x, y) \in f$. The element y is called 'the **image** of x under f' or 'the **value** of f at x', and the element x is called 'a **pre-image** of y'.

For $(x, y) \in f$, the element y is determined by x, and so we also use the notation $f(x)$, read as 'f of x', in place of y.

For example, instead of $y = 2x + 1$ we can write $f(x) = 2x + 1$. Then $f(5)$ means the y-value obtained when $x = 5$. Therefore $f(5) = 2 \times 5 + 1 = 11$.

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By incorporating this notation, we have an alternative way of writing functions:

- For the function with rule $y = x^2$ and domain $\mathbb{R}$, we write $f(x) = x^2,\ x \in \mathbb{R}$.
- For the function with rule $y = 2x - 1$ and domain $[0, 4]$, we write $f(x) = 2x - 1,\ x \in [0, 4]$.

If the domain is $\mathbb{R}$, we often just write the rule. For example: $f(x) = x^2$.

With this notation for functions, the domain of f is written as **dom** $\boldsymbol{f}$ and range of f as **ran** $\boldsymbol{f}$.

TI-84

Using the TI-Nspire CX non-CAS

- Use (menu) > **Actions** > **Define** to define the function $f(x) = 4x - 3$.
- To find the value of $f(-3)$, type $f(-3)$ followed by (enter).
- To evaluate $f(1)$, $f(2)$ and $f(3)$, type $f(\{1, 2, 3\})$ followed by (enter).

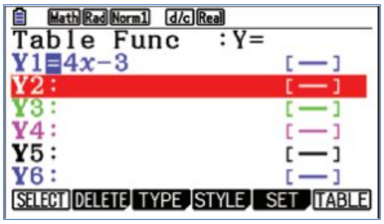
Using the Casio

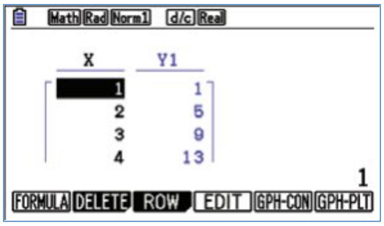
To display a table of values for the function $f(x) = 4x - 3$:

- Press (MENU) (7) to select **Table** mode.
- Enter the rule $y = 4x - 3$ in $Y1$:

  (4) (X,θ,T) (−) (3) (EXE)

- Select **Table** (F6).



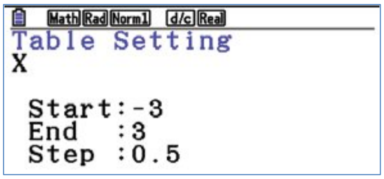


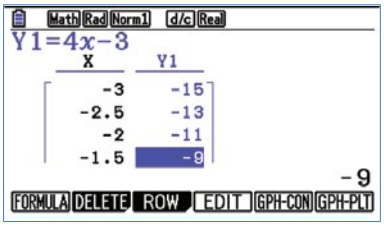
To change the x-values used in the table:

- Press (EXIT) to return to the function list.
- Select **Set** (F5) to adjust the Table Settings. For example, set the x-values to start at -3 and end at 3 with a step size of 0.5:

  ((−)) (3) (EXE) (3) (EXE) (0) (•) (5) (EXE) (EXIT)

- Select **Table** (F6). Use the cursor keys to move around the table.





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Example 5

For $f(x) = 2x^2 + x$, find:

a $f(3)$ **b** $f(-2)$ **c** $f(x-1)$

Solution

a $f(3) = 2(3)^2 + 3$
$= 21$

b $f(-2) = 2(-2)^2 - 2$
$= 6$

c $f(x-1) = 2(x-1)^2 + (x-1)$
$= 2(x^2 - 2x + 1) + x - 1$
$= 2x^2 - 3x + 1$

Example 6

For $g(x) = 3x^2 + 1$:

a Find $g(-2)$ and $g(4)$.

b Express each the following in terms of x:

i $g(-2x)$ **ii** $g(x-2)$ **iii** $g(x+2)$ **iv** $g(x^2)$

Solution

a $g(-2) = 3(-2)^2 + 1 = 13$ and $g(4) = 3(4)^2 + 1 = 49$

b **i** $g(-2x) = 3(-2x)^2 + 1$
$= 3 \times 4x^2 + 1$
$= 12x^2 + 1$

ii $g(x-2) = 3(x-2)^2 + 1$
$= 3(x^2 - 4x + 4) + 1$
$= 3x^2 - 12x + 13$

iii $g(x+2) = 3(x+2)^2 + 1$
$= 3(x^2 + 4x + 4) + 1$
$= 3x^2 + 12x + 13$

iv $g(x^2) = 3(x^2)^2 + 1$
$= 3x^4 + 1$

Example 7

Consider the function defined by $f(x) = 2x - 4$ for all $x \in \mathbb{R}$.

a Find the value of $f(2)$, $f(-1)$ and $f(t)$.

b For what values of t is $f(t) = t$?

c For what values of x is $f(x) \geq x$?

d Find the pre-image of 6.

Solution

a $f(2) = 2(2) - 4 = 0$
$f(-1) = 2(-1) - 4 = -6$
$f(t) = 2t - 4$

b $f(t) = t$
$2t - 4 = t$
$t - 4 = 0$
$\therefore \quad t = 4$

c $f(x) \geq x$
$2x - 4 \geq x$
$x - 4 \geq 0$
$\therefore \quad x \geq 4$

d $f(x) = 6$
$2x - 4 = 6$
$x = 5$

Thus 5 is the pre-image of 6.

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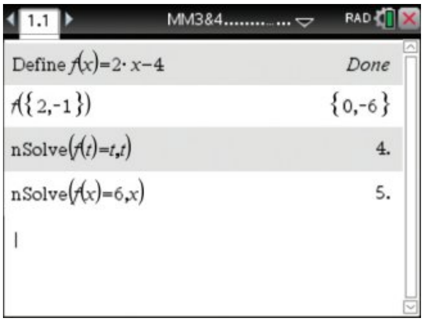
Using the TI-Nspire CX non-CAS

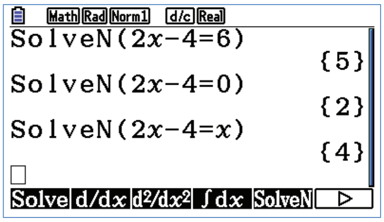
- Use (menu) > **Actions** > **Define** to define the function $f(x) = 2x - 4$.
- Find $f(2)$ and $f(-1)$ as shown.
- Use (menu) > **Algebra** > **Numerical Solve** to solve the equations $f(t) = t$ and $f(x) = 6$.



Using the Casio

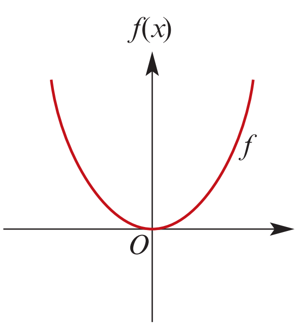
To solve equations involving the function $f(x) = 2x - 4$:

- Press (MENU) (1) to select **Run-Matrix** mode.
- Select the numerical solver by going to **Calculation** (OPTN) (F4), then **SolveN** (F5).
- To solve $f(x) = 6$, enter the equation $2x - 4 = 6$.
- The equations $f(x) = 0$ and $f(x) = x$ can be solved similarly as shown.

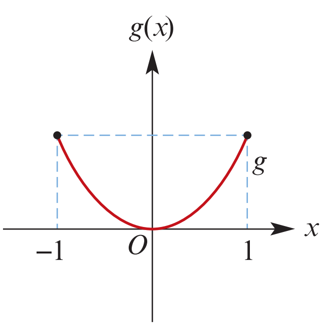


▶ Restriction of a function

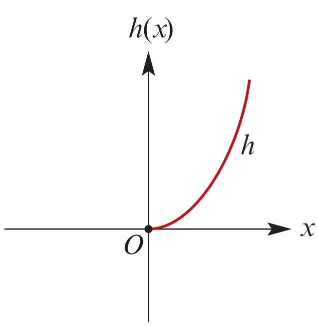
Consider the following functions:



$f(x) = x^2, \ x \in \mathbb{R}$



$g(x) = x^2, \ -1 \le x \le 1$



$h(x) = x^2, \ x \ge 0$

The different letters, f, g and h, used to name the functions emphasise the fact that there are three different functions, even though they each have the same rule. They are different because they are defined for different domains.

We call g and h **restrictions** of f, since their domains are subsets of the domain of f.

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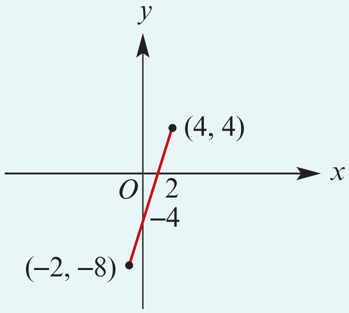
Example 8

For each of the following, sketch the graph and state the range:

a $f(x) = 2x - 4, \ x \in [-2, 4]$ **b** $g(x) = x^2, \ x \in (-1, 2]$

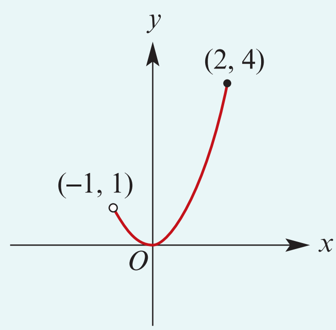
Solution

a



Range = $[-8, 4]$

b



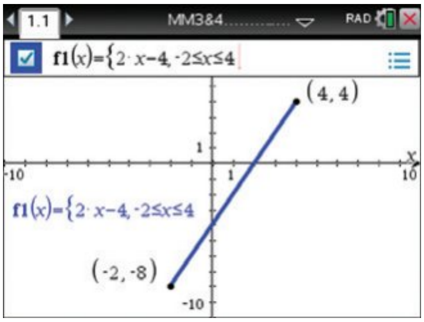
Range = $[0, 4]$

Using the TI-Nspire CX non-CAS

Domain restrictions can be entered with the function if required.

For example: $f1(x) = 2x - 4 \mid -2 \le x \le 4$

Note: The 'with' symbol | and the inequality signs can be accessed using (ctrl) (=).



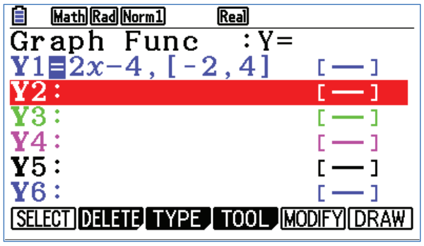
Using the Casio

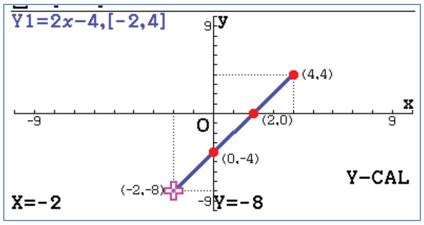
- Press (MENU) (5) to select **Graph** mode.
- Enter the rule $y = 2x - 4$ and the domain $[-2, 4]$ in $Y1$:

  (2) (X,θ,T) (−) (4) (,)

  (SHIFT) (+) ((−)) (2) (,) (4) (SHIFT) (−) (EXE)
- Select **Draw** (F6) to view the graph. Adjust the View Window (SHIFT) (F3) if required.
- To label key points on the graph, use the **G-Solve** menu (SHIFT) (F5).

Note: When defining a restricted function, always use square brackets to specify the domain (not round brackets).





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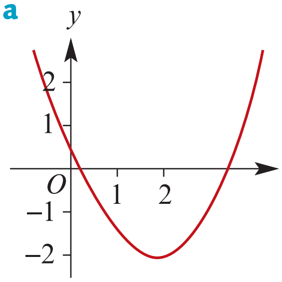
Section summary

- A **relation** is a set of ordered pairs.
  - The set of all the first coordinates of the ordered pairs is called the **domain**.
  - The set of all the second coordinates of the ordered pairs is called the **range**.
- Some relations may be defined by a rule relating the elements in the domain to their corresponding elements in the range: for example, $\{(x, y) : y = x + 1,\ x \geq 0\}$.
- A **function** is a relation such that for each x-value there is only one corresponding y-value.
- Vertical-line test If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a function.
- For an ordered pair (x, y) of a function f, we say that y is the **image** of x under f or that y is the value of f at x, and we say that x is a **pre-image** of y. Since the y-value is determined by the x-value, we use the notation $f(x)$, read as 'f of x', in place of y.
- Notation for defining functions: For example, we write $f(x) = 2x - 1,\ x \in [0, 4]$, to define a function f with domain $[0, 4]$ and rule $f(x) = 2x - 1$.
- A **restriction** of a function has the same rule but a 'smaller' domain.

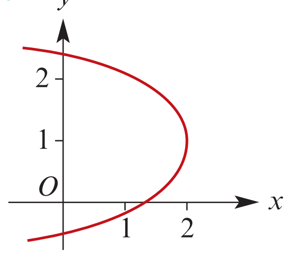
Exercise 1B

Skillsheet **1** State the domain and range for the relations represented by each of the following graphs:

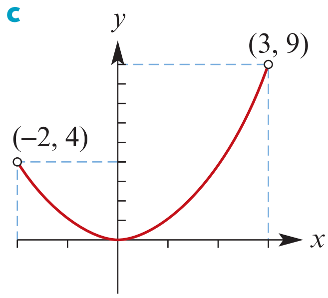
a



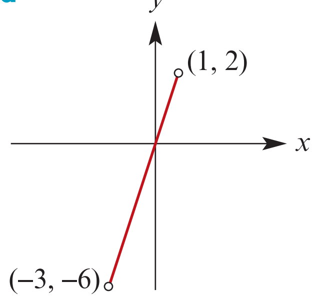
b



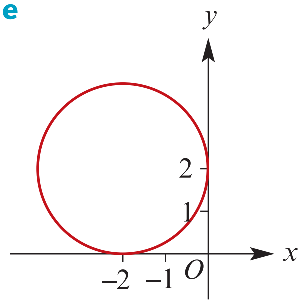
c



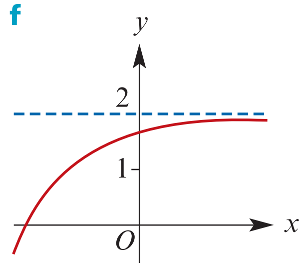
d



e



f



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Example 3 **2** Sketch a graph of each of the following relations and state its domain and range:

a $y = x^2 + 1$ **b** $x^2 + y^2 = 9$ **c** $3x + 12y = 24, \ x \geq 0$

d $y = \sqrt{2x}$ **e** $y = 5 - x, \ 0 \leq x \leq 5$ **f** $y = x^2 + 2, \ x \in [0, 4]$

g $y = 3x - 2, \ -1 \leq x \leq 2$ **h** $y = 4 - x^2$ **i** $\{ (x, y) : y \leq 1 - x \}$

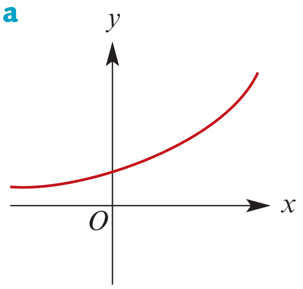
Example 4 **3** Which of the following relations are functions? State the domain and range for each.

a $\{(-1, 1), (-1, 2), (1, 2), (3, 4), (2, 3)\}$ **b** $\{(-2, 0), (-1, -1), (0, 3), (1, 5), (2, -4)\}$

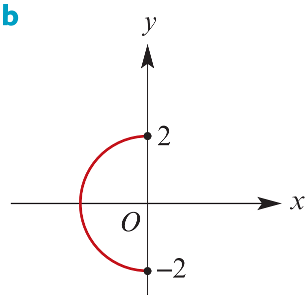
c $\{(-1, 1), (-1, 2), (-2, -2), (2, 4), (4, 6)\}$ **d** $\{(-1, 4), (0, 4), (1, 4), (2, 4), (3, 4)\}$

4 Each of the following is the graph of a relation. Which are the graph of a function?

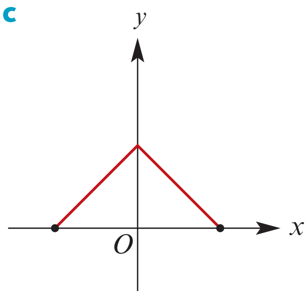
a



b



c

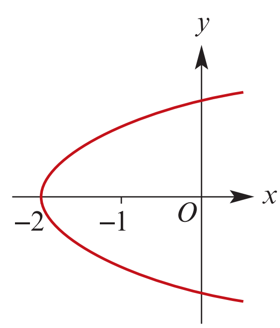


5 Which of the following relations are functions? State the domain and range for each.

a $\{ (x, 4) : x \in \mathbb{R} \}$ **b** $\{ (2, y) : y \in \mathbb{Z} \}$ **c** $y = -2x + 4$

d $y \geq 3x + 2$ **e** $x^2 + y^2 = 16$

6 The graph of the relation $y^2 = x + 2$ is shown on the right. From this relation, form two functions and specify the range of each.



Example 5 **7** Let $f(x) = 2x^2 + 4x$ and $g(x) = 2x^3 + 2x - 6$.

a Evaluate $f(-1)$, $f(2)$, $f(-3)$ and $f(2a)$.

b Evaluate $g(-1)$, $g(2)$, $g(3)$ and $g(a - 1)$.

Example 6 **8** Consider the function $g(x) = 3x^2 - 2$.

a Find $g(-2)$ and $g(4)$.

b Express the following in terms of x:

i $g(-2x)$ **ii** $g(x - 2)$ **iii** $g(x + 2)$ **iv** $g(x^2)$

Example 7 **9** Consider the function $f(x) = 2x - 3$. Find:

a the image of 3 **b** the pre-image of 11

c the value of x such that $f(x) = 4x$ **d** the values of x such that $f(x) > x$

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10 Consider the functions $g(x) = 6x + 7$ and $h(x) = 3x - 2$. Determine the values of x such that:

a $g(x) = h(x)$ **b** $g(x) > h(x)$ **c** $h(x) = 0$

Example 8 **11** Sketch the graph of each of the following and state the range of each:

a $y = x + 1,\ x \geq 2$ **b** $y = -x + 1,\ x \geq 2$
c $y = 2x + 1,\ x \geq -4$ **d** $y = 3x + 2,\ x < 3$
e $y = x + 1,\ x \in (-\infty, 3]$ **f** $y = 3x - 1,\ x \in [-2, 6]$
g $y = -3x - 1,\ x \in [-5, -1]$ **h** $y = 5x - 1,\ x \in (-2, 4)$

12 For $f(x) = 2x^2 - 6x + 1$ and $g(x) = 3 - 2x$:

a Evaluate $f(2)$, $f(-3)$ and $f(-2)$. **b** Evaluate $g(-2)$, $g(1)$ and $g(-3)$.

c Express the following in terms of a:

i $f(a)$ **ii** $f(a + 2)$ **iii** $g(-a)$ **iv** $g(2a)$
v $f(5 - a)$ **vi** $f(2a)$ **vii** $g(a) + f(a)$ **viii** $g(a) - f(a)$

13 For $f(x) = 3x^2 + x - 2$, determine the values of x such that:

a $f(x) = 0$ **b** $f(x) = x$ **c** $f(x) = -2$
d $f(x) > 0$ **e** $f(x) > x$ **f** $f(x) \leq -2$

14 For $f(x) = x^2 + x$, find:

a $f(-2)$ **b** $f(2)$
c $f(-a)$ in terms of a **d** $f(a) + f(-a)$ in terms of a
e $f(a) - f(-a)$ in terms of a **f** $f(a^2)$ in terms of a

15 For $g(x) = 3x - 2$, determine the values of x such that:

a $g(x) = 4$ **b** $g(x) > 4$ **c** $g(x) = a$
d $g(-x) = 6$ **e** $g(2x) = 4$ **f** $\dfrac{1}{g(x)} = 6$

16 Find the value of k for each of the following if $f(3) = 3$, where:

a $f(x) = kx - 1$ **b** $f(x) = x^2 - k$ **c** $f(x) = x^2 + kx + 1$
d $f(x) = \dfrac{k}{x}$ **e** $f(x) = kx^2$ **f** $f(x) = 1 - kx^2$

17 Find the values of x for which the given functions have the given value:

a $f(x) = 5x - 4,\ f(x) = 2$ **b** $f(x) = \dfrac{1}{x},\ f(x) = 5$
c $f(x) = \dfrac{1}{x^2},\ f(x) = 9$ **d** $f(x) = x + \dfrac{1}{x},\ f(x) = 2$
e $f(x) = (x + 1)(x - 2),\ f(x) = 0$

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1C Implied domains and types of functions

Implied domains

If the domain of a function is not specified, then the domain is the largest subset of $\mathbb{R}$ for which the rule is defined; this is called the **implied domain** or the **maximal domain**.

Thus, for the function $f(x) = \sqrt{x}$, the implied domain is $[0, \infty)$. We write:

$$f(x) = \sqrt{x}, \ x \in [0, \infty)$$

Example 9

Find the implied domain and the corresponding range for the functions with rules:

a $f(x) = 2x - 3$ **b** $f(x) = \dfrac{1}{(x-2)^2}$ **c** $f(x) = \sqrt{x+6}$ **d** $f(x) = \sqrt{4 - x^2}$

Solution

a $f(x) = 2x - 3$ is defined for all x. The implied domain is $\mathbb{R}$. The range is $\mathbb{R}$.

b $f(x) = \dfrac{1}{(x-2)^2}$ is defined for $x \neq 2$. The implied domain is $\mathbb{R} \setminus \{2\}$. The range is $\mathbb{R}^+$.

c $f(x) = \sqrt{x+6}$ is defined for $x + 6 \geq 0$, i.e. for $x \geq -6$.
Thus the implied domain is $[-6, \infty)$. The range is $[0, \infty)$.

d $f(x) = \sqrt{4 - x^2}$ is defined for $4 - x^2 \geq 0$, i.e. for $x^2 \leq 4$.
Thus the implied domain is $[-2, 2]$. The range is $[0, 2]$.

Example 10

Find the implied domain of the functions with the following rules:

a $f(x) = \dfrac{2}{2x - 3}$ **b** $g(x) = \sqrt{5 - x}$

c $h(x) = \sqrt{x - 5} + \sqrt{8 - x}$ **d** $f(x) = \sqrt{x^2 - 7x + 12}$

Solution

a $f(x)$ is defined when $2x - 3 \neq 0$, i.e. when $x \neq \frac{3}{2}$. Thus the implied domain is $\mathbb{R} \setminus \{\frac{3}{2}\}$.

b $g(x)$ is defined when $5 - x \geq 0$, i.e. when $x \leq 5$. Thus the implied domain is $(-\infty, 5]$.

c $h(x)$ is defined when $x - 5 \geq 0$ and $8 - x \geq 0$, i.e. when $x \geq 5$ and $x \leq 8$. Thus the implied domain is $[5, 8]$.

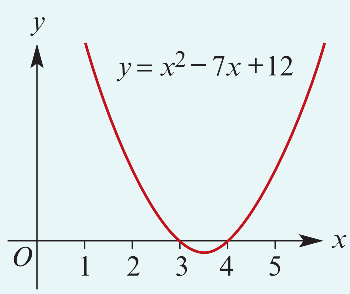
d $f(x)$ is defined when

$$x^2 - 7x + 12 \geq 0$$

which is equivalent to

$$(x - 3)(x - 4) \geq 0$$

Thus $f(x)$ is defined when $x \geq 4$ or $x \leq 3$.
The implied domain is $(-\infty, 3] \cup [4, \infty)$.



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▶ Piecewise-defined functions

Functions which have different rules for different subsets of their domain are called **piecewise-defined functions**. They are also known as **hybrid functions**.

Example 11

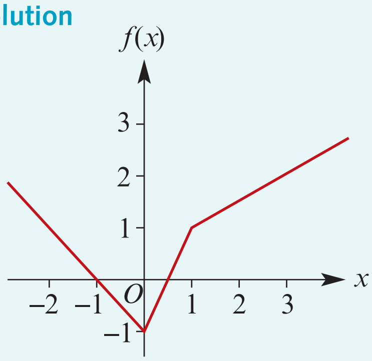
a Sketch the graph of the function f given by:

$$f(x) = \begin{cases} -x - 1 & \text{for } x < 0 \\ 2x - 1 & \text{for } 0 \le x \le 1 \\ \frac{1}{2}x + \frac{1}{2} & \text{for } x > 1 \end{cases}$$

b State the range of f.

Solution

a



b The range is $[-1, \infty)$.

Explanation

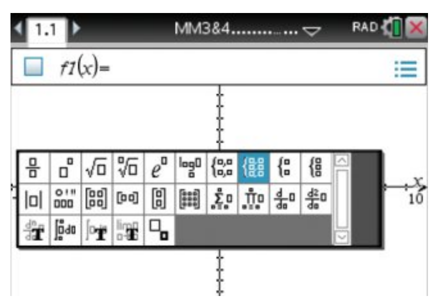
- The graph of $y = -x - 1$ is sketched for $x < 0$. Note that when $x = 0$, $y = -1$ for this rule.
- The graph of $y = 2x - 1$ is sketched for $0 \le x \le 1$. Note that when $x = 0$, $y = -1$ and when $x = 1$, $y = 1$ for this rule.
- The graph of $y = \frac{1}{2}x + \frac{1}{2}$ is sketched for $x > 1$. Note that when $x = 1$, $y = 1$ for this rule.

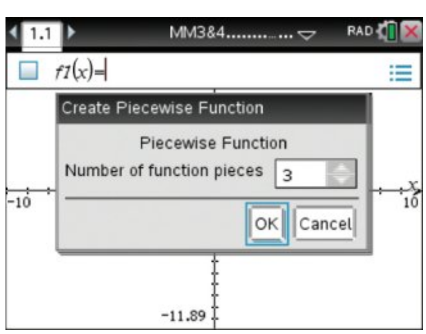
Note: For this function, the sections of the graph 'join up'. This is not always the case.

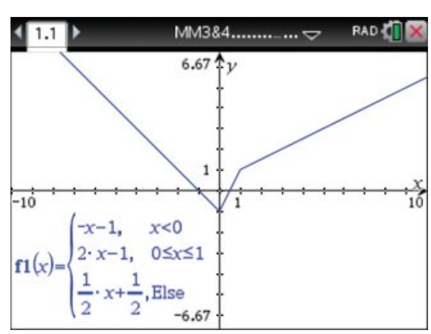
TI-84

Using the TI-Nspire CX non-CAS

- In a **Graphs** application with the cursor in the entry line, select the piecewise function template as shown. (Access the template from the 2D-template palette ⌈⁽ᴵᴵ⁾⌋.)
- If the domain of the last function piece is the remaining subset of ℝ, then leave the final condition blank and it will autofill as 'Else' when you press enter.







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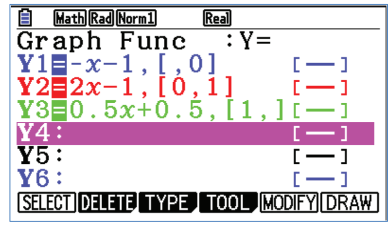
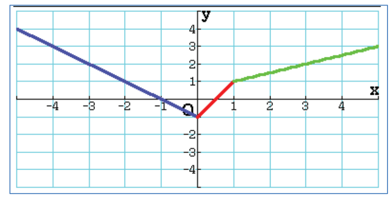
Using the Casio

- Press MENU 5 to select **Graph** mode.
- Enter the first rule, $y = -x - 1$ for $x < 0$, in $Y1$:

  (−) X,θ,T − 1 ,

  SHIFT + , 0 SHIFT − EXE
- Enter the second and third rules in $Y2$ and $Y3$ as shown.
- Select **Draw** F6 to view the graph. Adjust the View Window SHIFT F3 if required.





Note: The syntax for entering a function with a restricted domain is:
function rule, [*start x-value*, *end x-value*]

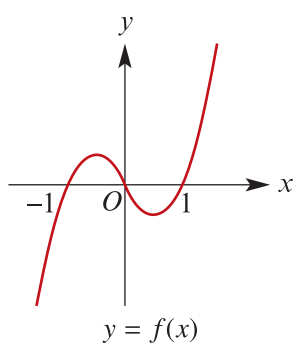
Odd and even functions

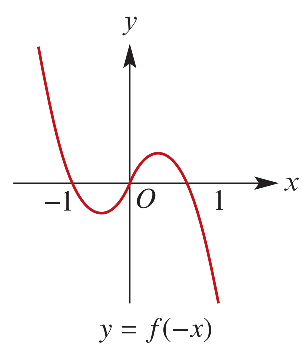
Odd functions

An **odd** function has the property that $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin: the graph remains unchanged after rotation of 180° about the origin.

For example, $f(x) = x^3 - x$ is an odd function, since

$$\begin{aligned} f(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -f(x) \end{aligned}$$



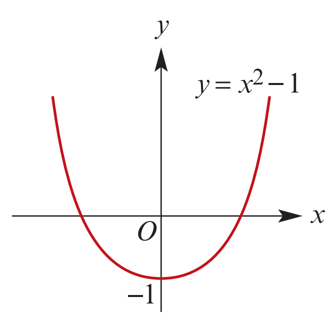


Even functions

An **even** function has the property that $f(-x) = f(x)$. The graph of an even function is symmetrical about the y-axis.

For example, $f(x) = x^2 - 1$ is an even function, since

$$\begin{aligned} f(-x) &= (-x)^2 - 1 \\ &= x^2 - 1 \\ &= f(x) \end{aligned}$$



The properties of odd and even functions often facilitate the sketching of graphs.

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Example 12

State whether each function is odd or even or neither:

a $f(x) = x^2 + 7$ **b** $f(x) = x^4 + x^2$ **c** $f(x) = -2x^3 + 7$

d $f(x) = \frac{1}{x}$ **e** $f(x) = \frac{1}{x-3}$ **f** $f(x) = x^5 + x^3 + x$

Solution

a $f(-a) = (-a)^2 + 7$
$= a^2 + 7$
$= f(a)$

The function is even.

b $f(-a) = (-a)^4 + (-a)^2$
$= a^4 + a^2$
$= f(a)$

The function is even.

c $f(-1) = -2(-1)^3 + 7 = 9$
but $f(1) = -2 + 7 = 5$
and $-f(1) = -5$

The function is neither even nor odd.

d $f(-a) = \frac{1}{-a}$
$= -\frac{1}{a}$
$= -f(a)$

The function is odd.

e $f(-1) = -\frac{1}{4}$
but $f(1) = -\frac{1}{2}$
and $-f(1) = \frac{1}{2}$

The function is neither even nor odd.

f $f(-a)$
$= (-a)^5 + (-a)^3 + (-a)$
$= -a^5 - a^3 - a$
$= -f(a)$

The function is odd.

Section summary

- When the domain of a function is not explicitly stated, it is assumed to consist of all real numbers for which the rule has meaning; this is called the **implied domain** or the **maximal domain** of the function.
- Functions which have different rules for different subsets of their domain are called **piecewise-defined functions**.
- A function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
- A function f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

Exercise 1C

Skillsheet

Example 9

SF

1 State the largest possible domain and range for the functions defined by each of the following rules:

a $y = 4 - x$ **b** $y = \sqrt{x}$ **c** $y = x^2 - 2$ **d** $y = \sqrt{16 - x^2}$

e $y = \frac{1}{x}$ **f** $y = 4 - 3x^2$ **g** $y = \sqrt{x - 3}$

2 Each of the following is the rule of a function. In each case, write down the implied domain and the range.

a $y = 3x + 2$ **b** $y = x^2 - 2$ **c** $f(x) = \sqrt{9 - x^2}$ **d** $g(x) = \frac{1}{x-1}$

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SF

Example 10

3 Find the implied domain for each of the following rules:

a $f(x) = \dfrac{1}{x-3}$ **b** $f(x) = \sqrt{x^2 - 3}$ **c** $g(x) = \sqrt{x^2 + 3}$

d $h(x) = \sqrt{x-4} + \sqrt{11-x}$ **e** $f(x) = \dfrac{x^2 - 1}{x+1}$ **f** $h(x) = \sqrt{x^2 - x - 2}$

g $f(x) = \dfrac{1}{(x+1)(x-2)}$ **h** $h(x) = \sqrt{\dfrac{x-1}{x+2}}$ **i** $f(x) = \sqrt{x - 3x^2}$

j $h(x) = \sqrt{25 - x^2}$ **k** $f(x) = \sqrt{x-3} + \sqrt{12-x}$

CF

Example 11

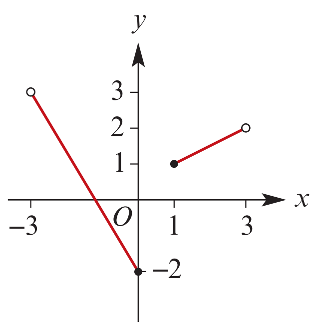
4 **a** Sketch the graph of the function

$$f(x) = \begin{cases} -2x - 2, & x < 0 \\ x - 2, & 0 \le x < 2 \\ 3x - 6, & x \ge 2 \end{cases}$$

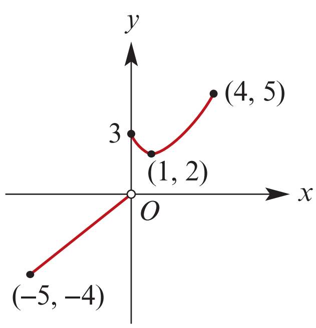
b What is the range of f?

5 For each of the following graphs, state the domain and range of the function:

a



b



6 **a** Sketch the graph of the function

$$f(x) = \begin{cases} 2x + 6, & 0 < x \le 2 \\ -x + 5, & -4 \le x \le 0 \\ -4, & x < -4 \end{cases}$$

b State the domain and range of f.

7 **a** Sketch the graph of the function

$$g(x) = \begin{cases} x^2 + 5, & x > 0 \\ 5 - x, & -3 \le x \le 0 \\ 8, & x < -3 \end{cases}$$

b State the range of g.

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8 Given that

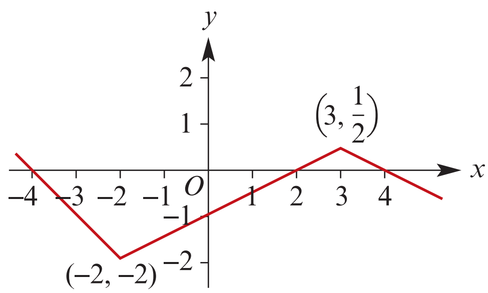
$$f(x) = \begin{cases} \dfrac{1}{x}, & x > 3 \\ 2x, & x \le 3 \end{cases}$$

find:

a $f(-4)$ **b** $f(0)$ **c** $f(4)$

d $f(a + 3)$ in terms of a **e** $f(2a)$ in terms of a **f** $f(a - 3)$ in terms of a

9 Specify the function illustrated by the graph.



Example 12 **10** State whether each of the following functions is odd, even or neither:

a $f(x) = x^4$ **b** $f(x) = x^5$ **c** $f(x) = x^4 - 3x$

d $f(x) = x^4 - 3x^2$ **e** $f(x) = x^5 - 2x^3$ **f** $f(x) = x^4 - 2x^5$

11 State whether each of the following functions is odd, even or neither:

a $f(x) = x^2 - 4$ **b** $f(x) = 2x^4 - x^2$ **c** $f(x) = -4x^3 + 7x$

d $f(x) = \dfrac{1}{2x}$ **e** $f(x) = \dfrac{1}{x + 5}$ **f** $f(x) = 3 + 2x^2$

g $f(x) = x^2 - 5x$ **h** $f(x) = 3^x$ **i** $f(x) = x^4 + x^2 + 2$

1D Combining functions

In this section we look at ways to combine two functions to form a new function, by taking a sum, product, quotient or composition. This enables us to consider quite complicated functions by 'breaking them apart' and viewing them as a combination of simpler functions. We will apply this idea to find the derivatives of functions in Chapters 7 and 8.

▶ Sums and products of functions

The domain of f is denoted by dom f and the domain of g by dom g. Let f and g be functions such that dom $f \cap$ dom $g \ne \emptyset$. The **sum**, $f + g$, and the **product**, fg, as functions on dom $f \cap$ dom g are defined by

$$(f + g)(x) = f(x) + g(x) \qquad \text{and} \qquad (fg)(x) = f(x)\,g(x)$$

The domain of both $f + g$ and fg is the intersection of the domains of f and g, i.e. the values of x for which both f and g are defined.

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Example 13

If $f(x) = \sqrt{x-2}$ for all $x \geq 2$ and $g(x) = \sqrt{4-x}$ for all $x \leq 4$, find:

a $f + g$ **b** $(f + g)(3)$ **c** fg **d** $(fg)(3)$

Solution

Note that dom $f \cap$ dom g $= [2, 4]$.

a
$$(f + g)(x) = f(x) + g(x) = \sqrt{x-2} + \sqrt{4-x}$$
$\text{dom}(f + g) = [2, 4]$

b
$$(f + g)(3) = \sqrt{3-2} + \sqrt{4-3} = 2$$

c
$$(fg)(x) = f(x)\,g(x) = \sqrt{(x-2)(4-x)}$$
$\text{dom}(fg) = [2, 4]$

d
$$(fg)(3) = \sqrt{(3-2)(4-3)} = 1$$

Quotients of functions

Let f and g be functions such that dom $f \cap$ dom $g \cap \{ x : g(x) \neq 0 \} \neq \varnothing$.

The **quotient** $\dfrac{f}{g}$ is defined by

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$$

The domain of the quotient is the intersection of the domains of f and g with the set of all real numbers x for which $g(x) \neq 0$.

Example 14

If $f(x) = x$ for all x and $g(x) = \sqrt{16-x}$ for all $x \leq 16$, find:

a $\dfrac{f}{g}$ **b** $\dfrac{f}{g}(7)$ **c** $\dfrac{g}{f}$ **d** $\dfrac{g}{f}(7)$

Solution

Note that dom $f \cap$ dom $g \cap \{ x : g(x) \neq 0 \} = (-\infty, 16)$.

a
$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x}{\sqrt{16-x}}$$
$$\text{dom}\left(\frac{f}{g}\right) = (-\infty, 16)$$

b
$$\frac{f}{g}(7) = \frac{f(7)}{g(7)} = \frac{7}{3}$$

Note that dom $g \cap$ dom $f \cap \{ x : f(x) \neq 0 \} = (-\infty, 16] \setminus \{0\}$.

c
$$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{16-x}}{x}$$
$$\text{dom}\left(\frac{g}{f}\right) = (-\infty, 16] \setminus \{0\}$$

d
$$\frac{g}{f}(7) = \frac{g(7)}{f(7)} = \frac{3}{7}$$

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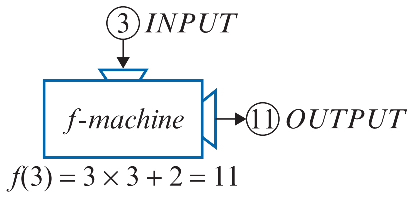
Composition of functions

A function may be considered to be similar to a machine for which the input (domain) is processed to produce an output (range). For example, the diagram on the right represents an 'f-machine' where $f(x) = 3x + 2$.



With many processes, more than one machine operation is required to produce an output.

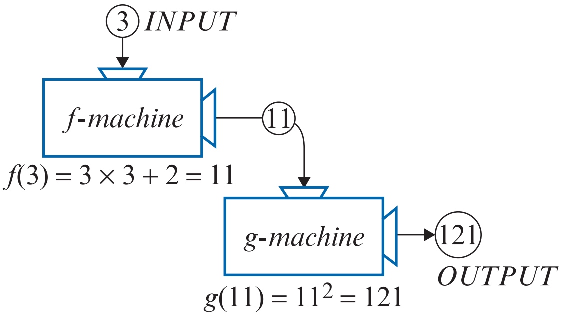
Suppose an output is the result of one function being applied after another.

For example: $f(x) = 3x + 2$

followed by $g(x) = x^2$

This is illustrated on the right.

A new function h is formed. The rule for h is $h(x) = (3x + 2)^2$.



The diagram shows $f(3) = 11$ and then $g(11) = 121$. This may be written:

$$h(3) = g(f(3)) = g(11) = 121$$

The new function h is said to be the **composition** of g with f. This is written $h = g \circ f$ (read 'composition of f followed by g') and the rule for h is given by $h(x) = g(f(x))$.

In the example we have considered:

$$\begin{aligned} h(x) &= g(f(x)) \\ &= g(3x + 2) \\ &= (3x + 2)^2 \end{aligned}$$

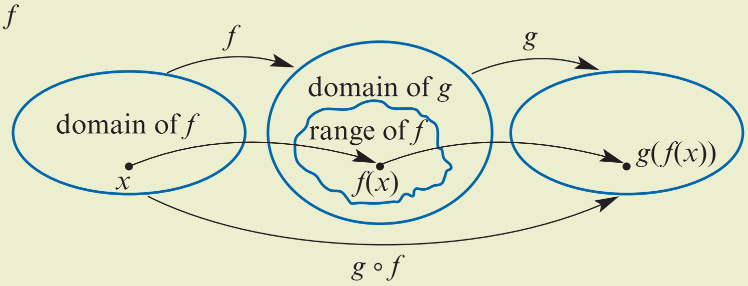
In general, for functions f and g such that

$$\operatorname{ran} f \subseteq \operatorname{dom} g$$

we define the **composite function** of g with f by

$$(g \circ f)(x) = g(f(x))$$

$$\operatorname{dom}(g \circ f) = \operatorname{dom} f$$



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Example 15

For the functions $f(x) = 2x - 1$ and $g(x) = 3x^2$, find:

a $f(g(x))$

b $g(f(x))$

c $f(f(x))$

d $g(g(x))$

Solution

a $f(g(x)) = f(3x^2)$
$= 2(3x^2) - 1$
$= 6x^2 - 1$

b $g(f(x)) = g(2x - 1)$
$= 3(2x - 1)^2$
$= 12x^2 - 12x + 3$

c $f(f(x)) = f(2x - 1)$
$= 2(2x - 1) - 1$
$= 4x - 3$

d $g(g(x)) = g(3x^2)$
$= 3(3x^2)^2$
$= 27x^4$

Note: It can be seen from this example that in general $f(g(x)) \neq g(f(x))$.

Example 16

Express each of the following as the composition of two functions:

a $h(x) = (2x + 1)^2$

b $h(x) = \sqrt{x^2}$

c $h(x) = (x^2 - 2)^n, \quad n \in \mathbb{N}$

Solution

a $h(x) = (2x + 1)^2$
Choose $f(x) = 2x + 1$
and $g(x) = x^2$.
Then $h(x) = g(f(x))$.

b $h(x) = \sqrt{x^2}$
Choose $f(x) = x^2$
and $g(x) = \sqrt{x}$.
Then $h(x) = g(f(x))$.

c $h(x) = (x^2 - 2)^n, \quad n \in \mathbb{N}$
Choose $f(x) = x^2 - 2$
and $g(x) = x^n$.
Then $h(x) = g(f(x))$.

Note: These are not the only possible answers, but the ‘natural’ choices have been made.

Section summary

Combining functions

	Rule	Domain
Sum	$(f + g)(x) = f(x) + g(x)$	$\text{dom}(f + g) = \text{dom}\, f \cap \text{dom}\, g$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	$\text{dom}(f \cdot g) = \text{dom}\, f \cap \text{dom}\, g$
Quotient	$\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)}$	$\text{dom}\left(\dfrac{f}{g}\right) = \text{dom}\, f \cap \text{dom}\, g \cap \{\, x : g(x) \neq 0 \,\}$
Composition	$(g \circ f)(x) = g(f(x))$	$\text{dom}(g \circ f) = \text{dom}\, f$ if $\text{ran}\, f \subseteq \text{dom}\, g$

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Exercise 1D

Skillsheet

SF

1 For each of the following, find $(f + g)(x)$ and $(fg)(x)$ and state the domain for both $f + g$ and fg:

Example 13

a $f(x) = 3x$ and $g(x) = x + 2$

b $f(x) = 1 - x^2$ for all $x \in [-2, 2]$ and $g(x) = x^2$ for all $x \in \mathbb{R}^+$

c $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{\sqrt{x}}$ for $x \in [1, \infty)$

d $f(x) = x^2$, $x \geq 0$, and $g(x) = \sqrt{4 - x}$, $0 \leq x \leq 4$

Example 14

2 For each pair of functions f and g from Question 1:

i find $\dfrac{f}{g}(x)$ and state the domain of $\dfrac{f}{g}$

ii find $\dfrac{g}{f}(x)$ and state the domain of $\dfrac{g}{f}$.

3 Functions f, g, h and k are defined by:

i $f(x) = x^2 + 1,\ x \in \mathbb{R}$

ii $g(x) = x,\ x \in \mathbb{R}$

iii $h(x) = \dfrac{1}{x^2},\ x \neq 0$

iv $k(x) = \dfrac{1}{x},\ x \neq 0$

a State which of the above functions are odd and which are even.

b Give the rules for the functions $f + h$, fh, $g + k$, gk, $f + g$ and fg, stating which are odd and which are even.

Example 15

4 For each of the following, find $f(g(x))$ and $g(f(x))$:

a $f(x) = 2x - 1,\ g(x) = 2x$

b $f(x) = 4x + 1,\ g(x) = 2x + 1$

c $f(x) = 2x - 1,\ g(x) = 2x - 3$

d $f(x) = 2x - 1,\ g(x) = x^2$

e $f(x) = 2x^2 + 1,\ g(x) = x - 5$

f $f(x) = 2x + 1,\ g(x) = x^2$

5 For the functions $f(x) = 2x - 1$ and $h(x) = 3x + 2$, find:

a $f(h(x))$ b $h(f(x))$ c $f(h(2))$ d $h(f(2))$

e $f(h(3))$ f $h(f(-1))$ g $f(h(0))$

6 For the functions $f(x) = x^2 + 2x$ and $h(x) = 3x + 1$, find:

a $f(h(x))$ b $h(f(x))$ c $f(h(3))$ d $h(f(3))$

e $f(h(0))$ f $h(f(0))$

Example 16

7 Express each of the following as the composition of two functions:

a $h(x) = (x^2 - 1)^4$

b $h(x) = \sqrt{x^4 + 3}$

c $h(x) = (x^2 - 2x)^n$ where $n \in \mathbb{N}$

d $h(x) = \dfrac{1}{2x + 3}$

e $h(x) = (x^2 - 2x)^3 - 2(x^2 - 2x)$

f $h(x) = 2(2x^2 + 1)^2 + 1$

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1E Power functions

We now consider functions with rules of the form $f(x) = x^r$, where r is a rational number. These functions are called **power functions**.

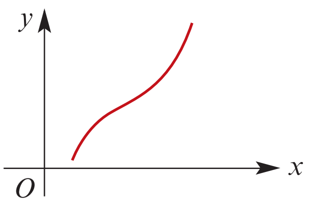
In this section, we look at power functions with rules such as

$$f(x) = x^4, \quad f(x) = x^{-4}, \quad f(x) = x^{\frac{1}{4}}, \quad f(x) = x^5, \quad f(x) = x^{-5}, \quad f(x) = x^{\frac{1}{3}}$$

You may like to investigate further by using your calculator to plot the graphs of more complicated power functions with rules such as $f(x) = x^{\frac{2}{3}}$ and $f(x) = x^{\frac{3}{2}}$.

▶ Increasing and decreasing functions

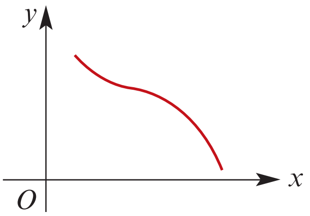
We say a function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.



For example:

- The graph opposite shows a strictly increasing function.
- A straight line with positive gradient is strictly increasing.
- The function $f(x) = x^2$, $x > 0$, is strictly increasing.

We say a function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

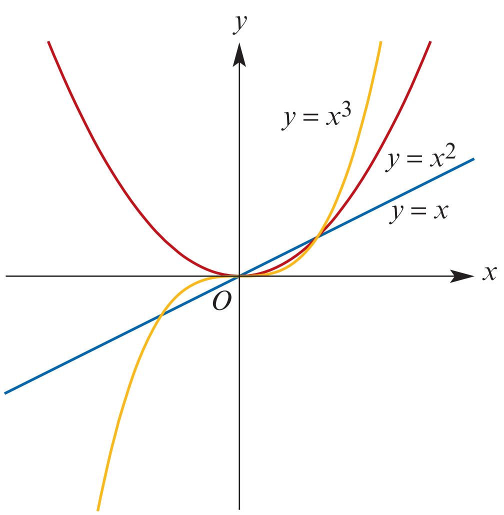


For example:

- The graph opposite shows a strictly decreasing function.
- A straight line with negative gradient is strictly decreasing.
- The function $f(x) = x^2$, $x < 0$, is strictly decreasing.

▶ Power functions with positive integer index

We start by considering power functions $f(x) = x^n$ where n is a positive integer.



Taking $n = 1, 2, 3$, we obtain the linear function $f(x) = x$, the quadratic function $f(x) = x^2$ and the cubic function $f(x) = x^3$.

We have studied these functions in Mathematical Methods Units 1 & 2 and have referred to them in the earlier sections of this chapter.

The general shape of the graph of $f(x) = x^n$ depends on whether the index n is odd or even.

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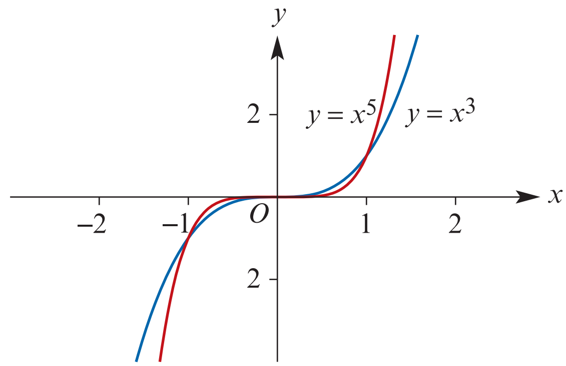
The function $f(x) = x^n$ where n is an odd positive integer

The graph has a similar shape to those shown below. The maximal domain is $\mathbb{R}$ and the range is $\mathbb{R}$.

Some properties of $f(x) = x^n$ where n is an odd positive integer:

- f is an odd function
- f is strictly increasing
- $f(0) = 0$, $f(1) = 1$ and $f(-1) = -1$
- as $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$.

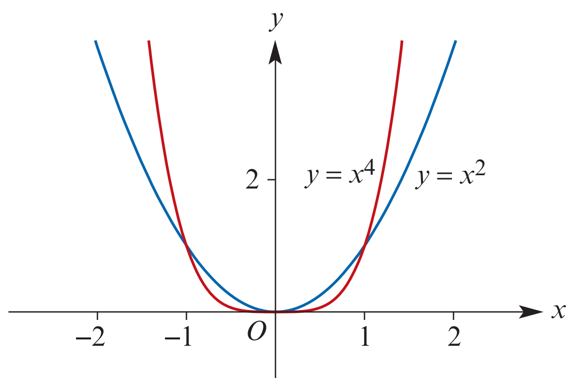


The function $f(x) = x^n$ where n is an even positive integer

The graph has a similar shape to those shown below. The maximal domain is $\mathbb{R}$ and the range is $[0, \infty)$.

Some properties of $f(x) = x^n$ where n is an even positive integer:

- f is an even function
- f is strictly increasing for $x > 0$
- f is strictly decreasing for $x < 0$
- $f(0) = 0$, $f(1) = 1$ and $f(-1) = 1$
- as $x \to \pm\infty$, $f(x) \to \infty$.



▶ Power functions with negative integer index

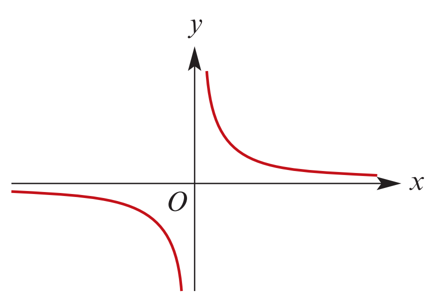
Again, the general shape of the graph depends on whether the index n is odd or even.

The function $f(x) = x^n$ where n is an odd negative integer

Taking $n = -1$, we obtain

$$f(x) = x^{-1} = \frac{1}{x}$$

The graph of this function is shown on the right. The graphs of functions of this type are all similar to this one.



In general, we consider the function $f(x) = x^{-k}$, where $k = 1, 3, 5, \ldots$

- the maximal domain is $\mathbb{R} \setminus \{0\}$ and the range is $\mathbb{R} \setminus \{0\}$
- f is an odd function
- there is a horizontal asymptote with equation $y = 0$
- there is a vertical asymptote with equation $x = 0$.

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Example 17

For the function f with rule $f(x) = \frac{1}{x^5}$:

a State the maximal domain and the corresponding range.

b Evaluate each of the following:

i $f(2)$ **ii** $f(-2)$ **iii** $f(\frac{1}{2})$ **iv** $f(-\frac{1}{2})$

c Sketch the graph without using your calculator.

Solution

a The maximal domain is $\mathbb{R} \setminus \{0\}$ and the range is $\mathbb{R} \setminus \{0\}$.

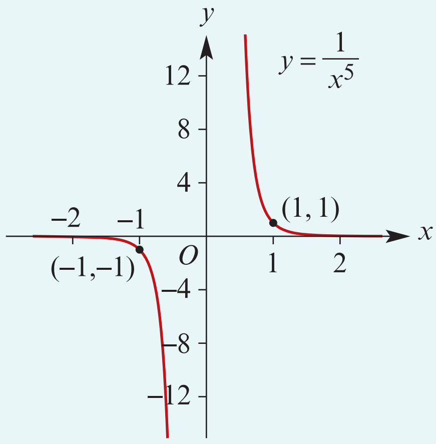
b **i** $f(2) = \frac{1}{2^5} = \frac{1}{32}$

ii $f(-2) = \frac{1}{(-2)^5} = -\frac{1}{32}$

iii $f(\frac{1}{2}) = \frac{1}{(\frac{1}{2})^5} = 32$

iv $f(-\frac{1}{2}) = \frac{1}{(-\frac{1}{2})^5} = -32$

c



Example 18

Let $f(x) = x^{-1}$ for $x \in \mathbb{R} \setminus \{0\}$ and $g(x) = x^{-3}$ for $x \in \mathbb{R} \setminus \{0\}$.

a Find the values of x for which $f(x) = g(x)$.

b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

Solution

a $f(x) = g(x)$

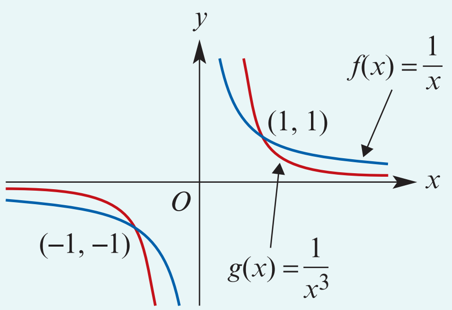
$x^{-1} = x^{-3}$

$\frac{1}{x} = \frac{1}{x^3}$

$x^2 = 1$

$\therefore \quad x = 1$ or $x = -1$

b



Note:

If $x > 1$, then $x^3 > x$ and so $\frac{1}{x} > \frac{1}{x^3}$.

If $0 < x < 1$, then $x^3 < x$ and so $\frac{1}{x} < \frac{1}{x^3}$.

If $x < -1$, then $x^3 < x$ and so $\frac{1}{x} < \frac{1}{x^3}$.

If $-1 < x < 0$, then $x^3 > x$ and so $\frac{1}{x} > \frac{1}{x^3}$.

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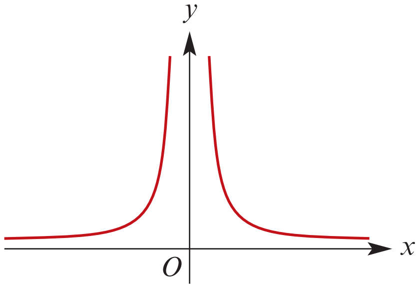
The function $f(x) = x^n$ where n is an even negative integer

Taking $n = -2$, we obtain

$$f(x) = x^{-2} = \frac{1}{x^2}$$

The graph of this function is shown on the right. The graphs of functions of this type are all similar to this one.



In general, we consider the function $f(x) = x^{-k}$, where $k = 2, 4, 6, \dots$

- the maximal domain $\mathbb{R} \setminus \{0\}$ and the range is $\mathbb{R}^+$
- f is an even function
- there is a horizontal asymptote with equation $y = 0$
- there is a vertical asymptote with equation $x = 0$.

▶ The function $f(x) = x^{\frac{1}{n}}$ where n is a positive integer

Let a be a positive real number and let $n \in \mathbb{N}$. Then $a^{\frac{1}{n}}$ is defined to be the nth root of a. That is, $a^{\frac{1}{n}}$ is the positive number whose nth power is a. We can also write this as $a^{\frac{1}{n}} = \sqrt[n]{a}$. For example: $9^{\frac{1}{2}} = 3$, since $3^2 = 9$.

We define $0^{\frac{1}{n}} = 0$, for each natural number n, since $0^n = 0$.

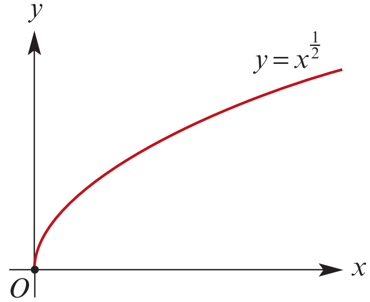
If n is odd, then we can also define $a^{\frac{1}{n}}$ when a is negative. If a is negative and n is odd, define $a^{\frac{1}{n}}$ to be the number whose nth power is a. For example: $(-8)^{\frac{1}{3}} = -2$, as $(-2)^3 = -8$.

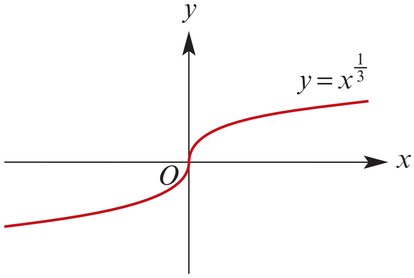
In all three cases we can write:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

In particular, $x^{\frac{1}{2}} = \sqrt{x}$.

Let $f(x) = x^{\frac{1}{n}}$. When n is even the maximal domain is $[0, \infty)$ and when n is odd the maximal domain is $\mathbb{R}$. The graphs of $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ and $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ are as shown.





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Example 19

Consider the functions $f(x) = x^{\frac{1}{3}}$ and $g(x) = x^{\frac{1}{2}}$, $x \geq 0$.

a Find the values of x for which $f(x) = g(x)$.

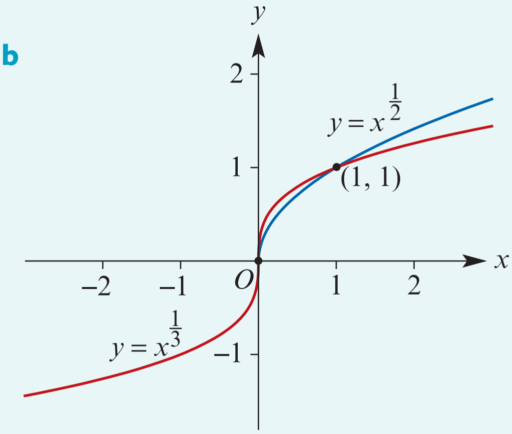
b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

Solution

a

$$f(x) = g(x)$$
$$x^{\frac{1}{3}} = x^{\frac{1}{2}}$$
$$x^{\frac{1}{3}} - x^{\frac{1}{2}} = 0$$
$$x^{\frac{1}{3}}\left(1 - x^{\frac{1}{6}}\right) = 0$$
$$\therefore \quad x = 0 \text{ or } 1 - x^{\frac{1}{6}} = 0$$
$$\therefore \quad x = 0 \text{ or } x = 1$$

b

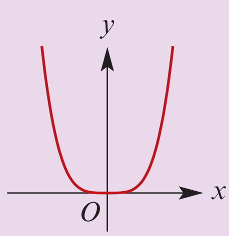


Section summary

- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
- A **power function** is a function f with rule $f(x) = x^r$, where r is a rational number.
- For a power function $f(x) = x^n$, where n is a non-zero integer, the general shape of the graph depends on whether n is positive or negative and whether n is even or odd:

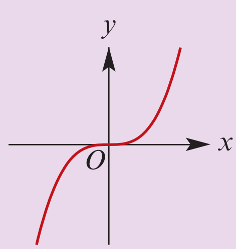
Even positive

$f(x) = x^4$



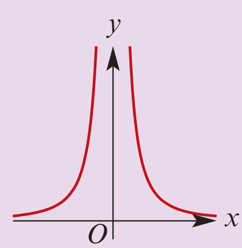
Odd positive

$f(x) = x^3$



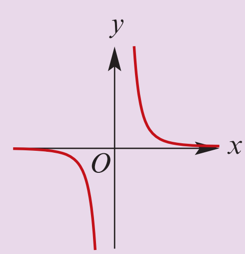
Even negative

$f(x) = x^{-2}$



Odd negative

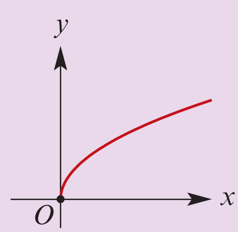
$f(x) = x^{-3}$



- For a power function $f(x) = x^{\frac{1}{n}}$, where n is a positive integer, the general shape of the graph depends on whether n is even or odd:

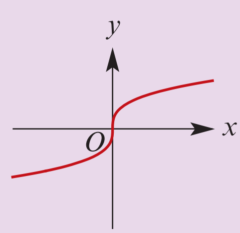
Even

$f(x) = x^{\frac{1}{2}}$



Odd

$f(x) = x^{\frac{1}{3}}$



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Exercise 1E

SF

Example 17 **1** For the function f with rule $f(x) = \frac{1}{x^4}$:

a State the maximal domain and the corresponding range.

b Evaluate each of the following:

i $f(2)$ **ii** $f(-2)$ **iii** $f(\frac{1}{2})$ **iv** $f(-\frac{1}{2})$

c Sketch the graph without using your calculator.

2 For each of the following, state whether the function is odd, even or neither:

a $f(x) = 2x^5$ **b** $f(x) = x^2 + 3$ **c** $f(x) = x^{\frac{1}{5}}$

d $f(x) = \frac{1}{x}$ **e** $f(x) = \frac{1}{x^2}$ **f** $f(x) = \sqrt[3]{x}$

Example 18 **3** Let $f(x) = x^{-2}$, $x \neq 0$, and $g(x) = x^{-4}$, $x \neq 0$.

a Find the values of x for which $f(x) = g(x)$.

b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

Example 19 **4** Let $f(x) = x^{\frac{1}{3}}$ and $g(x) = x^{\frac{1}{4}}$, $x \geq 0$.

a Find the values of x for which $f(x) = g(x)$.

b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

1F Applications of functions

In this section we use function notation in the solution of some problems.

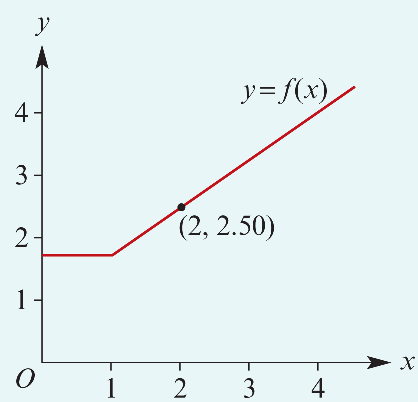
Example 20

The cost of a taxi trip in a particular city is \$1.75 up to and including 1 km. After 1 km the passenger pays an additional 75 cents per kilometre. Find the function f which describes this method of payment and sketch the graph of $y = f(x)$.

Solution

Let x denote the length of the trip in kilometres.
Then the cost in dollars is given by

$$f(x) = \begin{cases} 1.75 & \text{for } 0 \leq x \leq 1 \\ 1.75 + 0.75(x-1) & \text{for } x > 1 \end{cases}$$



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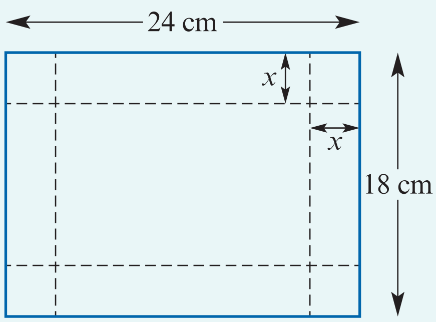
Example 21

A rectangular piece of cardboard has dimensions 18 cm by 24 cm. Four squares each x cm by x cm are cut from the corners. An open box is formed by folding up the flaps.

Find a function V which gives the volume of the box in terms of x, and state the domain of the function.



Solution

The dimensions of the box will be $24 - 2x$, $18 - 2x$ and x.

Thus the volume of the box is determined by the function

$$V(x) = (24 - 2x)(18 - 2x)x$$

For the box to be formed:

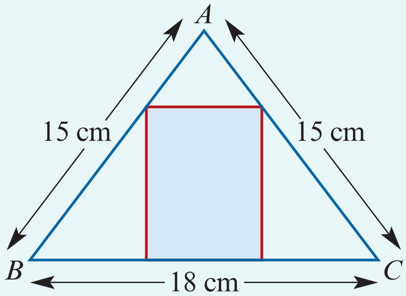
$$24 - 2x \geq 0 \quad \text{and} \quad 18 - 2x \geq 0 \quad \text{and} \quad x \geq 0$$

Therefore $x \leq 12$ and $x \leq 9$ and $x \geq 0$. The domain of V is $[0, 9]$.

Example 22

A rectangle is inscribed in an isosceles triangle with the dimensions as shown.

Find an area-of-the-rectangle function and state the domain.

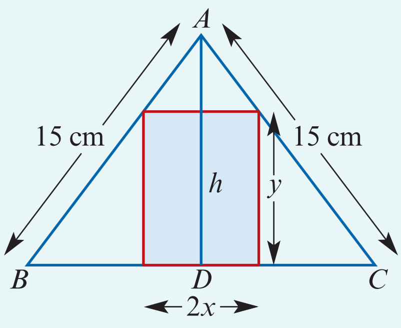


Solution

Let the height of the rectangle be y cm and the width $2x$ cm.

The height (h cm) of the triangle can be determined by Pythagoras' theorem:

$$h = \sqrt{15^2 - 9^2} = 12$$



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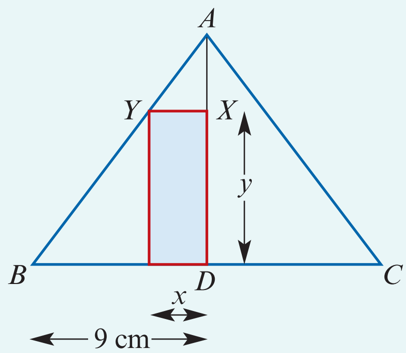
In the diagram opposite, the triangle AYX is similar to the triangle ABD. Therefore

$$\frac{x}{9} = \frac{12-y}{12}$$

$$\frac{12x}{9} = 12 - y$$

$$\therefore \quad y = 12 - \frac{12x}{9}$$



The area of the rectangle is $A = 2xy$, and so

$$A(x) = 2x\left(12 - \frac{12x}{9}\right) = \frac{24x}{9}(9-x)$$

For the rectangle to be formed, we need

$$x \ge 0 \quad \text{and} \quad 12 - \frac{12x}{9} \ge 0$$

$$\therefore \quad x \ge 0 \quad \text{and} \quad x \le 9$$

The domain is $[0, 9]$, and so the function is given by $A(x) = \frac{24x}{9}(9-x), \ x \in [0, 9]$.

Exercise 1F

CF

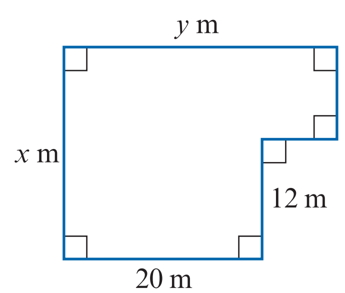
Example 20

1 The cost of a taxi trip in a particular city is \$4.00 up to and including 2 km. After 2 km the passenger pays an additional \$2.00 per kilometre. Find the function f which describes this method of payment and sketch the graph of $y = f(x)$, where x is the number of kilometres travelled. (Use a continuous model.)

Example 21

2 A rectangular piece of cardboard has dimensions 20 cm by 36 cm. Four squares each x cm by x cm are cut from the corners. An open box is formed by folding up the flaps. Find a function V which gives the volume of the box in terms of x, and state the domain for the function.

3 The dimensions of an enclosure are shown. The perimeter of the enclosure is 160 m.



a Find a rule for the area, A m^2, of the enclosure in terms of x.

b State a suitable domain of the function $A(x)$.

c Sketch the graph of A against x.

d Find the maximum possible area of the enclosure and state the corresponding values of x and y.

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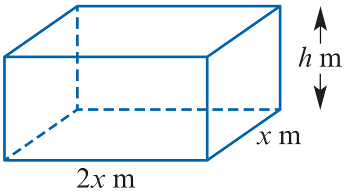
CF

4 A cuboid tank is open at the top and the internal dimensions of its base are x m and $2x$ m. The height is h m. The volume of the tank is V m^3 and the volume is fixed. Let S m^2 denote the internal surface area of the tank.



a Find S in terms of:

i x and h

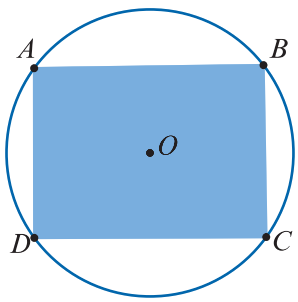
ii V and x

b State the maximal domain for the function defined by the rule in part **a ii**.

c If $2 \le x \le 15$, find the maximum value of S if $V = 1000$ m^3.

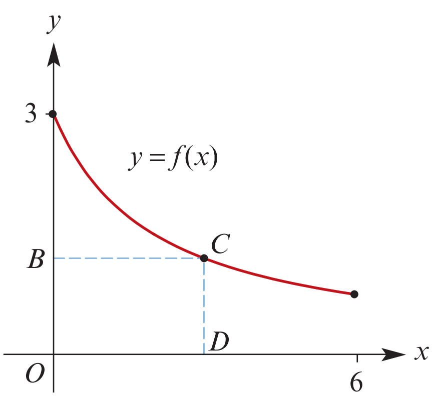
Example 22

5 A rectangle $ABCD$ is inscribed in a circle of radius a. Find an area-of-the-rectangle function and state the domain.



CU

6 Let $f(x) = \dfrac{6}{x+2}$ for $x \in [0, 6]$.
Rectangle $OBCD$ is formed so that the coordinates of C are $(a, f(a))$.



a Find an expression for the area-of-rectangle function A.

b State the implied domain and range of A.

c State the maximum value of $A(x)$ for $x \in [0, 6]$.

d Sketch the graph of $y = A(x)$ for $x \in [0, 6]$.

7 A man walks at a speed of 2 km/h for 45 minutes and then runs at 4 km/h for 30 minutes. Let S km be the distance the man has travelled after t minutes. The distance travelled can be described by

$$S(t) = \begin{cases} at & \text{if } 0 \le t \le c \\ bt + d & \text{if } c < t \le e \end{cases}$$

a Find the values a, b, c, d, e.

b Sketch the graph of $S(t)$ against t.

c State the range of the function.

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Chapter summary

AS
Nrich

Relations

- A **relation** is a set of ordered pairs.
- The **domain** is the set of all the first coordinates of the ordered pairs in the relation.
- The **range** is the set of all the second coordinates of the ordered pairs in the relation.

Functions

- A **function** is a relation such that no two ordered pairs in the relation have the same first coordinate.
- For each x in the domain of a function f, there is a unique element y in the range such that $(x, y) \in f$. The element y is called the **value** of f at x and is denoted by $f(x)$.
- When the domain of a function is not explicitly stated, it is assumed to consist of all real numbers for which the rule has meaning; this is called the **implied domain** or the **maximal domain** of the function.
- For a function f, the domain is denoted by **dom f** and the range by **ran f**.

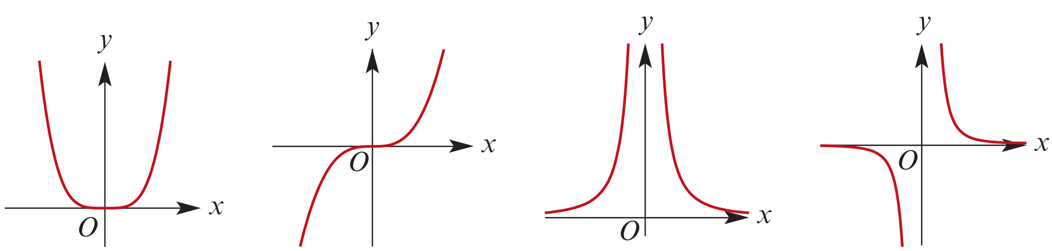
Combining functions

	Rule	Domain
Sum	$(f + g)(x) = f(x) + g(x)$	$\text{dom}(f + g) = \text{dom}\, f \cap \text{dom}\, g$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	$\text{dom}(f \cdot g) = \text{dom}\, f \cap \text{dom}\, g$
Quotient	$\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)}$	$\text{dom}\left(\dfrac{f}{g}\right) = \text{dom}\, f \cap \text{dom}\, g \cap \{\, x : g(x) \neq 0 \,\}$
Composition	$(g \circ f)(x) = g(f(x))$	$\text{dom}(g \circ f) = \text{dom}\, f$ if $\text{ran}\, f \subseteq \text{dom}\, g$

Types of functions

- A function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
- A function f is **even** if $f(-x) = f(x)$ for all x in the domain of f.
- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
- A **power function** is a function f with rule $f(x) = x^r$, where r is a rational number.
- For a power function $f(x) = x^n$, where n is a non-zero integer, the general shape of the graph depends on whether n is positive or negative and whether n is even or odd:

Even positive $f(x) = x^4$ | Odd positive $f(x) = x^3$ | Even negative $f(x) = x^{-2}$ | Odd negative $f(x) = x^{-3}$



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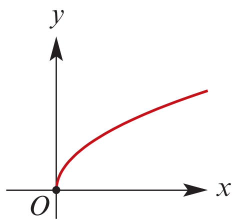
Review

- For a power function $f(x) = x^{\frac{1}{n}}$, where n is a positive integer, the general shape of the graph depends on whether n is even or odd:

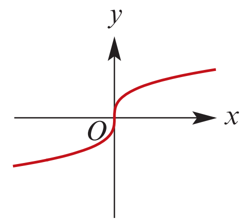
Even

$f(x) = x^{\frac{1}{2}}$



Odd

$f(x) = x^{\frac{1}{3}}$



Technology-free questions

SF

1 Sketch the graph of each of the following relations and state the implied domain and range:

a $f(x) = x^2 + 1$ **b** $f(x) = 2x - 6$ **c** $\{(x, y) : x^2 + y^2 = 25\}$
d $\{(x, y) : y \geq 2x + 1\}$ **e** $\{(x, y) : y < x - 3\}$

2 Consider the function defined by $g(x) = \frac{x+3}{2}$ for $x \in [0, 5]$.

a Sketch the graph of $y = g(x)$.
b State the range of g.
c Determine the value of x such that $g(x) = 4$.

3 For $g(x) = 5x + 1$, find the values of x such that:

a $g(x) = 2$ **b** $g(x) \leq 2$ **c** $\frac{1}{g(x)} = 2$

4 Sketch the graph of the function f given by

$$f(x) = \begin{cases} x + 1 & \text{for } x > 2 \\ x^2 - 1 & \text{for } 0 \leq x \leq 2 \\ -x^2 & \text{for } x < 0 \end{cases}$$

5 Find the implied domain for each of the following:

a $f(x) = \frac{1}{2x - 6}$ **b** $g(x) = \frac{1}{\sqrt{x^2 - 5}}$
c $h(x) = \frac{1}{(x-1)(x+2)}$ **d** $h(x) = \sqrt{25 - x^2}$
e $f(x) = \sqrt{x - 5} + \sqrt{15 - x}$ **f** $h(x) = \frac{1}{3x - 6}$

6 For $f(x) = (x + 2)^2$ and $g(x) = x - 3$, find $(f + g)(x)$ and $(fg)(x)$.

7 Let $f(x) = (x - 1)^2$ for $x \in [1, 5]$ and $g(x) = 2x$ for $x \in \mathbb{R}$. Find $f + g$ and fg.

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8 For $f(x) = 2x + 3$ and $g(x) = -x^2$, find:

a $(f + g)(x)$ **b** $(fg)(x)$ **c** the values of x such that $(f + g)(x) = 0$

9 For $f(x) = 2x + 3$ and $g(x) = -x^3$, find:

a $f(g(x))$ **b** $g(f(x))$ **c** $g(g(x))$ **d** $f(f(x))$

10 Express each of the following as the composition of two functions:

a $h(x) = (x^3 - 1)^{\frac{1}{3}}$ **b** $h(x) = \dfrac{1}{x^2 + 1}$ **c** $h(x) = \dfrac{1}{x^2 - 1}$

Multiple-choice questions

1 The maximal domain of the function with rule $f(x) = \sqrt{6 - 2x}$ is

A $(-\infty, 6]$ **B** $[3, \infty)$ **C** $(-\infty, 6]$ **D** $(3, \infty)$ **E** $(-\infty, 3]$

2 Let $f(x) = -x^2$ for $x \in [-1, 3)$. The range of f is

A $\mathbb{R}$ **B** $(-9, 0]$ **C** $(-\infty, 0]$ **D** $(-9, -1]$ **E** $[-9, 0]$

3 If $f(x) = 3x^2 + 2x$, then $f(2a) =$

A $20a^2 + 4a$ **B** $6a^2 + 2a$ **C** $6a^2 + 4a$ **D** $36a^2 + 4a$ **E** $12a^2 + 4a$

4 Let $f(x) = 10 - x$ for $x \in (a, b]$. The range of f is

A $(10 - a, 10 - b)$ **B** $(10 - a, 10 - b]$ **C** $(10 - b, 10 - a)$

D $(10 - b, 10 - a]$ **E** $[10 - b, 10 - a)$

5 For the function with rule

$$f(x) = \begin{cases} x^2 + 5 & x \geq 3 \\ -x + 6 & x < 3 \end{cases}$$

the value of $f(a + 3)$, where a is a negative real number, is

A $a^2 + 6a + 14$ **B** $-a + 9$ **C** $-a + 3$ **D** $a^2 + 14$ **E** $a^2 + 8a + 8$

6 The range of the function with rule $f(x) = x^2 + 2x - 6$ for $x \in [-2, 4)$ is

A $\mathbb{R}$ **B** $(-3, 18]$ **C** $(-6, 18)$ **D** $[0, 6]$ **E** $[-7, 18)$

7 The maximal domain and range of $f(x) = \dfrac{2x + 1}{x - 1}$ are

A $\mathbb{R} \setminus \{0\}$, $\mathbb{R} \setminus \{2\}$ **B** $\mathbb{R} \setminus \{1\}$, $\mathbb{R} \setminus \{-2\}$ **C** $\mathbb{R} \setminus \{1\}$, $\mathbb{R} \setminus \{2\}$

D $\mathbb{R} \setminus \{2\}$, $\mathbb{R} \setminus \{1\}$ **E** $\mathbb{R} \setminus \{-2\}$, $\mathbb{R} \setminus \{-1\}$

8 If $f(x) = 3x^2$ and $g(x) = 2x + 1$, then $f(g(a))$ is equal to

A $12a^2 + 3$ **B** $12a^2 + 12a + 3$ **C** $6a^2 + 1$

D $6a^2 + 4$ **E** $4a^2 + 4a + 1$

 ISBN 978-1-108-45164-2

Review

9 Let $f(x) = \sqrt{x+1}$, $x > -1$, and $g(x) = \sqrt{4-x}$, $x \le 4$. The domain of $f + g$ is

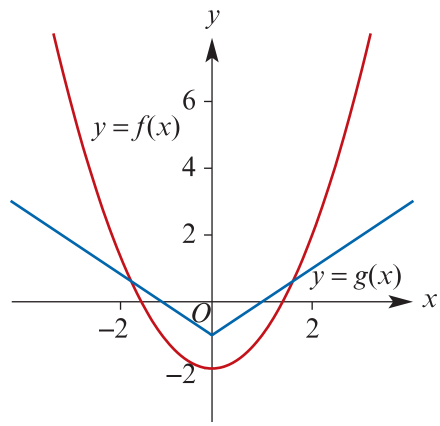
A $\mathbb{R}$ **B** $(-\infty, -1)$ **C** $(-1, 4]$ **D** $(-1, \infty)$ **E** $[-4, 1)$

10 Let $f(x) = 5 - x$ for $x \in D$. If the range of f is $[-2, 3)$, then the domain D is

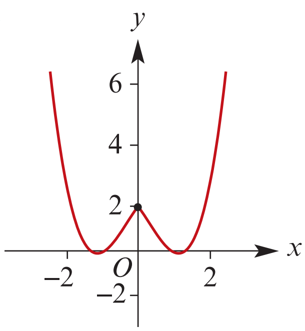
A $[-7, 2)$ **B** $(2, 7]$ **C** $\mathbb{R}$ **D** $[-2, 7)$ **E** $[2, 7)$

11 The graphs of $y = f(x)$ and $y = g(x)$ are as shown on the right.

Which one of the following best represents the graph of $y = f(g(x))$?



A



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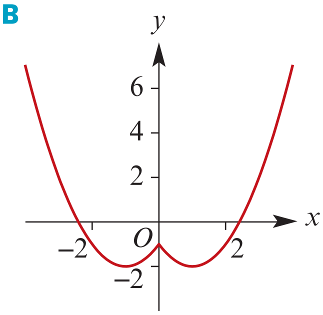

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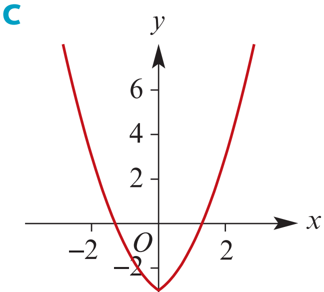

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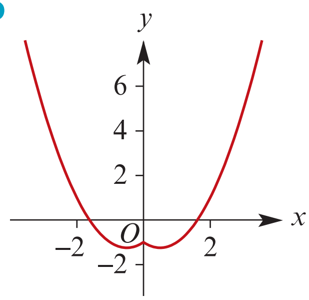

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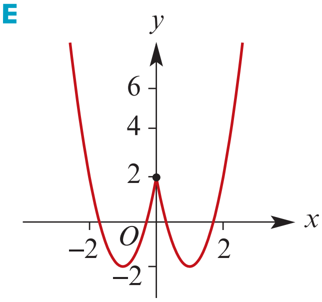

12 Which of the following functions is strictly increasing on the interval $(-\infty, -1]$?

A $f(x) = x^2$ **B** $f(x) = x^4$ **C** $f(x) = x^{\frac{1}{5}}$

D $f(x) = \sqrt{4-x}$ **E** $f(x) = -x^3$

13 For a function with rule $y = \dfrac{-2}{(x+3)^3} - 5$, the maximal domain and range are

A $x \ne 3,\ y \ne -5$ **B** $x \ne -5,\ y \ne -3$ **C** $x \ne -3,\ y \ne -5$

D $x \ne -2,\ y \ne -5$ **E** $x \ne -3,\ y \ne 5$

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14 A function with rule $f(x) = \frac{1}{x^4}$ can be defined on different domains. Which one of the following does not give the correct range for the given domain?

A dom $f = [-1, -0.5]$, ran $f = [1, 16]$

B dom $f = [-0.5, 0.5] \setminus \{0\}$, ran $f = [16, \infty)$

C dom $f = (-0.5, 0.5) \setminus \{0\}$, ran $f = (16, \infty)$

D dom $f = [-0.5, 1] \setminus \{0\}$, ran $f = [1, 16]$

E dom $f = [0.5, 1)$, ran $f = (1, 16]$

Extended-response questions

CF

1 Self-Travel, a car rental firm, has two methods of charging for car rental:

Method 1 $64 per day + 25 cents per kilometre

Method 2 $89 per day with unlimited travel.

a Write a rule for each method if x kilometres per day are travelled and the cost in dollars is C_1 using method 1 and C_2 using method 2.

b Draw the graph of each, using the same axes.

c Determine, from the graph, the distance that must be travelled per day if method 2 is cheaper than method 1.

2 Express the total surface area, S, of a cube as a function of:

a the length x of an edge

b the volume V of the cube.

3 Express the area, A, of an equilateral triangle as a function of:

a the length s of each side

b the altitude h.

CU

4 The base of a 3 m ladder leaning against a wall is x metres from the wall.

a Express the distance, d, from the top of the ladder to the ground as a function of x and sketch the graph of the function.

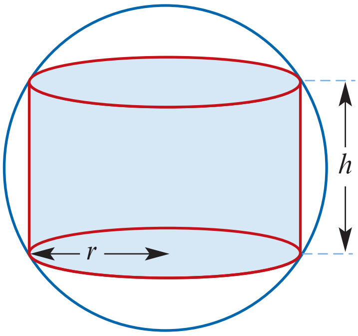
b State the domain and range of the function.

5 A car travels half the distance of a journey at an average speed of 80 km/h and half at an average speed of x km/h. Define a function, S, which gives the average speed for the total journey as a function of x.

6 A cylinder is inscribed in a sphere with a radius of length 6 cm.

a Define a function, V_1, which gives the volume of the cylinder as a function of its height, h. (State the rule and domain.)

b Define a function, V_2, which gives the volume of the cylinder as a function of the radius of the cylinder, r. (State the rule and domain.)



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CF

7 A function f is defined as follows:

$$f(x) = \begin{cases} x^2 - 4 & \text{for } x \in (-\infty, 2) \\ x & \text{for } x \in [2, \infty) \end{cases}$$

a Sketch the graph of f.

b Find the values of $f(-1)$ and $f(3)$.

c If $h(x) = 2x$, find $f(h(x))$ and $h(f(x))$.

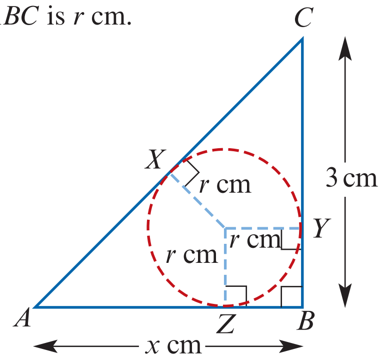
CU

8 Find the rule for the area, $A(t)$, enclosed by the graph of the function

$$f(x) = \begin{cases} 3x, & 0 \le x \le 1 \\ 3, & x > 1 \end{cases}$$

the x-axis, the y-axis and the vertical line $x = t$ (for $t > 0$). State the domain and range of the function A.

9 The radius of the incircle of the right-angled triangle ABC is r cm.



a Find:

i YB in terms of r

ii ZB in terms of r

iii AZ in terms of r and x

iv CY

b Using the geometric results $CY = CX$ and $AX = AZ$, find an expression for r in terms of x.

c **i** Find r when $x = 4$.

ii Find x when $r = 0.5$.

d Use a calculator to investigate the possible values that r can take.

10 Let $f(x) = \dfrac{px + q}{x + r}$ where $x \in \mathbb{R} \setminus \{-r, r\}$.

a If $f(x) = f(-x)$ for all x, show that $f(x) = p$ for $x \in \mathbb{R} \setminus \{-r, r\}$.

b If $f(-x) = -f(x)$ for $x \neq 0$, find the rule for $f(x)$ in terms of q.

c If $p = 3$, $q = 8$ and $r = -3$, find the values of x for which $f(x) = x$.

11 **a** Let $f(x) = \dfrac{x + 1}{x - 1}$.

i Find $f(2)$, $f(f(2))$ and $f(f(f(2)))$.

ii Find $f(f(x))$.

b Let $f(x) = \dfrac{x - 3}{x + 1}$. Find $f(f(x))$ and $f(f(f(x)))$.

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2 Coordinate geometry and transformations

Objectives

- To revise coordinate geometry:
  - finding the **distance** between two points
  - finding the **midpoint** of a line segment
  - calculating the **gradient** of a straight line
  - interpreting and using different forms of the **equation of a straight line**
  - finding the **angle of slope** of a straight line
  - determining the gradient of a line **perpendicular** to a given line.
- To introduce a notation for considering transformations of the plane, including **translations**, **dilations** from an axis and **reflections** in an axis.
- To use transformations to help with graph sketching.
- To consider transformations of power functions.
- To determine the rule for a function given sufficient information.

Much of the material presented in this chapter has been covered in Mathematical Methods Units 1 & 2. The chapter provides a framework for revision with worked examples and exercises.

Many graphs of functions can be described as transformations of graphs of other functions, or 'movements' of graphs about the Cartesian plane. For example, the graph of the function $y = -x^2$ can be considered as a reflection in the x-axis of the graph of the function $y = x^2$.

A good understanding of transformations, combined with knowledge of the 'simplest' function and its graph in each family, provides an important tool with which to sketch graphs and identify rules of more complicated functions.

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2A Linear coordinate geometry

In this section, we revise the concepts of linear coordinate geometry.

Example 1

A straight line passes through the points $A(-2, 6)$ and $B(4, 7)$. Find:

a the distance AB

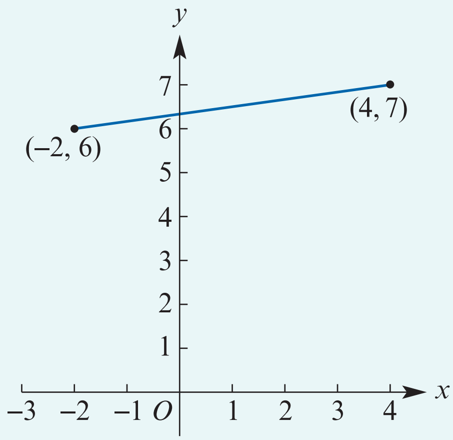
b the midpoint of line segment AB

c the gradient of line AB

d the equation of line AB

e the equation of the line parallel to AB which passes through the point $(1, 5)$

f the equation of the line perpendicular to AB which passes through the midpoint of AB.



Solution

a The distance AB is

$$\sqrt{(4-(-2))^2+(7-6)^2}=\sqrt{37}$$

Explanation: The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

b The midpoint of AB is

$$\left(\frac{-2+4}{2}, \frac{6+7}{2}\right)=\left(1, \frac{13}{2}\right)$$

Explanation: The line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ has midpoint $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

c The gradient of line AB is

$$\frac{7-6}{4-(-2)}=\frac{1}{6}$$

Explanation: Gradient

$$m=\frac{y_2-y_1}{x_2-x_1}$$

d The equation of line AB is

$$y-6=\frac{1}{6}(x-(-2))$$

which simplifies to $6y-x-38=0$.

Explanation: Equation of a straight line passing through a given point (x_1, y_1) and having gradient m is $y-y_1=m(x-x_1)$.

e Gradient $m=\frac{1}{6}$ and $(x_1, y_1)=(1, 5)$.

The line has equation

$$y-5=\frac{1}{6}(x-1)$$

which simplifies to $6y-x-29=0$.

Explanation: Parallel lines have the same gradient.

f A perpendicular line has gradient -6.

Thus the equation is

$$y-\frac{13}{2}=-6(x-1)$$

which simplifies to $2y+12x-25=0$.

Explanation: If two straight lines are perpendicular to each other, then the product of their gradients is -1.

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Example 2

A fruit and vegetable wholesaler sells 30 kg of hydroponic tomatoes for \$148.50 and sells 55 kg of hydroponic tomatoes for \$247.50. Find a linear model for the cost, \$$C$, of x kg of hydroponic tomatoes. How much would 20 kg of tomatoes cost?

Solution

Let $(x_1, C_1) = (30, 148.5)$ and $(x_2, C_2) = (55, 247.5)$.

The equation of the straight line is given by

$$C - C_1 = m(x - x_1) \quad \text{where } m = \frac{C_2 - C_1}{x_2 - x_1}$$

Now $m = \dfrac{247.5 - 148.5}{55 - 30} = 3.96$ and so

$$C - 148.5 = 3.96(x - 30)$$

Therefore the straight line has equation $C = 3.96x + 29.7$.

Substitute $x = 20$:

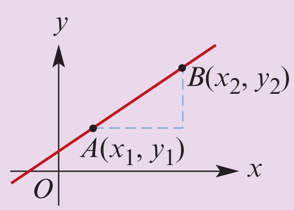
$$C = 3.96 \times 20 + 29.7 = 108.9$$

Hence it would cost \$108.90 to buy 20 kg of tomatoes.

The following is a summary of the material that is assumed to have been covered in Mathematical Methods Units 1 & 2.

Section summary

- For two points $A(x_1, y_1)$ and $B(x_2, y_2)$:
  - The **distance** between points A and B is given by $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
  - The **midpoint** of the line segment AB is the point with coordinates $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.
  - The **gradient** of the line AB is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.



- Different forms for the equation of a straight line:

$y = mx + c$	where m is the gradient and c is the y-axis intercept
$y - y_1 = m(x - x_1)$	where m is the gradient and (x_1, y_1) is a point on the line
$\dfrac{x}{a} + \dfrac{y}{b} = 1$	where $(a, 0)$ and $(0, b)$ are the axis intercepts

- The **angle of slope** of a straight line is found using $m = \tan\theta$, where m is the gradient and θ is the angle that the line makes with the positive direction of the x-axis.
- Two straight lines are **perpendicular** to each other if and only if the product of their gradients is -1, i.e. $m_1 m_2 = -1$. (Unless one line is vertical and the other horizontal.)

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Exercise 2A

Skillsheet SF

Example 1

1 A straight line passes through the points $A(-2, 6)$ and $B(4, -7)$. Find:

a the distance AB
b the midpoint of line segment AB
c the gradient of line AB
d the equation of line AB
e the equation of the line parallel to AB which passes through the point $(1, 5)$
f the equation of the line perpendicular to AB which passes through the midpoint of AB.

2 Find the coordinates of M, the midpoint of AB, where A and B have the following coordinates:

a $A(1, 4)$, $B(5, 11)$
b $A(-6, 4)$, $B(1, -8)$
c $A(-1, -6)$, $B(4, 7)$

3 If M is the midpoint of XY, find the coordinates of Y when X and M have the following coordinates:

a $X(-4, 5)$, $M(0, 6)$
b $X(-1, -4)$, $M(2, -3)$
c $X(6, -3)$, $M(4, 8)$

4 Use $y = mx + c$ to sketch the graph of each of the following:

a $y = 3x - 3$
b $y = -3x + 4$
c $3y + 2x = 12$
d $4x + 6y = 12$
e $3y - 6x = 18$
f $8x - 4y = 16$

5 Find the equations of the following straight lines:

a gradient +2, passing through $(4, 2)$
b gradient -3, passing through $(-3, 4)$
c passing through $(1, 3)$ and $(4, 7)$
d passing through $(-2, -3)$ and $(2, 5)$

6 Use the intercept form $\frac{x}{a} + \frac{y}{b} = 1$ to find the equation of the straight line passing through:

a $(-3, 0)$ and $(0, 2)$
b $(4, 0)$ and $(0, 6)$
c $(0, -2)$ and $(6, 0)$

7 Write the following in intercept form and hence draw their graphs:

a $3x + 6y = 12$
b $4y - 3x = 12$
c $\frac{3}{2}x - 3y = 9$

Example 2

8 A printing firm charges \$35 for printing 600 sheets of headed notepaper and \$46 for printing 800 sheets. Find a linear model for the charge, \$$C$, for printing n sheets. How much would they charge for printing 1000 sheets?

9 An electronic bank teller registered \$775 after it had counted 120 notes and \$975 after it had counted 160 notes.

a Find a formula for the sum registered (\$$C$) in terms of the number of notes (n) counted.
b Was there a sum already on the register when counting began?
c If so, how much?

10 Find the distance between each of the following pairs of points:

a $(2, 6)$, $(3, 4)$
b $(-2, -6)$, $(2, -8)$
c $(0, 4)$, $(3, 0)$

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SF

11 **a** Find the equation of the straight line which passes through the point $(1, 6)$ and is:

i parallel to the line with equation $y = 2x + 3$

ii perpendicular to the line with equation $y = 2x + 3$.

b Find the equation of the straight line which passes through the point $(2, 3)$ and is:

i parallel to the line with equation $4x + 2y = 10$

ii perpendicular to the line with equation $4x + 2y = 10$.

CF

12 Find the equation of the line which passes through the point of intersection of the lines $y = x$ and $x + y = 6$ and which is perpendicular to the line with equation $3x + 6y = 12$.

13 The length of the line segment joining $A(2, -1)$ and $B(5, y)$ is 5 units. Find y.

14 For each of the following, find the angle that the line joining the given points makes with the positive direction of the x-axis:

a $(-4, 1), \ (4, 6)$ **b** $(2, 3), \ (-4, 6)$ **c** $(5, 1), \ (-1, -8)$ **d** $(-4, 2), \ (2, -8)$

15 Find the acute angle between the lines $y = 2x + 4$ and $y = -3x + 6$.

16 Given the points $A(a, 3)$, $B(-2, 1)$ and $C(3, 2)$, find the possible values of a if the length of AB is twice the length of BC.

17 Three points have coordinates $A(1, 7)$, $B(7, 5)$ and $C(0, -2)$. Find:

a the equation of the perpendicular bisector of AB

b the point of intersection of this perpendicular bisector and BC.

18 The point (h, k) lies on the line $y = x + 1$ and is 5 units from the point $(0, 2)$. Write down two equations connecting h and k and hence find the possible values of h and k.

CU

19 P and Q are the points of intersection of the line $\frac{y}{2} + \frac{x}{3} = 1$ with the x- and y-axes respectively. The gradient of QR is $\frac{1}{2}$ and the point R has x-coordinate $2a$, where $a > 0$.

a Find the y-coordinate of R in terms of a.

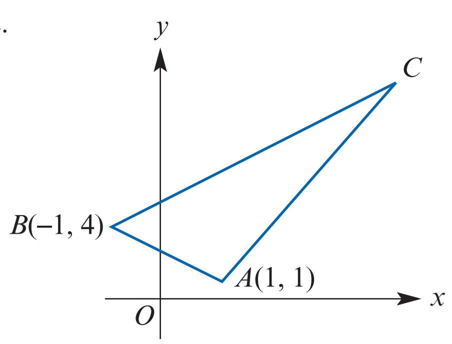
b Find the value of a if the gradient of PR is -2.

20 The figure shows a triangle ABC with $A(1, 1)$ and $B(-1, 4)$. The gradients of AB, AC and BC are $-3m$, $3m$ and m respectively.

a Find the value of m.

b Find the coordinates of C.

c Show that $AC = 2AB$.



21 $ABCD$ is a parallelogram, with vertices labelled anticlockwise, such that A and C are the points $(-1, 5)$ and $(5, 1)$ respectively.

a Find the coordinates of the midpoint of AC.

b Given that BD is parallel to the line $y + 5x = 2$, find the equation of BD.

c Given that BC is perpendicular to AC, find:

i the equation of BC **ii** the coordinates of B **iii** the coordinates of D.

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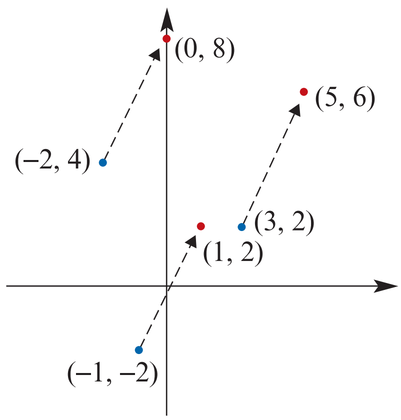
2B Translations

The **Cartesian plane** is represented by the set $\mathbb{R}^2$ of all ordered pairs of real numbers. That is, $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. The transformations considered in this book associate each ordered pair of $\mathbb{R}^2$ with a unique ordered pair. We can refer to them as examples of **transformations of the plane**.

► Notation

Consider the translation 2 units in the positive direction of the x-axis (to the right) and 4 units in the positive direction of the y-axis (up).



This can be described by the rule $(x, y) \to (x + 2, y + 4)$. This reads as '$(x, y)$ maps to $(x + 2, y + 4)$'.

For example, $(3, 2) \to (3 + 2, 2 + 4)$.

In applying this translation, it is useful to think of every point (x, y) in the plane as being mapped to a new point (x', y'). We can write:

$$x' = x + 2 \quad \text{and} \quad y' = y + 4$$

For positive numbers h and k:

- A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule

$$(x, y) \to (x + h, y + k)$$

or $\quad x' = x + h \quad \text{and} \quad y' = y + k$

- A translation of h units in the negative direction of the x-axis and k units in the negative direction of the y-axis is described by the rule

$$(x, y) \to (x - h, y - k)$$

or $\quad x' = x - h \quad \text{and} \quad y' = y - k$

Notes:

- Under a translation, if $(a', b') = (c', d')$, then $(a, b) = (c, d)$.
- For a translation $(x, y) \to (x + h, y + k)$, for each point $(a, b) \in \mathbb{R}^2$ there is a point (p, q) such that $(p, q) \to (a, b)$. (It is clear that $(a - h, b - k) \to (a, b)$ under this translation.)

► Applying translations to sketch graphs

A translation moves every point on the graph the same distance in the same direction.

Every translation of the plane can be described by giving two components:

- a translation parallel to the x-axis and
- a translation parallel to the y-axis.

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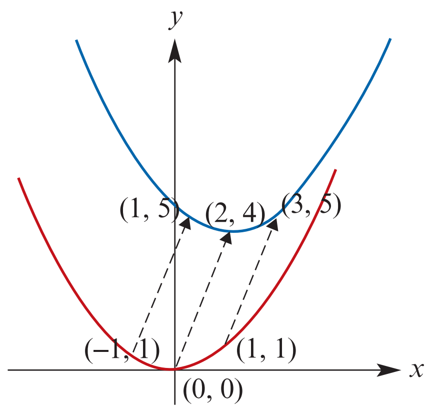
For example, consider a translation of 2 units in the positive direction of the x-axis and 4 units in the positive direction of the y-axis applied to the graph of $y = x^2$.

The set of points $\{ (x, y) : y = x^2 \}$ is translated according to the rule

$$(x, y) \to (x + 2, y + 4)$$

$$x' = x + 2 \quad \text{and} \quad y' = y + 4$$



For each point (x, y) there is a unique point (x', y') and vice versa.

We have $x = x' - 2$ and $y = y' - 4$.

This means the points on the curve with equation $y = x^2$ are mapped to the curve with equation $y' - 4 = (x' - 2)^2$.

Hence $\{ (x, y) : y = x^2 \}$ maps to $\{ (x', y') : y' - 4 = (x' - 2)^2 \}$.

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x - h)$ and graphing the result.

Proof A point (a, b) is on the graph of $y = f(x)$

$\Leftrightarrow \; f(a) = b$

$\Leftrightarrow \; f(a + h - h) = b$

$\Leftrightarrow \; f(a + h - h) = b + k - k$

$\Leftrightarrow \; (a + h, b + k)$ is a point on the graph of $y - k = f(x - h)$

Note: The double arrows indicate that the steps are reversible.

Example 3

Find the image of the curve with equation $y = \frac{1}{x}$ under a translation of 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

Solution

Let (x', y') be the image of the point (x, y).

Then $x' = x + 3$ and $y' = y - 2$.

Hence $x = x' - 3$ and $y = y' + 2$.

The graph of $y = \frac{1}{x}$ is mapped to $y' + 2 = \frac{1}{x' - 3}$.

The equation of the image can be written as

$$y = \frac{1}{x - 3} - 2$$

Explanation

The rule is $(x, y) \to (x + 3, y - 2)$.

Substitute $x = x' - 3$ and $y = y' + 2$ into the equation $y = \frac{1}{x}$.

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Recognising that a transformation has been applied makes it easy to sketch many graphs.

For example, to sketch the graph of $y = \sqrt{x-2}$, note that it is of the form $y = f(x-2)$, where $f(x) = \sqrt{x}$. The graph of $y = \sqrt{x}$ is translated 2 units in the positive direction of the x-axis.

Examples of two other functions to which this translation is applied are:

$$f(x) = x^2 \qquad f(x-2) = (x-2)^2$$

$$f(x) = \frac{1}{x} \qquad f(x-2) = \frac{1}{x-2}$$

Section summary

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x+h, y+k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x-h)$ and graphing the result.

Exercise 2B

SF

1 Find the image of the point $(-2, 5)$ after a mapping of a translation:

a of 1 unit in the positive direction of the x-axis and 2 units in the negative direction of the y-axis

b of 3 units in the negative direction of the x-axis and 5 units in the positive direction of the y-axis

c defined by the rule $(x, y) \to (x-3, y+2)$

d defined by the rule $(x, y) \to (x-1, y+1)$.

Example 3

2 Find the image of the curve with equation $y = \frac{1}{x}$ under:

a a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

b a translation of 2 units in the negative direction of the x-axis and 3 units in the positive direction of the y-axis

c a translation of $\frac{1}{2}$ unit in the positive direction of the x-axis and 4 units in the positive direction of the y-axis.

3 Sketch the graph of each of the following. Label asymptotes and axis intercepts, and state the domain and range.

a $y = \frac{1}{x} + 3$ **b** $y = \frac{1}{x^2} - 3$ **c** $y = \frac{1}{(x+2)^2}$

d $y = \sqrt{x-2}$ **e** $y = \frac{1}{x-1}$ **f** $y = \frac{1}{x} - 4$

g $y = \frac{1}{x+2}$ **h** $f(x) = \frac{1}{(x-3)^2}$ **i** $f(x) = \frac{1}{x-1} + 1$

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4 For $y = f(x) = \frac{1}{x}$, sketch the graph of each of the following. Label asymptotes and axis intercepts.

a $y = f(x - 1)$ **b** $y = f(x) + 1$ **c** $y = f(x + 3)$

d $y = f(x) - 3$ **e** $y = f(x + 1)$ **f** $y = f(x) - 1$

5 For each of the following, state a transformation which maps the graph of $y = f(x)$ to the graph of $y = f_1(x)$:

a $f(x) = x^2$, $f_1(x) = (x + 5)^2$ **b** $f(x) = \frac{1}{x}$, $f_1(x) = \frac{1}{x} + 2$

c $f(x) = \frac{1}{x^2}$, $f_1(x) = \frac{1}{x^2} + 4$ **d** $f(x) = \frac{1}{x^2} - 3$, $f_1(x) = \frac{1}{x^2}$

e $f(x) = \frac{1}{x - 3}$, $f_1(x) = \frac{1}{x}$

6 Write down the equation of the image when the graph of each of the functions below is transformed by:

i a translation of 7 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis

ii a translation of 2 units in the negative direction of the x-axis and 6 units in the negative direction of the y-axis

iii a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

iv a translation of 1 unit in the negative direction of the x-axis and 4 units in the positive direction of the y-axis.

a $y = x^{\frac{1}{4}}$ **b** $y = \sqrt[3]{x}$ **c** $y = \frac{1}{x^3}$ **d** $y = \frac{1}{x^4}$

7 Find the equation for the image of the graph of each of the following under the stated translation:

a $y = (x - 2)^2 + 3$ Translation: $(x, y) \to (x - 3, y + 2)$

b $y = 2(x + 3)^2 + 3$ Translation: $(x, y) \to (x + 3, y - 3)$

c $y = \frac{1}{(x - 2)^2} + 3$ Translation: $(x, y) \to (x + 4, y - 2)$

8 For each of the following, state a transformation which maps the graph of $y = f(x)$ to the graph of $y = f_1(x)$:

a $f(x) = \frac{1}{x^2}$, $f_1(x) = \frac{1}{(x - 2)^2} + 3$ **b** $f(x) = \frac{1}{x}$, $f_1(x) = \frac{1}{x + 2} - 3$

c $f(x) = \sqrt{x}$, $f_1(x) = \sqrt{x + 4} + 2$

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2C Dilations and reflections

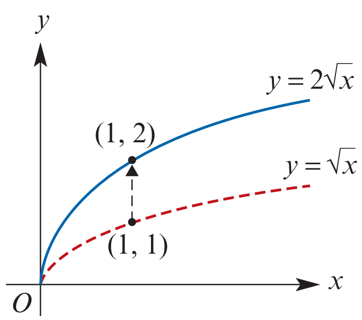
We can determine the equation of the image of a curve under a dilation or a reflection by following the same approach used for translations.

▶ Dilation from the x-axis

A dilation of factor 2 from the x-axis is defined by the rule $(x, y) \to (x, 2y)$. Hence the point with coordinates $(1, 1) \to (1, 2)$.

Consider the curve with equation $y = \sqrt{x}$ and the dilation of factor 2 from the x-axis.

- Let (x', y') be the image of the point with coordinates (x, y) on the curve.
- Hence $x' = x$ and $y' = 2y$, and thus $x = x'$ and $y = \dfrac{y'}{2}$.
- Substituting for x and y, we see that the curve with equation $y = \sqrt{x}$ maps to the curve with equation $\dfrac{y'}{2} = \sqrt{x'}$, i.e. the curve with equation $y = 2\sqrt{x}$.

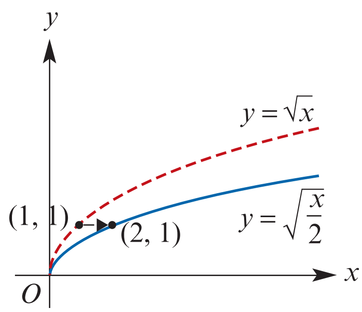


▶ Dilation from the y-axis

A dilation of factor 2 from the y-axis is defined by the rule $(x, y) \to (2x, y)$. Hence the point with coordinates $(1, 1) \to (2, 1)$.

Again, consider the curve with equation $y = \sqrt{x}$.

- Let (x', y') be the image of the point (x, y).
- Hence $x' = 2x$ and $y' = y$, and thus $x = \dfrac{x'}{2}$ and $y = y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{\dfrac{x'}{2}}$.



Example 4

Determine the rule of the image when the graph of $y = \dfrac{1}{x^2}$ is dilated by a factor of 4:

a from the x-axis **b** from the y-axis.

Solution

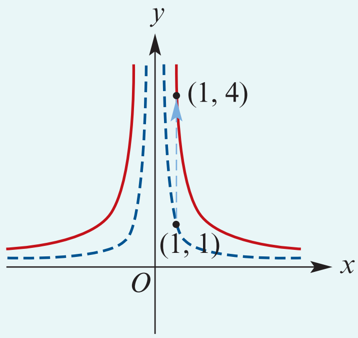
a $(x, y) \to (x, 4y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = x,\ y' = 4y$.

Rearranging gives $x = x',\ y = \dfrac{y'}{4}$.

Therefore $y = \dfrac{1}{x^2}$ becomes $\dfrac{y'}{4} = \dfrac{1}{(x')^2}$.

The rule of the transformed function is $y = \dfrac{4}{x^2}$.



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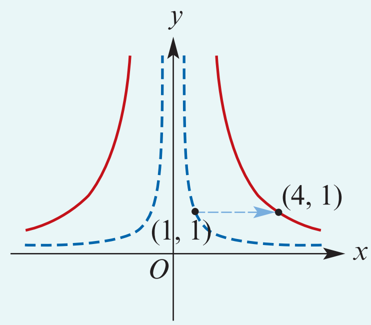
b $(x, y) \to (4x, y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = 4x$, $y' = y$.

Rearranging gives $x = \dfrac{x'}{4}$, $y = y'$.

Therefore $y = \dfrac{1}{x^2}$ becomes $y' = \dfrac{1}{(\frac{x'}{4})^2}$.

The rule of the transformed function is $y = \dfrac{16}{x^2}$.



Example 5

Determine the factor of dilation when the graph of $y = \sqrt{3x}$ is obtained by dilating the graph of $y = \sqrt{x}$:

a from the y-axis **b** from the x-axis.

Solution

a Note that a dilation from the y-axis 'changes' the x-values. So write the transformed function as

$$y' = \sqrt{3x'}$$

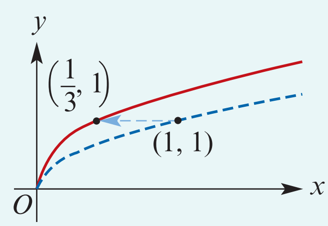
where (x', y') are the coordinates of the image of (x, y).

Therefore $x = 3x'$ and $y = y'$ ('changed' x).

Rearranging gives $x' = \dfrac{x}{3}$ and $y' = y$.

So the mapping is given by $(x, y) \to \left(\dfrac{x}{3}, y\right)$.

The graph of $y = \sqrt{x}$ is dilated by a factor of $\frac{1}{3}$ from the y-axis to produce the graph of $y = \sqrt{3x}$.



b Note that a dilation from the x-axis 'changes' the y-values. So write the transformed function as

$$\frac{y'}{\sqrt{3}} = \sqrt{x'}$$

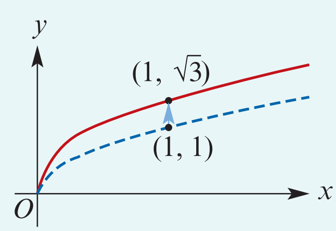
where (x', y') are the coordinates of the image of (x, y).

Therefore $x = x'$ and $y = \dfrac{y'}{\sqrt{3}}$ ('changed' y).

Rearranging gives $x' = x$ and $y' = \sqrt{3}\,y$.

So the mapping is given by $(x, y) \to (x, \sqrt{3}\,y)$.

The graph of $y = \sqrt{x}$ is dilated by a factor of $\sqrt{3}$ from the x-axis to produce the graph of $y = \sqrt{3x}$.



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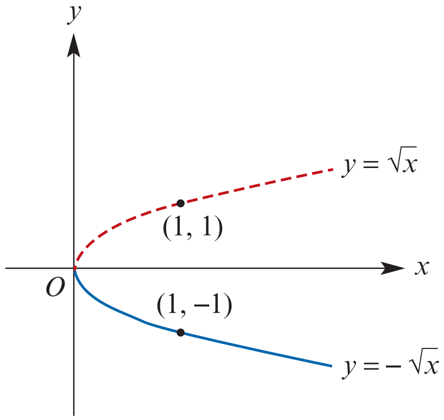
▶ Reflection in the x-axis

A reflection in the x-axis can be defined by the rule $(x, y) \to (x, -y)$. Hence the point with coordinates $(1, 1) \to (1, -1)$.

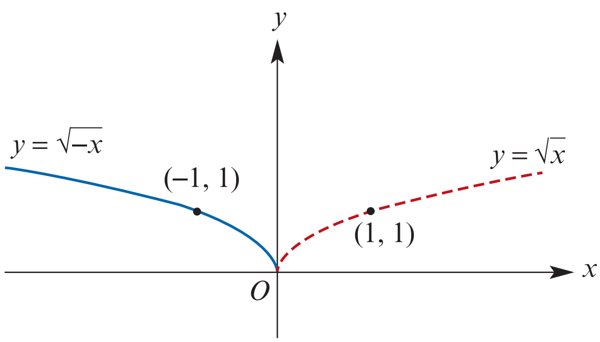
- Let (x', y') be the image of the point (x, y).
- Hence $x' = x$ and $y' = -y$, which gives $x = x'$ and $y = -y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $-y' = \sqrt{x'}$, i.e. the curve with equation $y = -\sqrt{x}$.



▶ Reflection in the y-axis

A reflection in the y-axis can be defined by the rule $(x, y) \to (-x, y)$. Hence the point with coordinates $(1, 1) \to (-1, 1)$.

- Let (x', y') be the image of the point (x, y).
- Hence $x' = -x$ and $y' = y$, which gives $x = -x'$ and $y = y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{-x'}$, i.e. the curve with equation $y = \sqrt{-x}$.



Section summary

Transformations of the graphs of functions

Mapping	Rule	Image of $y = f(x)$
Reflection in the x-axis	$x' = x,\ y' = -y$	$y = -f(x)$
Reflection in the y-axis	$x' = -x,\ y' = y$	$y = f(-x)$
Dilation of factor a from the y-axis	$x' = ax,\ y' = y$	$y = f\left(\frac{x}{a}\right)$
Dilation of factor b from the x-axis	$x' = x,\ y' = by$	$y = bf(x)$

Exercise 2C

Example 4 **1** Determine the rule of the image when the graph of $y = \frac{1}{x}$ is dilated by a factor of 3: SF

a from the x-axis **b** from the y-axis.

2 Determine the rule of the image when the graph of $y = \sqrt{x}$ is dilated by a factor of 2:

a from the x-axis **b** from the y-axis.

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SF

3 Determine the rule of the image when the graph of $y = x^3$ is dilated by a factor of 2:

a from the x-axis **b** from the y-axis.

4 Sketch the graph of each of the following:

a $y = \frac{4}{x}$ **b** $y = \frac{1}{2x}$ **c** $y = \sqrt{3x}$ **d** $y = \frac{2}{x^2}$

5 For $y = f(x) = \frac{1}{x^2}$, sketch the graph of each of the following:

a $y = f(2x)$ **b** $y = 2f(x)$ **c** $y = f\left(\frac{x}{2}\right)$ **d** $y = 3f(x)$

6 Sketch the graphs of each of the following on the one set of axes:

a $y = \frac{1}{x}$ **b** $y = \frac{3}{x}$ **c** $y = \frac{3}{2x}$

7 Sketch the graph of the function $f(x) = 3\sqrt{x}$ for $x \in \mathbb{R}^+$.

Example 5

8 Determine the factor of dilation when the graph of $y = \sqrt{5x}$ is obtained by dilating the graph of $y = \sqrt{x}$:

a from the y-axis **b** from the x-axis.

9 For each of the following, state a transformation which maps the graph of $y = f(x)$ to the graph of $y = f_1(x)$:

a $f(x) = \frac{1}{x^2}$, $f_1(x) = \frac{5}{x^2}$ **b** $f(x) = \sqrt{x}$, $f_1(x) = \sqrt{5x}$

c $f(x) = \sqrt{\frac{x}{3}}$, $f_1(x) = \sqrt{x}$ **d** $f(x) = \frac{1}{4x^2}$, $f_1(x) = \frac{1}{x^2}$

10 Write down the equation of the image when the graph of each of the functions below is transformed by:

i a dilation of factor 4 from the x-axis
ii a dilation of factor $\frac{2}{3}$ from the x-axis
iii a dilation of factor $\frac{1}{2}$ from the y-axis
iv a dilation of factor 5 from the y-axis.

a $y = x^2$ **b** $y = \frac{1}{x^2}$ **c** $y = \sqrt[3]{x}$ **d** $y = \frac{1}{x^4}$ **e** $y = x^{\frac{1}{5}}$

11 Find the equation of the image when the graph of $y = (x - 1)^2$ is reflected:

a in the x-axis **b** in the y-axis.

12 State a transformation which maps the graph of $y = \sqrt{x}$ to the graph of $y = \sqrt{-x}$.

13 Find the equation of the image when the graph of each of the functions below is transformed by:

i a reflection in the x-axis
ii a reflection in the y-axis.

a $y = x^3$ **b** $y = \sqrt[3]{x}$ **c** $y = \frac{1}{x^3}$ **d** $y = \frac{1}{x^4}$ **e** $y = x^{\frac{1}{4}}$

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2D Combinations of transformations

In the previous two sections, we considered three types of transformations separately. In the remainder of this chapter we look at situations where a graph may have been transformed by any combination of dilations, reflections and translations.

For example, first consider:

- a dilation of factor 2 from the x-axis
- followed by a reflection in the x-axis.

The rule becomes

$$(x, y) \to (x, 2y) \to (x, -2y)$$

First the dilation is applied and then the reflection. For example, $(1, 1) \to (1, 2) \to (1, -2)$.

Another example is:

- a dilation of factor 2 from the x-axis
- followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

The rule becomes

$$(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$$

First the dilation is applied and then the translation. For example, $(1, 1) \to (1, 2) \to (3, -1)$.

Example 6

Find the equation of the image of $y = \sqrt{x}$ under:

a a dilation of factor 2 from the x-axis followed by a reflection in the x-axis

b a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

Solution

a From the discussion above, the rule is $(x, y) \to (x, 2y) \to (x, -2y)$.

If (x, y) maps to (x', y'), then $x' = x$ and $y' = -2y$. Thus $x = x'$ and $y = \dfrac{y'}{-2}$.

So the image of $y = \sqrt{x}$ has equation

$$\frac{y'}{-2} = \sqrt{x'}$$

and hence $y' = -2\sqrt{x'}$. The equation can be written as $y = -2\sqrt{x}$.

b From the discussion above, the rule is $(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$.

If (x, y) maps to (x', y'), then $x' = x + 2$ and $y' = 2y - 3$. Thus $x = x' - 2$ and $y = \dfrac{y' + 3}{2}$.

So the image of $y = \sqrt{x}$ has equation

$$\frac{y' + 3}{2} = \sqrt{x' - 2}$$

and hence $y' = 2\sqrt{x' - 2} - 3$. The equation can be written as $y = 2\sqrt{x - 2} - 3$.

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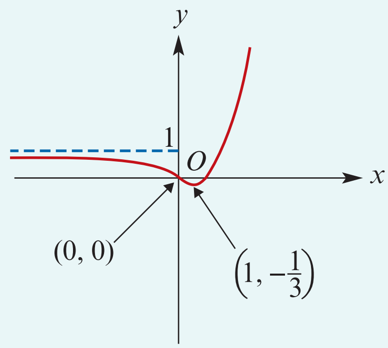
Example 7

Sketch the image of the graph shown under the following sequence of transformations:

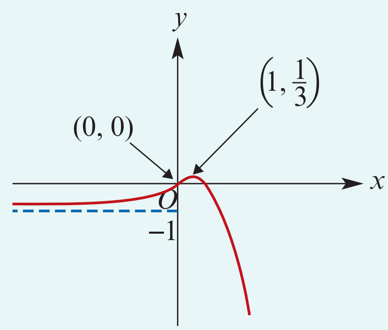
- a reflection in the x-axis
- a dilation of factor 3 from the x-axis
- a translation 2 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.



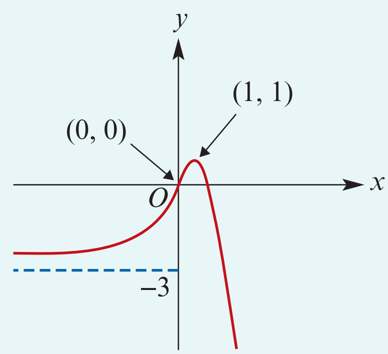
Solution

Consider each transformation in turn and sketch the graph at each stage.

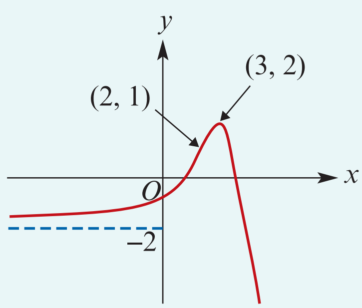
A reflection in the x-axis produces the graph shown on the right.



Next apply the dilation of factor 3 from the x-axis.



Finally, apply the translation 2 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.



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Example 8

For the graph of $y = x^2$:

a Sketch the graph of the image under the sequence of transformations:

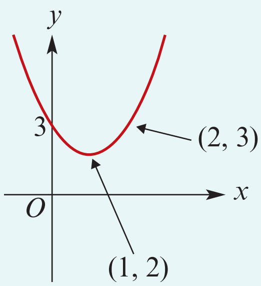
- a translation of 1 unit in the positive direction of the x-axis and 2 units in the positive direction of the y-axis
- a dilation of factor 2 from the y-axis
- a reflection in the x-axis.

b State the rule of the image.

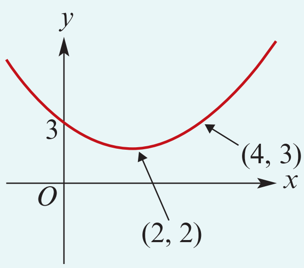
Solution

a Apply each transformation in turn and sketch the graph at each stage.

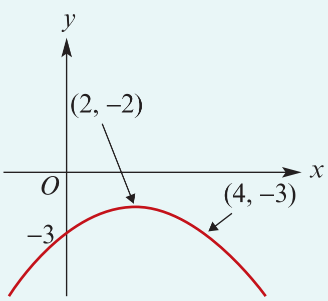
1 The translation:



2 The dilation of factor 2 from the y-axis:



3 The reflection in the x-axis:



b The mapping representing the sequence of transformations is

$$(x, y) \to (x + 1, y + 2) \to (2(x + 1), y + 2) \to (2(x + 1), -(y + 2))$$

Let (x', y') be the image of (x, y). Then $x' = 2(x + 1)$ and $y' = -(y + 2)$.

Rearranging gives $x = \frac{1}{2}(x' - 2)$ and $y = -y' - 2$.

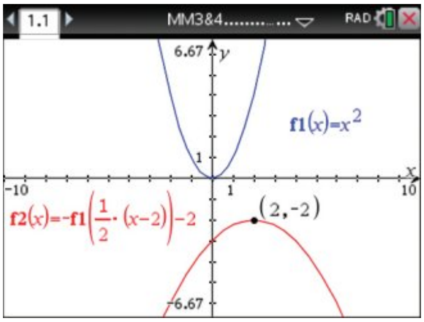
Therefore $y = x^2$ becomes $-y' - 2 = \left(\frac{1}{2}(x' - 2)\right)^2$.

The rule of the image is $y = -\frac{1}{4}(x - 2)^2 - 2$.

TI-84

Using the TI-Nspire CX non-CAS

- In a **Graphs** application, enter $f1(x) = x^2$.
- The rule for the transformed function can be entered as $f2(x) = -f1\left(\frac{1}{2}(x - 2)\right) - 2$.



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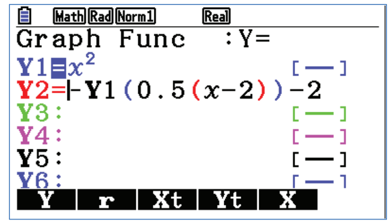
Cambridge University Press

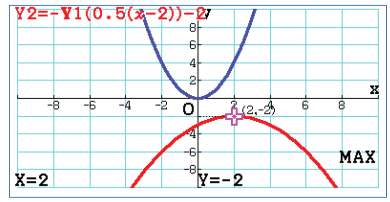
Using the Casio

- Press (MENU) (5) to select **Graph** mode.
- Enter the rule $y = x^2$ in $Y1$.
- Enter the rule for the transformed function in $Y2$ as shown.

Note: To obtain the function name $Y1$ in the rule for $Y2$, go to the Variable Data menu (VARS). Select **Graph** (F4), **Y** (F1); then press (1).

- Select **Draw** (F6) to view the graphs.





Section summary

The following method can be used to find the image of the graph of $y = f(x)$ under a sequence of transformations:

Step 1 Determine the rule $(x, y) \to (x', y')$ for the sequence of transformations.

Step 2 Write down formulas for x' and y' in terms of x and y.

Step 3 Transpose these formulas to express x and y in terms of x' and y'.

Step 4 Substitute these expressions for x and y into the equation $y = f(x)$.

Exercise 2D

Skillsheet
Example 6

1 Find the rule of the image when the graph of each of the functions listed below undergoes each of the following sequences of transformations:

i a dilation of factor 2 from the x-axis, followed by a translation 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

ii a dilation of factor 3 from the y-axis, followed by a translation 2 units in the negative direction of the x-axis and 4 units in the negative direction of the y-axis

iii a dilation of factor 2 from the x-axis, followed by a reflection in the y-axis.

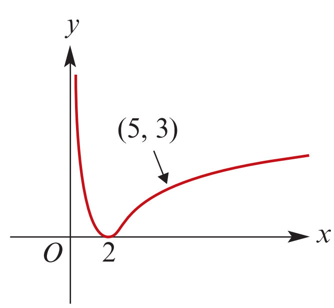
a $y = x^2$ **b** $y = \sqrt[3]{x}$ **c** $y = \dfrac{1}{x^2}$

CF

Example 7

2 Sketch the image of the graph shown under the following sequence of transformations:

- a reflection in the x-axis
- a dilation of factor 2 from the x-axis
- a translation 3 units in the positive direction of the x-axis and 4 units in the positive direction of the y-axis.



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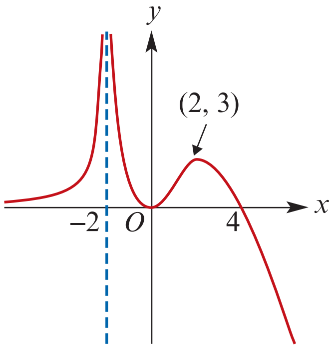
CF

3 Sketch the image of the graph shown under the following sequence of transformations:

- a reflection in the y-axis
- a translation 2 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis
- a dilation of factor 2 from the y-axis.



4 Find the rule of the image when the graph of each of the functions listed below undergoes each of the following sequences of transformations:

i a dilation of factor 2 from the x-axis, followed by a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis

ii a dilation of factor 2 from the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis

iii a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis

iv a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 from the x-axis

v a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 from the x-axis, followed by a reflection in the x-axis

vi a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis.

a $y = x^2$ **b** $y = \sqrt[3]{x}$ **c** $y = \dfrac{1}{x}$ **d** $y = \dfrac{1}{x^3}$ **e** $y = x^{-2}$

5 Find the rule of the image when the graph of $y = \sqrt{x}$ is translated 4 units in the negative direction of the x-axis, reflected in the x-axis and dilated by factor 3 from the y-axis.

Example 8

6 For the graph of $y = \dfrac{3}{x^2}$:

a Sketch the graph of the image under the sequence of transformations:

- a dilation of factor 2 from the x-axis
- a translation of 2 units in the negative direction of the x-axis and 1 unit in the negative direction of the y-axis
- a reflection in the x-axis.

b State the rule of the image.

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2E Using transformations to sketch graphs

By considering a rule for a graph as a combination of transformations of a more 'simple' rule, we can readily sketch graphs of many apparently 'complicated' functions.

Example 9

Identify a sequence of transformations that maps the graph of $y = \frac{1}{x}$ onto the graph of $y = \frac{4}{x+5} - 3$. Use this to sketch the graph of $y = \frac{4}{x+5} - 3$, stating the equations of asymptotes and the coordinates of axis intercepts.

Solution

Rearrange the equation of the transformed graph to have the same 'shape' as $y = \frac{1}{x}$:

$$\frac{y' + 3}{4} = \frac{1}{x' + 5}$$

where (x', y') are the coordinates of the image of (x, y).

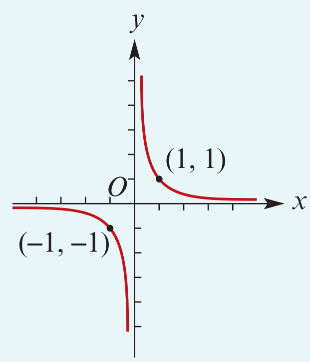
Therefore $x = x' + 5$ and $y = \frac{y' + 3}{4}$. Rearranging gives $x' = x - 5$ and $y' = 4y - 3$.

The mapping is $(x, y) \to (x - 5, 4y - 3)$, and so a sequence of transformations is:

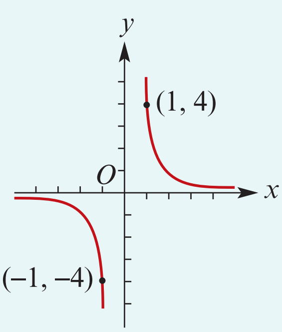
1. a dilation of factor 4 from the x-axis
2. a translation of 5 units in the negative direction of the x-axis
3. a translation of 3 units in the negative direction of the y-axis.

The original graph $y = \frac{1}{x}$ is shown on the right.

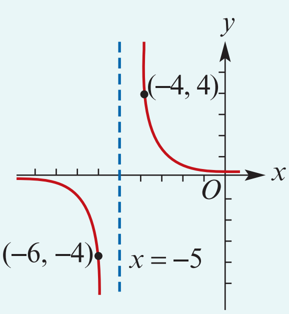
The effect of the transformations is shown below.



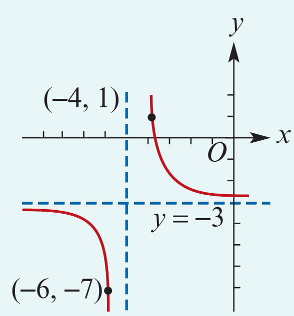
1 Dilation from x-axis:



2 Translation in negative direction of x-axis:



3 Translation in negative direction of y-axis:



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Find the axis intercepts in the usual way, as below. The transformed graph, with asymptotes and intercepts marked, is shown on the right.

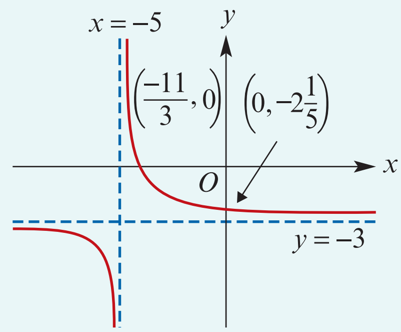
When $x = 0$, $y = \frac{4}{5} - 3 = -2\frac{1}{5}$

When $y = 0$, $\frac{4}{x+5} - 3 = 0$

$$4 = 3x + 15$$

$$3x = -11$$

$$\therefore \quad x = -\frac{11}{3}$$



$y = \frac{4}{x+5} - 3$

Once you have done a few of these types of exercises, you can identify the transformations more quickly by carefully observing the rule of the transformed graph and relating it to the 'shape' of the simplest function in its family. Consider the following examples.

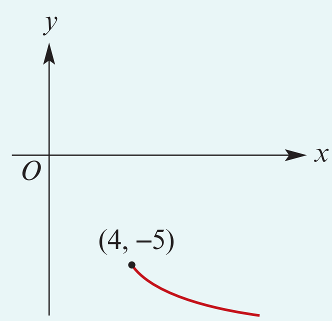
Example 10

Sketch the graph of $y = -\sqrt{x-4} - 5$.

Solution

The graph is obtained from the graph of $y = \sqrt{x}$ by:

- a reflection in the x-axis, followed by a translation of 5 units in the negative direction of the y-axis, and
- a translation of 4 units in the positive direction of the x-axis.



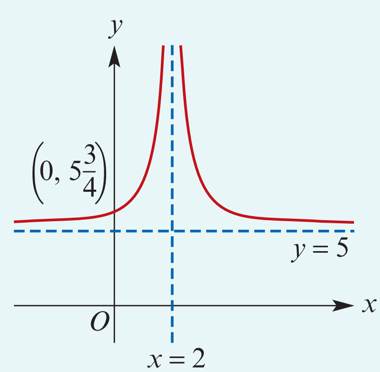
Example 11

Sketch the graph of $y = \frac{3}{(x-2)^2} + 5$.

Solution

This is obtained from the graph of $y = \frac{1}{x^2}$ by:

- a dilation of factor 3 from the x-axis, followed by a translation of 5 units in the positive direction of the y-axis, and
- a translation of 2 units in the positive direction of the x-axis.



 ISBN 978-1-108-45164-2

Section summary

In general, the function given by the equation

$$y = Af(n(x+c)) + b$$

where $b, c \in \mathbb{R}^+$ and $A, n \in \mathbb{R}^+$, represents a transformation of the graph of $y = f(x)$ by:

- a dilation of factor A from the x-axis, followed by a translation of b units in the positive direction of the y-axis, and
- a dilation of factor $\frac{1}{n}$ from the y-axis, followed by a translation of c units in the negative direction of the x-axis.

Similar statements can be made for $b, c \in \mathbb{R}^-$. The case where $A \in \mathbb{R}^-$ corresponds to a reflection in the x-axis and a dilation from the x-axis. The case where $n \in \mathbb{R}^-$ corresponds to a reflection in the y-axis and a dilation from the y-axis.

Exercise 2E

Skillsheet

Example 9

SF

1 Sketch the graph of each of the following. State the equations of asymptotes and the axis intercepts. State the range of each function.

a $f(x) = \dfrac{3}{x-1}$ **b** $g(x) = \dfrac{2}{x+1} - 1$ **c** $h(x) = \dfrac{3}{(x-2)^2}$

d $f(x) = \dfrac{2}{(x-1)^2} - 1$ **e** $h(x) = \dfrac{-1}{x-3}$ **f** $f(x) = \dfrac{-1}{x+2} + 3$

Example 10, 11

2 Sketch the graph of each of the following without using your calculator. State the range of each.

a $y = -\sqrt{x-3}$ **b** $y = -\sqrt{x-3} + 2$ **c** $y = \sqrt{2(x+3)}$

d $y = \dfrac{1}{2x-3}$ **e** $y = 5\sqrt{x+2}$ **f** $y = -5\sqrt{x+2} - 2$

g $y = \dfrac{-3}{x-2}$ **h** $y = \dfrac{-2}{(x+2)^2} - 4$ **i** $y = \dfrac{3}{2x} - 5$

j $y = \dfrac{5}{2x} + 5$ **k** $y = 2(x-3)^2 + 5$ **l** $y = \dfrac{4}{3-x} + 4$

3 Without using your calculator, sketch the graph of $f(x) = \dfrac{3x+2}{x+1}$.

Hint: Show that $f(x) = 3 - \dfrac{1}{x+1}$.

4 Without using your calculator, sketch the graph of $f(x) = \dfrac{4x-5}{2x+1}$.

Hint: Show that $f(x) = 2 - \dfrac{7}{2(x+\frac{1}{2})}$.

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2F Transformations of power functions

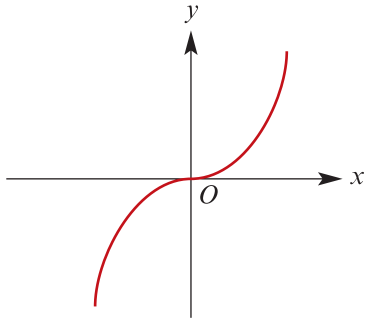
We recall that every quadratic polynomial function can be written in the turning point form $y = a(x - h)^2 + k$. This is not true for polynomials of higher degree. However, there are many polynomials that can be written as $y = a(x - h)^n + k$.

In Section 1E we introduced power functions, which include the functions $f(x) = x^n$, where n is a positive integer. In this section we look at transformations of these functions.

▶ The function $f(x) = x^n$ where n is an odd positive integer

Assume that n is an odd integer with $n \geq 3$. You will recall from Mathematical Methods Units 1 & 2 that the derivative function of $f(x) = x^n$ has rule

$$f'(x) = nx^{n-1}$$



Hence the gradient is zero when $x = 0$. Since n is odd and therefore $n - 1$ is even, we have $f'(x) = nx^{n-1} > 0$ for all $x \neq 0$. That is, the gradient of the graph of $y = f(x)$ is positive when $x \neq 0$ and is zero when $x = 0$. Recall that, for functions of this form, the stationary point at $(0, 0)$ is called a stationary point of inflection.

Transformations of $f(x) = x^n$ where n is an odd positive integer

Transformations of these functions result in graphs with rules of the form $y = a(x - h)^n + k$ where a, h and k are real constants.

Example 12

Sketch the graph of:

a $y = (x - 2)^3 + 1$ **b** $y = -(x - 1)^3 + 2$ **c** $y = 2(x + 1)^3 + 2$

Solution

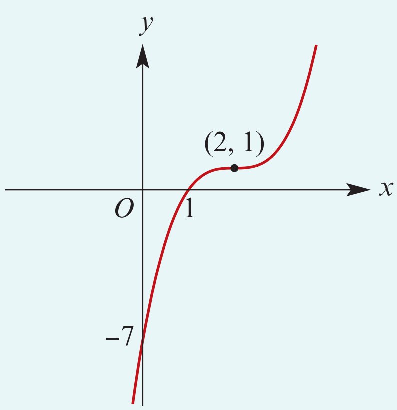
a The translation $(x, y) \to (x + 2, y + 1)$ maps the graph of $y = x^3$ onto the graph of $y = (x - 2)^3 + 1$.

So $(2, 1)$ is a point of zero gradient.

Find the axis intercepts:

- When $x = 0$, $y = (-2)^3 + 1 = -7$
- When $y = 0$,

$$\begin{aligned} 0 &= (x - 2)^3 + 1 \\ -1 &= (x - 2)^3 \\ -1 &= x - 2 \\ \therefore \quad x &= 1 \end{aligned}$$



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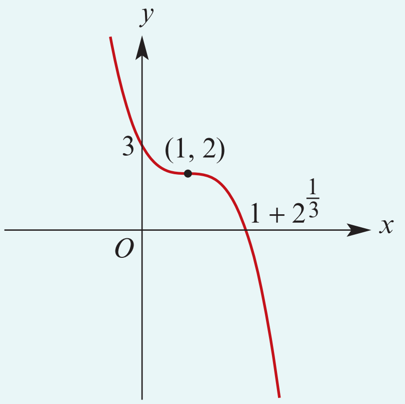
b A reflection in the x-axis followed by the translation $(x, y) \to (x + 1, y + 2)$ maps the graph of $y = x^3$ onto the graph of $y = -(x - 1)^3 + 2$.

So $(1, 2)$ is a point of zero gradient.

Find the axis intercepts:

- When $x = 0$, $y = -(-1)^3 + 2 = 3$
- When $y = 0$,

$$\begin{aligned} 0 &= -(x-1)^3 + 2 \\ (x-1)^3 &= 2 \\ x - 1 &= 2^{\frac{1}{3}} \\ \therefore \quad x &= 1 + 2^{\frac{1}{3}} \approx 2.26 \end{aligned}$$



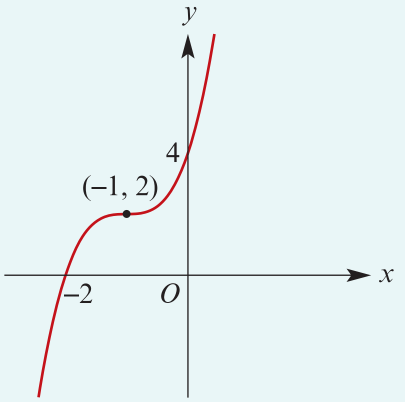
c A dilation of factor 2 from the x-axis followed by the translation $(x, y) \to (x - 1, y + 2)$ maps the graph of $y = x^3$ onto the graph of $y = 2(x + 1)^3 + 2$.

So $(-1, 2)$ is a point of zero gradient.

Find the axis intercepts:

- When $x = 0$, $y = 2 + 2 = 4$
- When $y = 0$,

$$\begin{aligned} 0 &= 2(x+1)^3 + 2 \\ -1 &= (x+1)^3 \\ -1 &= x + 1 \\ \therefore \quad x &= -2 \end{aligned}$$



Example 13

The graph of $y = a(x - h)^3 + k$ has a point of zero gradient at $(1, 1)$ and passes through the point $(0, 4)$. Find the values of a, h and k.

Solution

Since $(1, 1)$ is the point of zero gradient,

$$h = 1 \quad \text{and} \quad k = 1$$

So $y = a(x - 1)^3 + 1$ and, since the graph passes through $(0, 4)$,

$$\begin{aligned} 4 &= -a + 1 \\ \therefore \quad a &= -3 \end{aligned}$$

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Example 14

a Find the rule for the image of the graph of $y = x^5$ under the following sequence of transformations:

- reflection in the y-axis
- dilation of factor 2 from the y-axis
- translation 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

b Find a sequence of transformations which takes the graph of $y = x^5$ to the graph of $y = 6 - 2(x+5)^5$.

Solution

a $(x, y) \to (-x, y) \to (-2x, y) \to (-2x + 2, y + 3)$

Let (x', y') be the image of (x, y) under this transformation.

Then $x' = -2x + 2$ and $y' = y + 3$. Hence $x = \dfrac{x' - 2}{-2}$ and $y = y' - 3$.

Therefore the graph of $y = x^5$ maps to the graph of

$$y' - 3 = \left(\frac{x' - 2}{-2}\right)^5$$

i.e. to the graph of

$$y = -\frac{1}{32}(x - 2)^5 + 3$$

b Rearrange $y' = 6 - 2(x' + 5)^5$ to $\dfrac{y' - 6}{-2} = (x' + 5)^5$.

Therefore $y = \dfrac{y' - 6}{-2}$ and $x = x' + 5$, which gives $y' = -2y + 6$ and $x' = x - 5$.

The sequence of transformations is:

- reflection in the x-axis
- dilation of factor 2 from the x-axis
- translation 5 units in the negative direction of the x-axis and 6 units in the positive direction of the y-axis.

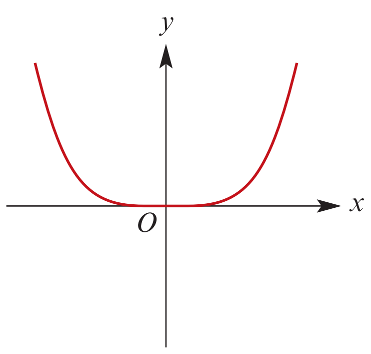
The function $f(x) = x^n$ where n is an even positive integer

Now assume that n is an even integer with $n \geq 2$. The derivative function of $f(x) = x^n$ has rule

$$f'(x) = nx^{n-1}$$

Hence the gradient is zero when $x = 0$. Since n is even and therefore $n - 1$ is odd, we have $f'(x) = nx^{n-1} > 0$ for all $x > 0$, and $f'(x) = nx^{n-1} < 0$ for all $x < 0$.

Thus the graph of $y = f(x)$ has a turning point at $(0, 0)$; this point is a local minimum.



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Section summary

A graph with rule of the form $y = a(x - h)^n + k$ can be obtained as a transformation of the graph of $y = x^n$.

Exercise 2F

Example 12

1 Sketch the graph of each of the following. State the coordinates of the point of zero gradient and the axis intercepts. SF

- **a** $f(x) = 2x^3$
- **b** $g(x) = -2x^3$
- **c** $h(x) = x^5 + 1$
- **d** $f(x) = x^3 - 4$
- **e** $f(x) = (x + 1)^3 - 8$
- **f** $f(x) = 2(x - 1)^3 - 2$
- **g** $g(x) = -2(x - 1)^3 + 2$
- **h** $h(x) = 3(x - 2)^3 - 4$
- **i** $f(x) = 2(x - 1)^3 + 2$
- **j** $h(x) = -2(x - 1)^3 - 4$
- **k** $f(x) = (x + 1)^5 - 32$
- **l** $f(x) = 2(x - 1)^5 - 2$

Example 13

2 The graph of $y = a(x - h)^3 + k$ has a point of zero gradient at $(0, 4)$ and passes through the point $(1, 1)$. Find the values of a, h and k. CF

3 Find the equation of the image of $y = x^3$ under each of the following transformations: SF

- **a** a dilation of factor 3 from the x-axis
- **b** a translation with rule $(x, y) \to (x - 1, y + 1)$
- **c** a reflection in the x-axis followed by the translation $(x, y) \to (x + 2, y - 3)$
- **d** a dilation of factor 2 from the x-axis followed by the translation $(x, y) \to (x - 1, y - 2)$
- **e** a dilation of factor 3 from the y-axis.

Example 14

4 **a** Find the rule for the image of the graph of $y = x^3$ under a reflection in the y-axis, followed by a dilation of factor 3 from the y-axis and then a translation 3 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis. CF

b Find a sequence of transformations which takes the graph of $y = x^3$ to the graph of $y = 4 - 3(x + 1)^3$.

5 **a** Find the rule for the image of the graph of $y = x^4$ under a reflection in the y-axis, followed by a dilation of factor 2 from the y-axis and then a translation 2 units in the negative direction of the x-axis and 1 unit in the negative direction of the y-axis.

b Find a sequence of transformations which takes the graph of $y = x^4$ to the graph of $y = 5 - 3(x + 1)^4$.

6 Sketch the graph of each of the following: SF

- **a** $y = 3(x - 1)^4 - 2$
- **b** $y = -2(x + 2)^4$
- **c** $y = (x - 2)^4 - 6$
- **d** $y = 2(x - 3)^4 - 1$
- **e** $y = 1 - (x + 4)^4$
- **f** $y = -3(x - 2)^4 - 3$

7 The graph of $y = a(x - h)^4 + k$ has a turning point at $(-2, 3)$ and passes through the point $(0, -6)$. Find the values of a, h and k. CF

8 The graph of $y = a(x - h)^4 + k$ has a turning point at $(1, 7)$ and passes through the point $(0, 23)$. Find the values of a, h and k.

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2G Determining the rule for a function from its graph

Given sufficient information about a curve, we can determine its rule. For example, if we know the coordinates of two points on a hyperbola of the form

$$y = \frac{a}{x} + b$$

then we can find the rule for the hyperbola, i.e. we can find the values of a and b.

Sometimes the rule has a more specific form. For example, the curve may be a dilation of $y = \sqrt{x}$. Then we know its rule is of the form $y = a\sqrt{x}$, and the coordinates of one point on the curve (with the exception of the origin) will be enough to determine the value of a.

Example 15

a The points (1, 5) and (4, 2) lie on a curve with equation $y = \frac{a}{x} + b$. Find the values of a and b.

b The points (2, 1) and (10, 6) lie on a curve with equation $y = a\sqrt{x-1} + b$. Find the values of a and b.

Solution

a When $x = 1$, $y = 5$ and so

$$5 = a + b \qquad (1)$$

When $x = 4$, $y = 2$ and so

$$2 = \frac{a}{4} + b \qquad (2)$$

Subtract (2) from (1):

$$3 = \frac{3a}{4}$$

$$\therefore \quad a = 4$$

Substitute in (1) to find b:

$$5 = 4 + b$$

$$\therefore \quad b = 1$$

The equation of the curve is

$$y = \frac{4}{x} + 1$$

b When $x = 2$, $y = 1$ and so

$$1 = a\sqrt{1} + b$$

i.e. $\quad 1 = a + b \qquad (1)$

When $x = 10$, $y = 6$ and so

$$6 = a\sqrt{9} + b$$

i.e. $\quad 6 = 3a + b \qquad (2)$

Subtract (1) from (2):

$$5 = 2a$$

$$\therefore \quad a = \frac{5}{2}$$

Substitute in (1) to find b:

$$1 = \frac{5}{2} + b$$

$$\therefore \quad b = -\frac{3}{2}$$

The equation of the curve is

$$y = \frac{5}{2}\sqrt{x-1} - \frac{3}{2}$$

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Exercise 2G

Skillsheet
Example 15a

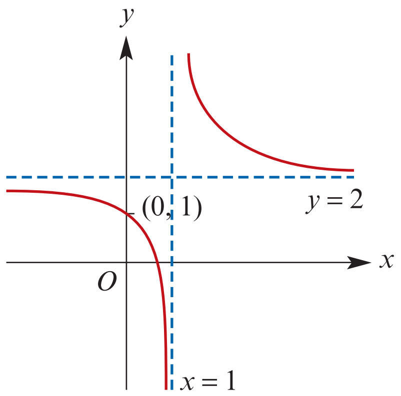
1 The points (1, 4) and (3, 1) lie on a curve with equation $y = \frac{a}{x} + b$. Find the values of a and b.

SF

2 The graph shown has the rule

$$y = \frac{A}{x+b} + B$$

Find the values of A, b and B.



Example 15b

3 The points (3, 1) and (11, 6) lie on a curve with equation $y = a\sqrt{x-2} + b$. Find the values of a and b.

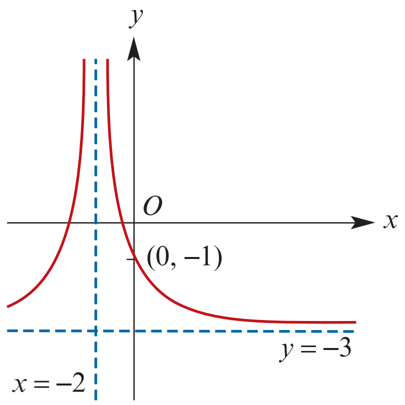
4 The points with coordinates (1, 5) and (16, 11) lie on a curve which has a rule of the form $y = A\sqrt{x} + B$. Find A and B.

5 The points with coordinates (1, 1) and (0.5, 7) lie on a curve which has a rule of the form $y = \frac{A}{x^2} + B$. Find the values of A and B.

6 The graph shown has the rule

$$y = \frac{A}{(x+b)^2} + B$$

Find the values of A, b and B.



7 The points with coordinates (1, −1) and $(2, \frac{3}{4})$ lie on a curve which has a rule of the form $y = \frac{a}{x^3} + b$. Find the values of a and b.

8 The points with coordinates (−1, 4) and (1, −8) lie on a curve which has a rule of the form $y = a\sqrt[3]{x} + b$. Find the values of a and b.

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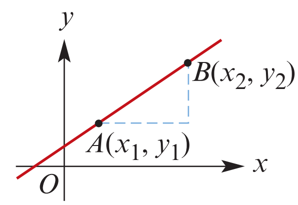
Chapter summary

AS
Nrich

Coordinate geometry

- For two points $A(x_1, y_1)$ and $B(x_2, y_2)$:
  - The **distance** between points A and B is given by $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
  - The **midpoint** of the line segment AB is the point with coordinates $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.
  - The **gradient** of the line AB is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.



- Different forms for the equation of a straight line:

$y = mx + c$	where m is the gradient and c is the y-axis intercept
$y - y_1 = m(x - x_1)$	where m is the gradient and (x_1, y_1) is a point on the line
$\dfrac{x}{a} + \dfrac{y}{b} = 1$	where $(a, 0)$ and $(0, b)$ are the axis intercepts

- The **angle of slope** of a straight line is found using $m = \tan\theta$, where m is the gradient and θ is the angle that the line makes with the positive direction of the x-axis.
- Two straight lines are **perpendicular** to each other if and only if the product of their gradients is -1, i.e. $m_1 m_2 = -1$. (Unless one line is vertical and the other horizontal.)

Transformations of the graphs of functions

Mapping	Rule	Image of $y = f(x)$
Reflection in the x-axis	$x' = x,\ y' = -y$	$y = -f(x)$
Reflection in the y-axis	$x' = -x,\ y' = y$	$y = f(-x)$
Dilation of factor a from the y-axis	$x' = ax,\ y' = y$	$y = f\left(\dfrac{x}{a}\right)$
Dilation of factor b from the x-axis	$x' = x,\ y' = by$	$y = bf(x)$
Translation	$x' = x + h,\ y' = y + k$	$y - k = f(x - h)$

Technology-free questions

SF

1 Sketch the graphs of the relations:

a $3y + 2x = 5$ **b** $x - y = 6$ **c** $\dfrac{x}{2} + \dfrac{y}{3} = 1$

2 Find the distance between the points with coordinates $(-1, 6)$ and $(2, 4)$.

3 Find the midpoint of the line segment AB joining the points $A(4, 6)$ and $B(-2, 8)$.

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Review

SF

4 **a** Find the equation of the straight line which passes through $(1, 3)$ and has gradient -2.

b Find the equation of the straight line which passes through $(1, 4)$ and $(3, 8)$.

c Find the equation of the straight line which is perpendicular to the line with equation $y = -2x + 6$ and which passes through the point $(1, 1)$.

d Find the equation of the straight line which is parallel to the line with equation $y = 6 - 2x$ and which passes through the point $(1, 1)$.

5 If M is the midpoint of XY, find the coordinates of Y when X and M have the following coordinates:

a $X(-6, 2),\ M(8, 3)$ **b** $X(-1, -4),\ M(2, -8)$

6 The length of the line segment joining $A(5, 12)$ and $B(10, y)$ is 13 units. Find y.

7 Sketch the graph of each of the following. Label any asymptotes and axis intercepts. State the range of each function.

a $f(x) = \dfrac{1}{x} - 3,\ x \in \mathbb{R} \setminus \{0\}$ **b** $f(x) = \dfrac{1}{x-2},\ x \in (2, \infty)$

c $f(x) = \dfrac{2}{x-1} - 3,\ x \in \mathbb{R} \setminus \{1\}$ **d** $f(x) = \dfrac{-3}{2-x} + 4,\ x \in (2, \infty)$

e $f(x) = 1 - \dfrac{1}{x-1},\ x \in \mathbb{R} \setminus \{1\}$

8 Sketch the graph of each of the following:

a $f(x) = 2\sqrt{x-3} + 1$ **b** $g(x) = \dfrac{3}{(x-2)^2} - 1$ **c** $h(x) = \dfrac{-3}{(x-2)^2} - 1$

9 Sketch the graph of each of the following. State the coordinates of the point of zero gradient and the axis intercepts.

a $f(x) = -2(x+1)^3$ **b** $g(x) = -2(x-1)^5 + 8$

c $h(x) = 2(x-2)^5 + 1$ **d** $f(x) = 4(x-1)^3 - 4$

10 The points with coordinates $(1, 6)$ and $(16, 12)$ lie on a curve which has a rule of the form $y = a\sqrt{x} + b$. Find a and b.

11 The points with coordinates $(1, 3)$ and $(3, 7)$ lie on a curve with equation of the form $y = \dfrac{a}{x} + b$. Find the values of a and b.

CF

12 **a** Find the rule for the image of the graph of $y = -x^2$ under the following sequence of transformations:

- reflection in the y-axis
- dilation of factor 2 from the y-axis
- translation 4 units in the positive direction of the x-axis and 6 units in the positive direction of the y-axis.

b Find a sequence of transformations which takes the graph of $y = x^4$ to the graph of $y = 6 - 4(x+1)^4$.

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13 Identify a sequence of transformations that maps the graph of $y = \frac{1}{x^2}$ onto the graph of $y = \frac{3}{(x-5)^2} + 3$. Use this to sketch the graph of $y = \frac{3}{(x-5)^2} + 3$, stating the equations of asymptotes and the coordinates of axis intercepts. CF

14 Find a sequence of transformations that takes the graph of $y = 2x^2 - 3$ to the graph of $y = x^2$.

15 Find a sequence of transformations that takes the graph of $y = 2(x-3)^3 + 4$ to the graph of $y = x^3$.

Multiple-choice questions

1 A straight line has gradient $-\frac{1}{2}$ and passes through $(1, 4)$. The equation of the line is

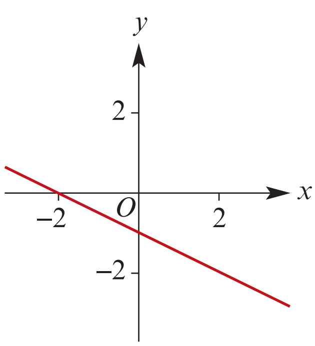
A $y = x + 4$ **B** $y = 2x + 2$ **C** $y = 2x + 4$
D $y = -\frac{1}{2}x + 4$ **E** $y = -\frac{1}{2}x + \frac{9}{2}$

2 The line $y = -2x + 4$ passes through a point $(a, 3)$. The value of a is

A $-\frac{1}{2}$ **B** 2 **C** $-\frac{7}{2}$ **D** -2 **E** $\frac{1}{2}$

3 The gradient of a line that is perpendicular to the line shown could be

A 1 **B** $\frac{1}{2}$ **C** $-\frac{1}{2}$
D 2 **E** -2



4 The coordinates of the midpoint of AB, where A has coordinates $(1, 7)$ and B has coordinates $(-3, 30)$, are

A $(-2, 3)$ **B** $(-1, 8)$ **C** $(-1, 18.5)$ **D** $(-1, 3)$ **E** $(-2, 8.5)$

5 The gradient of the line passing through $(3, -2)$ and $(-1, 10)$ is

A -3 **B** -2 **C** $-\frac{1}{3}$ **D** 4 **E** 3

6 If two lines $-2x + y - 3 = 0$ and $ax - 3y + 4 = 0$ are parallel, then a equals

A 6 **B** 2 **C** $\frac{1}{3}$ **D** $\frac{2}{3}$ **E** -6

7 A straight line passes through $(-1, -2)$ and $(3, 10)$. The equation of the line is

A $y = 3x - 1$ **B** $y = 3x - 4$ **C** $y = 3x + 1$ **D** $y = \frac{1}{3}x + 9$ **E** $y = 4x - 2$

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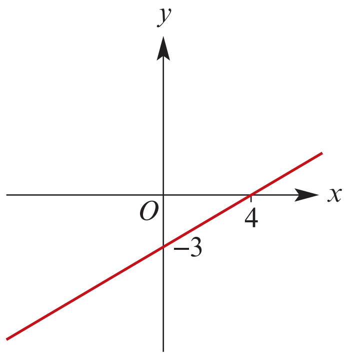
8 The length of the line segment connecting $(1, 4)$ and $(5, -2)$ is

A 10 **B** $2\sqrt{13}$ **C** 12 **D** 50 **E** $2\sqrt{5}$

9 The function with graph as shown has the rule



A $f(x) = 3x - 3$

B $f(x) = -\frac{3}{4}x - 3$

C $f(x) = \frac{3}{4}x - 3$

D $f(x) = \frac{4}{3}x - 3$

E $f(x) = 4x - 4$

10 The midpoint of the line segment joining $(0, -6)$ and $(4, d)$ is

A $\left(-2, \frac{d+6}{2}\right)$ **B** $\left(2, \frac{d+6}{2}\right)$ **C** $\left(\frac{d+6}{2}, 2\right)$ **D** $\left(2, \frac{d-6}{2}\right)$ **E** $\frac{d+6}{4}$

11 The gradient of a line perpendicular to the line through $(3, 0)$ and $(0, -6)$ is

A $\frac{1}{2}$ **B** -2 **C** $-\frac{1}{2}$ **D** 2 **E** 6

12 The point $P(3, -4)$ lies on the graph of a function f. The graph of f is translated 3 units up (parallel to the y-axis) and reflected in the x-axis. The coordinates of the final image of P are

A $(6, 4)$ **B** $(3, 1)$ **C** $(3, -1)$ **D** $(-3, 1)$ **E** $(3, 7)$

13 The graph of $y = x^3 + 4$ is translated 3 units 'down' and 2 units 'to the right'. The resulting graph has equation

A $y = (x-2)^3 + 2$ **B** $y = (x-2)^3 + 1$ **C** $y = (x-2)^3 + 5$

D $y = (x+2)^3 + 1$ **E** $y = (x+2)^3 + 6$

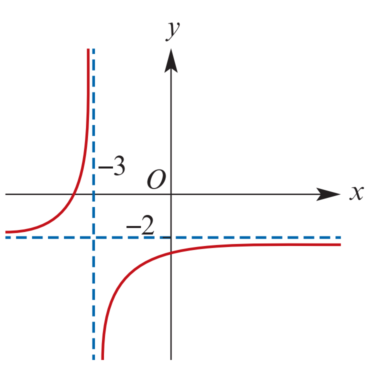
14 The graph of the function with rule $y = x^2$ is reflected in the x-axis and then translated 4 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis. The rule for the new function is

A $y = (-x+4)^2 - 3$ **B** $y = -(x-4)^2 + 3$ **C** $y = -(x-3)^2 + 4$

D $y = (-x-4)^2 + 3$ **E** $y = -(x+4)^2 - 3$

15 The graph of $y = \frac{a}{x+b} + c$ is shown on the right.

Possible values for a, b and c are



A $a = -1, \ b = 3, \ c = 2$

B $a = 1, \ b = 2, \ c = -3$

C $a = -1, \ b = -3, \ c = -2$

D $a = -1, \ b = 3, \ c = -2$

E $a = 1, \ b = 2, \ c = -3$

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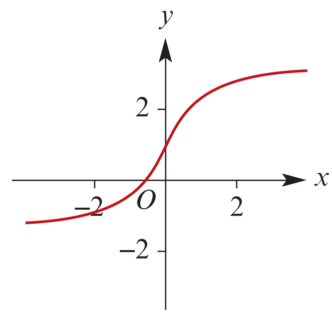
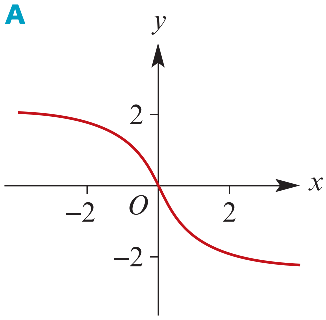
16 The graph of $y = f(x)$ is shown on the right.

Which one of the following could be the graph of $y = f(-x)$?



A



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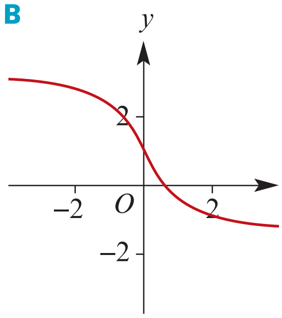

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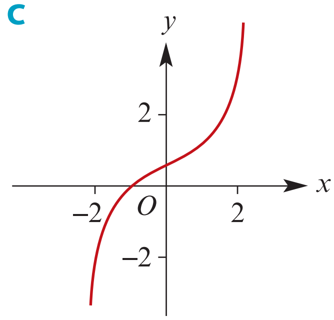

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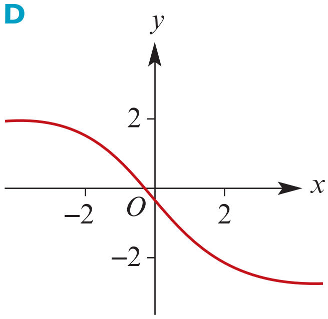
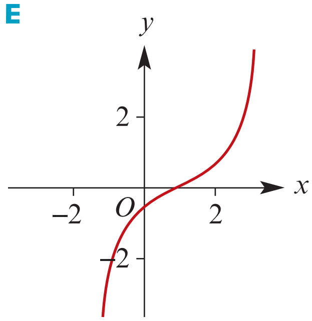


E



17 Let $f(x) = 3x - 2$ and $g(x) = x^2 - 4x + 2$. A sequence of transformations that takes the graph of $y = g(x)$ to the graph of $y = g(f(x))$ is

A a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation $\frac{2}{3}$ units in the positive direction of the x-axis

B a dilation of factor 3 from the y-axis followed by a translation 2 units in the negative direction of the x-axis

C a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation $\frac{1}{2}$ unit in the positive direction of the x-axis

D a dilation of factor 3 from the y-axis followed by a translation 2 units in the positive direction of the x-axis

E a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation 2 units in the positive direction of the x-axis

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Extended-response questions

1 A firm manufacturing jackets is capable of producing 100 jackets per day, but it can only sell all of these if the charge to wholesalers is no more than \$50 per jacket. On the other hand, at the current price of \$75 per jacket, only 50 can be sold per day. Assume that the graph of price, \$$P$, against number sold per day, N, is a straight line.

a Sketch the graph of P against N.

b Find the equation of the straight line.

c Use the equation to find:

i the price at which 88 jackets per day could be sold

ii the number of jackets that should be manufactured to sell at \$60 each.

2 A new town was built 10 years ago to house the workers of a woollen mill established in a remote country area. Three years after the town was built, it had a population of 12 000 people. Business in the wool trade steadily grew, and eight years after the town was built the population had swelled to 19 240.

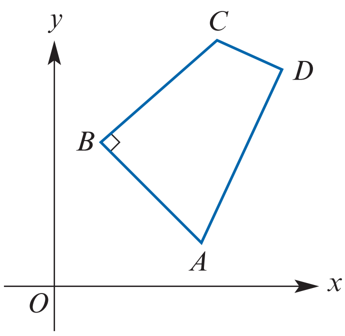
a Assuming the population growth can be modelled by a linear relationship, find a suitable relation for the population, p, in terms of t, the number of years since the town was built.

b Sketch the graph of p against t, and interpret the p-axis intercept.

c Find the current population of the town.

d Calculate the average rate of growth of the town.

3 $ABCD$ is a quadrilateral with angle ABC a right angle. The point D lies on the perpendicular bisector of AB. The coordinates of A and B are $(7, 2)$ and $(2, 5)$ respectively. The equation of line AD is $y = 4x - 26$.



a Find the equation of the perpendicular bisector of line segment AB.

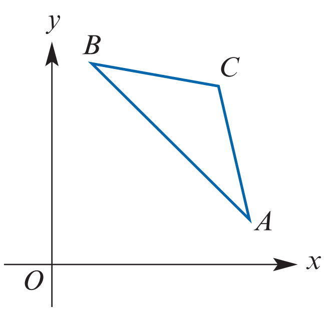
b Find the coordinates of point D.

c Find the gradient of line BC.

d Find the value of the second coordinate c of the point $C(8, c)$.

e Find the area of quadrilateral $ABCD$.

4 Triangle ABC is isosceles with $BC = AC$. The coordinates of the vertices are $A(6, 1)$ and $B(2, 8)$.



a Find the equation of the perpendicular bisector of AB.

b The x-coordinate of C is 3.5. Find the y-coordinate.

c Find the length of AB.

d Find the area of triangle ABC.

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Review

CF

5 If $A = (-4, 6)$ and $B = (6, -7)$, find:

a the coordinates of the midpoint of AB

b the length of AB

c the distance between A and B

d the equation of AB

e the equation of the perpendicular bisector of AB

f the coordinates of the point P on the line segment AB such that $AP : PB = 3 : 1$

g the coordinates of the point P on the line AB such that $AP : AB = 3 : 1$ and P is closer to point B than to point A.

CU

6 **a** **i** Find the dilation from the x-axis which takes $y = x^2$ to the parabola with its vertex at the origin that passes through the point $(25, 15)$.

ii State the rule which reflects this dilated parabola in the x-axis.

iii State the rule which takes the reflected parabola of part **ii** to a parabola with x-axis intercepts $(0, 0)$ and $(50, 0)$ and vertex $(25, 15)$.

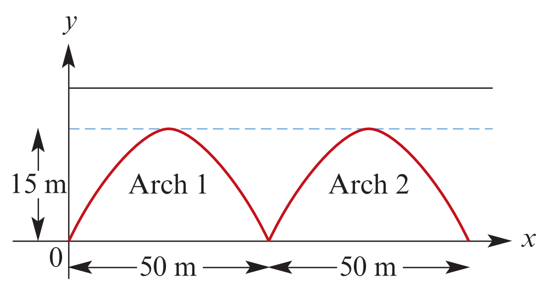
iv State the rule which takes the curve $y = x^2$ to the parabola defined in part **iii**.

b The plans for the entrance of a new building involve twin parabolic arches as shown in the diagram.

i From the results of part **a**, give the equation for the curve of arch 1.

ii Find the translation which maps the curve of arch 1 to the curve of arch 2.

iii Find the equation of the curve of arch 2.



c The architect wishes to have flexibility in her planning and so wants to develop an algorithm for determining the equations of the curves when each arch has width m metres and height n metres.

i Find the rule for the transformation which takes the graph of $y = x^2$ to the current arch 1 with these new dimensions.

ii Find the equation for the curve of arch 1.

iii Find the equation for the curve of arch 2.

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Chapter 3

3 Polynomial functions

Objectives

- To revise the properties of **quadratic functions**.
- To add, subtract and multiply polynomials.
- To be able to use the technique of **equating coefficients**.
- To **divide polynomials**.
- To use the **remainder theorem** and the **factor theorem** to identify the linear factors of cubic and quartic polynomials.
- To draw and use **sign diagrams**.
- To find the rules for given polynomial graphs.
- To apply **polynomial functions** to problem solving.

A polynomial function of degree 2 is called a **quadratic function**. The general rule for such a function is

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

A polynomial function of degree 3 is called a **cubic function**. The general rule for such a function is

$$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0$$

A polynomial function of degree 4 is called a **quartic function**. The general rule for such a function is

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0$$

In this chapter we revise quadratic functions, and build on our previous study of cubic and quartic functions.

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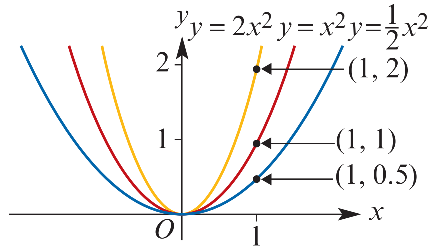
3A Quadratic functions

In this section, we revise material on quadratic functions covered in Mathematical Methods Units 1 & 2.

▶ Transformations of parabolas

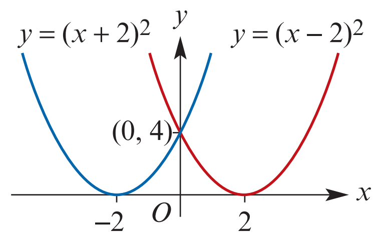
Dilation from the x-axis



For $a > 0$, the graph of the function $y = ax^2$ is obtained from the graph of $y = x^2$ by a dilation of factor a from the x-axis.

The graphs on the right are those of $y = x^2$, $y = 2x^2$ and $y = \frac{1}{2}x^2$, i.e. $a = 1$, 2 and $\frac{1}{2}$.

Translation parallel to the x-axis

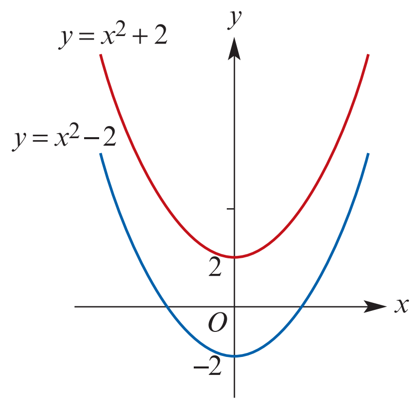


The graphs of $y = (x + 2)^2$ and $y = (x - 2)^2$ are shown.

For $h > 0$, the graph of $y = (x + h)^2$ is obtained from the graph of $y = x^2$ by a translation of h units in the negative direction of the x-axis.

For $h < 0$, the graph of $y = (x + h)^2$ is obtained from the graph of $y = x^2$ by a translation of $-h$ units in the positive direction of the x-axis.

Translation parallel to the y-axis

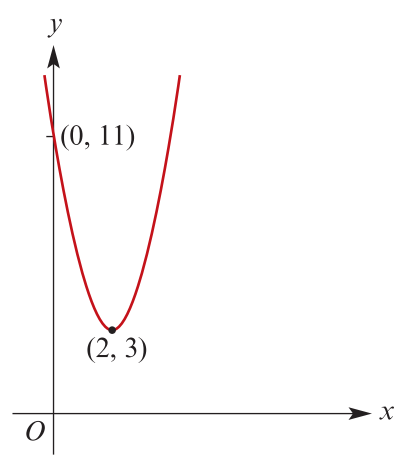


The graphs of $y = x^2 + 2$ and $y = x^2 - 2$ are shown.

For $k > 0$, the graph of $y = x^2 + k$ is obtained from the graph of $y = x^2$ by a translation of k units in the positive direction of the y-axis.

For $k < 0$, the translation is in the negative direction of the y-axis.

Combinations of transformations



The graph of the function

$$f(x) = 2(x - 2)^2 + 3$$

is obtained by transforming the graph of the function $f(x) = x^2$ by:

- dilation of factor 2 from the x-axis
- translation of 2 units in the positive direction of the x-axis
- translation of 3 units in the positive direction of the y-axis.

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▶ Graphing quadratics in turning point form

By applying dilations, reflections and translations to the basic parabola $y = x^2$, we can sketch the graph of any quadratic expressed in **turning point form** $y = a(x - h)^2 + k$:

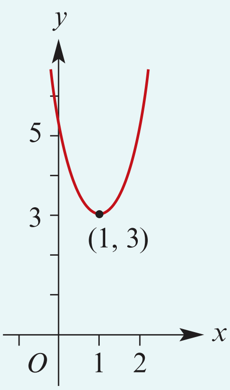
- If $a > 0$, the graph has a minimum point.
- If $a < 0$, the graph has a maximum point.
- The vertex is the point (h, k).
- The axis of symmetry is $x = h$.
- If h and k are positive, then the graph of $y = a(x - h)^2 + k$ is obtained from the graph of $y = ax^2$ by translating h units in the positive direction of the x-axis and k units in the positive direction of the y-axis.
- Similar results hold for different combinations of h and k positive and negative.

Example 1

Sketch the graph of $y = 2(x - 1)^2 + 3$.

Solution

The graph of $y = 2x^2$ is translated 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.



The vertex has coordinates $(1, 3)$.

The axis of symmetry is the line $x = 1$.

The graph will be narrower than $y = x^2$.

The range is $[3, \infty)$.

To add further detail to our graph, we can find the axis intercepts:

y-axis intercept

When $x = 0$, $y = 2(0 - 1)^2 + 3 = 5$.

x-axis intercepts

In this example, the minimum value of y is 3, and so y cannot be 0. Therefore this graph has no x-axis intercepts.

Note: Another way to see this is to let $y = 0$ and try to solve for x:

$$0 = 2(x - 1)^2 + 3$$
$$-3 = 2(x - 1)^2$$
$$-\tfrac{3}{2} = (x - 1)^2$$

As the square root of a negative number is not a real number, this equation has no real solutions.

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Example 2

Sketch the graph of $y = -(x + 1)^2 + 4$.

Solution

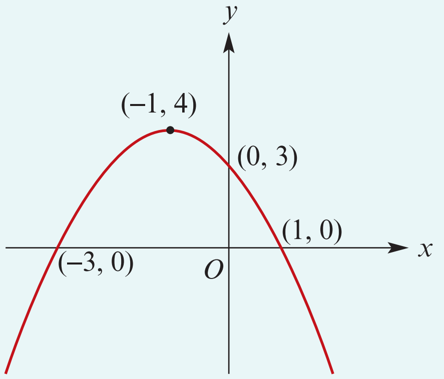
The vertex has coordinates $(-1, 4)$ and so the axis of symmetry is the line $x = -1$.

When $x = 0$, $y = -(0 + 1)^2 + 4 = 3$.
$\therefore$ the y-axis intercept is 3.

When $y = 0$,

$$-(x + 1)^2 + 4 = 0$$
$$(x + 1)^2 = 4$$
$$x + 1 = \pm 2$$
$$x = \pm 2 - 1$$

$\therefore$ the x-axis intercepts are 1 and -3.



► The axis of symmetry

For a quadratic function written in polynomial form $y = ax^2 + bx + c$, the axis of symmetry of its graph has the equation $x = -\dfrac{b}{2a}$.

Therefore the x-coordinate of the turning point is $-\dfrac{b}{2a}$. Substitute this value into the quadratic polynomial to find the y-coordinate of the turning point.

Example 3

For each of the following quadratic functions, use the axis of symmetry to find the turning point of the graph, express the function in the form $y = a(x - h)^2 + k$, and hence find the maximum or minimum value and the range:

a $y = x^2 - 4x + 3$ **b** $y = -2x^2 + 12x - 7$

Solution

a The x-coordinate of the turning point is 2.

When $x = 2$, $y = 4 - 8 + 3 = -1$.

The coordinates of the turning point are $(2, -1)$. Hence the equation is $y = (x - 2)^2 - 1$.

The minimum value is -1 and the range is $[-1, \infty)$.

Explanation

Here $a = 1$ and $b = -4$, so the axis of symmetry is $x = -\left(\dfrac{-4}{2}\right) = 2$.

For the turning point form $y = a(x - h)^2 + k$, we have found that $a = 1$, $h = 2$ and $k = -1$.

Since $a > 0$, the parabola has a minimum.

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b The x-coordinate of the turning point is 3.

When $x = 3$, $y = -2 \times (3)^2 + 12 \times 3 - 7 = 11$.

The coordinates of the turning point are $(3, 11)$. Hence the equation is $y = -2(x - 3)^2 + 11$.

The maximum value is 11 and the range is $(-\infty, 11]$.

Here $a = -2$ and $b = 12$, so the axis of symmetry is $x = -\left(\frac{12}{-4}\right) = 3$.

For the turning point form $y = a(x - h)^2 + k$, we have found that $a = -2$, $h = 3$ and $k = 11$.

Since $a < 0$, the parabola has a maximum.

Graphing quadratics in polynomial form

It is not essential to convert a quadratic to turning point form in order to sketch its graph.

For a quadratic in polynomial form, we can find the x- and y-axis intercepts and the axis of symmetry by other methods and use these details to sketch the graph.

Step 1 Find the y-axis intercept.

Step 2 Find the x-axis intercepts.

Step 3 Find the equation of the axis of symmetry.

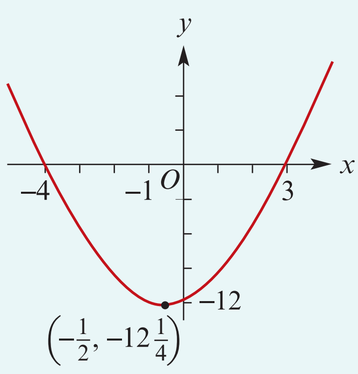
Step 4 Find the coordinates of the turning point.

Example 4

Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 + x - 12$.

Solution

Step 1 $c = -12$. Therefore the y-axis intercept is -12.



Step 2 Let $y = 0$. Then

$$0 = x^2 + x - 12$$
$$0 = (x + 4)(x - 3)$$
$$\therefore \quad x = -4 \quad \text{or} \quad x = 3$$

The x-axis intercepts are -4 and 3.

Step 3 The axis of symmetry is the line with equation

$$x = \frac{-4 + 3}{2} = -\frac{1}{2}$$

Step 4 When $x = -\frac{1}{2}$, $y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 12$
$= -12\frac{1}{4}$

The turning point has coordinates $(-\frac{1}{2}, -12\frac{1}{4})$.

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▶ Completing the square

By completing the square, all quadratics in polynomial form, $y = ax^2 + bx + c$, may be transposed into turning point form, $y = a(x - h)^2 + k$. We have seen that this can be used to sketch the graphs of quadratic polynomials.

To complete the square of $x^2 + bx + c$:

- Take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square $\frac{b^2}{4}$.

To complete the square of $ax^2 + bx + c$:

- First take out a as a factor and then complete the square inside the bracket.

Example 5

By completing the square, write the quadratic $f(x) = 2x^2 - 4x - 5$ in turning point form, and hence sketch the graph of $y = f(x)$.

Solution

$$\begin{aligned} f(x) &= 2x^2 - 4x - 5 \\ &= 2\left(x^2 - 2x - \frac{5}{2}\right) \\ &= 2\left(x^2 - 2x + 1 - 1 - \frac{5}{2}\right) \qquad \text{add and subtract } \left(\frac{b}{2}\right)^2 \text{ to 'complete the square'} \\ &= 2\left[(x^2 - 2x + 1) - \frac{7}{2}\right] \\ &= 2\left[(x - 1)^2 - \frac{7}{2}\right] \\ &= 2(x - 1)^2 - 7 \end{aligned}$$

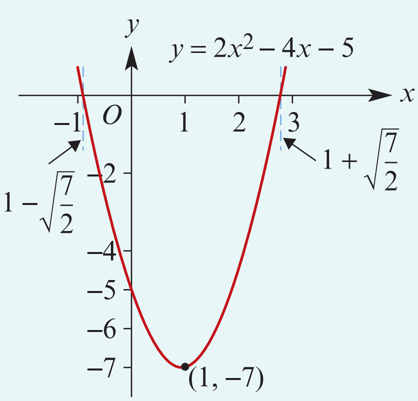
The x-axis intercepts can be determined after completing the square:

$$\begin{aligned} 2x^2 - 4x - 5 &= 0 \\ 2(x - 1)^2 - 7 &= 0 \\ (x - 1)^2 &= \frac{7}{2} \\ x - 1 &= \pm\sqrt{\frac{7}{2}} \end{aligned}$$

$$\therefore \quad x = 1 + \sqrt{\frac{7}{2}} \quad \text{or} \quad x = 1 - \sqrt{\frac{7}{2}}$$

This information can now be used to sketch the graph:

- The y-axis intercept is $c = -5$.
- The turning point is $(1, -7)$.
- The x-axis intercepts are $1 + \sqrt{\frac{7}{2}}$ and $1 - \sqrt{\frac{7}{2}}$.



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▶ The quadratic formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It should be noted that the equation of the axis of symmetry can be derived from this general formula: the axis of symmetry is the line with equation

$$x = -\frac{b}{2a}$$

Example 6

Sketch the graph of $f(x) = -3x^2 - 12x - 7$ by:

- finding the equation of the axis of symmetry
- finding the coordinates of the turning point
- using the general quadratic formula to find the x-axis intercepts.

Solution

Since $c = -7$, the y-axis intercept is -7.

Axis of symmetry
$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\left(\frac{-12}{2 \times (-3)}\right) \\ &= -2 \end{aligned}$$

Turning point

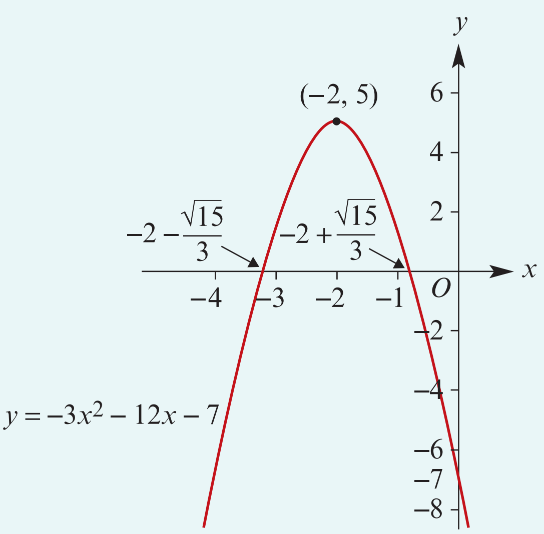
When $x = -2$, $y = -3(-2)^2 - 12(-2) - 7 = 5$.

The turning point coordinates are $(-2, 5)$.

x-axis intercepts

$$-3x^2 - 12x - 7 = 0$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(-3)(-7)}}{2(-3)} \\ &= \frac{12 \pm \sqrt{60}}{-6} \\ &= \frac{12 \pm 2\sqrt{15}}{-6} \\ &= -2 \pm \tfrac{1}{3}\sqrt{15} \end{aligned}$$



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▶ The discriminant

The **discriminant** Δ of a quadratic polynomial $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac$$

For the equation $ax^2 + bx + c = 0$:

- If $\Delta > 0$, there are two solutions.
- If $\Delta = 0$, there is one solution.
- If $\Delta < 0$, there are no real solutions.

For the equation $ax^2 + bx + c = 0$ where a, b and c are rational numbers:

- If Δ is a perfect square and $\Delta \neq 0$, then the equation has two rational solutions.
- If $\Delta = 0$, then the equation has one rational solution.
- If Δ is not a perfect square and $\Delta > 0$, then the equation has two irrational solutions.

Example 7

Without sketching graphs, determine whether the graph of each of the following functions crosses, touches or does not intersect the x-axis:

a $f(x) = 2x^2 - 4x - 6$

b $f(x) = -4x^2 + 12x - 9$

c $f(x) = 3x^2 - 2x + 8$

Solution	Explanation
a $\Delta = b^2 - 4ac$	
$= (-4)^2 - 4 \times 2 \times (-6)$	Here $a = 2$, $b = -4$, $c = -6$.
$= 16 + 48$	
$= 64 > 0$	
The graph crosses the x-axis twice.	As $\Delta > 0$, there are two x-axis intercepts.
b $\Delta = b^2 - 4ac$	
$= (12)^2 - 4 \times (-4) \times (-9)$	Here $a = -4$, $b = 12$, $c = -9$.
$= 144 - 144$	
$= 0$	
The graph touches the x-axis once.	As $\Delta = 0$, there is only one x-axis intercept.
c $\Delta = b^2 - 4ac$	
$= (-2)^2 - 4 \times 3 \times 8$	Here $a = 3$, $b = -2$, $c = 8$.
$= 4 - 96$	
$= -92 < 0$	
The graph does not intersect the x-axis.	As $\Delta < 0$, there are no x-axis intercepts.

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Example 8

Find the values of m for which the equation $3x^2 - 2mx + 3 = 0$ has:

a one solution **b** no solution **c** two distinct solutions.

Solution

For the quadratic $3x^2 - 2mx + 3$, the discriminant is $\Delta = 4m^2 - 36$.

a For one solution:

$$\Delta = 0$$

i.e. $4m^2 - 36 = 0$

$$m^2 = 9$$

$\therefore$ $m = \pm 3$

b For no solution:

$$\Delta < 0$$

i.e. $4m^2 - 36 < 0$

From the graph, this is equivalent to

$$-3 < m < 3$$

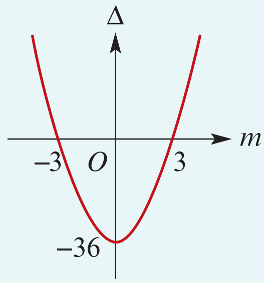
c For two distinct solutions:

$$\Delta > 0$$

i.e. $4m^2 - 36 > 0$

From the graph it can be seen that

$$m > 3 \quad \text{or} \quad m < -3$$



Section summary

- The graph of $y = a(x - h)^2 + k$ is a parabola congruent to the graph of $y = ax^2$. The vertex (or turning point) is the point (h, k). The axis of symmetry is $x = h$.
- The axis of symmetry of the graph of $y = ax^2 + bx + c$ has equation $x = -\dfrac{b}{2a}$.
- By completing the square, all quadratic functions in polynomial form $y = ax^2 + bx + c$ may be transposed into the turning point form $y = a(x - h)^2 + k$.
- To complete the square of $x^2 + bx + c$:
  - Take half the coefficient of x (that is, $\dfrac{b}{2}$) and add and subtract its square $\dfrac{b^2}{4}$.
- To complete the square of $ax^2 + bx + c$:
  - First take out a as a factor and then complete the square inside the bracket.
- The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

From the formula it can be seen that:

- If $b^2 - 4ac > 0$, there are two solutions.
- If $b^2 - 4ac = 0$, there is one solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

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Exercise 3A

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Example 1, 2

1 Sketch the graphs of the following functions:

a $f(x) = 2(x-1)^2$ **b** $f(x) = 2(x-1)^2 - 2$ **c** $f(x) = -2(x-1)^2$

d $f(x) = 4 - 2(x+1)^2$ **e** $f(x) = 4 + 2(x+\frac{1}{2})^2$ **f** $f(x) = 2(x+1)^2 - 1$

g $f(x) = 3(x-2)^2 - 4$ **h** $f(x) = (x+1)^2 - 1$ **i** $f(x) = 5x^2 - 1$

Example 3

2 For each of the following quadratic functions, use the axis of symmetry to find the turning point of the graph, express the function in the form $y = a(x-h)^2 + k$, and hence find the maximum or minimum value and the range:

a $f(x) = x^2 + 3x - 2$ **b** $f(x) = x^2 - 6x + 8$ **c** $f(x) = 2x^2 + 8x - 6$

d $f(x) = 2x^2 - 5x$ **e** $f(x) = 7 - 2x - 3x^2$ **f** $f(x) = -2x^2 + 9x + 11$

Example 4

3 Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of each of the following:

a $y = -x^2 + 2x$ **b** $y = x^2 - 6x + 8$ **c** $y = -x^2 - 5x - 6$

d $y = -2x^2 + 8x - 6$ **e** $y = 4x^2 - 12x + 9$ **f** $y = 6x^2 + 3x - 18$

Example 5

4 Sketch the graph of each of the following by first completing the square:

a $y = x^2 + 2x - 6$ **b** $y = x^2 - 4x - 10$ **c** $y = -x^2 - 5x - 3$

d $y = -2x^2 + 8x - 10$ **e** $y = x^2 - 7x + 3$

Example 6

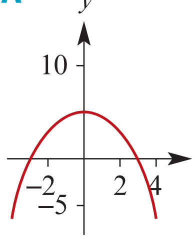
5 Sketch the graph of $f(x) = 3x^2 - 2x - 1$ by first finding the equation of the axis of symmetry, then finding the coordinates of the vertex, and finally using the quadratic formula to calculate the x-axis intercepts.

6 Match each of the following functions with the appropriate graph below:

a $y = \frac{1}{3}(x+4)(8-x)$ **b** $y = x^2 - \frac{x}{2} + 1$

c $y = -10 + 2(x-1)^2$ **d** $y = \frac{1}{2}(9 - x^2)$

A



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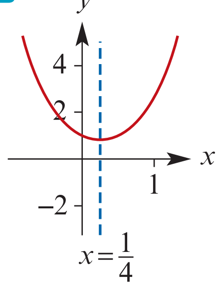

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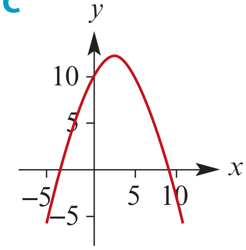

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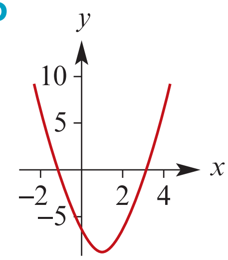

Example 7

7 Without sketching the graphs of the following functions, determine whether they cross, touch or do not intersect the x-axis:

a $f(x) = x^2 - 5x + 2$ **b** $f(x) = -4x^2 + 2x - 1$ **c** $f(x) = x^2 - 6x + 9$

d $f(x) = 8 - 3x - 2x^2$ **e** $f(x) = 3x^2 + 2x + 5$ **f** $f(x) = -x^2 - x - 1$

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Example 8 **8** For which values of m does the equation $mx^2 - 2mx + 3 = 0$ have:

a two solutions for x **b** one solution for x?

9 Find the value of m for which $(4m + 1)x^2 - 6mx + 4$ is a perfect square.

10 Find the values of a for which the equation $(a - 3)x^2 + 2ax + (a + 2) = 0$ has no solutions for x.

11 Prove that the equation $x^2 + (a + 1)x + (a - 2) = 0$ always has two distinct solutions.

12 Show that the equation $(k + 1)x^2 - 2x - k = 0$ has a solution for all values of k.

13 For which values of k does the equation $kx^2 - 2kx = 5$ have:

a two solutions for x **b** one solution for x?

14 For which values of k does the equation $(k - 3)x^2 + 2kx + (k + 2) = 0$ have:

a two solutions for x **b** one solution for x?

15 Show that the equation $ax^2 - (a + b)x + b = 0$ has a solution for all values of a and b.

CF

3B Determining the rule for a parabola

In this section, we revise methods for finding the rule of a quadratic function from its graph. The following three forms are useful. You will see others in the worked examples.

1 $y = a(x - e)(x - f)$ This can be used if two x-axis intercepts and the coordinates of one other point are known.

2 $y = a(x - h)^2 + k$ This can be used if the coordinates of the turning point and one other point are known.

3 $y = ax^2 + bx + c$ This can be used if the coordinates of three points on the parabola are known.

Example 9

A parabola has x-axis intercepts -3 and 4 and it passes through the point $(1, 24)$. Find the rule for this parabola.

Solution

$y = a(x + 3)(x - 4)$

When $x = 1$, $y = 24$. Thus

$$24 = a(1 + 3)(1 - 4)$$

$$24 = -12a$$

$$\therefore \quad a = -2$$

The rule is $y = -2(x + 3)(x - 4)$.

Explanation

Two x-axis intercepts are given. Therefore use the form $y = a(x - e)(x - f)$.

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Example 10

The coordinates of the turning point of a parabola are (2, 6) and the parabola passes through the point (3, 3). Find the rule for this parabola.

Solution

$y = a(x-2)^2 + 6$

When $x = 3$, $y = 3$. Thus

$$3 = a(3-2)^2 + 6$$
$$3 = a + 6$$
$$\therefore \quad a = -3$$

The rule is $y = -3(x-2)^2 + 6$.

Explanation

The coordinates of the turning point and one other point on the parabola are given. Therefore use $y = a(x-h)^2 + k$.

Example 11

A parabola passes through the points (1, 4), (0, 5) and (−1, 10). Find the rule for this parabola.

Solution

$y = ax^2 + bx + c$

When $x = 1$, $y = 4$.
When $x = 0$, $y = 5$.
When $x = -1$, $y = 10$.

Therefore

$$4 = a + b + c \qquad (1)$$
$$5 = c \qquad (2)$$
$$10 = a - b + c \qquad (3)$$

Substitute from equation (2) into equations (1) and (3):

$$-1 = a + b \qquad (1')$$
$$5 = a - b \qquad (3')$$

Add (1′) and (3′):

$$4 = 2a$$
$$\therefore \quad a = 2$$

Substitute into equation (1′):

$$-1 = 2 + b$$
$$\therefore \quad b = -3$$

The rule is $y = 2x^2 - 3x + 5$.

Explanation

The coordinates of three points on the parabola are given. Therefore we substitute values into the general polynomial form $y = ax^2 + bx + c$ to obtain three equations in three unknowns.

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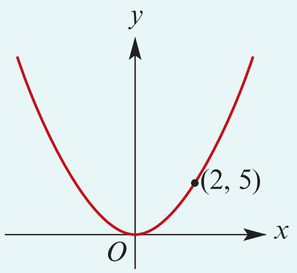
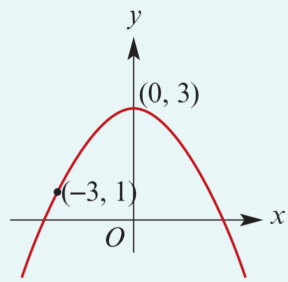
Example 12

Find the equation of each of the following parabolas:

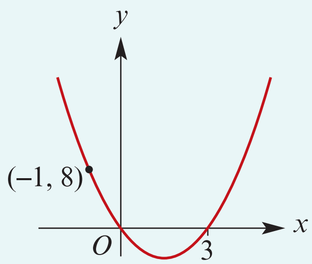
a



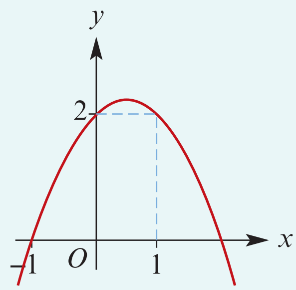
b



c



d



Solution

a This is of the form $y = ax^2$ (since the graph has its vertex at the origin).
As the point $(2, 5)$ is on the parabola,

$$5 = a(2)^2$$

$$\therefore \quad a = \frac{5}{4}$$

The rule is $y = \frac{5}{4}x^2$.

b This is of the form $y = ax^2 + c$ (since the graph is symmetric about the y-axis).

For $(0, 3)$: $\quad 3 = a(0)^2 + c$

$$\therefore \quad c = 3$$

For $(-3, 1)$: $\quad 1 = a(-3)^2 + 3$

$$1 = 9a + 3$$

$$\therefore \quad a = -\frac{2}{9}$$

The rule is $y = -\frac{2}{9}x^2 + 3$.

c This is of the form $y = ax(x - 3)$.
As the point $(-1, 8)$ is on the parabola,

$$8 = -a(-1 - 3)$$

$$8 = 4a$$

$$\therefore \quad a = 2$$

The rule is $y = 2x(x - 3)$.

d This is of the form $y = ax^2 + bx + c$.
The y-axis intercept is 2 and so $c = 2$.
As $(-1, 0)$ and $(1, 2)$ are on the parabola,

$$0 = a - b + 2 \qquad (1)$$

$$2 = a + b + 2 \qquad (2)$$

Add equations (1) and (2):

$$2 = 2a + 4$$

$$2a = -2$$

$$\therefore \quad a = -1$$

Substitute $a = -1$ in (1) to obtain $b = 1$.
The rule is $y = -x^2 + x + 2$.

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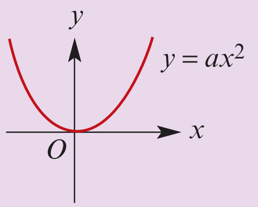
Section summary

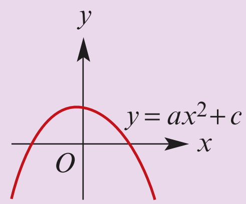
To find a quadratic rule to fit given points, first choose the best form of quadratic expression to work with. Then substitute in the coordinates of the known points to determine the unknown parameters. Some possible forms are given here:

i



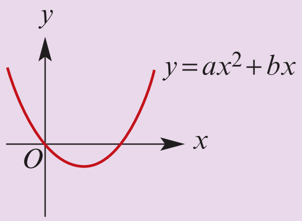
One point is needed to determine a.

ii



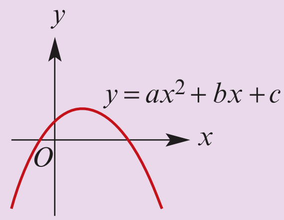
Two points are needed to determine a and c.

iii



Two points are needed to determine a and b.

iv



Three points are needed to determine a, b and c.

Exercise 3B

SF

Skillsheet

Example 9

1 A parabola has x-axis intercepts -3 and -2 and it passes through the point $(1, -24)$. Find the rule for this parabola.

2 A parabola has x-axis intercepts -3 and $-\frac{3}{2}$ and it passes through the point $(1, 20)$. Find the rule for this parabola.

Example 10

3 The coordinates of the turning point of a parabola are $(-2, 4)$ and the parabola passes through the point $(4, 58)$. Find the rule for this parabola.

4 The coordinates of the turning point of a parabola are $(-2, -3)$ and the parabola passes through the point $(-3, -5)$. Find the rule for this parabola.

Example 11

5 A parabola passes through the points $(1, 19)$, $(0, 18)$ and $(-1, 7)$. Find the rule for this parabola.

6 A parabola passes through the points $(2, -14)$, $(0, 10)$ and $(-4, 10)$. Find the rule for this parabola.

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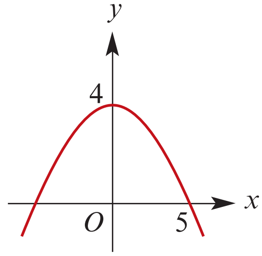
Example 12

7 Determine the equation of each of the following parabolas:

a



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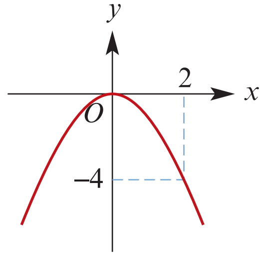

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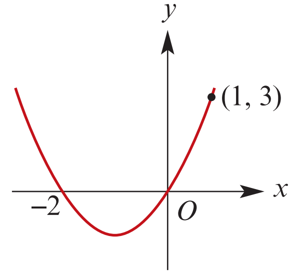

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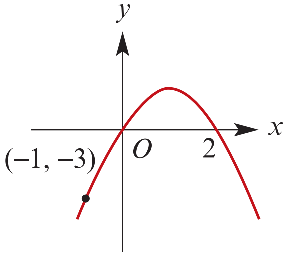

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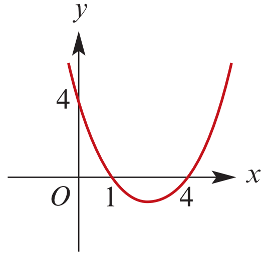

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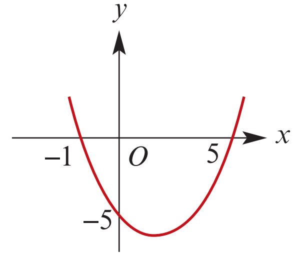
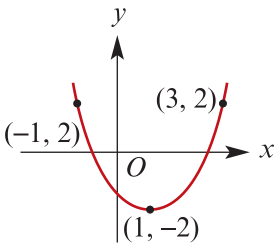
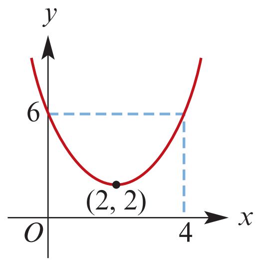


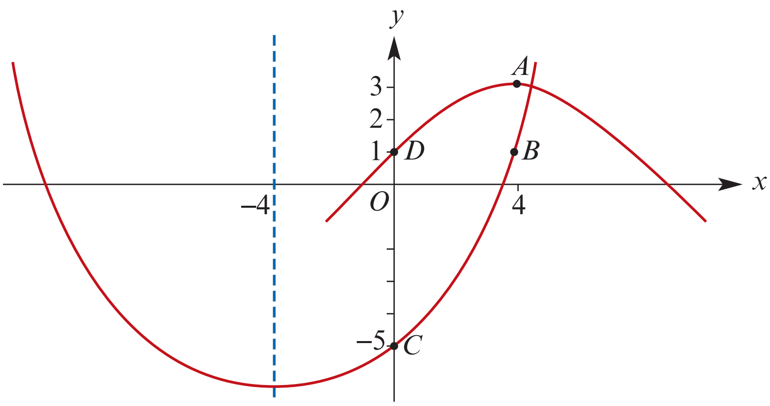
g



h



8 Find quadratic expressions for the two curves in the diagram, given that the coefficient of x in each case is 1. The marked points are $A(4, 3)$, $B(4, 1)$, $C(0, -5)$ and $D(0, 1)$.



9 The graph of the quadratic function $f(x) = A(x + b)^2 + B$ has a vertex at $(-2, 4)$ and passes through the point $(0, 8)$. Find the values of A, b and B.

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3C The language of polynomials

- A **polynomial function** is a function that can be written in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$.
- The number 0 is called the **zero polynomial**.
- The **leading term**, $a_n x^n$, of a polynomial is the term of highest index among those terms with a non-zero coefficient.
- The **degree of a polynomial** is the index n of the leading term.
- A **monic polynomial** is a polynomial whose leading term has coefficient 1.
- The **constant term** is the term of index 0. (This is the term not involving x.)

Example 13

Let $P(x) = x^4 - 3x^3 - 2$. Find:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$ **d** $P(-2)$

Solution

a $P(1) = 1^4 - 3 \times 1^3 - 2$
$= 1 - 3 - 2$
$= -4$

b $P(-1) = (-1)^4 - 3 \times (-1)^3 - 2$
$= 1 + 3 - 2$
$= 2$

c $P(2) = 2^4 - 3 \times 2^3 - 2$
$= 16 - 24 - 2$
$= -10$

d $P(-2) = (-2)^4 - 3 \times (-2)^3 - 2$
$= 16 + 24 - 2$
$= 38$

Example 14

a Let $P(x) = 2x^4 - x^3 + 2cx + 6$. If $P(1) = 21$, find the value of c.

b Let $Q(x) = 2x^6 - x^3 + ax^2 + bx + 20$. If $Q(-1) = Q(2) = 0$, find the values of a and b.

Solution

a $P(x) = 2x^4 - x^3 + 2cx + 6$ and $P(1) = 21$.

$P(1) = 2(1)^4 - (1)^3 + 2c + 6$
$= 2 - 1 + 2c + 6$
$= 7 + 2c$

Since $P(1) = 21$,

$7 + 2c = 21$
$\therefore \quad c = 7$

Explanation

We will substitute $x = 1$ into $P(x)$ to form an equation and solve.

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b $Q(x) = 2x^6 - x^3 + ax^2 + bx + 20$ and $Q(-1) = Q(2) = 0$.	First find $Q(-1)$ and $Q(2)$ in terms of a and b.
$Q(-1) = 2(-1)^6 - (-1)^3 + a(-1)^2 - b + 20$ $= 2 + 1 + a - b + 20$ $= 23 + a - b$	
$Q(2) = 2(2)^6 - (2)^3 + a(2)^2 + 2b + 20$ $= 128 - 8 + 4a + 2b + 20$ $= 140 + 4a + 2b$	
Since $Q(-1) = Q(2) = 0$, this gives $23 + a - b = 0$ (1) $140 + 4a + 2b = 0$ (2)	Form simultaneous equations in a and b by putting $Q(-1) = 0$ and $Q(2) = 0$.
Divide (2) by 2: $70 + 2a + b = 0$ (3)	
Add (1) and (3): $93 + 3a = 0$ $\therefore \quad a = -31$	
Substitute in (1) to obtain $b = -8$.	

► The arithmetic of polynomials

The operations of addition, subtraction and multiplication for polynomials are naturally defined, as shown in the following examples.

Let $P(x) = x^3 + 3x^2 + 2$ and $Q(x) = 2x^2 + 4$. Then

$$\begin{aligned} P(x) + Q(x) &= (x^3 + 3x^2 + 2) + (2x^2 + 4) \\ &= x^3 + 5x^2 + 6 \end{aligned}$$

$$\begin{aligned} P(x) - Q(x) &= (x^3 + 3x^2 + 2) - (2x^2 + 4) \\ &= x^3 + x^2 - 2 \end{aligned}$$

$$\begin{aligned} P(x)Q(x) &= (x^3 + 3x^2 + 2)(2x^2 + 4) \\ &= (x^3 + 3x^2 + 2) \times 2x^2 + (x^3 + 3x^2 + 2) \times 4 \\ &= 2x^5 + 6x^4 + 4x^2 + 4x^3 + 12x^2 + 8 \\ &= 2x^5 + 6x^4 + 4x^3 + 16x^2 + 8 \end{aligned}$$

The sum, difference and product of two polynomials is a polynomial.

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Example 15

Let $P(x) = x^3 - 6x + 3$ and $Q(x) = x^2 - 3x + 1$. Find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $P(x)Q(x)$

Solution

a $P(x) + Q(x)$

$$= x^3 - 6x + 3 + x^2 - 3x + 1$$
$$= x^3 + x^2 - 6x - 3x + 3 + 1$$
$$= x^3 + x^2 - 9x + 4$$

b $P(x) - Q(x)$

$$= x^3 - 6x + 3 - (x^2 - 3x + 1)$$
$$= x^3 - 6x + 3 - x^2 + 3x - 1$$
$$= x^3 - x^2 - 6x + 3x + 3 - 1$$
$$= x^3 - x^2 - 3x + 2$$

c
$$\begin{aligned} P(x)Q(x) &= (x^3 - 6x + 3)(x^2 - 3x + 1) \\ &= x^3(x^2 - 3x + 1) - 6x(x^2 - 3x + 1) + 3(x^2 - 3x + 1) \\ &= x^5 - 3x^4 + x^3 - 6x^3 + 18x^2 - 6x + 3x^2 - 9x + 3 \\ &= x^5 - 3x^4 + (x^3 - 6x^3) + (18x^2 + 3x^2) - (6x + 9x) + 3 \\ &= x^5 - 3x^4 - 5x^3 + 21x^2 - 15x + 3 \end{aligned}$$

We use the notation $\deg(f)$ to denote the degree of a polynomial f. For $f, g \neq 0$, we have

$$\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$$
$$\deg(f \times g) = \deg(f) + \deg(g)$$

▶ Equating coefficients

Two polynomials P and Q are equal only if their corresponding coefficients are equal. For two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, they are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

For example, if

$$P(x) = 4x^3 + 5x^2 - x + 3 \quad \text{and} \quad Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$$

then $P(x) = Q(x)$ if and only if $b_3 = 4$, $b_2 = 5$, $b_1 = -1$ and $b_0 = 3$.

Example 16

The polynomial $P(x) = x^3 + 3x^2 + 2x + 1$ can be written in the form $(x - 2)(x^2 + bx + c) + r$ where b, c and r are real numbers. Find the values of b, c and r.

Solution

Expand the required form:

$$\begin{aligned} (x - 2)(x^2 + bx + c) + r &= x(x^2 + bx + c) - 2(x^2 + bx + c) + r \\ &= x^3 + bx^2 + cx - 2x^2 - 2bx - 2c + r \\ &= x^3 + (b - 2)x^2 + (c - 2b)x - 2c + r \end{aligned}$$

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If $x^3 + 3x^2 + 2x + 1 = x^3 + (b - 2)x^2 + (c - 2b)x - 2c + r$ for all real numbers x, then by equating coefficients:

coefficient of x^2	$3 = b - 2$	$\therefore\ b = 5$
coefficient of x	$2 = c - 2b$	$\therefore\ c = 2b + 2 = 12$
constant term	$1 = -2c + r$	$\therefore\ r = 2c + 1 = 25$

Hence $b = 5$, $c = 12$ and $r = 25$. This means that

$$\begin{aligned} P(x) &= (x - 2)(x^2 + bx + c) + r \\ &= (x - 2)(x^2 + 5x + 12) + 25 \end{aligned}$$

Example 17

a If $x^3 + 3x^2 + 3x + 8 = a(x + 1)^3 + b$ for all $x \in \mathbb{R}$, find the values of a and b.

b Show that $x^3 + 6x^2 + 6x + 8$ cannot be written in the form $a(x + c)^3 + b$ for real numbers a, b and c.

Solution

a Expand the right-hand side of the equation:

$$\begin{aligned} a(x + 1)^3 + b &= a(x^3 + 3x^2 + 3x + 1) + b \\ &= ax^3 + 3ax^2 + 3ax + a + b \end{aligned}$$

If $x^3 + 3x^2 + 3x + 8 = ax^3 + 3ax^2 + 3ax + a + b$ for all $x \in \mathbb{R}$, then by equating coefficients:

coefficient of x^3	$1 = a$
coefficient of x^2	$3 = 3a$
coefficient of x	$3 = 3a$
constant term	$8 = a + b$

Hence $a = 1$ and $b = 7$.

b Expand the proposed form:

$$\begin{aligned} a(x + c)^3 + b &= a(x^3 + 3cx^2 + 3c^2x + c^3) + b \\ &= ax^3 + 3cax^2 + 3c^2ax + c^3a + b \end{aligned}$$

Suppose $x^3 + 6x^2 + 6x + 8 = ax^3 + 3cax^2 + 3c^2ax + c^3a + b$ for all $x \in \mathbb{R}$. Then

coefficient of x^3	$1 = a$	(1)
coefficient of x^2	$6 = 3ca$	(2)
coefficient of x	$6 = 3c^2a$	(3)
constant term	$8 = c^3a + b$	(4)

From (1), we have $a = 1$. So from (2), we have $c = 2$.
But substituting $a = 1$ and $c = 2$ into (3) gives $6 = 12$, which is a contradiction.

ISBN 978-1-108-45164-2

Section summary

- A **polynomial function** is a function that can be written in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$.
The **leading term** is $a_n x^n$ (the term of highest index) and the **constant term** is a_0 (the term not involving x).
- The **degree of a polynomial** is the index n of the leading term.
- The sum, difference and product of two polynomials is a polynomial. Division does not always lead to another polynomial.
- Two polynomials P and Q are equal only if their corresponding coefficients are equal. Two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

Exercise 3C

SF

Example 13

1 Let $P(x) = x^3 - 2x^2 + 3x + 1$. Find:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$ **d** $P(-2)$ **e** $P(\frac{1}{2})$ **f** $P(-\frac{1}{2})$

2 Let $P(x) = x^3 + 3x^2 - 4x + 6$. Find:

a $P(0)$ **b** $P(1)$ **c** $P(2)$ **d** $P(-1)$ **e** $P(a)$ **f** $P(2a)$

Example 14

3 **a** Let $P(x) = x^3 + 3x^2 - ax - 30$. If $P(2) = 0$, find the value of a.
b Let $P(x) = x^3 + ax^2 + 5x - 14$. If $P(3) = 68$, find the value of a.
c Let $P(x) = x^4 - x^3 - 2x + c$. If $P(1) = 6$, find the value of c.
d Let $P(x) = 2x^6 - 5x^3 + ax^2 + bx + 12$. If $P(-1) = P(2) = 0$, find a and b.
e Let $P(x) = x^5 - 2x^4 + ax^3 + bx^2 + 12x - 36$. If $P(3) = P(1) = 0$, find a and b.

Example 15

4 Let $f(x) = 2x^3 - x^2 + 3x$, $g(x) = 2 - x$ and $h(x) = x^2 + 2x$. Simplify each of the following:

a $f(x) + g(x)$ **b** $f(x) + h(x)$ **c** $f(x) - g(x)$
d $3f(x)$ **e** $f(x)\,g(x)$ **f** $g(x)\,h(x)$
g $f(x) + g(x) + h(x)$ **h** $f(x)\,h(x)$

5 Expand each of the following products and collect like terms:

a $(x - 2)(x^2 - 3x + 4)$ **b** $(x - 5)(x^2 - 2x + 3)$ **c** $(x + 1)(2x^2 - 3x - 4)$
d $(x + 2)(x^2 + bx + c)$ **e** $(2x - 1)(x^2 - 4x - 3)$

CF

Example 16

6 It is known that $x^3 - x^2 - 6x - 4 = (x + 1)(x^2 + bx + c)$ for all values of x, for suitable values of b and c.

a Expand $(x + 1)(x^2 + bx + c)$ and collect like terms.
b Find b and c by equating coefficients.
c Hence write $x^3 - x^2 - 6x - 4$ as a product of three linear factors.

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Example 17

7 **a** If $2x^3 - 18x^2 + 54x - 49 = a(x-3)^3 + b$ for all $x \in \mathbb{R}$, find the values of a and b.

b If $-2x^3 + 18x^2 - 54x + 52 = a(x+c)^3 + b$ for all $x \in \mathbb{R}$, find the values of a, b and c.

c Show that $x^3 - 5x^2 - 2x + 24$ cannot be written in the form $a(x+c)^3 + b$ for real numbers a, b and c.

8 Find the values of A and B such that $A(x+3) + B(x+2) = 4x + 9$ for all real numbers x.

9 Find the values of A, B and C in each of the following:

a $x^2 - 4x + 10 = A(x+B)^2 + C$ for all $x \in \mathbb{R}$

b $4x^2 - 12x + 14 = A(x+B)^2 + C$ for all $x \in \mathbb{R}$

c $x^3 - 9x^2 + 27x - 22 = A(x+B)^3 + C$ for all $x \in \mathbb{R}$.

CF

3D Division and factorisation of polynomials

The division of polynomials was introduced in Mathematical Methods Units 1 & 2.

When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

$$P(x) = D(x)Q(x) + R(x)$$

and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.

Here $P(x)$ is the **dividend** and $D(x)$ is the **divisor**.

The following example illustrates the process of dividing.

Example 18

Divide $x^3 + x^2 - 14x - 24$ by $x + 2$.

Solution

$$\begin{array}{r|l} & x^2 - x - 12 \\ \hline x+2 & x^3 + x^2 - 14x - 24 \\ & x^3 + 2x^2 \\ \hline & \quad -x^2 - 14x - 24 \\ & \quad -x^2 - 2x \\ \hline & \qquad -12x - 24 \\ & \qquad -12x - 24 \\ \hline & \qquad\qquad 0 \end{array}$$

Explanation

- Divide x, from $x + 2$, into the leading term x^3 to get x^2.
- Multiply x^2 by $x + 2$ to give $x^3 + 2x^2$.
- Subtract from $x^3 + x^2 - 14x - 24$, leaving $-x^2 - 14x - 24$.
- Now divide x, from $x + 2$, into $-x^2$ to get $-x$.
- Multiply $-x$ by $x + 2$ to give $-x^2 - 2x$.
- Subtract from $-x^2 - 14x - 24$, leaving $-12x - 24$.
- Divide x into $-12x$ to get -12.
- Multiply -12 by $x + 2$ to give $-12x - 24$.
- Subtract from $-12x - 24$, leaving remainder of 0.

In this example we see that $x + 2$ is a factor of $x^3 + x^2 - 14x - 24$, as the remainder is zero.
Thus $(x^3 + x^2 - 14x - 24) \div (x + 2) = x^2 - x - 12$ with zero remainder.

$$\therefore \quad \frac{x^3 + x^2 - 14x - 24}{x + 2} = x^2 - x - 12$$

ISBN 978-1-108-45164-2

Example 19

Divide $3x^4 - 9x^2 + 27x - 8$ by $x - 2$.

Solution

$$
\begin{array}{r l}
 & 3x^3 + 6x^2 + 3x + 33 \\
x - 2 \,\big) & 3x^4 + 0x^3 - 9x^2 + 27x - 8 \\
 & 3x^4 - 6x^3 \\ \hline
 & \quad 6x^3 - 9x^2 + 27x - 8 \\
 & \quad 6x^3 - 12x^2 \\ \hline
 & \qquad 3x^2 + 27x - 8 \\
 & \qquad 3x^2 - 6x \\ \hline
 & \qquad\quad 33x - 8 \\
 & \qquad\quad 33x - 66 \\ \hline
 & \qquad\qquad 58
\end{array}
$$

Therefore

$$3x^4 - 9x^2 + 27x - 8 = (x - 2)(3x^3 + 6x^2 + 3x + 33) + 58$$

or, equivalently,

$$\frac{3x^4 - 9x^2 + 27x - 8}{x - 2} = 3x^3 + 6x^2 + 3x + 33 + \frac{58}{x - 2}$$

In this example, the dividend is $3x^4 - 9x^2 + 27x - 8$, the divisor is $x - 2$, and the remainder is 58.

A second method for division, called **equating coefficients**, can be seen in the explanation column of the next example.

Example 20

Divide $3x^3 + 2x^2 - x - 2$ by $2x + 1$.

Solution

$$
\begin{array}{r l}
 & \tfrac{3}{2}x^2 + \tfrac{1}{4}x - \tfrac{5}{8} \\
2x + 1 \,\big) & 3x^3 + 2x^2 - x - 2 \\
 & 3x^3 + \tfrac{3}{2}x^2 \\ \hline
 & \quad \tfrac{1}{2}x^2 - x - 2 \\
 & \quad \tfrac{1}{2}x^2 + \tfrac{1}{4}x \\ \hline
 & \qquad -\tfrac{5}{4}x - 2 \\
 & \qquad -\tfrac{5}{4}x - \tfrac{5}{8} \\ \hline
 & \qquad\qquad -1\tfrac{3}{8}
\end{array}
$$

Explanation

We show the alternative method here.

First write the identity
$3x^3 + 2x^2 - x - 2 = (2x + 1)(ax^2 + bx + c) + r$

Equate coefficients of x^3:
$3 = 2a$. Therefore $a = \frac{3}{2}$.

Equate coefficients of x^2:
$2 = a + 2b$. Therefore $b = \frac{1}{2}(2 - \frac{3}{2}) = \frac{1}{4}$.

Equate coefficients of x:
$-1 = 2c + b$. Therefore $c = \frac{1}{2}(-1 - \frac{1}{4}) = -\frac{5}{8}$.

Equate constant terms:
$-2 = c + r$. Therefore $r = -2 + \frac{5}{8} = -\frac{11}{8}$.

ISBN 978-1-108-45164-2

Dividing by a non-linear polynomial

We give one example of dividing by a non-linear polynomial. The technique is exactly the same as when dividing by a linear polynomial.

Example 21

Divide $3x^3 - 2x^2 + 3x - 4$ by $x^2 - 1$.

Solution

$$\begin{array}{r}
3x - 2 \\
x^2 + 0x - 1 \,\big)\, \overline{3x^3 - 2x^2 + 3x - 4} \\
\underline{3x^3 + 0x^2 - 3x } \\
-2x^2 + 6x - 4 \\
\underline{-2x^2 + 0x + 2} \\
6x - 6
\end{array}$$

Therefore

$$3x^3 - 2x^2 + 3x - 4 = (x^2 - 1)(3x - 2) + 6x - 6$$

or, equivalently,

$$\frac{3x^3 - 2x^2 + 3x - 4}{x^2 - 1} = 3x - 2 + \frac{6x - 6}{x^2 - 1}$$

Explanation

We write $x^2 - 1$ as $x^2 + 0x - 1$.

The remainder theorem and the factor theorem

The following two results are recalled from Mathematical Methods Units 1 & 2.

The remainder theorem

Suppose that, when the polynomial $P(x)$ is divided by $x - \alpha$, the quotient is $Q(x)$ and the remainder is R. Then

$$P(x) = (x - \alpha)Q(x) + R$$

Now, as the two expressions are equal for all values of x, they are equal for $x = \alpha$.

$$P(\alpha) = (\alpha - \alpha)Q(\alpha) + R$$

$$\therefore \quad P(\alpha) = R$$

i.e. when $P(x)$ is divided by $x - \alpha$, the remainder R is equal to $P(\alpha)$. We therefore have

$$P(x) = (x - \alpha)Q(x) + P(\alpha)$$

More generally:

Remainder theorem

When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.

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Example 22

Find the remainder when $P(x) = 3x^3 + 2x^2 + x + 1$ is divided by $2x + 1$.

Solution

By the remainder theorem, the remainder is

$$P\left(-\frac{1}{2}\right) = 3\left(-\frac{1}{2}\right)^3 + 2\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) + 1$$

$$= -\frac{3}{8} + \frac{2}{4} - \frac{1}{2} + 1 = \frac{5}{8}$$

The factor theorem

Now, in order for $x - \alpha$ to be a factor of the polynomial $P(x)$, the remainder must be zero. We state this result as the **factor theorem**.

Factor theorem

For a polynomial $P(x)$:

- If $P(\alpha) = 0$, then $x - \alpha$ is a factor of $P(x)$.
- Conversely, if $x - \alpha$ is a factor of $P(x)$, then $P(\alpha) = 0$.

More generally:

- If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
- Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.

Example 23

Given that $x + 1$ and $x - 2$ are factors of $6x^4 - x^3 + ax^2 - 6x + b$, find the values of a and b.

Solution

Let $P(x) = 6x^4 - x^3 + ax^2 - 6x + b$.

By the factor theorem, we have $P(-1) = 0$ and $P(2) = 0$. Hence

$$6 + 1 + a + 6 + b = 0 \qquad (1)$$

$$96 - 8 + 4a - 12 + b = 0 \qquad (2)$$

Rearranging gives:

$$a + b = -13 \qquad (1')$$

$$4a + b = -76 \qquad (2')$$

Subtract (1′) from (2′):

$$3a = -63$$

Therefore $a = -21$ and, from (1′), $b = 8$.

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Example 24

Show that $x + 1$ is a factor of $x^3 - 4x^2 + x + 6$ and hence find the other linear factors.

Solution

Let $P(x) = x^3 - 4x^2 + x + 6$

Then $P(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6$

$= 0$

Thus $x + 1$ is a factor (by the factor theorem).

Divide by $x + 1$ to find the other factor:

$$\begin{array}{r|l} & x^2 - 5x + 6 \\ x+1 & x^3 - 4x^2 + x + 6 \\ & x^3 + x^2 \\ \hline & -5x^2 + x + 6 \\ & -5x^2 - 5x \\ \hline & 6x + 6 \\ & 6x + 6 \\ \hline & 0 \end{array}$$

$\therefore\ x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$

$= (x + 1)(x - 3)(x - 2)$

The linear factors of $x^3 - 4x^2 + x + 6$ are $(x + 1)$, $(x - 3)$ and $(x - 2)$.

Explanation

We can use the factor theorem to find one factor, and then divide to find the other two linear factors.

Here is an alternative method:

Once we have found that $x + 1$ is a factor, we know that we can write

$$x^3 - 4x^2 + x + 6 = (x + 1)(x^2 + bx + c)$$

By equating constant terms, we have $6 = 1 \times c$. Hence $c = 6$.

By equating coefficients of x^2, we have $-4 = 1 + b$. Hence $b = -5$.

$\therefore\ x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$

Sums and differences of cubes

If $P(x) = x^3 - a^3$, then $x - a$ is a factor and so by division:

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

Replacing a with $-a$ in this equation gives:

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

Example 25

Factorise:

a $8x^3 + 64$

b $125a^3 - b^3$

Solution

a $8x^3 + 64 = (2x)^3 + (4)^3$

$= (2x + 4)(4x^2 - 8x + 16)$

b $125a^3 - b^3 = (5a)^3 - b^3$

$= (5a - b)(25a^2 + 5ab + b^2)$

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Solving polynomial equations

The factor theorem may be used in the solution of equations.

Example 26

Factorise $P(x) = x^3 - 4x^2 - 11x + 30$ and hence solve the equation $x^3 - 4x^2 - 11x + 30 = 0$.

Solution

$$P(1) = 1 - 4 - 11 + 30 \neq 0$$
$$P(-1) = -1 - 4 + 11 + 30 \neq 0$$
$$P(2) = 8 - 16 - 22 + 30 = 0$$

Therefore $x - 2$ is a factor.

Dividing $x^3 - 4x^2 - 11x + 30$ by $x - 2$ gives

$$\begin{aligned} P(x) &= (x-2)(x^2 - 2x - 15) \\ &= (x-2)(x-5)(x+3) \end{aligned}$$

Now we see that $P(x) = 0$ if and only if

$$x - 2 = 0 \quad \text{or} \quad x - 5 = 0 \quad \text{or} \quad x + 3 = 0$$
$$\therefore \quad x = 2 \quad \text{or} \quad x = 5 \quad \text{or} \quad x = -3$$

Section summary

- **Division of polynomials**
  When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that
  $$P(x) = D(x)Q(x) + R(x)$$
  and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.
- Two methods for dividing polynomials are long division and equating coefficients.
- **Remainder theorem**
  When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.
- **Factor theorem**
  - If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
  - Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.
- A cubic polynomial can be factorised by using the factor theorem to find the first linear factor and then using polynomial division or the method of equating coefficients to complete the factorisation.
- Difference of two cubes: $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
- Sum of two cubes: $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

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Exercise 3D

Skillsheet
Example 18

1 For each of the following, divide the first polynomial by the second:

a $x^3 - x^2 - 14x + 24,\ x + 4$ **b** $2x^3 + x^2 - 25x + 12,\ x - 3$

SF

Example 19

2 For each of the following, divide the first polynomial by the second:

a $x^3 - x^2 - 15x + 25,\ x + 3$ **b** $2x^3 - 4x + 12,\ x - 3$

Example 20

3 For each of the following, divide the first polynomial by the second:

a $2x^3 - 2x^2 - 15x + 25,\ 2x + 3$ **b** $4x^3 + 6x^2 - 4x + 12,\ 2x - 3$

4 For each of the following, divide the first expression by the second:

a $2x^3 - 7x^2 + 15x - 3,\ x - 3$ **b** $5x^5 + 13x^4 - 2x^2 - 6,\ x + 1$

Example 21

5 For each of the following, divide the first expression by the second:

a $x^4 - 9x^3 + 25x^2 - 8x - 2,\ x^2 - 2$ **b** $x^4 + x^3 + x^2 - x - 2,\ x^2 - 1$

CF

Example 22

6 **a** Find the remainder when $x^3 + 3x - 2$ is divided by $x + 2$.

b Find the value of a for which $(1 - 2a)x^2 + 5ax + (a - 1)(a - 8)$ is divisible by $x - 2$ but not by $x - 1$.

7 Given that $f(x) = 6x^3 + 5x^2 - 17x - 6$:

SF

a Find the remainder when $f(x)$ is divided by $x - 2$.

b Find the remainder when $f(x)$ is divided by $x + 2$.

c Factorise $f(x)$ completely.

8 **a** Prove that the expression $x^3 + (k - 1)x^2 + (k - 9)x - 7$ is divisible by $x + 1$ for all values of k.

b Find the value of k for which the expression has a remainder of 12 when divided by $x - 2$.

Example 23

9 The polynomial $f(x) = 2x^3 + ax^2 - bx + 3$ has a factor $x + 3$. When $f(x)$ is divided by $x - 2$, the remainder is 15.

CF

a Calculate the values of a and b.

b Find the other two linear factors of $f(x)$.

10 The expression $4x^3 + ax^2 - 5x + b$ leaves remainders of -8 and 10 when divided by $2x - 3$ and $x - 3$ respectively. Calculate the values of a and b.

11 Find the remainder when $(x + 1)^4$ is divided by $x - 2$.

SF

12 Let $P(x) = x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1$.

CF

a Show that neither $x - 1$ nor $x + 1$ is a factor of $P(x)$.

b Given that $P(x)$ can be written in the form $(x^2 - 1)Q(x) + ax + b$, where $Q(x)$ is a polynomial and a and b are constants, hence or otherwise, find the remainder when $P(x)$ is divided by $x^2 - 1$.

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Example 24 **13** Show that $x + 1$ is a factor of $2x^3 - 5x^2 - 4x + 3$ and find the other linear factors. SF

14 **a** Show that both $x - \sqrt{3}$ and $x + \sqrt{3}$ are factors of $x^4 + x^3 - x^2 - 3x - 6$. CF

b Hence write down one quadratic factor of $x^4 + x^3 - x^2 - 3x - 6$, and find a second quadratic factor.

Example 25 **15** Factorise each of the following: SF

a $8a^3 + 27b^3$ **b** $64 - a^3$

c $125x^3 + 64y^3$ **d** $(a - b)^3 + (a + b)^3$

Example 26 **16** Solve each of the following equations for x:

a $(2 - x)(x + 4)(x - 2)(x - 3) = 0$ **b** $x^3(2 - x) = 0$

c $(2x - 1)^3(2 - x) = 0$ **d** $(x + 2)^3(x - 2)^2 = 0$

e $x^4 - 4x^2 = 0$ **f** $x^4 - 9x^2 = 0$

g $12x^4 + 11x^3 - 26x^2 + x + 2 = 0$ **h** $x^4 + 2x^3 - 3x^2 - 4x + 4 = 0$

i $6x^4 - 5x^3 - 20x^2 + 25x - 6 = 0$

17 Find the x-axis intercepts and y-axis intercept of the graph of each of the following:

a $y = x^3 - x^2 - 2x$ **b** $y = x^3 - 2x^2 - 5x + 6$

c $y = x^3 - 4x^2 + x + 6$ **d** $y = 2x^3 - 5x^2 + x + 2$

e $y = x^3 + 2x^2 - x - 2$ **f** $y = 3x^3 - 4x^2 - 13x - 6$

g $y = 5x^3 + 12x^2 - 36x - 16$ **h** $y = 6x^3 - 5x^2 - 2x + 1$

i $y = 2x^3 - 3x^2 - 29x - 30$

18 The expressions $px^4 - 5x + q$ and $x^4 - 2x^3 - px^2 - qx - 8$ have a common factor $x - 2$. Find the values of p and q. CF

19 Find the remainder when $f(x) = x^4 - x^3 + 5x^2 + 4x - 36$ is divided by $x + 1$. SF

20 Factorise each of the following polynomials, using a calculator to help find at least one linear factor:

a $x^3 - 11x^2 - 125x + 1287$ **b** $x^3 - 9x^2 - 121x + 1089$

c $2x^3 - 9x^2 - 242x + 1089$ **d** $4x^3 - 367x + 1287$

21 Factorise each of the following:

a $x^4 - x^3 - 43x^2 + x + 42$ **b** $x^4 + 4x^3 - 27x - 108$

22 Factorise each of the following polynomials, using a calculator to help find at least one linear factor:

a $2x^4 - 25x^3 + 57x^2 + 9x + 405$ **b** $x^4 + 13x^3 + 40x^2 + 81x + 405$

c $x^4 + 3x^3 - 4x^2 + 3x - 135$ **d** $x^4 + 4x^3 - 35x^2 - 78x + 360$

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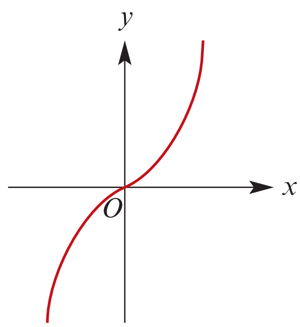
3E The general cubic function

Not all cubic functions can be written in the form $f(x) = a(x-h)^3 + k$. In this section we consider the general cubic function. The form of a general cubic function is

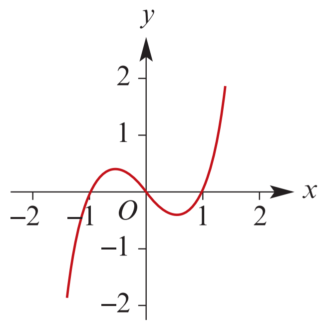
$$f(x) = ax^3 + bx^2 + cx + d, \quad \text{where } a \neq 0$$

It is impossible to fully investigate cubic functions without the use of calculus. Cubic functions will be revisited in Chapter 8.

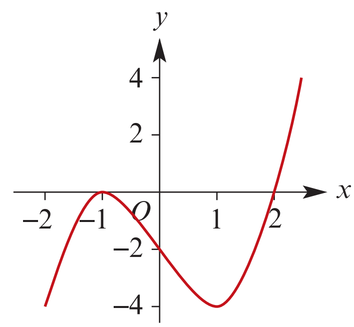
The 'shapes' of cubic graphs vary. Below is a gallery of cubic graphs, demonstrating the variety of 'shapes' that are possible.



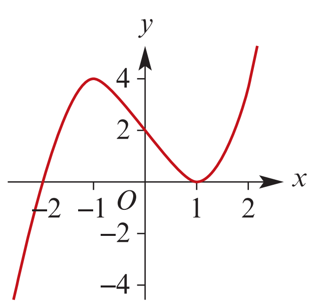
$f(x) = x^3 + x$



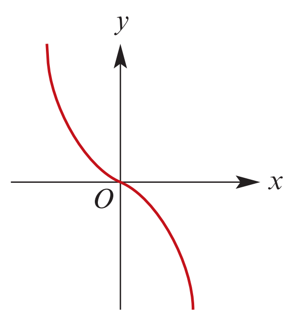
$f(x) = x^3 - x$



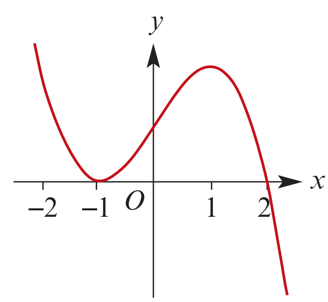
$f(x) = x^3 - 3x - 2$



$f(x) = x^3 - 3x + 2$



$f(x) = -x^3 - x$



$f(x) = -x^3 + 3x + 2$

Notes:

- A cubic graph can have one, two or three x-axis intercepts.
- Not all cubic graphs have a stationary point. For example, the graph of $f(x) = x^3 + x$ shown above has no points of zero gradient.
- The turning points do not occur symmetrically between consecutive x-axis intercepts as they do for quadratics. Differential calculus must be used to determine them.
- If a cubic graph has a turning point on the x-axis, this corresponds to a **repeated factor**. For example, the graph of $f(x) = x^3 - 3x - 2$ shown above has a turning point at $(-1, 0)$. The factorisation is $f(x) = (x+1)^2(x-2)$.

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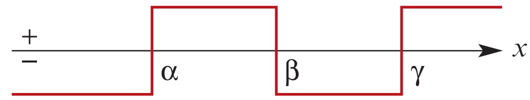
Sign diagrams

A **sign diagram** is a number-line diagram that shows when an expression is positive or negative. For a cubic function with rule $f(x) = (x - \alpha)(x - \beta)(x - \gamma)$, where $\alpha < \beta < \gamma$, the sign diagram is as shown.



Example 27

Draw a sign diagram for the cubic function $f(x) = x^3 - 4x^2 - 11x + 30$.

Solution

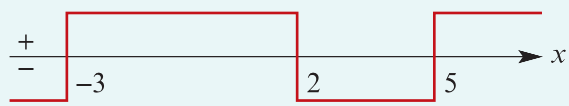
From Example 26, we have

$$f(x) = (x + 3)(x - 2)(x - 5)$$

Therefore $f(-3) = f(2) = f(5) = 0$. We note that

$$f(x) > 0 \quad \text{for } x > 5$$
$$f(x) < 0 \quad \text{for } 2 < x < 5$$
$$f(x) > 0 \quad \text{for } -3 < x < 2$$
$$f(x) < 0 \quad \text{for } x < -3$$

Hence the sign diagram may be drawn as shown.



Example 28

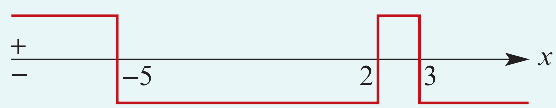
For the cubic function with rule $f(x) = -x^3 + 19x - 30$:

a Sketch the graph of $y = f(x)$ using a calculator to find the coordinates of the turning points, correct to two decimal places.

b Sketch the graph of $y = \frac{1}{2}f(x - 1)$.

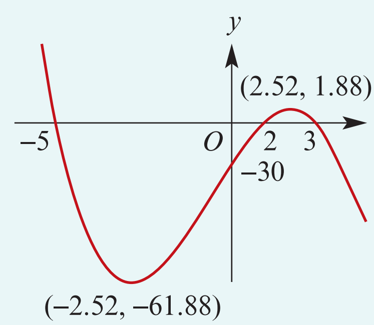
Solution

a $f(x) = -x^3 + 19x - 30$
$= (3 - x)(x - 2)(x + 5)$
$= -(x + 5)(x - 2)(x - 3)$



The x-axis intercepts are at $x = -5$, $x = 2$ and $x = 3$ and the y-axis intercept is at $y = -30$.

The turning points can be found using a graphics calculator. The method is described following this example.



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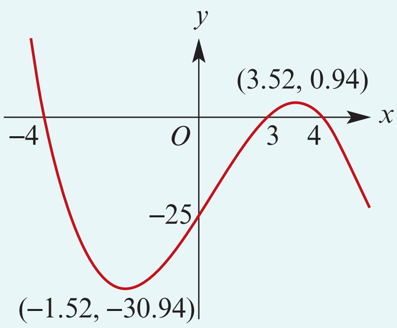
b The rule for the transformation is

$$(x, y) \to \left(x + 1, \frac{1}{2}y\right)$$

This is a dilation of factor $\frac{1}{2}$ from the x-axis followed by a translation 1 unit to the right.
Transformations of the turning points:

$$(2.52, 1.88) \to (3.52, 0.94)$$
$$(-2.52, -61.88) \to (-1.52, -30.94)$$

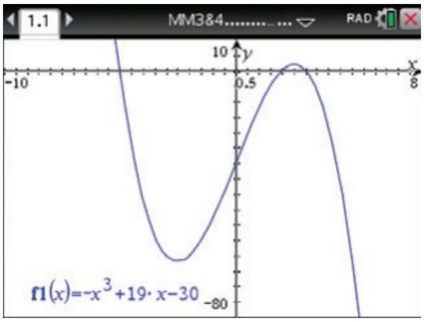


TI-84

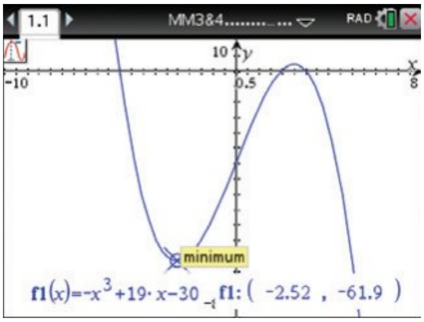
Using the TI-Nspire CX non-CAS

To add detail to the graph, the coordinates of the turning points can be found with a calculator.

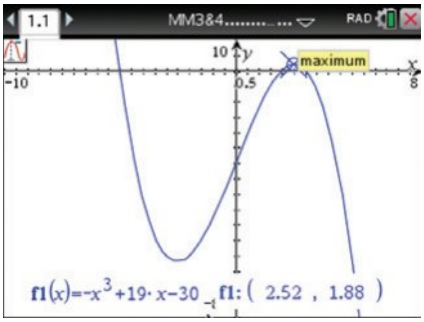
- Enter the function in a **Graphs** page.
- Use (menu) > **Window/Zoom** > **Window Settings** to set an appropriate window.
- Use either (menu) > **Trace** > **Graph Trace** or (menu) > **Analyze Graph** > **Maximum or Minimum** to display the approximate (decimal) coordinates of key points on the graph.



- In **Graph Trace**, the tracing point (×) can be moved either by using the arrow keys (◄ ►) or by typing a specific x-value then (enter). When the tracing point reaches a local minimum, it displays 'minimum'.
- Pressing (enter) will paste the coordinates to the point on the graph.
- Press (esc) to exit the command.



- Here **Graph Trace** has been used to find the turning points of the cubic function.
- If you use **Analyze Graph** instead, select the lower bound by moving to the left of the key point and clicking ( ) and then select the upper bound by moving to the right (►) of the key point and clicking.



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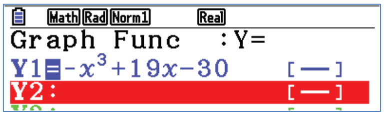
Using the Casio

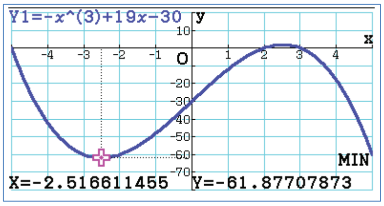
a To add detail to the graph, the coordinates of the turning points can be found with a calculator.

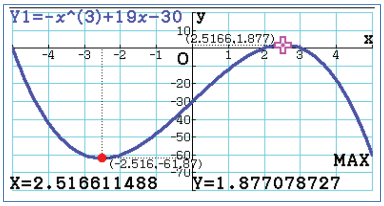
- Press (MENU) (5) to select **Graph** mode.
- Enter the rule $y = -x^3 + 19x - 30$ in $Y1$:

  ((–)) (X,θ,T) (∧) (3) ► (+) (1) (9) (X,θ,T)
  (–) (3) (0) (EXE)

- Adjust the View Window (SHIFT) (F3) for $-5 \le x \le 5$ and $-80 \le y \le 20$ with y-scale 10.
- Select **Draw** (F6) to view the graph.
- To find the local minimum, go to the **G-Solve** menu (SHIFT) (F5) and select **Minimum** (F3).
- For the local maximum, go to **G-Solve** and select **Maximum** (F2).

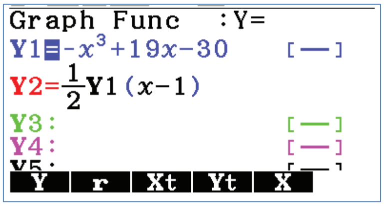


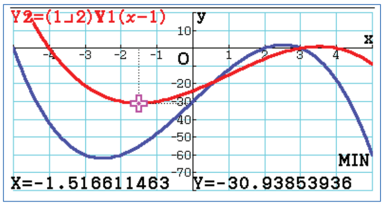




b To graph the transformed function:

- Press (EXIT) to return to the function list.
- Enter the rule for the transformed function in $Y2$ as shown.
- Select **Draw** (F6) to view the graphs.
- The coordinates of the turning points can be found as above.





Section summary

- The graph of a cubic function can have one, two or three x-axis intercepts.
- The graph of a cubic function can have zero, one or two stationary points.
- To sketch a cubic in factorised form $y = a(x - \alpha)(x - \beta)(x - \gamma)$:
  - Find the y-axis intercept.
  - Find the x-axis intercepts.
  - Prepare a sign diagram.
  - Consider the y-values as x increases to the right of all x-axis intercepts.
  - Consider the y-values as x decreases to the left of all x-axis intercepts.
- If there is a repeated factor to the power 2, then the y-values have the same sign immediately to the left and right of the corresponding x-axis intercept.

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Exercise 3E

SF

Example 27 **1** Draw a sign diagram for each of the following expressions:

a $(3 - x)(x - 1)(x - 6)$ **b** $(3 + x)(x - 1)(x + 6)$
c $(x - 5)(x + 1)(2x - 6)$ **d** $(4 - x)(5 - x)(1 - 2x)$
e $(x - 5)^2(x - 4)$ **f** $(x - 5)^2(4 - x)$

2 First factorise and then draw a sign diagram for each of the following expressions:

a $x^3 - 4x^2 + x + 6$ **b** $4x^3 + 3x^2 - 16x - 12$
c $x^3 - 7x^2 + 4x + 12$ **d** $2x^3 + 3x^2 - 11x - 6$

Example 28 **3** **a** Use a calculator to plot the graph of $y = f(x)$ where $f(x) = x^3 - 2x^2 + 1$.

b On the same screen, plot the graphs of:

i $y = f(x - 2)$ **ii** $y = f(x + 2)$ **iii** $y = 3f(x)$

4 **a** Use a calculator to plot the graph of $y = f(x)$ where $f(x) = x^3 + x^2 - 4x + 2$.

b On the same screen, plot the graphs of:

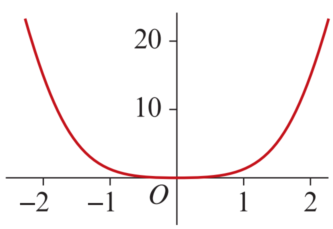
i $y = f(2x)$ **ii** $y = f\left(\frac{x}{2}\right)$ **iii** $y = 2f(x)$

3F Polynomials of higher degree

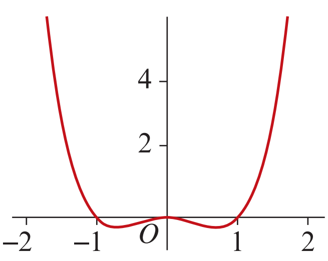
The general form for a quartic function is

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad \text{where } a \neq 0$$

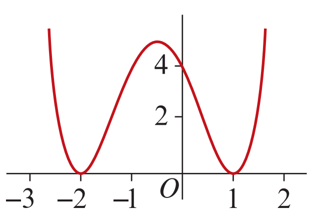
A gallery of quartic functions is shown below.



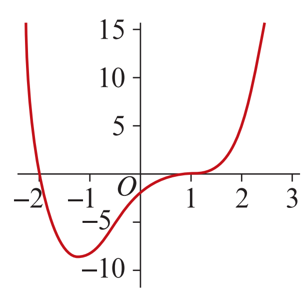
$f(x) = x^4$



$f(x) = x^4 - x^2$



$f(x) = (x - 1)^2(x + 2)^2$



$f(x) = (x - 1)^3(x + 2)$

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The techniques that have been developed for cubic functions may now be applied to quartic functions and to polynomial functions of higher degree in general.

For a polynomial $P(x)$ of degree n, there are at most n solutions to the equation $P(x) = 0$. Therefore the graph of $y = P(x)$ has at most n x-axis intercepts.

The graph of a polynomial of even degree may have no x-axis intercepts: for example, $P(x) = x^2 + 1$. But the graph of a polynomial of odd degree must have at least one x-axis intercept.

Example 29

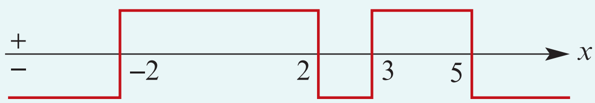
Draw a sign diagram for each quartic expression:

a $(2 - x)(x + 2)(x - 3)(x - 5)$

b $x^4 + x^2 - 2$

Solution

a



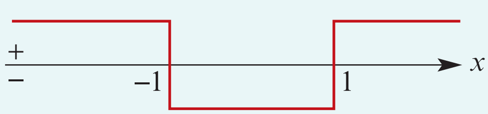
b Let $P(x) = x^4 + x^2 - 2$.

Then $P(1) = 1 + 1 - 2 = 0$.

Thus $x - 1$ is a factor.

$$
\begin{array}{r|l}
 & x^3 + x^2 + 2x + 2 \\
\hline
x - 1 & x^4 + 0x^3 + x^2 + 0x - 2 \\
 & x^4 - x^3 \\
\hline
 & \quad x^3 + x^2 + 0x - 2 \\
 & \quad x^3 - x^2 \\
\hline
 & \qquad 2x^2 + 0x - 2 \\
 & \qquad 2x^2 - 2x \\
\hline
 & \qquad\quad 2x - 2 \\
 & \qquad\quad 2x - 2 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

$$
\begin{aligned}
\therefore \quad P(x) &= (x - 1)(x^3 + x^2 + 2x + 2) \\
&= (x - 1)[x^2(x + 1) + 2(x + 1)] \\
&= (x - 1)(x + 1)(x^2 + 2)
\end{aligned}
$$



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Example 30

For $p(x) = x^4 - 2x^2 + 1$, find the coordinates of the points where the graph of $y = p(x)$ intersects the x- and y-axes, and hence sketch the graph.

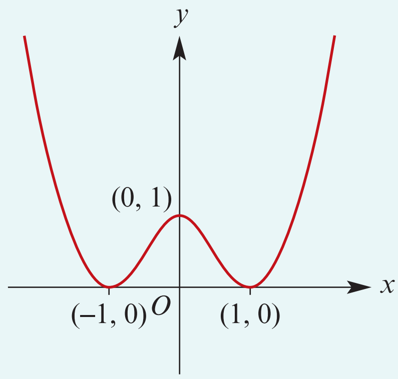
Solution

Note that

$$p(x) = (x^2)^2 - 2(x^2) + 1$$
$$= (x^2 - 1)^2$$
$$= [(x - 1)(x + 1)]^2$$
$$= (x - 1)^2(x + 1)^2$$

Therefore the x-axis intercepts are 1 and −1.

When $x = 0$, $y = 1$. So the y-axis intercept is 1.



Explanation

Alternatively, we can factorise $p(x)$ by using the factor theorem and division.

Note that

$$p(1) = 1 - 2 + 1 = 0$$

Therefore $x - 1$ is a factor.

$$p(x) = (x - 1)(x^3 + x^2 - x - 1)$$
$$= (x - 1)[x^2(x + 1) - (x + 1)]$$
$$= (x - 1)(x + 1)(x^2 - 1)$$
$$= (x - 1)^2(x + 1)^2$$

Section summary

- The graph of a quartic function can have zero, one, two, three or four x-axis intercepts.
- The graph of a quartic function can have one, two or three stationary points.
- To sketch a quartic in factorised form $y = a(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$:
  - Find the y-axis intercept.
  - Find the x-axis intercepts.
  - Prepare a sign diagram.
  - Consider the y-values as x increases to the right of all x-axis intercepts.
  - Consider the y-values as x decreases to the left of all x-axis intercepts.
- If there is a repeated factor to an even power, then the y-values have the same sign immediately to the left and right of the corresponding x-axis intercept.

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Exercise 3F

SF

Example 29 **1** Draw a sign diagram for each quartic expression:

a $(3 - x)(x + 4)(x - 5)(x - 1)$ **b** $x^4 - 2x^3 - 3x^2 + 4x + 4$

CF

Example 30 **2** For $h(x) = 81x^4 - 72x^2 + 16$, find the coordinates of the points where the graph of $y = h(x)$ intersects the x- and y-axes, and hence sketch the graph.
Hint: First express $h(x)$ as the square of a quadratic expression.

SF

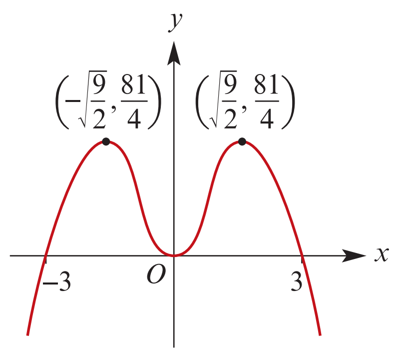
3 **a** Use a calculator to plot the graph of $y = f(x)$, where $f(x) = x^4 - 2x^3 + x + 1$.

b On the same screen, plot the graphs of:

i $y = f(x - 2)$ **ii** $y = f(2x)$ **iii** $y = f\left(\frac{x}{2}\right)$

CF

4 The graph of $y = 9x^2 - x^4$ is as shown. Sketch the graph of each of the following by applying suitable transformations:



a $y = 9(x - 1)^2 - (x - 1)^4$

b $y = 18x^2 - 2x^4$

c $y = 18(x + 1)^2 - 2(x + 1)^4$

d $y = 9x^2 - x^4 - \frac{81}{4}$

e $y = 9x^2 - x^4 + 1$

(Do not find the x-axis intercepts for part **e**.)

SF

5 Sketch the graph of $f(x) = x^6 - x^2$. (Use a calculator to find the stationary points.)

6 Sketch the graph of $f(x) = x^5 - x^3$. (Use a calculator to find the stationary points.)

3G Determining the rule for the graph of a polynomial

A straight line is determined by any two points on the line. More generally, the graph of a polynomial function of degree n is completely determined by any $n + 1$ points on the curve.

For example, for a cubic function with rule $y = f(x)$, if it is known that $f(a_1) = b_1$, $f(a_2) = b_2$, $f(a_3) = b_3$ and $f(a_4) = b_4$, then the rule can be determined.

Finding the rule for a parabola has been discussed in Section 3B.

The method for finding the rule from a graph of a cubic function will depend on what information is given in the graph.

If the cubic function has rule of the form $f(x) = a(x - h)^3 + k$ and the point of inflection (h, k) is given, then one other point needs to be known in order to find the value of a.

For those that are not of this form, the information given may be some or all of the x-axis intercepts as well as the coordinates of other points including possibly the y-axis intercept.

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Example 31

a A cubic function has rule of the form $y = a(x-2)^3 + 2$. The point $(3, 10)$ is on the graph of the function. Find the value of a.

b A cubic function has rule of the form $y = a(x-1)(x+2)(x-4)$. The point $(5, 16)$ is on the graph of the function. Find the value of a.

c A cubic function has rule of the form $f(x) = ax^3 + bx$. The points $(1, 16)$ and $(2, 30)$ are on the graph of the function. Find the values of a and b.

Solution

a $y = a(x-2)^3 + 2$

When $x = 3$, $y = 10$. Solve for a:

$$10 = a(3-2)^3 + 2$$
$$8 = a \times 1^3$$
$$\therefore \quad a = 8$$

b $y = a(x-1)(x+2)(x-4)$

When $x = 5$, $y = 16$ and so

$$16 = a(5-1)(5+2)(5-4)$$
$$16 = 28a$$
$$\therefore \quad a = \frac{4}{7}$$

c $f(x) = ax^3 + bx$

We know $f(1) = 16$ and $f(2) = 30$:

$$16 = a + b \qquad (1)$$
$$30 = a(2)^3 + 2b \qquad (2)$$

Multiply (1) by 2 and subtract from (2):

$$-2 = 6a$$
$$\therefore \quad a = -\frac{1}{3}$$

Substitute in (1):

$$16 = -\frac{1}{3} + b$$
$$\therefore \quad b = \frac{49}{3}$$

Explanation

In each of these problems, we substitute the given values to find the unknowns.

a The coordinates of the point of inflection of a graph which is a translation of $y = ax^3$ are known and the coordinates of one further point are known.

b Three x-axis intercepts are known and the coordinates of a fourth point are known.

c Form simultaneous equations in a and b.

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Example 32

For the cubic function with rule $f(x) = ax^3 + bx^2 + cx + d$, it is known that the points with coordinates $(-1, -18)$, $(0, -5)$, $(1, -4)$ and $(2, -9)$ lie on the graph. Find the values of a, b, c and d.

Solution

The following equations can be formed:

$$-a + b - c + d = -18 \qquad (1)$$
$$d = -5 \qquad (2)$$
$$a + b + c + d = -4 \qquad (3)$$
$$8a + 4b + 2c + d = -9 \qquad (4)$$

Adding (1) and (3) gives

$$2b + 2d = -22$$

Since $d = -5$, we obtain $b = -6$.

There are now only two unknowns.

Equations (3) and (4) become:

$$a + c = 7 \qquad (3')$$
$$8a + 2c = 20 \qquad (4')$$

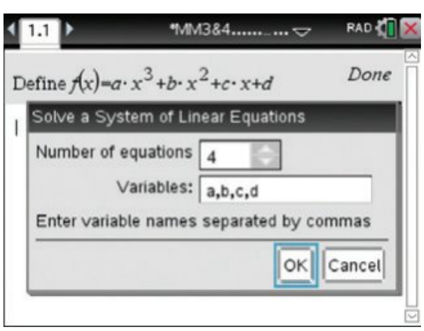
Multiply (3′) by 2 and subtract from (4′) to obtain

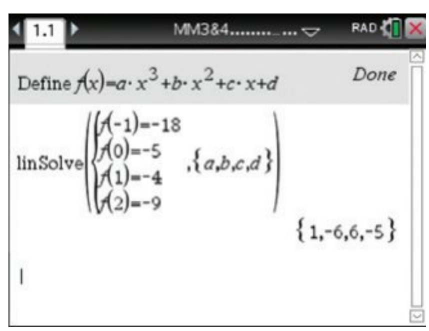
$$6a = 6$$

Thus $a = 1$ and $c = 6$.

Using the TI-Nspire CX non-CAS

- Define $f(x) = ax^3 + bx^2 + cx + d$.
- Use the simultaneous equations template (menu > **Algebra** > **Solve System of Linear Equations**) to solve for a, b, c, d given that $f(-1) = -18$, $f(0) = -5$, $f(1) = -4$ and $f(2) = -9$.





- Hence $a = 1$, $b = -6$, $c = 6$ and $d = -5$.

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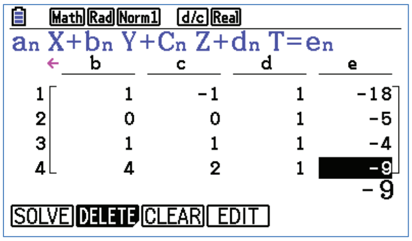
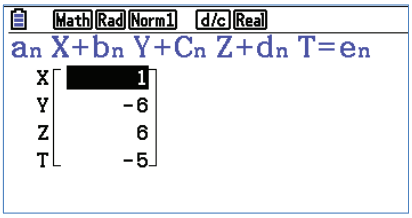
Using the Casio

- Select **Equation** mode (MENU) (ALPHA) (X,θ,T).
- Select **Simultaneous** (F1), then select four unknowns (F3).
- Enter the coefficients of the four equations in the table as shown:

$$-a + b - c + d = -18$$
$$d = -5$$
$$a + b + c + d = -4$$
$$8a + 4b + 2c + d = -9$$

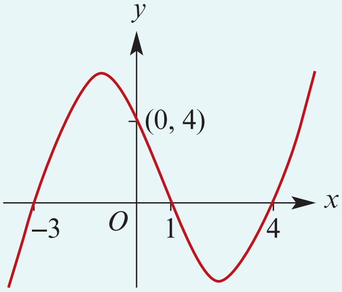
- Select **Solve** (F1).
- Hence $a = 1$, $b = -6$, $c = 6$ and $d = -5$.





Example 33

The graph shown is that of a cubic function. Find the rule for this cubic function.



Solution

From the graph, the function has a rule of the form

$$y = a(x - 4)(x - 1)(x + 3)$$

The point $(0, 4)$ is on the graph. Hence

$$4 = a(-4)(-1)3$$
$$4 = 12a$$
$$\therefore \quad a = \frac{1}{3}$$

The rule for the function is

$$y = \frac{1}{3}(x - 4)(x - 1)(x + 3)$$

Explanation

The x-axis intercepts are -3, 1 and 4.

So $x + 3$, $x - 1$ and $x - 4$ are linear factors.

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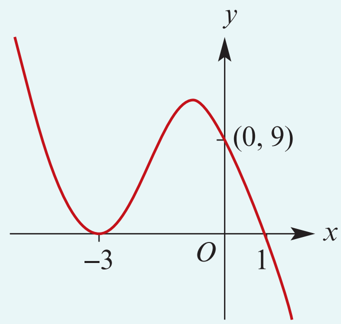
Example 34

The graph shown is that of a cubic function. Find the rule for this cubic function.



Solution

From the graph, the function is of the form

$$y = k(x - 1)(x + 3)^2$$

The point (0, 9) is on the graph. Hence

$$9 = k(-1)(9)$$

$$\therefore \quad k = -1$$

The rule is $y = -(x - 1)(x + 3)^2$.

Explanation

The graph touches the x-axis at $x = -3$. Therefore $x + 3$ is a repeated factor.

Exercise 3G

Skillsheet

Example 31

SF

1 **a** A cubic function has rule of the form $y = a(x - 5)^3 - 2$. The point (4, 0) is on the graph of the function. Find the value of a.

b A cubic function has rule of the form $y = a(x - 1)(x + 1)(x + 2)$. The point (3, 120) is on the graph of the function. Find the value of a.

c A cubic function has rule of the form $f(x) = ax^3 + bx$. The points (2, −20) and (−1, 20) are on the graph of the function. Find the values of a and b.

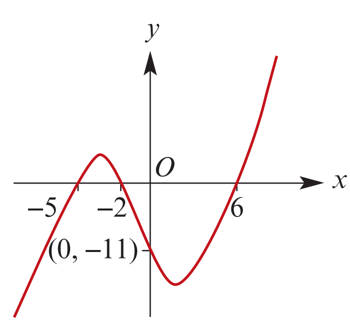
Example 32

CF

2 For the cubic function with rule $f(x) = ax^3 + bx^2 + cx + d$, it is known that the points with coordinates (−1, 14), (0, 5), (1, 0) and (2, −19) lie on the graph of the cubic. Find the values of a, b, c and d.

Example 33

3 Determine the rule for the cubic function with the graph shown.



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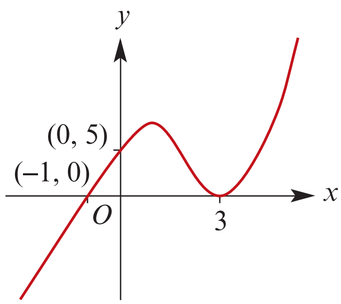
Example 34

4 Determine the rule for the cubic function with the graph shown.

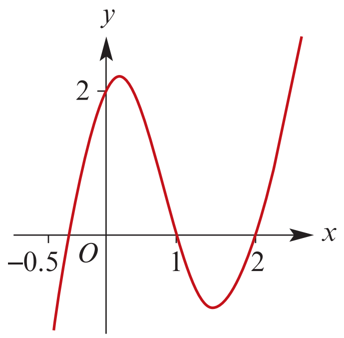


5 Find the rule for the cubic function that passes through the following points:

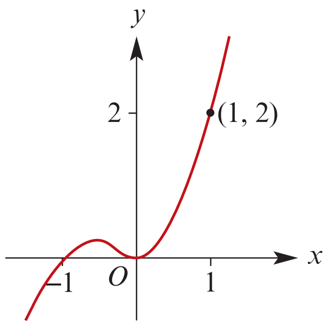
a $(0, 1), (1, 3), (-1, -1)$ and $(2, 11)$
b $(0, 1), (1, 1), (-1, 1)$ and $(2, 7)$
c $(0, -2), (1, 0), (-1, -6)$ and $(2, 12)$

6 Find expressions which define the following cubic curves:

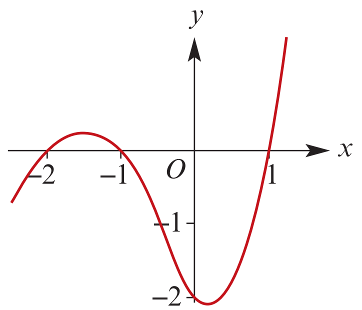
a



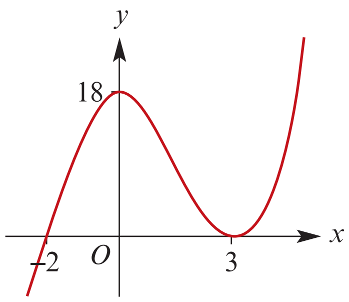
b



c



d



7 Find the rule of the cubic function for which the graph passes through the points with coordinates:

a $(0, 135), (1, 156), (2, 115), (3, 0)$
b $(-2, -203), (0, 13), (1, 25), (2, -11)$

8 Find the rule of the quartic function for which the graph passes through the points with coordinates:

a $(-1, 43), (0, 40), (2, 70), (6, 1618), (10, 670)$
b $(-3, 119), (-2, 32), (-1, 9), (0, 8), (1, 11)$
c $(-3, 6), (-1, 2), (1, 2), (3, 66), (6, 1227)$

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3H Solution of literal equations and systems of equations

▶ Literal equations

A literal equation in x is an equation whose solution will be expressed in terms of pronumerals rather than numbers.

For the equation $2x + 5 = 7$, the solution is $x = 1$.

For the literal equation $ax + b = c$, the solution is $x = \dfrac{c - b}{a}$.

Example 35

Solve each of the following literal equations for x:

a $ax + b = cx + d$ **b** $x^2 + kx + k = 0$ **c** $x^3 - 3ax^2 + 2a^2x = 0$

Solution

a
$$ax + b = cx + d$$
$$ax - cx = d - b$$
$$x(a - c) = d - b$$
$$\therefore \quad x = \frac{d - b}{a - c}$$

b The quadratic formula gives

$$x = \frac{-k \pm \sqrt{k^2 - 4k}}{2}$$

A real solution exists only for $k^2 - 4k \geq 0$, that is, for $k \geq 4$ or $k \leq 0$.

c
$$x^3 - 3ax^2 + 2a^2x = 0$$
$$x(x^2 - 3ax + 2a^2) = 0$$
$$x(x - a)(x - 2a) = 0$$

Hence $x = 0$ or $x = a$ or $x = 2a$.

In the next example, we use the following two facts about power functions:

- If n is an odd natural number, then $b^n = a$ is equivalent to $b = a^{\frac{1}{n}}$.
- If n is an even natural number, then $b^n = a$ is equivalent to $b = \pm a^{\frac{1}{n}}$, where $a \geq 0$.

Note that care must be taken with even powers: for example, $x^2 = 2$ is equivalent to $x = \pm\sqrt{2}$.

Example 36

Solve each of the following equations for x:

a $ax^3 - b = c$ **b** $a(x + b)^3 = c$ **c** $x^4 = c$, where $c > 0$

d $ax^{\frac{1}{5}} = b$ **e** $x^5 - c = d$

Solution

a
$$ax^3 - b = c$$
$$ax^3 = b + c$$
$$x^3 = \frac{b + c}{a}$$
$$\therefore \quad x = \left(\frac{b + c}{a}\right)^{\frac{1}{3}}$$

b
$$a(x + b)^3 = c$$
$$(x + b)^3 = \frac{c}{a}$$
$$x + b = \left(\frac{c}{a}\right)^{\frac{1}{3}}$$
$$\therefore \quad x = \left(\frac{c}{a}\right)^{\frac{1}{3}} - b$$

c
$$x^4 = c$$
$$\therefore \quad x = \sqrt[4]{c} \text{ or } x = -\sqrt[4]{c}$$

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d $ax^{\frac{1}{5}} = b$

$x^{\frac{1}{5}} = \frac{b}{a}$

$\therefore \quad x = \left(\frac{b}{a}\right)^5$

e $x^5 - c = d$

$x^5 = c + d$

$\therefore \quad x = (c + d)^{\frac{1}{5}}$

Simultaneous equations

In Mathematical Methods Units 1 & 2, you have solved simultaneous linear equations to find the coordinates of the point of intersection of two straight lines. In this section, we look at methods for finding the coordinates of the points of intersection of different graphs.

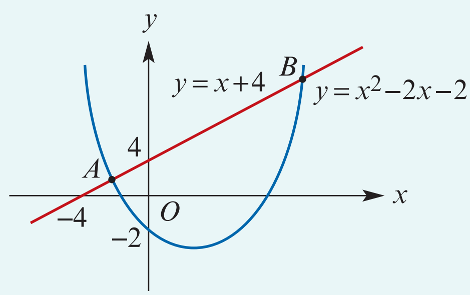
Example 37

Find the coordinates of the points of intersection of the parabola with equation $y = x^2 - 2x - 2$ and the straight line with equation $y = x + 4$.

Solution

Equate the two expressions for y:

$$x^2 - 2x - 2 = x + 4$$
$$x^2 - 3x - 6 = 0$$
$$\therefore \quad x = \frac{3 \pm \sqrt{9 - 4 \times (-6)}}{2}$$
$$= \frac{3 \pm \sqrt{33}}{2}$$



The points of intersection are $A\left(\frac{3 - \sqrt{33}}{2}, \frac{11 - \sqrt{33}}{2}\right)$ and $B\left(\frac{3 + \sqrt{33}}{2}, \frac{11 + \sqrt{33}}{2}\right)$.

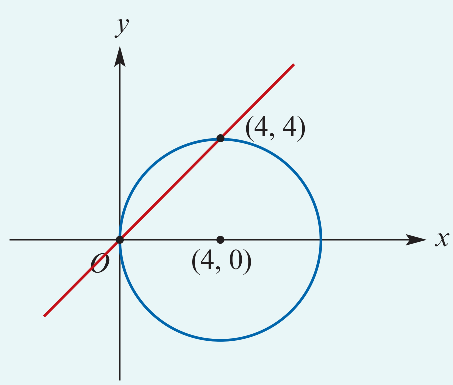
Example 38

Find the points of intersection of the circle with equation $(x - 4)^2 + y^2 = 16$ and the line with equation $x - y = 0$.

Solution

Rearrange $x - y = 0$ to make y the subject.
Substitute $y = x$ into the equation of the circle:

$$(x - 4)^2 + x^2 = 16$$
$$x^2 - 8x + 16 + x^2 = 16$$
$$2x^2 - 8x = 0$$
$$2x(x - 4) = 0$$
$$\therefore \quad x = 0 \text{ or } x = 4$$



The points of intersection are (0, 0) and (4, 4).

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Example 39

Find the point of contact of the line with equation $\frac{1}{9}x + y = \frac{2}{3}$ and the curve with equation $xy = 1$.

Solution

Rewrite the equations as $y = -\frac{1}{9}x + \frac{2}{3}$ and $y = \frac{1}{x}$.

Equate the expressions for y:

$$-\frac{1}{9}x + \frac{2}{3} = \frac{1}{x}$$

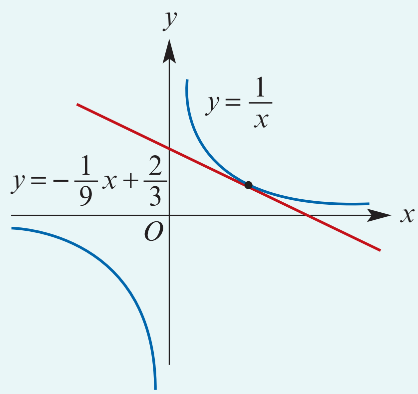
$$-x^2 + 6x = 9$$

$$x^2 - 6x + 9 = 0$$

$$(x-3)^2 = 0$$

$$\therefore \quad x = 3$$

The point of intersection is $(3, \frac{1}{3})$.



Exercise 3H

SF

Example 35 **1** Solve each of the following literal equations for x:

a $px = qx - 6$ **b** $mx - n = nx + m$ **c** $\frac{1}{x-a} = \frac{a}{x}$

d $kx^2 + x + k = 0$ **e** $x^3 - 7ax^2 + 12a^2x = 0$ **f** $x(x^3 - a) = 0$

g $x^2 - kx + k = 0$ **h** $x^3 - ax = 0$ **i** $x^4 - a^4 = 0$

j $(x-a)^5(x-b) = 0$ **k** $(a-x)^4(a-x^3)(x^2-a) = 0$

Example 36 **2** Solve each of the following equations for x:

a $ax^3 + b = 2c$ **b** $ax^2 - b = c$, where $a, b, c > 0$

c $a - bx^2 = c$, where $a > c$ and $b > 0$ **d** $x^{\frac{1}{3}} = a$

e $x^{\frac{1}{n}} + c = a$, where $n \in \mathbb{N}$ and $a > c$ **f** $a(x - 2b)^3 = c$

g $ax^{\frac{1}{3}} = b$ **h** $x^3 - c = d$

Example 37 **3** Find the coordinates of the points of intersection for each of the following:

a $y = x^2$, $y = x$ **b** $y - 2x^2 = 0$, $y - x = 0$ **c** $y = x^2 - x$, $y = 2x + 1$

Example 38 **4** Find the coordinates of the points of intersection for each of the following:

a $x^2 + y^2 = 178$, $x + y = 16$ **b** $x^2 + y^2 = 125$, $x + y = 15$ **c** $x^2 + y^2 = 185$, $x - y = 3$

d $x^2 + y^2 = 97$, $x + y = 13$ **e** $x^2 + y^2 = 106$, $x - y = 4$

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Example 39

5 Find the coordinates of the points of intersection for each of the following:

a $x + y = 28$
$xy = 187$

b $x + y = 51$
$xy = 518$

c $x - y = 5$
$xy = 126$

SF

6 Find the coordinates of the points of intersection of the straight line with equation $y = 2x$ and the circle with equation $(x - 5)^2 + y^2 = 25$.

7 Find the coordinates of the points of intersection of the curves with equations $y = \dfrac{1}{x-2} + 3$ and $y = x$.

8 Find the coordinates of the points of intersection of the line with equation $\dfrac{y}{4} - \dfrac{x}{5} = 1$ and the circle with equation $x^2 + 4x + y^2 = 12$.

9 Find the coordinates of the points of intersection of the curve $y = \dfrac{1}{x+2} - 3$ and the line $y = -x$.

10 Find the coordinates of the point where the line with equation $4y = 9x + 4$ touches the parabola with equation $y^2 = 9x$.

11 Find the coordinates of the points of intersection of the curve with equation $y = \dfrac{2}{x-2}$ and the line $y = x - 1$.

12 Solve the simultaneous equations:

a $5x - 4y = 7$ and $xy = 6$

b $2x + 3y = 37$ and $xy = 45$

c $5x - 3y = 18$ and $xy = 24$

CF

13 What is the condition for $x^2 + ax + b$ to be divisible by $x + c$?

14 Find the equations of the lines that pass through the point $(1, 7)$ and touch the parabola $y = -3x^2 + 5x + 2$.
Hint: Form a quadratic equation and consider when the discriminant Δ is zero.

15 Find the values of m for which the line $y = mx - 8$ intersects the parabola $y = x^2 - 5x + m$ twice.

16 The line $y = x + c$ meets the hyperbola $y = \dfrac{9}{2-x}$ once. Find the possible values of c.

17 **a** Solve the simultaneous equations $y = mx$ and $y = \dfrac{1}{x} + 5$ for x in terms of m.

b Find the value of m for which the graphs of $y = mx$ and $y = \dfrac{1}{x} + 5$ touch, and give the coordinates of this point.

c For which values of m do the graphs not meet?

18 Show that, if the line with equation $y = kx + b$ touches the curve $y = x^2 + x + 4$, then $k^2 - 2k + 4b - 15 = 0$. Hence find the equations of such lines that also pass through the point $(0, 3)$.

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Review

Chapter summary

AS Nrich

Quadratic polynomials

- Turning point form
  - By completing the square, all quadratic functions in polynomial form $y = ax^2 + bx + c$ may be transposed into turning point form $y = a(x - h)^2 + k$.
  - The graph of $y = a(x - h)^2 + k$ is a parabola congruent to the graph of $y = ax^2$. The vertex (or turning point) is the point (h, k). The axis of symmetry is $x = h$.
- Axis of symmetry
  The axis of symmetry of the graph of the quadratic function $y = ax^2 + bx + c$ is the line with equation $x = -\dfrac{b}{2a}$.
- Quadratic formula
  The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  From the formula it can be seen that:
  - If $b^2 - 4ac > 0$, there are two solutions.
  - If $b^2 - 4ac = 0$, there is one solution.
  - If $b^2 - 4ac < 0$, there are no real solutions.

  The quantity $\Delta = b^2 - 4ac$ is called the **discriminant** of the quadratic $ax^2 + bx + c$.

Polynomials in general

- A **polynomial function** is a function that can be written in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$.
  The **leading term** is $a_n x^n$ (the term of highest index) and the **constant term** is a_0 (the term not involving x).
- The **degree of a polynomial** is the index n of the leading term.
  - Polynomials of degree 1 are called **linear** functions.
  - Polynomials of degree 2 are called **quadratic** functions.
  - Polynomials of degree 3 are called **cubic** functions.
  - Polynomials of degree 4 are called **quartic** functions.
- The sum, difference and product of two polynomials is a polynomial. Division does not always lead to another polynomial.
- Two polynomials P and Q are equal only if their corresponding coefficients are equal. Two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

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- Division of polynomials

  When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

  $$P(x) = D(x)Q(x) + R(x)$$

  and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.

  Two methods for dividing polynomials are **long division** and **equating coefficients**.
- Remainder theorem

  When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.
- Factor theorem
  - If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
  - Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.
- A cubic polynomial can be factorised by using the factor theorem to find the first linear factor and then using polynomial division or the method of equating coefficients to complete the factorisation.
- Difference and sum of two cubes
  - $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
  - $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

Technology-free questions

SF

1 Sketch the graph of each of the following quadratic functions. Clearly indicate coordinates of the vertex and the axis intercepts.

a $h(x) = 3(x - 1)^2 + 2$ **b** $h(x) = (x - 1)^2 - 9$ **c** $f(x) = x^2 - x + 6$
d $f(x) = x^2 - x - 6$ **e** $f(x) = 2x^2 - x + 5$ **f** $h(x) = 2x^2 - x - 1$

2 The points with coordinates (1, 1) and (2, 5) lie on a parabola with equation of the form $y = ax^2 + b$. Find the values of a and b.

3 Solve the equation $3x^2 - 2x - 10 = 0$ by using the quadratic formula.

4 Sketch the graph of each of the following. State the coordinates of the point of zero gradient and the axis intercepts.

a $f(x) = 2(x - 1)^3 - 16$ **b** $g(x) = -(x + 1)^3 + 8$ **c** $h(x) = -(x + 2)^3 - 1$
d $f(x) = (x + 3)^3 - 1$ **e** $f(x) = 1 - (2x - 1)^3$

5 Express each of the following in turning point form:

a $x^2 + 4x$ **b** $3x^2 + 6x$ **c** $x^2 - 4x + 6$
d $2x^2 - 6x - 4$ **e** $2x^2 - 7x - 4$ **f** $-x^2 + 3x - 4$

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SF

6 Draw a sign diagram for each of the following:

a $y = (x + 2)(2 - x)(x + 1)$ **b** $y = (x - 3)(x + 1)(x - 1)$

c $y = x^3 + 7x^2 + 14x + 8$ **d** $y = 3x^3 + 10x^2 + x - 6$

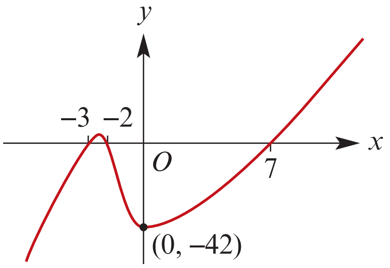
7 Without actually dividing, find the remainder when the first polynomial is divided by the second:

a $x^3 + 3x^2 - 4x + 2,\ \ x + 1$

b $x^3 - 3x^2 - x + 6,\ \ x - 2$

c $2x^3 + 3x^2 - 3x - 2,\ \ x + 2$

8 Determine the rule for the cubic function shown in the graph.



9 Factorise each of the following:

a $x^3 + 2x^2 - 5x - 6$ **b** $x^3 - 3x^2 - x + 3$

c $x^4 - x^3 - 7x^2 + x + 6$ **d** $x^3 + 2x^2 - 4x + 1$

10 Find the quotient and remainder when $x^2 + 4$ is divided by $x^2 - 2x + 2$.

11 Find the value of a for which $x - 2$ is a factor of $3x^3 + ax^2 + x - 2$.

CF

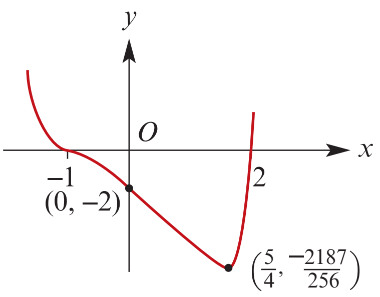
12 The graph of $f(x) = (x + 1)^3(x - 2)$ is shown. Sketch the graph of:

a $y = f(x - 1)$

b $y = f(x + 1)$

c $y = f(2x)$

d $y = f(x) + 2$



13 For what value of k is $2x^2 - kx + 8$ a perfect square?

SF

14 Find the coordinates of the points of intersection of the graph of $y = 2x + 3$ with the graph of $y = x^2 + 3x - 9$.

15 Find constants a, b and c such that $3x^2 - 5x + 1 = a(x + b)^2 + c$ holds for all values of x.

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Review SF CF

16 Expand $(3 + 4x)^3$.

17 Given that $x^3 - 2x^2 + 5 = ax(x - 1)^2 + b(x - 1) + c$ for all real numbers x, find the values of a, b and c.

18 Find the values of p for which the equation $4x^2 - 2px + p + 3 = 0$ has no real solutions.

19 Find the rule for the cubic function, the graph of which passes through the points $(1, 1)$, $(2, 4)$, $(3, 9)$ and $(0, 6)$.

Multiple-choice questions

1 By completing the square, the expression $5x^2 - 10x - 2$ can be written in turning point form $a(x - h)^2 + k$ as

A $(5x + 1)^2 + 5$ **B** $(5x - 1)^2 - 5$ **C** $5(x - 1)^2 - 5$
D $5(x + 1)^2 - 2$ **E** $5(x - 1)^2 - 7$

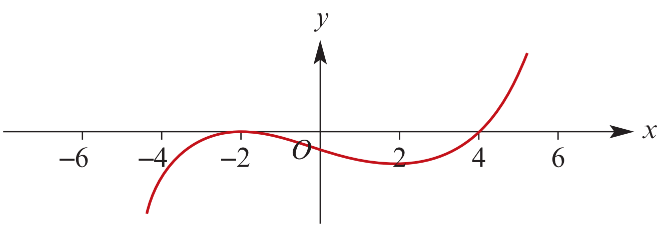
2 For which value(s) of m does the equation $mx^2 + 6x - 3 = 0$ have two real solutions?

A $m = -3$ **B** $m = 3$ **C** $m = 0$ **D** $m > -3$ **E** $m < -3$

3 $x^3 + 27$ is equal to

A $(x + 3)^3$ **B** $(x - 3)^3$ **C** $(x + 3)(x^2 - 6x + 9)$
D $(x - 3)(x^2 + 3x + 9)$ **E** $(x + 3)(x^2 - 3x + 9)$

4 The equation of the graph shown on the right is



A $y = x(x - 2)(x + 4)$
B $y = x(x + 2)(x - 4)$
C $y = (x + 2)^2(x - 4)$
D $y = (x + 2)(x - 4)^2$
E $y = (x + 2)^2(x - 4)^2$

5 If $x - 1$ is a factor of $x^3 + 3x^2 - 2ax + 1$, then the value of a is

A 2 **B** 5 **C** $\frac{2}{5}$ **D** $-\frac{2}{5}$ **E** $\frac{5}{2}$

6 $6x^2 - 8xy - 8y^2$ is equal to

A $(3x + 2y)(2x - 4y)$ **B** $(3x - 2y)(6x + 4y)$ **C** $(6x - 4y)(x + 2y)$
D $(3x - 2y)(2x + 4y)$ **E** $(6x + y)(x - 8y)$

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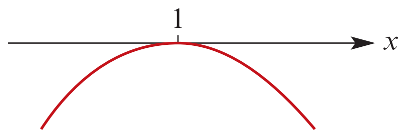
Review

7 The diagram shows a part of the graph of a cubic polynomial function f, near the point $(1, 0)$. Which of the following could be the rule for f?

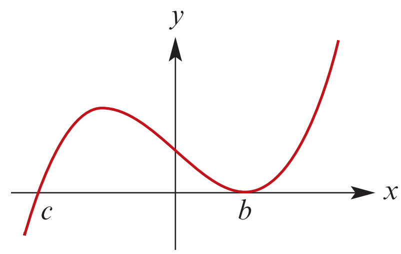


A $f(x) = x^2(x-1)$ **B** $f(x) = (x-1)^3$ **C** $f(x) = -x(x-1)^2$
D $f(x) = x(x-1)^2$ **E** $f(x) = -x(x+1)^2$

8 The coordinates of the turning point of the graph of the function $p(x) = 3((x-2)^2 + 4)$ are

A $(-2, 12)$ **B** $(-2, 4)$ **C** $(2, -12)$ **D** $(2, 4)$ **E** $(2, 12)$

9 The diagram shows part of the graph of a polynomial function. A possible equation for the graph is



A $y = (x+c)(x-b)^2$
B $y = (x-b)(x-c)^2$
C $y = (x-c)(b-x)^2$
D $y = -(x-c)(b-x)^2$
E $y = (x+b)^2(x-c)$

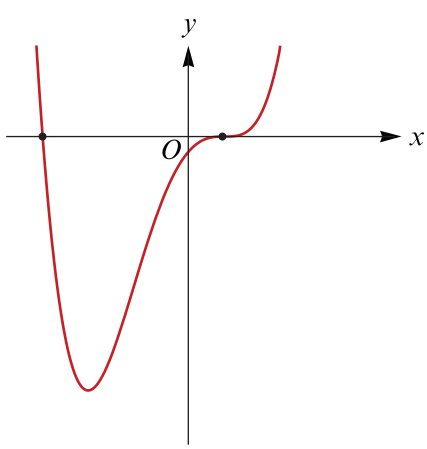
10 The number of solutions of the equation $(x^2 + a)(x-b)(x+c) = 0$, where $a, b, c \in \mathbb{R}^+$, is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

11 The graph of $y = kx - 3$ meets the graph of $y = -x^2 + 2x - 12$ at two distinct points for

A $k \in [-4, 8]$ **B** $k \in \{-4, -8\}$ **C** $k \in (-\infty, -4) \cup (8, \infty)$
D $k \in (-4, 8)$ **E** $k \in (-\infty, -8) \cup (4, \infty)$

12 The function f is a quartic polynomial. Its graph is shown on the right. It has x-axis intercepts at $(a, 0)$ and $(b, 0)$, where $a > 0$ and $b < 0$. A possible rule for this function is



A $f(x) = (x-a)^2(x+b)^2$
B $f(x) = (x-a)^3(x-b)$
C $f(x) = (x-a)(x-b)^2$
D $f(x) = (x+a)^2(x-b)^2$
E $f(x) = (x-b)^3(x-a)$

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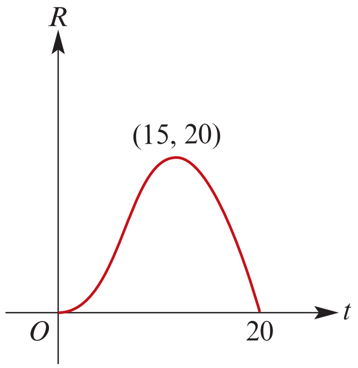
CU

Extended-response questions

1 The rate of flow of water, R mL/min, into a vessel is described by the quartic expression

$$R = kt^3(20 - t), \quad \text{for } 0 \le t \le 20$$

where t minutes is the time elapsed from the beginning of the flow. The graph is shown.



a Find the value of k.

b Find the rate of flow when $t = 10$.

c The flow is adjusted so that the new expression for the flow is

$$R_{new} = 2kt^3(20 - t), \quad \text{for } 0 \le t \le 20$$

i Sketch the graph of R_{new} against t for $0 \le t \le 20$.

ii Find the rate of flow when $t = 10$.

d Water is allowed to run from the vessel and it is found that the rate of flow from the vessel is given by

$$R_{out} = -k(t - 20)^3(40 - t), \quad \text{for } 20 \le t \le 40$$

i Sketch the graph of R_{out} against t for $20 \le t \le 40$.

ii Find the rate of flow when $t = 30$.

Hints: The graph of R_{new} against t is given by a dilation of factor 2 from the x-axis. The graph of R_{out} against t is given by the translation with rule $(t, R) \to (t + 20, R)$ followed by a reflection in the t-axis.

2 A large gas container is being deflated. The volume V (in m^3) at time t hours is given by

$$V = 4(9 - t)^3, \quad \text{for } 0 \le t \le 9$$

a Find the volume when:

i $t = 0$ **ii** $t = 9$

b Sketch the graph of V against t for $0 \le t \le 9$.

c At what time is the volume 512 m^3?

3 A hemispherical bowl of radius 6 cm contains water. The volume of water in the hemispherical bowl, where the depth of the water is x cm, is given by

$$V = \frac{1}{3}\pi x^2(18 - x) \text{ cm}^3$$

a Find the volume of water when:

i $x = 2$ **ii** $x = 3$ **iii** $x = 4$

b Find the volume when the hemispherical bowl is full.

c Sketch the graph of V against x.

d Find the depth of water when the volume is equal to $\frac{325\pi}{3}$ cm^3.

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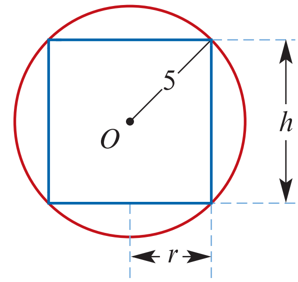
Review

4 A metal worker is required to cut a circular cylinder from a solid sphere of radius 5 cm. A cross-section of the sphere and the cylinder is shown in the diagram.



a Express r in terms of h, where r cm is the radius of the cylinder and h cm is the height of the cylinder. Hence show that the volume, V cm^3, of the cylinder is given by $V = \frac{1}{4}\pi h(100 - h^2)$.

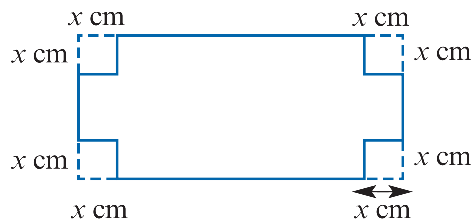
b Sketch the graph of V against h for $0 < h < 10$.

Hint: The coordinates of the maximum point are approximately $(5.77, 302.3)$.

c Find the volume of the cylinder if $h = 6$.

d Find the height and radius of the cylinder if the volume of the cylinder is 48π cm^3.

5 An open tank is to be made from a sheet of metal 84 cm by 40 cm by cutting congruent squares of side length x cm from each of the corners.



a Find the volume, V cm^3, of the box in terms of x.

b State the maximal domain for V when it is considered as a function of x.

c Plot the graph of V against x using a calculator.

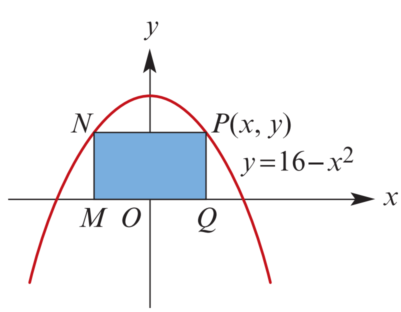
d Find the volume of the tank when:

i $x = 2$ **ii** $x = 6$ **iii** $x = 8$ **iv** $x = 10$

e Find the value(s) of x, correct to two decimal places, for which the capacity of the tank is 10 litres.

f Find, correct to two decimal places, the maximum capacity of the tank in cubic centimetres.

6 A rectangle is defined by vertices N and $P(x, y)$ on the curve with equation $y = 16 - x^2$ and vertices M and Q on the x-axis.



a **i** Find the area, A, of the rectangle in terms of x.

ii State the implied domain for the function defined by the rule given in part **i**.

b **i** Find the value of A when $x = 3$.

ii Find the value of x, correct to two decimal places, when $A = 25$.

c A cuboid has volume V given by the rule $V = xA$.

i Find V in terms of x.

ii Find the value of x, correct to two decimal places, such that $V = 100$.

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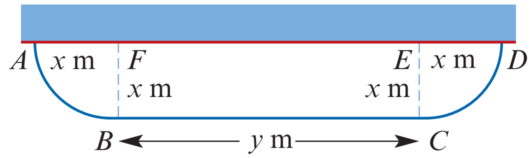
7 The plan of a garden adjoining a wall is shown. The rectangle $BCEF$ is of length y m and width x m. The borders of the two end sections are quarter circles of radius x m and centres at E and F.



A fence is erected along the curves AB and CD and the straight line BC.

a Find the area, A m^2, of the garden in terms of x and y.

b If the length of the fence is 100 m, find:

i y in terms of x

ii A in terms of x

iii the maximal domain of the function with the rule obtained in part **ii**.

c Find, correct to two decimal places, the value(s) of x if the area of the garden is to be 1000 m^2.

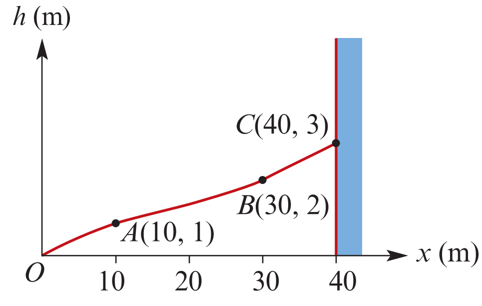
d It is decided to build the garden up to a height of $\frac{x}{50}$ metres. If the length of the fence is 100 m, find correct to two decimal places:

i the volume, V m^3, of soil needed in terms of x

ii the volume of soil needed for a garden of area 1000 m^2

iii the value(s) of x for which 500 m^3 of soil is required.

8 A mound of earth is piled up against a wall. The cross-section is as shown. The coordinates of several points on the surface are given.



a Find the rule of the cubic function for which the graph passes through the points O, A, B and C.

b For what value of x is the height of the mound 1.5 metres?

c The coefficient of x^3 for the function is 'small'. Consider the quadratic formed when the x^3 term is deleted. Compare the graph of the resulting quadratic function with the graph of the cubic function.

d The mound moves and the curve describing the cross-section now passes through the points $O(0, 0)$, $A(10, 0.3)$, $B(30, 2.7)$ and $D(40, 2.8)$. Find the rule of the cubic function for which the graph passes through these points.

e Let $y = f(x)$ be the function obtained in part **a**.

i Sketch the graph of the piecewise-defined function

$$g(x) = \begin{cases} f(x) & \text{for } 0 \le x \le 40 \\ f(80 - x) & \text{for } 40 < x \le 80 \end{cases}$$

ii Comment on the appearance of the graph of $y = g(x)$.

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Chapter 4

4 Trigonometric functions

Objectives

- To measure angles in **degrees** and **radians**.
- To define the trigonometric functions **sine**, **cosine** and **tangent**.
- To explore the **symmetry properties** of trigonometric functions.
- To find **exact values** of trigonometric functions.
- To **sketch graphs** of trigonometric functions.
- To **solve equations** involving trigonometric functions.
- To apply trigonometric functions in modelling **periodic motion**.

Following on from our study of polynomial functions, we meet a further three important functions in this chapter. Again we use the notation developed in Chapter 1 for describing functions and their properties.

In this chapter we revise and extend our consideration of the functions sine, cosine and tangent. The first two of these functions have the real numbers as their domain, and the third the real numbers without the odd multiples of $\frac{\pi}{2}$.

An important property of these three functions is that they are periodic. That is, they each repeat their values in regular intervals or periods. In general, a function f is **periodic** if there is a positive constant a such that $f(x + a) = f(x)$. The sine and cosine functions each have period 2π, while the tangent function has period π.

The sine and cosine functions are used to model wave motion, and are therefore central to the application of mathematics to any problem in which periodic motion is involved – from the motion of the tides and ocean waves to sound waves and modern telecommunications.

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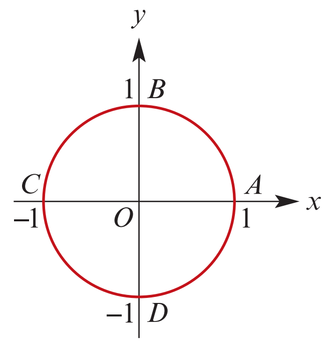
4A Measuring angles in degrees and radians

The diagram shows a **unit circle**, i.e. a circle of radius 1 unit.



The circumference of the unit circle $= 2\pi \times 1$

$= 2\pi$ units

Thus, the distance in an anticlockwise direction around the circle from

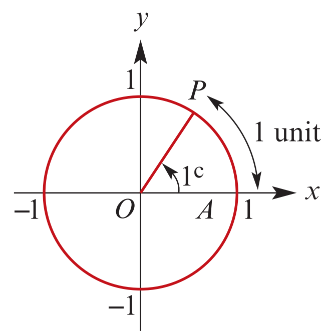
$$A \text{ to } B = \frac{\pi}{2} \text{ units}$$

$$A \text{ to } C = \pi \text{ units}$$

$$A \text{ to } D = \frac{3\pi}{2} \text{ units}$$

▶ Definition of a radian

In moving around the circle a distance of 1 unit from A to P, the angle POA is defined. The measure of this angle is 1 radian.



> One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.

▶ Degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$$2\pi^c = 360^\circ$$

$$\therefore \quad \pi^c = 180^\circ$$

$$\therefore \quad 1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

Example 1

Convert 30° to radians.

Solution	Explanation
$1^\circ = \dfrac{\pi^c}{180}$	
$\therefore \quad 30^\circ = \dfrac{30 \times \pi}{180} = \dfrac{\pi^c}{6}$	Multiply by $\dfrac{\pi}{180}$ and simplify by cancelling.

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Example 2

Convert $\frac{\pi^c}{4}$ to degrees.

Solution

$1^c = \frac{180°}{\pi}$

$\therefore \quad \frac{\pi^c}{4} = \frac{\pi \times 180}{4 \times \pi} = 45°$

Explanation

Multiply by $\frac{180}{\pi}$ and simplify by cancelling.

Note: Often the symbol for radians, ^{c}, is omitted.
For example, the angle 45° is written as $\frac{\pi}{4}$ rather than $\frac{\pi^c}{4}$.

Section summary

- One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.
- To convert:
  - degrees to radians, multiply by $\frac{\pi}{180}$
  - radians to degrees, multiply by $\frac{180}{\pi}$.

Exercise 4A

SF

Example 1 **1** Express the following angles in radian measure in terms of π:

a 50° **b** 136° **c** 250°
d 340° **e** 420° **f** 490°

Example 2 **2** Express, in degrees, the angles with the following radian measures:

a $\frac{\pi}{3}$ **b** $\frac{5\pi}{6}$ **c** $\frac{4\pi}{3}$
d $\frac{7\pi}{9}$ **e** 3.5π **f** $\frac{7\pi}{5}$

3 Use a calculator to convert each of the following angles from radians to degrees:

a 0.8 **b** 1.64 **c** 2.5
d 3.96 **e** 4.18 **f** 5.95

4 Use a calculator to express each of the following in radian measure. (Give your answer correct to two decimal places.)

a 37° **b** 74° **c** 115°
d 122.25° **e** 340° **f** 132.5°

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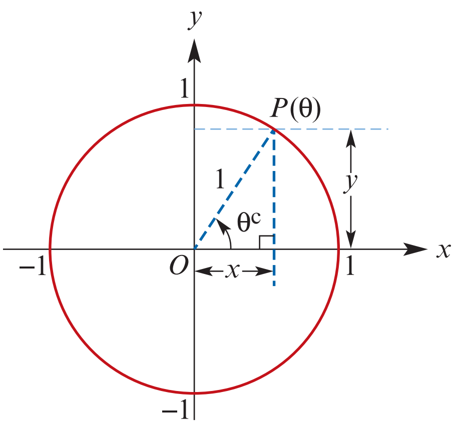
4B Defining sine, cosine and tangent

The point P on the unit circle corresponding to an angle θ is written $P(\theta)$.

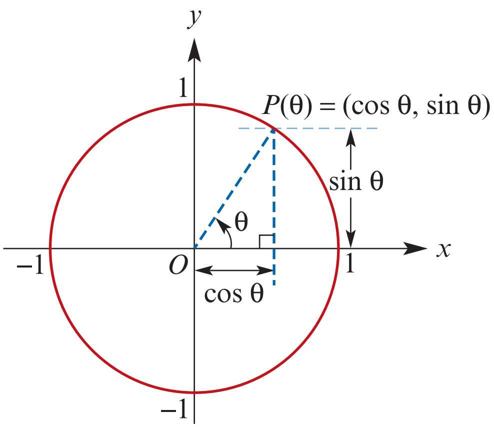


The x-coordinate of $P(\theta)$ is determined by the angle θ. Similarly, the y-coordinate of $P(\theta)$ is determined by the angle θ. So we can define two functions, called sine and cosine, as follows:

> The x-coordinate of $P(\theta)$ is given by
>
> $x = \text{cosine}\,\theta, \quad \text{for } \theta \in \mathbb{R}$
>
> The y-coordinate of $P(\theta)$ is given by
>
> $y = \text{sine}\,\theta, \quad \text{for } \theta \in \mathbb{R}$

These functions are usually written in an abbreviated form as follows:

$$x = \cos\theta$$
$$y = \sin\theta$$

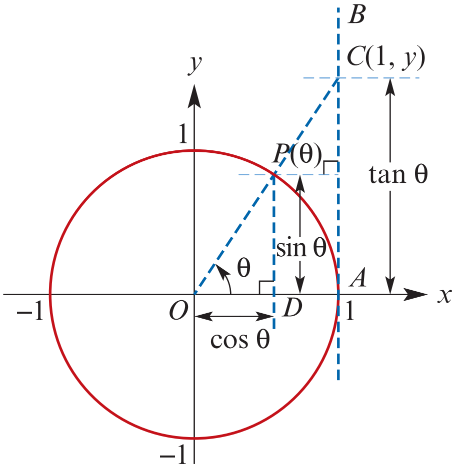


Hence the coordinates of $P(\theta)$ are $(\cos\theta, \sin\theta)$.

Note: Adding 2π to the angle results in a return to the same point on the unit circle. Thus $\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$.

Again consider the unit circle.

If we draw a tangent to the unit circle at A, then the y-coordinate of C, the point of intersection of the line OP and the tangent, is called **tangent** θ (abbreviated to $\tan\theta$).



By considering the similar triangles OPD and OCA:

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Note that $\tan\theta$ is undefined when $\cos\theta = 0$.

Hence $\tan\theta$ is undefined when $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$

Note: Adding π to the angle does not change the line OP. Thus $\tan(\pi + \theta) = \tan\theta$.

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From the periodicity of the trigonometric functions:

- $\sin(2k\pi + \theta) = \sin\theta$, for all integers k
- $\cos(2k\pi + \theta) = \cos\theta$, for all integers k
- $\tan(k\pi + \theta) = \tan\theta$, for all integers k.

Example 3

Evaluate each of the following:

a $\sin\left(\frac{3\pi}{2}\right)$ **b** $\sin\left(-\frac{3\pi}{2}\right)$ **c** $\cos\left(\frac{5\pi}{2}\right)$ **d** $\cos\left(-\frac{\pi}{2}\right)$

e $\cos\left(\frac{23\pi}{2}\right)$ **f** $\sin\left(\frac{55\pi}{2}\right)$ **g** $\tan(55\pi)$ **h** $\tan\left(\frac{15\pi}{2}\right)$

Solution

a $\sin\left(\frac{3\pi}{2}\right) = -1$

Explanation: since $P\left(\frac{3\pi}{2}\right)$ has coordinates $(0, -1)$.

b $\sin\left(-\frac{3\pi}{2}\right) = 1$

Explanation: since $P\left(-\frac{3\pi}{2}\right)$ has coordinates $(0, 1)$.

c $\cos\left(\frac{5\pi}{2}\right) = \cos\left(2\pi + \frac{\pi}{2}\right) = 0$

Explanation: since $P\left(\frac{\pi}{2}\right)$ has coordinates $(0, 1)$.

d $\cos\left(-\frac{\pi}{2}\right) = 0$

Explanation: since $P\left(-\frac{\pi}{2}\right)$ has coordinates $(0, -1)$.

e $\cos\left(\frac{23\pi}{2}\right) = \cos\left(10\pi + \frac{3\pi}{2}\right) = 0$

Explanation: since $P\left(\frac{3\pi}{2}\right)$ has coordinates $(0, -1)$.

f $\sin\left(\frac{55\pi}{2}\right) = \sin\left(26\pi + \frac{3\pi}{2}\right) = -1$

Explanation: since $P\left(\frac{3\pi}{2}\right)$ has coordinates $(0, -1)$.

g $\tan(55\pi) = 0$

Explanation: since $\tan(k\pi) = 0$, for any integer k.

h $\tan\left(\frac{15\pi}{2}\right)$ is undefined

Explanation: since $\tan\left(\frac{k\pi}{2}\right)$ is undefined for any odd integer k.

Example 4

Evaluate using a calculator. (Give answers to two decimal places.)

a $\tan 1.3$ **b** $\sin 1.8$ **c** $\cos(-2.6)$ **d** $\sin 3.8$

e $\tan(-2.8)$ **f** $\tan 59°$ **g** $\tan 138°$

Solution

a $\tan 1.3 = 3.60$ **b** $\sin 1.8 = 0.97$

c $\cos(-2.6) = -0.86$ **d** $\sin 3.8 = -0.61$

e $\tan(-2.8) = 0.36$ **f** $\tan 59° = 1.66$

g $\tan 138° = -0.90$

Explanation

Your calculator should be in radian mode for **a**–**e** and in degree mode for **f** and **g**.

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▶ Exact values of trigonometric functions

A calculator can be used to find the values of the trigonometric functions for different values of θ. For many values of θ the calculator gives an approximation. We consider some values of θ such that sin, cos and tan can be calculated exactly.

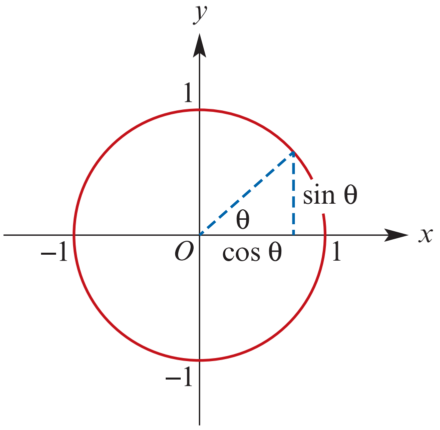
Exact values for 0 (0°) and $\frac{\pi}{2}$ (90°)

From the unit circle:

$\sin 0^\circ = 0$ $\quad$ $\sin 90^\circ = 1$

$\cos 0^\circ = 1$ $\quad$ $\cos 90^\circ = 0$

$\tan 0^\circ = 0$ $\quad$ $\tan 90^\circ$ is undefined



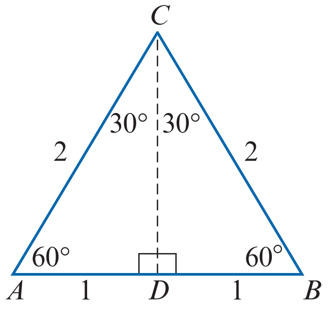
Exact values for $\frac{\pi}{6}$ (30°) and $\frac{\pi}{3}$ (60°)

Consider an equilateral triangle ABC of side length 2 units.
In $\triangle ACD$, by Pythagoras' theorem, $CD = \sqrt{AC^2 - AD^2} = \sqrt{3}$.

$\sin 30^\circ = \frac{AD}{AC} = \frac{1}{2}$ $\quad$ $\sin 60^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2}$

$\cos 30^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2}$ $\quad$ $\cos 60^\circ = \frac{AD}{AC} = \frac{1}{2}$

$\tan 30^\circ = \frac{AD}{CD} = \frac{1}{\sqrt{3}}$ $\quad$ $\tan 60^\circ = \frac{CD}{AD} = \frac{\sqrt{3}}{1} = \sqrt{3}$



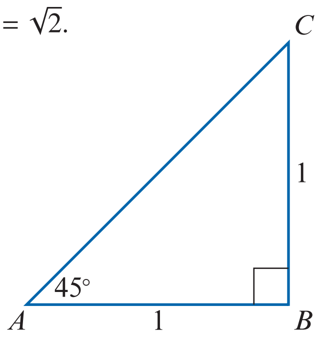
Exact values for $\frac{\pi}{4}$ (45°)

For the triangle ABC shown on the right, we have $AC = \sqrt{1^2 + 1^2} = \sqrt{2}$.

$\sin 45^\circ = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$

$\cos 45^\circ = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$

$\tan 45^\circ = \frac{BC}{AB} = 1$



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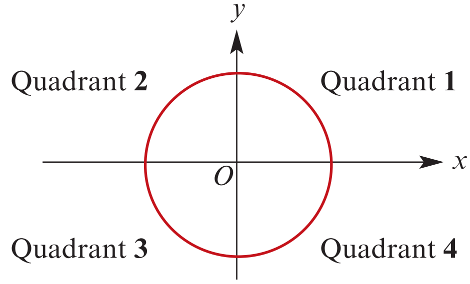
▶ Symmetry properties of trigonometric functions

The coordinate axes divide the unit circle into four quadrants. The quadrants can be numbered, anticlockwise from the positive direction of the x-axis, as shown.



Using symmetry, we can determine relationships between the trigonometric functions for angles in different quadrants:

Quadrant 2

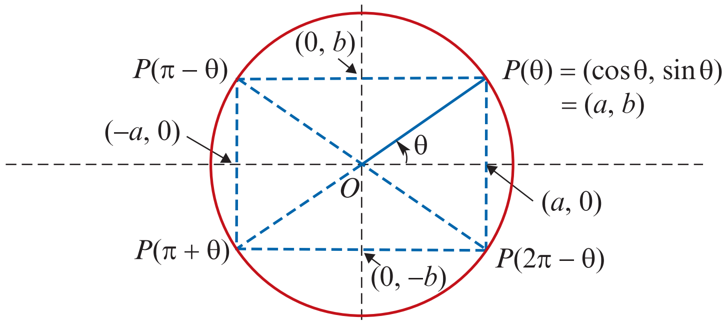
By symmetry:

$\sin(\pi - \theta) = b = \sin\theta$

$\cos(\pi - \theta) = -a = -\cos\theta$

$\tan(\pi - \theta) = \dfrac{b}{-a} = -\tan\theta$

Quadrant 1



Quadrant 3

$\sin(\pi + \theta) = -b = -\sin\theta$

$\cos(\pi + \theta) = -a = -\cos\theta$

$\tan(\pi + \theta) = \dfrac{-b}{-a} = \tan\theta$

Quadrant 4

$\sin(2\pi - \theta) = -b = -\sin\theta$

$\cos(2\pi - \theta) = a = \cos\theta$

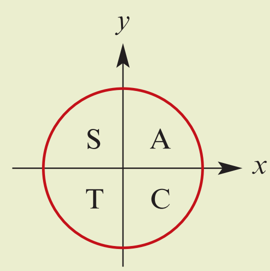
$\tan(2\pi - \theta) = \dfrac{-b}{a} = -\tan\theta$

Note: These relationships are true for all values of θ.

Signs of trigonometric functions

Using the symmetry properties, the signs of sin, cos and tan for the four quadrants can be summarised as follows:

1st quadrant	all are positive	(A)
2nd quadrant	sin is positive	(S)
3rd quadrant	tan is positive	(T)
4th quadrant	cos is positive	(C)



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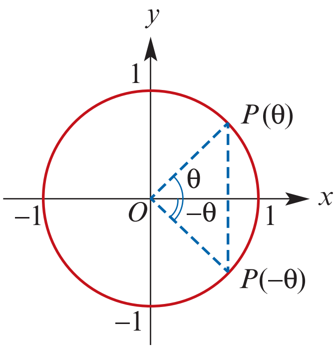
Negative of angles

By symmetry:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



Therefore:

- sin is an odd function
- cos is an even function
- tan is an odd function.

Example 5

Evaluate:

a $\cos\left(\frac{5\pi}{4}\right)$ **b** $\sin\left(\frac{11\pi}{6}\right)$ **c** $\cos\left(\frac{200\pi}{3}\right)$ **d** $\tan\left(\frac{52\pi}{6}\right)$

Solution

a $\cos\left(\frac{5\pi}{4}\right)$

$= \cos\left(\pi + \frac{\pi}{4}\right)$

$= -\cos\left(\frac{\pi}{4}\right)$

$= -\frac{1}{\sqrt{2}}$

b $\sin\left(\frac{11\pi}{6}\right)$

$= \sin\left(2\pi - \frac{\pi}{6}\right)$

$= -\sin\left(\frac{\pi}{6}\right)$

$= -\frac{1}{2}$

c $\cos\left(\frac{200\pi}{3}\right)$

$= \cos\left(66\pi + \frac{2\pi}{3}\right)$

$= \cos\left(\frac{2\pi}{3}\right)$

$= \cos\left(\pi - \frac{\pi}{3}\right)$

$= -\cos\left(\frac{\pi}{3}\right)$

$= -\frac{1}{2}$

d $\tan\left(\frac{52\pi}{6}\right)$

$= \tan\left(8\pi + \frac{2\pi}{3}\right)$

$= \tan\left(\frac{2\pi}{3}\right)$

$= \tan\left(\pi - \frac{\pi}{3}\right)$

$= -\tan\left(\frac{\pi}{3}\right)$

$= -\sqrt{3}$

Example 6

If $\sin x = 0.6$, find the value of:

a $\sin(\pi - x)$ **b** $\sin(\pi + x)$ **c** $\sin(2\pi - x)$ **d** $\sin(-x)$

Solution

a $\sin(\pi - x)$

$= \sin x$

$= 0.6$

b $\sin(\pi + x)$

$= -\sin x$

$= -0.6$

c $\sin(2\pi - x)$

$= -\sin x$

$= -0.6$

d $\sin(-x)$

$= -\sin x$

$= -0.6$

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Example 7

If $\cos x^\circ = 0.8$, find the value of:

a $\cos(180 - x)^\circ$ **b** $\cos(180 + x)^\circ$ **c** $\cos(360 - x)^\circ$ **d** $\cos(-x)^\circ$

Solution

a $\cos(180 - x)^\circ$
$= -\cos x^\circ$
$= -0.8$

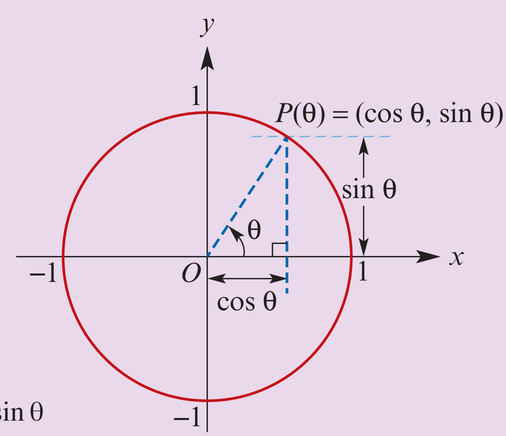
b $\cos(180 + x)^\circ$
$= -\cos x^\circ$
$= -0.8$

c $\cos(360 - x)^\circ$
$= \cos x^\circ$
$= 0.8$

d $\cos(-x)^\circ$
$= \cos x^\circ$
$= 0.8$

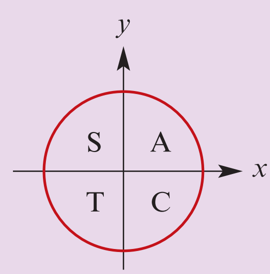
Section summary

- $P(\theta) = (\cos\theta, \sin\theta)$
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$
- The trigonometric functions are periodic:
  - $\sin(2\pi + \theta) = \sin\theta$
  - $\cos(2\pi + \theta) = \cos\theta$
  - $\tan(\pi + \theta) = \tan\theta$



- Negative of angles:
  - sin is an odd function, i.e. $\sin(-\theta) = -\sin\theta$
  - cos is an even function, i.e. $\cos(-\theta) = \cos\theta$
  - tan is an odd function, i.e. $\tan(-\theta) = -\tan\theta$
- Memory aids:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	undefined



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Exercise 4B

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Example 3

1 Evaluate each of the following:

a $\sin(3\pi)$ **b** $\cos\left(-\frac{5\pi}{2}\right)$ **c** $\sin\left(\frac{7\pi}{2}\right)$ **d** $\cos(3\pi)$

e $\sin(-4\pi)$ **f** $\tan(-\pi)$ **g** $\tan(2\pi)$ **h** $\tan(-2\pi)$

i $\cos(23\pi)$ **j** $\cos\left(\frac{49\pi}{2}\right)$ **k** $\cos(35\pi)$ **l** $\cos\left(\frac{-45\pi}{2}\right)$

m $\tan(24\pi)$ **n** $\cos(20\pi)$

Example 4

2 Evaluate each of the following using a calculator. (Give answers correct to two decimal places.)

a $\sin 1.7$ **b** $\sin 2.6$ **c** $\sin 4.2$ **d** $\cos 0.4$

e $\cos 2.3$ **f** $\cos(-1.8)$ **g** $\sin(-1.7)$ **h** $\sin(-3.6)$

i $\tan 1.6$ **j** $\tan(-1.2)$ **k** $\tan 3.9$ **l** $\tan(-2.5)$

Example 5

3 Write down the exact values of:

a $\sin\left(\frac{3\pi}{4}\right)$ **b** $\cos\left(\frac{2\pi}{3}\right)$ **c** $\cos\left(\frac{7\pi}{6}\right)$ **d** $\sin\left(\frac{5\pi}{6}\right)$

e $\cos\left(\frac{4\pi}{3}\right)$ **f** $\sin\left(\frac{5\pi}{4}\right)$ **g** $\sin\left(\frac{7\pi}{4}\right)$ **h** $\cos\left(\frac{5\pi}{3}\right)$

i $\cos\left(\frac{11\pi}{3}\right)$ **j** $\sin\left(\frac{200\pi}{3}\right)$ **k** $\cos\left(-\frac{11\pi}{3}\right)$ **l** $\sin\left(\frac{25\pi}{3}\right)$

m $\sin\left(-\frac{13\pi}{4}\right)$ **n** $\cos\left(-\frac{20\pi}{3}\right)$ **o** $\sin\left(\frac{67\pi}{4}\right)$ **p** $\cos\left(\frac{68\pi}{3}\right)$

q $\tan\left(\frac{11\pi}{3}\right)$ **r** $\tan\left(\frac{200\pi}{3}\right)$ **s** $\tan\left(-\frac{11\pi}{6}\right)$ **t** $\tan\left(\frac{25\pi}{3}\right)$

u $\tan\left(-\frac{13\pi}{4}\right)$ **v** $\tan\left(-\frac{25\pi}{6}\right)$

Example 6

4 If $\sin\theta = 0.52$, $\cos x = 0.68$ and $\tan\alpha = 0.4$, find the value of:

a $\sin(\pi - \theta)$ **b** $\cos(\pi + x)$ **c** $\sin(2\pi + \theta)$ **d** $\tan(\pi + \alpha)$

e $\sin(\pi + \theta)$ **f** $\cos(2\pi - x)$ **g** $\tan(2\pi - \alpha)$ **h** $\cos(\pi - x)$

i $\sin(-\theta)$ **j** $\cos(-x)$ **k** $\tan(-\alpha)$

5 If $\sin\theta = 0.4$, $\cos x = 0.7$ and $\tan\alpha = 1.2$, find the value of:

a $\sin(\pi - \theta)$ **b** $\cos(\pi + x)$ **c** $\sin(2\pi + \theta)$ **d** $\tan(\pi + \alpha)$

e $\sin(\pi + \theta)$ **f** $\cos(2\pi - x)$ **g** $\tan(2\pi - \alpha)$ **h** $\cos(\pi - x)$

i $\sin(-\theta)$ **j** $\cos(-x)$ **k** $\tan(-\alpha)$

Example 7

6 Without using a calculator, evaluate the sin, cos and tan of each of the following:

a $150°$ **b** $225°$ **c** $405°$ **d** $-120°$

e $-315°$ **f** $-30°$

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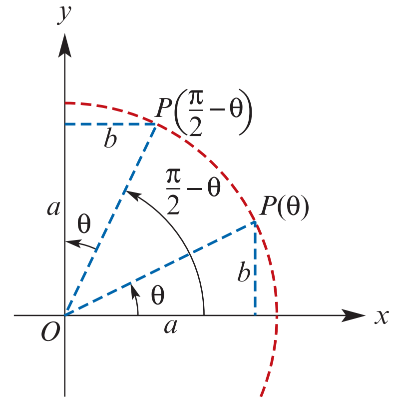
4C Further symmetry properties and the Pythagorean identity

Complementary relationships

From the diagram to the right:

$$\sin\left(\frac{\pi}{2} - \theta\right) = a = \cos\theta$$

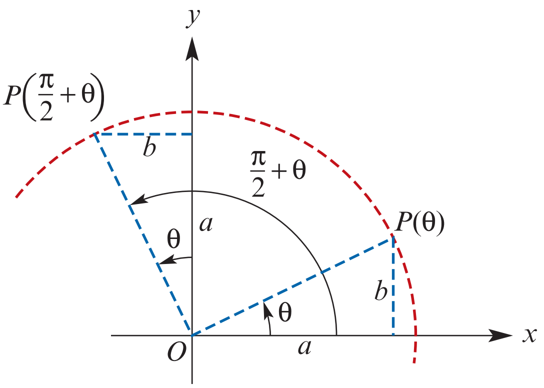
$$\cos\left(\frac{\pi}{2} - \theta\right) = b = \sin\theta$$



From the diagram to the right:

$$\sin\left(\frac{\pi}{2} + \theta\right) = a = \cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -b = -\sin\theta$$



Example 8

If $\sin\theta = 0.3$ and $\cos\psi = 0.8$, find the value of:

a $\sin\left(\frac{\pi}{2} - \psi\right)$ **b** $\cos\left(\frac{\pi}{2} + \theta\right)$

Solution

a $\sin\left(\frac{\pi}{2} - \psi\right) = \cos\psi = 0.8$

b $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta = -0.3$

The Pythagorean identity

Consider a point, $P(\theta)$, on the unit circle.

By Pythagoras' theorem,

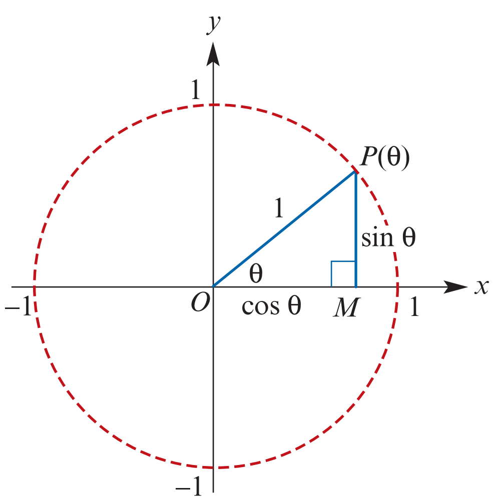
$$OP^2 = OM^2 + MP^2$$

$$\therefore \quad 1 = (\cos\theta)^2 + (\sin\theta)^2$$

Now $(\cos\theta)^2$ and $(\sin\theta)^2$ may be written as $\cos^2\theta$ and $\sin^2\theta$. Thus we obtain:

$$\cos^2\theta + \sin^2\theta = 1$$

This holds for all values of θ, and is called the **Pythagorean identity**.



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Example 9

Given that $\sin x = \frac{3}{5}$ and $\frac{\pi}{2} < x < \pi$, find:

a $\cos x$

b $\tan x$

Solution

a Substitute $\sin x = \frac{3}{5}$ into the Pythagorean identity:

$$\cos^2 x + \sin^2 x = 1$$

$$\cos^2 x + \frac{9}{25} = 1$$

$$\cos^2 x = 1 - \frac{9}{25}$$

$$= \frac{16}{25}$$

Therefore $\cos x = \pm\frac{4}{5}$. But x is in the 2nd quadrant, and so $\cos x = -\frac{4}{5}$.

b Using part **a**, we have

$$\tan x = \frac{\sin x}{\cos x}$$

$$= \frac{3}{5} \div \left(-\frac{4}{5}\right)$$

$$= \frac{3}{5} \times \left(-\frac{5}{4}\right)$$

$$= -\frac{3}{4}$$

Section summary

- Complementary relationships

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

- Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

Exercise 4C

Example 8

SF

1 If $\sin x = 0.3$, $\cos\alpha = 0.6$ and $\tan\theta = 0.7$, find the value of:

a $\cos(-\alpha)$ **b** $\sin\left(\frac{\pi}{2} + \alpha\right)$ **c** $\tan(-\theta)$

d $\cos\left(\frac{\pi}{2} - x\right)$ **e** $\sin(-x)$ **f** $\tan\left(\frac{\pi}{2} - \theta\right)$

g $\cos\left(\frac{\pi}{2} + x\right)$ **h** $\sin\left(\frac{\pi}{2} - \alpha\right)$ **i** $\sin\left(\frac{3\pi}{2} + \alpha\right)$

j $\cos\left(\frac{3\pi}{2} - x\right)$ **k** $\tan\left(\frac{3\pi}{2} - \theta\right)$ **l** $\cos\left(\frac{5\pi}{2} - x\right)$

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Example 9 **2** **a** Given that $\cos x = \frac{3}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

b Given that $\sin x = \frac{5}{13}$ and $\frac{\pi}{2} < x < \pi$, find $\cos x$ and $\tan x$.

c Given that $\cos x = \frac{1}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

d Given that $\sin x = -\frac{12}{13}$ and $\pi < x < \frac{3\pi}{2}$, find $\cos x$ and $\tan x$.

e Given that $\cos x = \frac{4}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

f Given that $\sin x = -\frac{5}{13}$ and $\pi < x < \frac{3\pi}{2}$, find $\cos x$ and $\tan x$.

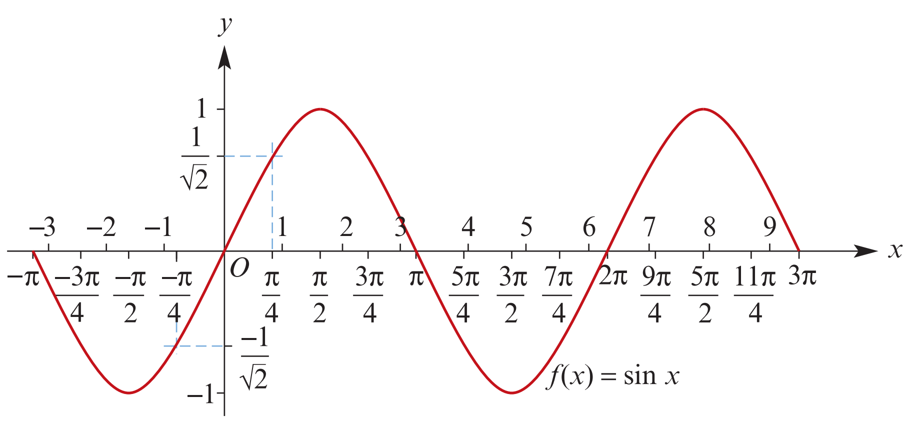
g Given that $\cos x = \frac{8}{10}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

CF

4D Graphs of sine and cosine

▶ Graph of the sine function

A calculator can be used to plot the graph of $f(x) = \sin x$ for $-\pi \le x \le 3\pi$. Note that radian mode must be selected.



Observations from the graph of $y = \sin x$

- The graph repeats itself after an interval of 2π units.
  A function which repeats itself regularly is called a **periodic** function, and the interval between the repetitions is called the **period** of the function (also called the wavelength).
  Thus $y = \sin x$ has a period of 2π units.
- The maximum and minimum values of $\sin x$ are 1 and -1 respectively.
  The distance between the mean position and the maximum position is called the **amplitude**. The graph of $y = \sin x$ has an amplitude of 1.

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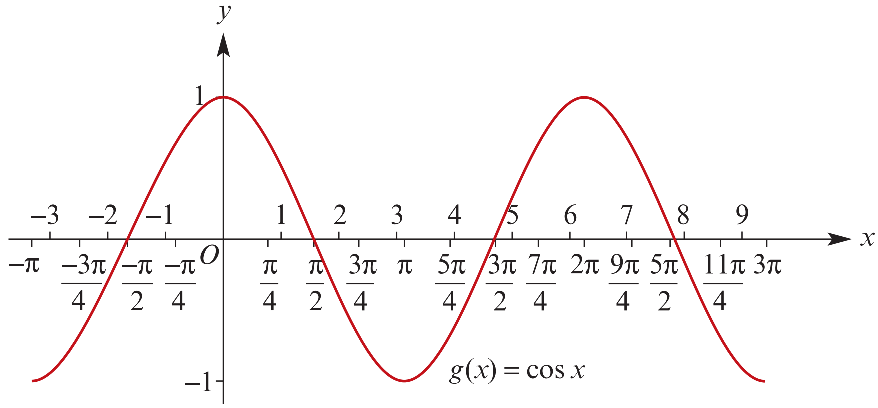
▶ Graph of the cosine function

The graph of $g(x) = \cos x$ is shown below for $-\pi \le x \le 3\pi$.



Observations from the graph of $y = \cos x$

- The period is 2π.
- The amplitude is 1.
- The graph of $y = \cos x$ is the graph of $y = \sin x$ translated $\frac{\pi}{2}$ units in the negative direction of the x-axis.

▶ Sketch graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$

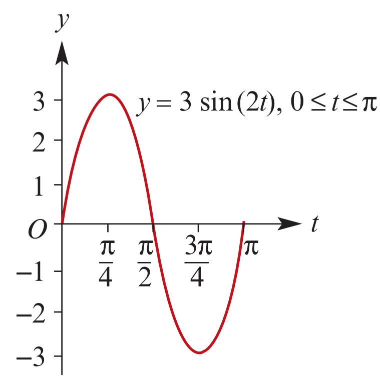
The graphs of functions of the forms $y = a\sin(nt)$ and $y = a\cos(nt)$ are transformations of the graphs of $y = \sin t$ and $y = \cos t$ respectively. We first consider the case where a and n are positive numbers.

Transformations: dilations

Graph of $y = 3\sin(2t)$ The image of the graph of $y = \sin t$ under a dilation of factor 3 from the t-axis and a dilation of factor $\frac{1}{2}$ from the y-axis is $y = 3\sin(2t)$.

Note: Let $f(t) = \sin t$. Then the graph of $y = f(t)$ is transformed to the graph of $y = 3f(2t)$. The point with coordinates (t, y) is mapped to the point with coordinates $\left(\frac{t}{2}, 3y\right)$.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$y = 3\sin(2t)$	0	3	0	−3	0



We make the following observations about the graph of $y = 3\sin(2t)$:

- amplitude is 3
- period is π

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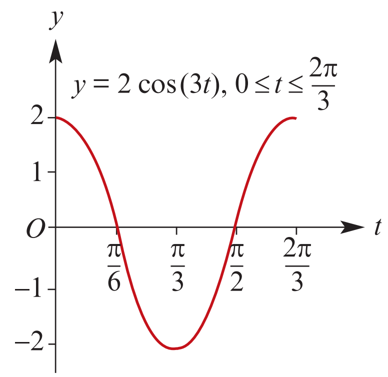
Graph of $y = 2\cos(3t)$ The image of the graph of $y = \cos t$ under a dilation of factor 2 from the t-axis and a dilation of factor $\frac{1}{3}$ from the y-axis is $y = 2\cos(3t)$.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$y = 2\cos(3t)$	2	0	−2	0	2



We make the following observations about the graph of $y = 2\cos(3t)$:

- amplitude is 2
- period is $\frac{2\pi}{3}$

Amplitude and period Comparing these results with those for $y = \sin t$ and $y = \cos t$, the following general rules can be stated for a and n positive:

Function	Amplitude	Period
$y = a\sin(nt)$	a	$\frac{2\pi}{n}$
$y = a\cos(nt)$	a	$\frac{2\pi}{n}$

Example 10

For each of the following functions with domain $\mathbb{R}$, state the amplitude and period:

a $f(t) = 2\sin(3t)$ **b** $f(t) = -\frac{1}{2}\sin\left(\frac{t}{2}\right)$ **c** $f(t) = 4\cos(3\pi t)$

Solution

a Amplitude is 2

Period is $\frac{2\pi}{3}$

b Amplitude is $\frac{1}{2}$

Period is $2\pi \div \frac{1}{2} = 4\pi$

c Amplitude is 4

Period is $\frac{2\pi}{3\pi} = \frac{2}{3}$

Graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$

For a and n positive numbers, the graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$ are obtained from the graphs of $y = \sin t$ and $y = \cos t$, respectively, by a dilation of factor a from the t-axis and a dilation of factor $\frac{1}{n}$ from the y-axis.

The point with coordinates (t, y) is mapped to the point with coordinates $\left(\frac{t}{n}, ay\right)$.

The following are important properties of both of the functions $f(t) = a\sin(nt)$ and $g(t) = a\cos(nt)$:

- The period is $\frac{2\pi}{n}$.
- The amplitude is a.
- The maximal domain is $\mathbb{R}$.
- The range is $[-a, a]$.

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Example 11

For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 3\sin(2x)$ **b** $g(x) = 4\sin\left(\frac{x}{2}\right)$

Solution

a The graph of $y = 3\sin(2x)$ is obtained from the graph of $y = \sin x$ by a dilation of factor 3 from the x-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

The function $g(x) = 3\sin(2x)$ has amplitude 3 and period $\frac{2\pi}{2} = \pi$.

b The graph of $y = 4\sin\left(\frac{x}{2}\right)$ is obtained from the graph of $y = \sin x$ by a dilation of factor 4 from the x-axis and a dilation of factor 2 from the y-axis.

The function $g(x) = 4\sin\left(\frac{x}{2}\right)$ has amplitude 4 and period $2\pi \div \frac{1}{2} = 4\pi$.

Example 12

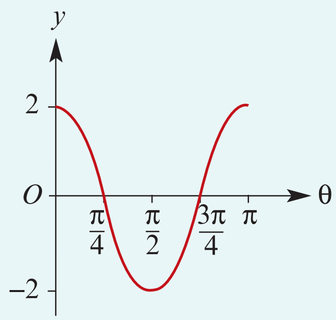
Sketch the graph of each of the following functions:

a $y = 2\cos(2\theta)$ **b** $y = \frac{1}{2}\sin\left(\frac{x}{2}\right)$

In each case, show one complete cycle.

Solution

a



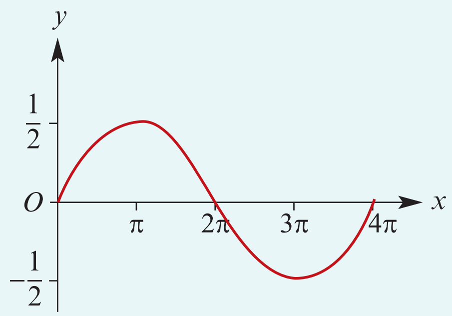
Explanation

The amplitude is 2.

The period is $\frac{2\pi}{2} = \pi$.

The graph of $y = 2\cos(2\theta)$ is obtained from the graph of $y = \cos\theta$ by a dilation of factor 2 from the θ-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

b



The amplitude is $\frac{1}{2}$.

The period is $2\pi \div \frac{1}{2} = 4\pi$.

The graph of $y = \frac{1}{2}\sin(\frac{x}{2})$ is obtained from the graph of $y = \sin x$ by a dilation of factor $\frac{1}{2}$ from the x-axis and a dilation of factor 2 from the y-axis.

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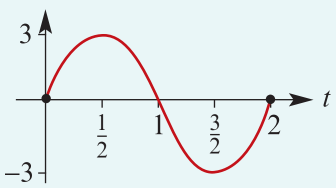
Example 13

Sketch the graph of the function with rule $f(t) = 3\sin(\pi t)$, where $t \in [0, 2]$.

Solution



Explanation

The amplitude is 3.

The period is $2\pi \div \pi = 2$.

The graph of $f(t) = 3\sin(\pi t)$ is obtained from the graph of $y = \sin t$ by a dilation of factor 3 from the t-axis and a dilation of factor $\frac{1}{\pi}$ from the y-axis.

Transformations: reflections

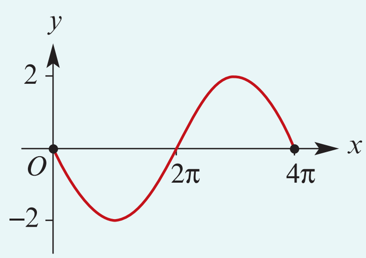
Example 14

Sketch the following graphs for $x \in [0, 4\pi]$:

a $f(x) = -2\sin\left(\frac{x}{2}\right)$ **b** $y = -\cos(2x)$

Solution

a

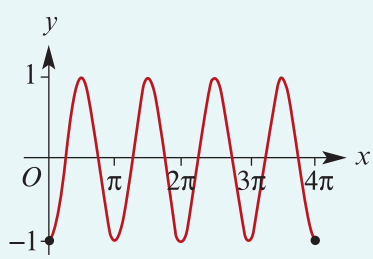


Explanation

The graph of $f(x) = -2\sin(\frac{x}{2})$ is obtained from the graph of $y = 2\sin(\frac{x}{2})$ by a reflection in the x-axis.

The amplitude is 2 and the period is 4π.

b



The graph of $y = -\cos(2x)$ is obtained from the graph of $y = \cos(2x)$ by a reflection in the x-axis.

The amplitude is 1 and the period is π.

Note: Recall that sin is an odd function and cos is an even function (i.e. $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$). When reflected in the y-axis, the graph of $y = \sin x$ transforms onto the graph of $y = -\sin x$, and the graph of $y = \cos x$ transforms onto itself.

Section summary

For positive numbers a and n, the graphs of $y = a\sin(nt)$, $y = -a\sin(nt)$, $y = a\cos(nt)$ and $y = -a\cos(nt)$ all have the following properties:

- The period is $\frac{2\pi}{n}$.
- The amplitude is a.
- The maximal domain is $\mathbb{R}$.
- The range is $[-a, a]$.

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Exercise 4D

SF

Example 10

1 Write down the period and the amplitude of each of the following:

a $3\sin\theta$ **b** $5\sin(3\theta)$ **c** $\frac{1}{2}\cos(2\theta)$ **d** $2\sin\left(\frac{1}{3}\theta\right)$

e $3\cos(4\theta)$ **f** $\frac{1}{2}\sin\theta$ **g** $3\cos\left(\frac{1}{2}\theta\right)$ **h** $2\sin\left(\frac{2\theta}{3}\right)$

Example 11

2 For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 4\sin(3x)$ **b** $g(x) = 5\sin\left(\frac{x}{3}\right)$

c $g(x) = 6\sin\left(\frac{x}{2}\right)$ **d** $g(x) = 4\sin(5x)$

3 For each of the following, give a sequence of transformations which takes the graph of $y = \cos x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 2\cos(3x)$ **b** $g(x) = 3\cos\left(\frac{x}{4}\right)$

c $g(x) = 6\cos\left(\frac{x}{5}\right)$ **d** $g(x) = 3\cos(7x)$

Example 12

4 Sketch the graph of each of the following, showing one complete cycle. State the amplitude and period.

a $y = 2\sin(3\theta)$ **b** $y = 2\cos(2\theta)$ **c** $y = 3\sin\left(\frac{1}{3}\theta\right)$

d $y = \frac{1}{3}\cos(2\theta)$ **e** $y = 3\sin(4\theta)$ **f** $y = 4\cos\left(\frac{1}{4}\theta\right)$

Example 13

5 Sketch the graph of $f(t) = 3\sin(2\pi t)$ for $0 \le t \le 1$.

6 Sketch the graph of $f(t) = 3\sin\left(\frac{\pi t}{2}\right)$ for $0 \le t \le 1$.

7 Sketch the graph of $f(x) = 5\cos(3x)$ for $0 \le x \le \pi$.

8 Sketch the graph of $f(x) = \frac{1}{2}\sin(2x)$ for $-\pi \le x \le 2\pi$.

9 Sketch the graph of $f(x) = 2\cos\left(\frac{3x}{2}\right)$ for $0 \le x \le 2\pi$.

Example 14

10 Sketch the graph of $f(x) = -3\cos\left(\frac{x}{2}\right)$ for $0 \le x \le 4\pi$.

CF

11 Find the equation of the image of the graph of $y = \sin x$ under a dilation of factor 2 from the x-axis followed by a dilation of factor 3 from the y-axis.

12 Find the equation of the image of the graph of $y = \cos x$ under a dilation of factor $\frac{1}{2}$ from the x-axis followed by a dilation of factor 3 from the y-axis.

13 Find the equation of the image of the graph of $y = \sin x$ under a dilation of factor $\frac{1}{2}$ from the x-axis followed by a dilation of factor 2 from the y-axis.

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4E Solution of trigonometric equations

In this section we revise methods for solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$.

► Solving equations of the form $\sin t = b$ and $\cos t = b$

First we look at the techniques for solving equations of the form $\sin t = b$ and $\cos t = b$. These same techniques will be applied to solve more complicated trigonometric equations later in this section.

Example 15

Find all solutions of the equation $\sin\theta = \frac{1}{2}$ for $\theta \in [0, 4\pi]$.

Solution

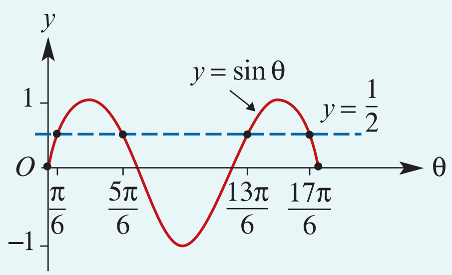
The solution for $\theta \in \left[0, \frac{\pi}{2}\right]$ is $\theta = \frac{\pi}{6}$.

The second solution is $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

The third solution is $\theta = 2\pi + \frac{\pi}{6} = \frac{13\pi}{6}$.

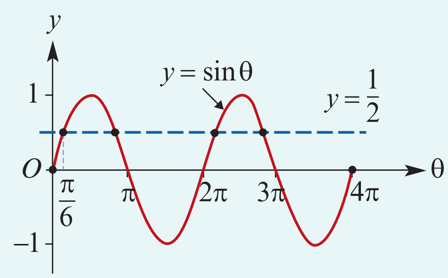
The fourth solution is $\theta = 2\pi + \frac{5\pi}{6} = \frac{17\pi}{6}$.

These four solutions are shown on the graph below.



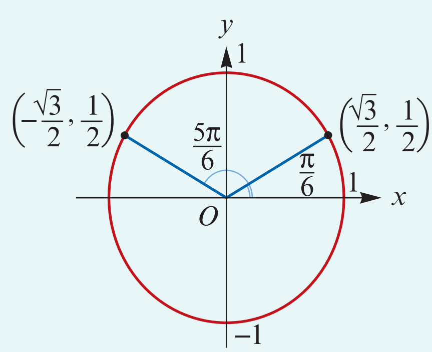
Explanation

By sketching a graph, we can see that there are four solutions in the interval $[0, 4\pi]$.



The first solution can be obtained from a knowledge of exact values or by using $\sin^{-1}$ on your calculator.

The second solution is obtained using symmetry. The sine function is positive in the 2nd quadrant and $\sin(\pi - \theta) = \sin\theta$.



Further solutions are found by adding 2π, since $\sin\theta = \sin(2\pi + \theta)$.

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Example 16

Find two values of x:

a $\sin x = -0.3$ with $0 \le x \le 2\pi$ **b** $\cos x^\circ = -0.7$ with $0^\circ \le x^\circ \le 360^\circ$

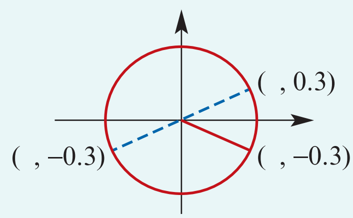
Solution

a First solve the equation $\sin \alpha = 0.3$ for $\alpha \in \left[0, \frac{\pi}{2}\right]$. Use your calculator to find the solution $\alpha = 0.30469\ldots$

Now the value of $\sin x$ is negative for $P(x)$ in the 3rd and 4th quadrants. From the symmetry relationships (or from the graph of $y = \sin x$):

3rd quadrant: $x = \pi + 0.30469\ldots$
$= 3.446$ (to 3 d.p.)

4th quadrant: $x = 2\pi - 0.30469\ldots$
$= 5.978$ (to 3 d.p.)



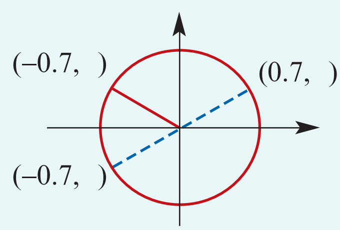
$\therefore$ If $\sin x = -0.3$, then $x = 3.446$ or $x = 5.978$.

b First solve the equation $\cos \alpha^\circ = 0.7$ for $\alpha^\circ \in [0^\circ, 90^\circ]$. Use your calculator to find the solution $\alpha^\circ = 45.57^\circ$.

Now the value of $\cos x^\circ$ is negative for $P(x^\circ)$ in the 2nd and 3rd quadrants.

2nd quadrant: $x^\circ = 180^\circ - 45.57^\circ$
$= 134.43^\circ$

3rd quadrant: $x^\circ = 180^\circ + 45.57^\circ$
$= 225.57^\circ$



$\therefore$ If $\cos x^\circ = -0.7$, then $x^\circ = 134.43^\circ$ or $x^\circ = 225.57^\circ$.

Example 17

Find all the values of θ° between 0° and 360° for which:

a $\cos \theta^\circ = \frac{\sqrt{3}}{2}$ **b** $\sin \theta^\circ = -\frac{1}{2}$ **c** $\cos \theta^\circ - \frac{1}{\sqrt{2}} = 0$

Solution	*Explanation*
a $\cos \theta^\circ = \frac{\sqrt{3}}{2}$	
$\theta^\circ = 30^\circ$ or $\theta^\circ = 360^\circ - 30^\circ$	$\cos \theta^\circ$ is positive, and so $P(\theta^\circ)$ lies in the 1st or 4th quadrant.
$\theta^\circ = 30^\circ$ or $\theta^\circ = 330^\circ$	$\cos(360^\circ - \theta^\circ) = \cos \theta^\circ$
b $\sin \theta^\circ = -\frac{1}{2}$	$\sin \theta^\circ$ is negative, and so $P(\theta^\circ)$ lies in the 3rd or 4th quadrant.
$\theta^\circ = 180^\circ + 30^\circ$ or $\theta^\circ = 360^\circ - 30^\circ$	$\sin(180^\circ + \theta^\circ) = -\sin \theta^\circ$
$\theta^\circ = 210^\circ$ or $\theta^\circ = 330^\circ$	$\sin(360^\circ - \theta^\circ) = -\sin \theta^\circ$

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c $\cos\theta^\circ - \dfrac{1}{\sqrt{2}} = 0$

$\therefore \quad \cos\theta^\circ = \dfrac{1}{\sqrt{2}}$

$\theta^\circ = 45^\circ \quad \text{or} \quad \theta^\circ = 360^\circ - 45^\circ$

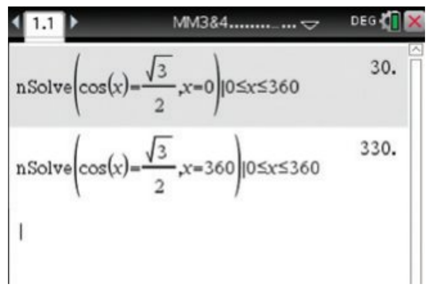
$\theta^\circ = 45^\circ \quad \text{or} \quad \theta^\circ = 315^\circ$

$\cos\theta^\circ$ is positive, and so $P(\theta^\circ)$ lies in the 1st or 4th quadrant.

TI-84

Using the TI-Nspire CX non-CAS

For Example 17a, make sure the calculator is in degree mode and use (menu) > **Algebra** > **Numerical Solve** as shown.



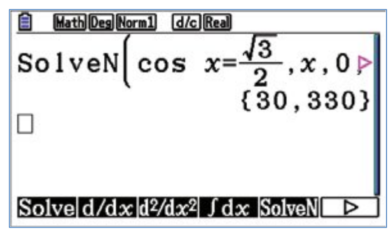
Note: Two different guess values are used to find the two solutions. It is advisable to plot the graph over the given domain to determine the number of solutions.

Using the Casio

For Example 17a:

- In **Run-Matrix** mode, ensure that the angle setting is Degrees ((SHIFT) (MENU)).
- Go to the numerical solver **SolveN** (OPTN) (F4) (F5).
- Complete the equation and domain by entering:

$$\cos x = \frac{\sqrt{3}}{2}, x, 0, 360)$$



▶ Solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$

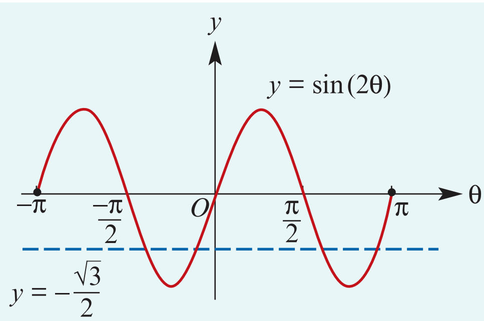
The techniques introduced above can be applied in a more general situation. This is achieved by a simple substitution, as shown in the following example.

Example 18

Solve the equation $\sin(2\theta) = -\dfrac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$.

Solution

It is clear from the graph that there are four solutions.



To solve the equation, let $x = 2\theta$.

Note: If $\theta \in [-\pi, \pi]$, then we have $x = 2\theta \in [-2\pi, 2\pi]$.

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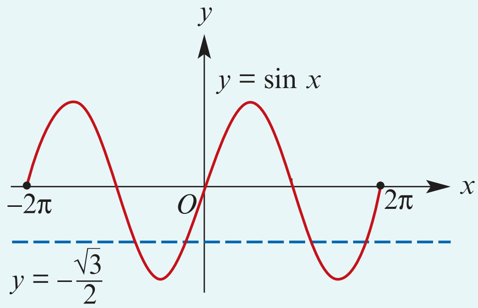
Now consider the equation

$$\sin x = -\frac{\sqrt{3}}{2} \quad \text{for } x \in [-2\pi, 2\pi]$$



The 1st quadrant solution to the equation $\sin\alpha = \frac{\sqrt{3}}{2}$ is $\alpha = \frac{\pi}{3}$.

Using symmetry, the solutions to

$\sin x = -\frac{\sqrt{3}}{2}$ for $x \in [0, 2\pi]$ are

$x = \pi + \frac{\pi}{3}$ and $x = 2\pi - \frac{\pi}{3}$

i.e. $x = \frac{4\pi}{3}$ and $x = \frac{5\pi}{3}$

The other two solutions (obtained by subtracting 2π) are $x = \frac{4\pi}{3} - 2\pi$ and $x = \frac{5\pi}{3} - 2\pi$.

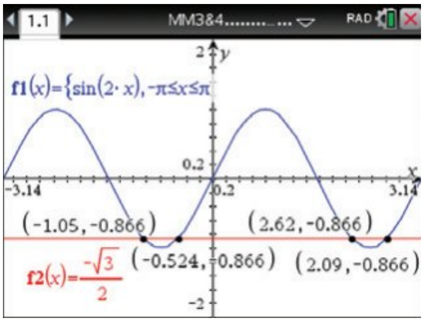
$\therefore$ The required solutions for x are $-\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

$\therefore$ The required solutions for θ are $-\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}$ and $\frac{5\pi}{6}$.

Using the TI-Nspire CX non-CAS

TI-84

- Ensure that the calculator is in radian mode.
- In a **Graphs** application, plot the graphs of:
  - $f1(x) = \sin(2x) \mid -\pi \le x \le \pi$
  - $f2(x) = -\frac{\sqrt{3}}{2}$
- Use (menu) > **Geometry** > **Points & Lines** > **Intersection Point(s)**.
- The solutions are −1.05, −0.524, 2.09 and 2.62, correct to three significant figures.

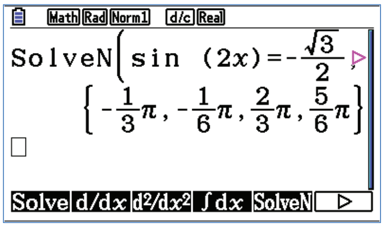


Note: You can also use (menu) > **Algebra** > **Numerical Solve**, but you will need to try several guess values to obtain all four solutions.

Using the Casio

- In **Run-Matrix** mode, ensure that the angle setting is Radians ((SHIFT) (MENU)).
- Go to the numerical solver **SolveN** (OPTN) (F4) (F5).
- Complete the equation and domain by entering:

$$\sin(2x) = -\frac{\sqrt{3}}{2}, x, -\pi, \pi)$$



ISBN 978-1-108-45164-2

Cambridge University Press

Section summary

- For solving equations of the form $\sin t = b$ and $\cos t = b$:
  - First find the solutions in the interval $[0, 2\pi]$. This can be done using your knowledge of exact values and symmetry properties, or with the aid of a calculator.
  - Further solutions can be found by adding and subtracting multiples of 2π.
- For solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$:
  - First substitute $x = nt$. Work out the interval in which solutions for x are required. Then proceed as in the case above to solve for x.
  - Once the solutions for x are found, the solutions for t can be found.

  For example: To solve $\sin(3t) = \frac{1}{2}$ for $t \in [0, 2\pi]$, first let $x = 3t$. The equation becomes $\sin x = \frac{1}{2}$ and the required solutions for x are in the interval $[0, 6\pi]$.

Exercise 4E

Skillsheet SF

Example 15

1 Solve each of the following for $x \in [0, 4\pi]$:

a $\sin x = \dfrac{1}{\sqrt{2}}$ **b** $\cos x = \dfrac{\sqrt{3}}{2}$ **c** $\sin x = \dfrac{-\sqrt{3}}{2}$ **d** $\cos x = -1$

2 Solve each of the following for $x \in [-\pi, \pi]$:

a $\sin x = -\dfrac{1}{2}$ **b** $\cos x = \dfrac{\sqrt{3}}{2}$ **c** $\cos x = \dfrac{-\sqrt{3}}{2}$ **d** $\sin x = 1$

3 Solve each of the following for $x \in [0, 2\pi]$:

a $\sqrt{2}\sin x - 1 = 0$ **b** $\sqrt{2}\cos x + 1 = 0$ **c** $2\cos x + \sqrt{3} = 0$

d $2\sin x + 1 = 0$ **e** $1 - \sqrt{2}\cos x = 0$ **f** $4\cos x + 2 = 0$

Example 16

4 Find all values of x between 0 and 2π for which:

a $\sin x = 0.6$ **b** $\cos x = 0.8$ **c** $\sin x = -0.45$ **d** $\cos x = -0.2$

5 Find all values of $\theta°$ between 0° and 360° for which:

a $\sin\theta° = 0.3$ **b** $\cos\theta° = 0.4$ **c** $\sin\theta° = -0.8$ **d** $\cos\theta° = -0.5$

Example 17

6 Without using a calculator, find all the values of θ between 0 and 360 for each of the following:

a $\cos\theta° = \dfrac{1}{2}$ **b** $\sin\theta° = \dfrac{\sqrt{3}}{2}$ **c** $\sin\theta° = -\dfrac{1}{\sqrt{2}}$

d $2\cos\theta° + 1 = 0$ **e** $2\sin\theta° = \sqrt{3}$ **f** $2\cos\theta° = -\sqrt{3}$

Example 18

7 Solve the following equations for $\theta \in [0, 2\pi]$:

a $\sin(2\theta) = -\dfrac{1}{2}$ **b** $\cos(2\theta) = \dfrac{\sqrt{3}}{2}$ **c** $\sin(2\theta) = \dfrac{1}{2}$

d $\sin(3\theta) = -\dfrac{1}{\sqrt{2}}$ **e** $\cos(2\theta) = -\dfrac{\sqrt{3}}{2}$ **f** $\sin(2\theta) = -\dfrac{1}{\sqrt{2}}$

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SF

8 Without using a calculator, find all the values of x between 0 and 2π for each of the following:

a $2\cos(3x) + \sqrt{3} = 0$ **b** $4\sin(2x) - 2 = 0$ **c** $\sqrt{2}\cos(3x) - 1 = 0$

d $10\sin(3x) - 5 = 0$ **e** $2\sin(2x) = \sqrt{2}$ **f** $4\cos(3x) = -2\sqrt{3}$

g $-2\sin(3x) = \sqrt{2}$ **h** $4\cos(2x) = -2$ **i** $-2\cos(2x) = \sqrt{2}$

9 Solve the following equations for $\theta \in [0, 2\pi]$:

a $\sin(2\theta) = -0.8$ **b** $\sin(2\theta) = -0.6$ **c** $\cos(2\theta) = 0.4$ **d** $\cos(3\theta) = 0.6$

4F Sketch graphs of $y = a \sin n(t \pm \varepsilon)$ and $y = a \cos n(t \pm \varepsilon)$

In this section, we consider translations of graphs of functions of the form $f(t) = a\sin(nt)$ and $g(t) = a\cos(nt)$ in the direction of the t-axis.

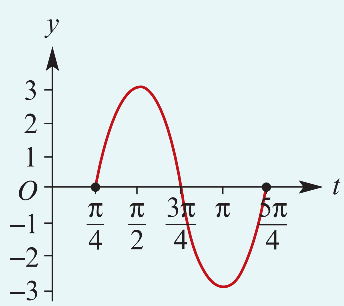
Example 19

On separate axes, draw the graphs of the following functions. Use a calculator to help establish the shape. Set the window appropriately by noting the range and period.

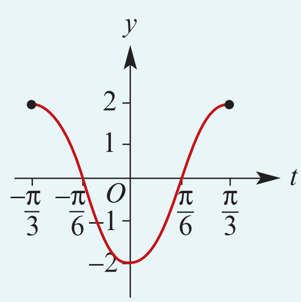
a $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$, $\frac{\pi}{4} \le t \le \frac{5\pi}{4}$ **b** $y = 2\cos 3\left(t + \frac{\pi}{3}\right)$, $-\frac{\pi}{3} \le t \le \frac{\pi}{3}$

Solution

a The range is $[-3, 3]$ and the period is π.



b The range is $[-2, 2]$ and the period is $\frac{2\pi}{3}$.



Observations from the example

a The graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$ is the same shape as $y = 3\sin(2t)$, but is translated $\frac{\pi}{4}$ units in the positive direction of the t-axis.

b The graph of $y = 2\cos 3\left(t + \frac{\pi}{3}\right)$ is the same shape as $y = 2\cos(3t)$, but is translated $\frac{\pi}{3}$ units in the negative direction of the t-axis.

The effect of $\pm\varepsilon$ is to translate the graph parallel to the t-axis. (Here $\pm\varepsilon$ is called the phase.)

Note: The techniques of Section 2E can be used to find the sequence of transformations.

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Example 20

Define the function $f(x) = \sin\left(x - \frac{\pi}{3}\right)$ for $0 \le x \le 2\pi$.

a Find $f(0)$ and $f(2\pi)$. **b** Sketch the graph of f.

Solution

a $f(0) = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ and $f(2\pi) = \sin\left(2\pi - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

b The graph of $y = \sin\left(x - \frac{\pi}{3}\right)$ is the graph of $y = \sin x$ translated $\frac{\pi}{3}$ units to the right.

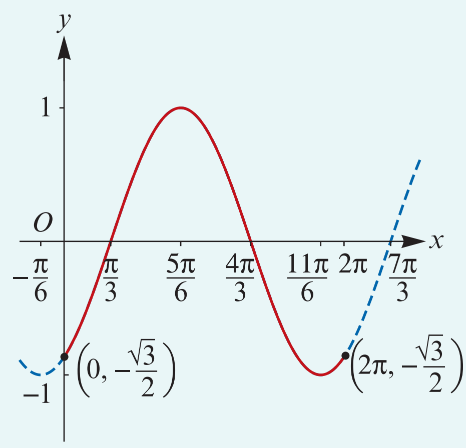
The period of f is 2π and the amplitude is 1.

The endpoints are $\left(0, -\frac{\sqrt{3}}{2}\right)$ and $\left(2\pi, -\frac{\sqrt{3}}{2}\right)$.

There are many ways to proceed from here.

Method 1

- Start at $x = \frac{\pi}{3}$ and start to draw one cycle stopping at $x = 2\pi$.
- Each 'loop' of the graph is of length π and each 'half loop' is of length $\frac{\pi}{2}$.
- The 'half loop' going back from $x = \frac{\pi}{3}$ would end at $\frac{\pi}{3} - \frac{\pi}{2} = -\frac{\pi}{6}$.



Method 2

Find the x-axis intercepts by solving the equation $\sin\left(x - \frac{\pi}{3}\right) = 0$. Use symmetry to find the coordinates of the maximum and minimum points.

Section summary

The graphs of $y = a \sin n(t \pm \varepsilon)$ and $y = a \cos n(t \pm \varepsilon)$ are translations of the graphs of $y = a \sin(nt)$ and $y = a \cos(nt)$ respectively.

The graphs are translated $\mp\varepsilon$ units parallel to the t-axis, where $\pm\varepsilon$ is called the phase.

Exercise 4F

Example 19

1 Sketch the graph of each of the following, showing one complete cycle. State the period and amplitude, and the greatest and least values of y. SF

a $y = 3\sin\left(\theta - \frac{\pi}{2}\right)$ **b** $y = \sin 2(\theta + \pi)$ **c** $y = 2\sin 3\left(\theta + \frac{\pi}{4}\right)$

d $y = \sqrt{3}\sin 2\left(\theta - \frac{\pi}{2}\right)$ **e** $y = 3\sin(2x - \pi)$ **f** $y = 2\cos 3\left(\theta + \frac{\pi}{4}\right)$

g $y = \sqrt{2}\sin 2\left(\theta - \frac{\pi}{3}\right)$ **h** $y = -3\sin\left(2x + \frac{\pi}{3}\right)$ **i** $y = -3\cos 2\left(\theta + \frac{\pi}{2}\right)$

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Example 20 **2** Define the function $f(x) = \cos\left(x - \frac{\pi}{3}\right)$ for $0 \le x \le 2\pi$.

a Find $f(0)$ and $f(2\pi)$. **b** Sketch the graph of f.

3 Define the function $f(x) = \sin 2\left(x - \frac{\pi}{3}\right)$ for $0 \le x \le 2\pi$.

a Find $f(0)$ and $f(2\pi)$. **b** Sketch the graph of f.

4 Define the function $f(x) = \sin 3\left(x + \frac{\pi}{4}\right)$ for $x \in [-\pi, \pi]$.

a Find $f(-\pi)$ and $f(\pi)$. **b** Sketch the graph of f.

SF

5 Find the equation of the image of $y = \sin x$ for each of the following transformations:

a dilation of factor 2 from the y-axis followed by dilation of factor 3 from the x-axis

b dilation of factor $\frac{1}{2}$ from the y-axis followed by dilation of factor 3 from the x-axis

c dilation of factor 3 from the y-axis followed by dilation of factor 2 from the x-axis

d dilation of factor $\frac{1}{2}$ from the y-axis followed by translation of $\frac{\pi}{3}$ units in the positive direction of the x-axis

e dilation of factor 2 from the y-axis followed by translation of $\frac{\pi}{3}$ units in the negative direction of the x-axis.

CF

4G Sketch graphs of $y = a \sin n(t \pm \varepsilon) \pm b$ and $y = a \cos n(t \pm \varepsilon) \pm b$

In general, the effect of $\pm b$ is to translate the graph $\pm b$ units parallel to the y-axis.

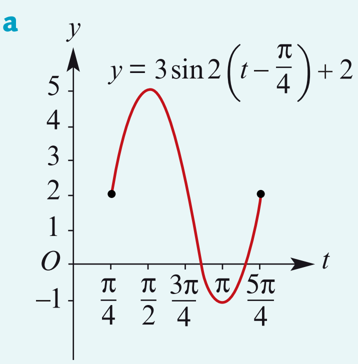
Example 21

Sketch each of the following graphs. Use a calculator to help establish the shape.

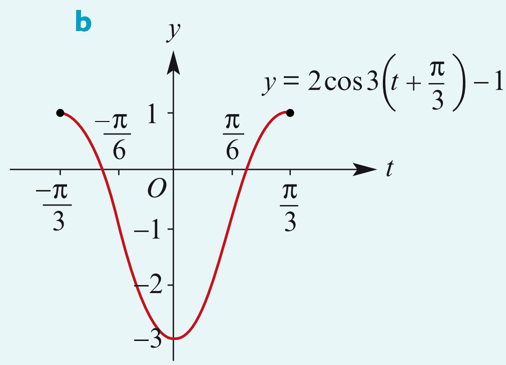
a $y = 3 \sin 2\left(t - \frac{\pi}{4}\right) + 2, \quad \frac{\pi}{4} \le t \le \frac{5\pi}{4}$ **b** $y = 2 \cos 3\left(t + \frac{\pi}{3}\right) - 1, \quad -\frac{\pi}{3} \le t \le \frac{\pi}{3}$

Solution

a



b



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Finding axis intercepts

Example 22

Sketch the graph of each of the following for $x \in [0, 2\pi]$. Clearly indicate axis intercepts.

a $y = \sqrt{2}\sin(x) + 1$ **b** $y = 2\cos(2x) - 1$ **c** $y = 2\sin 2\left(x - \frac{\pi}{3}\right) - \sqrt{3}$

Solution

a To determine the x-axis intercepts, solve the equation $\sqrt{2}\sin(x) + 1 = 0$.

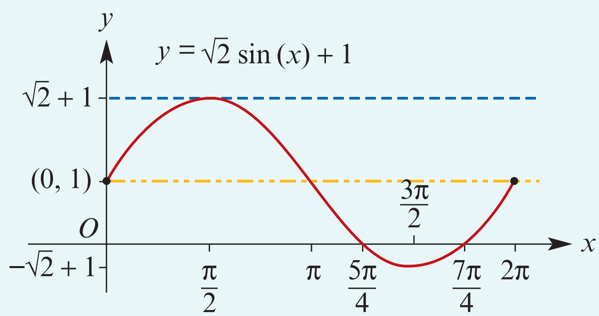
$$\sqrt{2}\sin(x) + 1 = 0$$

$$\therefore \quad \sin x = -\frac{1}{\sqrt{2}}$$

$$\therefore x = \pi + \frac{\pi}{4} \text{ or } 2\pi - \frac{\pi}{4}$$

$$\therefore x = \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

The x-axis intercepts are $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.



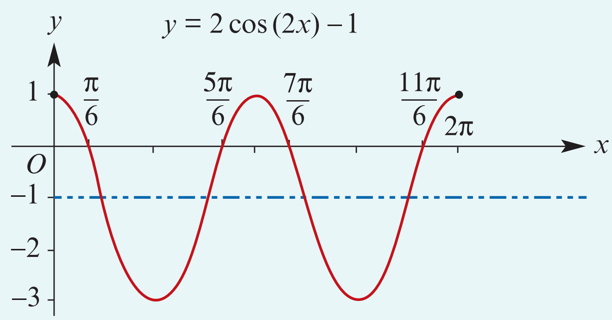
b $$2\cos(2x) - 1 = 0$$

$$\therefore \quad \cos(2x) = \frac{1}{2}$$

$$\therefore 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \text{ or } \frac{11\pi}{3}$$

$$\therefore \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$

The x-axis intercepts are $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.



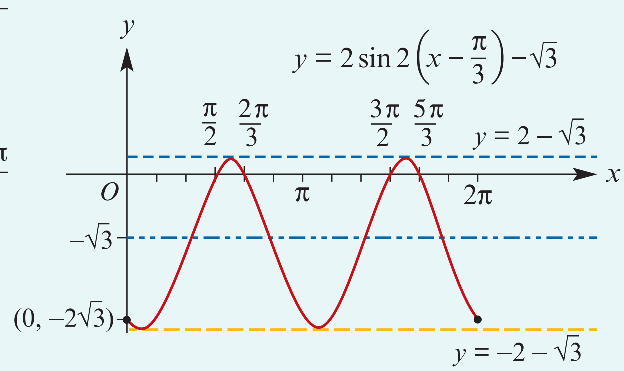
c $$\sin 2\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\therefore 2\left(x - \frac{\pi}{3}\right) = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3} \text{ or } \frac{8\pi}{3}$$

$$\therefore \quad x - \frac{\pi}{3} = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6} \text{ or } \frac{4\pi}{3}$$

$$\therefore \quad x = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2} \text{ or } \frac{5\pi}{3}$$

The x-axis intercepts are $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}$ and $\frac{5\pi}{3}$.



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Section summary

The graphs of $y = a \sin n(t \pm \varepsilon) \pm b$ and $y = a \cos n(t \pm \varepsilon) \pm b$ are translations of the graphs of $y = a \sin(nt)$ and $y = a \cos(nt)$ respectively.

The graphs are translated $\mp\varepsilon$ units parallel to the t-axis, where $\pm\varepsilon$ is called the phase. They are also translated $\pm b$ units parallel to the y-axis.

Exercise 4G

Skillsheet Example 21

SF

1 Sketch each of the following graphs. Use a calculator to help establish the shape. Label the endpoints with their coordinates.

a $y = 2\sin 2\left(t - \frac{\pi}{3}\right) + 2, \quad \frac{\pi}{3} \le t \le \frac{4\pi}{3}$ **b** $y = 2\cos 3\left(t + \frac{\pi}{4}\right) - 1, \quad -\frac{\pi}{4} \le t \le \frac{5\pi}{12}$

Example 22

2 Sketch the graph of each of the following for $x \in [0, 2\pi]$. List the x-axis intercepts of each graph for this interval.

a $y = 2\sin(x) + 1$ **b** $y = 2\sin(2x) - \sqrt{3}$ **c** $y = \sqrt{2}\cos(x) + 1$

d $y = 2\sin(2x) - 2$ **e** $y = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right) + 1$

3 Sketch the graph of each of the following for $x \in [-\pi, 2\pi]$:

a $y = 2\sin(3x) - 2$ **b** $y = 2\cos 3\left(x - \frac{\pi}{4}\right)$ **c** $y = 2\sin(2x) - 3$

d $y = 2\cos(2x) + 1$ **e** $y = 2\cos 2\left(x - \frac{\pi}{3}\right) - 1$ **f** $y = 2\sin 2\left(x + \frac{\pi}{6}\right) + 1$

CF

4 Sketch the graph of each of the following, showing one complete cycle. State the period, amplitude and range in each case.

a $y = 2\sin\left(\theta - \frac{\pi}{3}\right)$ **b** $y = \sin 2(\theta - \pi)$ **c** $y = 3\sin 2\left(\theta + \frac{\pi}{4}\right)$

d $y = \sqrt{3}\sin 3\left(\theta - \frac{\pi}{2}\right)$ **e** $y = 2\sin(3x) + 1$ **f** $y = 3\cos 2\left(x + \frac{\pi}{2}\right) - 1$

g $y = \sqrt{2}\sin 2\left(\theta - \frac{\pi}{6}\right) + 2$ **h** $y = 3 - 4\sin(2x)$ **i** $y = 2 - 3\cos 2\left(\theta - \frac{\pi}{2}\right)$

5 Find the equation of the image of the graph of $y = \cos x$ under:

a a dilation of factor $\frac{1}{2}$ from the x-axis, followed by a dilation of factor 3 from the y-axis, followed by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis

b a dilation of factor 2 from the x-axis, followed by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis

c a dilation of factor $\frac{1}{3}$ from the x-axis, followed by a reflection in the x-axis, then followed by a translation of $\frac{\pi}{3}$ units in the positive direction of the x-axis.

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CF

6 Give a sequence of transformations that takes the graph of $y = \sin x$ to the graph of:

a $y = -3\sin(2x)$ **b** $y = -3\sin 2\left(x - \frac{\pi}{3}\right)$

c $y = 3\sin 2\left(x - \frac{\pi}{3}\right) + 2$ **d** $y = 5 - 2\sin 2\left(x - \frac{\pi}{3}\right)$

7 Sketch the graph of each of the following for $x \in [0, 2\pi]$. List the x-axis intercepts of each graph for this interval.

a $y = 2\cos x + 1$ **b** $y = 2\cos(2x) - \sqrt{3}$ **c** $y = \sqrt{2}\cos x - 1$

d $y = 2\cos x - 2$ **e** $y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) + 1$

8 Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 2\sin\left(x - \frac{\pi}{4}\right) + 1$ **b** $y = 1 - 2\sin x$ **c** $y = 2\cos 3\left(x - \frac{\pi}{4}\right)$

d $y = 2\cos\left(3x - \frac{\pi}{4}\right)$ **e** $y = 1 - \cos(2x)$ **f** $y = -1 - \sin x$

4H Determining rules for graphs of trigonometric functions

In the previous chapter, we introduced procedures for finding the rule for a graph known to come from a polynomial function. In this section, we find rules for graphs of functions known to be of the form $f(t) = A\sin(nt + \varepsilon) + b$.

Example 23

A function has rule $f(t) = A\sin(nt)$. The amplitude is 6; the period is 10.

Find A and n and sketch the graph of $y = f(t)$ for $0 \le t \le 10$.

Solution

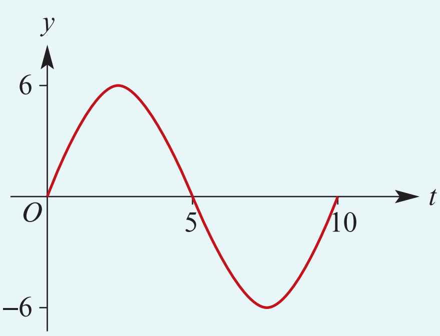
$$\text{Period} = \frac{2\pi}{n} = 10$$

$$\therefore \quad n = \frac{\pi}{5}$$

The amplitude is 6 and therefore $A = 6$.

The function has rule

$$f(t) = 6\sin\left(\frac{\pi t}{5}\right)$$



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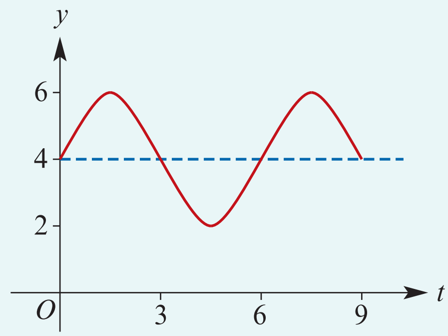
Example 24

a The graph shown on the right has a rule of the form $y = A\sin(nt) + b$. Find the values of A, n and b.



b A function with rule $y = A\sin(nt) + b$ has range $[-2, 4]$ and period 3. Find the values of A, n and b.

Solution

a The amplitude $A = 2$.

The period is 6. Therefore $\dfrac{2\pi}{n} = 6$ and so $n = \dfrac{\pi}{3}$.

The 'centreline' has equation $y = 4$ and so $b = 4$.

Hence the rule is $y = 2\sin\left(\dfrac{\pi t}{3}\right) + 4$.

b The amplitude $A = \frac{1}{2}(4 - (-2)) = 3$.

The period is 3. Therefore $\dfrac{2\pi}{n} = 3$ and so $n = \dfrac{2\pi}{3}$.

The 'centreline' has equation $y = 1$ and so $b = 1$.

Hence the rule is $y = 3\sin\left(\dfrac{2\pi t}{3}\right) + 1$.

Example 25

A function with rule $y = A\sin(nt + \varepsilon)$ has the following properties:

- range $= [-2, 2]$
- period $= 6$
- when $t = 4$, $y = 0$.

Find values for A, n and ε.

Solution

Since the range is $[-2, 2]$, the amplitude $A = 2$.

Since the period is 6, we have $\dfrac{2\pi}{n} = 6$, which implies $n = \dfrac{\pi}{3}$.

Hence $y = 2\sin\left(\dfrac{\pi}{3}t + \varepsilon\right)$. When $t = 4$, $y = 0$ and so

$$2\sin\left(\frac{4\pi}{3} + \varepsilon\right) = 0$$

$$\sin\left(\frac{4\pi}{3} + \varepsilon\right) = 0$$

Therefore $\dfrac{4\pi}{3} + \varepsilon = 0$ or $\pm\pi$ or $\pm 2\pi$ or ...

We choose the simplest solution, which is $\varepsilon = -\dfrac{4\pi}{3}$.

The rule $y = 2\sin\left(\dfrac{\pi}{3}t - \dfrac{4\pi}{3}\right)$ satisfies the three properties.

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Exercise 4H

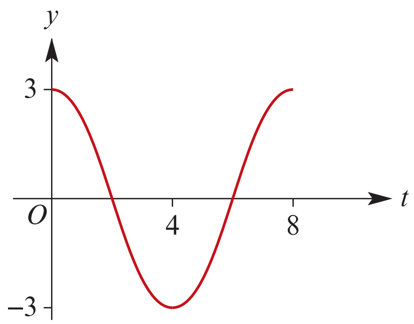
Example 23 **1** **a** A function has rule $f(t) = A\sin(nt)$. The amplitude is 4; the period is 6. Find A and n and sketch the graph of $y = f(t)$ for $0 \le t \le 6$. Assume $A > 0$, $n > 0$.

b A function has rule $f(t) = A\sin(nt)$. The amplitude is 2; the period is 7. Find A and n and sketch the graph of $y = f(t)$ for $0 \le t \le 7$. Assume $A > 0$, $n > 0$.

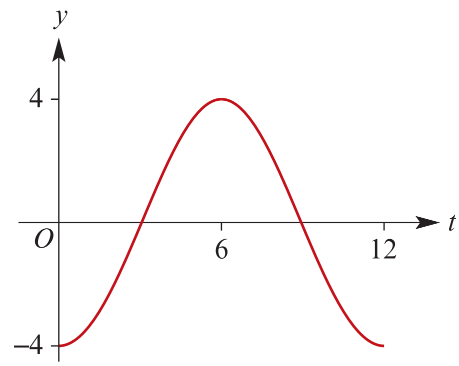
c A function has rule $f(t) = A\cos(nt)$. The amplitude is 3; the period is 5. Find A and n and sketch the graph of $y = f(t)$ for $0 \le t \le 5$. Assume $A > 0$, $n > 0$.

Example 24a **2** For each of the following graphs, the function has a rule of the form $y = A\cos(nt)$. Find the values of A and n.

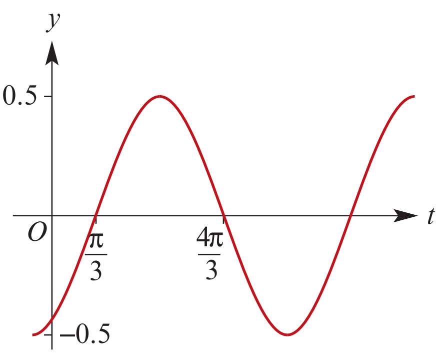
a



b



3 The graph shown has a rule of the form $y = A\sin(t + \varepsilon)$.
Find possible values for A and ε.



SF

Example 24b **4** A function with rule $y = A\sin(nt) + b$ has range $[2, 8]$ and period $\frac{2\pi}{3}$. Find A, n and b.

CF

Example 25 **5** A function with rule $y = A\sin(nt + \varepsilon)$ has the following three properties:

- range $= [-4, 4]$
- period $= 8$
- when $t = 2$, $y = 0$.

Find values for A, n and ε.

6 A function with rule $y = A\sin(nt + \varepsilon)$ has range $[-2, 2]$ and period 6, and when $t = 1$, $y = 1$. Find possible values for A, n and ε.

7 A function with rule $y = A\sin(nt + \varepsilon) + d$ has range $[-2, 6]$ and period 8, and when $t = 2$, $y = 2$. Find possible values for A, n, d and ε.

8 A function with rule $y = A\sin(nt + \varepsilon) + d$ has range $[0, 4]$ and period 6, and when $t = 1$, $y = 3$. Find possible values for A, n, d and ε.

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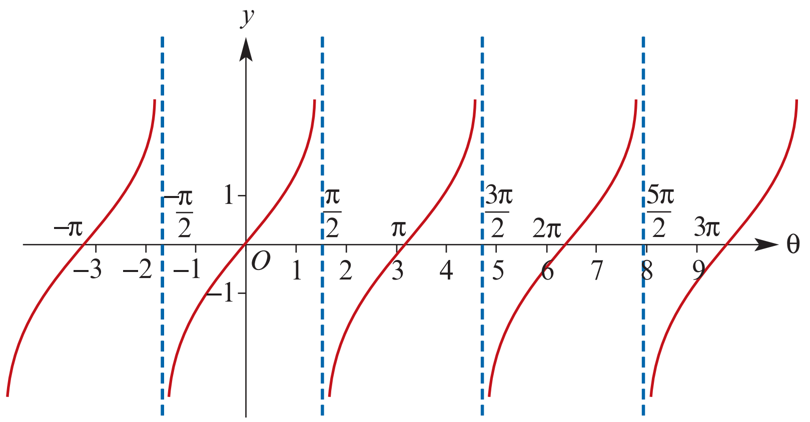
4I The tangent function

The tangent function is given by

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{for } \cos\theta \neq 0$$

A table of values for $y = \tan\theta$ is given below:

θ	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	1	ud	−1	0	1	ud	−1	0	1	ud	−1	0	1	ud	−1	0



Note: There are vertical asymptotes at $\theta = \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ and $\frac{5\pi}{2}$.

Observations from the graph of $y = \tan\theta$

- The graph repeats itself every π units, i.e. the period of tan is π.
- The range of tan is $\mathbb{R}$.
- The vertical asymptotes have equations $\theta = \frac{(2k+1)\pi}{2}$ where $k \in \mathbb{Z}$.
- The axis intercepts are at $\theta = k\pi$ where $k \in \mathbb{Z}$.

Graph of $y = a\tan(nt)$

For a and n positive numbers, the graph of $y = a\tan(nt)$ is obtained from the graph of $y = \tan t$ by a dilation of factor a from the t-axis and a dilation of factor $\frac{1}{n}$ from the y-axis.

The following are important properties of $f(t) = a\tan(nt)$:

- The period is $\frac{\pi}{n}$.
- The range is $\mathbb{R}$.
- The vertical asymptotes have equations $t = \frac{(2k+1)\pi}{2n}$ where $k \in \mathbb{Z}$.
- The axis intercepts are at $t = \frac{k\pi}{n}$ where $k \in \mathbb{Z}$.

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Example 26

Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

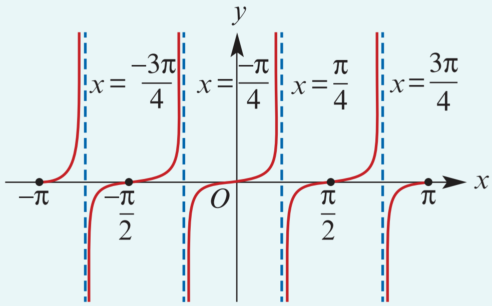
a $y = 3\tan(2x)$ **b** $y = -2\tan(3x)$

Solution

a Period $= \dfrac{\pi}{n} = \dfrac{\pi}{2}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{4}, \ k \in \mathbb{Z}$

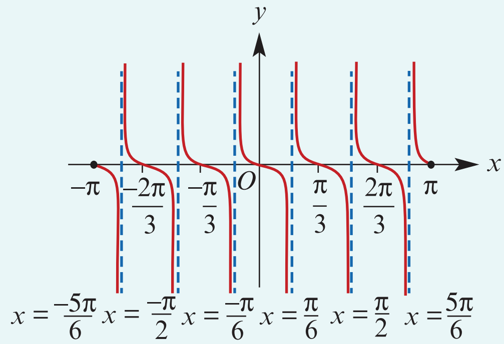
Axis intercepts: $x = \dfrac{k\pi}{2}, \ k \in \mathbb{Z}$



b Period $= \dfrac{\pi}{n} = \dfrac{\pi}{3}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{6}, \ k \in \mathbb{Z}$

Axis intercepts: $x = \dfrac{k\pi}{3}, \ k \in \mathbb{Z}$



Example 27

Sketch the graph of $y = 3\tan\left(2x - \dfrac{\pi}{3}\right)$ for $\dfrac{\pi}{6} \le x \le \dfrac{13\pi}{6}$.

Solution

Consider $y = 3\tan 2\left(x - \dfrac{\pi}{6}\right)$.

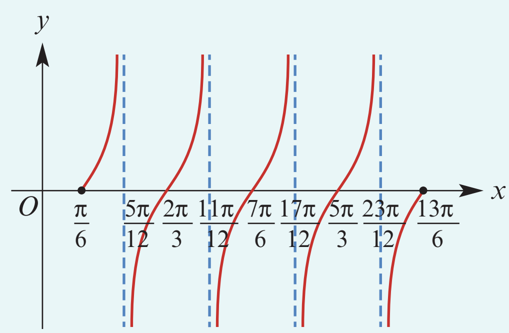
The graph is the image of $y = \tan x$ under:

- a dilation of factor 3 from the x-axis
- a dilation of factor $\frac{1}{2}$ from the y-axis
- a translation of $\dfrac{\pi}{6}$ units in the positive direction of the x-axis.

Period $= \dfrac{\pi}{2}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{4} + \dfrac{\pi}{6} = \dfrac{(6k+5)\pi}{12}, \ k \in \mathbb{Z}$

Axis intercepts: $x = \dfrac{k\pi}{2} + \dfrac{\pi}{6} = \dfrac{(3k+1)\pi}{6}, \ k \in \mathbb{Z}$



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Solution of equations involving the tangent function

We now consider the solution of equations involving the tangent function. We will then apply this to finding the x-axis intercepts for graphs of the tangent function which have been translated parallel to the y-axis.

We recall the following exact values:

$$\tan 0 = 0, \qquad \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}, \qquad \tan\left(\frac{\pi}{4}\right) = 1, \qquad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

and the symmetry properties:

- $\tan(\pi + \theta) = \tan\theta$
- $\tan(-\theta) = -\tan\theta$

Example 28

Solve the equation $3\tan(2x) = \sqrt{3}$ for $x \in (0, 2\pi)$.

Solution

$$3\tan(2x) = \sqrt{3}$$

$$\tan(2x) = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

$$\therefore \quad 2x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{19\pi}{6}$$

$$x = \frac{\pi}{12} \text{ or } \frac{7\pi}{12} \text{ or } \frac{13\pi}{12} \text{ or } \frac{19\pi}{12}$$

Explanation

Since we want solutions for x in $(0, 2\pi)$, we find solutions for $2x$ in $(0, 4\pi)$.

Once we have found one solution for $2x$, we can obtain all other solutions by adding and subtracting multiples of π.

Example 29

Solve the equation $\tan\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right) = -1$ for $x \in [-2\pi, 2\pi]$.

Solution

$$\tan\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right) = -1$$

implies

$$\frac{1}{2}\left(x - \frac{\pi}{4}\right) = \frac{-\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$x - \frac{\pi}{4} = \frac{-\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\therefore \quad x = \frac{-\pi}{4} \text{ or } \frac{7\pi}{4}$$

Explanation

Note that

$$x \in [-2\pi, 2\pi] \quad \Leftrightarrow \quad x - \frac{\pi}{4} \in \left[-\frac{9\pi}{4}, \frac{7\pi}{4}\right]$$

$$\Leftrightarrow \quad \frac{1}{2}\left(x - \frac{\pi}{4}\right) \in \left[-\frac{9\pi}{8}, \frac{7\pi}{8}\right]$$

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Example 30

Sketch the graph of $y = 3\tan\left(2x - \frac{\pi}{3}\right) + \sqrt{3}$ for $\frac{\pi}{6} \le x \le \frac{13\pi}{6}$.

Solution

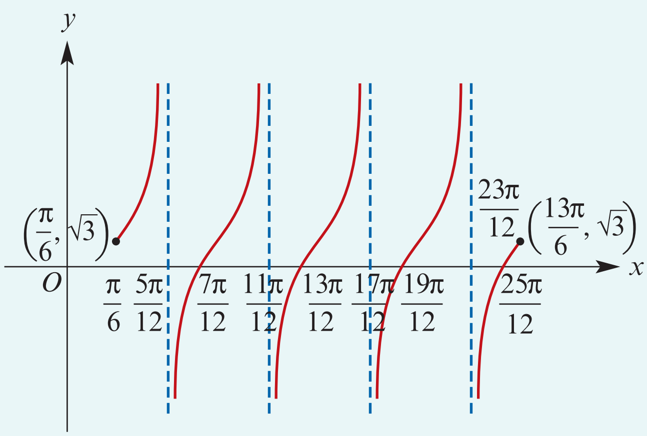
First write the equation as

$$y = 3\tan 2\left(x - \frac{\pi}{6}\right) + \sqrt{3}$$

The graph is the image of $y = \tan x$ under:

- a dilation of factor 3 from the x-axis
- a dilation of factor $\frac{1}{2}$ from the y-axis
- a translation of $\frac{\pi}{6}$ units in the positive direction of the x-axis
- a translation of $\sqrt{3}$ units in the positive direction of the y-axis.

The graph can be obtained from the graph in Example 27 by a translation $\sqrt{3}$ units in the positive direction of the y-axis.



To find the x-axis intercepts, solve the equation $3\tan\left(2x - \frac{\pi}{3}\right) + \sqrt{3} = 0$ for $\frac{\pi}{6} \le x \le \frac{13\pi}{6}$.

The x-axis intercepts are $\frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$ and $\frac{25\pi}{12}$.

▶ Solution of equations of the form $\sin(nx) = k\cos(nx)$

We can find the coordinates of the points of intersection of certain sine and cosine graphs by using the following observation:

If $\sin(nx) = k\cos(nx)$, then $\tan(nx) = k$.

This is obtained by dividing both sides of the equation $\sin(nx) = k\cos(nx)$ by $\cos(nx)$, for $\cos(nx) \ne 0$.

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Example 31

On the same set of axes, sketch the graphs of $y = \sin x$ and $y = \cos x$ for $x \in [0, 2\pi]$ and find the coordinates of the points of intersection.

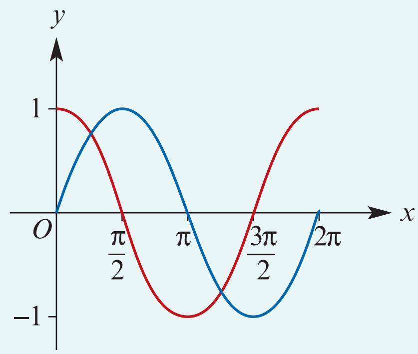
Solution

$\sin x = \cos x$ implies $\tan x = 1$

$$\therefore \quad x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

The coordinates of the points of intersection are

$$\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$$



Example 32

Solve the equation $\sin(2x) = \cos(2x)$ for $x \in [0, 2\pi]$.

Solution

$\sin(2x) = \cos(2x)$ implies $\tan(2x) = 1$

$$\therefore \quad 2x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \text{ or } \frac{9\pi}{4} \text{ or } \frac{13\pi}{4}$$

$$\therefore \quad x = \frac{\pi}{8} \text{ or } \frac{5\pi}{8} \text{ or } \frac{9\pi}{8} \text{ or } \frac{13\pi}{8}$$

Section summary

- The tangent function is given by $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$.
  - The period is π.
  - The vertical asymptotes have equations $\theta = \dfrac{(2k+1)\pi}{2}$ where $k \in \mathbb{Z}$.
  - The axis intercepts are at $\theta = k\pi$ where $k \in \mathbb{Z}$.
- Useful symmetry properties:
  - $\tan(\pi + \theta) = \tan\theta$
  - $\tan(-\theta) = -\tan\theta$

Exercise 4I

SF

1 State the period for each of the following:

a $\tan(3\theta)$ **b** $\tan\left(\frac{\theta}{2}\right)$ **c** $\tan\left(\frac{3\theta}{2}\right)$ **d** $\tan(\pi\theta)$ **e** $\tan\left(\frac{\pi\theta}{2}\right)$

Example 26

2 Sketch the graph of each of the following for $x \in (0, 2\pi)$:

a $y = \tan(2x)$ **b** $y = 2\tan(3x)$ **c** $y = -2\tan(2x)$

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Example 27 **3** Sketch the graph of each of the following for $x \in (0, 2\pi)$:

a $y = 2\tan\left(x + \frac{\pi}{4}\right)$ **b** $y = 2\tan 3\left(x + \frac{\pi}{2}\right)$ **c** $y = 3\tan 2\left(x - \frac{\pi}{4}\right)$

Example 28 **4** Solve:

a $\tan(2x) = 1$ for $x \in (0, 2\pi)$ **b** $\tan(2x) = -1$ for $x \in (-\pi, \pi)$

c $\tan(2x) = -\sqrt{3}$ for $x \in (-\pi, \pi)$ **d** $\tan(2x) = \sqrt{3}$ for $x \in (-\pi, \pi)$

e $\tan(2x) = \frac{1}{\sqrt{3}}$ for $x \in (-\pi, \pi)$

Example 29 **5** Solve the equation $\tan 2\left(x - \frac{\pi}{3}\right) = 1$ for $x \in [0, 2\pi]$.

6 Solve the equation $\tan\left(x - \frac{\pi}{4}\right) = \sqrt{3}$ for $x \in [0, 2\pi]$.

Example 30 **7** Sketch the graph of each of the following for $x \in (0, 2\pi)$:

a $y = 3\tan x + 1$ **b** $y = 2\tan\left(x + \frac{\pi}{2}\right) + 1$ **c** $y = 3\tan 2\left(x - \frac{\pi}{4}\right) - 2$

8 Sketch the graph of $y = -2\tan(\pi\theta)$ for $-2 \le \theta \le 2$.

9 Sketch the graph of $y = \tan(-2\theta)$ for $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$.

Example 31 **10** **a** On the same set of axes, sketch the graphs of $y = \cos(2x)$ and $y = -\sin(2x)$ for $x \in [-\pi, \pi]$.

b Find the coordinates of the points of intersection.

Example 32 **11** Solve each of the following equations for $x \in [0, 2\pi]$:

a $\sqrt{3}\sin x = \cos x$ **b** $\sin(4x) = \cos(4x)$ **c** $\sqrt{3}\sin(2x) = \cos(2x)$

d $-\sqrt{3}\sin(2x) = \cos(2x)$ **e** $\sin(3x) = -\cos(3x)$ **f** $\sin x = 0.5\cos x$

g $\sin x = 2\cos x$ **h** $\sin(2x) = -\cos(2x)$ **i** $\cos(3x) = \sqrt{3}\sin(3x)$

j $\sin(3x) = \sqrt{3}\cos(3x)$

12 **a** On the same set of axes, sketch the graphs of $y = \cos x$ and $y = \sqrt{3}\sin x$ for $x \in [0, 2\pi]$.

b Find the coordinates of the points of intersection.

13 Solve each of the following equations for $0 \le x \le 2\pi$:

a $\tan\left(2x - \frac{\pi}{4}\right) = \sqrt{3}$ **b** $3\tan(2x) = -\sqrt{3}$ **c** $\tan\left(3x - \frac{\pi}{6}\right) = -1$

14 A function with rule $y = A\tan(nt)$ has the following properties:

- the asymptotes have equations $t = \frac{(2k+1)\pi}{6}$ where $k \in \mathbb{Z}$
- when $t = \frac{\pi}{12}$, $y = 5$.

Find values for A and n.

15 A function with rule $y = A\tan(nt)$ has period 2 and, when $t = \frac{1}{2}$, $y = 6$. Find values for A and n.

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4J Applications of trigonometric functions

A **sinusoidal function** has a rule of the form $y = a\sin(nt + \varepsilon) + b$ or, equivalently, of the form $y = a\cos(nt + \varepsilon) + b$. Such functions can be used to model periodic motion.

Example 33

A wheel is mounted on a wall and rotates such that the distance, d cm, of a particular point P on the wheel from the ground is given by the rule

$$d = 100 - 60\cos\left(\frac{4\pi}{3}t\right)$$

where t is the time in seconds.

a How far is the point P above the ground when $t = 0$?

b How long does it take for the wheel to rotate once?

c Find the maximum and minimum distances of the point P above the ground.

d Sketch the graph of d against t.

e In the first rotation, find the intervals of time when the point P is less than 70 cm above the ground.

Solution

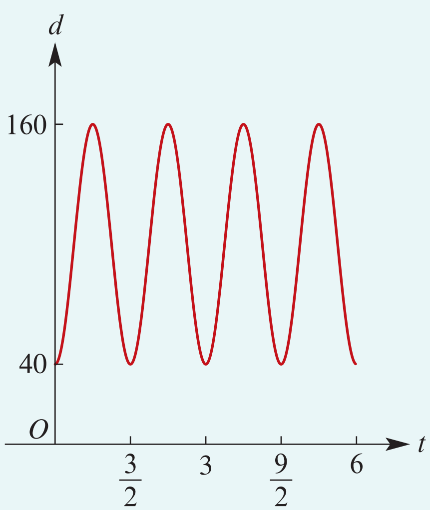
a When $t = 0$, $d = 100 - 60 \times 1 = 40$. The point is 40 cm above the ground.

b The period is $2\pi \div \frac{4\pi}{3} = \frac{3}{2}$. The wheel takes $\frac{3}{2}$ seconds to rotate once.

c The minimum occurs when $\cos\left(\frac{4\pi}{3}t\right) = 1$, which gives $d = 100 - 60 = 40$. Hence the minimum distance is 40 cm.

The maximum occurs when $\cos\left(\frac{4\pi}{3}t\right) = -1$, which gives $d = 100 + 60 = 160$. Hence the maximum distance is 160 cm.

d



e

$$100 - 60\cos\left(\frac{4\pi}{3}t\right) = 70$$

$$-60\cos\left(\frac{4\pi}{3}t\right) = -30$$

$$\cos\left(\frac{4\pi}{3}t\right) = \frac{1}{2}$$

$$\therefore \quad \frac{4\pi}{3}t = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$

$$t = \frac{1}{4} \text{ or } \frac{5}{4}$$

From the graph, the distance is less than 70 cm for $0 \le t < \frac{1}{4}$ and for $\frac{5}{4} < t \le \frac{3}{2}$.

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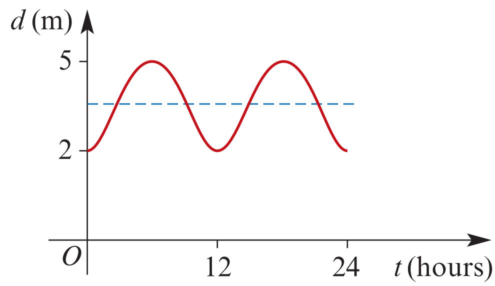
Exercise 4J

Example 33

1 The graph shows the distance, $d(t)$, of the tip of the hour hand of a large clock from the ceiling at time t hours.



a The function d is sinusoidal. Find:

i the amplitude

ii the period

iii the rule for $d(t)$

iv the length of the hour hand.

b At what times is the distance less than 3.5 metres from the ceiling?

2 In a tidal river, the time between high tides is 12 hours. The average depth of water at a point in the river is 5 m; at high tide the depth is 8 m. Assume that the depth of water, $h(t)$ m, at this point is given by

$$h(t) = A\sin(nt + \varepsilon) + b$$

where t is the number of hours after noon. At noon there is a high tide.

a Find the values of A, n, b and ε.

b At what times is the depth of the water 6 m?

c Sketch the graph of $y = h(t)$ for $0 \le t \le 24$.

3 Passengers on a ferris wheel access their seats from a platform 5 m above the ground. As each seat is filled, the ferris wheel moves around so that the next seat can be filled. Once all seats are filled, the ride begins and lasts for 6 minutes. The height, h m, of Isobel's seat above the ground t seconds after the ride has begun is given by $h = 15\sin(10t - 45)° + 16.5$.

a Use a calculator to sketch the graph of h against t for the first 2 minutes of the ride.

b How far above the ground is Isobel's seat at the commencement of the ride?

c After how many seconds does Isobel's seat pass the access platform?

d How many times will her seat pass the access platform in the first 2 minutes?

e How many times will her seat pass the access platform during the entire ride?

Due to a malfunction, the ferris wheel stops abruptly 1 minute 40 seconds into the ride.

f How far above the ground is Isobel stranded?

g If Isobel's brother Hamish had a seat 1.5 m above the ground at the commencement of the ride, how far above the ground is Hamish stranded?

CF

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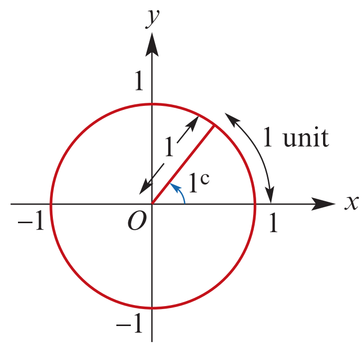
Chapter summary

AS
Nrich

- **Definition of a radian**

  One radian (written 1^c) is the angle formed at the centre of the unit circle by an arc of length 1 unit.

  $$1^c = \frac{180^\circ}{\pi} \qquad 1^\circ = \frac{\pi^c}{180}$$



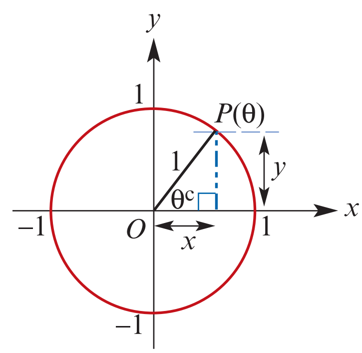
- **Sine and cosine functions**

  x-coordinate of $P(\theta)$ on unit circle:

  $$x = \cos\theta, \qquad \theta \in \mathbb{R}$$

  y-coordinate of $P(\theta)$ on unit circle:

  $$y = \sin\theta, \qquad \theta \in \mathbb{R}$$



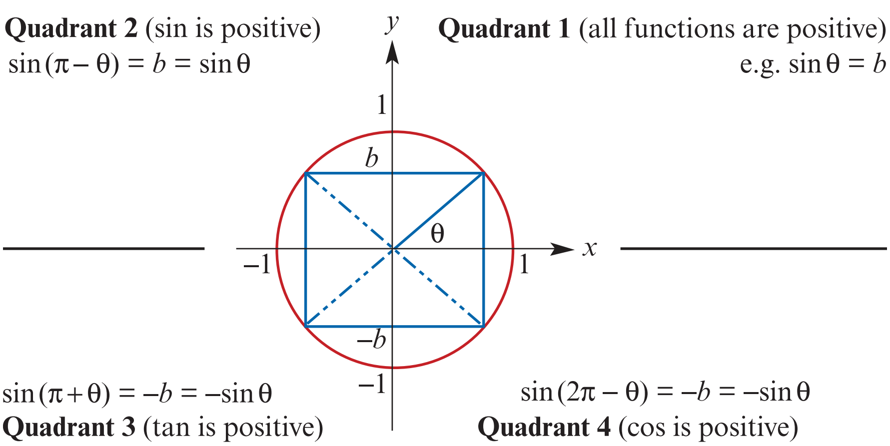
- **Tangent function**

  $$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{for } \cos\theta \neq 0$$

- **Symmetry properties of trigonometric functions**

  Quadrant 2 (sin is positive)
  $\sin(\pi - \theta) = b = \sin\theta$

  Quadrant 1 (all functions are positive)
  e.g. $\sin\theta = b$



  $\sin(\pi + \theta) = -b = -\sin\theta$
  Quadrant 3 (tan is positive)

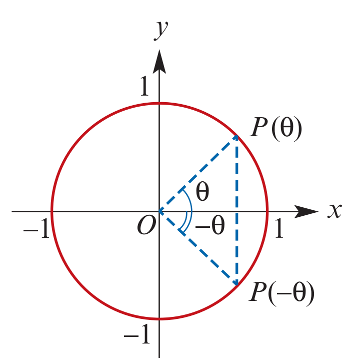
  $\sin(2\pi - \theta) = -b = -\sin\theta$
  Quadrant 4 (cos is positive)

  Negative angles:

  $$\sin(-\theta) = -\sin\theta$$

  $$\cos(-\theta) = \cos\theta$$

  $$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



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Complementary angles:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \qquad \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

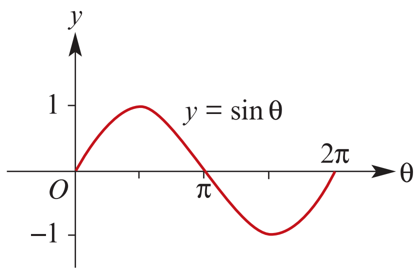
- Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

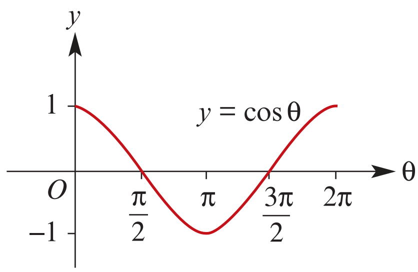
- Exact values of trigonometric functions

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

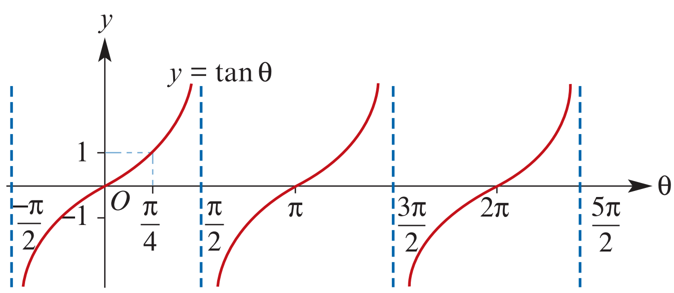
- Graphs of trigonometric functions



Amplitude = 1
Period = 2π



Amplitude = 1
Period = 2π



Amplitude is undefined
Period = π

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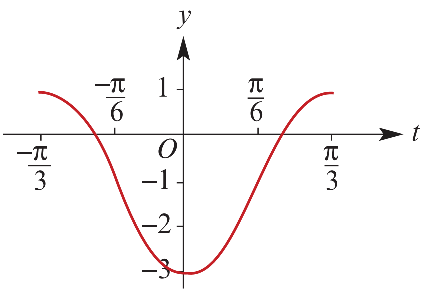
Review

- Transformations of sine and cosine graphs:
  $y = a \sin n(t \pm \varepsilon) \pm b$ and $y = a \cos n(t \pm \varepsilon) \pm b$

  e.g. $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1$



  - Amplitude, $a = 2$
  - Period $= \frac{2\pi}{n} = \frac{2\pi}{3}$
  - The graph is the same shape as $y = 2\cos(3t)$ but is translated $\frac{\pi}{3}$ units in the negative direction of the t-axis and 1 unit in the negative direction of the y-axis.
- Transformations of the graph of $y = \tan t$
  e.g. $y = a \tan n(t + \varepsilon) + b$, where $n > 0$
  - Period $= \frac{\pi}{n}$
  - Asymptotes: $t = \frac{(2k+1)\pi}{2n} - \varepsilon$, where $k \in \mathbb{Z}$
- Solution of trigonometric equations
  e.g. Solve $\cos x^\circ = -0.7$ for $x \in [0, 360]$.
  First look at the 1st quadrant: If $\cos \alpha^\circ = 0.7$, then $\alpha = 45.6$.
  Since $\cos x^\circ$ is negative for $P(x^\circ)$ in the 2nd and 3rd quadrants, the solutions are

$$x = 180 - 45.6 = 134.4 \quad \text{and} \quad x = 180 + 45.6 = 225.6$$

Technology-free questions

SF

1 Find:

a $\sin(2\pi - \theta)$ if $\sin\theta = 0.4$
b $\cos(-\theta)$ if $\cos\theta = -0.6$
c $\tan(\pi + \theta)$ if $\tan\theta = 2$
d $\sin(\pi + \theta)$ if $\sin\theta = 0.7$
e $\sin\left(\frac{\pi}{2} - \theta\right)$ if $\cos\theta = \frac{1}{5}$
f $\cos\theta$ if $\sin\theta = \frac{4}{5}$ and $0 < \theta < \frac{\pi}{2}$

2 Solve each of the following equations for $x \in [-\pi, 2\pi]$:

a $\sin x = \frac{1}{2}$
b $2\cos x = -1$
c $2\cos x = \sqrt{3}$
d $\sqrt{2}\sin x + 1 = 0$
e $4\sin x + 2 = 0$
f $\sin(2x) + 1 = 0$
g $\cos(2x) = \frac{-1}{\sqrt{2}}$
h $2\sin(3x) - 1 = 0$

3 Sketch the graph of each of the following, showing one cycle. Clearly label axis intercepts.

a $f(x) = \sin(3x)$
b $f(x) = 2\sin(2x) - 1$
c $g(x) = 2\sin(2x) + 1$
d $f(x) = 2\sin\left(x - \frac{\pi}{4}\right)$
e $f(x) = 2\sin\left(\frac{\pi x}{3}\right)$
f $h(x) = 2\cos\left(\frac{\pi x}{4}\right)$

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Review

SF

4 Solve each of the following equations for $x \in [0, 360]$:

a $\sin x° = 0.5$ **b** $\cos(2x)° = 0$ **c** $2\sin x° = -\sqrt{3}$

d $\sin(2x + 60)° = \frac{-\sqrt{3}}{2}$ **e** $2\sin(\frac{1}{2}x)° = \sqrt{3}$

CF

5 Sketch the graph of each of the following, showing one cycle. Clearly label axis intercepts.

a $y = 2\sin\left(x + \frac{\pi}{3}\right) + 2$ **b** $y = -2\sin\left(x + \frac{\pi}{3}\right) + 1$

c $y = 2\sin\left(x - \frac{\pi}{4}\right) + \sqrt{3}$ **d** $y = -3\sin x$

e $y = \sin\left(x - \frac{\pi}{6}\right) + 3$ **f** $y = 2\sin\left(x - \frac{\pi}{2}\right) + 1$

6 Sketch, on the same set of axes, the curves $y = \cos x$ and $y = \sin(2x)$ for the interval $0 \le x \le 2\pi$, labelling each curve carefully. State the number of solutions in this interval for each of the following equations:

a $\sin(2x) = 0.6$ **b** $\sin(2x) = \cos x$ **c** $\sin(2x) - \cos x = 1$

SF

7 Sketch on separate axes for $0° \le x° \le 360°$:

a $y = 3\cos x°$ **b** $y = \cos(2x)°$ **c** $y = \cos(x - 30)°$

8 Solve each of the following for $x \in [-\pi, \pi]$:

a $\tan x = \sqrt{3}$ **b** $\tan x = -1$

c $\tan(2x) = -1$ **d** $\tan(2x) + \sqrt{3} = 0$

CF

9 Solve the equation $\sin x = \sqrt{3}\cos x$ for $x \in [-\pi, \pi]$.

10 The graphs of $y = a\cos x$ and $y = \sin x$, where a is a real constant, have a point of intersection at $x = \frac{\pi}{6}$.

a Find the value of a.

b If $x \in [-\pi, \pi]$, find the x-coordinate(s) of the other point(s) of intersection of the two graphs.

Multiple-choice questions

1 The period of the graph of $y = 3\sin\left(\frac{1}{2}x - \pi\right) + 4$ is

A π **B** 3 **C** 4π **D** $\pi + 4$ **E** 2π

2 The range of the graph of $y = f(x)$, where $f(x) = 5\cos\left(2x - \frac{\pi}{3}\right) - 7$, is

A $[-12, -2]$ **B** $[-7, 7]$ **C** $(-2, 5)$ **D** $[-2, 5]$ **E** $[-2, 12]$

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3 Let $f(x) = a\sin(bx) + c$ for all $x \in \mathbb{R}$, where a, b and c are positive constants. The function f has period

A a **B** b **C** $\frac{2\pi}{a}$ **D** $\frac{2\pi}{b}$ **E** $\frac{b}{2\pi}$

4 Define the function f by $f(x) = \cos(3x) - 1$ for $x \in \left[0, \frac{2\pi}{9}\right)$. The range of f is

A $(-1, 0)$ **B** $(-2, 0]$ **C** $\left(-\frac{3}{2}, 0\right]$ **D** $\left[-\frac{3}{2}, 0\right)$ **E** $[-2, 0)$

5 The vertical distance of a point on a wheel from the ground as it rotates is given by $D(t) = 3 - 3\sin(6\pi t)$, where t is the time in seconds. The time in seconds for a full rotation of the wheel is

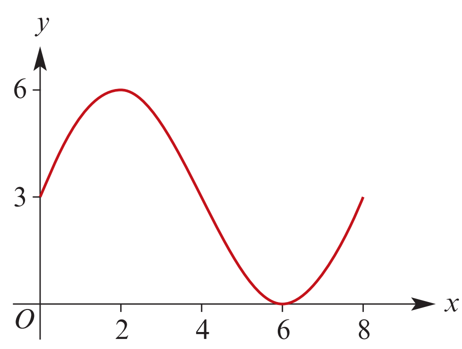
A $\frac{1}{6\pi}$ **B** $\frac{1}{3}$ **C** 6π **D** $\frac{1}{3\pi}$ **E** 3

6 The equation of the image of $y = \cos x$ under a transformation of a dilation of factor 2 from the x-axis, followed by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis is

A $y = \cos\left(\frac{1}{2}x + \frac{\pi}{4}\right)$ **B** $y = \cos\left(\frac{1}{2}x - \frac{\pi}{4}\right)$ **C** $y = 2\cos\left(x + \frac{\pi}{4}\right)$

D $y = 2\sin\left(x - \frac{\pi}{4}\right)$ **E** $y = 2\cos\left(x - \frac{\pi}{4}\right)$

7 Which of the following is likely to be the rule for the graph of the trigonometric function shown?



A $y = 3 + 3\cos\left(\frac{\pi x}{4}\right)$

B $y = 3 + 3\sin\left(\frac{\pi x}{4}\right)$

C $y = 3 + 3\sin(4\pi x)$

D $y = 3 + 3\cos\left(\frac{x}{4}\right)$

E $y = 3 + 3\sin\left(\frac{x}{4}\right)$

8 Consider the function $f(x) = \cos(3x) - 2$, $x \in \left[0, \frac{\pi}{2}\right]$. If the graph of f is transformed by a reflection in the x-axis followed by a dilation of factor 3 from the y-axis, then the resulting graph is defined by

A $g(x) = 6 - 3\cos(3x),\ x \in \left[0, \frac{\pi}{2}\right]$ **B** $g(x) = 3\cos x - 6,\ x \in \left[0, \frac{\pi}{2}\right]$

C $g(x) = 2 - \cos x,\ x \in \left[0, \frac{3\pi}{2}\right]$ **D** $g(x) = \cos(-x) - 2,\ x \in \left[0, \frac{\pi}{2}\right]$

E $g(x) = \cos(9x) + 1,\ x \in \left[0, \frac{3\pi}{2}\right]$

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Review

9 The equation $3\sin x - 1 = b$, where b is a positive real number, has one solution in the interval $(0, 2\pi)$. The value of b is

A 2 **B** 0.2 **C** 3 **D** 5 **E** 6

10 Let $f(x) = p\cos(5x) + q$, where $p > 0$. Then $f(x) \leq 0$ for all values of x if

A $q \geq 0$ **B** $-p \leq q \leq p$ **C** $p \leq -q$ **D** $p \geq q$ **E** $-q \leq p$

Extended-response questions

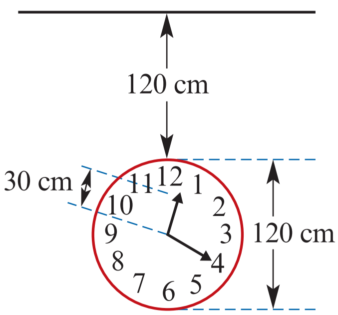
1 In a tidal river, the time between high tide and low tide is 6 hours. The average depth of water at a point in the river is 4 metres; at high tide the depth is 5 metres. CU

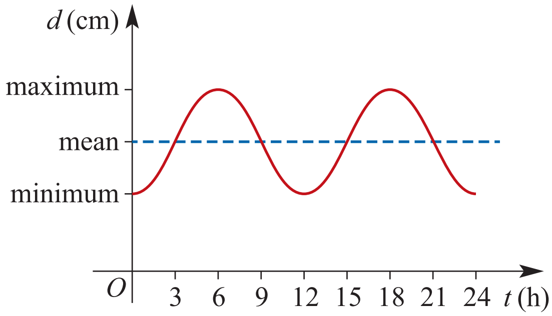
a Sketch the graph of the depth of water at the point for the time interval from 0 to 24 hours if the relationship between time and depth is sinusoidal and there is high tide at noon.

b If a boat requires a depth of 4 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?

c If a boat requires a depth of 3.5 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?

2 A clock hangs 120 cm below a ceiling. The diameter of the clock is 120 cm, and the length of the hour hand is 30 cm. The graph shows the distance from the ceiling to the tip of the hour hand over a 24-hour period.





a What are the values for the maximum, minimum and mean distance?

b An equation that determines this curve is of the form

$$y = A\sin(nt + \varepsilon) + b$$

Find suitable values of A, n, ε and b.

c Find the distance from the ceiling to the tip of the hour hand at:

i 2 a.m. **ii** 11 p.m.

d Find the times in the morning at which the tip of the hour hand is 200 cm below the ceiling.

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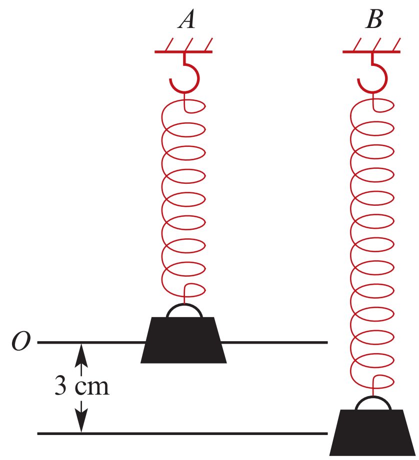
3 A weight is suspended from a spring as shown. The weight is pulled down 3 cm from O and released. The vertical displacement from O at time t is described by a function of the form

$$y = a\cos(nt)$$

where y cm is the vertical displacement at time t seconds. The following data were recorded.



t	0	0.5	1
y	−3	3	−3

It was also noted that the centre of the weight went no further than 3 cm from the centre O.

a Find the values of a and n.

b Sketch the graph of y against t.

c Find when the centre of the weight is first:

i 1.5 cm above O **ii** 1.5 cm below O.

d When does the weight first reach a point 1 cm below O?

4 The manager of a reservoir and its catchment area has noted that the inflow of water into the reservoir is very predictable and in fact models the inflow using a function with rule of the form

$$y = a\sin(nt + \varepsilon) + b$$

The following observations were made:

- The average inflow is 100 000 m^3/day.
- The minimum daily inflow is 80 000 m^3/day.
- The maximum daily inflow is 120 000 m^3/day, and this occurs on 1 May ($t = 121$) each year.

a Find the values of a, b and n and the smallest possible positive value for ε.

b Sketch the graph of y against t.

c Find the times of year when the inflow per day is:

i 90 000 m^3/day **ii** 110 000 m^3/day

d Find the inflow rate on 1 June.

5 The number of hours of daylight at a point on the Antarctic Circle is given approximately by $d = 12 + 12\cos\left(\frac{1}{6}\pi\left(t + \frac{1}{3}\right)\right)$, where t is the number of months that have elapsed since 1 January.

a **i** Find d on 21 June ($t \approx 5.7$).

ii Find d on 21 March ($t \approx 2.7$).

b When will there be 5 hours of daylight?

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CU

6 The depth, $D(t)$ m, of water at the entrance to a harbour at t hours after midnight on a particular day is given by $D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right)$, $0 \le t \le 24$.

a Sketch the graph of $y = D(t)$ for $0 \le t \le 24$.

b Find the values of t for which $D(t) \ge 8.5$.

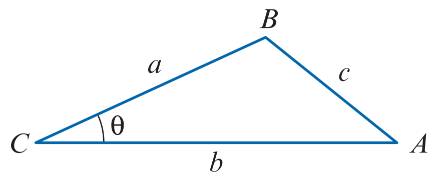
c Boats that need a depth of w m are permitted to enter the harbour only if the depth of the water at the entrance is at least w m for a continuous period of 1 hour. Find, correct to one decimal place, the largest value of w that satisfies this condition.

7 The area of a triangle is given by

$$A = \frac{1}{2}ab\sin\theta$$

and the perimeter is given by

$$P = a + b + \sqrt{a^2 + b^2 - 2ab\cos\theta}$$



a For $a = b = 10$ and $\theta = \frac{\pi}{3}$, find:

i the area of the triangle

ii the perimeter of the triangle.

b For $a = b = 10$, find the value(s) of θ for which $A = P$. (Give value(s) correct to two decimal places.)

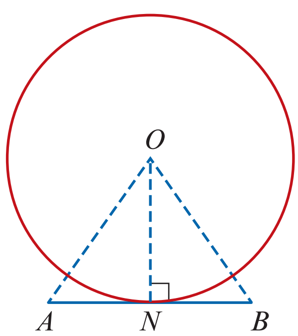
c Show graphically that, if $a = b = 6$, then $P > A$ for all θ.

d Assume $\theta = \frac{\pi}{2}$. If $a = 6$, find the value of b such that $A = P$.

e For $a = 10$ and $b = 6$, find the value(s) of θ for which $A = P$.

f If $a = b$ and $\theta = \frac{\pi}{3}$, find the value of a such that $A = P$.

8 AB is one side of a regular n-sided polygon that circumscribes a circle (i.e. each edge of the polygon is tangent to the circle). The circle has radius 1.



a Show that the area of triangle OAB is $\tan\left(\frac{\pi}{n}\right)$.

b Show that the area, A, of the polygon is given by $A = n\tan\left(\frac{\pi}{n}\right)$.

c Use a calculator to help sketch the graph of $A(x) = x\tan\left(\frac{\pi}{x}\right)$ for $x \ge 3$. Label the horizontal asymptote.

d What is the difference in area of the polygon and the circle when:

i $n = 3$ **ii** $n = 4$ **iii** $n = 12$ **iv** $n = 50$?

e State the area of an n-sided polygon that circumscribes a circle of radius r cm.

f **i** Find a formula for the area of an n-sided regular polygon that can be inscribed in a circle of radius 1.

ii Sketch the graph of this function for $x \ge 3$.

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Revision

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5 Revision of preliminary topics

5A Technology-free questions

1 State the maximal domain and range of each of the following:

a $f(x) = \frac{1}{x} + 2$ **b** $f(x) = 3 - 2\sqrt{3x - 2}$ **c** $f(x) = \frac{4}{(x-2)^2} + 3$

d $h(x) = 4 - \frac{3}{x-2}$ **e** $f(x) = \sqrt{x-2} - 5$

2 Solve the equation $\sin\left(\frac{3x}{2}\right) = \frac{1}{2}$ for $-\pi \le x \le \pi$.

3 **a** State the range and period of the function $h(x) = 5 - 3\cos\left(\frac{\pi x}{3}\right)$, $x \in \mathbb{R}$.

b Solve the equation $\cos\left(2x + \frac{\pi}{6}\right) = \frac{1}{2}$ for $x \in [0, \pi]$.

4 If a graph has rule $y = \frac{a}{x^2} + b$ and passes through the points $(1, -1)$ and $(-2, \frac{1}{2})$, find the values of a and b.

5 Find the value(s) of m for which the equation $x^2 + mx + 2 = 0$ has:

a one solution **b** two solutions **c** no solution.

6 Two points A and B have coordinates $(a, -2)$ and $(3, 1)$.

a Find the value(s) of a if:

i the midpoint of AB is $(0, -\frac{1}{2})$

ii the length of AB is $\sqrt{13}$

iii the gradient of AB is $\frac{1}{2}$.

b Find the equation of the line passing through A and B if $a = -2$, and find the angle the line makes with the positive direction of the x-axis.

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7 Let $f(x) = 2 - x$ and $g(x) = \sqrt{2x - 3}$.

a Find:

i $f(-2)$ **ii** $g(4)$ **iii** $f(2a)$ **iv** $g(a - 1)$

b Find the values of x for which:

i $f(x) = 10$ **ii** $g(x) = 10$ **iii** $f(2x) > 0$

SF

8 Let $f(x) = 4x - 3$ and $g(x) = x^2 + 2x$.

a Find:

i $f(g(x))$ **ii** $g(f(x))$

b Find a transformation that takes the graph of $y = g(x)$ to the graph of $y = g(f(x))$.

c Find a transformation that takes the graph of $y = x^2$ to the graph of $y = g(x)$.

CF

9 Solve $\cos\left(x - \frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ for $x \in [0, 2\pi]$.

SF

10 Solve each of the following inequalities for x:

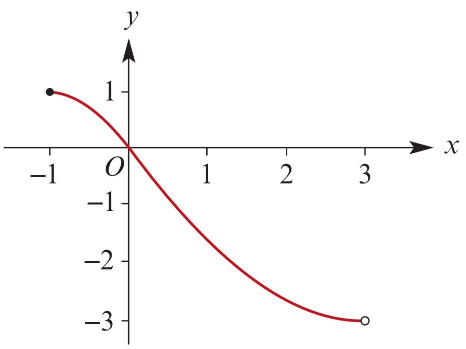
a $2x^3 - 3x^2 - 11x + 6 \geq 0$ **b** $-x^3 + 4x^2 - 4x > 0$

CF

5B Multiple-choice questions

1 The domain of the function whose graph is shown is

A $(-3, 1]$
B $(-1, 3]$
C $[1, 3]$
D $[-1, 3)$
E $(-1, 3)$



2 The implied (largest possible) domain for the function with the rule $y = \frac{1}{\sqrt{2 - x}}$ is

A $\mathbb{R} \setminus \{2\}$ **B** $(-\infty, 2)$ **C** $(2, \infty)$ **D** $(-\infty, 2]$ **E** $\mathbb{R}^+$

3 If $f(x) = \frac{x}{x - 1}$, then $f\left(-\frac{1}{a}\right)$ can be simplified as

A $\frac{1}{-1 - a}$ **B** -1 **C** 0 **D** $\frac{a^2}{1 - a}$ **E** $\frac{1}{a + 1}$

4 If $f(x) = \sin(2x)$ and $g(x) = 2\sin x$, then the value of $(f + g)\left(\frac{3\pi}{2}\right)$ is

A 2 **B** 0 **C** −1 **D** 1 **E** −2

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5 If $f(x) = 3x + 2$ and $g(x) = 2x^2$, then $f(g(3))$ equals

A 36 **B** 20 **C** 56 **D** 144 **E** 29

6 The implied domain for the function with rule $y = \sqrt{4 - x^2}$ is

A $[2, \infty)$ **B** $\{ x : -2 < x < 2 \}$ **C** $[-2, 2]$
D $(-\infty, 2)$ **E** $\mathbb{R}^+$

7 If $f(x) = 3x^2$, $0 \le x \le 6$, and $g(x) = \sqrt{2 - x}$, $x \le 2$, then the domain of $f + g$ is

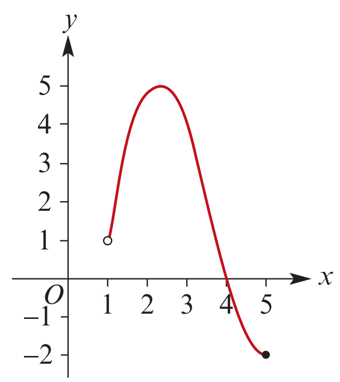
A $[0, 2]$ **B** $[0, 6]$ **C** $(-\infty, 2]$ **D** $[0, \infty)$ **E** $[2, 6]$

8 If $g(x) = 2x^2 + 1$ and $f(x) = 3x + 2$, then the product function fg has rule

A $(fg)(x) = 2x^2 + 3x + 3$ **B** $(fg)(x) = 6x^3 + 4x^2 + 3x + 2$
C $(fg)(x) = 6x^3 + 3$ **D** $(fg)(x) = 6x^3 + 2x^2 + 3$
E $(fg)(x) = 6x^3 + 2$

9 The domain of the function whose graph is shown is

A $[1, 5]$
B $(1, 5]$
C $(-2, 5]$
D $(1, 5)$
E $(-2, 5)$



10 The graph shown has the rule

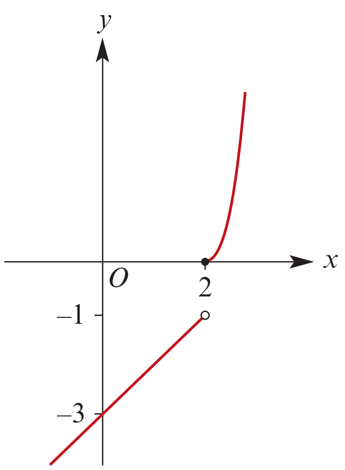
A $y = \begin{cases} (x-2)^2, & x \ge 2 \\ x - 3, & x < 2 \end{cases}$

B $y = \begin{cases} x-2, & x \ge 2 \\ x - 3, & x < 2 \end{cases}$

C $y = \begin{cases} (2-x)^2, & x \ge 2 \\ 2x - 3, & x < 2 \end{cases}$

D $y = \begin{cases} (x-2)^2, & x < 2 \\ 2x - 3, & x \ge 2 \end{cases}$

E $y = \begin{cases} (x-2)^2, & x \ge 2 \\ 2x - 3, & x < 2 \end{cases}$



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11 The graph shows

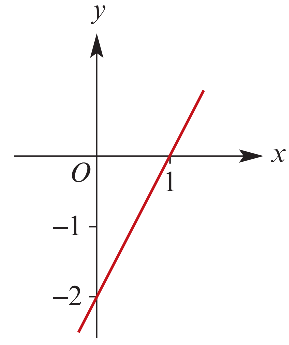
A $y + 2 = x$

B $y = 2x - 2$

C $y + 2x + 2 = 0$

D $y = -2x + 2$

E $y - 2 = x$



12 The equation of the line that passes through the points $(-2, 3)$ and $(4, 0)$ is

A $2y = x + 4$ **B** $y = -\frac{1}{2}x - 2$ **C** $2y + x = 4$

D $y = \frac{1}{2}x - 2$ **E** $2y - x = 4$

13 The straight line with equation $y = \frac{4}{5}x - 4$ meets the x-axis at A and the y-axis at B. If O is the origin, the area of the triangle OAB is

A $3\frac{1}{5}$ square units **B** $9\frac{2}{5}$ square units **C** 10 square units

D 15 square units **E** 20 square units

14 The graphs of the relations $7x - 6y = 20$ and $3x + 4y = 2$ are drawn on the same pair of axes. The x-coordinate of the point of intersection is

A -2 **B** -1 **C** 1 **D** 2 **E** 3

15 Which one of the following is an even function of x?

A $f(x) = 3x + 1$ **B** $f(x) = x^3 - x$ **C** $f(x) = (1 - x)^2$

D $f(x) = -x^2$ **E** $f(x) = x^3 + x^2$

16 Which one of the following is *not* an even function?

A $f(x) = 4x^2$ **B** $f(x) = \cos^2 x$ **C** $f(x) = \cos x$

D $f(x) = 4x^2 - 3$ **E** $f(x) = (x - 2)^2$

17 A possible equation for the graph shown is

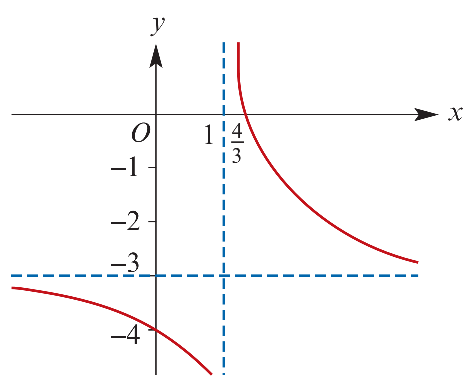
A $y - 3 = \dfrac{1}{x - 1}$

B $y + 3 = \dfrac{1}{x + 1}$

C $y - 3 = \dfrac{1}{x + 1}$

D $y - 4 = \dfrac{1}{x + 1}$

E $y = \dfrac{1}{x - 1} - 3$



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18 The function given by $f(x) = \dfrac{1}{x+3} - 2$ has the range

A $\mathbb{R} \setminus \{-2\}$ **B** $\mathbb{R}$ **C** $\mathbb{R} \setminus \{3\}$ **D** $\mathbb{R} \setminus \{2\}$ **E** $\mathbb{R} \setminus \{-3\}$

19 A parabola has its vertex at $(2, 3)$. A possible equation for this parabola is

A $y = (x+2)^2 + 3$ **B** $y = (x-2)^2 - 3$ **C** $y = (x+2)^2 - 3$
D $y = (x-2)^2 + 3$ **E** $y = 3 - (x+2)^2$

20 The graph of $y = 3\sqrt{x+2}$ can be obtained from the graph of $y = \sqrt{x}$ by

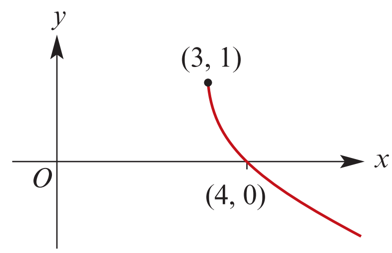
A a translation $(x, y) \to (x-2, y)$ followed by a dilation of factor 3 from the x-axis
B a translation $(x, y) \to (x+2, y)$ followed by a dilation of factor $\frac{1}{3}$ from the x-axis
C a translation $(x, y) \to (x+3, y)$ followed by a dilation of factor 3 from the y-axis
D a translation $(x, y) \to (x-2, y)$ followed by a dilation of factor 3 from the y-axis
E a translation $(x, y) \to (x+2, y)$ followed by a dilation of factor 3 from the y-axis

21 A function with rule $f(x) = 3\sqrt{x-2} + 1$ has maximal domain

A $(-\infty, 2)$ **B** $[1, \infty)$ **C** $(2, \infty)$ **D** $[-2, \infty)$ **E** $[2, \infty)$

22 A possible equation for the graph shown is

A $y = 2\sqrt{x-3} + 1$
B $y = -2\sqrt{x-3} + 1$
C $y = \sqrt{x-3} + 1$
D $y = -\sqrt{x-3} + 1$
E $y = -2\sqrt{x-3} + 2$



23 Let $f(x) = \dfrac{3}{(x-2)^2} + 4$ for $x \neq 2$. The range of f is

A $(3, 4]$ **B** $(-\infty, 4)$ **C** $[3, 4)$ **D** $[4, \infty)$ **E** $(4, \infty)$

24 If $3x^2 + kx + 1 = 0$ when $x = 1$, then k equals

A -4 **B** -1 **C** 1 **D** 4 **E** 0

25 The quadratic equation with solutions 5 and -7 is

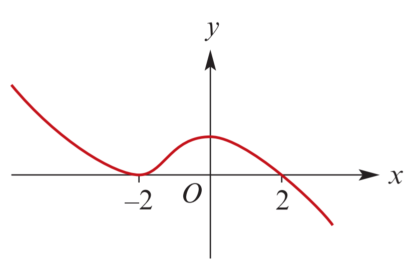
A $x^2 + 2x - 35 = 0$ **B** $x^2 - 2x - 35 = 0$ **C** $x^2 + 12x - 35 = 0$
D $x^2 - 12x - 35 = 0$ **E** $-x^2 + 12x + 35 = 0$

26 If $x^3 - 5x^2 + x + k$ is divisible by $x + 1$, then k equals

A -7 **B** -5 **C** -2 **D** 5 **E** 7

27 A possible equation for the graph shown is

A $y = x(x-2)(x+2)$
B $y = -x(x+2)(x-2)$
C $y = -(x+2)^2(x-2)$
D $y = (x-2)^2(x+2)$
E $y = x(x-2)^2$



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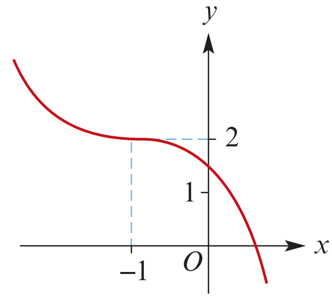
Revision

28 The graph shown is



A $y + 2 = -2(x + 1)^3$

B $y - 2 = 2(x - 1)^3$

C $y = x^3 + 2$

D $y = -\frac{1}{2}(x + 1)^3 + 2$

E $y = 2(x - 1)^3 - 2$

29 $P(x) = x^3 + 2x^2 - 5x - 6$ has the factorisation

A $(x - 1)(x - 2)(x + 3)$ **B** $(x + 1)(x + 2)(x + 3)$ **C** $(x + 1)(x - 2)(x + 3)$

D $(x + 1)(x - 2)(x - 3)$ **E** $(x - 1)(x - 2)(x - 3)$

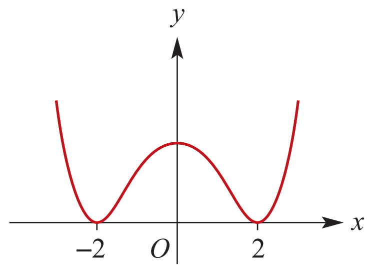
30 Let $P(x) = 2x^3 - 2x^2 + 3x + 1$. When $P(x)$ is divided by $x - 2$, the remainder is

A 31 **B** 15 **C** 1 **D** −2 **E** −29

31 If $x^3 + 2x^2 + ax - 4$ has remainder 1 when divided by $x + 1$, then a equals

A −8 **B** −4 **C** −2 **D** 0 **E** 2

32 Which of these equations is represented by the graph shown?



A $y = (x + 2)^2(x - 2)$

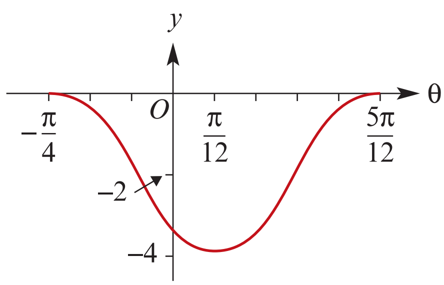
B $y = 16 - x^4$

C $y = (x^2 - 4)^2$

D $y = (x + 2)^2(2 - x)$

E $y = x^4 - 16$

33 A possible equation for the graph shown is



A $y = 2\cos 3\left(\theta + \frac{\pi}{4}\right) - 4$

B $y = 2\cos 2\left(\theta + \frac{\pi}{4}\right) - 2$

C $y = 2\sin 3\left(\theta + \frac{\pi}{4}\right) - 2$

D $y = 2\cos 3\left(\theta + \frac{\pi}{4}\right) - 2$

E $y = 2\cos 3\left(\theta - \frac{\pi}{4}\right) - 2$

34 The function $f(x) = 2 - 3\cos 2\left(x + \frac{\pi}{2}\right)$, $x \in \mathbb{R}$, has range

A $[-3, 5]$ **B** $[2, 5]$ **C** $\mathbb{R}$ **D** $[-1, 5]$ **E** $[-3, 2]$

35 Two values between 0 and 2π for which $2\sin\theta + \sqrt{3} = 0$ are

A $\frac{\pi}{3}, \frac{2\pi}{3}$ **B** $60°, 240°$ **C** $\frac{2\pi}{3}, \frac{5\pi}{3}$ **D** $\frac{4\pi}{3}, \frac{5\pi}{3}$ **E** $\frac{7\pi}{6}, \frac{11\pi}{6}$

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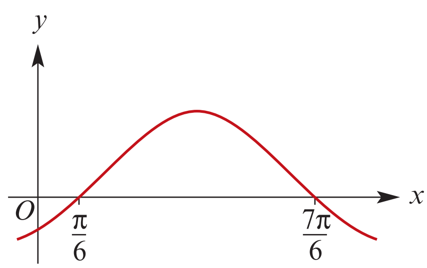
36 A possible equation for the graph shown is



A $y = \sin\left(x - \frac{\pi}{6}\right)$

B $y = \sin\left(x + \frac{\pi}{6}\right)$

C $y = -\sin\left(x - \frac{\pi}{6}\right)$

D $y = \cos\left(x - \frac{\pi}{6}\right)$

E $y = \cos\left(x + \frac{\pi}{6}\right)$

37 The function $f(x) = 3\sin(2x)$, $x \in \mathbb{R}$, has

A amplitude 3 and period π

B amplitude 2 and period $\frac{\pi}{2}$

C amplitude 1 and period $\frac{\pi}{2}$

D amplitude $\frac{3}{2}$ and period 2π

E amplitude $1\frac{1}{2}$ and period 2π

38 The function $f(x) = 3\sin(2x)$, $x \in \mathbb{R}$, has range

A $[0, 3]$ **B** $[-2, 2]$ **C** $[2, 3]$ **D** $[-3, 3]$ **E** $[-1, 5]$

39 It is known that the graph of the function with rule $y = 2ax + \cos(2x)$ has an x-axis intercept when $x = \pi$. The value of a is

A 2 **B** $\frac{1}{2\pi}$ **C** 2π **D** -2π **E** $\frac{-1}{2\pi}$

40 Consider the polynomial $p(x) = (x - 2a)^2(x + a)(x^2 + a)$ where $a > 0$. The equation $p(x) = 0$ has exactly

A 1 distinct real solution

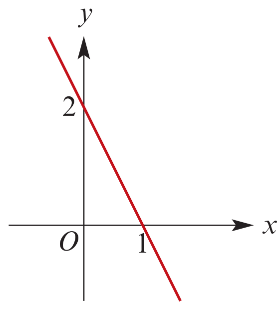
B 2 distinct real solutions

C 3 distinct real solutions

D 4 distinct real solutions

E 5 distinct real solutions

41 The gradient of a straight line perpendicular to the line shown is



A 2 **B** -2 **C** $\frac{-1}{2}$

D $\frac{1}{2}$ **E** 3

42 Let $f(x) = a\sin(bx + c)$ for all $x \in \mathbb{R}$, where a, b and c are positive constants. The function f has period

A a **B** b **C** c **D** $\frac{2\pi}{a}$ **E** $\frac{2\pi}{b}$

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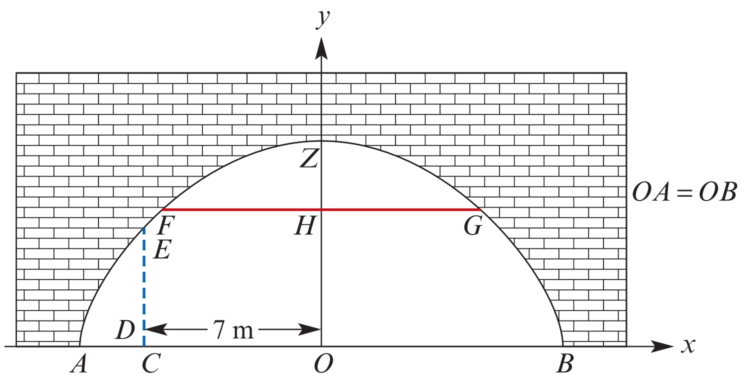
Revision

5C Extended-response questions

CU

1 An arch is constructed as shown.



The height of the arch is 9 metres ($OZ = 9$ m). The width of the arch is 20 metres ($AB = 20$ m). The equation of the curve is of the form $y = ax^2 + b$, taking axes as shown.

a Find the values of a and b.

b A man of height 1.8 m stands at C ($OC = 7$ m). How far above his head is the point E on the arch? (That is, find the distance DE.)

c A horizontal bar FG is placed across the arch as shown. The height, OH, of the bar above the ground is 6.3 m. Find the length of the bar.

2 **a** The expression $2x^3 + ax^2 - 72x - 18$ leaves a remainder of 17 when divided by $x + 5$. Determine the value of a.

b Solve the equation $2x^3 = x^2 + 5x + 2$.

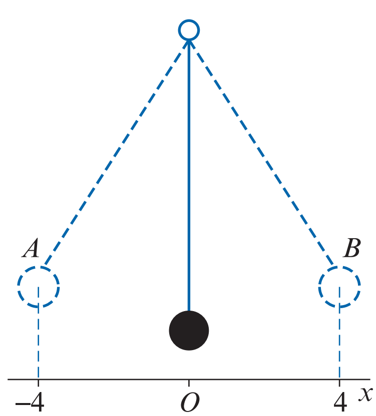
c **i** Given that the expression $x^2 - 5x + 7$ leaves the same remainder whether divided by $x - b$ or $x - c$, where $b \neq c$, show that $b + c = 5$.

ii Given further that $4bc = 21$ and $b > c$, find the values of b and c.

3 As a pendulum swings, its horizontal position, x cm, measured from the central position, varies from -4 cm (at A) to 4 cm (at B).

The position x is given by the rule

$$x = -4\sin(\pi t)$$



a Sketch the graph of x against t for $t \in [0, 2]$.

b Find the horizontal position of the pendulum for:

i $t = 0$ **ii** $t = \frac{1}{2}$ **iii** $t = 1$

c Find the first time that the pendulum has horizontal position $x = 2$.

d Find the period of the pendulum, i.e. the time it takes to go from A to B and back to A.

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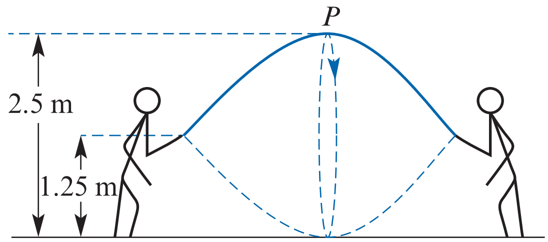
4 Two people are rotating a skipping rope. The rope is held 1.25 m above the ground. It reaches a height of 2.5 m above the ground, and just touches the ground.

The vertical position, y m, of the point P on the rope at time t seconds is given by the rule

$$y = -1.25\cos(2\pi t) + 1.25$$



a Find y when:

i $t = 0$ **ii** $t = \frac{1}{2}$ **iii** $t = 1$

b How long does it take for one rotation of the rope?

c Sketch the graph of y against t.

d Find the first time that the point P on the rope reaches a height of 2 m above the ground.

5 A rugby ball is kicked so that it leaves the player's foot with a velocity of V m/s. The total horizontal distance travelled by the ball, x m, is given by

$$x = \frac{V^2 \sin(2\alpha)}{10}$$

where α is the angle of projection.

a Find the horizontal distance travelled by the ball if $V = 25$ m/s and $\alpha = 45°$.

b For $V = 20$, sketch the graph of x against α for $0° \le \alpha \le 90°$.

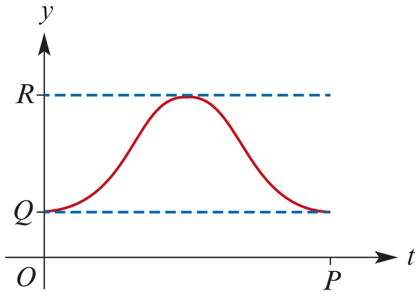
c If the ball goes 30 m and the initial velocity is 20 m/s, find the angle of projection.

6 **a** The graph is of one complete cycle of

$$y = h - k\cos\left(\frac{\pi t}{6}\right)$$

i How many units long is OP?

ii Express OQ and OR in terms of h and k.



b For a certain city in the northern hemisphere, the number of hours of daylight on the 21st day of each month is given by the table:

x	Dec	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
y	7.5	8.2	9.9	12.0	14.2	15.8	16.5	15.9	14.3	12.0	9.8	8.1	7.5

Using suitable scales, plot these points and draw a curve through them. Call December month 0, January month 1, etc., and treat all months as of equal length.

c Find the values of h and k so that your graph is approximately that of

$$y = h - k\cos\left(\frac{\pi t}{6}\right)$$

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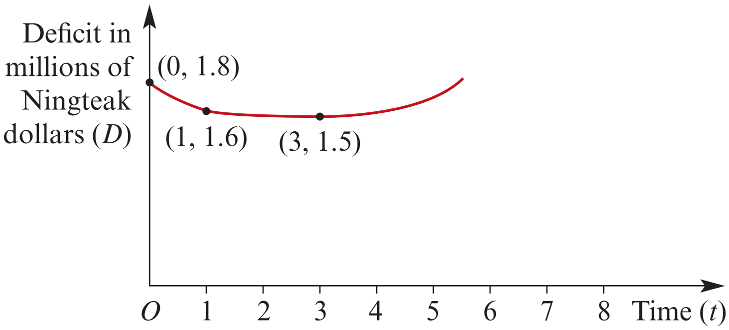
Revision

CU

7 The deficit of a government department in Ningteak, a small monarchy east of Africa, is continually assessed over a period of 8 years. The following graph shows the deficit over these 8 years.



The graph is read as follows: The deficit at the beginning of the 8-year period is \$1.8 million. At the end of the third year the deficit is \$1.5 million, and this is the smallest deficit for the period $0 \le t \le 8$.

a Find the rule for D in terms of t, assuming that it is of the form $D = at^2 + bt + c$.

b Use this model to predict the deficit at the end of 8 years.

8 The rate of rainfall, R mm per hour, was recorded during a very rainy day in North Queensland. The recorded data are given in the table.

Time	Rainfall
4 a.m.	7.5 mm per hour
8 a.m.	9.0 mm per hour
10 a.m.	8.0 mm per hour

Assume a quadratic rule of the form

$$R = at^2 + bt + c$$

is applicable for $0 \le t \le 12$, where $t = 0$ is 4 a.m.

Use the quadratic model to predict the rate of rainfall at noon. At what time was the rate of rainfall greatest?

5D Degree-of-difficulty classified questions

▶ Simple familiar questions

1 Consider the function defined by $f(x) = 2x^2 - 12x + 7$ for all $x \in \mathbb{R}$.

a Express $f(x)$ in the form $a(x-h)^2 + k$, where a, h and k are integers.

b Determine the minimum value of $f(x)$.

c Write down the range of the function f.

2 Let $g(x) = \sqrt{1 - x^2}$ for $-1 \le x \le 1$. Calculate:

a $g(1)$ **b** $g(0)$ **c** $g\left(\frac{1}{2}\right)$ **d** $g\left(\frac{1}{\sqrt{2}}\right)$ **e** $g\left(\frac{\sqrt{3}}{2}\right)$

3 Let $f(x) = x^2 + 3x - 4$ and $g(x) = 2x - 3$. Solve the equation $f(x) = g(x)$ for x.

4 Let $f(x) = (x-1)^2$ for $-2 \le x \le 4$. State the range of f.

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5 Define $f(x) = 8\sin(5x)$ for all $x \in \mathbb{R}$.

a State the period of f.

b State the amplitude of f.

c Describe a sequence of transformations that maps:

i the graph of $y = \sin x$ to the graph of $y = f(x)$

ii the graph of $y = \cos x$ to the graph of $y = f(x)$.

6 Given that $\sin A = \dfrac{1}{3}$, where A is an acute angle, find the exact value of:

a $\cos A$ **b** $\tan A$

7 Find all values of x between 0° and 360° that satisfy $\cos(3x) = \dfrac{1}{2}$.

8 Solve the equation $1 - 2\cos\left(\dfrac{\pi x}{6}\right) = 0$ for $-12 \le x \le 12$.

9 Factorise $x^3 - \dfrac{y^3}{27}$.

10 Factorise $x^4 + 2x^3 + 2x^2 + x - 6$.

11 State the implied domain for the function with rule $g(x) = \sqrt{1 - 5x}$.

12 Suppose that functions f, g and h satisfy $g(x) = \dfrac{1}{2}f(x + 4)$ and $h(x) = 2g(5x - 11) + 3$ for all $x \in \mathbb{R}$. Write a rule for the function h in terms of the function f.

13 Let $f(t) = 4\sin\left(\dfrac{\pi t}{6}\right) + 2$ for $t \ge 0$.

a State the period and amplitude of f.

b State the maximum and minimum values of f.

c Solve the equation $f(t) = 2$ for $0 \le t \le 12$.

d Sketch the graph of $y = f(t)$ for $0 \le t \le 12$.

► Complex familiar questions

1 State the implied domain of the function $f(x) = \sqrt{(x - 2)(x + 4)}$.

2 The polynomial $P(x) = 3x^3 + ax^2 + bx - 12$ is exactly divisible by $x^2 + 7x + 12$.

a Find the values of a and b.

b Find the remainder when $P(x)$ is divided by $x + 2$.

3 Find the value of n such that dividing $x^{2n} - 8x^n + 10$ by $x - 2$ gives remainder 10.

4 Consider the three points $A(-3, 5)$, $B(1, -2)$ and $C(5, \lambda)$.

a Let M be the midpoint of BC. Find the coordinates of M in terms of λ.

b Find the values of λ for which AM is perpendicular to BC.

5 Find all non-zero values of a for which the equation $2ax^2 + 2(a^2 + a)x + 3(a + 1) = 0$ has only one solution.

 ISBN 978-1-108-45164-2

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6 Let $P(x) = x^4 + ax^3 + bx^2 - 12x + 4$, where a and b are real constants.

a Find the values of a and b given that $P(x) = 0$ when $x = 1$ and when $x = 2$.

b Use the values of a and b obtained in part **a**. By factorising $P(x)$, show that the equation $P(x) = 0$ has no real solutions other than $x = 1$ and $x = 2$.

7 Find the values of a, b and c such that the identity $3x^2 - 8x - 2 = a + b(x-2) + c(x-2)^2$ holds for all real numbers x.

8 Let $h(x) = x^3 + ax + b$. Given that the equation $h(x) = x$ has solutions $x = 2$ and $x = 3$, find the values of a and b.

9 The line $y = 2x - 3$ is parallel to the line $y = ax + 7$ and perpendicular to the line $y = bx + 7$.

a State the values of a and b.

b Find the perpendicular distance between the two parallel lines.

10 The line with equation $y = x + 5$ is the perpendicular bisector of the line segment joining the points $P(3, 10)$ and $Q(\alpha, \beta)$.

a Determine the coordinates of the midpoint of PQ in terms of α and β.

b Find the values of α and β.

11 The equation $x^2 + 2(1 - p)x + p^2 + 5 = 0$ is satisfied by only one value of x. Find the value of p and the corresponding value of x.

12 Given that $x^3 - 2x^2 - 3x - 11$ and $x^3 - x^2 - 9$ leave the same remainder when divided by $x + a$, find the possible values of a.

13 Two taxi services use the following different systems for charging for a journey:

Speedy Taxi	Initial charge of \$12, plus a charge of 35 cents for each 200 m travelled
Thrifty Taxi	Flat fee of \$30 for travelling up to 24 km, plus a charge of \$1.50 for each kilometre travelled beyond 24 km

a Let $S(d)$ be the cost (in dollars) of a journey of d km in a Speedy Taxi. Show that

$$S(d) = 1.75d + 12 \quad \text{for } d \geq 0$$

b Let $T(d)$ be the cost (in dollars) of a journey of d km in a Thrifty Taxi. Show that

$$T(d) = \begin{cases} 30 & \text{for } 0 \leq d \leq 24 \\ 1.5d - 6 & \text{for } d > 24 \end{cases}$$

c On the same coordinate axes, sketch graphs to represent $S(d)$ and $T(d)$.

d Find the cost of a journey of:

i 17 km in a Speedy Taxi

ii 15 km in a Thrifty Taxi

e Karen wants to travel a distance of 45 km. Which taxi service will be cheaper?

f Find the distances for which a Thrifty Taxi is the cheaper option.

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► Complex unfamiliar questions

1 **a** Find the minimum integer value of a such that $ax^2 + 7x + 3$ is positive for all x.

b Find the minimum integer value of b such that $-3x^2 + bx - 4$ is negative for all x.

2 Consider the following equation in the variable x:

$$\frac{\lambda}{2x} = \frac{a}{x+c} + \frac{b}{x-c}$$

where λ, a, b and c are real constants with $c \neq 0$.

a First assume that $a \neq b$ and that the equation has only one solution. Find the product of the two possible values of λ in terms of a and b.

b Discuss what happens when $a = b$ and the equation has only one solution.

3 Consider the quadratic equation $ax^2 + bx + c = 0$, where a, b and c are integers.

a Show that if $a + b + c = 0$, then $b^2 - 4ac$ is a perfect square.

b Show that if $b - a - c = 0$, then $b^2 - 4ac$ is a perfect square.

c Explain how these two results can help you to choose the coefficients of a quadratic equation so that it has rational solutions.

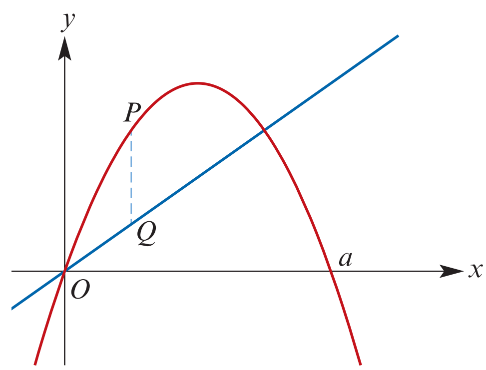
d Give several examples of quadratic equations with rational solutions.

e Give an example to show that not all quadratic equations with rational solutions satisfy one of the two properties $a + b + c = 0$ or $b - a - c = 0$.

4 The parabola $y = ax - x^2$ and line $y = x$ are shown, where a is a constant with $a > 1$.

Point $P(x, y)$ is on the parabola and point $Q(x, x)$ is on the line, for $0 \le x \le a - 1$.

Let h be the length of line segment PQ. Then h is the 'vertical distance' between the two graphs for a given value of x.



a Find an expression for h in terms of x.

b Find the value of x (expressed in terms of a) for which h is a maximum.

c Find this maximum value of h in terms of a.

d Find the value of a for which the maximum value of h is:

i $\frac{1}{4}$ **ii** 1 **iii** 5 **iv** 9 **v** 10

5 Let $f(x) = x^2 - ax$ and $g(x) = bx - x^2$, where a and b are constants with $0 < a < b$.

a Find the coordinates of the points of intersection of the graphs of f and g.

b Sketch the graphs of f and g on the one set of axes.

Consider points $P(x, f(x))$ and $Q(x, g(x))$ for $0 \le x \le \dfrac{a+b}{2}$.

c Find an expression for the distance PQ in terms of x, a and b.

d Hence find the maximum possible distance PQ in terms of a and b.

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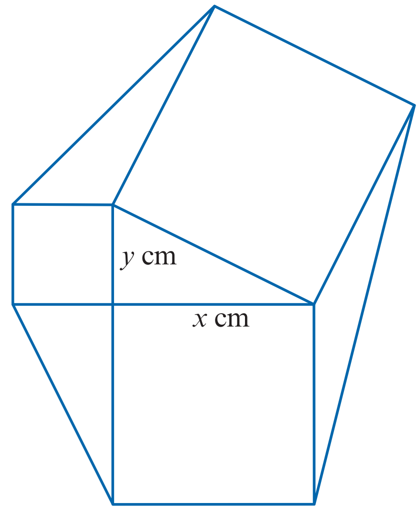
Revision

6 The two shorter sides of a right-angled triangle have lengths x cm and y cm. A square is constructed on each side of the triangle and then a hexagon is constructed as shown.



a Show that the area, A cm^2, of the hexagon is given by $A = 2(x^2 + xy + y^2)$.

b Given that $x + y = 7$, find the minimum area of the hexagon and the values of x and y for which this occurs.

c Given that $x + y = a$, where a is a positive real number, find the minimum area of the hexagon and the values of x and y for which this occurs.

d Comment on this general result.

7 A ball is projected from ground level at an angle of 45° to the horizontal with an initial speed of v m/s. The path of the ball is described by the equation

$$y = x - \frac{10x^2}{v^2}$$

where the unit is metres. The ball is projected from the point $O(0, 0)$ and lands at the point $A(a, 0)$.

a Find a in terms of v.

b Given that the ball's initial speed is $v = 25$ m/s, determine:

i the value of a

ii the coordinates of the point where the ball reaches its maximum height

iii the values of x at which the height of the ball is 10 m.

c Now assume that the ball is projected at an angle of 60° to the horizontal. The equation of the path is

$$y = \sqrt{3}x - \frac{20x^2}{v^2}$$

Given that the ball's initial speed is $v = 30$ m/s, determine each of the following correct to two decimal places:

i the coordinates of the point where the ball reaches its maximum height

ii the value of a, where the ball is projected from $O(0, 0)$ and lands at $A(a, 0)$.

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6 Exponential and logarithmic functions

Objectives

- To revise **exponential functions**.
- To introduce **Euler's number** e and the exponential function $f(x) = e^x$.
- To understand **logarithms** and use the **logarithm laws**.
- To introduce the **natural logarithm** function.
- To graph **logarithmic functions** and transformations of these functions.
- To solve equations involving exponential and logarithmic functions.
- To apply exponential and logarithmic functions in modelling.
- To understand **logarithmic scales**.
- To model data using regression with a graphics calculator.

In this chapter, we revise exponential functions and their applications, including radioactive decay and population growth. We also introduce the exponential function $f(x) = e^x$, which has many interesting properties. In particular, this function is its own derivative. That is, $f'(x) = f(x)$. This will be discussed in Chapter 8. Here we define the number e as

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

We will show that limits such as this arise in the consideration of compound interest.

A **logarithm function** has a rule of the form $f(x) = \log_a x$, where the base a is a positive real number other than 1. These functions are the 'reverse' of exponential functions, and have historical and practical importance. For example, we use logarithms in measurement scales such as the Richter scale (earthquakes), decibels (noise) and pH (acidity).

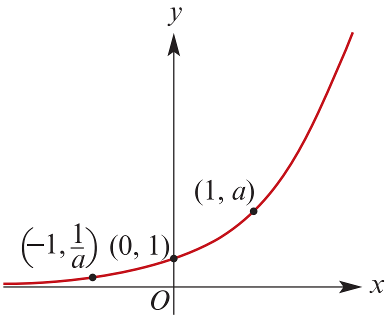
This chapter covers Unit 3 Topic 1: The logarithmic function 2.

ISBN 978-1-108-45164-2

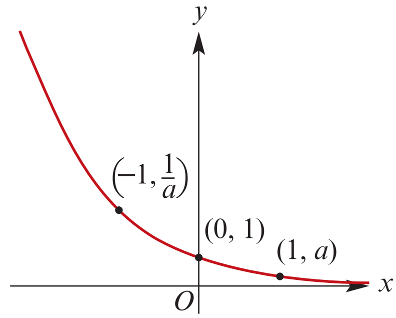
6A Revision of exponential functions

The function $f(x) = a^x$, where $a \in \mathbb{R}^+ \setminus \{1\}$, is an **exponential function**. The shape of the graph depends on whether $a > 1$ or $0 < a < 1$.

Graph of $f(x) = a^x$ for $a > 1$



Graph of $f(x) = a^x$ for $0 < a < 1$



- Key values are $f(-1) = \frac{1}{a}$, $f(0) = 1$ and $f(1) = a$.
- The maximal domain is $\mathbb{R}$ and the range is $\mathbb{R}^+$.
- The x-axis is a horizontal asymptote.

An exponential function with $a > 1$ is strictly increasing, and an exponential function with $0 < a < 1$ is strictly decreasing.

▶ Graphing transformations of $f(x) = a^x$

Translations

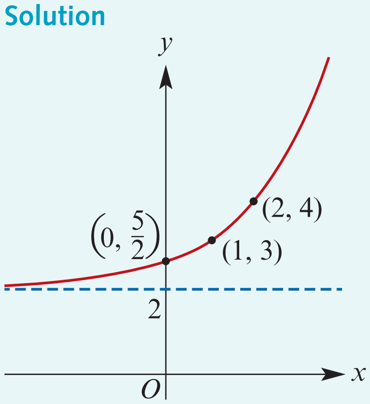
If the translation $(x, y) \to (x + h, y + k)$ is applied to the graph of $y = a^x$, then the image has equation $y = a^{x-h} + k$.

- The horizontal asymptote of the image has equation $y = k$.
- The range of the image is (k, ∞).

Example 1

Sketch the graph and state the range of $y = 2^{x-1} + 2$.

Solution



The range of the function is $(2, \infty)$.

Explanation

The graph of $y = 2^x$ is translated 1 unit in the positive direction of the x-axis and 2 units in the positive direction of the y-axis.

The mapping is $(x, y) \to (x + 1, y + 2)$.

Translation of key points:

- $(-1, \frac{1}{2}) \to (0, \frac{5}{2})$
- $(0, 1) \to (1, 3)$
- $(1, 2) \to (2, 4)$

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Reflections

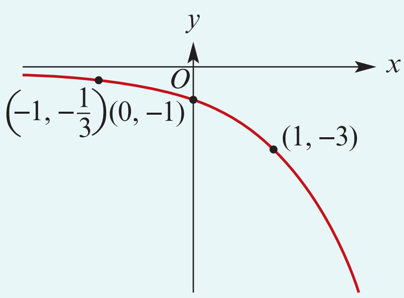
If a **reflection in the x-axis**, given by the mapping $(x, y) \to (x, -y)$, is applied to the graph of $y = a^x$, then the image has equation $y = -a^x$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(-\infty, 0)$.

Example 2

Sketch the graph of $y = -3^x$.

Solution



Explanation

The graph of $y = 3^x$ is reflected in the x-axis.

The mapping is $(x, y) \to (x, -y)$.

Reflection of key points:

- $(-1, \frac{1}{3}) \to (-1, -\frac{1}{3})$
- $(0, 1) \to (0, -1)$
- $(1, 3) \to (1, -3)$

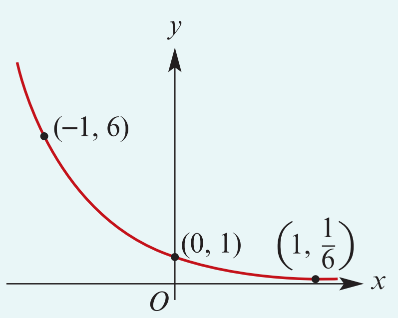
If a **reflection in the y-axis**, given by the mapping $(x, y) \to (-x, y)$, is applied to the graph of $y = a^x$, then the image has equation $y = a^{-x}$. This can also be written as $y = \dfrac{1}{a^x}$ or $y = \left(\dfrac{1}{a}\right)^x$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(0, \infty)$.

Example 3

Sketch the graph of $y = 6^{-x}$.

Solution



Explanation

The graph of $y = 6^x$ is reflected in the y-axis.

The mapping is $(x, y) \to (-x, y)$.

Reflection of key points:

- $(-1, \frac{1}{6}) \to (1, \frac{1}{6})$
- $(0, 1) \to (0, 1)$
- $(1, 6) \to (-1, 6)$

Dilations

For $k > 0$, if a **dilation of factor k from the x-axis**, given by the mapping $(x, y) \to (x, ky)$, is applied to the graph of $y = a^x$, then the image has equation $y = ka^x$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(0, \infty)$.

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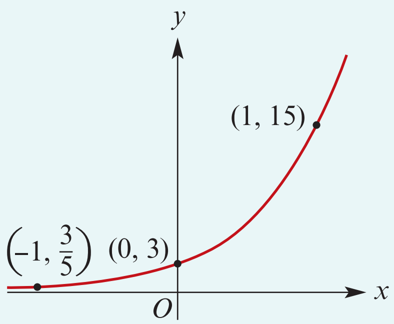
Example 4

Sketch the graph of each of the following:

a $y = 3 \times 5^x$ **b** $y = 0.2 \times 8^x$

Solution

a



Explanation

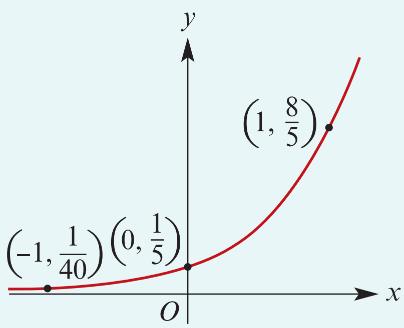
The graph of $y = 5^x$ is dilated by factor 3 from the x-axis.

The mapping is $(x, y) \to (x, 3y)$.

Dilation of key points:

- $(-1, \frac{1}{5}) \to (-1, \frac{3}{5})$
- $(0, 1) \to (0, 3)$
- $(1, 5) \to (1, 15)$

b



The graph of $y = 8^x$ is dilated by factor $\frac{1}{5}$ from the x-axis.

The mapping is $(x, y) \to (x, \frac{1}{5}y)$.

Dilation of key points:

- $(-1, \frac{1}{8}) \to (-1, \frac{1}{40})$
- $(0, 1) \to (0, \frac{1}{5})$
- $(1, 8) \to (1, \frac{8}{5})$

For $k > 0$, if a **dilation of factor k from the y-axis**, given by the mapping $(x, y) \to (kx, y)$, is applied to the graph of $y = a^x$, then the image has equation $y = a^{\frac{x}{k}}$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(0, \infty)$.

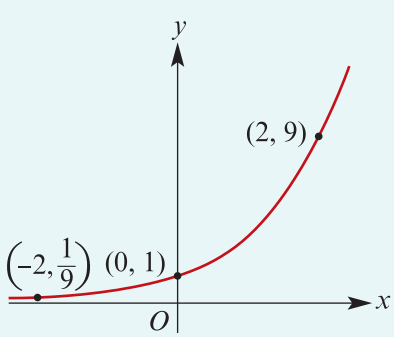
Example 5

Sketch the graph of each of the following:

a $y = 9^{\frac{x}{2}}$ **b** $y = 3^{2x}$

Solution

a



Explanation

The graph of $y = 9^x$ is dilated by factor 2 from the y-axis.

The mapping is $(x, y) \to (2x, y)$.

Dilation of key points:

- $(-1, \frac{1}{9}) \to (-2, \frac{1}{9})$
- $(0, 1) \to (0, 1)$
- $(1, 9) \to (2, 9)$

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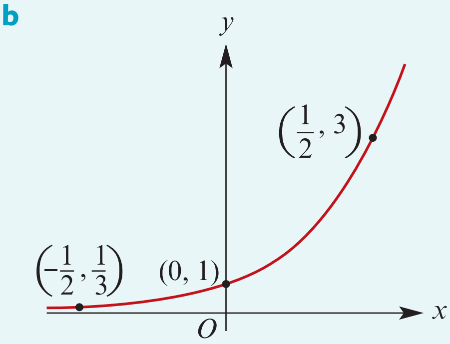
b



The graph of $y = 3^x$ is dilated by factor $\frac{1}{2}$ from the y-axis.

The mapping is $(x, y) \to (\frac{1}{2}x, y)$.

Dilation of key points:

- $(-1, \frac{1}{3}) \to (-\frac{1}{2}, \frac{1}{3})$
- $(0, 1) \to (0, 1)$
- $(1, 3) \to (\frac{1}{2}, 3)$

Note: Since $9^{\frac{x}{2}} = \left(9^{\frac{1}{2}}\right)^x = 3^x$, the graph of $y = 9^{\frac{x}{2}}$ is the same as the graph of $y = 3^x$. Similarly, the graph of $y = 3^{2x}$ is the same as the graph of $y = 9^x$.

A translation parallel to the x-axis results in a dilation from the x-axis. For example, if the graph of $y = 5^x$ is translated 3 units in the positive direction of the x-axis, then the image is the graph of $y = 5^{x-3}$, which can be written $y = 5^{-3} \times 5^x$. Hence, a translation of 3 units in the positive direction of the x-axis is equivalent to a dilation of factor 5^{-3} from the x-axis.

Combinations of transformations

We have seen translations, reflections and dilations applied to exponential graphs. In the following example we consider combinations of these transformations.

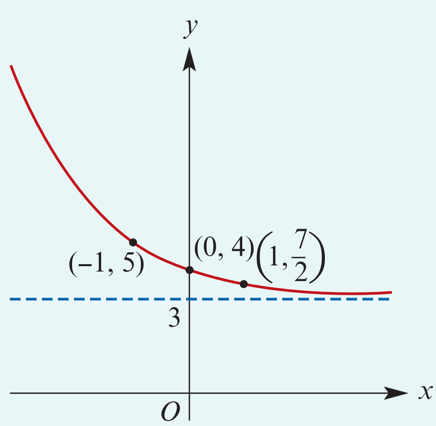
Example 6

Sketch the graph and state the range of each of the following:

a $y = 2^{-x} + 3$ **b** $y = 4^{3x} - 1$ **c** $y = -10^{x-1} - 2$

Solution

a



Graph of $y = 2^{-x} + 3$:

- The asymptote has equation $y = 3$.
- The y-axis intercept is $2^0 + 3 = 4$.
- The range of the function is $(3, \infty)$.

Explanation

The graph of $y = 2^{-x} + 3$ is obtained from the graph of $y = 2^x$ by a reflection in the y-axis followed by a translation 3 units in the positive direction of the y-axis.

The mapping is $(x, y) \to (-x, y + 3)$.

For example:

$(-1, \frac{1}{2}) \to (1, \frac{7}{2})$

$(0, 1) \to (0, 4)$

$(1, 2) \to (-1, 5)$

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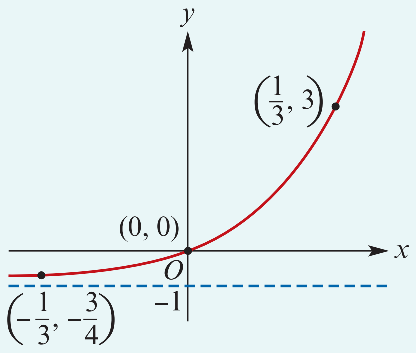
b



Graph of $y = 4^{3x} - 1$:

- The asymptote has equation $y = -1$.
- The y-axis intercept is $4^0 - 1 = 0$.
- The range of the function is $(-1, \infty)$.

The graph of $y = 4^{3x} - 1$ is obtained from the graph of $y = 4^x$ by a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation 1 unit in the negative direction of the y-axis.

The mapping is $(x, y) \to (\frac{1}{3}x, y - 1)$.

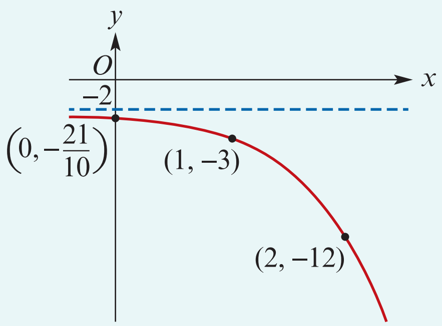
For example:

$(-1, \frac{1}{4}) \to (-\frac{1}{3}, -\frac{3}{4})$

$(0, 1) \to (0, 0)$

$(1, 4) \to (\frac{1}{3}, 3)$

c



Graph of $y = -10^{x-1} - 2$:

- The asymptote has equation $y = -2$.
- The y-axis intercept is $-10^{-1} - 2 = -\frac{21}{10}$.
- The range of the function is $(-\infty, -2)$.

The graph of $y = -10^{x-1} - 2$ is obtained from the graph of $y = 10^x$ by a reflection in the x-axis followed by a translation 1 unit in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

The mapping is $(x, y) \to (x + 1, -y - 2)$.

For example:

$(-1, \frac{1}{10}) \to (0, -\frac{21}{10})$

$(0, 1) \to (1, -3)$

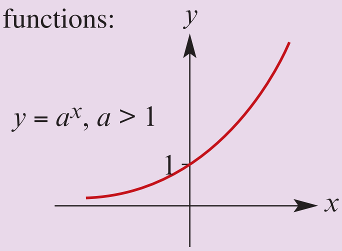
$(1, 10) \to (2, -12)$

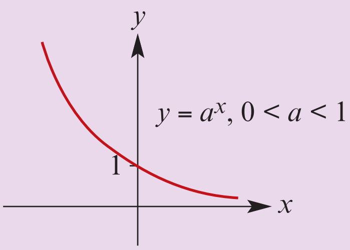
Note: For each graph in Example 6, we can use the method for determining transformations from Chapter 2. Here we show the method for part **c**:

- Write the equation as $y' = -10^{x'-1} - 2$.
- Rearrange to $-y' - 2 = 10^{x'-1}$.
- We choose to write $y = -y' - 2$ and $x = x' - 1$.
- Hence $y' = -y - 2$ and $x' = x + 1$.

Section summary

- Graphs of exponential functions:





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- For $a \in \mathbb{R}^+ \setminus \{1\}$, the graph of $y = a^x$ has the following properties:
  - The x-axis is an asymptote.
  - The y-axis intercept is 1.
  - The y-values are always positive.
  - There is no x-axis intercept.
- Transformations can be applied to exponential functions. For example, the graph of

$$y = a^{b(x-h)} + k, \quad \text{where } b > 0$$

can be obtained from the graph of $y = a^x$ by a dilation of factor $\frac{1}{b}$ from the y-axis followed by the translation $(x, y) \to (x + h, y + k)$.

Exercise 6A

SF

Example 1

1 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 2^{x+1} - 2$ **b** $y = 2^{x-3} - 1$ **c** $y = 2^{x+2} - 1$ **d** $y = 2^{x-2} + 2$

Example 2, 3

2 For each of the following, use the one set of axes to sketch the two graphs (labelling asymptotes):

a $y = 2^x$ and $y = 3^x$ **b** $y = 2^{-x}$ and $y = 3^{-x}$

c $y = 5^x$ and $y = -5^x$ **d** $y = 1.5^x$ and $y = -1.5^x$

Example 4, 5

3 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 3 \times 2^x$ **b** $y = \frac{1}{2} \times 5^x$ **c** $y = 2^{3x}$ **d** $y = 2^{\frac{x}{3}}$

Example 6

4 Sketch the graph and state the range of each of the following:

a $y = 3^{-x} + 2$ **b** $y = 2^{5x} - 4$ **c** $y = -10^{x-2} - 2$

5 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 3^x$ **b** $y = 3^x + 1$ **c** $y = 1 - 3^x$

d $y = (\frac{1}{3})^x$ **e** $y = 3^{-x} + 2$ **f** $y = (\frac{1}{3})^x - 1$

6 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = (\frac{1}{2})^{x-2}$ **b** $y = (\frac{1}{2})^x - 1$ **c** $y = (\frac{1}{2})^{x-2} + 1$

CF

7 For $f(x) = 2^x$, sketch the graph of each of the following, labelling asymptotes where appropriate:

a $y = f(x + 1)$ **b** $y = f(x) + 1$ **c** $y = f(-x) + 2$

d $y = -f(x) - 1$ **e** $y = f(3x)$ **f** $y = f\left(\frac{x}{2}\right)$

g $y = 2f(x - 1) + 1$ **h** $y = f(x - 2)$

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SF

8 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 10^x - 1$ **b** $y = 10^{\frac{x}{10}} + 1$ **c** $y = 2 \times 10^x - 20$

d $y = 1 - 10^{-x}$ **e** $y = 10^{x+1} + 3$ **f** $y = 2 \times 10^{\frac{x}{10}} + 4$

CF

9 A bank offers cash loans at 0.04% interest per day, compounded daily. A loan of \$10 000 is taken and the interest payable at the end of x days is given by $C_1 = 10\,000\,[(1.0004)^x - 1]$.

a Plot the graph of C_1 against x.

b Find the interest at the end of:

i 100 days **ii** 300 days.

c After how many days is the interest payable \$1000?

d A loan company offers \$10 000 and charges a fee of \$4.25 per day. The amount charged after x days is given by $C_2 = 4.25x$.

i Plot the graph of C_2 against x (using the same window as in part **a**).

ii Find the smallest value of x for which $C_2 < C_1$.

10 If you invest \$100 at an interest rate of 2% per day, compounded daily, then after x days the amount of money you have (in dollars) is given by $y = 100(1.02)^x$. For how many days would you have to invest to double your money?

11 **a** **i** Graph $y = 2^x$, $y = 3^x$ and $y = 5^x$ on the same set of axes.

ii For what values of x is $2^x > 3^x > 5^x$?

iii For what values of x is $2^x < 3^x < 5^x$?

iv For what values of x is $2^x = 3^x = 5^x$?

b Repeat part **a** for $y = (\frac{1}{2})^x$, $y = (\frac{1}{3})^x$ and $y = (\frac{1}{5})^x$.

c Use your answers to parts **a** and **b** to sketch the graph of $y = a^x$ for:

i $a > 1$ **ii** $a = 1$ **iii** $0 < a < 1$

6B The exponential function $f(x) = e^x$

In the previous section, we explored the family of exponential functions $f(x) = a^x$, where $a \in \mathbb{R}^+ \setminus \{1\}$. One particular member of this family is of great importance in mathematics.

This function has the rule $f(x) = e^x$, where e is Euler's number, named after the eighteenth century Swiss mathematician Leonhard Euler.

Euler's number is defined as follows.

Euler's number

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

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To see what the value of e might be, we could try large values of n and use a calculator to evaluate $(1 + \frac{1}{n})^n$, as shown in the table on the right.

As n is taken larger and larger, it can be seen that $(1 + \frac{1}{n})^n$ approaches a limiting value ($\approx 2.718\ 28$).

n	$(1 + \frac{1}{n})^n$
100	$(1.01)^{100} = 2.704\ 813\ldots$
1000	$(1.001)^{1000} = 2.716\ 923\ldots$
10 000	$(1.0001)^{10\,000} = 2.718\ 145\ldots$
100 000	$(1.000\ 01)^{100\,000} = 2.718\ 268\ldots$
1 000 000	$(1.000\ 001)^{1\,000\,000} = 2.718\ 280\ldots$

Like π, the number e is irrational:

$$e = 2.718\ 281\ 828\ 459\ 045\ldots$$

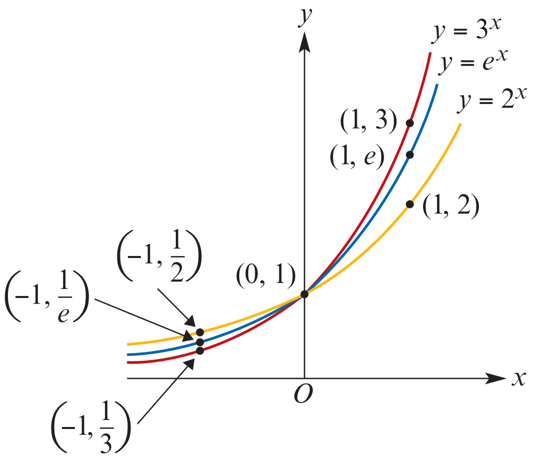
The function $f(x) = e^x$ is very important in mathematics. In Chapter 8 you will find that it has the remarkable property that $f'(x) = f(x)$. That is, the derivative of e^x is e^x.

Note: The function e^x can be found on your calculator.

Graphing $f(x) = e^x$

The graph of $y = e^x$ is as shown.

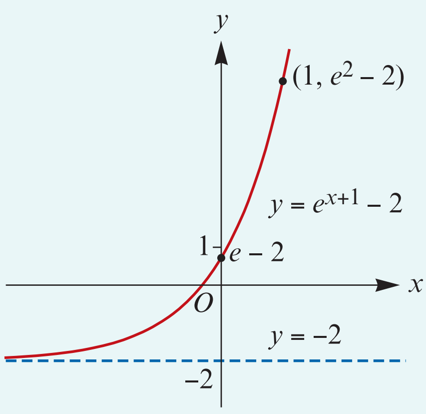
The graphs of $y = 2^x$ and $y = 3^x$ are shown on the same set of axes.



Example 7

Sketch the graph of $y = e^{x+1} - 2$.

Solution



The asymptote has equation $y = -2$.
The y-axis intercept is $e - 2$.

Explanation

To find the transformation:

- Write the image as $y' + 2 = e^{x'+1}$.
- We can choose $y = y' + 2$ and $x = x' + 1$.
- Hence $y' = y - 2$ and $x' = x - 1$.

The mapping is

$$(x, y) \to (x - 1, y - 2)$$

which is a translation of 1 unit in the negative direction of the x-axis and 2 units in the negative direction of the y-axis.

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▶ Compound interest

Assume that you invest \$$P$ at an annual interest rate r. If the interest is compounded only once per year, then the balance of your investment after t years is given by $A = P(1 + r)^t$.

Now assume that the interest is compounded n times per year. The interest rate in each period is $\frac{r}{n}$. The balance at the end of one year is $P\left(1 + \frac{r}{n}\right)^n$, and the balance at the end of t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(\left(1 + \frac{r}{n}\right)^{\frac{n}{r}}\right)^{rt}$$

We recognise that

$$\lim_{\frac{n}{r}\to\infty}\left(1 + \frac{r}{n}\right)^{\frac{n}{r}} = e$$

So, as $n \to \infty$, we can write $A = Pe^{rt}$.

For example, if \$1000 is invested for one year at 5%, the resulting amount is \$1050. However, if the interest is compounded 'continuously', then the amount is given by

$$A = Pe^{rt} = 1000 \times e^{0.05} = 1000 \times 1.051\,271\ldots \approx 1051.27$$

That is, the balance after one year is \$1051.27.

Note: In the continuous model $A = Pe^{rt}$, the constant r is known as the **relative growth rate**. We consider more examples involving constant relative growth rates in Section 6H. We will also revisit relative growth rates in Chapter 8 when we study differentiation.

Section summary

Euler's number is the natural base for exponential functions:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$
$$= 2.718\,281\ldots$$

Exercise 6B

Skillsheet
Example 7

1 Sketch the graph of each of the following and state the range:

- **a** $f(x) = e^x + 1$
- **b** $f(x) = 1 - e^x$
- **c** $f(x) = 1 - e^{-x}$
- **d** $f(x) = e^{-2x}$
- **e** $f(x) = e^{x-1} - 2$
- **f** $f(x) = 2e^x$
- **g** $h(x) = 2(1 + e^x)$
- **h** $h(x) = 2(1 - e^{-x})$
- **i** $g(x) = 2e^{-x} + 1$
- **j** $h(x) = 2e^{x-1}$
- **k** $f(x) = 3e^{x+1} - 2$
- **l** $h(x) = 2 - 3e^x$

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CF

2 For each of the following, give a sequence of transformations that maps the graph of $y = e^x$ to the graph of $y = f_1(x)$:

a $f_1(x) = e^{x+2} - 3$
b $f_1(x) = 3e^{x+1} - 4$
c $f_1(x) = 5e^{2x+1}$
d $f_1(x) = 2 - e^{x-1}$
e $f_1(x) = 3 - 2e^{x+2}$
f $f_1(x) = 4e^{2x} - 1$

3 Find the rule of the image when the graph of $f(x) = e^x$ undergoes each of the following sequences of transformations:

a a dilation of factor 2 from the x-axis, followed by a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis
b a dilation of factor 2 from the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis
c a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis
d a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 from the x-axis
e a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 from the x-axis, followed by a reflection in the x-axis
f a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis.

4 For each of the following, give a sequence of transformations that maps the graph of $y = f_1(x)$ to the graph of $y = e^x$:

a $f_1(x) = e^{x+2} - 3$
b $f_1(x) = 3e^{x+1} - 4$
c $f_1(x) = 5e^{2x+1}$
d $f_1(x) = 2 - e^{x-1}$
e $f_1(x) = 3 - 2e^{x+2}$
f $f_1(x) = 4e^{2x} - 1$

SF

5 Solve each of the following equations using a graphics calculator. Give answers correct to three decimal places.

a $e^x = x + 2$
b $e^{-x} = x + 2$
c $x^2 = e^x$
d $x^3 = e^x$

CF

6 **a** Using a calculator, plot the graph of $y = f(x)$ where $f(x) = e^x$.
b Using the same screen, plot the graphs of:
i $y = f(x - 2)$ **ii** $y = f\left(\dfrac{x}{3}\right)$ **iii** $y = f(-x)$

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6C Revision of exponential equations

One method for solving exponential equations is to use the following property of exponential functions:

$a^x = a^y$ implies $x = y$, for $a \in \mathbb{R}^+ \setminus \{1\}$

Example 8

Find the value of x for which:

a $4^x = 256$

b $3^{x-1} = 81$

Solution

a
$$4^x = 256$$
$$4^x = 4^4$$
$$\therefore \quad x = 4$$

b
$$3^{x-1} = 81$$
$$3^{x-1} = 3^4$$
$$\therefore \quad x - 1 = 4$$
$$x = 5$$

When solving an exponential equation, you may also need to use the index laws.

Index laws

For all positive numbers a and b and all real numbers x and y:

- $a^x \times a^y = a^{x+y}$
- $a^x \div a^y = a^{x-y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
- $a^{-x} = \frac{1}{a^x}$
- $a^x = \frac{1}{a^{-x}}$
- $a^0 = 1$

Note: More generally, each index law applies for real numbers a and b provided both sides of the equation are defined. For example: $a^m \times a^n = a^{m+n}$ for $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}$.

Example 9

Find the value of x for which $5^{2x-4} = 25^{-x+2}$.

Solution	Explanation
$5^{2x-4} = 25^{-x+2}$ $= (5^2)^{-x+2}$ $= 5^{-2x+4}$	Express both sides of the equation as powers with base 5.
$\therefore \quad 2x - 4 = -2x + 4$ $4x = 8$ $x = 2$	Use the fact that $5^a = 5^b$ implies $a = b$.

To solve the equations in the next example, we must recognise that they will become quadratic equations once we make a substitution.

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Example 10

Solve for x:

a $9^x = 12 \times 3^x - 27$

b $3^{2x} = 27 - 6 \times 3^x$

Solution

a We can write the equation as

$$(3^x)^2 = 12 \times 3^x - 27$$

Let $y = 3^x$. The equation becomes

$$y^2 = 12y - 27$$
$$y^2 - 12y + 27 = 0$$
$$(y - 3)(y - 9) = 0$$

$$\therefore \quad y = 3 \quad \text{or} \quad y = 9$$
$$3^x = 3 \quad \text{or} \quad 3^x = 3^2$$
$$x = 1 \quad \text{or} \quad x = 2$$

b We can write the equation as

$$(3^x)^2 = 27 - 6 \times 3^x$$

Let $y = 3^x$. The equation becomes

$$y^2 = 27 - 6y$$
$$y^2 + 6y - 27 = 0$$
$$(y - 3)(y + 9) = 0$$

$$\therefore \quad y = 3 \quad \text{or} \quad y = -9$$
$$3^x = 3 \quad \text{or} \quad 3^x = -9$$

The only solution is $x = 1$, since $3^x > 0$ for all x.

Section summary

- One method for solving an exponential equation, without using a calculator, is first to express both sides of the equation as powers with the same base and then to equate the indices (since $a^x = a^y$ implies $x = y$, for any base $a \in \mathbb{R}^+ \setminus \{1\}$).
  For example: $2^{x+1} = 8 \Leftrightarrow 2^{x+1} = 2^3 \Leftrightarrow x + 1 = 3 \Leftrightarrow x = 2$
- Equations such as $3^{2x} - 6 \times 3^x - 27 = 0$ can be solved by making a substitution. In this case, substitute $y = 3^x$ to obtain a quadratic equation in y.

Exercise 6C

SF

1 Simplify the following expressions:

a $3x^2y^3 \times 2x^4y^6$
b $\dfrac{12x^8}{4x^2}$
c $18x^2y^3 \div (3x^4y)$
d $(4x^4y^2)^2 \div (2(x^2y)^4)$
e $(4x^0)^2$
f $15(x^5y^{-2})^4 \div (3(x^4y)^{-2})$
g $\dfrac{3(2x^2y^3)^4}{2x^3y^2}$
h $(8x^3y^6)^{\frac{1}{3}}$
i $\dfrac{x^2 + y^2}{x^{-2} + y^{-2}}$

Example 8

2 Solve for x in each of the following:

a $3^x = 81$
b $81^x = 9$
c $2^x = 256$
d $625^x = 5$
e $32^x = 8$
f $5^x = 125$
g $16^x = 1024$
h $2^{-x} = \dfrac{1}{64}$
i $5^{-x} = \dfrac{1}{625}$

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Example 9

3 Solve for n in each of the following:

a $5^{2n} \times 25^{2n-1} = 625$ **b** $4^{2n-2} = 1$ **c** $4^{2n-1} = \frac{1}{256}$

d $\frac{3^{n-2}}{9^{2-n}} = 27$ **e** $2^{2n-2} \times 4^{-3n} = 64$ **f** $2^{n-4} = 8^{4-n}$

g $27^{n-2} = 9^{3n+2}$ **h** $8^{6n+2} = 8^{4n-1}$ **i** $125^{4-n} = 5^{6-2n}$

j $2^{n-1} \times 4^{2n+1} = 16$ **k** $(27 \times 3^n)^n = 27^n \times 3^{\frac{1}{4}}$

Example 10

4 Solve for x:

a $3^{2x} - 2(3^x) - 3 = 0$ **b** $5^{2x} - 23(5^x) - 50 = 0$ **c** $5^{2x} - 10(5^x) + 25 = 0$

d $2^{2x} = 6(2^x) - 8$ **e** $8(3^x) - 6 = 2(3^{2x})$ **f** $2^{2x} - 20(2^x) = -64$

g $4^{2x} - 5(4^x) = -4$ **h** $3(3^{2x}) = 28(3^x) - 9$ **i** $7(7^{2x}) = 8(7^x) - 1$

6D Logarithms and the logarithm laws

Consider the statement

$$2^3 = 8$$

This may be written in an alternative form:

$$\log_2 8 = 3$$

which is read as 'the logarithm of 8 to the base 2 is equal to 3'.

For $a \in \mathbb{R}^+ \setminus \{1\}$, the **logarithm function** with base a is defined as follows:

$a^x = y$ is equivalent to $\log_a y = x$

Note: Since a^x is positive, the expression $\log_a y$ is only defined when y is positive.

Further examples:

- $3^2 = 9$ is equivalent to $\log_3 9 = 2$
- $10^4 = 10\,000$ is equivalent to $\log_{10} 10\,000 = 4$
- $a^0 = 1$ is equivalent to $\log_a 1 = 0$

Example 11

Without the aid of a calculator, evaluate the following:

a $\log_2 32$ **b** $\log_3 81$

Solution

a Let $\log_2 32 = x$

Then $2^x = 32$

$2^x = 2^5$

Therefore $x = 5$, giving $\log_2 32 = 5$.

b Let $\log_3 81 = x$

Then $3^x = 81$

$3^x = 3^4$

Therefore $x = 4$, giving $\log_3 81 = 4$.

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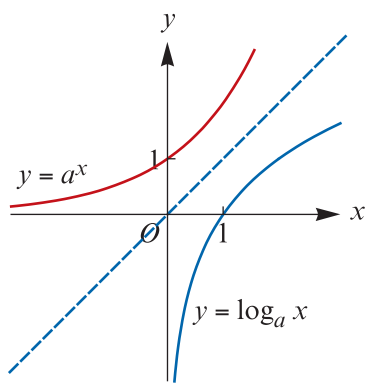
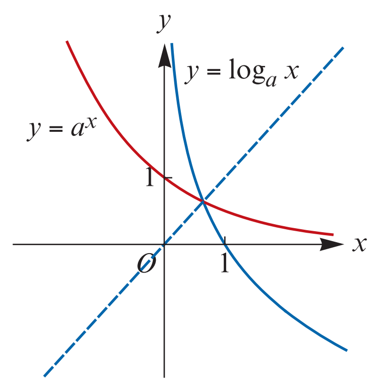
Note: To find $\log_2 32$, we ask 'What power of 2 gives 32?'
To find $\log_3 81$, we ask 'What power of 3 gives 81?'

▶ Inverse relationship between logarithms and exponentials

When the graph of $y = a^x$ is reflected in the line $y = x$, we obtain the graph of $x = a^y$. By the defining property of logarithms, this corresponds exactly to the graph of $y = \log_a x$.



$a > 1$



$0 < a < 1$

The logarithm function with base a is the 'reverse' of the exponential function with base a. This can be stated more precisely as follows:

Inverse relationship

Let $a \in \mathbb{R}^+ \setminus \{1\}$. The function with rule $y = \log_a x$, for $x > 0$, and the function with rule $y = a^x$, for $x \in \mathbb{R}$, are inverse to each other in the following way:

- $\log_a(a^x) = x$ for all x
- $a^{\log_a x} = x$ for all positive values of x

▶ The natural logarithm

Earlier in the chapter we defined the number e and the important exponential function $y = e^x$. The corresponding logarithmic function is $y = \log_e x$.

The logarithm function with base e is known as the **natural logarithm** function and is commonly written as $y = \ln x$. That is,

$$\ln x = \log_e x$$

Natural logarithm

The natural logarithm is defined as follows:

$$e^x = y \quad \text{is equivalent to} \quad \ln y = x$$

The natural logarithm function $y = \ln x$, for $x > 0$, and the exponential function $y = e^x$ are inverse to each other in the following way:

- $\ln(e^x) = x$ for all x
- $e^{\ln x} = x$ for all positive values of x

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Logarithm laws

The index laws are used to establish rules for computations with logarithms.

Law 1: Logarithm of a product

The logarithm of a product is the sum of their logarithms:

$$\log_a(mn) = \log_a m + \log_a n$$

Proof Let $\log_a m = x$ and $\log_a n = y$, where m and n are positive real numbers. Then $a^x = m$ and $a^y = n$, and therefore

$$mn = a^x \times a^y = a^{x+y} \qquad \text{(using the first index law)}$$

Hence $\log_a(mn) = x + y = \log_a m + \log_a n$.

For example:

$$\begin{aligned}\log_{10} 200 + \log_{10} 5 &= \log_{10}(200 \times 5)\\ &= \log_{10} 1000 = 3\end{aligned}$$

Law 2: Logarithm of a quotient

The logarithm of a quotient is the difference of their logarithms:

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Proof Let $\log_a m = x$ and $\log_a n = y$, where m and n are positive real numbers. Then as before $a^x = m$ and $a^y = n$, and therefore

$$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y} \qquad \text{(using the second index law)}$$

Hence $\log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n$.

For example:

$$\begin{aligned}\log_2 32 - \log_2 8 &= \log_2\left(\frac{32}{8}\right)\\ &= \log_2 4 = 2\end{aligned}$$

Law 3: Logarithm of a power

$$\log_a(m^p) = p \log_a m$$

Proof Let $\log_a m = x$. Then $a^x = m$, and therefore

$$m^p = (a^x)^p = a^{xp} \qquad \text{(using the third index law)}$$

Hence $\log_a(m^p) = xp = p \log_a m$.

For example:

$$\log_2 32 = \log_2(2^5) = 5$$

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Law 4: Logarithm of $\frac{1}{m}$

$$\log_a(m^{-1}) = -\log_a m$$

Proof Use logarithm law 3 with $p = -1$.

For example:

$$\log_a\left(\tfrac{1}{2}\right) = \log_a(2^{-1}) = -\log_a 2$$

Law 5

$$\log_a 1 = 0 \quad \text{and} \quad \log_a a = 1$$

Proof Since $a^0 = 1$, we have $\log_a 1 = 0$. Since $a^1 = a$, we have $\log_a a = 1$.

Example 12

Express each of the following as the logarithm of a single term:

a $\log_{10} 5 + 2\log_{10} 7$

b $2\ln 3 + \ln 16 - 2\ln\left(\frac{6}{5}\right)$

Solution

a $\log_{10} 5 + 2\log_{10} 7$

$= \log_{10} 5 + \log_{10}(7^2)$

$= \log_{10} 5 + \log_{10} 49$

$= \log_{10}(5 \times 49)$

$= \log_{10} 245$

b $2\ln 3 + \ln 16 - 2\ln\left(\frac{6}{5}\right)$

$= \ln(3^2) + \ln 16 - \ln\left(\frac{6}{5}\right)^2$

$= \ln 9 + \ln 16 - \ln\left(\frac{36}{25}\right)$

$= \ln\left(9 \times 16 \times \frac{25}{36}\right)$

$= \ln 100$

Note: For part **b**, remember that $\ln x = \log_e x$.

▶ Logarithmic equations

Example 13

Solve each of the following equations for x:

a $\log_2 x = 5$

b $\log_2(2x - 1) = 4$

c $\ln(3x + 1) = 0$

Solution

a $\log_2 x = 5$

$x = 2^5$

$\therefore \quad x = 32$

b $\log_2(2x - 1) = 4$

$2x - 1 = 2^4$

$2x = 17$

$\therefore \quad x = \frac{17}{2}$

c $\ln(3x + 1) = 0$

$3x + 1 = e^0$

$3x = 1 - 1$

$\therefore \quad x = 0$

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Example 14

Solve each of the following equations for x:

a $\ln(x - 1) + \ln(x + 2) = \ln(6x - 8)$

b $\log_2 x - \log_2(7 - 2x) = \log_2 6$

Solution

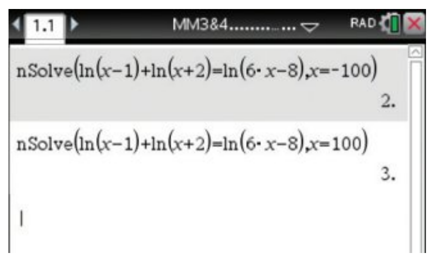
a
$$\begin{aligned}
\ln(x - 1) + \ln(x + 2) &= \ln(6x - 8)\\
\ln((x - 1)(x + 2)) &= \ln(6x - 8)\\
x^2 + x - 2 &= 6x - 8\\
x^2 - 5x + 6 &= 0\\
(x - 3)(x - 2) &= 0\\
\therefore \quad x = 3 &\text{ or } x = 2
\end{aligned}$$

Note: The solutions must satisfy $x - 1 > 0$, $x + 2 > 0$ and $6x - 8 > 0$. Therefore both of these solutions are allowable.

b
$$\begin{aligned}
\log_2 x - \log_2(7 - 2x) &= \log_2 6\\
\log_2\left(\frac{x}{7 - 2x}\right) &= \log_2 6\\
\frac{x}{7 - 2x} &= 6\\
x &= 42 - 12x\\
13x &= 42\\
\therefore \quad x &= \frac{42}{13}
\end{aligned}$$

Using the TI-Nspire CX non-CAS

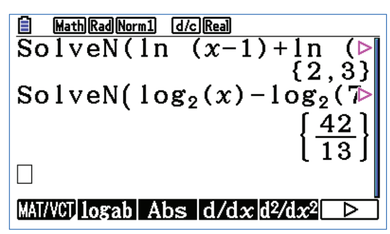
- Use menu > **Algebra** > **Numerical Solve** as shown. (Note the two extreme guess values used to obtain the two solutions.)
- Check the number of solutions by plotting the graphs of $f1(x) = \ln(x - 1) + \ln(x + 2)$ and $f2(x) = \ln(6x - 8)$, and then finding the intersection points.



Note: For the natural logarithm (base e), press ctrl e^x. For logarithms with other bases, press ctrl 10^x and complete the template.

Using the Casio

- In **Run-Matrix** mode, select the numerical solver (**Calculation** OPTN F4, **SolveN** F5).
- Enter the equation and press EXE.



Note: For the natural logarithm (base e), press ln. For logarithms with other bases, use the $\log_{\square}(\square)$ template, which is accessed from the **Math** menu F4 by selecting **logab** F2.

Example 15

Solve each of the following equations for x:

a $\ln(2x + 1) - \ln(x - 1) = 4$

b $\ln(x - 1) + \ln(x + 1) = 1$

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Solution

a

$$\ln(2x+1) - \ln(x-1) = 4$$
$$\ln\left(\frac{2x+1}{x-1}\right) = 4$$
$$\frac{2x+1}{x-1} = e^4$$
$$2x+1 = e^4(x-1)$$
$$(2-e^4)x = -(e^4+1)$$
$$\therefore \quad x = \frac{e^4+1}{e^4-2}$$

b

$$\ln(x-1) + \ln(x+1) = 1$$
$$\ln((x-1)(x+1)) = 1$$
$$\ln(x^2-1) = 1$$
$$x^2 - 1 = e$$
$$\therefore \quad x = \pm\sqrt{e+1}$$

But the original equation is not defined for $x = -\sqrt{e+1}$ and so the only solution is $x = \sqrt{e+1}$.

Example 16

Solve the equation $\log_x 27 = \frac{3}{2}$ for x.

Solution

$\log_x 27 = \frac{3}{2}$ is equivalent to

$$x^{\frac{3}{2}} = 27$$
$$(\sqrt{x})^3 = 3^3$$
$$\sqrt{x} = 3$$
$$\therefore \quad x = 9$$

Section summary

- For $a \in \mathbb{R}^+ \setminus \{1\}$, the logarithm function base a is defined as follows:

  $a^x = y$ is equivalent to $\log_a y = x$

- To evaluate $\log_a y$ ask the question: 'What power of a gives y?'
- Let $a \in \mathbb{R}^+ \setminus \{1\}$. The logarithmic function $y = \log_a x$, for $x > 0$, and the exponential function $y = a^x$, for $x \in \mathbb{R}$, are inverse to each other in the following way:
  - $\log_a(a^x) = x$ for all x
  - $a^{\log_a x} = x$ for all positive values of x
- **The natural logarithm**

  The function $y = \log_e x$ is commonly written as $y = \ln x$. Therefore

  $e^x = y$ is equivalent to $\ln y = x$

  The natural logarithm function $y = \ln x$, for $x > 0$, and the exponential function $y = e^x$, for $x \in \mathbb{R}$, are inverse to each other in the following way:
  - $\ln(e^x) = x$ for all x
  - $e^{\ln x} = x$ for all positive values of x
- **Logarithm laws**

  1 $\log_a(mn) = \log_a m + \log_a n$

  2 $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

  3 $\log_a(m^p) = p\log_a m$

  4 $\log_a(m^{-1}) = -\log_a m$

  5 $\log_a 1 = 0$ and $\log_a a = 1$

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Exercise 6D

Skillsheet SF

1 Evaluate each of the following: (Example 11)

a $\log_{10} 1000$ **b** $\log_2\left(\frac{1}{16}\right)$ **c** $\log_{10} 0.001$

d $\log_2 64$ **e** $\log_{10} 1\,000\,000$ **f** $\log_2\left(\frac{1}{128}\right)$

2 Express each of the following as the logarithm of a single term: (Example 12)

a $\log_{10} 2 + \log_{10} 3$ **b** $\log_{10} 32 - \log_{10} 8$ **c** $\ln 10 + \ln 100 + \ln 1000$

d $\ln\left(\frac{1}{2}\right) + \ln 14$ **e** $\ln\left(\frac{1}{3}\right) + \ln\left(\frac{1}{4}\right) + \ln\left(\frac{1}{5}\right)$ **f** $\ln(uv) + \ln(uv^2) + \ln(uv^3)$

g $2 \ln x + 5 \ln x$ **h** $\ln(x + y) + \ln(x - y) - \ln(x^2 - y^2)$

3 Solve each of the following equations for x: (Example 13)

a $\log_{10} x = 2$ **b** $2 \log_2 x = 8$ **c** $\ln(x - 5) = 0$

d $\log_2 x = 6$ **e** $2 \ln(x + 5) = 6$ **f** $\ln(2x) = 0$

g $\ln(2x + 3) = 0$ **h** $\log_{10} x = -3$ **i** $2 \log_2(x - 4) = 10$

4 Solve each of the following equations for x: (Example 14)

a $\log_{10} x = \log_{10} 3 + \log_{10} 5$ **b** $\ln x = \ln 15 - \ln 3$

c $\ln x = \frac{2}{3} \ln 8$ **d** $\ln x + \ln(2x - 1) = 0$

e $2 \ln x - \ln(x - 1) = \ln(x + 3)$

5 Express each of the following as the logarithm of a single term:

a $\log_{10} 9 + \log_{10} 3$ **b** $\log_2 24 - \log_2 6$ **c** $\frac{1}{2} \log_{10} a - \frac{1}{2} \log_{10} b$

d $1 + \log_{10} a - \frac{1}{3} \log_{10} b$ **e** $\frac{1}{2} \log_{10} 36 - \frac{1}{3} \log_{10} 27 - \frac{2}{3} \log_{10} 64$

6 Without using your calculator, evaluate each of the following:

a $\log_{10} 5 + \log_{10} 2$ **b** $\log_{10} 5 + 3 \log_{10} 2 - \log_{10} 4$

c $\log_2 \sqrt{2} + \log_2 1 + 2 \log_2 2$ **d** $2 \log_{10} 5 + 2 \log_{10} 2 + 1$

e $4 \log_{10} 2 - \log_{10} 16$

CF

7 Simplify the following expressions:

a $\log_3\left(\frac{1}{3^x}\right)$ **b** $\log_2 x - 2 \log_2 y + \log_2(xy^2)$ **c** $\ln(x^2 - y^2) - \ln(x - y) - \ln(x + y)$

8 Solve each of the following equations for x: (Example 15)

a $\ln(x^2 - 2x + 8) = 2 \ln x$ **b** $\ln(5x) - \ln(3 - 2x) = 1$

9 Solve each of the following equations for x:

a $\ln x + \ln(3x + 1) = 1$ **b** $8e^{-x} - e^x = 2$

10 Solve each of the following equations for x: (Example 16)

a $\log_x 81 = 4$ **b** $\log_x\left(\frac{1}{32}\right) = 5$

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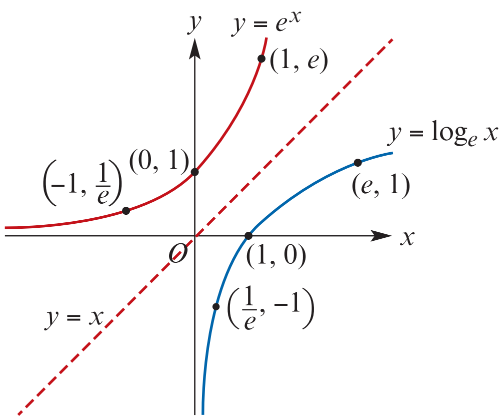
11 Solve $2\ln x + \ln 4 = \ln(9x - 2)$.

12 Given that $\log_a N = \frac{1}{2}(\log_a 24 - \log_a 0.375 - 6\log_a 3)$, find the value of N.

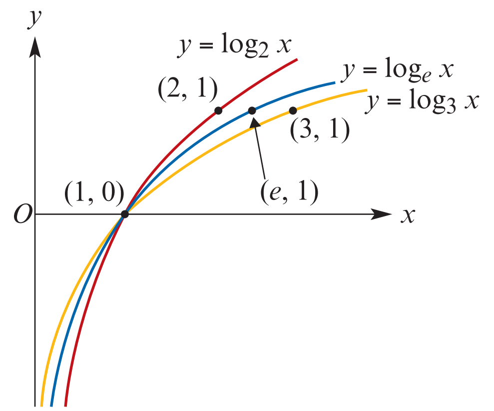
CF

6E Graphing logarithmic functions

The graphs of $y = e^x$ and $y = \log_e x$ are shown on the one set of axes.



The graphs of $y = \log_2 x$, $y = \log_e x$ and $y = \log_3 x$ are shown on the one set of axes.



Note: Recall that we usually write the natural logarithm function $y = \log_e x$ as $y = \ln x$.

For each base $a \in \mathbb{R}^+ \setminus \{1\}$, the graph of $f(x) = \log_a x$ has the following features:

- Key values are $f\left(\frac{1}{a}\right) = -1$, $f(1) = 0$ and $f(a) = 1$.
- The maximal domain is $\mathbb{R}^+$ and the range is $\mathbb{R}$.
- The y-axis is a vertical asymptote.

A logarithmic function with $a > 1$ is strictly increasing, and a logarithmic function with $0 < a < 1$ is strictly decreasing.

▶ Graphing transformations of $f(x) = \log_a x$

We now look at transformations applied to the graph of $f(x) = \log_a x$ where $a > 1$.

Example 17

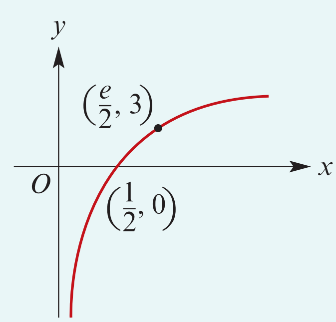
Sketch the graph of $y = 3\ln(2x)$.

Solution

This is obtained from the graph of $y = \ln x$ by a dilation of factor 3 from the x-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

The mapping is $(x, y) \to (\frac{1}{2}x, 3y)$.

- $(1, 0) \to (\frac{1}{2}, 0)$
- $(e, 1) \to (\frac{1}{2}e, 3)$



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Example 18

Sketch the graph and state the implied domain of each of the following:

a $y = \log_2(x - 5) + 1$ **b** $y = -\log_3(x + 4)$

Solution

a The graph of $y = \log_2(x - 5) + 1$ is obtained from the graph of $y = \log_2 x$ by a translation of 5 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.

The mapping is $(x, y) \to (x + 5, y + 1)$.

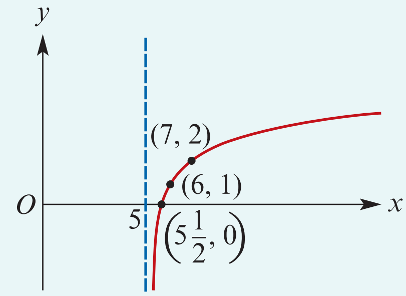
- $(1, 0) \to (6, 1)$
- $(2, 1) \to (7, 2)$

The asymptote has equation $x = 5$.

When $y = 0$,

$$\log_2(x - 5) + 1 = 0$$
$$\log_2(x - 5) = -1$$
$$x - 5 = 2^{-1}$$
$$\therefore \quad x = 5\tfrac{1}{2}$$

The domain of the function is $(5, \infty)$.



b The graph of $y = -\log_3(x + 4)$ is obtained from the graph of $y = \log_3 x$ by a reflection in the x-axis and a translation of 4 units in the negative direction of the x-axis.

The mapping is $(x, y) \to (x - 4, -y)$.

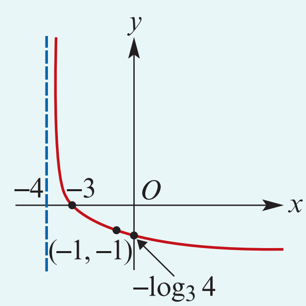
- $(1, 0) \to (-3, 0)$
- $(3, 1) \to (-1, -1)$

The asymptote has equation $x = -4$.

When $x = 0$,

$$y = -\log_3(0 + 4)$$
$$= -\log_3 4$$

The domain of the function is $(-4, \infty)$.



Example 19

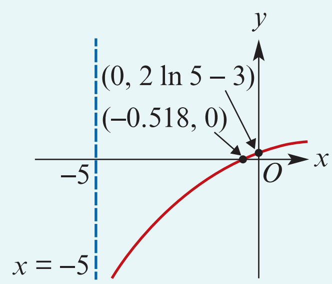
Sketch the graph of $y = 2\ln(x + 5) - 3$ and state the implied domain.

Solution

The graph of $y = 2\ln(x + 5) - 3$ is obtained from the graph of $y = \ln x$ by a dilation of factor 2 from the x-axis followed by a translation of 5 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis.

The equation of the asymptote is $x = -5$.

The domain of the function is $(-5, \infty)$.



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Axis intercepts

When $x = 0$, $\quad y = 2\ln(0+5) - 3$

$\qquad\qquad\quad = 2\ln 5 - 3$

When $y = 0$, $\quad 2\ln(x+5) - 3 = 0$

$$\ln(x+5) = \frac{3}{2}$$

$$x + 5 = e^{\frac{3}{2}}$$

$$\therefore \quad x = e^{\frac{3}{2}} - 5$$

► Exponential and logarithmic graphs with different bases

It is often useful to know how to go from one base to another.

To change the base of $\log_a x$ from a to b (where $a, b > 0$ and $a, b \neq 1$), we use the definition that $y = \log_a x$ implies $a^y = x$. Taking $\log_b$ of both sides:

$$\log_b(a^y) = \log_b x$$

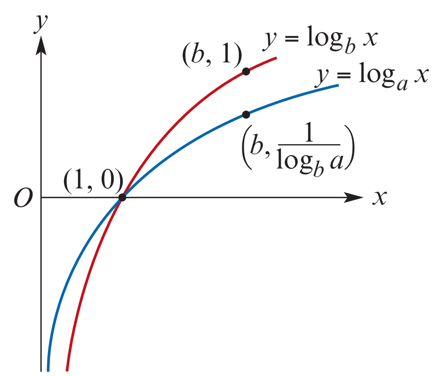
$$y \log_b a = \log_b x$$

$$y = \frac{\log_b x}{\log_b a}$$

Since $y = \log_a x$, this gives:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

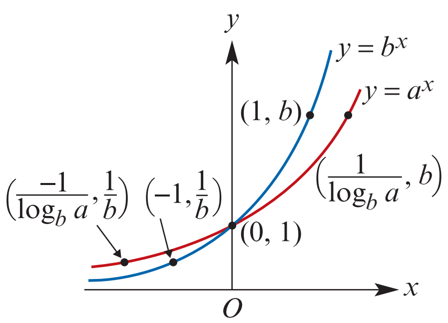
Hence the graph of $y = \log_a x$ can be obtained from the graph of $y = \log_b x$ by a dilation of factor $\dfrac{1}{\log_b a}$ from the x-axis.



Using the inverse relationship between logarithms and exponentials, we can write $a = b^{\log_b a}$.
This gives:

$$a^x = b^{(\log_b a)x}$$

Hence the graph of $y = a^x$ can be obtained from the graph of $y = b^x$ by a dilation of factor $\dfrac{1}{\log_b a}$ from the y-axis.



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Example 20

Find a transformation that takes the graph of $y = 2^x$ to the graph of $y = e^x$.

Solution

We can write $e = 2^{\log_2 e}$ and so

$$e^x = (2^{\log_2 e})^x$$
$$= 2^{(\log_2 e)x}$$

The graph of $y = e^x$ is the image of the graph of $y = 2^x$ under a dilation of factor $\dfrac{1}{\log_2 e}$ from the y-axis.

Section summary

Let $a \in \mathbb{R}^+ \setminus \{1\}$.

- The function $y = \log_a x$ has domain $\mathbb{R}^+$ and range $\mathbb{R}$.
- The graph of $y = \log_a x$ is the reflection of the graph of $y = a^x$ in the line $y = x$, and has the following properties:
  - The y-axis is an asymptote.
  - The x-values are always positive.
  - The x-axis intercept is 1.
  - There is no y-axis intercept.

Exercise 6E

SF

Example 17

1 Sketch the graph of each of the following:

a $y = 2\ln(3x)$ **b** $y = 4\ln(5x)$ **c** $y = 2\ln(4x)$ **d** $y = 3\ln\left(\dfrac{x}{2}\right)$

Example 18

2 For each of the following functions, sketch the graph (labelling axis intercepts and asymptotes) and state the maximal domain and range:

a $y = 2\ln(x-3)$ **b** $y = \ln(x+3) - 2$ **c** $y = 2\ln(x+1) - 1$
d $y = 2 + \ln(3x-2)$ **e** $y = -2\ln(x+2)$ **f** $y = -2\ln(x-2)$
g $y = 1 - \ln(x+1)$ **h** $y = \ln(2-x)$ **i** $y + 1 = \ln(4-3x)$

Example 19

3 Sketch the graph of each of the following. Label the axis intercepts and asymptotes. State the implied domain of each function.

a $y = \log_2(2x)$ **b** $y = \log_{10}(x-5)$ **c** $y = -\log_{10} x$
d $y = \log_{10}(-x)$ **e** $y = \log_{10}(5-x)$ **f** $y = 2\log_2(2x) + 2$
g $y = -2\log_2(3x)$ **h** $y = \log_{10}(-x-5) + 2$ **i** $y = 4\log_2(-3x)$
j $y = 2\log_2(2-x) - 6$ **k** $y = \ln(2x-1)$ **l** $y = -\ln(3-2x)$

4 Solve each of the following equations using a calculator. Give answers correct to three decimal places.

a $-x + 2 = \ln x$ **b** $\dfrac{1}{3}\ln(2x+1) = -\dfrac{1}{2}x + 1$

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SF

5 a Using a calculator, plot the graph of $y = f(x)$ where $f(x) = \ln x$.

b Using the same screen, plot the graphs of:

i $y = f(-x)$ ii $y = -f(x)$ iii $y = f\left(\frac{x}{3}\right)$ iv $y = f(3x)$

Example 20

6 Find a transformation that takes the graph of $y = 3^x$ to the graph of $y = e^x$.

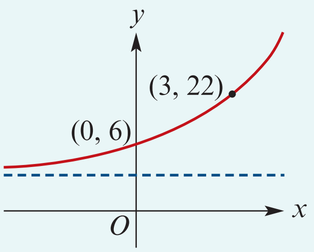
7 Find a transformation that takes the graph of $y = e^x$ to the graph of $y = 2^x$.

6F Determining rules for graphs of exponential and logarithmic functions

In previous chapters, we have determined the rules for graphs of various types of functions, including polynomial functions and trigonometric functions. In this chapter, we consider similar questions for exponential and logarithmic functions.

Example 21

The rule for the function with the graph shown is of the form $y = ae^x + b$. Find the values of a and b.



Solution

When $x = 0$, $y = 6$ and when $x = 3$, $y = 22$:

$$6 = ae^0 + b \qquad (1)$$

$$22 = ae^3 + b \qquad (2)$$

Subtract (1) from (2):

$$16 = a(e^3 - e^0)$$

$$16 = a(e^3 - 1)$$

$$\therefore \quad a = \frac{16}{e^3 - 1}$$

From equation (1):

$$b = 6 - a$$

$$= 6 - \frac{16}{e^3 - 1}$$

$$= \frac{6e^3 - 22}{e^3 - 1}$$

The function has rule $y = \left(\frac{16}{e^3 - 1}\right)e^x + \frac{6e^3 - 22}{e^3 - 1}$.

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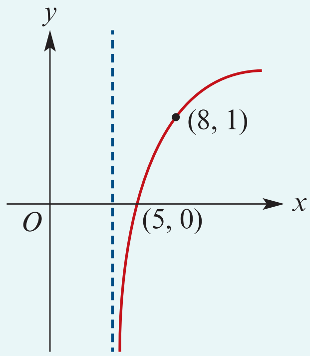
Example 22

The rule for the function with the graph shown is of the form $y = a\ln(x + b)$. Find the values of a and b.



Solution

$$0 = a\ln(5 + b) \qquad (1)$$
$$1 = a\ln(8 + b) \qquad (2)$$

From (1): $\ln(5 + b) = 0$

$$5 + b = e^0$$
$$\therefore \quad b = -4$$

Substitute in (2): $1 = a\ln 4$

$$\therefore \quad a = \frac{1}{\ln 4}$$

The rule is $y = \dfrac{1}{\ln 4}\ln(x - 4)$.

Explanation

Form two equations in a and b by substituting into the rule $y = a\ln(x + b)$:

- $y = 0$ when $x = 5$
- $y = 1$ when $x = 8$

Example 23

Given that $y = Ae^{bt}$ with $y = 6$ when $t = 1$ and $y = 8$ when $t = 2$, find A and b.

Solution

$$6 = Ae^b \qquad (1)$$
$$8 = Ae^{2b} \qquad (2)$$

Divide (2) by (1): $\dfrac{4}{3} = e^b$

$$\therefore \quad b = \ln\frac{4}{3}$$

Substitute in (1): $6 = Ae^{\ln\frac{4}{3}}$

$$6 = \frac{4}{3}A$$
$$\therefore \quad A = \frac{18}{4} = \frac{9}{2}$$

The rule is $y = \dfrac{9}{2}e^{(\ln\frac{4}{3})t}$.

Explanation

Form two equations in A and b by substituting into the rule $y = Ae^{bt}$:

- $y = 6$ when $t = 1$
- $y = 8$ when $t = 2$

$e^{\ln\frac{4}{3}} = \dfrac{4}{3}$ since $e^{\ln a} = a$ for all $a > 0$

Note that $y = \dfrac{9}{2}\left(e^{\ln\frac{4}{3}}\right)^t = \dfrac{9}{2}\left(\dfrac{4}{3}\right)^t$

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Exercise 6F

SF

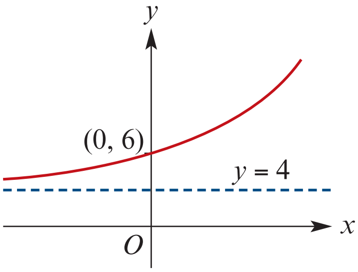
Example 21 **1** An exponential function has rule $y = a \times e^x + b$ and the points with coordinates $(0, 5)$ and $(4, 11)$ are on the graph of the function. Find the values of a and b.

Example 22 **2** A logarithmic function has rule $y = a \ln(x + b)$ and the points with coordinates $(5, 0)$ and $(10, 2)$ are on the graph of the function. Find the values of a and b.

3 The graph shown has rule

$$y = ae^x + b$$

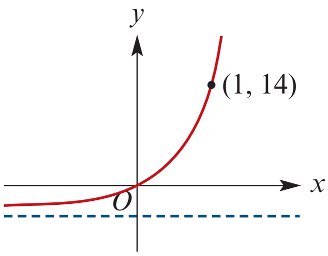
Find the values of a and b.



4 The rule for the function for which the graph is shown is of the form

$$y = ae^x + b$$

Find the values of a and b.

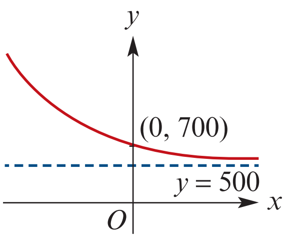


Example 23 **5** Find the values of a and b such that the graph of $y = ae^{-bx}$ goes through the points $(3, 50)$ and $(6, 10)$.

6 The rule for the function f is of the form

$$f(x) = ae^{-x} + b$$

Find the values of a and b.

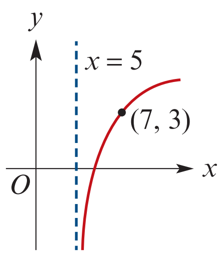


7 Find the values of a and b such that the graph of $y = a \log_2 x + b$ goes through the points $(8, 10)$ and $(32, 14)$.

8 The rule of the graph shown is of the form

$$y = a \log_2(x - b)$$

Find the values of a and b.



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9 Find the values of a and b such that the graph of $y = ae^{bx}$ goes through the points $(3, 10)$ and $(6, 50)$.

10 Find the values of a and b such that the graph of $y = a\log_2(x - b)$ passes through the points $(5, 2)$ and $(7, 4)$.

11 The points $(3, 10)$ and $(5, 12)$ lie on the graph of $y = a\ln(x - b) + c$. The graph has a vertical asymptote with equation $x = 1$. Find the values of a, b and c.

12 The graph of the function with rule $f(x) = a\ln(-x) + b$ passes through the points $(-2, 6)$ and $(-4, 8)$. Find the values of a and b.

6G Solving equations involving exponential and logarithmic functions

▶ Using logarithms to solve exponential equations

If $a \in \mathbb{R}^+ \setminus \{1\}$, then the statements $a^x = b$ and $\log_a b = x$ are equivalent. This defining property of logarithms may be used in the solution of exponential equations.

Example 24

Solve for x if $2^x = 11$, expressing the answer to two decimal places.

Solution

$$2^x = 11 \Leftrightarrow x = \log_2 11$$
$$= 3.45943\ldots$$

Therefore $x \approx 3.46$ correct to two decimal places.

Example 25

Solve $3^{2x-1} = 28$, expressing the answer to three decimal places.

Solution

$$3^{2x-1} = 28 \Leftrightarrow 2x - 1 = \log_3 28$$

Thus $2x - 1 = \log_3 28$

$$2x = \log_3 28 + 1$$
$$x = \frac{1}{2}(\log_3 28 + 1)$$
$$\approx 2.017 \quad \text{correct to three decimal places}$$

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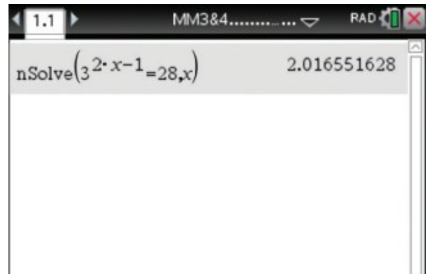
TI-84

Using the TI-Nspire CX non-CAS

- Use (menu) > **Algebra** > **Numerical Solve** and complete as shown.
- Round to three decimal places as required: $x = 2.017$.

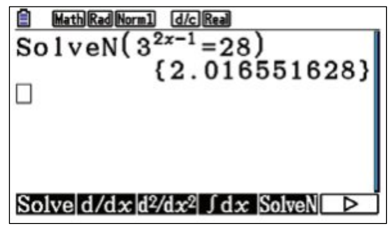
Hint: Check the number of solutions by finding the intersection points of the graphs of $f1(x) = 3^{2x-1}$ and $f2(x) = 28$.



Using the Casio

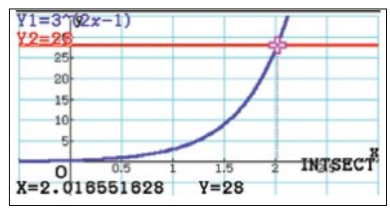
Method 1: Using the numerical solver

- In **Run-Matrix** mode, use the numerical solver (**Calculation** (OPTN) (F4), **SolveN** (F5)).



Method 2: Using Graph mode

- In **Graph** mode, enter the rules $y = 3^{2x-1}$ and $y = 28$ in $Y1$ and $Y2$ respectively.
- Select **Draw** (F6) to view the graph. Adjust the View Window (SHIFT) (F3) if required.
- To find the intersection point, go to the **G-Solve** menu (SHIFT) (F5) and select **Intersection** (F5).



Example 26

Solve the inequality $0.7^x \geq 0.3$.

Solution

Taking $\log_{10}$ of both sides:

$$\log_{10}(0.7^x) \geq \log_{10} 0.3$$

$$x \log_{10} 0.7 \geq \log_{10} 0.3$$

$$\therefore \quad x \leq \frac{\log_{10} 0.3}{\log_{10} 0.7} \qquad \text{(direction of inequality reversed since } \log_{10} 0.7 < 0\text{)}$$

Alternatively, we can solve the inequality $0.7^x \geq 0.3$ directly as follows:

Note that $0 < 0.7 < 1$ and thus $y = 0.7^x$ is strictly decreasing. Therefore the inequality $0.7^x \geq 0.3$ holds for $x \leq \log_{0.7} 0.3$.

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Solving logarithmic equations

We solved equations involving logarithms in Section 6D. We consider further examples here.

Example 27

Solve each of the following equations for x:

a $\log_2(3x - 4) = 5$

b $\log_3\left(\frac{x-3}{2}\right) = -1$

Solution

a
$$\begin{aligned} \log_2(3x - 4) &= 5 \\ 3x - 4 &= 2^5 \\ 3x - 4 &= 32 \\ 3x &= 36 \\ \therefore \quad x &= 12 \end{aligned}$$

b
$$\begin{aligned} \log_3\left(\frac{x-3}{2}\right) &= -1 \\ \frac{x-3}{2} &= 3^{-1} \\ x - 3 &= \frac{2}{3} \\ \therefore \quad x &= \frac{11}{3} \end{aligned}$$

Example 28

Solve $\log_{10} 25 + \log_{10} x - \log_{10}(x - 1) = 2$.

Solution

$$\begin{aligned} \log_{10} 25 + \log_{10} x - \log_{10}(x - 1) &= 2 \\ \log_{10}\left(\frac{25x}{x-1}\right) &= 2 \\ \frac{25x}{x-1} &= 10^2 \\ \frac{25x}{x-1} &= 100 \\ x &= 4x - 4 \\ \therefore \quad x &= \frac{4}{3} \end{aligned}$$

Example 29

If $\log_2 6 = k \log_2 3 + 1$, find the value of k.

Solution

$$\begin{aligned} \log_2 6 &= k \log_2 3 + 1 \\ &= \log_2(3^k) + \log_2 2 \\ &= \log_2(2 \times 3^k) \\ \therefore \quad 6 &= 2 \times 3^k \\ 3 &= 3^k \\ k &= 1 \end{aligned}$$

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Transposing equations

The inverse relationship between logarithms and exponentials can be used when solving and rearranging equations:

- $\log_a(a^x) = x$ for all $x \in \mathbb{R}$
- $a^{\log_a x} = x$ for all $x \in \mathbb{R}^+$

Example 30

Rewrite the equation $y = 2\ln(x) + 3$ with x as the subject.

Solution

$$y = 2\ln(x) + 3$$

$$\frac{y-3}{2} = \ln x$$

$$\therefore \quad x = e^{\frac{y-3}{2}}$$

Example 31

Rearrange the equation $x = 2\ln(y - 1) + 3$ to express y as a function of x.

Solution

$$x = 2\ln(y - 1) + 3$$

$$\frac{x-3}{2} = \ln(y - 1)$$

$$y - 1 = e^{\frac{x-3}{2}}$$

$$\therefore \quad y = e^{\frac{x-3}{2}} + 1$$

Example 32

Rewrite the equation $P = Ae^{kt}$ with t as the subject.

Solution

$$P = Ae^{kt}$$

Take logarithms with base e of both sides:

$$\begin{aligned} \ln P &= \ln(Ae^{kt}) \\ &= \ln A + \ln(e^{kt}) \\ &= \ln A + kt \end{aligned}$$

$$\begin{aligned} \therefore \quad t &= \frac{1}{k}(\ln P - \ln A) \\ &= \frac{1}{k}\ln\left(\frac{P}{A}\right) \end{aligned}$$

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Section summary

- If $a \in \mathbb{R}^+ \setminus \{1\}$, then the statements $a^x = b$ and $\log_a b = x$ are equivalent. This defining property of logarithms may be used in the solution of exponential equations and inequalities. For example:
  - $2^x = 5 \Leftrightarrow x = \log_2 5$
  - $2^x \geq 5 \Leftrightarrow x \geq \log_2 5$
  - $0.3^x = 5 \Leftrightarrow x = \log_{0.3} 5$
  - $0.3^x \geq 5 \Leftrightarrow x \leq \log_{0.3} 5$
- An exponential inequality may also be solved by taking $\log_a$ of both sides. For $a > 1$, the direction of the inequality stays the same (as $y = \log_a x$ is strictly increasing). For $0 < a < 1$, the direction of the inequality reverses (as $y = \log_a x$ is strictly decreasing).
- The inverse relationship between logarithms and exponentials is helpful when solving and rearranging equations:
  - $\log_a(a^x) = x$ for all $x \in \mathbb{R}$
  - $a^{\log_a x} = x$ for all $x \in \mathbb{R}^+$

Exercise 6G

SF

Example 24, 25

1 Use your calculator to solve each of the following equations, correct to two decimal places:

a $2^x = 6$ **b** $3^x = 0.7$ **c** $3^x = 11$ **d** $4^x = 5$

e $2^{-x} = 5$ **f** $0.2^x = 3$ **g** $5^x = 3^{x-1}$ **h** $8^x = 2005^{x+1}$

i $3^{x-1} = 8$ **j** $0.3^{x+2} = 0.7$ **k** $2^{x-1} = 3^{x+1}$ **l** $1.4^{x+2} = 25(0.9^x)$

m $5^x = 2^{2x-2}$ **n** $2^{\frac{1}{2}(x+2)} = 3^{x-1}$ **o** $2^{x+1} \times 3^{x-1} = 100$

2 Solve for x using a graphics calculator. Express answers correct to two decimal places.

a $2^x < 7$ **b** $3^x > 6$ **c** $0.2^x > 3$ **d** $0.2^x \leq 0.4$

3 Solve each of the following equations for x. Give exact answers.

a $2^x = 5$ **b** $3^{2x-1} = 8$ **c** $7^{3x+1} = 20$

d $3^x = 7$ **e** $3^x = 6$ **f** $5^x = 6$

g $3^{2x} - 3^{x+2} + 8 = 0$ **h** $5^{2x} - 4 \times 5^x - 5 = 0$

Example 26

4 Solve each of the following inequalities for x. Give exact answers.

a $7^x > 52$ **b** $3^{2x-1} < 40$ **c** $4^{3x+1} \geq 5$

d $3^{x-5} \leq 30$ **e** $3^x < 106$ **f** $5^x < 0.6$

CF

Example 27

5 Solve each of the following equations for x:

a $\log_2(2x - 3) = 4$ **b** $\log_5\left(\frac{x-3}{5}\right) = -2$

Example 28

6 Solve each of the following equations for x:

a $\log_2(x + 2) - \log_2 x = 2$ **b** $\log_{10} 60 + \log_{10}(2x) - \log_{10}(x - 2) = 3$

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7 **a** Solve $2\log_{10} 15 + \log_{10}(5 - x) - \log_{10}(4x) = 2$.
b Solve $\log_{10}(1 - 2x) - 2\log_{10} x = 1 - \log_{10}(2 - 5x)$.
c Solve $\log_{10}(3x + 2) + 6\log_{10} 2 = 2 + \log_{10}(2x + 1)$.

CF

Example 29 **8** **a** If $\log_2 8 = k\log_2 7 + 2$, find the value of k.
b If $\log_2 7 - x\log_2 7 = 4$, find the value of x.
c If $\ln 7 - x\ln 14 = 1$, find the value of x.

Example 30 **9** Rewrite the equation $y = 3\ln(x) - 4$ with x as the subject.

SF

Example 31 **10** For each of the following functions $y = f(x)$, rearrange the rule to show that x can be expressed as a function of y:

a $y = \ln(2x)$ **b** $y = 3\ln(2x) + 1$ **c** $y = e^x + 2$
d $y = e^{x+2}$ **e** $y = \ln(2x + 1)$ **f** $y = 4\ln(3x + 2)$
g $y = \log_{10}(x + 1)$ **h** $y = 2e^{x-1}$

CF

11 **a** Using a calculator, for each of the following plot the graphs of $y = f(x)$ and $y = g(x)$, together with the line $y = x$, on the one set of axes:
i $f(x) = \ln x$ and $g(x) = e^x$
ii $f(x) = 2\ln(x) + 3$ and $g(x) = e^{\frac{x-3}{2}}$
iii $f(x) = \log_{10} x$ and $g(x) = 10^x$
b Use your answers to part **a** to comment on the relationship in general between the graphs of $f(x) = a\log_b(x) + c$ and $g(x) = b^{\frac{x-c}{a}}$.

Example 32 **12** Rewrite the equation $P = Ae^{-kt} + b$ with t as the subject.

SF

13 For each of the following formulas, make the pronumeral in brackets the subject:

a $y = 2\ln(x) + 5$ (x) **b** $P = Ae^{-6x}$ (x)
c $y = ax^n$ (n) **d** $y = 5 \times 10^x$ (x)
e $y = 5 - 3\ln(2x)$ (x) **f** $y = 6x^{2n}$ (n)
g $y = \ln(2x - 1)$ (x) **h** $y = 5(1 - e^{-x})$ (x)

14 **a** If $a\log_2 7 = 3 - \log_6 14$, find the value of a, correct to three significant figures.
b If $\log_3 18 = \log_{11} k$, find the value of k, correct to one decimal place.

15 Prove that if $\log_r p = q$ and $\log_q r = p$, then $\log_q p = pq$.

CF

16 If $u = \log_9 x$, find in terms of u:
a x **b** $\log_9(3x)$ **c** $\log_x 81$

17 Solve the equation $\log_5 x = 16\log_x 5$.

18 Given that $q^p = 25$, find $\log_5 q$ in terms of p.

19 **a** Prove that $\log_a x = \dfrac{1}{\log_x a}$.
b Hence solve $\log_x 8 + \log_8 x = \dfrac{13}{6}$ for x, where x is an integer greater than 1.

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6H Applications of exponential functions

If the rate at which a quantity increases or decreases is proportional to its current value, then the quantity obeys the **law of exponential change**.

Let A be the quantity at time t. Then

$$A = A_0 e^{kt}$$

where A_0 is the initial quantity and k is a constant (called the **relative growth rate**).

If $k > 0$, the model represents **growth**:

- growth of cells
- population growth
- continuously compounded interest

If $k < 0$, the model represents **decay**:

- radioactive decay
- cooling of materials

An equivalent way to write this model is as $A = A_0 b^t$, where we take $b = e^k$. In this form, growth corresponds to $b > 1$ and decay corresponds to $b < 1$.

Cell growth

Suppose a particular type of bacteria cell divides into two new cells every T_D minutes. Let N_0 be the initial number of cells of this type. After t minutes the number of cells, N, is given by

$$N = N_0 2^{\frac{t}{T_D}}$$

where T_D is called the **generation time**.

Example 33

What is the generation time of a bacterial population that increases from 5000 cells to 100 000 cells in four hours of growth?

Solution

In this example, $N_0 = 5000$ and $N = 100\,000$ when $t = 240$.

Hence $100\,000 = 5000 \times 2^{\frac{240}{T_D}}$

$$20 = 2^{\frac{240}{T_D}}$$

Thus $T_D = \dfrac{240}{\log_2 20} \approx 55.53$ (correct to two decimal places).

The generation time is approximately 55.53 minutes.

Radioactive decay

Radioactive materials decay such that the amount of radioactive material, A, present at time t (in years) is given by

$$A = A_0 e^{-kt}$$

where A_0 is the initial amount and k is a positive constant that depends on the type of material. A radioactive substance is often described in terms of its **half-life**, which is the time required for half the material to decay.

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Example 34

After 1000 years, a sample of radium-226 has decayed to 64.7% of its original mass. Find the half-life of radium-226.

Solution

We use the formula $A = A_0 e^{-kt}$. When $t = 1000$, $A = 0.647A_0$. Thus

$$\begin{aligned} 0.647A_0 &= A_0 e^{-1000k} \\ 0.647 &= e^{-1000k} \\ -1000k &= \ln 0.647 \\ k &= \frac{-\ln 0.647}{1000} \approx 0.000435 \end{aligned}$$

To find the half-life, we consider when $A = \frac{1}{2}A_0$:

$$\begin{aligned} A_0 e^{-kt} &= \tfrac{1}{2}A_0 \\ e^{-kt} &= \tfrac{1}{2} \\ -kt &= \ln(\tfrac{1}{2}) \\ t &= -\frac{\ln(\frac{1}{2})}{k} \approx 1591.95 \end{aligned}$$

The half-life of radium-226 is approximately 1592 years.

Population growth

It is sometimes possible to model population growth through exponential models.

Example 35

The population of a town was 8000 at the beginning of 2007 and 15 000 at the end of 2014. Assume that the growth is exponential.

a Find the population at the end of 2016.

b In what year will the population be double that of 2014?

Solution

Let P be the population at time t years (measured from 1 January 2007). Then

$$P = 8000e^{kt}$$

At the end of 2014, $t = 8$ and $P = 15\,000$. Therefore

$$\begin{aligned} 15\,000 &= 8000e^{8k} \\ \frac{15}{8} &= e^{8k} \\ k &= \frac{1}{8}\ln\left(\frac{15}{8}\right) \approx 0.079 \end{aligned}$$

The relative rate of increase is 7.9% per annum.

Note: The approximation 0.079 was not used in the calculations which follow. The value for k was held in the calculator.

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a When $t = 10$, $P = 8000e^{10k}$

$$\approx 17\,552.6049$$
$$\approx 17\,550$$

The population is approximately 17 550.

b When does $P = 30\,000$? Consider the equation

$$30\,000 = 8000e^{kt}$$
$$\frac{30\,000}{8000} = e^{kt}$$
$$\frac{15}{4} = e^{kt}$$
$$\therefore \quad t = \frac{1}{k}\ln\left(\frac{15}{4}\right)$$
$$\approx 16.82$$

The population reaches 30 000 approximately 16.82 years after the beginning of 2007, i.e. during the year 2023.

Example 36

There are approximately ten times as many red kangaroos as grey kangaroos in a certain area. If the population of grey kangaroos increases at a rate of 11% per annum while that of the red kangaroos decreases at 5% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

Solution

Let G_0 be the population of grey kangaroos at the start.

Then the number of grey kangaroos after n years is $G = G_0(1.11)^n$, and the number of red kangaroos after n years is $R = 10G_0(0.95)^n$.

When the proportions are reversed:

$$G = 10R$$
$$G_0(1.11)^n = 10 \times 10G_0(0.95)^n$$
$$(1.11)^n = 100(0.95)^n$$

Taking logarithms base e of both sides:

$$\ln((1.11)^n) = \ln(100(0.95)^n)$$
$$n\ln 1.11 = \ln 100 + n\ln 0.95$$
$$\therefore \quad n = \frac{\ln 100}{\ln 1.11 - \ln 0.95}$$
$$\approx 29.6$$

i.e. the proportions of the kangaroo populations will be reversed after 30 years.

 ISBN 978-1-108-45164-2

Section summary

There are many situations in which a varying quantity can be modelled by an exponential function. Let A be the quantity at time t. Then

$$A = A_0 e^{kt}$$

where A_0 is the initial quantity and k is a constant (called the **relative growth rate**). Growth corresponds to $k > 0$, and decay corresponds to $k < 0$.

Exercise 6H

Skillsheet

Example 33

CF

1 A population of 1000 E. coli bacteria doubles every 15 minutes.

a Determine the formula for the number of bacteria at time t minutes.

b How long will it take for the population to reach 10 000? (Give your answer to the nearest minute.)

2 In the initial period of its life, a particular species of tree grows in the manner described by the rule $d = d_0 10^{mt}$, where d is the diameter (in cm) of the tree t years after the beginning of this period. The diameter is 52 cm after 1 year, and 80 cm after 3 years. Calculate the values of the constants d_0 and m.

3 The number of people, N, who have a particular disease at time t years is given by $N = N_0 e^{kt}$.

a If the number is initially 20 000 and the number decreases by 20% each year, find:

i the value of N_0 **ii** the value of k.

b How long does it take until only 5000 people are infected?

Example 34

4 Polonium-210 is a radioactive substance. The decay of polonium-210 is described by the formula $M = M_0 e^{-kt}$, where M is the mass in grams of polonium-210 left after t days, and M_0 and k are constants. At $t = 0$, $M = 10$ g and at $t = 140$, $M = 5$ g.

a Find the values of M_0 and k.

b What will be the mass of the polonium-210 after 70 days?

c After how many days is the mass remaining 2 g?

5 A quantity A of radium at time t years is given by $A = A_0 e^{-kt}$, where k is a positive constant and A_0 is the amount of radium at time $t = 0$.

a Given that $A = \frac{1}{2}A_0$ when $t = 1690$ years, calculate k.

b After how many years does only 20% of the original amount remain? Give your answer to the nearest year.

6 The half-life of plutonium-239 is 24 000 years. If 20 grams are present now, how long will it take until only 20% of the original sample remains? (Give your answer to the nearest year.)

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CF

7 Carbon-14 is a radioactive substance with a half-life of 5730 years. It is used to determine the age of ancient objects. A Babylonian cloth fragment now has 40% of the carbon-14 that it contained originally. How old is the fragment of cloth?

Example 35

8 The population of a town was 10 000 at the beginning of 2002 and 15 000 at the end of 2014. Assume that the growth is exponential.

a Find the population at the end of 2017.

b In what year will the population be double that of 2014?

Example 36

9 There are approximately five times as many magpies as currawongs in a certain area. If the population of currawongs increases at a rate of 12% per annum while that of the magpies decreases at 6% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

10 The pressure in the Earth's atmosphere decreases exponentially as you rise above the surface. The pressure in millibars at a height of h kilometres is given approximately by the function $P(h) = 1000 \times 10^{-0.05428h}$.

a Find the pressure at a height of 4 km. (Give your answer to the nearest millibar.)

b Find the height at which the pressure is 450 millibars. (Give your answer to the nearest metre.)

11 A biological culture contains 500 000 bacteria at 12 p.m. on Sunday. The culture increases by 10% every hour. At what time will the culture exceed 4 million bacteria?

12 When a liquid is placed into a refrigerator, its temperature T°C at time t minutes is given by the formula $T = T_0 e^{-kt}$. The temperature is initially 100°C and drops to 40°C in 5 minutes. Find the temperature of the liquid after 15 minutes.

13 The number of bacteria in a certain culture at time t weeks is given by the rule $N = N_0 e^{kt}$. If when $t = 2$, $N = 101$ and when $t = 4$, $N = 203$, calculate the values of N_0 and k.

14 Five kilograms of sugar is gradually dissolved in a vat of water. After t hours, the amount, S kg, of undissolved sugar remaining is given by $S = 5 \times e^{-kt}$.

a Calculate k given that $S = 3.2$ when $t = 2$.

b At what time will there be 1 kg of sugar remaining?

15 The number of bacteria, N, in a culture increases exponentially with time according to the rule $N = a \times b^t$, where time t is measured in hours. When observation started, there were 1000 bacteria, and 5 hours later there were 15 000 bacteria.

a Find the values of a and b.

b Find, to the nearest minute, when there were 5000 bacteria.

c Find, to the nearest minute, when the number of bacteria first exceeds 1 000 000.

d How many bacteria would there be 12 hours after the first observation?

ISBN 978-1-108-45164-2

6I Applications of logarithmic functions

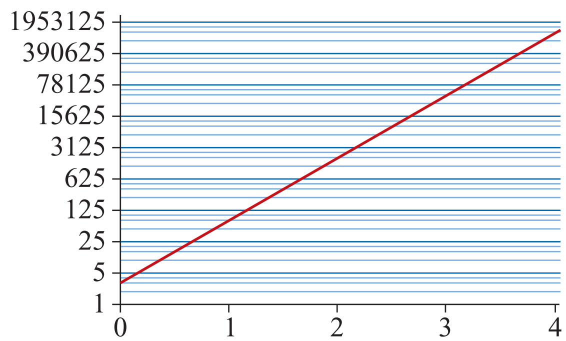
A **logarithmic scale** is a scale of measurement that uses the logarithm of a quantity. Familiar examples of logarithmic scales include the Richter scale (earthquakes), decibels (noise) and pH (acidity). In this section, we will show why such scales are useful and study these three examples. We will also look at some other applications of logarithmic functions.

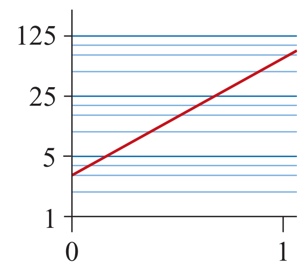
► Charts with logarithmic scales

In the chart below we show an example for which the vertical axis has equally spaced increments that are labelled $1 = 5^0$, $5 = 5^1$, $25 = 5^2$, $125 = 5^3$, . . . instead of 0, 1, 2, 3, ...

The major horizontal grid lines, which are equally spaced, are the integer powers of 5.

The straight line is the result of graphing $y = 3 \times 5^{2x}$ but with the vertical axis having a logarithmic scale (base 5).





The bottom-left corner of the chart is shown 'blown up' on the right.

- The horizontal grid lines between 1 and 5 represent 2, 3 and 4. Their positions are determined by $\log_5 2$, $\log_5 3$ and $\log_5 4$.
- The horizontal grid lines between 5 and 25 represent 10, 15 and 20. Their positions are determined by $\log_5 10$, $\log_5 15$ and $\log_5 20$.
- The horizontal grid lines between 25 and 125 represent 50, 75 and 100. Their positions are determined by $\log_5 50$, $\log_5 75$ and $\log_5 100$.

Notice that there is the same gap between the grid lines for 1 and 2, for 5 and 10, and for 25 and 50. This is because

$$\log_5 2 - \log_5 1 = \log_5 2 \qquad \log_5 10 - \log_5 5 = \log_5 2 \qquad \log_5 50 - \log_5 25 = \log_5 2$$

This is not what we are used to from working with linear scales.

- For a **linear scale**, the change between two values is determined by the difference between the values. That is, a change from 1 to 2 is the same as a change from 6 to 7. We use linear scales to measure temperature (degrees Celsius) and length (metres) and in standard Cartesian graphs.
- For a **logarithmic scale**, the change between two values is determined by the ratio of the values. That is, a change from 1 to 2 (ratio of 1 : 2) would be perceived as the same amount of increase as a change from 6 to 12 (also a ratio of 1 : 2).

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The ratio property of logarithmic scales is explained by the logarithm law

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

The difference between $\log_a x$ and $\log_a y$ depends only on the ratio $x : y$.

Presentation of data on a logarithmic scale can be helpful when the data covers a large range of values. The use of the logarithms of the values rather than the actual values reduces a wide range to a more manageable size.

▶ Graphing with logarithmic scales

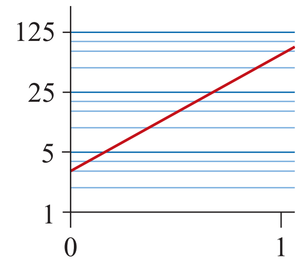
We will see how to sketch the graph of an exponential function using a logarithmic scale for the vertical axis and a linear scale for the horizontal axis.

Let $y = b \times a^{mx}$, where a, b and m are positive real numbers. Then taking logarithms base a of both sides gives

$$\log_a y = \log_a b + mx$$

So the graph of $\log_a y$ against x is a straight line with gradient m and with $(\log_a y)$-axis intercept at $(0, \log_a b)$.

In the case of the exponential function $y = 3 \times 5^{2x}$, the equation for the straight line becomes $\log_5 y = \log_5 3 + 2x$. On the chart, we have labelled the vertical axis with powers of 5 rather than the logarithm.



This straight line has gradient 2. The line through points $(0.5, 15)$ and $(1, 75)$ on this chart has actual gradient

$$\frac{\log_5 75 - \log_5 15}{1 - 0.5} = 2\log_5\left(\frac{75}{15}\right) = 2\log_5 5 = 2$$

Of course any two points on the line will give this result.

Example 37

Let $y = 4 \times 3^{2x}$, for $x \geq 0$.

a If $\log_3 y = mx + c$, give the values of m and c.

b Sketch the graph of $\log_3 y$ against x.

c Sketch the graph of $\log_3 y$ against x labelling your vertical axis with powers of 3.

Solution

a Take logarithms base 3 of both sides of the equation:

$$\log_3 y = \log_3(4 \times 3^{2x})$$
$$\log_3 y = \log_3 4 + \log_3(3^{2x})$$
$$\log_3 y = 2x + \log_3 4$$

Therefore $m = 2$ and $c = \log_3 4$.

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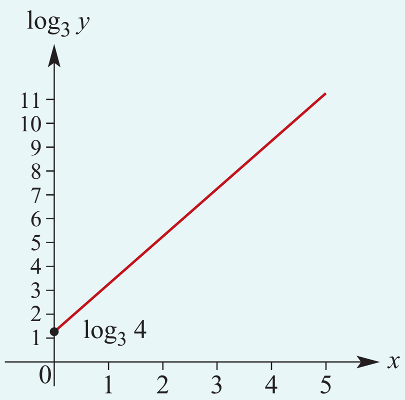
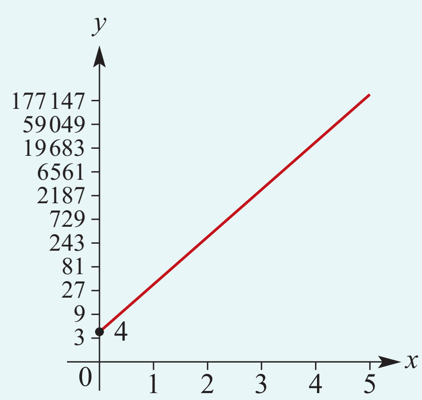
b



c



Note: For part **c**, using the logarithmic scale on the vertical axis means that the axis intercept is at $3^{\log_3 4} = 4$.

Applications of logarithmic scales

We now look at three uses of logarithmic scales.

Decibels

A **decibel** is defined as one-tenth of a bel, which is named after Alexander Graham Bell, the inventor of the telephone.

The decibel is a logarithmic scale for measuring 'loudness of noise'. The intensity of a sound in decibels can be defined by

$$dB = 10\log_{10}(P \times 10^{16})$$

where P is the power of the sound in watt/cm^2.

Example 38

A power mower generates a noise of 96 dB and a conversation in a restaurant generates noise of 60 dB. Find the power in watt/cm^2 for each of these.

Solution

- **Power mower**

  We have $96 = 10\log_{10}(P \times 10^{16})$.
  Hence

  $$\log_{10}(P \times 10^{16}) = 9.6$$
  $$P \times 10^{16} = 10^{9.6}$$
  $$P = 10^{-6.4}$$

  The power is $10^{-6.4}$ watt/cm^2.

- **Conversation**

  We have $60 = 10\log_{10}(P \times 10^{16})$.
  Hence

  $$\log_{10}(P \times 10^{16}) = 6$$
  $$P \times 10^{16} = 10^{6}$$
  $$P = 10^{-10}$$

  The power is 10^{-10} watt/cm^2.

Note: The maximum intensity which the ear can tolerate is about 10^{-4} watt/cm^2, which corresponds to a loudness level of about 120 dB.

ISBN 978-1-108-45164-2

The Richter scale

Earthquake intensity is often reported on the **Richter scale**. The formula is

$$R = \log_{10}\left(\frac{a}{T}\right) + B$$

where a is the amplitude of the ground motion, T is the period of the seismic wave, and B is a term that allows for the weakening of the seismic wave with increasing distance from the epicentre of the earthquake.

Example 39

Assume that, for a particular earthquake, we have $a = 10$, $T = 1$ and $B = 6.8$. Find the earthquake's magnitude on the Richter scale.

Solution

$$R = \log_{10}\left(\frac{10}{1}\right) + 6.8 = 7.8$$

Example 40

Early in the twentieth century an earthquake in San Francisco registered 8.3 on the Richter scale. In the same year, another earthquake was recorded in South America that was four times stronger. We will take this to mean that the value of $\frac{a}{T}$ for South America is four times that for San Francisco. What was the magnitude of the earthquake in South America? Assume both were measured at the same distance from the epicentre, and so the constant B is the same for both.

Solution

For San Francisco:

$$8.3 = \log_{10}\left(\frac{a_1}{T_1}\right) + B \qquad (1)$$

Hence, for South America:

$$\begin{aligned} R &= \log_{10}\left(\frac{a_2}{T_2}\right) + B \\ &= \log_{10}\left(\frac{4a_1}{T_1}\right) + B \\ &= \log_{10} 4 + \log_{10}\left(\frac{a_1}{T_1}\right) + B \\ &= \log_{10} 4 + 8.3 \qquad \text{using equation (1)} \\ &\approx 8.9 \end{aligned}$$

The magnitude was 8.9.

Note: Although the earthquake in South America was four times stronger, the magnitude on the Richter scale only increased by $\log_{10} 4 \approx 0.6$.

ISBN 978-1-108-45164-2

The pH scale

The pH scale for measuring the acidity of a solution is logarithmic. The pH of a solution is determined by the concentration of hydronium ions, $[H_3O^+]$, in the solution. (Concentration is measured in moles per litre.) The definition is

$$pH = \log_{10}\left(\frac{1}{[H_3O^+]}\right) = -\log_{10}([H_3O^+])$$

Vinegar has pH around 3, and bananas have pH in the interval $[4.5, 4.7]$. Strong hydrochloric acid solutions are approximately pH 0 and strong sodium hydroxide solutions are approximately pH 14.

Example 41

The pH of blood normally lies in the interval $[7.37, 7.44]$. Find the range for the concentration of hydronium ions.

Solution

For pH 7.37, we have

$$\begin{aligned} -\log_{10}([H_3O^+]) &= 7.37 \\ \log_{10}([H_3O^+]) &= -7.37 \\ [H_3O^+] &= 10^{-7.37} \text{ moles per litre} \end{aligned}$$

Now, for pH 7.44, we have

$$\begin{aligned} \log_{10}([H_3O^+]) &= -7.44 \\ [H_3O^+] &= 10^{-7.44} \text{ moles per litre} \end{aligned}$$

So the concentration of hydronium ions lies in the interval

$$[10^{-7.44}, 10^{-7.37}]$$

We can write this interval as

$$[3.63 \times 10^{-8}, 4.27 \times 10^{-8}]$$

correct to three significant figures.

▶ Modelling with logarithmic functions

The logarithm function with base a is the 'reverse' of the exponential function with base a:

$$y = a^x \quad \text{is equivalent to} \quad x = \log_a y$$

This enables us to model using a logarithmic function instead of an exponential function when we are interested in the 'other' variable. For example, we showed in Example 32 that the exponential model $P = Ae^{kt}$ can be rewritten as

$$t = \frac{1}{k}\ln\left(\frac{P}{A}\right)$$

That is, we can express t in terms of P, instead of expressing P in terms of t.

Cambridge Senior Maths for Queensland Mathematical Methods 3&4 ISBN 978-1-108-45164-2

Example 42

A mug of boiling water is placed into a refrigerator. The time, t minutes, at which the temperature of the water is T°C can be modelled by the formula

$$t = c(\ln T_0 - \ln T) \quad \text{for } 0 < T \le T_0$$

where c and T_0 are positive constants. The temperature of the water is initially 100°C and drops to 40°C in 5 minutes.

a Find the values of c and T_0.

b Find the time at which the temperature of the water is 30°C.

Solution

a When $t = 0$, $T = 100$.
When $t = 5$, $T = 40$.
This gives the two equations:

$$0 = c(\ln T_0 - \ln 100) \qquad (1)$$

$$5 = c(\ln T_0 - \ln 40) \qquad (2)$$

From equation (1):

$$\ln T_0 = \ln 100$$

$$\therefore \quad T_0 = 100$$

From equation (2):

$$5 = c(\ln 100 - \ln 40)$$

$$5 = c\ln\left(\frac{5}{2}\right)$$

$$\therefore \quad c = \frac{5}{\ln(\frac{5}{2})}$$

Hence the formula is

$$t = \frac{5}{\ln(\frac{5}{2})}(\ln 100 - \ln T)$$

b When $T = 30$, we have

$$t = \frac{5}{\ln(\frac{5}{2})}(\ln 100 - \ln 30)$$

$$= \frac{5}{\ln(\frac{5}{2})} \times \ln\left(\frac{10}{3}\right)$$

$$\approx 6.57$$

The temperature of the water is 30°C after approximately 6.57 minutes.

Note: The logarithmic model in this example corresponds to an exponential model of the form $T = T_0e^{-kt}$. In many ways, the exponential model is easier to work with.

Section summary

- **Linear scale** The change between two values is determined by the difference between the values. That is, a change from 1 to 2 is the same as a change from 6 to 7. We use linear scales for temperature and length.
- **Logarithmic scale** The change between two values is determined by the ratio of the values. That is, a change from 1 to 2 (ratio of 1 : 2) is the same as a change from 6 to 12 (also a ratio of 1 : 2). We use logarithmic scales for noise and acidity.

ISBN 978-1-108-45164-2

- Consider the exponential function $y = b \times a^{mx}$, where a, b and m are positive real numbers. Taking logarithms base a of both sides gives

  $$\log_a y = \log_a b + mx$$

  So the graph of $\log_a y$ against x is a straight line with gradient m and with $(\log_a y)$-axis intercept at $(0, \log_a b)$.

Exercise 6I

Example 37

1 Let $y = 3 \times 4^{2x}$, for $x \geq 0$.

a If $\log_4 y = mx + c$, give the values of m and c.

b Sketch the graph of $\log_4 y$ against x.

c Sketch the graph of $\log_4 y$ against x labelling your vertical axis with powers of 4.

2 Let $y = 2 \times 5^{3x}$, for $x \geq 0$.

a If $\log_5 y = mx + c$, give the values of m and c.

b Sketch the graph of $\log_5 y$ against x.

c Sketch the graph of $\log_5 y$ against x labelling your vertical axis with powers of 5.

Example 38

3 A busy street generates noise of 70 dB and a quiet car generates noise of 50 dB. Find the power in watt/cm^2 for each of these.

4 Use the formula $dB = 10 \log_{10}(P \times 10^{16})$ to answer the following:

a If P is increased by a factor of 2, what is the effect on dB?

b If P is increased by a factor of 10, what is the effect on dB?

c If dB is increased by a factor of 3, what is the effect on P?

d For what value of P is $dB = 0$?

e For what value of P is $dB = 100$?

5 If $dB_1 - dB_2 = \lambda$, find P_1 in terms of P_2.

Example 39

6 Find the magnitude on the Richter scale of an earthquake with $a = 10$, $T = 2$ and $B = 5$.

Example 40

7 An earthquake in Turkey registered 7.3 on the Richter scale. In the same year, an earthquake in Greece had a quarter of this strength. We will take this to mean that the value of $\frac{a}{T}$ for Greece is one-quarter that for Turkey. What was the magnitude of the earthquake in Greece? Assume both were measured at the same distance from the epicentre, and so the constant B is the same for both.

Example 41

8 The pH of a soft drink normally lies in the interval $[2.0, 4.0]$. Find the range for the concentration of hydronium ions.

CF

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CF

9 Pure water has pH 7.

a What is the concentration of hydronium ions in water?

b A solution is made by diluting 1 litre of an acid of pH 2 with 1 litre of water. What is the pH of the solution?

c A solution is made by diluting 100 mL of an acid of pH 1.3 with water. If the solution has pH 2.5, how much water was added?

d A solution is made by mixing 500 mL of an acid of pH 2.5 with 400 mL of an acid of pH 3. What is the pH of the solution?

10 The energy, E joules, released by an earthquake of magnitude R is given by the formula

$$\log_{10} E = 4.4 + 1.5R$$

a Find R in terms of E.

b Find the amount of energy released by an earthquake of magnitude:

i 5 **ii** 6 **iii** 7 **iv** 8 **v** 8.6

c What is the magnitude of an earthquake that releases 10^{13} joules of energy?

11 A siren generates a noise of 100 dB. If you have two sirens, the power is doubled. What is the decibel level of the noise of two sirens?

12 The cost, \$$C$, of manufacturing x units of a particular product is given by

$$C = 1.5x \log_{10} x + 2000 \quad \text{for } 1 \le x \le 3070$$

a Find the cost when:

i $x = 500$ **ii** $x = 1000$ **iii** $x = 2000$ **iv** $x = 3000$

b Write down a formula that gives the cost per unit in terms of x.

c Find the cost per unit when:

i $x = 500$ **ii** $x = 1000$ **iii** $x = 2000$ **iv** $x = 3000$

Example 42 **13** A cup of hot milk is placed into a refrigerator. The time, t minutes, at which the temperature of the milk is T°C can be modelled by

$$t = c(\ln T_0 - \ln T) \quad \text{for } 0 < T \le T_0$$

where c and T_0 are positive constants. The temperature of the milk is initially 80°C and drops to 50°C in 4 minutes.

a Find the values of c and T_0.

b Find the time at which the temperature is 40°C.

c Find the time at which the temperature is 30°C.

ISBN 978-1-108-45164-2

14 Suppose that the graph of $\ln y$ against x is a straight line that crosses the x-axis at 3 and crosses the $(\ln y)$-axis at 2. Find an equation connecting y and x of the form: CF

a $y = pe^{kx}$, where p and k are constants

b $y = pq^x$, where p and q are constants.

15 When carbon dioxide is absorbed into the bloodstream, it lowers the pH. We can model the pH of the bloodstream using the equation CU

$$pH = 6.1 + \log_{10}\left(\frac{800}{x}\right)$$

where x is the partial pressure of carbon dioxide in the bloodstream (measured in torr).

a Find the pH if the partial pressure of carbon dioxide is 50.

b Find the partial pressure of carbon dioxide if the pH is 7.2.

16 A contagious disease spreads through a flock of 2000 sheep. The time, t days, that it takes for n sheep to become infected is modelled by

$$t = 4\ln\left(\frac{1999n}{2000 - n}\right)$$

a Find the number of days that it takes for the disease to spread to:

i 50 sheep **ii** 100 sheep **iii** 1000 sheep **iv** 1500 sheep

b Express n in terms of t.

17 A group of students are given a list of unfamiliar words to memorise. Suppose that the percentage of words that a student remembers, $P\%$, depends on the amount of time spent studying the words, t minutes, according to the rule

$$P = 16\ln t + 30 \quad \text{for } 1 \le t \le 60$$

a Sketch the graph of P against t.

b What is the percentage of words recalled when the study time is 20 minutes?

c How much study time would a student need in order to recall 80% of the words?

d Find t in terms of P.

18 Newton's law of cooling Let T_0 be the initial temperature of an object and let T_s be the temperature of the environment surrounding the object (where T_s is assumed to be constant). The temperature of the object at time t is given by

$$T = T_s + (T_0 - T_s)e^{-kt}$$

where k is a constant. Make t the subject of this formula, assuming that $T_0 \ge T > T_s$.

ISBN 978-1-108-45164-2

6J Modelling data using a graphics calculator

Scientists, economists and people in many other professions need to interpret data. This is often done by displaying the data in tabular or graphical form, and then finding a rule that 'fits' the data. In this section, we look at fitting sets of data using a graphics calculator.

The investigation of how well a rule fits the data requires statistical analysis, which is not part of this course. The examples in this section are therefore confined to finding a suitable rule. A rule that fits the data is called a **mathematical model**. The process of fitting a model to data is called **regression**.

▶ Regression using a graphics calculator

The following example shows how to attempt linear regression by hand.

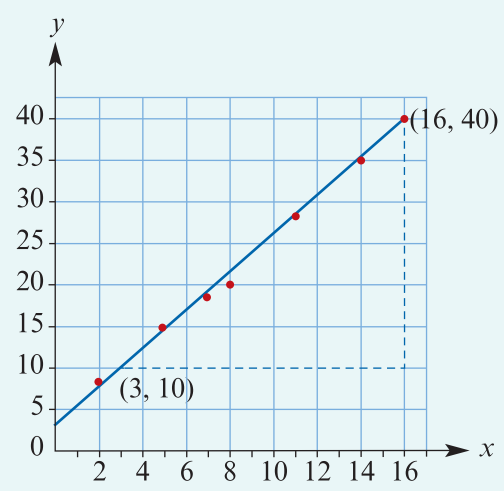
Example 43

The table gives observed values of variables x and y, which are claimed to be related by a linear equation $y = ax + b$. Without using technology, find possible values for a and b.

x	2	5	7	8	11	14	16
y	8	15	18	20	28	35	40

Solution

Plot the data points and draw a 'line of best fit'. Use two points on the line to find its equation.



Here we obtain

$$y - 10 = \frac{30}{13}(x - 3)$$

So the rule $y = 2.31x + 3.08$ provides a model for the experimental data.

Note: The **residual** for a given x-value is the difference between the y-values from the data and the model.
A table of residuals can give an indication as to how well the model fits the data.

x	2	5	7	8	11	14	16
y (data)	8	15	18	20	28	35	40
y (model)	7.70	14.63	19.25	21.56	28.49	35.42	40.04
Residual	0.30	0.37	−1.25	−1.56	−0.49	−0.42	−0.04

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The next example shows how to use a graphics calculator to fit a linear model to data. The same steps can be followed to fit other models to data as appropriate (including quadratic, sinusoidal, exponential and logarithmic models).

Note: When you use your graphics calculator to fit a linear model to data, it will also give the value of r^2, called the **coefficient of determination**. This is a statistical measure of how well the model fits the data. If $r^2 = 1$, then the model fits the data perfectly. The closer that r^2 is to 1, the better the fit.

Example 44

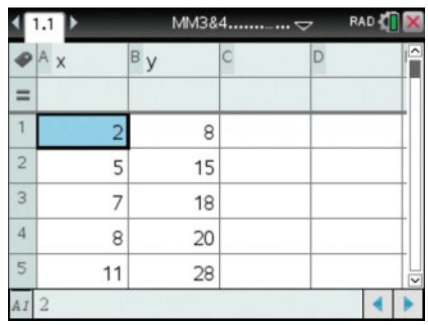
The table gives observed values of variables x and y, which are claimed to be related by a linear equation $y = ax + b$. Using technology, find possible values for a and b.

x	2	5	7	8	11	14	16
y	8	15	18	20	28	35	40

Using the TI-Nspire CX non-CAS

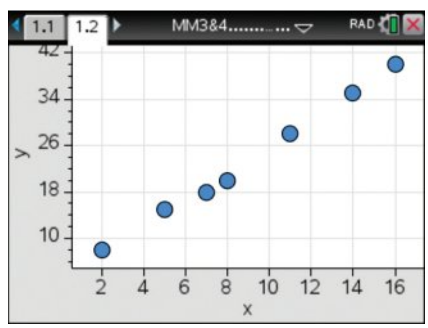
Step 1: Enter the data

- Insert a **Lists & Spreadsheet** page.
- Enter the data in lists named x and y as shown.



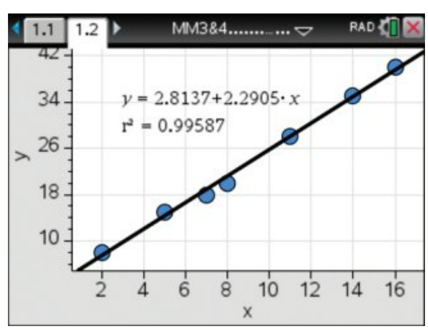
Step 2: Plot the data

- Insert a **Data & Statistics** page.
- Click on the 'Click to add variable' box on the x-axis and select x.
- Repeat for the y-axis and select y.



Step 3: Fit a model to the data

- Determine the linear regression model using menu > **Analyze** > **Regression** > **Show Linear (a+bx)**. The plot and the superimposed regression line are displayed as shown.
- The equation of the regression line is $y = 2.29x + 2.81$.



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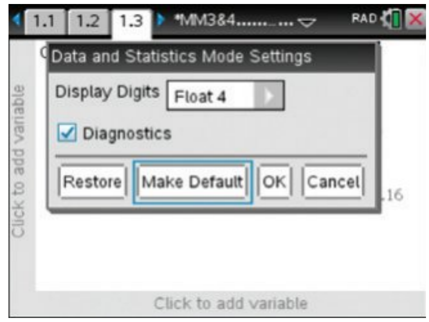
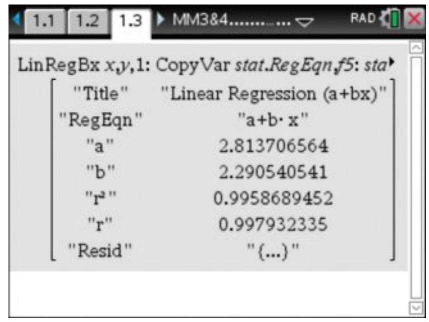
Statistical information

- If the value of r^2 is not showing on the graph, then you can turn on **Diagnostics** as follows:
  - In a **Data & Statistics** page, go to (menu) > **Settings** > **Settings**.
  - Tick the option **Diagnostics** and select **Make Default**.



- A full display of the statistical information can be shown on a **Calculator** page using (menu) > **Statistics** > **Stat Calculations** > **Linear Regression (a+bx)**.



Other regression models

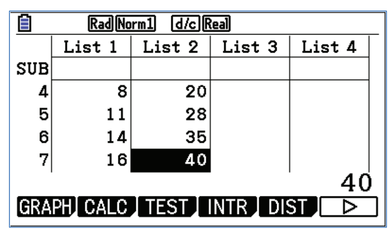
The same method can be used to fit different types of curves to data. In Step 3, choose a different option from (menu) > **Analyze** > **Regression**. Some of the options include:

- **Show Quadratic**
- **Show Exponential**
- **Show Logarithmic**

Using the Casio

Step 1: Enter the data

- Press (MENU) (2) to select **Statistics** mode.
- Enter the x-values in List1 and the y-values in List2 as shown.

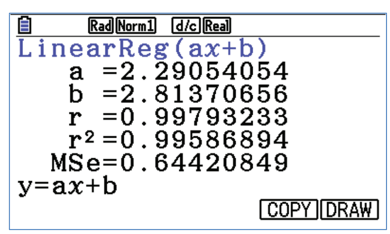


Step 2: Plot the data

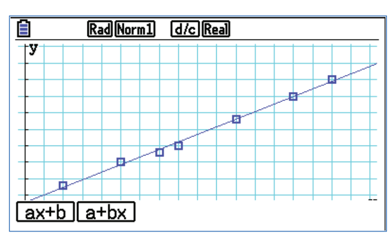
- Select **Graph** (F1), then **Graph1** (F1).

Step 3: Fit a model to the data

- To fit a line $y = ax + b$ to the data, go to **Calculation** (F1), **X** (F2), then **ax+b** (F1).
- The equation of the regression line is $y = 2.29x + 2.81$.



- Select **Draw** (F6) to view the graph of the data with the superimposed regression line.



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Other regression models

The same method can be used to fit different types of curves to data. In Step 3, choose a different option from the **Calculation** menu. Some options are given in the table below.

Model	Form	Option
Linear	$y = ax + b$	**X** (F2), **ax+b** (F1)
Quadratic	$y = ax^2 + bx + c$	**X²** (F4)
Exponential	$y = ab^x$	**Exp** (F6) (F3), **ab^x** (F2)
Exponential	$y = ae^{bx}$	**Exp** (F6) (F3), **ae^bx** (F1)
Logarithmic	$y = a + b \ln x$	**Log** (F6) (F2)

► Changing to 'linear form'

Sometimes we can understand a set of data better by 'transforming' the variables. This is illustrated by the next example.

Example 45

An experiment was conducted to find the relationship between two variables x and y. The results are shown in the table.

x	0	1	2	3	4	5
y	1	3	8	19	33	51

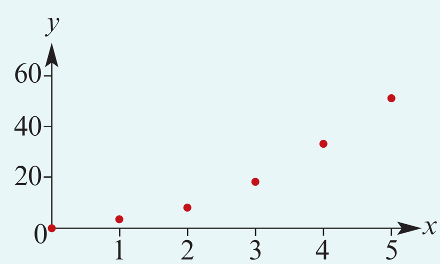
a Plot the graph of y against x.

b Plot the graph of y against x^2.

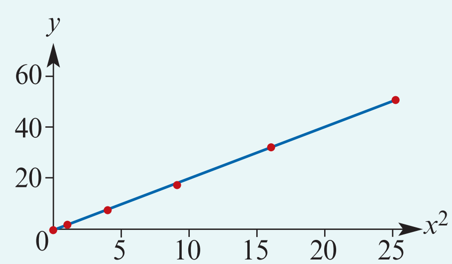
c Fit a rule of the form $y = ax^2 + b$.

Solution

a Graph of y against x:



b Graph of y against x^2:



c The graph of y against x^2 is 'approximately linear'. This suggests that the data can be modelled by a rule of the form $y = ax^2 + b$.

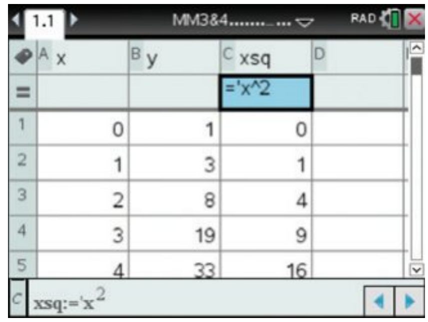
By fitting a straight line to the graph in part **b**, we obtain the rule

$$y = 2x^2 + 0.7$$

ISBN 978-1-108-45164-2

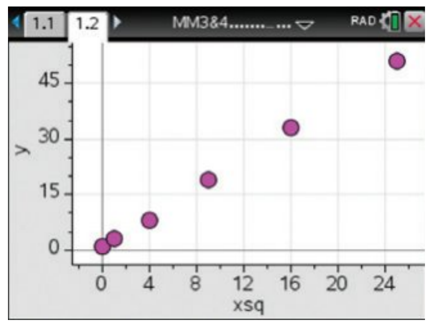
Using the TI-Nspire CX non-CAS

- Insert a **Lists & Spreadsheet** page.
- Enter the data in lists named x and y.
- Label the third list as xsq (for x^2).
- To generate the third list, place the cursor in the formula line as shown and enter the formula = x^2. To obtain x in this formula, press (var) and select x from the dropdown list.
- Press (enter) to populate the third list.

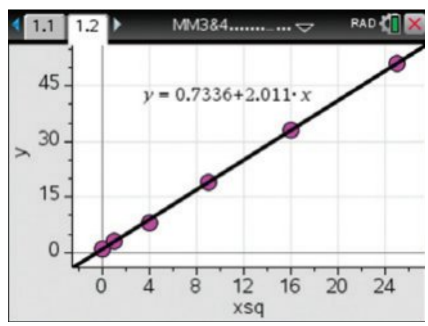


Note: If you obtain x in the formula by using (X) instead of (var), you will be prompted as to whether you mean column X or variable x. Select variable x.

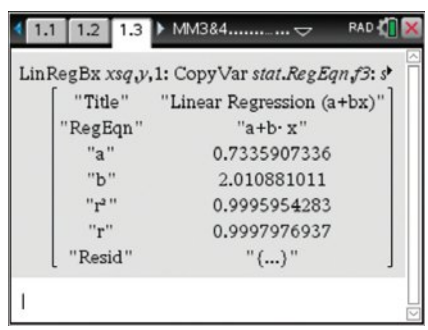
- Insert a **Data & Statistics** page.
- Click on the 'Click to add variable' box on the x-axis and select xsq.
- Repeat for the y-axis and select y.



- Determine the linear regression model using (menu) > **Analyze** > **Regression** > **Show Linear (a+bx)**. The plot and the superimposed regression line are displayed as shown.
- Hence the data is modelled by the rule $y = 2.011x^2 + 0.734$.



- A full display of the statistical information can be shown on a **Calculator** page using (menu) > **Statistics** > **Stat Calculations** > **Linear Regression (a+bx)**.



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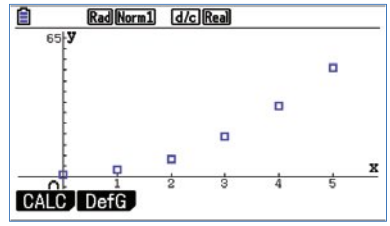
Using the Casio

a To graph y against x:

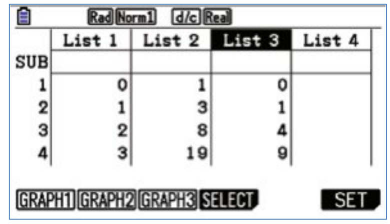
- Press (MENU) (2) to select **Statistics** mode.
- Enter the x-values in List1 and the y-values in List2.
- Select **Graph** (F1), **Graph1** (F1).



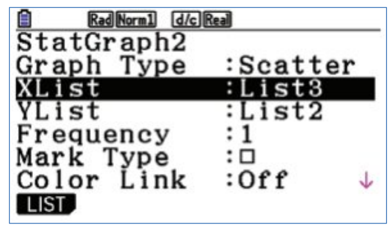
b To graph y against x^2:

- Move the cursor to the top of List3.
- To generate List3, enter the formula List1^2 as follows:

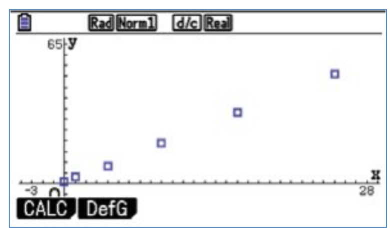
  (SHIFT) (1) (1) (x^2) (EXE)



- Change the settings for Graph2 so that it is the graph of List2 (y) against List3 (x^2) as follows:
  - Select **Graph** (F1), **Set** (F6), **Graph2** (F2).
  - Move the cursor down to **XList**.
  - Change the setting to List3: (F1) (3) (EXE)
  - Press (EXIT).

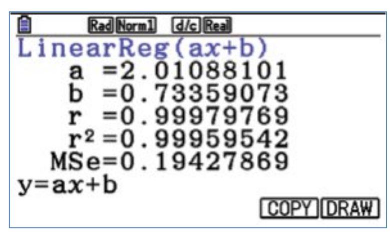


- Select **Graph2** (F2) to view the graph.

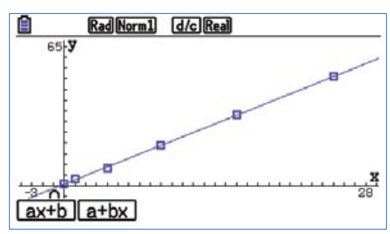


c To determine the rule $y = ax^2 + b$:

- Select linear regression using **Calculation** (F1), **X** (F2), **ax+b** (F1).
- Hence the data is modelled by the rule $y = 2.011x^2 + 0.734$.



- Select **Draw** (F6) to view the graph.



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Using logarithmic scales

In Section 6I, we saw how to convert an exponential graph into a straight-line graph by using a logarithmic scale. In the next two examples, we apply similar ideas to modelling data. We consider the following two transformations.

- **Exponential model**
  Let $y = a \times e^{mx}$, where a and m are constants with $a > 0$. By taking logarithms base e of both sides, we obtain $\ln y = \ln a + mx$. The graph of $\ln y$ against x is linear.
- **Power model**
  Let $y = a \times x^m$ for $x > 0$, where a and m are constants with $a > 0$. By taking logarithms base e of both sides, we obtain $\ln y = \ln a + m \ln x$. The graph of $\ln y$ against $\ln x$ is linear.

Example 46

The data in the table gives the concentration of a gas, C, at time t seconds during thermal decomposition. (Concentration is measured in moles per millilitre.)

t	0	20	40	60	80
C	22.90	12.29	6.89	3.68	2.16

Given that a rule $C = ae^{-bt}$ can be used to model the data:

a Find a corresponding linear rule.

b Draw a linear graph from the data.

c Estimate the values of a and b.

Solution

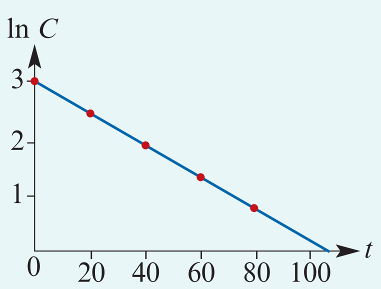
a Take logarithms base e of both sides of the rule $C = ae^{-bt}$:

$$\ln C = \ln(ae^{-bt})$$
$$\ln C = \ln a + \ln(e^{-bt})$$
$$\therefore \quad \ln C = \ln a - bt$$

A table of values for $\ln C$ against t is shown below.

t	0	20	40	60	80
$\ln C$	3.13	2.51	1.93	1.30	0.77

b Graph of $\ln C$ against t:



c By fitting a straight line to the graph in part **b**, we obtain

$$\ln C = 3.114 - 0.03t$$
$$\therefore \quad C = e^{3.114-0.03t}$$
$$= e^{3.114}e^{-0.03t}$$
$$\approx 22.5e^{-0.03t}$$

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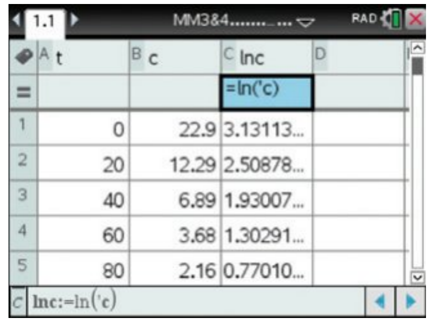
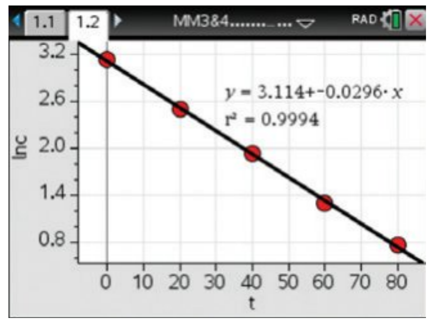
TI-84

Using the TI-Nspire CX non-CAS

- Insert a **Lists & Spreadsheet** page and enter the data in lists named t and c.
- Label the third list as *lnc* (for $\ln c$).
- To generate the third list, place the cursor in the formula line as shown and enter the formula = ln(c). To obtain c in this formula, press (var) and select c.
- Press (enter) to populate the third list.



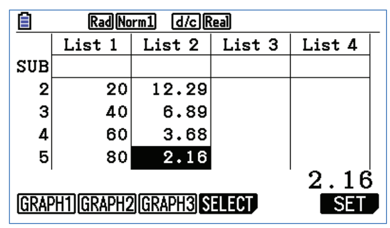
- Insert a **Data & Statistics** page. Add t on the x-axis and add *lnc* on the y-axis.
- Determine the linear regression model using (menu) > **Analyze** > **Regression** > **Show Linear (a+bx)**.
- The rule is $\ln C = 3.114 - 0.03t$, which gives $C = 22.5e^{-0.03t}$.

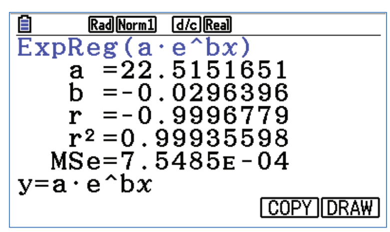


Using the Casio

Here we use exponential regression directly.

- Press (MENU) (2) to select **Statistics** mode.
- Enter the t-values in List1 and the C-values in List2 as shown.
- To plot the data, select **Graph** (F1), **Graph1** (F1).
- To fit a curve $y = ae^{bx}$ to the data, use **Calculation** (F1), **Exp** (F6) (F3), **ae^{bx}** (F1).
- The rule is $C = 22.5e^{-0.03t}$.





Note: Using exponential regression ($y = ab^x$) for Example 46 would give the rule $C = 22.5 \times 0.971^t$, which is not in the required form. To change to base e, we write

$$0.971 = e^k$$
$$\ln 0.971 = k$$
$$\therefore \quad k \approx -0.03$$

The rule becomes $C = 22.5e^{-0.03t}$.

 ISBN 978-1-108-45164-2

Example 47

An experiment involved estimating the number of microbes, N, in a culture after t days. The results are shown in the table.

t	3	5	10	15	20
N	1350	3000	7500	13 500	24 000

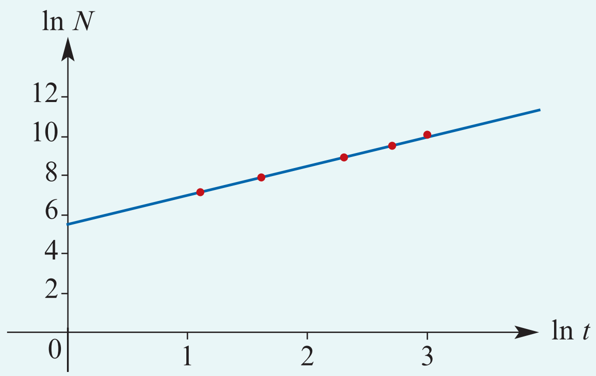
a Using technology, plot the graph of $\ln N$ against $\ln t$.

b Find estimates (two decimal places) for constants a and b such that $\ln N = a \ln t + b$.

c Hence find estimates for constants A and B such that $N = At^B$.

Solution

a



b By fitting a straight line to the graph in part **a**, we obtain

$$\ln N = 1.47 \ln t + 5.59$$

c Rearrange the rule from part **b**:

$$\ln N - 1.47 \ln t = 5.59$$

$$\ln\left(\frac{N}{t^{1.47}}\right) = 5.59$$

$$\frac{N}{t^{1.47}} = e^{5.59}$$

$$N = e^{5.59} \times t^{1.47}$$

$$\therefore \quad N = 267.7 \times t^{1.47}$$

Exercise 6J

Example 44

1 The table gives observed values of variables P and Q, which are claimed to be related by a linear equation $P = aQ + b$. Using technology, find possible values for a and b.

Q	2.5	3.5	4.4	5.8	7.5	9.6	12.0	15.1
P	13.6	17.6	22.2	28.8	35.5	47.4	56.1	74.6

SF

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SF

2 Find a quadratic model $y = ax^2 + bx + c$ for the following data set.

x	0	1	2	3	4	5
y	6	8.2	10.8	13.5	16.5	20

3 Find an exponential model $y = a \times b^x$ for each data set:

a

x	3.1	4.6	12.7	13.8	19.6
y	2.0771	2.9028	17.693	22.615	82.503

b

x	1	2	3	4	5
y	1.155	1.213	1.273	1.337	1.404

4 Find a logarithmic model $y = a + b \ln x$ for each data set:

a

x	1.5	6.5	11.5	16.5	21.5
y	3.41	4.87	5.44	5.80	6.07

b

x	2	4.5	7	9.5	12
y	11.12	18.02	21.77	24.37	26.35

c

x	3	8.5	14	19.5	25
y	16.7	26.6	31.3	34.5	36.8

Example 45

CF

5 The ABC Cable Company will manufacture cable of any given diameter. The price per metre for several different diameters is given in the following table.

Diameter, d mm	5	10	15	20	25
Price, \$$P$ per metre	10	22	40	62	95

a Plot the graph of P against d^2.

b Hence find a rule for P in terms of d.

SF

6 Find a sinusoidal model $y = a\sin(bx + c) + d$ for each data set:

a

x	0	0.2	0.4	0.6	0.8
y	0	1.77	2.85	2.85	1.77

b

x	0	0.2	0.4	0.6	0.8
y	5	2.18	0.34	0.13	1.62

Example 46

CF

7 The number of bacteria in a culture at time t days is given by $N = ae^{kt}$. Find the values of a and k using the following data.

t	1	2	3	4	5	6
N	1065	1515	2145	3045	4320	6120

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Example 47 **8** An experiment involved estimating the number of microbes, N, in a culture after t days. The results are shown in the table. CF

t	3	5	10	15	20
N	140	630	5000	16 900	40 100

a Using technology, plot the graph of $\ln N$ against $\ln t$.

b Find estimates (two decimal places) for constants a and b such that $\ln N = a \ln t + b$.

c Hence find estimates for constants A and B such that $N = Bt^A$.

9 A veterinarian is working for a politician who owns a large piggery. The veterinarian is interested in increasing the weight of the politician's pigs. She increases the amount of a particular chemical in the pigs' diet by feeding them tablets. Different groups of pigs are given different numbers of tablets. After a month, the veterinarian produces the following table of results, where N is the number of tablets per day and P% is the average percentage weight gain over a month.

N	0	1	2	3	4	5	6	7
P	2.25	12.75	20.25	24.75	26.25	24.75	20.25	12.75

a Using a graphics calculator, plot the graph of P against N.

b The data can be modelled by a rule of the form $P = aN^2 + bN + c$. Find values for a, b and c.

10 In 1618, the German astronomer Johannes Kepler discovered a relationship between the mean distance from the Sun to a planet and the period of the orbit. (As the measure of mean distance, we use the length in astronomical units of the semi-major axis of the approximately elliptical orbit.) CU

Planet	Mean distance (D AU)	Period of orbit (T days)
Mercury	0.387 10	87.9693
Venus	0.723 33	224.7008
Earth	1	365.2564
Mars	1.523 66	686.9796
Jupiter	5.303 36	4332.8201
Saturn	9.537 07	10 775.599
Uranus	19.1913	30 687.153
Neptune	30.0690	60 190.03

a Using the data in the table, plot the graph of $\ln T$ against $\ln D$.

b Find a rule connecting $\ln T$ and $\ln D$.

c Transform this rule into an equation of the form $T = a \times D^b$.

d Square both sides of this equation to discover Kepler's result.

 ISBN 978-1-108-45164-2

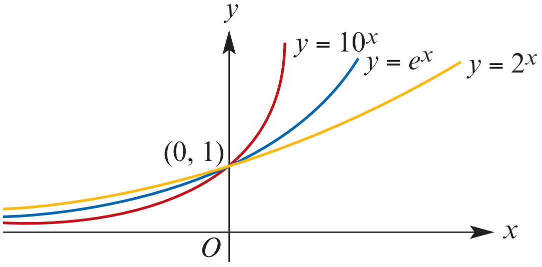
Review

Chapter summary

AS

Nrich

- Sketch graphs of the form $y = a^x$ and transformations of these graphs.



- **Index laws**

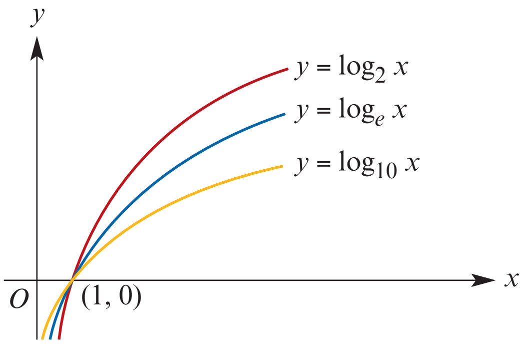
$$a^m \times a^n = a^{m+n} \qquad a^m \div a^n = a^{m-n} \qquad (a^m)^n = a^{mn}$$

- **Logarithms**

For $a \in \mathbb{R}^+ \setminus \{1\}$, the logarithm function with base a is defined as follows:

$$a^x = y \quad \text{is equivalent to} \quad \log_a y = x$$

- Sketch graphs of the form $y = \log_a x$ and transformations of these graphs.



- **The natural logarithm**

The function $y = \log_e x$ is commonly written as $y = \ln x$.

- **Logarithm laws**

$$\log_a(mn) = \log_a m + \log_a n \qquad \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$\log_a\left(\frac{1}{n}\right) = -\log_a n \qquad \log_a(m^p) = p\log_a m$$

- **Change of base**

$$\log_a x = \frac{\log_b x}{\log_b a} \quad \text{and} \quad a^x = b^{(\log_b a)x}$$

- **Inverse relationship** The functions $f(x) = a^x$, for $x \in \mathbb{R}$, and $g(x) = \log_a x$, for $x \in \mathbb{R}^+$, are inverse to each other in the following way:
  - $\log_a(a^x) = x$ for all $x \in \mathbb{R}$
  - $a^{\log_a x} = x$ for all $x \in \mathbb{R}^+$

- **Law of exponential change** If the rate at which the quantity A increases or decreases is proportional to its current value, then the value of A at time t is given by

$$A = A_0 e^{kt}$$

where A_0 is the initial quantity and k is a constant (called the **relative growth rate**). Growth corresponds to $k > 0$, and decay corresponds to $k < 0$.

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Review

Technology-free questions

SF

1 Sketch the graph of each of the following. Label asymptotes and axis intercepts.

a $f(x) = e^x - 2$ **b** $g(x) = 10^{-x} + 1$ **c** $h(x) = \frac{1}{2}(e^x - 1)$

d $f(x) = 2 - e^{-x}$ **e** $f(x) = \ln(2x + 1)$ **f** $h(x) = \ln(x - 1) + 1$

g $g(x) = -\ln(x - 1)$ **h** $f(x) = -\ln(1 - x)$

2 For each of the following, find y in terms of x:

a $\ln y = \ln x + 2$ **b** $\log_{10} y = \log_{10} x + 1$ **c** $\log_2 y = 3\log_2 x + 4$

d $\log_{10} y = -1 + 5\log_{10} x$ **e** $\ln y = 3 - \ln x$ **f** $\ln y = 2x - 3$

3 Solve each of the following equations for x, expressing your answers in terms of logarithms with base e:

a $3^x = 11$ **b** $2^x = 0.8$ **c** $2^x = 3^{x+1}$

CF

4 Solve each of the following for x:

a $2^{2x} - 2^x - 2 = 0$ **b** $\ln(3x - 1) = 0$

c $\log_{10}(2x) + 1 = 0$ **d** $10^{2x} - 7 \times 10^x + 12 = 0$

SF

5 The graph of the function with rule $y = 3\log_2(x + 1) + 2$ intersects the axes at the points $(a, 0)$ and $(0, b)$. Find the exact values of a and b.

6 The graph of $y = 5\log_{10}(x + 1)$ passes through the point $(k, 6)$. Find the value of k.

7 Find the exact value of x for which $4e^{3x} = 287$.

8 Find x in terms of a, where $3\log_a x = 3 + \log_a 8$.

9 Given that $y = \log_3(x - 4)$, express x as a function of y.

CF

10 The graph of the function with rule $f(x) = e^{2x} - 3ke^x + 5$ intersects the axes at $(0, 0)$ and $(a, 0)$ and has a horizontal asymptote at $y = b$. Find the exact values of a, b and k.

SF

11 Given that $y = e^{3x} - 4$, express x as a function of y.

CU

12 Show that, if $3^x = 4^y = 12^z$, then $z = \dfrac{xy}{x + y}$.

CF

13 Evaluate $2\log_2 12 + 3\log_2 5 - \log_2 15 - \log_2 150$.

14 **a** Given that $\log_p 7 + \log_p k = 0$, find k.

b Given that $4\log_q 3 + 2\log_q 2 - \log_q 144 = 2$, find q.

SF

15 Given that $\ln y = a + b\ln x$, where a and b are constants, find y in terms of x.

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16 Let $f(x) = e^x + e^{-x}$ and $g(x) = e^x - e^{-x}$.

a Show that f is an even function.

b Find $f(u) + f(-u)$.

c Find $f(u) - f(-u)$.

d Find $[f(u)]^2 - 2$.

e Show that g is an odd function.

f Find $f(x) + g(x)$, $f(x) - g(x)$ and $f(x) \cdot g(x)$.

Multiple-choice questions

1 If $4\log_b(x^2) = \log_b 16 + 8$, then x is equal to

A b^4 **B** ± 6 **C** $\pm\sqrt{2}\,b$ **D** 2^6 **E** $2b^4$

2 The expression $\ln(4e^{3x})$ is equal to

A $\ln(e^{12x})$ **B** $\ln(12) + x$ **C** $3x\ln(4)$ **D** $\ln(4) + 3x$ **E** $12x$

3 The expression $3^{\log_3(x-4)}$ is equal to

A $\dfrac{x}{4}$ **B** $x - 4$ **C** $3(x-4)$ **D** $3^x - 3^4$ **E** $\log_3 x - \log_3 4$

4 If $x = 5$ is a solution of the equation $\log_{10}(kx - 3) = 2$, then the exact value of k is

A $\dfrac{103}{5}$ **B** $\dfrac{\log_{10} 2 + 3}{5}$ **C** 2 **D** 5 **E** 21

5 $3^{4\log_3(x)+\log_3(4x)}$ is equal to

A $8x$ **B** $x^4 + 4x$ **C** $4x^5$ **D** 3^{8x} **E** $\log_3(4x^5)$

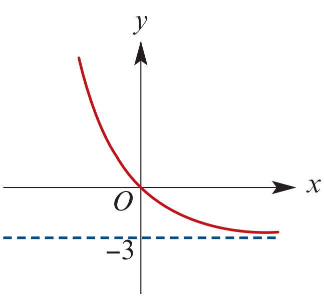
6 The solution of the equation $3x = 10^{-0.3x}$ is closest to

A 0.83 **B** 0.28 **C** 0 **D** 0.30 **E** 0.91

7 The graph of the function with rule $y = ae^{-x} + b$ is shown on the right.
The values of a and b respectively are

A $3,\ -3$ **B** $-3,\ 3$ **C** $-3,\ -3$

D $0,\ -3$ **E** $-3,\ 0$



8 Which one of the following statements is not true about the graph of the function defined by $f(x) = \log_5 x$ for $x > 0$?

A The domain is $\mathbb{R}^+$.

B The range is $\mathbb{R}$.

C It passes through the point $(5, 0)$.

D It has a vertical asymptote with equation $x = 0$.

E The slope of the tangent at any point on the graph is positive.

ISBN 978-1-108-45164-2

Review

9 If $3\log_2 x - 7\log_2(x-1) = 2 + \log_2 y$, then y is equal to

A $\dfrac{3x}{28(x-1)}$ **B** $\dfrac{1}{4x^4}$ **C** $3 - 4x$

D $\dfrac{x^3}{4(x-1)^7}$ **E** $x^3 - (x-1)^7 - 4$

10 The graph of the function $f(x) = e^{2x} - 12$ intersects the graph of $g(x) = -e^x$ where

A $x = \ln 3$ **B** $x = \ln 2$ **C** $x = \ln 7$ **D** $x = \ln 4$ **E** $x = \ln 5$

11 Let the rule for a function g be $g(x) = \ln((x-4)^2)$. For the function g, the maximal domain and range are

A $\mathbb{R}$, $\mathbb{R}$ **B** $(4, \infty)$, $\mathbb{R}^+$ **C** $\mathbb{R} \setminus \{4\}$, $\mathbb{R}$

D $\mathbb{R} \setminus \{2\}$, $\mathbb{R} \setminus \{0\}$ **E** $(-\infty, 4)$, $\mathbb{R}$

12 The maximal domain for a function with rule $f(x) = \ln((x-3)^2) + 6$ is

A $(3, \infty)$ **B** $[3, \infty)$ **C** $\mathbb{R} \setminus \{3\}$ **D** $\mathbb{R}^+$ **E** $(\infty, 3)$

13 When rewritten with x as the subject, the equation $y = e^{3x+4}$ becomes

A $x = -3\ln(3y - 4)$ **B** $x = \dfrac{\ln(y) - 4}{3}$ **C** $x = -3\ln\left(\dfrac{y-4}{3}\right)$

D $x = \ln(3y - 4)$ **E** $x = \ln(3y) - 4$

14 If $f(x) = 2\ln(3x)$ and $f(6x) = \ln(y)$, then

A $y = 18x$ **B** $y = \dfrac{x}{3}$ **C** $y = 6x^2$ **D** $y = 324x^2$ **E** $y = 36x^2$

Extended-response questions

CF

1 A liquid cools from its original temperature of 90°C to a temperature of T°C in x minutes. Given that $T = 90(0.98)^x$, find:

a the value of T when $x = 10$ **b** the value of x when $T = 27$.

2 The barometric pressure P (in centimetres of mercury) at a height h km above sea level is given by $P = 75(10^{-0.15h})$. Find:

a P when $h = 0$ **b** P when $h = 10$ **c** h when $P = 60$.

3 The value, $\$V$, of a particular car can be modelled by the equation $V = ke^{-\lambda t}$, where t years is the age of the car. The car's original price was \$22 497, and after 1 year it is valued at \$18 000.

a State the value of k and calculate λ, giving your answer to two decimal places.

b Find the value of the car when it is 3 years old.

4 A radioactive substance is decaying such that the amount, A g, at time t years is given by the formula $A = A_0 e^{kt}$. If when $t = 1$, $A = 60.7$ and when $t = 6$, $A = 5$, find the values of the constants A_0 and k.

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5 There are two species of insects living in a suburb: the *Asla bibla* and the *Cutus pius*. The number of *Asla bibla* alive at time t days after 1 January 2000 is given by

$$N_A(t) = 10\,000 + 1000t, \quad 0 \le t \le 15$$

The number of *Cutus pius* alive at time t days after 1 January 2000 is given by

$$N_C(t) = 8000 + 3 \times 2^t, \quad 0 \le t \le 15$$

a With a calculator, plot the graphs of $y = N_A(t)$ and $y = N_C(t)$ on the one screen.

b **i** Find the coordinates of the point of intersection of the two graphs.

ii At what time is $N_A(t) = N_C(t)$?

iii What is the number of each species of insect at this time?

c **i** Show that $N_A(t) = N_C(t)$ if and only if $t = 3\log_2 10 + \log_2\left(\frac{2+t}{3}\right)$.

ii Plot the graphs of $y = x$ and $y = 3\log_2 10 + \log_2\left(\frac{2+x}{3}\right)$ and find the coordinates of the point of intersection.

d It is found by observation that the model for *Cutus pius* does not quite work. It is known that the model for the population of *Asla bibla* is satisfactory. The form of the model for *Cutus pius* is $N_C(t) = 8000 + c \times 2^t$. Find the value of c, correct to two decimal places, if it is known that $N_A(15) = N_C(15)$.

6 The growth of a population of bacteria is modelled by the formula $n = A(1 - e^{-Bt})$, where n is the size of the population at time t hours, and A and B are positive constants.

a When $t = 2$, $n = 10\,000$ and when $t = 4$, $n = 15\,000$.

i Show that $2e^{-4B} - 3e^{-2B} + 1 = 0$.

ii Use the substitution $a = e^{-2B}$ to show that $2a^2 - 3a + 1 = 0$.

iii Solve this equation for a.

iv Find the exact value of B.

v Find the exact value of A.

b Sketch the graph of n against t.

c After how many hours is the population of bacteria 18 000?

7 Newton's law of cooling for an object in a medium of constant temperature states

$$T - T_s = (T_0 - T_s)\,e^{-kt}$$

where:

- T is the temperature (in °C) of the object at time t (in minutes)
- T_s is the temperature of the surrounding medium
- T_0 is the initial temperature of the object.

An egg at 96°C is placed to cool in a sink of water at 15°C. After 5 minutes the egg's temperature is 40°C. (Assume that the temperature of the water does not change.)

a Find the value of k.

b Find the temperature of the egg when $t = 10$.

c How long does it take for the egg to reach a temperature of 30°C?

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8 In a chemical reaction, the amount of time (t minutes) that it takes for x grams of a substance to have reacted is given by

$$t = 5\ln\left(\frac{8}{8-x}\right) \quad \text{for } 0 \le x < 8$$

a Find the time that it takes for the amount of reacted substance to reach:

i 2 g **ii** 4 g **iii** 6 g

b Find the amount of reacted substance after 7 minutes.

c Find x in terms of t.

9 It is conjectured that the area affected by an earthquake, A km^2, is related to the magnitude of the earthquake on the Richter scale, R, by the formula

$$R = 2.3\log_{10}(A + 4800) - 7.5 \quad \text{for } 1 \le R \le 8$$

a Make A the subject of this formula.

b Determine the area affected by an earthquake of magnitude 7.

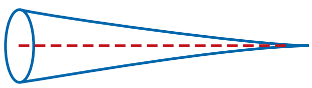
c Determine the magnitude of an earthquake that will affect twice as much area as an earthquake of magnitude 7.

d Assume that the boundary of the affected area is a circle (with centre at the epicentre of the earthquake). Find the radius of this circle when:

i $R = 5$ **ii** $R = 7$ **iii** $R = 8$

10 The process used to manufacture a certain glass fibre means that its cross-sectional area gradually decreases along its length. Let A mm^2 be the cross-sectional area at a distance of d cm from the start of the fibre.

d (cm)	0	1	3	7	15	31
A (mm^2)	3	2.75				



a Using the information in the table, find the values of constants α and β such that $A = \alpha\log_2(d + 1) + \beta$. Then use your rule for A to fill in the rest of the table.

b What is the theoretical maximum length of a glass fibre that can be manufactured in this way?

c Suppose that such a glass fibre is useable only if its cross-sectional area stays above 1.2 mm^2. What is the maximum length of a glass fibre that meets this restriction? (Answer correct to the nearest centimetre.)

11 A parasite is growing on a large tree. The data in the table gives the size, s cm^3, of the parasite after d days.

d	14	19	23	26	28	30	33	35	37	41
s	1.85	4.25	7.50	11.00	14.50	18.95	26.65	38.00	49.75	84.50

Using a graphics calculator:

a Plot the graph of s against d.

b Plot the graph of $\ln s$ against $\ln d$.

c Plot the graph of $\ln s$ against d.

d Find a model that fits the data.

ISBN 978-1-108-45164-2

7 Refresher on differentiation

Objectives

- ▶ To understand the concept of **limit**.
- ▶ To understand the definition of **differentiation**.
- ▶ To understand and use the notation for the **derivative** of a function.
- ▶ To find the **gradient** of a tangent to the graph of a polynomial function by calculating its derivative.
- ▶ To deduce the **graph of the derivative** from the graph of a function.

Throughout this course, we have been using different types of functions as mathematical models of the relationship between two variables. We have used the idea that one variable, say y, is a function of another variable, say x.

When establishing and applying such a mathematical model, it is important to understand how the relationship between the two variables is changing. For example, if x increases, does y also increase, or does it decrease, or remain unaltered? And, if it does change, does it do so consistently, quickly, slowly, indefinitely, etc.?

We can study rates of change using differential calculus. It is believed that calculus was discovered independently in the late seventeenth century by two great mathematicians: Isaac Newton and Gottfried Leibniz. Like most scientific breakthroughs, the discovery of calculus did not arise out of a vacuum. In fact, many mathematicians and philosophers going back to ancient times made discoveries relating to calculus.

In this chapter, we revise some of the important ideas and results from your study of differential calculus in Mathematical Methods Units 1 & 2.

ISBN 978-1-108-45164-2

7A The derivative

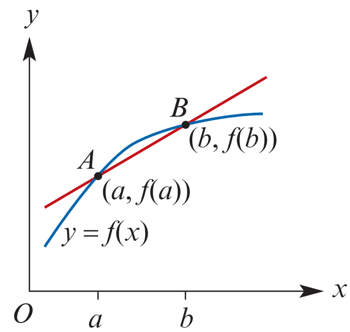
We begin this chapter by recalling the definition of average rate of change from Mathematical Methods Units 1 & 2.

► Average rate of change

For any function $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the line through the two points $A(a, f(a))$ and $B(b, f(b))$.

That is:

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$



Example 1

Find the average rate of change of the function with rule $f(x) = x^2 - 2x + 5$ as x changes from 1 to 5.

Solution

$$\text{Average rate of change} = \frac{\text{change in } y}{\text{change in } x}$$

$f(1) = (1)^2 - 2(1) + 5 = 4$

$f(5) = (5)^2 - 2(5) + 5 = 20$

$$\text{Average rate of change} = \frac{20 - 4}{5 - 1}$$

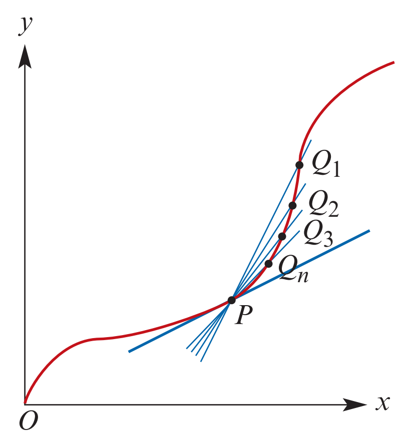
$$= 4$$

► The tangent to a curve at a point

We first recall that a **chord** of a curve is a line segment joining points P and Q on the curve. A **secant** is a line through points P and Q on the curve.

The **instantaneous rate of change** at P can be defined by considering what happens when we look at a sequence of secants $PQ_1, PQ_2, PQ_3, \ldots, PQ_n, \ldots$, where the points Q_i get closer and closer to P.

Here we first focus our attention on the gradient of the tangent at P.



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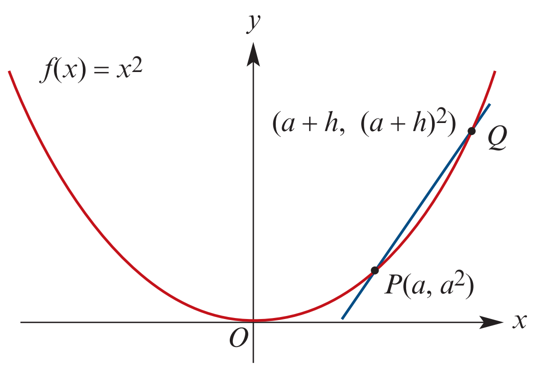
Consider the function $f(x) = x^2$.

The gradient of the secant PQ shown on the graph is

$$\text{gradient of } PQ = \frac{(a+h)^2 - a^2}{a+h-a}$$
$$= \frac{a^2 + 2ah + h^2 - a^2}{h}$$
$$= 2a + h$$



The limit of $2a + h$ as h approaches 0 is $2a$, and so the gradient of the tangent at P is said to be $2a$.

Note: This also can be interpreted as the instantaneous rate of change of f at $(a, f(a))$.

The straight line that passes through the point P and has gradient $2a$ is called the **tangent** to the curve at P.

It can be seen that there is nothing special about a here. The same calculation works for any real number x. The gradient of the tangent to the graph of $y = x^2$ at any point x is $2x$.

We say that the **derivative of x^2 with respect to x is $2x$**, or more briefly, we can say that the **derivative of x^2 is $2x$**.

Limit notation

The notation for the limit of $2x + h$ as h approaches 0 is

$$\lim_{h \to 0} (2x + h)$$

The derivative of a function with rule $f(x)$ may be found by:

1. finding an expression for the gradient of the line through $P(x, f(x))$ and $Q(x + h, f(x + h))$
2. finding the limit of this expression as h approaches 0.

Example 2

Consider the function $f(x) = x^3$. By first finding the gradient of the secant through $P(2, 8)$ and $Q(2 + h, (2 + h)^3)$, find the gradient of the tangent to the curve at the point $(2, 8)$.

Solution

$$\text{Gradient of } PQ = \frac{(2+h)^3 - 8}{2+h-2}$$
$$= \frac{8 + 12h + 6h^2 + h^3 - 8}{h}$$
$$= \frac{12h + 6h^2 + h^3}{h}$$
$$= 12 + 6h + h^2$$

The gradient of the tangent line at $(2, 8)$ is $\lim_{h \to 0} (12 + 6h + h^2) = 12$.

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The following example provides practice in determining limits.

Example 3

Find:

a $\lim_{h\to 0}(22x^2 + 20xh + h)$

b $\lim_{h\to 0}\dfrac{3x^2h + 2h^2}{h}$

c $\lim_{h\to 0} 12x$

d $\lim_{h\to 0} 4$

Solution

a $\lim_{h\to 0}(22x^2 + 20xh + h) = 22x^2$

b $$\lim_{h\to 0}\frac{3x^2h + 2h^2}{h} = \lim_{h\to 0}(3x^2 + 2h) = 3x^2$$

c $\lim_{h\to 0} 12x = 12x$

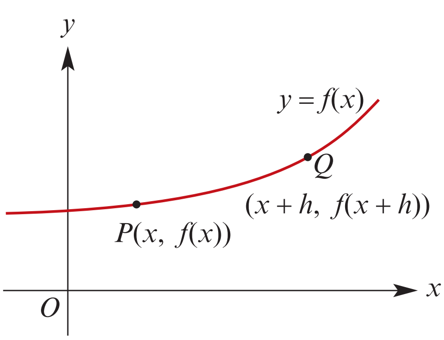
d $\lim_{h\to 0} 4 = 4$

Definition of the derivative

In general, consider the graph of $y = f(x)$, where f is a function.

$$\text{Gradient of secant } PQ = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h}$$

The gradient of the tangent to the graph of $y = f(x)$ at the point $P(x, f(x))$ is the limit of this expression as h approaches 0.



Derivative of a function

- The **derivative** of the function f is denoted f' and is defined by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

- The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Warning: This definition of the derivative assumes that the limit exists. For polynomial functions, such limits always exist. But it is not true that for every function you can find the derivative at every point of its domain. This is discussed further in Section 7D.

ISBN 978-1-108-45164-2

Differentiation by first principles

Determining the derivative of a function by evaluating the limit is called **differentiation by first principles**.

Example 4

Find $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ for each of the following:

a $f(x) = 3x^2 + 2x + 2$

b $f(x) = 2 - x^3$

Solution

a
$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{3(x+h)^2 + 2(x+h) + 2 - (3x^2 + 2x + 2)}{h} \\ &= \frac{3x^2 + 6xh + 3h^2 + 2x + 2h + 2 - 3x^2 - 2x - 2}{h} \\ &= \frac{6xh + 3h^2 + 2h}{h} \\ &= 6x + 3h + 2\end{aligned}$$

Therefore

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (6x + 3h + 2) = 6x + 2$$

b
$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{2 - (x+h)^3 - (2 - x^3)}{h} \\ &= \frac{2 - (x^3 + 3x^2h + 3xh^2 + h^3) - 2 + x^3}{h} \\ &= \frac{-3x^2h - 3xh^2 - h^3}{h} \\ &= -3x^2 - 3xh - h^2\end{aligned}$$

Therefore

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (-3x^2 - 3xh - h^2) = -3x^2$$

Section summary

- The **derivative** of the function f is denoted f' and is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

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Exercise 7A

SF

Example 1 **1** Find the average rate of change of the function with rule $f(x) = -x^2 + 2x + 1$ as x changes from -1 to 4.

2 Find the average rate of change of the function with rule $f(x) = 6 - x^3$ as x changes from -1 to 1.

Example 2 **3** For the curve with equation $y = x^2 + 5x$:

a Find the gradient of the secant through points P and Q, where P is the point $(2, 14)$ and Q is the point $(2 + h, (2 + h)^2 + 5(2 + h))$.

b From the result of **a**, find the gradient of the tangent to the curve at the point $(2, 14)$.

Example 3 **4** Find:

a $\lim_{h \to 0} \dfrac{4x^2h^2 + xh + h}{h}$

b $\lim_{h \to 0} \dfrac{2x^3h - 2xh^2 + h}{h}$

c $\lim_{h \to 0} (40 - 50h)$

d $\lim_{h \to 0} 5h$

e $\lim_{h \to 0} 5$

f $\lim_{h \to 0} \dfrac{30h^2x^2 + 20h^2x + h}{h}$

g $\lim_{h \to 0} \dfrac{3h^2x^3 + 2hx + h}{h}$

h $\lim_{h \to 0} 3x$

i $\lim_{h \to 0} \dfrac{3x^3h - 5x^2h^2 + xh}{h}$

j $\lim_{h \to 0} (6x - 7h)$

5 For the curve with equation $y = x^3 - x$:

a Find the gradient of the chord PQ, where P is the point $(1, 0)$ and Q is the point $(1 + h, (1 + h)^3 - (1 + h))$.

b From the result of **a**, find the gradient of the tangent to the curve at the point $(1, 0)$.

6 If $f(x) = x^2 - 2$, simplify $\dfrac{f(x + h) - f(x)}{h}$. Hence find the derivative of $x^2 - 2$.

7 Let P and Q be points on the curve $y = x^2 + 2x + 5$ at which $x = 2$ and $x = 2 + h$ respectively. Express the gradient of the line PQ in terms of h, and hence find the gradient of the tangent to the curve $y = x^2 + 2x + 5$ at $x = 2$.

Example 4 **8** For each of the following, find $f'(x)$ by finding $\lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$:

a $f(x) = 5x^2$

b $f(x) = 3x + 2$

c $f(x) = 5$

d $f(x) = 3x^2 + 4x + 3$

e $f(x) = 5x^3 - 5$

f $f(x) = 5x^2 - 6x$

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7B Rules for differentiation

The derivative of x^n where n is a positive integer

Differentiating from first principles gives the following:

- For $f(x) = x$, $f'(x) = 1$.
- For $f(x) = x^2$, $f'(x) = 2x$.
- For $f(x) = x^3$, $f'(x) = 3x^2$.

This suggests the following general result:

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \dots$

Proof We can prove this result using the binomial theorem (discussed in Appendix A).

Let $f(x) = x^n$, where $n \in \mathbb{N}$ with $n \geq 2$.

Then
$$\begin{aligned} f(x+h) - f(x) &= (x+h)^n - x^n \\ &= x^n + {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \dots + {}^nC_{n-1} xh^{n-1} + h^n - x^n \\ &= {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \dots + {}^nC_{n-1} xh^{n-1} + h^n \\ &= nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \dots + {}^nC_{n-1} xh^{n-1} + h^n \end{aligned}$$

and so
$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{1}{h}\left(nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \dots + {}^nC_{n-1} xh^{n-1} + h^n\right) \\ &= nx^{n-1} + {}^nC_2 x^{n-2}h + \dots + {}^nC_{n-1} xh^{n-2} + h^{n-1} \end{aligned}$$

Thus
$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0}\left(nx^{n-1} + {}^nC_2 x^{n-2}h + \dots + {}^nC_{n-1} xh^{n-2} + h^{n-1}\right) \\ &= nx^{n-1} \end{aligned}$$

The derivative of a polynomial function

The following results are very useful when finding the derivative of a polynomial function.

- Constant function: If $f(x) = c$, then $f'(x) = 0$.
- Multiple: If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
  For example: if $f(x) = 5x^2$, then $f'(x) = 5(2x) = 10x$.
- Sum: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
  For example: if $f(x) = x^2 + 2x$, then $f'(x) = 2x + 2$.
- Difference: If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.
  For example: if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$.

We will revise the rules for the derivatives of products and quotients in Chapter 8.

The process of finding the derivative function is called **differentiation**.

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Example 5

Find the derivative of $x^5 - 2x^3 + 2$, i.e. differentiate $x^5 - 2x^3 + 2$ with respect to x.

Solution

Let $f(x) = x^5 - 2x^3 + 2$

Then $f'(x) = 5x^4 - 2(3x^2) + 0$

$= 5x^4 - 6x^2$

Explanation

We use the following results:

- the derivative of x^n is nx^{n-1}
- the derivative of a number is 0
- the multiple, sum and difference rules.

Example 6

Find the derivative of $f(x) = 3x^3 - 6x^2 + 1$ and thus find $f'(1)$.

Solution

Let $f(x) = 3x^3 - 6x^2 + 1$

Then $f'(x) = 3(3x^2) - 6(2x) + 0$

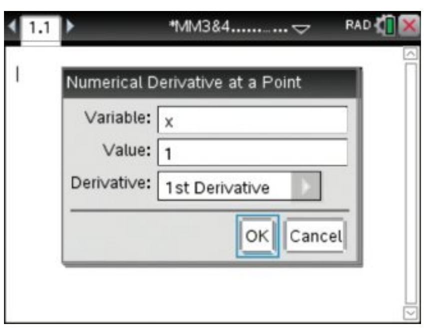
$= 9x^2 - 12x$

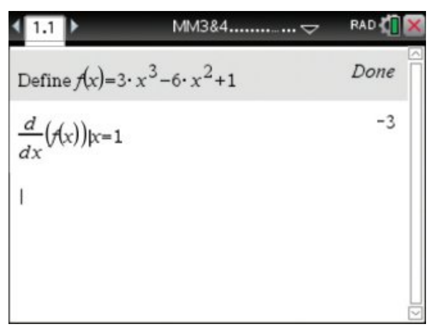
$\therefore \quad f'(1) = 9 - 12 = -3$

TI-84

Using the TI-Nspire CX non-CAS

- In a **Calculator** application, define $f(x) = 3x^3 - 6x^2 + 1$.
- To find the value of the derivative at $x = 1$, use menu > **Calculus** > **Numerical Derivative at a Point** and complete as shown.





Note: The derivative template can also be accessed from the 2D-template palette or by using shift –.

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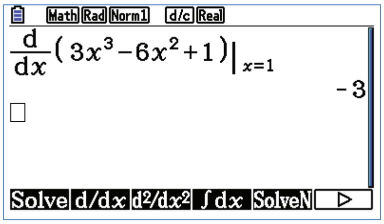
Using the Casio

To find the derivative of $3x^3 - 6x^2 + 1$ at $x = 1$:

- Press (MENU) (1) to select **Run-Matrix** mode.
- Go to **Calculation** (OPTN) (F4), then **d/dx** (F2).
- Enter the expression $3x^3 - 6x^2 + 1$ and the x-value 1:

  (3) (X,θ,T) (∧) (3) ► (−) (6) (X,θ,T) (x²) (+) (1)

  ► (1) (EXE)



Finding the gradient of a tangent line

We discussed the tangent line at a point on a graph in Section 7A. We recall the following:

> The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Example 7

For the curve determined by the rule $f(x) = 3x^3 - 6x^2 + 1$, find the gradient of the tangent line to the curve at the point $(1, -2)$.

Solution

Now $f'(x) = 9x^2 - 12x$ and so $f'(1) = 9 - 12 = -3$.

The gradient of the tangent line at the point $(1, -2)$ is -3.

► Alternative notations

It was mentioned in the introduction to this chapter that the German mathematician Gottfried Leibniz was one of the two people to whom the discovery of calculus is attributed. A form of the notation he introduced is still in use today.

Leibniz notation

An alternative notation for the derivative is the following:

> If $y = x^3$, then the derivative can be denoted by $\dfrac{dy}{dx}$, and so we write $\dfrac{dy}{dx} = 3x^2$.

In general, if y is a function of x, then the derivative of y with respect to x is denoted by $\dfrac{dy}{dx}$.

Similarly, if z is a function of t, then the derivative of z with respect to t is denoted $\dfrac{dz}{dt}$.

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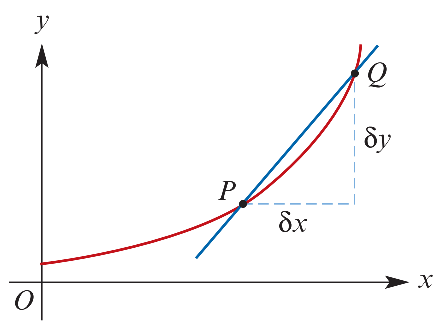

Warning: In Leibniz notation, the symbol d is not a factor and cannot be cancelled.

This notation came about because, in the eighteenth century, the standard diagram for finding the limiting gradient was labelled as shown:

- δx means a small difference in x
- δy means a small difference in y

where δ (delta) is the lowercase Greek letter d.

Example 8

a If $y = t^2$, find $\frac{dy}{dt}$.

b If $x = t^3 + t$, find $\frac{dx}{dt}$.

c If $z = \frac{1}{3}x^3 + x^2$, find $\frac{dz}{dx}$.

Solution

a
$$y = t^2$$
$$\frac{dy}{dt} = 2t$$

b
$$x = t^3 + t$$
$$\frac{dx}{dt} = 3t^2 + 1$$

c
$$z = \frac{1}{3}x^3 + x^2$$
$$\frac{dz}{dx} = x^2 + 2x$$

Example 9

a For $y = (x+3)^2$, find $\frac{dy}{dx}$.

b For $z = (2t-1)^2(t+2)$, find $\frac{dz}{dt}$.

c For $y = \frac{x^2+3x}{x}$, find $\frac{dy}{dx}$.

d Differentiate $y = 2x^3 - 1$ with respect to x.

Solution

a First write $y = (x+3)^2$ in expanded form:

$$y = x^2 + 6x + 9$$
$$\therefore \quad \frac{dy}{dx} = 2x + 6$$

b Expanding:

$$\begin{aligned} z &= (4t^2 - 4t + 1)(t+2) \\ &= 4t^3 - 4t^2 + t + 8t^2 - 8t + 2 \\ &= 4t^3 + 4t^2 - 7t + 2 \end{aligned}$$
$$\therefore \quad \frac{dz}{dt} = 12t^2 + 8t - 7$$

c First simplify:

$$y = x + 3 \quad (\text{for } x \neq 0)$$
$$\therefore \quad \frac{dy}{dx} = 1 \quad (\text{for } x \neq 0)$$

d
$$y = 2x^3 - 1$$
$$\therefore \quad \frac{dy}{dx} = 6x^2$$

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Operator notation

'Find the derivative of $2x^2 - 4x$ with respect to x' can also be written as 'find $\frac{d}{dx}(2x^2 - 4x)$'.

In general: $\frac{d}{dx}(f(x)) = f'(x)$.

Example 10

Find:

a $\frac{d}{dx}(5x - 4x^3)$ **b** $\frac{d}{dz}(5z^2 - 4z)$ **c** $\frac{d}{dz}(6z^3 - 4z^2)$

Solution

a $\frac{d}{dx}(5x - 4x^3)$
$= 5 - 12x^2$

b $\frac{d}{dz}(5z^2 - 4z)$
$= 10z - 4$

c $\frac{d}{dz}(6z^3 - 4z^2)$
$= 18z^2 - 8z$

Example 11

For each of the following curves, find the coordinates of the points on the curve at which the gradient of the tangent line at that point has the given value:

a $y = x^3$, gradient = 8

b $y = x^2 - 4x + 2$, gradient = 0

c $y = 4 - x^3$, gradient = −6

Solution

a $y = x^3$ implies $\frac{dy}{dx} = 3x^2$

$\therefore \quad 3x^2 = 8$

$\therefore \quad x = \pm\sqrt{\frac{8}{3}} = \frac{\pm 2\sqrt{6}}{3}$

The points are $\left(\frac{2\sqrt{6}}{3}, \frac{16\sqrt{6}}{9}\right)$ and $\left(\frac{-2\sqrt{6}}{3}, \frac{-16\sqrt{6}}{9}\right)$.

b $y = x^2 - 4x + 2$ implies $\frac{dy}{dx} = 2x - 4$

$\therefore \quad 2x - 4 = 0$

$\therefore \quad x = 2$

The only point is $(2, -2)$.

c $y = 4 - x^3$ implies $\frac{dy}{dx} = -3x^2$

$\therefore \quad -3x^2 = -6$

$\therefore \quad x^2 = 2$

$\therefore \quad x = \pm\sqrt{2}$

The points are $\left(2^{\frac{1}{2}}, 4 - 2^{\frac{3}{2}}\right)$ and $\left(-2^{\frac{1}{2}}, 4 + 2^{\frac{3}{2}}\right)$.

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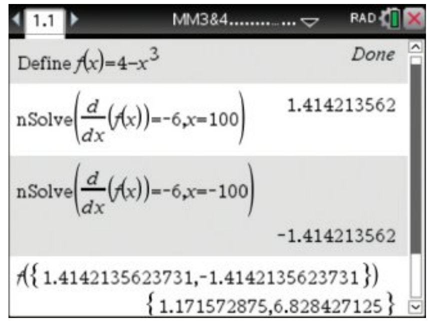
Using the TI-Nspire CX non-CAS

The calculator can be used to obtain an approximate solution for Example 11c.

- Define $f(x) = 4 - x^3$.
- Solve the equation $\frac{d}{dx}(f(x)) = -6$.
- Substitute in $f(x)$ to find the y-coordinates.
- The points are $(1.41, 1.17)$ and $(-1.41, 6.83)$, correct to two decimal places.



Note: Check the number of solutions by finding the intersection points of the graphs of $f1(x) = \frac{d}{dx}(f(x))$ and $f2(x) = -6$.

Using the Casio

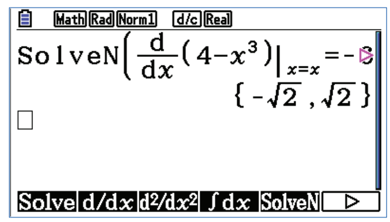
Method 1: Using the numerical solver

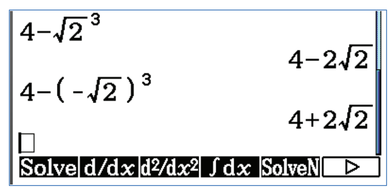
For Example 11c:

- Press MENU 1 to select **Run-Matrix** mode.
- Select the numerical solver by going to **Calculation** OPTN F4, then **SolveN** F5.
- Enter the equation $\frac{d}{dx}(4 - x^3) = -6$ as follows:

  F2 4 − X,θ,T ∧ 3 ► ► X,θ,T ►
  SHIFT • (−) 6) EXE

- Substitute each x-value into the rule $y = 4 - x^3$ to find the corresponding y-value.
- The points are $(\sqrt{2}, 4 - 2\sqrt{2})$ and $(-\sqrt{2}, 4 + 2\sqrt{2})$.





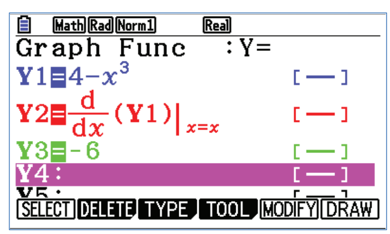
Method 2: Using Graph mode

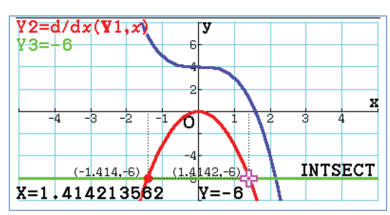
For Example 11c:

- Press MENU 5 to select **Graph** mode.
- Enter the rule $y = 4 - x^3$ in $Y1$.
- Enter the derivative of $Y1$ in $Y2$ as follows:

  OPTN F2 F1 F1 1 ► X,θ,T EXE

- Enter the rule $y = -6$ in $Y3$.
- Select **Draw** F6 to view the graphs.
- Adjust the View Window if required.
- Go to **G-Solve** SHIFT F5, then **Intersection** F5. Select the graph of $Y2$ and the graph of $Y3$.





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► An angle associated with the gradient of a curve at a point

The gradient of a curve at a point is the gradient of the tangent at that point. A straight line, the tangent, is associated with each point on the curve.

If α is the angle a straight line makes with the positive direction of the x-axis, then the gradient, m, of the straight line is equal to $\tan\alpha$. That is, $m = \tan\alpha$.

For example, if $\alpha = 135°$, then $\tan\alpha = -1$ and so the gradient is -1.

Example 12

Find the coordinates of the points on the curve with equation $y = x^2 - 7x + 8$ at which the tangent line:

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = -2x + 6$.

Solution

a $\dfrac{dy}{dx} = 2x - 7$

$2x - 7 = 1 \quad (\text{as } \tan 45° = 1)$

$2x = 8$

$\therefore \quad x = 4$

$y = 4^2 - 7 \times 4 + 8 = -4$

The coordinates are $(4, -4)$.

b The line $y = -2x + 6$ has gradient -2.

$2x - 7 = -2$

$2x = 5$

$\therefore \quad x = \dfrac{5}{2}$

The coordinates are $\left(\dfrac{5}{2}, -\dfrac{13}{4}\right)$.

► Increasing and decreasing functions

We have discussed strictly increasing and strictly decreasing functions in Chapter 1.

- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

We have the following very important results.

If $f'(x) > 0$, for all x in the interval, then the function is strictly increasing.
(Think of the tangents at each point – they each have positive gradient.)

If $f'(x) < 0$, for all x in the interval, then the function is strictly decreasing.
(Think of the tangents at each point – they each have negative gradient.)

Warning: The function $f(x) = x^3$, $x \in \mathbb{R}$, is strictly increasing, but $f'(0) = 0$. This means that *strictly increasing does not imply* $f'(x) > 0$.

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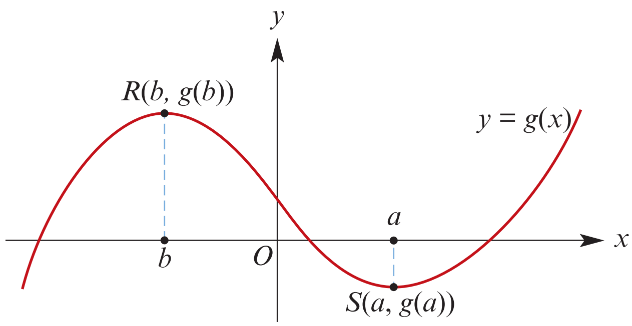
▶ Sign of the derivative

Gradients of tangents can, of course, be negative or zero. They are not always positive.

At a point $(a, g(a))$ on the graph of $y = g(x)$, the gradient of the tangent is $g'(a)$.



Some features of the graph shown are:

- For $x < b$, the gradient of any tangent is positive, i.e. $g'(x) > 0$.
- For $x = b$, the gradient of the tangent is zero, i.e. $g'(b) = 0$.
- For $b < x < a$, the gradient of any tangent is negative, i.e. $g'(x) < 0$.
- For $x = a$, the gradient of the tangent is zero, i.e. $g'(a) = 0$.
- For $x > a$, the gradient of any tangent is positive, i.e. $g'(x) > 0$.

Note: This function g is strictly decreasing on the open interval (b, a), but it is also strictly decreasing on the closed interval $[b, a]$. Similarly, the function g is strictly increasing on the intervals $[a, \infty)$ and $(-\infty, b]$.

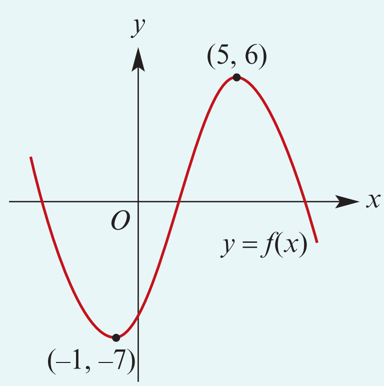
Example 13

For the graph of $y = f(x)$ shown, find the values of x for which:

a $f'(x) > 0$

b $f'(x) < 0$

c $f'(x) = 0$



Solution

a $f'(x) > 0$ for $-1 < x < 5$

b $f'(x) < 0$ for $x < -1$ or $x > 5$

c $f'(x) = 0$ for $x = -1$ or $x = 5$

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Section summary

- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \ldots$
- Constant function: If $f(x) = c$, then $f'(x) = 0$.
- Multiple: If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
- Sum: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
- Difference: If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.
- Angle of inclination of tangent
  - A straight line, the tangent, is associated with each point on a smooth curve.
  - If α is the angle that a straight line makes with the positive direction of the x-axis, then the gradient of the line is given by $m = \tan\alpha$.
- Increasing and decreasing functions
  - A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
  - A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
  - If $f'(x) > 0$ for all x in the interval, then the function is strictly increasing.
  - If $f'(x) < 0$ for all x in the interval, then the function is strictly decreasing.

Exercise 7B

SF

Skillsheet Example 5

1 For each of the following, find the derivative with respect to x:

a x^5 **b** $4x^7$ **c** $6x$

d $5x^2 - 4x + 3$ **e** $4x^3 + 6x^2 + 2x - 4$ **f** $5x^4 + 3x^3$

g $-2x^2 + 4x + 6$ **h** $6x^3 - 2x^2 + 4x - 6$

Example 6

2 For each of the following, find the derivative of $f(x)$ and thus find $f'(1)$:

a $f(x) = 2x^3 - 5x^2 + 1$ **b** $f(x) = -2x^3 - x^2 - 1$

c $f(x) = x^4 - 2x^3 + 1$ **d** $f(x) = x^5 - 3x^3 + 2$

Example 7

3 **a** For the curve determined by the rule $f(x) = 2x^3 - 5x^2 + 2$, find the gradient of the tangent line to the curve at the point $(1, -1)$.

b For the curve determined by the rule $f(x) = -2x^3 - 3x^2 + 2$, find the gradient of the tangent line to the curve at the point $(2, -26)$.

Example 8

4 **a** If $y = t^3$, find $\dfrac{dy}{dt}$.

b If $x = t^3 - t^2$, find $\dfrac{dx}{dt}$.

c If $z = \dfrac{1}{4}x^4 + 3x^3$, find $\dfrac{dz}{dx}$.

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Example 9 **5** For each of the following, find $\frac{dy}{dx}$: SF

a $y = -2x$

b $y = 7$

c $y = 5x^3 - 3x^2 + 2x + 1$

d $y = \frac{2}{5}(x^3 - 4x + 6)$

e $y = (2x + 1)(x - 3)$

f $y = 3x(2x - 4)$

g $y = \frac{10x^7 + 2x^2}{x^2}, \ x \neq 0$

h $y = \frac{9x^4 + 3x^2}{x}, \ x \neq 0$

Example 10 **6** Find:

a $\frac{d}{dx}(2x^2 - 5x^3)$

b $\frac{d}{dz}(-2z^2 - 6z)$

c $\frac{d}{dz}(6z^3 - 4z^2 + 3)$

d $\frac{d}{dx}(-2x - 5x^3)$

e $\frac{d}{dz}(-2z^2 - 6z + 7)$

f $\frac{d}{dz}(-z^3 - 4z^2 + 3)$

Example 11 **7** Find the coordinates of the points on the curves given by the following equations at which the gradient has the given value:

a $y = 2x^2 - 4x + 1$, gradient $= -6$

b $y = 4x^3$, gradient $= 48$

c $y = x(5 - x)$, gradient $= 1$

d $y = x^3 - 3x^2$, gradient $= 0$

Example 12 **8** Find the coordinates of the points on the curve with equation $y = 2x^2 - 3x + 8$ at which the tangent line: CF

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = 2x + 8$.

9 Find the value of x such that the tangent line to the curve $f(x) = x^2 - x$ at $(x, f(x))$:

a makes an angle of 45° with the positive direction of the x-axis

b makes an angle of 135° with the positive direction of the x-axis

c makes an angle of 60° with the positive direction of the x-axis

d makes an angle of 30° with the positive direction of the x-axis

e makes an angle of 120° with the positive direction of the x-axis.

10 For each of the following, find the angle that the tangent line to the curve $y = f(x)$ makes with the positive direction of the x-axis at the given point: SF

a $y = x^2 + 3x, \ (1, 4)$

b $y = -x^2 + 2x, \ (1, 1)$

c $y = x^3 + x, \ (0, 0)$

d $y = -x^3 - x, \ (0, 0)$

e $y = x^4 - x^2, \ (1, 0)$

f $y = x^4 - x^2, \ (-1, 0)$

11 **a** Differentiate $y = (2x - 1)^2$ with respect to x. CF

b For $y = \frac{x^3 + 2x^2}{x}$, $x \neq 0$, find $\frac{dy}{dx}$.

c Given that $y = 2x^3 - 6x^2 + 18x$, find $\frac{dy}{dx}$. Hence show that $\frac{dy}{dx} > 0$ for all x.

d Given that $y = \frac{x^3}{3} - x^2 + x$, find $\frac{dy}{dx}$. Hence show that $\frac{dy}{dx} \geq 0$ for all x.

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SF

12 At the points on the following curves corresponding to the given values of x, find the y-coordinate and the gradient:

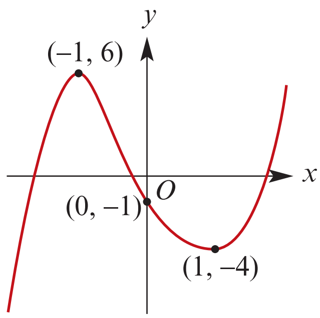
a $y = x^2 + 2x + 1, \ x = 3$
b $y = x^2 - x - 1, \ x = 0$
c $y = 2x^2 - 4x, \ x = -1$
d $y = (2x + 1)(3x - 1)(x + 2), \ x = 4$
e $y = (2x + 5)(3 - 5x)(x + 1), \ x = 1$
f $y = (2x - 5)^2, \ x = 2\frac{1}{2}$

13 For the function $f(x) = 3(x - 1)^2$, find the value(s) of x for which:

a $f(x) = 0$
b $f'(x) = 0$
c $f'(x) > 0$
d $f'(x) < 0$
e $f'(x) = 10$
f $f(x) = 27$

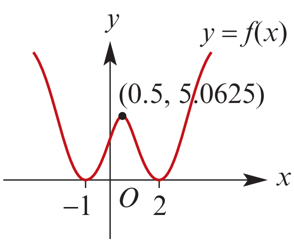
Example 13 **14** For the graph of $y = h(x)$ shown, find the values of x such that:

a $h'(x) > 0$
b $h'(x) < 0$
c $h'(x) = 0$



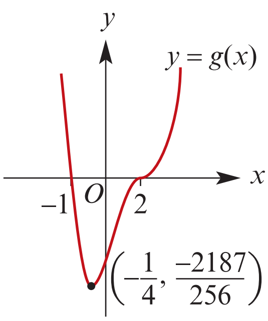
15 For the graph of $y = f(x)$ shown, find the values of x such that:

a $f'(x) > 0$
b $f'(x) < 0$
c $f'(x) = 0$



16 For the graph of $y = g(x)$ shown, find the values of x such that:

a $g'(x) > 0$
b $g'(x) < 0$
c $g'(x) = 0$



CF

17 Find the coordinates of the points on the parabola $y = x^2 - 4x - 8$ at which:

a the gradient is zero
b the tangent is parallel to $y = 2x + 6$
c the tangent is parallel to $3x + 2y = 8$.

18 **a** Show that $f(x) = x^3$, $x \in \mathbb{R}$, is a strictly increasing function by showing that $f'(x) > 0$, for all non-zero x, and showing that, if $b > 0$, then $f(b) > f(0)$ and, if $0 > b$, then $f(0) > f(b)$.

b Show that $f(x) = -x^3$, $x \in \mathbb{R}$, is a strictly decreasing function.

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CF

19 **a** Show that $f(x) = x^2$, $x \in [0, \infty)$, is a strictly increasing function.
b Show that $f(x) = x^2$, $x \in (-\infty, 0]$, is a strictly decreasing function.

20 For the function $f(x) = x^2 - x - 12$, show that the largest interval for which f is strictly increasing is $[\frac{1}{2}, \infty)$.

21 For each of the following, find the largest interval for which the function is strictly decreasing:
a $y = x^2 + 2x$ **b** $y = -x^2 + 4x$ **c** $y = 2x^2 + 3$ **d** $y = -2x^2 + 6x$

7C Differentiating x^n where n is a negative integer

In the previous sections we have seen how to differentiate polynomial functions. In this section we add to the family of functions that we can differentiate. In particular, we will consider functions which involve linear combinations of powers of x, where the indices may be negative integers.

e.g. $f(x) = x^{-1}$ for $x \neq 0$

$f(x) = 2x + 3 + x^{-2}$ for $x \neq 0$

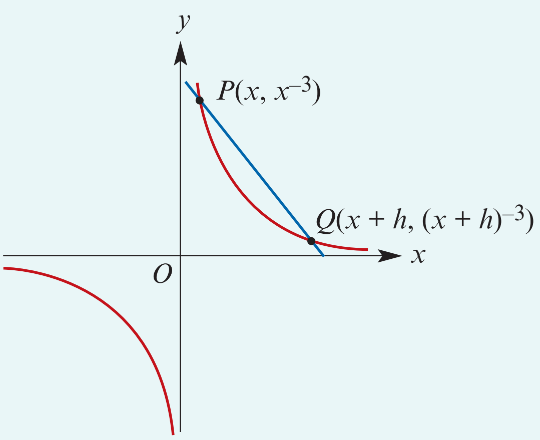
Example 14

Define the function $f(x) = x^{-3}$ for $x \neq 0$. Find $f'(x)$ by first principles.

Solution

The gradient of secant PQ is given by

$$\frac{(x+h)^{-3} - x^{-3}}{h}$$
$$= \frac{x^3 - (x+h)^3}{(x+h)^3 x^3} \times \frac{1}{h}$$
$$= \frac{x^3 - (x^3 + 3x^2h + 3xh^2 + h^3)}{(x+h)^3 x^3} \times \frac{1}{h}$$
$$= \frac{-3x^2h - 3xh^2 - h^3}{(x+h)^3 x^3} \times \frac{1}{h}$$
$$= \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3}$$



So the gradient of the curve at P is given by

$$\lim_{h \to 0} \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3} = \frac{-3x^2}{x^6} = -3x^{-4}$$

Hence $f'(x) = -3x^{-4}$.

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We are now in a position to state a generalisation of the result we found in Section 7B. This result can be proved by again using the binomial theorem.

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

When n is positive, we take the domain of f to be $\mathbb{R}$, and when n is negative, we take the domain of f to be $\mathbb{R} \setminus \{0\}$.

Example 15

Find the derivative of $x^4 - 2x^{-3} + x^{-1} + 2$, $x \neq 0$.

Solution

If $\quad f(x) = x^4 - 2x^{-3} + x^{-1} + 2 \qquad$ (for $x \neq 0$)

then $\quad f'(x) = 4x^3 - 2(-3x^{-4}) + (-x^{-2}) + 0$

$\qquad = 4x^3 + 6x^{-4} - x^{-2} \qquad$ (for $x \neq 0$)

Example 16

Find the derivative of $f(x) = 3x^2 - 6x^{-2} + 1$, $x \neq 0$.

Solution

$f'(x) = 3(2x) - 6(-2x^{-3}) + 0$

$\qquad = 6x + 12x^{-3} \qquad$ (for $x \neq 0$)

Example 17

Find the gradient of the tangent to the curve $y = x^2 + \dfrac{1}{x}$ at the point $(1, 2)$.

Solution

$$\frac{dy}{dx} = 2x + (-x^{-2})$$

$$= 2x - x^{-2}$$

When $x = 1$, $\dfrac{dy}{dx} = 2 - 1 = 1$. The gradient of the curve is 1 at the point $(1, 2)$.

Example 18

Show that the derivative of the function $f(x) = x^{-3}$, $x \neq 0$, is always negative.

Solution

$f'(x) = -3x^{-4} = -\dfrac{3}{x^4} \qquad$ (for $x \neq 0$)

Since x^4 is positive for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$.

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Section summary

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

Exercise 7C

SF

1 **a** Sketch the graph of $f(x) = \dfrac{2}{x^2}$, $x \neq 0$.

b Let P be the point $(1, 2)$ and Q the point $(1 + h, f(1 + h))$. Find the gradient of the secant PQ.

c Hence find the gradient of the tangent to the curve $y = \dfrac{2}{x^2}$ at $(1, 2)$.

Example 14

2 **a** Let $f(x) = \dfrac{1}{x-3}$, $x \neq 3$. Find $f'(x)$ by first principles.

b Let $f(x) = \dfrac{1}{x+2}$, $x \neq -2$. Find $f'(x)$ by first principles.

CF

3 Let $f(x) = x^{-4}$. Find $f'(x)$ by first principles.

Hint: Remember that $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.

Example 15, 16

SF

4 Differentiate each of the following with respect to x:

a $3x^{-2} + 5x^{-1} + 6$ **b** $\dfrac{5}{x^3} + 6x^2$ **c** $\dfrac{-5}{x^3} + \dfrac{4}{x^2} + 1$

d $6x^{-3} + 3x^{-2}$ **e** $\dfrac{4x^2 + 2x}{x^2}$

5 Find the derivative of each of the following:

a $\dfrac{2z^2 - 4z}{z^2}$, $z \neq 0$ **b** $\dfrac{6 + z}{z^3}$, $z \neq 0$ **c** $16 - z^{-3}$, $z \neq 0$

d $\dfrac{4z + z^3 - z^4}{z^2}$, $z \neq 0$ **e** $\dfrac{6z^2 - 2z}{z^4}$, $z \neq 0$ **f** $\dfrac{6}{x} - 3x^2$, $x \neq 0$

Example 17

6 Find the gradient of the tangent to each of the following curves at the stated point:

a $y = x^{-2} + x^3$, $x \neq 0$, at $(2, 8\frac{1}{4})$ **b** $y = x^{-2} - \dfrac{1}{x}$, $x \neq 0$, at $(4, \frac{1}{2})$

c $y = x^{-2} - \dfrac{1}{x}$, $x \neq 0$, at $(1, 0)$ **d** $y = x(x^{-1} + x^2 - x^{-3})$, $x \neq 0$, at $(1, 1)$

Example 18

7 Show that the derivative of the function $f(x) = -2x^{-5}$, $x \neq 0$, is always positive.

8 Find the x-coordinates of the points on the curve $y = \dfrac{x^2 - 1}{x}$ at which the gradient of the curve is 5.

CF

9 Given that the curve $y = ax^2 + \dfrac{b}{x}$ has a gradient of -5 at the point $(2, -2)$, find the values of a and b.

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10 Find the gradient of the curve $y = \dfrac{2x - 4}{x^2}$ at the point where the curve crosses the x-axis. SF

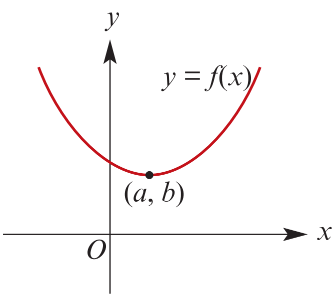
11 The gradient of the curve $y = \dfrac{a}{x} + bx^2$ at the point $(3, 6)$ is 7. Find the values of a and b. CF

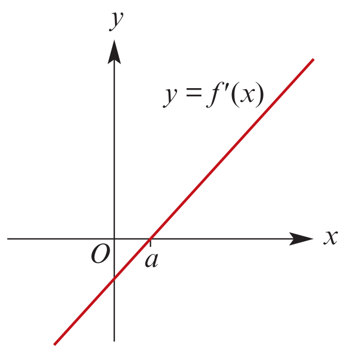
12 For the curve with equation $y = \dfrac{5}{3}x + kx^2 - \dfrac{8}{9}x^3$, calculate the possible values of k such that the tangents at the points with x-coordinates 1 and $-\frac{1}{2}$ are perpendicular. CU

7D The graph of the derivative function

First consider the quadratic function with rule $y = f(x)$ shown in the graph on the left. The vertex is at the point with coordinates (a, b).

- For $x < a$, $f'(x) < 0$.
- For $x = a$, $f'(x) = 0$.
- For $x > a$, $f'(x) > 0$.



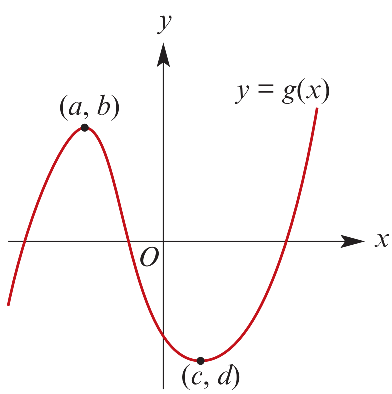


The graph of the derivative function with rule $y = f'(x)$ is therefore as shown on the right.

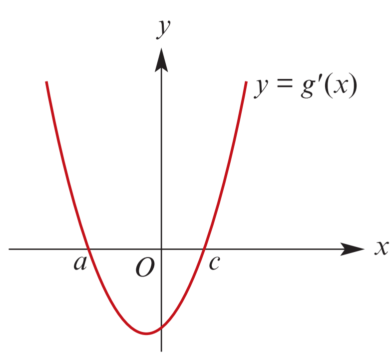
The derivative f' is known to be linear as f is quadratic.

Now consider the cubic function with rule $y = g(x)$ shown in the graph.

- For $x < a$, $g'(x) > 0$.
- For $x = a$, $g'(x) = 0$.
- For $a < x < c$, $g'(x) < 0$.
- For $x = c$, $g'(x) = 0$.
- For $x > c$, $g'(x) > 0$.



The graph of the derivative function with rule $y = g'(x)$ is therefore as shown to the right. The derivative g' is known to be quadratic as g is cubic.



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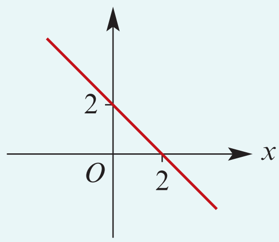
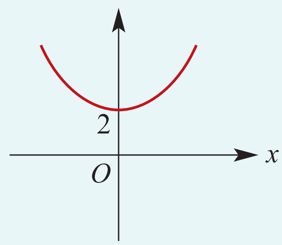
Example 19

Sketch the graph of the derivative function for each of the functions of the graphs shown:

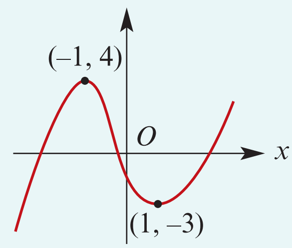
a



b



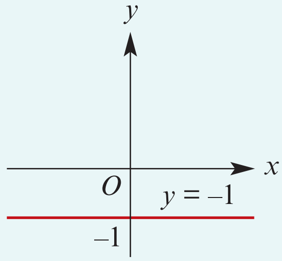
c



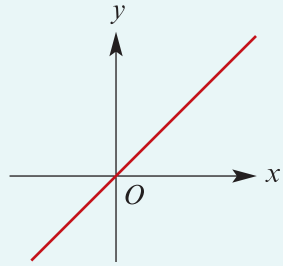
Solution

Note: Not all features of the graphs are known.

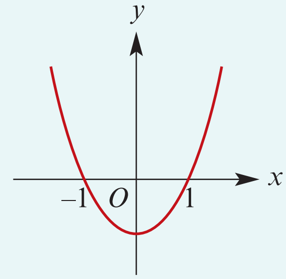
a



b



c



▶ Continuous functions

In this course, we only require an intuitive understanding of continuity. Informally, we say that a function f is continuous at $x = a$ if the graph of $y = f(x)$ can be drawn through the point with coordinates $(a, f(a))$ without a break.

We can give a more formal definition of continuity using limits as follows.

A function f is **continuous** at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

Most of the functions that we study in this course are continuous at each point of their domains. However, we have considered piecewise-defined functions that are not continuous.

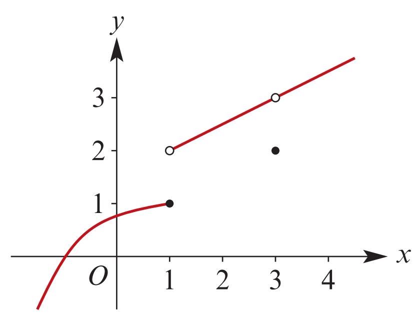
For example, the graph of a piecewise-defined function f is shown on the right. We see that:

- $f(x) \to 1$ as $x \to 1$ from the left
- $f(x) \to 2$ as $x \to 1$ from the right.

Since $f(x)$ does not approach a unique value as $x \to 1$, we say that $\lim_{x \to 1} f(x)$ does not exist.

The function f has a discontinuity at $x = 1$.

There is also a discontinuity at $x = 3$. In this case, we have $\lim_{x \to 3} f(x) = 3$, but $f(3) = 2$.



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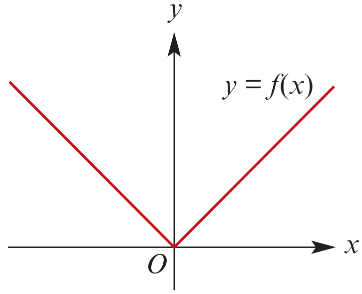
When is a function differentiable?

A function f is said to be **differentiable** at $x = a$ if $\lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

Many of the functions that we study in this course are differentiable at each point of their domains. However, this is not true for all functions.

For example, consider the function

$$f(x) = \sqrt{x^2} = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$



The gradient of the secant through the points $(0, 0)$ and $(0 + h, f(0 + h))$ is given by

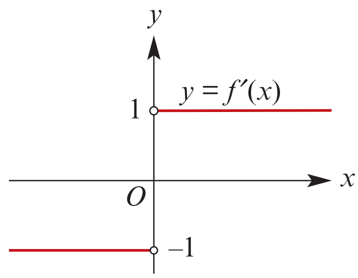
$$\frac{f(0+h) - f(0)}{h} = \frac{f(h)}{h} = \begin{cases} 1 & \text{for } h > 0 \\ -1 & \text{for } h < 0 \end{cases}$$

The gradient does not approach a unique value as $h \to 0$, and so we say $\lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$ does not exist. The function f is not differentiable at $x = 0$.

The graph of f has gradient 1 to the right of 0, and gradient -1 to the left of 0. Therefore the derivative function is given by

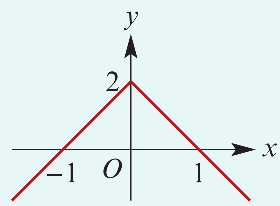
$$f'(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases}$$

The domain of f' is $\mathbb{R} \setminus \{0\}$. The graph of f' is shown on the right.



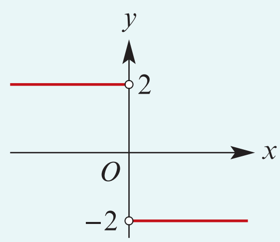
Example 20

Draw a sketch graph of f' where the graph of f is as illustrated. Indicate where f' is not defined.



Solution

The derivative does not exist at $x = 0$, i.e. the function is not differentiable at $x = 0$.



Note: If a function is differentiable at $x = a$, then it is also continuous at $x = a$. However, the converse is not true. For example, the function f from Example 20 is continuous at $x = 0$, but not differentiable at $x = 0$.

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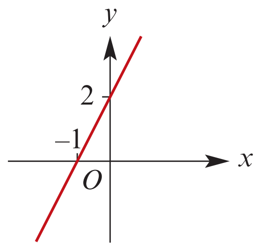
Exercise 7D

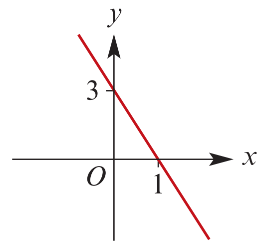
SF

Example 19 **1** Sketch the graph of the derivative function for each of the following functions:

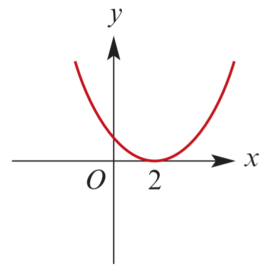
a



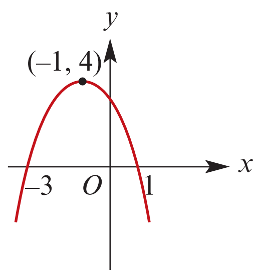
b



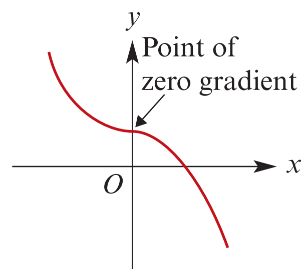
c



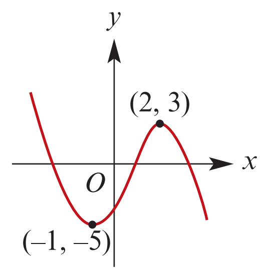
d



e

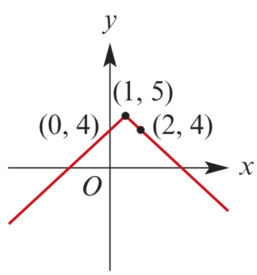


f

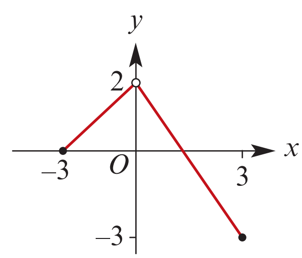


Example 20 **2** Sketch the graph of the derivative function for each of the following functions:

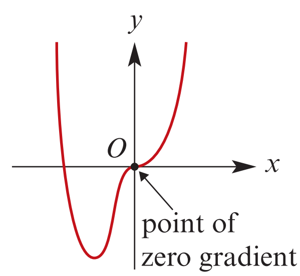
a



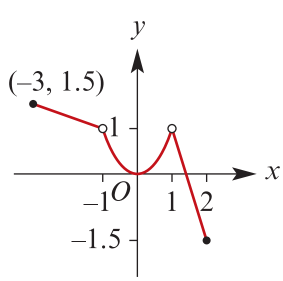
b



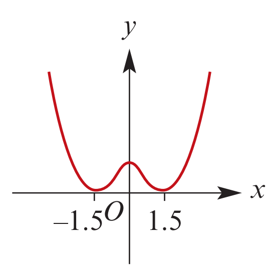
c



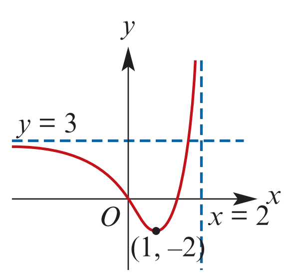
d



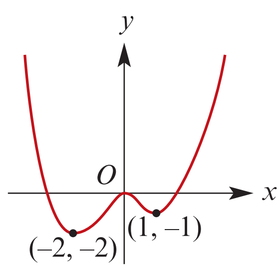
e



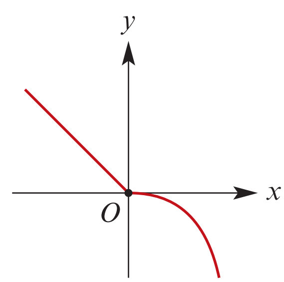
f



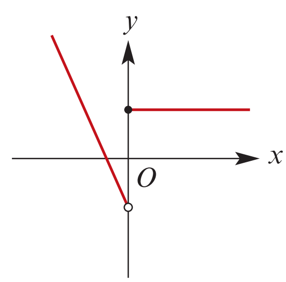
g



h



i



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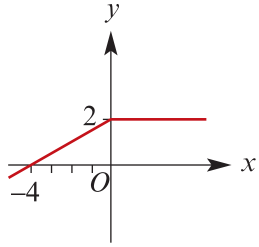
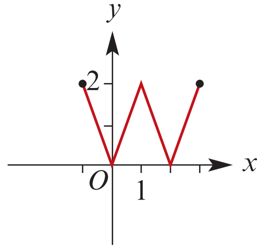
SF

3 Match the graphs of the functions **a**–**f** with the graphs of their derivatives **A**–**F**:

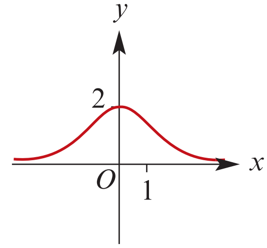
a



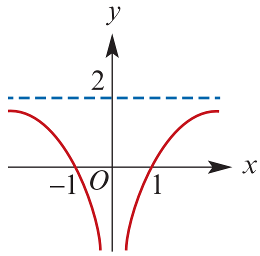
b



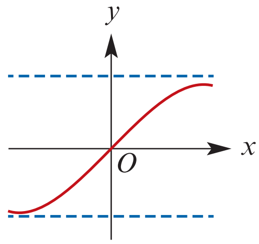
c



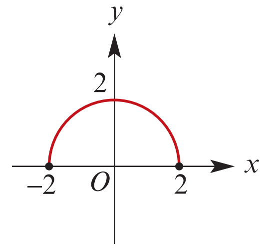
d



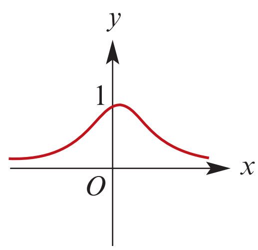
e



f



A



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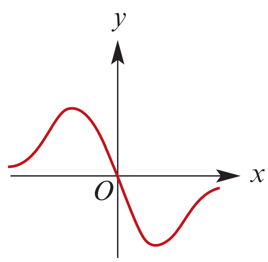

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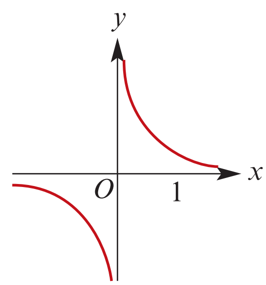

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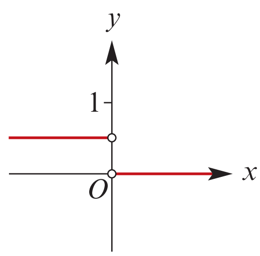

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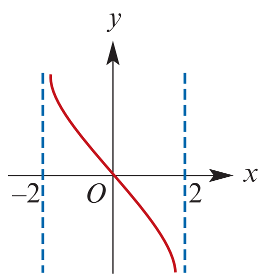
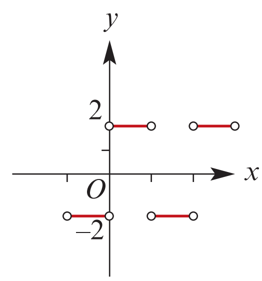


F



4 **a** Use a calculator to plot the graph of $y = f(x)$ where $f(x) = (x^2 - 2x)^2$.

b Using the same screen, plot the graph of $y = f'(x)$. (Do not attempt to determine the rule for $f'(x)$ first.)

c Use a calculator to determine $f'(x)$ for:

i $x = 0$ **ii** $x = 2$ **iii** $x = 1$ **iv** $x = 4$

d For $0 \le x \le 1$, find the value of x for which:

i $f(x)$ is a maximum **ii** $f'(x)$ is a maximum.

5 For each of the following functions, plot the graphs of the function and its derivative on the same screen. Comment.

a $f(x) = \dfrac{x^3}{3} - x^2 + x + 1$ **b** $g(x) = x^3 + 2x + 1$

ISBN 978-1-108-45164-2

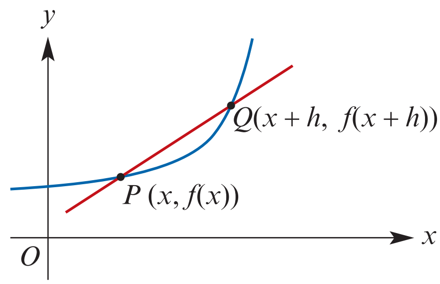
Review

Chapter summary

AS

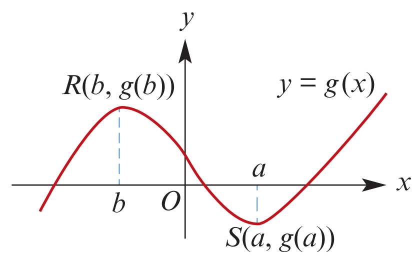
The derivative

- The notation for the limit as h approaches 0 is $\lim\limits_{h \to 0}$.
- For the graph of $y = f(x)$:
  - The gradient of the secant PQ is given by
    $$\frac{f(x+h) - f(x)}{h}$$
  - The gradient of the tangent to the graph at the point P is given by
    $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



- The **derivative** of the function f is denoted f' and is defined by
  $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
- At a point $(a, g(a))$ on the curve $y = g(x)$, the gradient is $g'(a)$.

  For the graph shown:
  - $g'(x) > 0$ for $x < b$ and for $x > a$
  - $g'(x) < 0$ for $b < x < a$
  - $g'(x) = 0$ for $x = b$ and for $x = a$.



Basic derivatives

- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.
- For $f(x) = c$, $f'(x) = 0$, where c is a constant.

Rules for differentiation

- If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
- If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.

Technology-free questions

1 Points P and Q lie on the curve $y = x^3$. The x-coordinates of P and Q are 2 and $2 + h$ respectively. SF

a Find the gradient of the secant PQ.

b Hence find the gradient of the tangent to the curve $y = x^3$ at the point P.

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SF

2 For $y = x^2 + 1$:

a Find the average rate of change of y with respect to x over the interval $[3, 5]$.

b Find the instantaneous rate of change of y with respect to x when $x = -4$.

3 Find the derivative of each of the following functions:

a $f(x) = 5 - 3x$ **b** $f(x) = 4x^2 - 2x + 11$ **c** $f(x) = \frac{1}{2}x(x + 3)$

d $f(x) = \frac{5}{x^2} - \frac{3}{x} + x^2$ **e** $f(x) = \frac{x^3 + 4}{x}$ **f** $f(x) = \frac{(x^2 - 1)(x^2 - 5)}{x^4}$

4 Let $f(x) = 2x^2 - 3x + 5$. Find:

a $f'(x)$ **b** $f'(0)$ **c** the value of x such that $f'(x) = 1$

5 Let $f(x) = x^3 + 3x^2 - 1$. Find the values of x for which:

a $f'(x) = 0$ **b** $f'(x) > 0$ **c** $f'(x) < 0$

6 Let $y = 1 - x^2$. Prove that $x\frac{dy}{dx} + 2 = 2y$ for all values of x.

7 Let $A = 4\pi r^2$. Calculate the value of $\frac{dA}{dr}$ when $r = 3$.

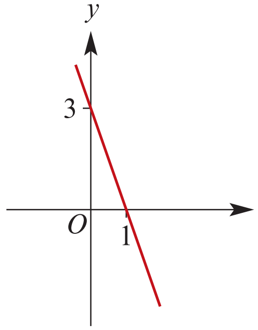
8 At what point on the graph of $y = 1.8x^2$ is the gradient equal to 1?

9 If $y = 3x^2 - 4x + 7$, find the value of x such that $\frac{dy}{dx} = 0$.

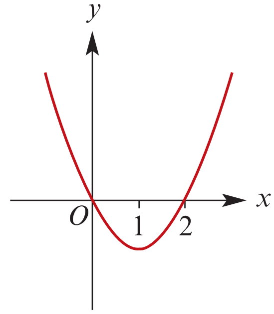
10 Let $y = x^4$. Prove that $x\frac{dy}{dx} = 4y$.

11 Sketch the graph of the derivative function for each of the following functions:

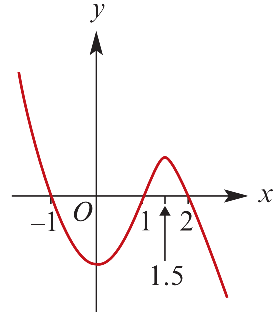
a



b



c



12 Let $f(x) = 3 + 6x^2 - 2x^3$. Determine the values of x for which the graph of $y = f(x)$ has a positive gradient.

13 For what value(s) of x do the graphs of $y = x^3$ and $y = x^3 + x^2 + x - 2$ have the same gradient?

CF

14 The graph of $y = bx^2 - cx$ crosses the x-axis at the point $(4, 0)$. The gradient at this point is 1. Find the values of b and c.

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Review

Multiple-choice questions

1 If $f(x) = \dfrac{4x^4 - 12x^2}{3x}$, then $f'(x)$ is equal to

A $\dfrac{16x^3 - 24x}{3}$ **B** $4x^2 - 4$ **C** $\dfrac{16x^3 - 24x}{3x}$ **D** $4x^2 - 8x$ **E** $\dfrac{8x^2 - 16x}{3x}$

2 Let $f(x) = x^5 + x^3 + x$. The value of $f'(1)$ is

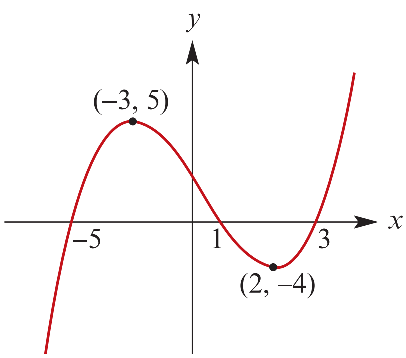
A 0 **B** 1 **C** 2 **D** 9 **E** −3

3 If $f(x) = \dfrac{3}{x}$, then $\dfrac{f(x+h) - f(x)}{h}$ is equal to

A $\dfrac{-3}{x(x+h)}$ **B** $\dfrac{3}{x^2}$ **C** $\dfrac{-3}{x^2}$ **D** $\dfrac{-3}{h(x+h)}$ **E** $f'(x)$

4 For the graph shown, the gradient is positive for

A $-3 < x < 2$
B $-3 \le x \le 2$
C $x < -3$ or $x > 2$
D $x \le -3$ or $x \ge 2$
E $-3 \le x \le 3$



5 If $f(x) = 4x(2 - 3x)$, then $f'(x) < 0$ for

A $x < \dfrac{1}{3}$ **B** $0 < x < \dfrac{2}{3}$ **C** $x = \dfrac{1}{3}$ **D** $x > \dfrac{1}{3}$ **E** $x = 0, \dfrac{2}{3}$

6 The point on the curve defined by the equation $y = (x + 3)(x - 2)$ at which the gradient is −7 has coordinates

A (−4, 6) **B** (−4, 0) **C** (−3, 0) **D** (−3, −5) **E** (−2, 0)

7 If $\dfrac{f(3+h) - f(3)}{h} = h^2 + 9h + 27$, then $f'(3)$ equals

A 3 **B** 9 **C** 18 **D** 27 **E** 63

8 The function $y = ax^2 - bx$ has zero gradient only for $x = 2$. The x-axis intercepts of the graph of this function are

A $\frac{1}{2}, -\frac{1}{2}$ **B** 0, 4 **C** 0, −4 **D** $0, \frac{1}{2}$ **E** $0, -\frac{1}{2}$

9 The function $f(x) = x^3 + 3x^2 - 9x + 7$ is increasing only when

A $x > 0$ **B** $-3 < x < 1$ **C** $x < -1$ or $x > 3$
D $x < -3$ or $x > 1$ **E** $-1 < x < 3$

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Chapter 8

8 Further differentiation and applications

Objectives

- To understand and use the **chain rule**.
- To differentiate **rational powers**.
- To differentiate **exponential functions** and **natural logarithmic functions**.
- To differentiate **trigonometric functions**.
- To understand and use the **product rule** and the **quotient rule**.
- To find the equations of the **tangent** and the **normal** at a given point on a curve.
- To apply differentiation in **rates of change** problems.
- To apply differentiation to **motion in a straight line**.
- To be able to find the **stationary points** on the graph of a function.
- To apply differentiation to **graph sketching**.

In this chapter, we continue our study of differentiation. We revise the chain rule, the product rule and the quotient rule, and we introduce differentiation of exponential, logarithmic and trigonometric functions.

The most important aspect of differential calculus is that it gives us the ability to consider how one quantity changes with respect to another. A useful example is the rate of change of position with respect to time for an object moving in a straight line. This rate of change is the velocity of the object.

In the final two sections of this chapter, we apply differentiation to graph sketching. We will develop further differentiation techniques to help with graph sketching in Chapter 12.

This chapter covers Unit 3 Topic 2: Further differentiation and applications 2.

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8A The chain rule

An expression such as $q(x) = (x^3 + 1)^2$ may be differentiated by expanding and then differentiating each term separately. This method is a great deal more tiresome for an expression such as $q(x) = (x^3 + 1)^{30}$.

We can express $q(x) = (x^3 + 1)^2$ as the composition of two simpler functions defined by

$$u = g(x) = x^3 + 1 \quad \text{and} \quad y = f(u) = u^2$$

which are 'chained' together:

$$x \xrightarrow{g} u \xrightarrow{f} y$$

That is, $q(x) = (x^3 + 1)^2 = f(g(x))$, and so q is expressed as the composition $f \circ g$.

The chain rule gives a method of differentiating such functions.

The chain rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $q(x) = f(g(x))$ is differentiable at x and

$$q'(x) = f'(g(x))\, g'(x)$$

Or using Leibniz notation, where $u = g(x)$ and $y = f(u)$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Proof To find the derivative of the composite function $q(x) = f(g(x))$ at $x = a$, consider the secant through the points $(a, f(g(a)))$ and $(a + h, f(g(a + h)))$. The gradient of this secant is

$$\frac{f(g(a+h)) - f(g(a))}{h}$$

We carry out the trick of multiplying the numerator and the denominator by $g(a + h) - g(a)$. This gives

$$\frac{f(g(a+h)) - f(g(a))}{h} \times \frac{g(a+h) - g(a)}{g(a+h) - g(a)}$$

provided $g(a + h) - g(a) \neq 0$.

Now write $b = g(a)$ and $b + k = g(a + h)$ so that $k = g(a + h) - g(a)$. The expression for the gradient becomes

$$\frac{f(b+k) - f(b)}{k} \times \frac{g(a+h) - g(a)}{h}$$

The function g is continuous, since its derivative exists, and therefore

$$\lim_{h \to 0} k = \lim_{h \to 0} \left[g(a+h) - g(a) \right] = 0$$

Thus, as h approaches 0, so does k. Hence $q'(a) = f'(g(a))\, g'(a)$.

Note that this proof does not hold for a function g such that $g(a + h) - g(a) = 0$ for arbitrarily chosen small h. However, a fully rigorous proof is beyond the scope of this course.

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Example 1

Differentiate $y = (4x^3 - 5x)^{-2}$.

Solution

The differentiation is undertaken using both notations:

Let $u = 4x^3 - 5x$

Then $y = u^{-2}$

We have

$$\frac{dy}{du} = -2u^{-3}$$

$$\frac{du}{dx} = 12x^2 - 5$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= -2u^{-3} \cdot (12x^2 - 5)$$
$$= \frac{-2(12x^2 - 5)}{(4x^3 - 5x)^3}$$

Let $h(x) = 4x^3 - 5x$

and $g(x) = x^{-2}$

Then $f(x) = g(h(x))$

We have

$$h'(x) = 12x^2 - 5$$

$$g'(x) = -2x^{-3}$$

Therefore

$$f'(x) = g'(h(x))\, h'(x)$$
$$= -2(h(x))^{-3}\, h'(x)$$
$$= -2(4x^3 - 5x)^{-3} \times (12x^2 - 5)$$
$$= \frac{-2(12x^2 - 5)}{(4x^3 - 5x)^3}$$

Example 2

Find the gradient of the tangent to the curve with equation $y = \dfrac{16}{3x^2 + 1}$ at the point $(1, 4)$.

Solution

Let $u = 3x^2 + 1$ then $y = 16u^{-1}$

So $\dfrac{du}{dx} = 6x$ and $\dfrac{dy}{du} = -16u^{-2}$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= -16u^{-2} \cdot 6x$$
$$= \frac{-96x}{(3x^2 + 1)^2}$$

$\therefore$ At $x = 1$, the gradient is $\dfrac{-96}{16} = -6$.

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Section summary

The chain rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $q(x) = f(g(x))$ is differentiable at x and

$$q'(x) = f'(g(x))\,g'(x)$$

Or using Leibniz notation, where $u = g(x)$ and $y = f(u)$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Exercise 8A

SF

Example 1

1 Differentiate each of the following with respect to x:

a $(x^2+1)^4$ **b** $(2x^2-3)^5$

c $(6x+1)^4$ **d** $(ax+b)^n$

e $(ax^2+b)^n$ **f** $(1-x^2)^{-3}$

g $\left(x^2-\dfrac{1}{x^2}\right)^{-3}$ **h** $(1-x)^{-1}$

2 Differentiate each of the following with respect to x:

a $(x^2+2x+1)^3$ **b** $(x^3+2x^2+x)^4$

c $\left(6x^3+\dfrac{2}{x}\right)^4$ **d** $(x^2+2x+1)^{-2}$

Example 2

3 Find the gradient of the tangent to the curve with equation $y = \dfrac{16}{3x^3+x}$ at the point $(1, 4)$.

4 Find the gradient of the tangent to the curve with equation $y = \dfrac{1}{x^2+1}$ at the points $(1, \frac{1}{2})$ and $(-1, \frac{1}{2})$.

CF

5 Given that $f'(x) = \sqrt{3x+4}$ and $g(x) = x^2 - 1$, find $F'(x)$ where $F(x) = f(g(x))$.

6 Differentiate each of the following with respect to x, giving the answer in terms of $f(x)$ and $f'(x)$:

a $[f(x)]^n$, where n is a positive integer

b $\dfrac{1}{f(x)}$, where $f(x) \neq 0$

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8B Differentiating rational powers

Before using the chain rule to differentiate rational powers, we will show how to differentiate $x^{\frac{1}{2}}$ and $x^{\frac{1}{3}}$ by first principles.

Example 3

Differentiate each of the following by first principles:

a $f(x) = x^{\frac{1}{2}},\ x > 0$ **b** $g(x) = x^{\frac{1}{3}},\ x \neq 0$

Solution

a
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\\ &= \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}\end{aligned}$$

$$\therefore\quad f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

b We use the identity

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

By observing that $\left(a^{\frac{1}{3}}\right)^3 = a$ and $\left(b^{\frac{1}{3}}\right)^3 = b$, we obtain

$$a - b = \left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\right)$$

and therefore

$$a^{\frac{1}{3}} - b^{\frac{1}{3}} = \frac{a-b}{a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}}$$

We now have

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h}\\ &= \frac{x+h-x}{h\left((x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}\right)}\\ &= \frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}}\end{aligned}$$

Hence

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}} = \frac{1}{3x^{\frac{2}{3}}}$$

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Note: We can prove that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$ for $n \geq 2$. We could use this result to find the derivative of $x^{\frac{1}{n}}$ by first principles, but instead we will use the chain rule.

Using the chain rule

Suppose that two variables x and y are related in such a way that $y = f(x)$ and $x = g(y)$. Then $y = f(g(y))$ and using the chain rule gives

$$1 = \frac{dy}{dy} = \frac{dy}{dx} \cdot \frac{dx}{dy}$$

Thus $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \text{for } \frac{dx}{dy} \neq 0$$

Now consider $y = x^{\frac{1}{n}}$, where $n \in \mathbb{Z} \setminus \{0\}$ and $x > 0$.

We have $x = y^n$ and so $\frac{dx}{dy} = ny^{n-1}$. Therefore

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{ny^{n-1}} = \frac{1}{n\left(x^{\frac{1}{n}}\right)^{n-1}} = \frac{1}{n}x^{\frac{1}{n}-1}$$

For $y = x^{\frac{1}{n}}$, $\frac{dy}{dx} = \frac{1}{n}x^{\frac{1}{n}-1}$, where $n \in \mathbb{Z} \setminus \{0\}$ and $x > 0$.

This result may now be extended to rational powers.

Let $y = x^{\frac{p}{q}}$, where $p, q \in \mathbb{Z} \setminus \{0\}$.

Write $y = \left(x^{\frac{1}{q}}\right)^p$. Let $u = x^{\frac{1}{q}}$. Then $y = u^p$. The chain rule yields

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= pu^{p-1} \cdot \frac{1}{q}x^{\frac{1}{q}-1} \\
&= p\left(x^{\frac{1}{q}}\right)^{p-1} \cdot \frac{1}{q}x^{\frac{1}{q}-1} \\
&= \frac{p}{q}x^{\frac{p}{q}-\frac{1}{q}}x^{\frac{1}{q}-1} \\
&= \frac{p}{q}x^{\frac{p}{q}-1}
\end{aligned}$$

Thus the result for integer powers has been extended to rational powers. In fact, the analogous result holds for any non-zero real power:

For $f(x) = x^a$, $f'(x) = ax^{a-1}$, where $a \in \mathbb{R} \setminus \{0\}$ and $x > 0$.

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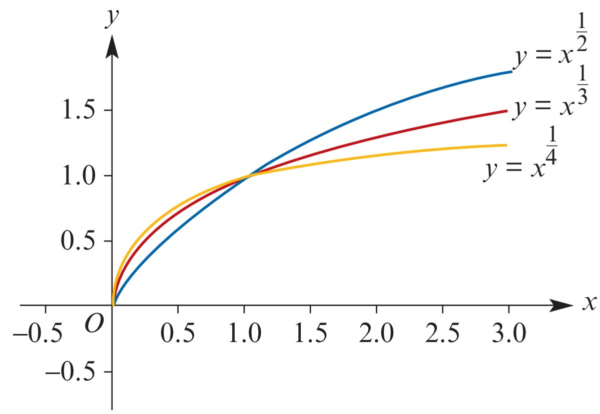
This result is stated for $x > 0$, as $(-3)^{\frac{1}{2}}$ is not defined, although $(-2)^{\frac{1}{3}}$ is defined.

The graphs of $y = x^{\frac{1}{2}}$, $y = x^{\frac{1}{3}}$ and $y = x^{\frac{1}{4}}$ are shown.

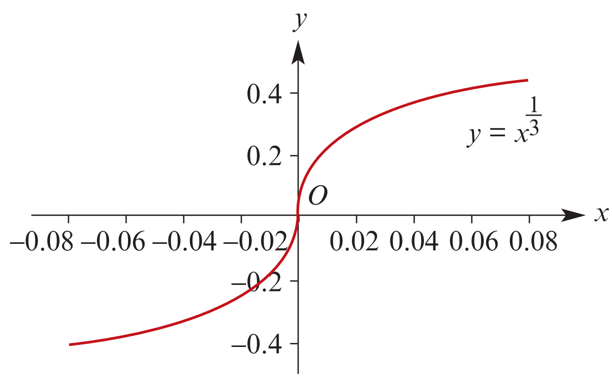
The domain of each has been taken to be $\mathbb{R}^+$.



The figure to the right is the graph of the function $f(x) = x^{\frac{1}{3}}$ for $x \in \mathbb{R}$.

Note that the values shown here are $-0.08 \leq x \leq 0.08$.

From this it can be seen that the tangent to $y = x^{\frac{1}{3}}$ at the origin is on the y-axis.



Example 4

Find the derivative of each of the following with respect to x:

a $2x^{-\frac{1}{5}} + 3x^{\frac{2}{7}}$

b $\sqrt[3]{x^2 + 2x}$

Solution

a
$$\frac{d}{dx}\left(2x^{-\frac{1}{5}} + 3x^{\frac{2}{7}}\right)$$
$$= 2\left(\frac{-1}{5}x^{-\frac{6}{5}}\right) + 3\left(\frac{2}{7}x^{-\frac{5}{7}}\right)$$
$$= -\frac{2}{5}x^{-\frac{6}{5}} + \frac{6}{7}x^{-\frac{5}{7}}$$

b
$$\frac{d}{dx}\left(\sqrt[3]{x^2 + 2x}\right)$$
$$= \frac{d}{dx}\left((x^2 + 2x)^{\frac{1}{3}}\right)$$
$$= \frac{1}{3}(x^2 + 2x)^{-\frac{2}{3}}(2x + 2) \qquad \text{(chain rule)}$$
$$= \frac{2x + 2}{3\sqrt[3]{(x^2 + 2x)^2}}$$

Section summary

For any non-zero rational number $r = \frac{p}{q}$, if $f(x) = x^r$, then $f'(x) = rx^{r-1}$.

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Exercise 8B

SF

Example 3 **1** Differentiate $2x^{\frac{1}{2}}$ by first principles.

Example 4a **2** Find the derivative of each of the following with respect to x:

a $x^{\frac{1}{5}}$ **b** $x^{\frac{5}{2}}$ **c** $x^{\frac{5}{2}} - x^{\frac{3}{2}}, \ x > 0$

d $3x^{\frac{1}{2}} - 4x^{\frac{5}{3}}$ **e** $x^{-\frac{6}{7}}$ **f** $x^{-\frac{1}{4}} + 4x^{\frac{1}{2}}$

3 Find the gradient of the tangent to the curve for each of the following at the stated value for x:

a $f(x) = x^{\frac{1}{3}}$ where $x = 27$ **b** $f(x) = x^{\frac{1}{3}}$ where $x = -8$

c $f(x) = x^{\frac{2}{3}}$ where $x = 27$ **d** $f(x) = x^{\frac{5}{4}}$ where $x = 16$

Example 4b **4** Find the derivative of each of the following with respect to x:

a $\sqrt{2x+1}$ **b** $\sqrt{4-3x}$ **c** $\sqrt{x^2+2}$

d $\sqrt[3]{4-3x}$ **e** $\dfrac{x^2+2}{\sqrt{x}}$ **f** $3\sqrt{x}\,(x^2+2x)$

5 **a** Show that $\dfrac{d}{dx}\left(\sqrt{x^2 \pm a^2}\right) = \dfrac{x}{\sqrt{x^2 \pm a^2}}$.

b Show that $\dfrac{d}{dx}\left(\sqrt{a^2 - x^2}\right) = \dfrac{-x}{\sqrt{a^2 - x^2}}$.

CF

6 If $y = (x + \sqrt{x^2+1})^2$, show that $\dfrac{dy}{dx} = \dfrac{2y}{\sqrt{x^2+1}}$.

SF

7 Find the derivative with respect to x of each of the following:

a $\sqrt{x^2+2}$ **b** $\sqrt[3]{x^2-5x}$ **c** $\sqrt[5]{x^2+2x}$

8C Differentiation of e^x

In this section we investigate the derivative of functions of the form $f(x) = a^x$. We will see that Euler's number e has the special property that $f'(x) = f(x)$ where $f(x) = e^x$.

First consider $f(x) = 2^x$.

To find the derivative of f we recall that:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{2^{x+h} - 2^x}{h} \\ &= 2^x \lim_{h\to 0} \frac{2^h - 1}{h} \\ &= 2^x f'(0) \end{aligned}$$

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We can investigate this limit numerically to find that $f'(0) \approx 0.6931$ and therefore

$$f'(x) \approx 0.6931 \times 2^x$$

Now consider $g(x) = 3^x$. Then, as for f, it may be shown that $g'(x) = 3^x g'(0)$. We find that $g'(0) \approx 1.0986$ and hence

$$g'(x) \approx 1.0986 \times 3^x$$

The question now arises:

> Can we find a number b between 2 and 3 such that, if $f(x) = b^x$, then $f'(0) = 1$ and therefore $f'(x) = b^x$?

Using a calculator or a spreadsheet, we can investigate the limit as $h \to 0$ of $\dfrac{b^h - 1}{h}$, for various values of b between 2 and 3.

This investigation is carried out in the spreadsheet shown on the right.

Start by taking values for b between 2.71 and 2.72 (first table) and finding $f'(0)$ for each of these values. From these results it may be seen that the required value of b lies between 2.718 and 2.719.

The investigation is continued with values of b between 2.718 and 2.719 (second table). From this the required value of b is seen to lie between 2.7182 and 2.7183.

b	$f'(0)$
2.710	0.996949
2.711	0.997318
2.712	0.997686
2.713	0.998055
2.714	0.998424
2.715	0.998792
2.716	0.999160
2.717	0.999528
2.718	0.999896
2.719	1.000264
2.720	1.000632

b	$f'(0)$
2.7180	0.999896
2.7181	0.999933
2.7182	0.999970
2.7183	1.000007
2.7184	1.000043
2.7185	1.000080
2.7186	1.000117
2.7187	1.000154
2.7188	1.000191
2.7189	1.000227
2.7190	1.000264

The required value of b is in fact Euler's number e, which was introduced in Chapter 6.

Our results can be recorded:

> If $f(x) = e^x$, then $f'(x) = e^x$.

Next consider $y = e^{kx}$ where $k \in \mathbb{R}$. The chain rule can be used to find the derivative:

Let $u = kx$. Then $y = e^u$. The chain rule yields

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^u \cdot k \\ &= ke^{kx}\end{aligned}$$

> If $f(x) = e^{kx}$, then $f'(x) = ke^{kx}$, where $k \in \mathbb{R}$.

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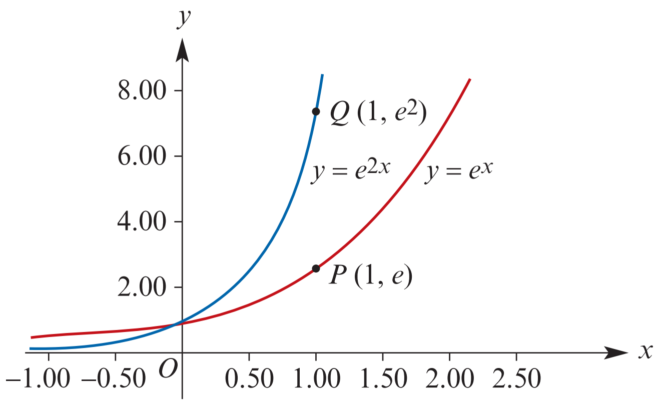
The graph illustrates the case where $k = 2$:

- the gradient of $y = e^x$ at the point $P(1, e)$ is e
- the gradient of $y = e^{2x}$ at the point $Q(1, e^2)$ is $2e^2$.



Example 5

Find the derivative of each of the following with respect to x:

a e^{3x} **b** e^{-2x} **c** e^{2x+1} **d** $\dfrac{1}{e^{2x}} + e^{3x}$

Solution

a Let $y = e^{3x}$. Then $\dfrac{dy}{dx} = 3e^{3x}$.

b Let $y = e^{-2x}$. Then $\dfrac{dy}{dx} = -2e^{-2x}$.

c Let $y = e^{2x+1}$. Then

$$y = e^{2x} \cdot e \qquad \text{(index laws)}$$
$$= e \cdot e^{2x}$$
$$\therefore \quad \frac{dy}{dx} = 2e \cdot e^{2x}$$
$$= 2e^{2x+1}$$

d Let $y = \dfrac{1}{e^{2x}} + e^{3x}$. Then

$$y = e^{-2x} + e^{3x}$$
$$\therefore \quad \frac{dy}{dx} = -2e^{-2x} + 3e^{3x}$$

Example 6

Find the derivative of each of the following with respect to x:

a e^{x^2} **b** e^{x^2+4x}

Solution

a Let $y = e^{x^2}$ and $u = x^2$.
Then $y = e^u$ and the chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= e^u \cdot 2x$$
$$= 2xe^{x^2}$$

b Let $y = e^{x^2+4x}$ and $u = x^2 + 4x$.
Then $y = e^u$ and the chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= e^u(2x + 4)$$
$$= (2x + 4)\, e^{x^2+4x}$$

In general, for $h(x) = e^{f(x)}$, the chain rule gives $h'(x) = f'(x)\, e^{f(x)}$.

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Example 7

Find the gradient of the tangent to the curve $y = e^{2x} + 4$ at the point:

a $(0, 5)$ **b** $(1, e^2 + 4)$

Solution

We have $\frac{dy}{dx} = 2e^{2x}$.

a When $x = 0$, $\frac{dy}{dx} = 2$.

The gradient at $(0, 5)$ is 2.

b When $x = 1$, $\frac{dy}{dx} = 2e^2$.

The gradient at $(1, e^2 + 4)$ is $2e^2$.

Example 8

For each of the following, first find the derivative with respect to x. Then evaluate the derivative at $x = 2$, given that $f(2) = 0$, $f'(2) = 4$ and $f'(e^2) = 5$.

a $e^{f(x)}$ **b** $f(e^x)$

Solution

a Let $y = e^{f(x)}$ and $u = f(x)$. Then $y = e^u$.

By the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= e^u f'(x)$$
$$= e^{f(x)} f'(x)$$

When $x = 2$, $\frac{dy}{dx} = e^0 \times 4 = 4$.

b Let $y = f(e^x)$ and $u = e^x$. Then $y = f(u)$.

By the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= f'(u) \cdot e^x$$
$$= f'(e^x) \cdot e^x$$

When $x = 2$, $\frac{dy}{dx} = f'(e^2) \cdot e^2 = 5e^2$.

Section summary

If $f(x) = e^{kx}$, then $f'(x) = ke^{kx}$, where $k \in \mathbb{R}$.

Exercise 8C

SF

Example 5

1 Find the derivative of each of the following with respect to x:

a e^{5x} **b** $7e^{-3x}$ **c** $3e^{-4x} + e^x - x^2$

d $\dfrac{e^{2x} - e^x + 1}{e^x}$ **e** $\dfrac{4e^{2x} - 2e^x + 1}{2e^{2x}}$ **f** $e^{2x} + e^4 + e^{-2x}$

Example 6

2 Find the derivative of each of the following with respect to x:

a e^{-2x^3} **b** $e^{x^2} + 3x + 1$ **c** $e^{x^2-4x} + 3x + 1$

d $e^{x^2-2x+3} - x$ **e** $e^{\frac{1}{x}}$, $x \neq 0$ **f** $e^{x^{\frac{1}{2}}}$

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Example 7

3 Find the gradient of the tangent to the curve $y = e^{\frac{x}{2}} + 4x$ at the point:

a $(0, 1)$ **b** $(1, e^{\frac{1}{2}} + 4)$

4 Find the gradient of the tangent to the curve $y = e^{x^2+3x} + 2x$ at the point:

a $(0, 1)$ **b** $(1, e^4 + 2)$

Example 8

5 Find the derivative with respect to x of:

a $e^{2f(x)}$ **b** $f(e^{2x})$

6 Find the derivative with respect to x of:

a $(e^{2x} - 1)^4$ **b** $e^{\sqrt{x}}$ **c** $\sqrt{e^x - 1}$

d $e^{x^{\frac{2}{3}}}$ **e** $e^{(x-1)(x-2)}$ **f** e^{e^x}

SF

8D Differentiation of the natural logarithm function

For the function with rule $f(x) = e^x$, we have seen that $f'(x) = e^x$.

This will be used to find the derivative of the function with rule $g(x) = \ln(kx)$, where k is a positive constant.

Let $y = \ln(kx)$ and solve for x:

$$e^y = kx$$

$$\therefore \quad x = \frac{1}{k}e^y$$

From our observation above:

$$\frac{dx}{dy} = \frac{1}{k}e^y$$

Since $e^y = kx$, this gives

$$\frac{dx}{dy} = \frac{kx}{k} = x$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x}$$

If $f(x) = \ln(kx)$, then $f'(x) = \frac{1}{x}$, where $k \in \mathbb{R}^+$.

Note: Here both the function f and its derivative f' have domain $\mathbb{R}^+$.

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Example 9

Find the derivative of each of the following with respect to x:

a $\ln(5x), \ x > 0$

b $\ln(5x + 3), \ x > -\frac{3}{5}$

Solution

a Let $y = \ln(5x)$ for $x > 0$.

Then $\frac{dy}{dx} = \frac{1}{x}$.

Alternatively, let $u = 5x$. Then $y = \ln u$ and the chain rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot 5 \\ &= \frac{5}{u} \\ &= \frac{1}{x}\end{aligned}$$

b Let $y = \ln(5x + 3)$ for $x > -\frac{3}{5}$.

Let $u = 5x + 3$. Then $y = \ln u$ and the chain rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{u} \cdot 5 \\ &= \frac{5}{u} \\ &= \frac{5}{5x + 3}\end{aligned}$$

In general, for $y = \ln(ax + b)$, the chain rule gives $\frac{dy}{dx} = \frac{a}{ax + b}$.

Example 10

Differentiate each of the following with respect to x:

a $\ln(x^2 + 2)$

b $(\ln x)^2, \ x > 0$

Solution

a We use the chain rule.

Let $y = \ln(x^2 + 2)$ and $u = x^2 + 2$.

Then $y = \ln u$.

$$\begin{aligned}\therefore \quad \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{u} \cdot 2x \\ &= \frac{2x}{x^2 + 2}\end{aligned}$$

b We use the chain rule.

Let $y = (\ln x)^2$ and $u = \ln x$.

Then $y = u^2$.

$$\begin{aligned}\therefore \quad \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 2u \cdot \frac{1}{x} \\ &= \frac{2 \ln x}{x}\end{aligned}$$

Section summary

- If $y = \ln(ax + b)$, then $\frac{dy}{dx} = \frac{a}{ax + b}$.
- If $h(x) = \ln(f(x))$, then the chain rule gives $h'(x) = \frac{f'(x)}{f(x)}$.

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Exercise 8D

Example 9

SF

1 Find the derivative of each of the following with respect to x:

a $y = 2\ln x$ **b** $y = 2\ln(2x)$ **c** $y = x^2 + 3\ln(2x)$

d $y = 3\ln x + \dfrac{1}{x}$ **e** $y = 3\ln(4x) + x$ **f** $y = \ln(x + 1)$

g $y = \ln(2x + 4)$ **h** $y = \ln(3x - 1)$ **i** $y = \ln(6x - 1)$

Example 10

2 Find the derivative of each of the following with respect to x:

a $y = \ln(x^3)$ **b** $y = (\ln x)^3$ **c** $y = \ln(x^2 + x - 1)$

d $y = \ln(x^3 + x^2)$ **e** $y = \ln((2x + 3)^2)$ **f** $y = \ln((3 - 2x)^2)$

3 For each of the following, find $f'(x)$:

a $f(x) = \ln(x^2 + 1)$ **b** $f(x) = \ln(e^x)$

4 Find the y-coordinate and the gradient of the tangent to the curve at the point corresponding to the given value of x:

a $y = \ln x$, $x > 0$, at $x = e$ **b** $y = \ln(x^2 + 1)$ at $x = e$

c $y = \ln(-x)$, $x < 0$, at $x = -e$ **d** $y = x + \ln x$ at $x = 1$

e $y = \ln(x^2 - 2x + 2)$ at $x = 1$ **f** $y = \ln(2x - 1)$ at $x = \frac{3}{2}$

5 Find $f'(1)$ if $f(x) = \ln(\sqrt{x^2 + 1})$.

6 Differentiate $\ln(1 + x + x^2)$.

7 If $f(x) = \ln(x^2 + 1)$, find $f'(3)$.

CF

8 Given that $f(0) = 2$ and $f'(0) = 4$, find $\dfrac{d}{dx}\big(\ln(f(x))\big)$ when $x = 0$.

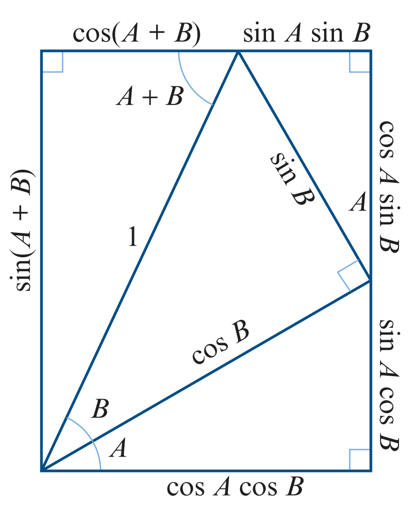
8E Differentiation of trigonometric functions

In this section we find the derivatives of the sine and cosine functions.

In the proof for the derivative of sine, we will use the angle-sum identity

$$\sin(A + B) = \sin A\ \cos B + \cos A\ \sin B$$

You may have seen this identity in Specialist Mathematics Units 1 & 2. The diagram gives a geometric proof of the identity in the special case where the angles A, B and $A + B$ are acute.



Note: The derivatives that we establish for $\sin x$ and $\cos x$ only apply when the angle x is measured in radians.

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▶ The derivative of sin(*ax* + *b*)

If $f(x) = \sin x$, then $f'(x) = \cos x$.

Proof Consider points $P(x, \sin x)$ and $Q(x + h, \sin(x + h))$ on the graph of $f(x) = \sin x$. The gradient of the secant PQ is

$$\frac{\sin(x+h) - \sin x}{h} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$
$$= \frac{\sin x \cdot (\cos h - 1)}{h} + \frac{\cos x \sin h}{h}$$

We now consider what happens as $h \to 0$. We use two limit results (the second limit is proved below and the first limit then follows using a trigonometric identity):

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0 \quad \text{and} \quad \lim_{h \to 0} \frac{\sin h}{h} = 1$$

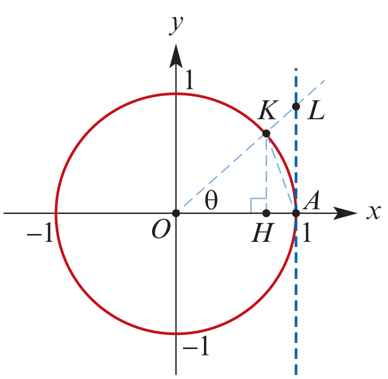
Therefore

$$f'(x) = \lim_{h \to 0} \left(\frac{\sin x \cdot (\cos h - 1)}{h} + \frac{\cos x \sin h}{h} \right)$$
$$= \sin x \times 0 + \cos x \times 1$$
$$= \cos x$$

We now prove the following result.

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

Proof Let K be a point on the unit circle as shown, and let $\angle KOH = \theta$. The coordinates of K are $(\cos\theta, \sin\theta)$. Point H is on the x-axis such that $\angle KHO$ is a right angle.



Draw a tangent to the circle at $A(1, 0)$. The line OK intersects this tangent at $L(1, \tan\theta)$.

The area of sector OAK is $\frac{1}{2}\theta$.

Thus area $\triangle OAK \le \frac{1}{2}\theta \le$ area $\triangle OAL$

i.e. $\frac{1}{2}OA \cdot HK \le \frac{1}{2}\theta \le \frac{1}{2}OA \cdot AL$

This implies that $\sin\theta \le \theta \le \tan\theta$.

For $0 < \theta < \frac{\pi}{2}$, we have $\sin\theta > 0$, and so we can divide both inequalities by $\sin\theta$ to obtain

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

As θ approaches 0, the value of $\cos\theta$ approaches 1, and so $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

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We now turn our attention to the function $f(x) = \sin(ax + b)$, where a and b are constants. We use the chain rule to determine $f'(x)$.

Let $y = \sin(ax + b)$ and let $u = ax + b$. Then $y = \sin u$ and therefore

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot a = a\cos(ax + b)$$

If $f(x) = \sin(ax + b)$, then $f'(x) = a\cos(ax + b)$.

The derivative of $\cos(ax + b)$

We next find the derivative of $\cos(ax + b)$. We first note the following:

$$\cos x = \sin\left(\frac{\pi}{2} - x\right) \quad \text{and} \quad \sin x = \cos\left(\frac{\pi}{2} - x\right)$$

These results will be used in the following way.

Let $y = \cos x = \sin\left(\frac{\pi}{2} - x\right)$ and let $u = \frac{\pi}{2} - x$. Then $y = \sin u$.

The chain rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot (-1)\\ &= -\cos\left(\frac{\pi}{2} - x\right)\\ &= -\sin x\end{aligned}$$

We have the following results:

- If $f(x) = \cos x$, then $f'(x) = -\sin x$.
- If $f(x) = \cos(ax + b)$, then $f'(x) = -a\sin(ax + b)$.

Example 11

Find the derivative with respect to x of each of the following:

a $\sin(2x)$ **b** $\sin^2(2x)$ **c** $\cos^3(4x + 1)$

Solution

a Let $y = \sin(2x)$. Then $\frac{dy}{dx} = 2\cos(2x)$.

b Let $y = \sin^2(2x)$ and $u = \sin(2x)$.
Then $y = u^2$. Using the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= 2u \cdot 2\cos(2x)\\ &= 4u\cos(2x)\\ &= 4\sin(2x)\cos(2x)\end{aligned}$$

c Let $y = \cos^3(4x + 1)$ and $u = \cos(4x + 1)$.
Then $y = u^3$. Using the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= 3u^2 \cdot (-4)\sin(4x + 1)\\ &= -12u^2\sin(4x + 1)\\ &= -12\cos^2(4x + 1)\sin(4x + 1)\end{aligned}$$

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Example 12

Consider the curve $y = \cos x$. Find the y-coordinate and the gradient of the tangent at the point on this curve where:

a $x = \frac{\pi}{4}$

b $x = \frac{\pi}{2}$

Solution

Let $y = \cos x$. Then $\frac{dy}{dx} = -\sin x$.

a When $x = \frac{\pi}{4}$, we have

$$y = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\frac{dy}{dx} = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

So the gradient at $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ is $-\frac{1}{\sqrt{2}}$.

b When $x = \frac{\pi}{2}$, we have

$$y = \cos\left(\frac{\pi}{2}\right) = 0$$

$$\frac{dy}{dx} = -\sin\left(\frac{\pi}{2}\right) = -1$$

So the gradient at $\left(\frac{\pi}{2}, 0\right)$ is -1.

Section summary

- If $f(x) = \sin(ax + b)$, then $f'(x) = a\cos(ax + b)$.
- If $f(x) = \cos(ax + b)$, then $f'(x) = -a\sin(ax + b)$.

Exercise 8E

SF

Example 11

1 Find the derivative with respect to x of each of the following:

a $\sin(5x)$ **b** $\cos(5x)$ **c** $\sin^2 x$ **d** $\cos(x^2 + 1)$

e $\sin^2\left(x - \frac{\pi}{4}\right)$ **f** $\cos^2\left(x - \frac{\pi}{3}\right)$ **g** $\sin^3\left(2x + \frac{\pi}{6}\right)$ **h** $\cos^3\left(2x - \frac{\pi}{4}\right)$

Example 12

2 Find the y-coordinate and the gradient of the tangent at the points on the following curves corresponding to the given values of x:

a $y = \sin(2x)$ at $x = \frac{\pi}{8}$ **b** $y = \sin(3x)$ at $x = \frac{\pi}{6}$ **c** $y = 1 + \sin(3x)$ at $x = \frac{\pi}{6}$

d $y = \cos^2(2x)$ at $x = \frac{\pi}{4}$ **e** $y = \sin^2(2x)$ at $x = \frac{\pi}{4}$ **f** $y = \sin(4x)$ at $x = \frac{\pi}{4}$

3 For each of the following, find $f'(x)$:

a $f(x) = 5\cos x - 2\sin(3x)$ **b** $f(x) = \cos x + \sin x$

4 Find the derivative of each of the following. (Change degrees to radians first.)

a $2\cos x^\circ$ **b** $3\sin x^\circ$ **c** $\sin(3x)^\circ$

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5 Find $\dfrac{dy}{dx}$ for each of the following:

a $y = -\ln(\cos x)$ **b** $y = -\ln(\sin x)$ **c** $y = e^{2\sin x}$ **d** $y = e^{\cos(2x)}$

SF

8F The product rule

In the next two sections, we revise two more rules for differentiation. The first of these is the product rule.

Let $f(x) = u(x) \cdot v(x)$. If $u'(x)$ and $v'(x)$ exist, then

$$f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$$

For example, consider $f(x) = (x^2 + 3x)(4x + 5)$. Then f is the product of two functions u and v, where $u(x) = x^2 + 3x$ and $v(x) = 4x + 5$. The product rule gives:

$$\begin{aligned} f'(x) &= u(x) \cdot v'(x) + v(x) \cdot u'(x) \\ &= (x^2 + 3x) \cdot 4 + (4x + 5) \cdot (2x + 3) \\ &= 4x^2 + 12x + 8x^2 + 22x + 15 \\ &= 12x^2 + 34x + 15 \end{aligned}$$

This could also have been found by expanding the brackets and then differentiating.

The product rule (using function notation)

Let $f(x) = u(x) \cdot v(x)$. If $u'(x)$ and $v'(x)$ exist, then

$$f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$$

Proof By the definition of the derivative of f, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{u(x+h)\, v(x+h) - u(x)\, v(x)}{h} \end{aligned}$$

Adding and subtracting $u(x+h)\, v(x)$:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{u(x+h)\, v(x+h) - u(x)\, v(x) + [u(x+h)\, v(x) - u(x+h)\, v(x)]}{h} \\ &= \lim_{h \to 0} \left[u(x+h) \cdot \left(\frac{v(x+h) - v(x)}{h} \right) + v(x) \cdot \left(\frac{u(x+h) - u(x)}{h} \right) \right] \end{aligned}$$

Since u and v are differentiable, we obtain

$$\begin{aligned} f'(x) &= \lim_{h \to 0} u(x+h) \cdot \lim_{h \to 0} \left(\frac{v(x+h) - v(x)}{h} \right) + \lim_{h \to 0} v(x) \cdot \lim_{h \to 0} \left(\frac{u(x+h) - u(x)}{h} \right) \\ &= u(x) \cdot v'(x) + v(x) \cdot u'(x) \end{aligned}$$

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We can state the product rule in Leibniz notation and give a geometric interpretation.

The product rule (using Leibniz notation)

If $y = uv$, where u and v are functions of x, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

In the following figure, the white region represents $y = uv$ and the shaded region δy, as explained below.

δv	$u\delta v$	$\delta u\delta v$
v	uv	$v\delta u$
	u	δu

$$\begin{aligned}\delta y &= (u + \delta u)(v + \delta v) - uv\\ &= uv + u\delta v + v\delta u + \delta u\delta v - uv\\ &= u\delta v + v\delta u + \delta u\delta v\end{aligned}$$

$$\therefore\quad \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\frac{\delta v}{\delta x}\delta x$$

In the limit, as $\delta x \to 0$, we have

$$\frac{\delta u}{\delta x} = \frac{du}{dx}, \qquad \frac{\delta v}{\delta x} = \frac{dv}{dx} \qquad \text{and} \qquad \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

Therefore

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example 13

Differentiate each of the following with respect to x:

a $(2x^2 + 1)(5x^3 + 16)$

b $x^3(3x - 5)^4$

Solution

a Let $y = (2x^2 + 1)(5x^3 + 16)$. Let $u = 2x^2 + 1$ and $v = 5x^3 + 16$.

Then $\dfrac{du}{dx} = 4x$ and $\dfrac{dv}{dx} = 15x^2$.

The product rule gives:

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= (2x^2 + 1)\cdot 15x^2 + (5x^3 + 16)\cdot 4x\\ &= 30x^4 + 15x^2 + 20x^4 + 64x\\ &= 50x^4 + 15x^2 + 64x\end{aligned}$$

ISBN 978-1-108-45164-2

Cambridge University Press

b Let $y = x^3(3x-5)^4$. Let $u = x^3$ and $v = (3x-5)^4$.

Then $\frac{du}{dx} = 3x^2$ and $\frac{dv}{dx} = 12(3x-5)^3$ using the chain rule.

The product rule gives:

$$\begin{aligned}\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} &= 12x^3(3x-5)^3 + (3x-5)^4 \cdot 3x^2\\ &= (3x-5)^3[12x^3 + 3x^2(3x-5)]\\ &= (3x-5)^3[12x^3 + 9x^3 - 15x^2]\\ &= (3x-5)^3(21x^3 - 15x^2)\\ &= 3x^2(7x-5)(3x-5)^3\end{aligned}$$

Example 14

For $f(x) = x^{-3}(10x^2-5)^3$, find $f'(x)$.

Solution

Let $u(x) = x^{-3}$ and $v(x) = (10x^2-5)^3$.

Then $u'(x) = -3x^{-4}$ and $v'(x) = 60x(10x^2-5)^2$ using the chain rule.

By the product rule:

$$\begin{aligned}f'(x) &= x^{-3} \cdot 60x(10x^2-5)^2 + (10x^2-5)^3 \cdot (-3x^{-4})\\ &= (10x^2-5)^2\,[60x^{-2} + (10x^2-5)\cdot(-3x^{-4})]\\ &= (10x^2-5)^2\left(\frac{60x^2 - 30x^2 + 15}{x^4}\right)\\ &= \frac{(10x^2-5)^2(30x^2+15)}{x^4}\end{aligned}$$

Example 15

Differentiate each of the following with respect to x:

a $e^x(2x^2+1)$

b $e^x\sqrt{x-1}$

Solution

a Use the product rule.

Let $y = e^x(2x^2+1)$. Then

$$\begin{aligned}\frac{dy}{dx} &= e^x(2x^2+1) + 4xe^x\\ &= e^x(2x^2+4x+1)\end{aligned}$$

b Use the product rule and the chain rule.

Let $y = e^x\sqrt{x-1}$. Then

$$\begin{aligned}\frac{dy}{dx} &= e^x\sqrt{x-1} + \frac{1}{2}e^x(x-1)^{-\frac{1}{2}}\\ &= e^x\sqrt{x-1} + \frac{e^x}{2\sqrt{x-1}}\\ &= \frac{2e^x(x-1)+e^x}{2\sqrt{x-1}}\\ &= \frac{2xe^x - e^x}{2\sqrt{x-1}}\end{aligned}$$

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Example 16

Find the derivative of each of the following with respect to x:

a $2x^2 \sin(2x)$ **b** $e^{2x} \sin(2x + 1)$ **c** $\cos(4x) \sin(2x)$

Solution

a Let $y = 2x^2 \sin(2x)$.

Applying the product rule:

$$\frac{dy}{dx} = 4x \sin(2x) + 4x^2 \cos(2x)$$

b Let $y = e^{2x} \sin(2x + 1)$.

Applying the product rule:

$$\begin{aligned}\frac{dy}{dx} &= 2e^{2x} \sin(2x + 1) + 2e^{2x} \cos(2x + 1) \\ &= 2e^{2x} [\sin(2x + 1) + \cos(2x + 1)]\end{aligned}$$

c Let $y = \cos(4x) \sin(2x)$. Then the product rule gives

$$\frac{dy}{dx} = -4 \sin(4x) \sin(2x) + 2 \cos(2x) \cos(4x)$$

Section summary

- **The product rule (using function notation)**
  If $h(x) = f(x)g(x)$, then $h'(x) = f(x)g'(x) + f'(x)g(x)$.
- **The product rule (using Leibniz notation)**
  If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

Exercise 8F

SF

Example 13, 14

1 Find the derivative of each of the following with respect to x, using the product rule:

a $(2x^2 + 6)(2x^3 + 1)$ **b** $3x^{\frac{1}{2}}(2x + 1)$ **c** $3x(2x - 1)^3$

d $4x^2(2x^2 + 1)^2$ **e** $(3x + 1)^{\frac{3}{2}}(2x + 4)$ **f** $(x^2 + 1)\sqrt{2x - 4}$

g $x^3(3x^2 + 2x + 1)^{-1}$ **h** $x^4\sqrt{2x^2 - 1}$ **i** $x^2\sqrt[3]{x^2 + 2x}$

j $x^{-2}(5x^2 - 4)^3$ **k** $x^{-3}(x^3 - 4)^2$ **l** $x^3\sqrt[5]{x^3 - x}$

Example 15

2 Find $f'(x)$ for each of the following:

a $f(x) = e^x(x^2 + 1)$ **b** $f(x) = e^{2x}(x^3 + 3x + 1)$

c $f(x) = e^{4x+1}(x + 1)^2$ **d** $f(x) = e^{-4x}\sqrt{x + 1}, \ x \geq -1$

3 For each of the following, find $f'(x)$:

a $f(x) = x \ln x, \ x > 0$ **b** $f(x) = 2x^2 \ln x, \ x > 0$

c $f(x) = e^x \ln x, \ x > 0$ **d** $f(x) = x \ln(-x), \ x < 0$

4 Differentiate each of the following with respect to x:

a $x^4 e^{-2x}$ **b** e^{2x+3} **c** $(e^{2x} + x)^{\frac{3}{2}}$ **d** $\dfrac{1}{x}e^x$

e $e^{\frac{1}{2}x^2}$ **f** $(x^2 + 2x + 2)e^{-x}$

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SF

5 Find each of the following:

a $\dfrac{d}{dx}\left(e^x f(x)\right)$ **b** $\dfrac{d}{dx}\left(\dfrac{e^x}{f(x)}\right)$ **c** $\dfrac{d}{dx}\left(e^{f(x)}\right)$ **d** $\dfrac{d}{dx}\left(e^x (f(x))^2\right)$

Example 16

6 Differentiate each of the following with respect to x:

a $x^3 \cos x$ **b** $(1 + x^2)\cos x$ **c** $e^{-x}\sin x$ **d** $6x\cos x$

e $\sin(3x)\cos(4x)$ **f** $12x\sin x$ **g** $x^2 e^{\sin x}$ **h** $x^2\cos^2 x$

7 For each of the following, find $f'(\pi)$:

a $f(x) = e^x \sin x$ **b** $f(x) = \cos^2(2x)$

CF

8 Given that $f(1) = 2$ and $f'(1) = 4$, find the derivative of $f(x)\ln(x)$ when $x = 1$.

8G The quotient rule

Let $f(x) = \dfrac{u(x)}{v(x)}$, where $v(x) \neq 0$. If $u'(x)$ and $v'(x)$ exist, then

$$f'(x) = \frac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{[v(x)]^2}$$

For example, if

$$f(x) = \frac{x^3 + 2x}{x^5 + 2}$$

then f can be considered as a quotient of two functions u and v, where $u(x) = x^3 + 2x$ and $v(x) = x^5 + 2$. The quotient rule gives

$$\begin{aligned} f'(x) &= \frac{(x^5+2)(3x^2+2) - (x^3+2x)5x^4}{(x^5+2)^2} \\ &= \frac{3x^7 + 6x^2 + 2x^5 + 4 - 5x^7 - 10x^5}{(x^5+2)^2} \\ &= \frac{-2x^7 - 8x^5 + 6x^2 + 4}{(x^5+2)^2} \end{aligned}$$

The quotient rule (using function notation)

Let $f(x) = \dfrac{u(x)}{v(x)}$, where $v(x) \neq 0$. If $u'(x)$ and $v'(x)$ exist, then

$$f'(x) = \frac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{[v(x)]^2}$$

Proof The quotient rule can be proved from first principles, but instead we will use the product rule and the chain rule.

We can write $f(x) = u(x)\cdot g(x)$, where $g(x) = [v(x)]^{-1}$. Using the chain rule, we have

$$g'(x) = -[v(x)]^{-2}\cdot v'(x)$$

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Therefore, using the product rule, we obtain

$$\begin{aligned} f'(x) &= u(x) \cdot g'(x) + g(x) \cdot u'(x) \\ &= -u(x) \cdot [v(x)]^{-2} \cdot v'(x) + [v(x)]^{-1} \cdot u'(x) \\ &= [v(x)]^{-2}\big(-u(x) \cdot v'(x) + v(x) \cdot u'(x)\big) \\ &= \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2} \end{aligned}$$

The quotient rule (using Leibniz notation)

If $y = \frac{u}{v}$, where u and v are functions of x and $v \neq 0$, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 17

Find the derivative of $\frac{x-2}{x^2+4x+1}$ with respect to x.

Solution

Let $y = \frac{x-2}{x^2+4x+1}$. The quotient rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2+4x+1-(x-2)(2x+4)}{(x^2+4x+1)^2} \\ &= \frac{x^2+4x+1-(2x^2-8)}{(x^2+4x+1)^2} \\ &= \frac{-x^2+4x+9}{(x^2+4x+1)^2} \end{aligned}$$

Example 18

Differentiate each of the following with respect to x:

a $\frac{e^x}{e^{2x}+1}$

b $\frac{\sin x}{x+1}$, $x \neq -1$

Solution

a Let $y = \frac{e^x}{e^{2x}+1}$.

Applying the quotient rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{(e^{2x}+1)e^x - e^x \cdot 2e^{2x}}{(e^{2x}+1)^2} \\ &= \frac{e^{3x}+e^x-2e^{3x}}{(e^{2x}+1)^2} \\ &= \frac{e^x-e^{3x}}{(e^{2x}+1)^2} \end{aligned}$$

b Let $y = \frac{\sin x}{x+1}$ for $x \neq -1$.

Applying the quotient rule:

$$\frac{dy}{dx} = \frac{(x+1)\cos x - \sin x}{(x+1)^2}$$

ISBN 978-1-108-45164-2

Section summary

- The quotient rule (using function notation)

  If $h(x) = \dfrac{f(x)}{g(x)}$, then $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.

- The quotient rule (using Leibniz notation)

  If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

Exercise 8G

SF

Example 17

1 Find the derivative of each of the following with respect to x:

a $\dfrac{x}{x+4}$ **b** $\dfrac{x^2-1}{x^2+1}$ **c** $\dfrac{x^{\frac{1}{2}}}{1+x}$ **d** $\dfrac{(x+2)^3}{x^2+1}$

e $\dfrac{x-1}{x^2+2}$ **f** $\dfrac{x^2+1}{x^2-1}$ **g** $\dfrac{3x^2+2x+1}{x^2+x+1}$ **h** $\dfrac{2x+1}{2x^3+2x}$

2 Find the y-coordinate and the gradient at the point on the curve corresponding to the given value of x:

a $y = (2x+1)^4x^2$ at $x = 1$ **b** $y = x^2\sqrt{x+1}$ at $x = 0$ **c** $y = x^2(2x+1)^{\frac{1}{2}}$ at $x = 0$

d $y = \dfrac{x}{x^2+1}$ at $x = 1$ **e** $y = \dfrac{2x+1}{x^2+1}$ at $x = 1$

3 For each of the following, find $f'(x)$:

a $f(x) = (x+1)\sqrt{x^2+1}$ **b** $f(x) = (x^2+1)\sqrt{x^3+1}, \ x > -1$

c $f(x) = \dfrac{2x+1}{x+3}, \ x \neq -3$

Example 18

4 For each of the following, find $f'(x)$:

a $f(x) = \dfrac{e^x}{e^{3x}+3}$ **b** $f(x) = \dfrac{\cos x}{x+1}, \ x \neq -1$ **c** $f(x) = \dfrac{\ln x}{x+1}, \ x > 0$

5 For each of the following, find $f'(x)$:

a $f(x) = \dfrac{\ln x}{x}, \ x > 0$ **b** $f(x) = \dfrac{\ln x}{x^2+1}, \ x > 0$

6 Find $f'(x)$ for each of the following:

a $f(x) = \dfrac{e^{3x}}{e^{3x}+3}$ **b** $f(x) = \dfrac{e^x+1}{e^x-1}$ **c** $f(x) = \dfrac{e^{2x}+2}{e^{2x}-2}$

7 For each of the following, find $f'(\pi)$:

a $f(x) = \dfrac{2x}{\cos x}$ **b** $f(x) = \dfrac{3x^2+1}{\cos x}$ **c** $f(x) = \dfrac{e^x}{\cos x}$ **d** $f(x) = \dfrac{\sin x}{x}$

CF

8 Derivative of tangent Let $f(x) = \tan x = \dfrac{\sin x}{\cos x}$. Show that $f'(x) = \dfrac{1}{\cos^2 x}$.

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8H Tangents and normals

The derivative of a function is a new function that gives the measure of the gradient of the tangent at each point on the curve. Having the gradient, we can find the equation of the tangent line at a given point on the curve.

Suppose that (x_1, y_1) is a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent at (x_1, y_1) is given by

$$y - y_1 = f'(x_1)(x - x_1)$$

Example 19

Find the equation of the tangent to the curve $y = x^3 + \frac{1}{2}x^2$ at the point $x = 1$.

Solution

When $x = 1$, $y = \frac{3}{2}$, and so $\left(1, \frac{3}{2}\right)$ is a point on the tangent.

Since $\frac{dy}{dx} = 3x^2 + x$, the gradient of the tangent at $x = 1$ is 4.

Hence the equation of the tangent is

$$y - \frac{3}{2} = 4(x - 1)$$

i.e. $$y = 4x - \frac{5}{2}$$

The **normal** to a curve at a point on the curve is the line that passes through the point and is perpendicular to the tangent at that point.

Recall from Chapter 2 that two lines with gradients m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.

Thus, if a tangent has gradient m, the normal has gradient $-\frac{1}{m}$.

Example 20

Find the equation of the normal to the curve with equation $y = x^3 - 2x^2$ at the point $(1, -1)$.

Solution

The point $(1, -1)$ is on the normal.

Since $\frac{dy}{dx} = 3x^2 - 4x$, the gradient of the normal at $x = 1$ is $\frac{-1}{-1} = 1$.

Hence the equation of the normal is

$$y - (-1) = 1(x - 1)$$

i.e. $$y = x - 2$$

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Example 21

Find the equation of the tangent to the curve with equation $y = x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$ at the point on the graph where $x = 4$.

Solution

Let $y = x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$. Then $\dfrac{dy}{dx} = \dfrac{3}{2}x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$.

When $x = 4$,

$$y = 4^{\frac{3}{2}} - 4 \times 4^{\frac{1}{2}} = 0$$

and $$\frac{dy}{dx} = \frac{3}{2} \times 4^{\frac{1}{2}} - 2 \times 4^{-\frac{1}{2}} = 2$$

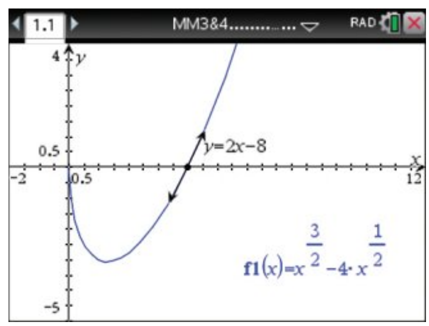
Hence the equation of the tangent is

$$y - 0 = 2(x - 4)$$

i.e. $$y = 2x - 8$$

Using the TI-Nspire CX non-CAS

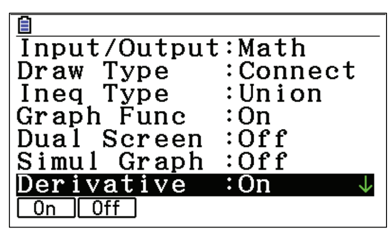
- In a **Graphs** application, plot the graph of $f1(x) = x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$.
- To add a tangent line to the graph, use (menu) > **Geometry** > **Points & Lines** > **Tangent**. Click on the curve and press (( ). Then type in the x-value 4 and press (enter).
- Hence the tangent line at $x = 4$ has the equation $y = 2x - 8$.



Using the Casio

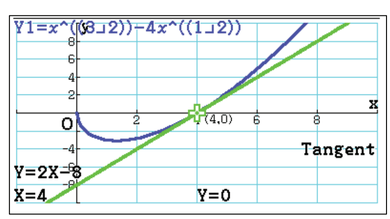
To change the derivative setting:

- In **Graph** mode, go to the set-up screen by pressing (SHIFT) (MENU).
- Use the cursor key ▼ to move down to **Derivative**. Press (F1) to change the setting to **On**. Then press (EXIT).



To draw a tangent line on a graph:

- Plot the graph of $y = x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$.
- Select **Sketch** (SHIFT) (F4), then **Tangent** (F2).
- Enter the x-value 4 and press (EXE) twice.
- Hence the tangent line at $x = 4$ has the equation $y = 2x - 8$.



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Example 22

Find the equation of the tangent to the graph of $y = \sin x$ at the point where $x = \frac{\pi}{3}$.

Solution

Let $y = \sin x$. Then $\frac{dy}{dx} = \cos x$. When $x = \frac{\pi}{3}$, $y = \frac{\sqrt{3}}{2}$ and $\frac{dy}{dx} = \frac{1}{2}$.

Therefore the equation of the tangent is

$$y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

i.e. $$y = \frac{x}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

Example 23

Find the equations of the tangent and normal to the graph of $y = -\cos x$ at the point $\left(\frac{\pi}{2}, 0\right)$.

Solution

First find the gradient of the curve at this point:

$\frac{dy}{dx} = \sin x$ and so, when $x = \frac{\pi}{2}$, $\frac{dy}{dx} = 1$.

The equation of the tangent is

$$y - 0 = 1\left(x - \frac{\pi}{2}\right)$$

i.e. $$y = x - \frac{\pi}{2}$$

The gradient of the normal is -1 and therefore the equation of the normal is

$$y - 0 = -1\left(x - \frac{\pi}{2}\right)$$

i.e. $$y = -x + \frac{\pi}{2}$$

The following example shows two situations in which we can view a graph as having a 'vertical tangent line' at a point where the derivative is not defined.

Example 24

Find the equation of the tangent to:

a $f(x) = x^{\frac{1}{3}}$ where $x = 0$ **b** $f(x) = x^{\frac{2}{3}}$ where $x = 0$.

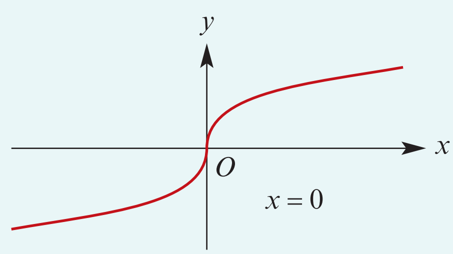
Solution

a The derivative of f is not defined at $x = 0$.

For $x \in \mathbb{R} \setminus \{0\}$, $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$.

It is clear that f is continuous at $x = 0$ and that $f'(x) \to \infty$ as $x \to 0$.

The graph has a **vertical tangent** at $x = 0$.



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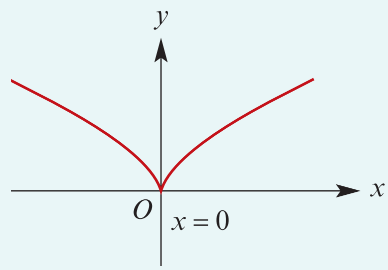
b $f(x) = x^{\frac{2}{3}}$

The derivative of f is not defined at $x = 0$.

For $x \in \mathbb{R} \setminus \{0\}$, $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$.

It is clear that f is continuous at $x = 0$ and that $f'(x) \to \infty$ as $x \to 0$ from the right and $f'(x) \to -\infty$ as $x \to 0$ from the left.

There is a **cusp** at $x = 0$, and the graph of $y = f(x)$ has a **vertical tangent** at $x = 0$.



Section summary

- Equation of a tangent line
  Suppose (x_1, y_1) is a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent to the curve at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.
- Gradient of a normal line
  If a tangent has gradient m, the normal has gradient $-\frac{1}{m}$.

Exercise 8H

SF

Example 19 **1** Find the equation of the tangent to the curve $y = x^2 - 1$ at the point $(2, 3)$.

Example 20 **2** Find the equation of the normal to the curve $y = x^2 + 3x - 1$ at the point where the curve cuts the y-axis.

3 Find the equations of the normals to the curve $y = x^2 - 5x + 6$ at the points where it cuts the x-axis.

4 Find the equations of the tangent and the normal to the curve $y = (2x + 1)^9$ at the point $(0, 1)$.

5 Find the coordinates of the point on $y = x^2 - 5$ at which the curve has gradient 3. Hence find the value of c for which the line $y = 3x + c$ is tangent to $y = x^2 - 5$.

6 Find the equations of **i** the tangent and **ii** the normal at the point corresponding to the given x-value on each of the following curves:

a $y = x^2 - 2;\ x = 1$

b $y = x^2 - 3x - 1;\ x = 0$

c $y = \frac{1}{x};\ x = -1$

d $y = (x - 2)(x^2 + 1);\ x = -1$

e $y = \sqrt{3x + 1};\ x = 0$

f $y = \sqrt{x};\ x = 1$

g $y = x^{\frac{2}{3}} + 1;\ x = 1$

h $y = x^3 - 8x;\ x = 2$

i $y = x^3 - 3x^2 + 2;\ x = 2$

j $y = 2x^3 + x^2 - 4x + 1;\ x = 1$

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Example 21

7 Use a graphics calculator to find the equation of the tangent to the curve with equation $y = 4x^{\frac{5}{2}} - 8x^{\frac{3}{2}}$ at the point on the graph where $x = 4$.

SF

8 Find the equation of the tangent at the point corresponding to the given x-value on each of the following curves:

a $y = \dfrac{x^2 - 1}{x^2 + 1}$; $x = 0$

b $y = \sqrt{3x^2 + 1}$; $x = 1$

c $y = \dfrac{1}{2x - 1}$; $x = 0$

d $y = \dfrac{1}{(2x - 1)^2}$; $x = 1$

CF

Example 22, 23

9 Find the equation of the tangent to each of the following curves at the given x-value:

a $y = \sin(2x)$; $x = 0$

b $y = \cos(2x)$; $x = \dfrac{\pi}{2}$

c $y = \cos(2x) + \sin x$; $x = \pi$

d $y = \sin x + x\sin(2x)$; $x = 0$

SF

10 For each function, find the equation of the tangent to the graph at the given value of x:

a $f(x) = e^x + e^{-x}$; $x = 0$

b $f(x) = \dfrac{e^x - e^{-x}}{2}$; $x = 0$

c $f(x) = x^2e^{2x}$; $x = 1$

d $f(x) = e^{\sqrt{x}}$; $x = 1$

e $f(x) = xe^{x^2}$; $x = 1$

f $f(x) = x^2e^{-x}$; $x = 2$

11 **a** Find the equation of the tangent to the graph of $f(x) = \ln x$ at the point $(1, 0)$.

b Find the equation of the tangent to the graph of $f(x) = \ln(2x)$ at the point $\left(\dfrac{1}{2}, 0\right)$.

c Find the equation of the tangent to the graph of $f(x) = \ln(kx)$ at the point $\left(\dfrac{1}{k}, 0\right)$, where $k \in \mathbb{R}^+$.

Example 24

12 Find the equation of the tangent at the point where $y = 0$ for each of the following curves:

a $y = x^{\frac{1}{5}}$

b $y = x^{\frac{3}{5}}$

c $y = (x - 4)^{\frac{1}{3}}$

d $y = (x + 5)^{\frac{2}{3}}$

e $y = (2x + 1)^{\frac{1}{3}}$

f $y = (x + 5)^{\frac{4}{5}}$

13 The tangent to the curve with equation $y = 2e^x$ at the point $(a, 2e^a)$ passes through the origin. Find the value of a.

CF

14 The tangent to the curve with equation $y = \ln x$ at the point $(a, \ln a)$ passes through the origin. Find the value of a.

15 The tangent to the curve with equation $y = x^2 + 2x$ at the point $(a, a^2 + 2a)$ passes through the origin. Find the value of a.

16 The tangent to the curve with equation $y = x^3 + x$ at the point $(a, a^3 + a)$ passes through the point $(1, 1)$. Find the value of a.

17 The tangent to the curve with equation $y = \tan(2x) = \dfrac{\sin(2x)}{\cos(2x)}$ at the point where $x = \dfrac{\pi}{8}$ meets the y-axis at the point A. Find the distance OA, where O is the origin.

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8I Rates of change

The derivative was defined geometrically in the previous chapter. However, the process of differentiation may also be used to tackle many kinds of problems involving rates of change.

For the function with rule $f(x)$:

- The **average rate of change** for $x \in [a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$.
- The **instantaneous rate of change** of f with respect to x when $x = a$ is given by $f'(a)$.

The derivative $\dfrac{dy}{dx}$ gives the instantaneous rate of change of y with respect to x.

- If $\dfrac{dy}{dx} > 0$, then y is increasing as x increases.
- If $\dfrac{dy}{dx} < 0$, then y is decreasing as x increases.

Example 25

For the function with rule $f(x) = x^2 + 2x$, find:

a the average rate of change for $x \in [2, 3]$

b the average rate of change for the interval $[2, 2 + h]$

c the instantaneous rate of change of f with respect to x when $x = 2$.

Solution

a Average rate of change $= \dfrac{f(3) - f(2)}{3 - 2} = 15 - 8 = 7$

b Average rate of change
$$\begin{aligned} &= \frac{f(2 + h) - f(2)}{2 + h - 2} \\ &= \frac{(2 + h)^2 + 2(2 + h) - 8}{h} \\ &= \frac{4 + 4h + h^2 + 4 + 2h - 8}{h} \\ &= \frac{6h + h^2}{h} = 6 + h \end{aligned}$$

c The derivative is $f'(x) = 2x + 2$. When $x = 2$, the instantaneous rate of change is $f'(2) = 6$. This can also be seen from the result of part **b**.

Example 26

A balloon develops a microscopic leak and gradually decreases in volume. Its volume, V cm^3, at time t seconds is $V = 600 - 10t - \dfrac{1}{100}t^2$, $t \geq 0$.

a Find the rate of change of volume after:

i 10 seconds **ii** 20 seconds

b For how long could the model be valid?

ISBN 978-1-108-45164-2

Solution

a $\dfrac{dV}{dt} = -10 - \dfrac{t}{50}$

i When $t = 10$, $\dfrac{dV}{dt} = -10\frac{1}{5}$

i.e. the volume is decreasing at a rate of $10\frac{1}{5}$ cm^3 per second.

ii When $t = 20$, $\dfrac{dV}{dt} = -10\frac{2}{5}$

i.e. the volume is decreasing at a rate of $10\frac{2}{5}$ cm^3 per second.

b The model will not be meaningful when $V < 0$. Consider $V = 0$:

$$600 - 10t - \frac{1}{100}t^2 = 0$$

$$\therefore \quad t = 100(\sqrt{31} - 5) \quad \text{or} \quad t = -100(\sqrt{31} + 5)$$

The model may be suitable for $0 \le t \le 100(\sqrt{31} - 5)$.

Example 27

A pot of liquid is put on the stove. When the temperature of the liquid reaches 80°C, the pot is taken off the stove and placed on the kitchen bench. The temperature in the kitchen is 20°C. The temperature of the liquid, T°C, at time t minutes is given by

$$T = 20 + 60e^{-0.3t}$$

a Find the rate of change of temperature with respect to time in terms of T.

b Find the rate of change of temperature with respect to time when:

i $T = 80$ **ii** $T = 30$

Solution

a By rearranging $T = 20 + 60e^{-0.3t}$, we see that $e^{-0.3t} = \dfrac{T - 20}{60}$.

Now $\quad T = 20 + 60e^{-0.3t}$

$$\therefore \quad \frac{dT}{dt} = -18e^{-0.3t}$$

Hence
$$\begin{aligned}\frac{dT}{dt} &= -18\left(\frac{T-20}{60}\right)\\ &= -3\left(\frac{T-20}{10}\right)\\ &= 0.3(20 - T)\end{aligned}$$

b **i** When $T = 80$, $\dfrac{dT}{dt} = 0.3(20 - 80)$
$= -18$

The liquid is cooling at a rate of 18°C per minute.

ii When $T = 30$, $\dfrac{dT}{dt} = 0.3(20 - 30)$
$= -3$

The liquid is cooling at a rate of 3°C per minute.

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Example 28

It is suggested that the height, $h(t)$ metres, of the tide above mean sea level on 1 January at Warnung is given approximately by the rule

$$h(t) = 4\sin\left(\frac{\pi t}{6}\right)$$

where t is the number of hours after midnight.

a Draw the graph of $y = h(t)$ for $0 \le t \le 24$.

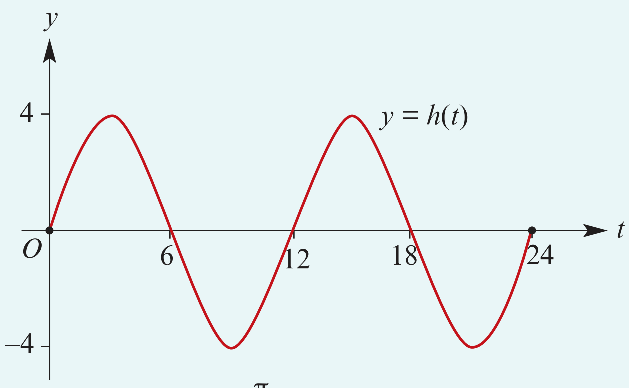
b Find the average rate of change of the height of the tide with respect to time between 3 a.m. and 9 a.m.

c Find the rate of change of the height of the tide at time t hours after midnight.

d Find the times at which the magnitude of this rate of change reaches its maximum value, and state the maximum value.

Solution

a



Note: Period $= 2\pi \div \frac{\pi}{6} = 12$

b Average rate of change $= \dfrac{h(9) - h(3)}{9 - 3} = \dfrac{-4 - 4}{6} = -\dfrac{4}{3}$ metres per hour

c $h'(t) = \dfrac{2\pi}{3}\cos\left(\dfrac{\pi t}{6}\right)$

d The magnitude of $h'(t)$ is at a maximum when:

$$\cos\left(\frac{\pi t}{6}\right) = \pm 1$$

$$\frac{\pi t}{6} = 0, \pi, 2\pi, 3\pi \text{ or } 4\pi$$

$$\therefore \quad t = 0, 6, 12, 18 \text{ or } 24$$

The times are 12 a.m., 6 a.m., 12 p.m., 6 p.m. and 12 a.m.

The maximum magnitude of $h'(t)$ is $\dfrac{2\pi}{3}$ metres per hour.

ISBN 978-1-108-45164-2

Relative growth rate

The **relative growth rate** of a function f is defined to be $\dfrac{f'(x)}{f(x)}$.

For an exponential function of the form $f(x) = Ae^{kx}$, the relative growth rate is

$$\frac{f'(x)}{f(x)} = \frac{kAe^{kx}}{Ae^{kx}} = k$$

Hence the relative growth rate of $f(x) = Ae^{kx}$ is the constant k.

Example 29

The growth of a population of small marsupials can be modelled by a rule of the form $n(t) = Ae^{kt}$, where $n(t)$ is the size of the population after t years.

a If the initial population is 600 and the relative growth rate is 5%, find the rule for $n(t)$.

b When is the population increasing at a rate of 50 per year? (Give the value of t correct to one decimal place.)

Solution

a The relative growth rate is 5%, so $k = 0.05$.

Since $n(0) = 600$, we obtain

$$600 = Ae^{0.05\times 0}$$

$$\therefore \quad A = 600$$

Hence $n(t) = 600e^{0.05t}$.

b Solve the equation $n'(t) = 50$ for t:

$$n'(t) = 50$$

$$30e^{0.05t} = 50$$

$$e^{0.05t} = \frac{5}{3}$$

$$0.05t = \ln\left(\frac{5}{3}\right)$$

$$\therefore \quad t \approx 10.2$$

The growth rate is 50 per year after 10.2 years.

Section summary

For the function with rule $f(x)$:

- The average rate of change for $x \in [a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$.
- The instantaneous rate of change of f with respect to x when $x = a$ is given by $f'(a)$.

Exercise 8I

Skillsheet

Example 25

SF

1 For the function with rule $f(x) = 3x^2 + 6x$, find:

a the average rate of change for $x \in [2, 3]$

b the average rate of change for the interval $[2, 2 + h]$

c the instantaneous rate of change of f with respect to x when $x = 2$.

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SF

2 Express each of the following in symbols:

a the rate of change of volume (V) with respect to time (t)

b the rate of change of surface area (S) of a sphere with respect to radius (r)

c the rate of change of volume (V) of a cube with respect to edge length (x)

Example 26

3 If your interest (I) in Mathematical Methods can be expressed as

$$I = \frac{4}{(t+1)^2}$$

where t is the time in days measured from the first day of Term 1, how fast is your interest waning when $t = 10$?

CF

4 A reservoir is being emptied and the quantity of water, V m^3, remaining in the reservoir t days after it starts to empty is given by

$$V(t) = 10^3(90 - t)^3$$

a At what rate is the reservoir being emptied at time t?

b How long does it take to empty the reservoir?

c What is the volume of water in the reservoir when $t = 0$?

d After what time is the reservoir being emptied at 3×10^5 m^3/day?

e Sketch the graph of $V(t)$ against t.

f Sketch the graph of $V'(t)$ against t.

5 A coffee percolator allows 1000 mL of water to flow into a filter in 20 minutes. The volume which has flowed into the filter at time t minutes is given by

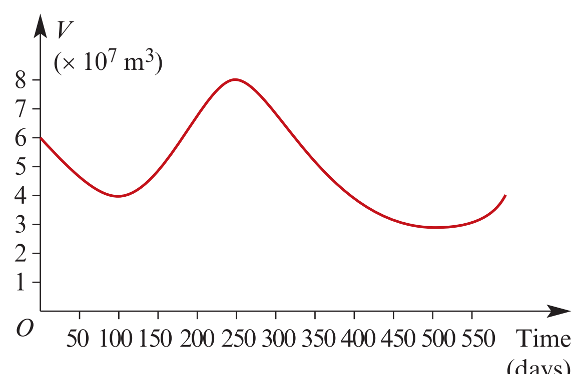
$$V(t) = \frac{1}{160}\left(5t^4 - \frac{t^5}{5}\right), \quad 0 \le t \le 20$$

a At what rate is the water flowing into the filter at time t minutes?

b Sketch the graph of $\frac{dV}{dt}$ against t for $0 \le t \le 20$.

c When is the rate of flow greatest?

6 The graph shows the volume, V m^3, of water in a reservoir at time t days.



a At what times is the rate of flow from the reservoir 0 m^3/day?

b Find an estimate for the rate of flow at $t = 200$.

c Find the average rate of flow for the interval [100, 250].

d State the times for which there is net flow into the reservoir.

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7 A car tyre is inflated to a pressure of 30 units. Eight hours later it is found to have deflated to a pressure of 10 units. The pressure, P, at time t hours is given by

$$P = P_0 e^{-\lambda t}$$

a Find the values of P_0 and λ.

b At what time would the pressure be 8 units?

c Find the rate of loss of pressure at:

i time $t = 0$ **ii** time $t = 8$

Example 27

8 A liquid is heated to a temperature of 90°C and then allowed to cool in a room in which the temperature is 15°C. While the liquid is cooling, its temperature, T°C, at time t minutes is given by $T = 15 + 75e^{-0.3t}$.

a Find the rate of change of temperature with respect to time in terms of T.

b Find the rate of change of temperature with respect to time when:

i $T = 90$ **ii** $T = 60$ **iii** $T = 30$

Example 28

9 The water level on a beach wall is given by

$$d(t) = 6 + 4\cos\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$$

where t is the number of hours after midnight and d is the depth of the water in metres.

a Sketch the graph of $d(t)$ for $0 \le t \le 24$.

b What is the earliest time of day at which the water is at its highest?

c Find the average rate of change of the depth of the water with respect to time between 3 a.m. and 9 a.m.

d Find the rate of change of the depth of the water at time t hours after midnight.

e Find the times at which the magnitude of this rate of change reaches its maximum value, and state the maximum value.

10 If $y = 3x + 2\cos x$, find $\frac{dy}{dx}$ and hence show that y increases as x increases.

11 The volume of water in a reservoir at time t is given by $V(t) = 3 + 2\sin\left(\frac{t}{4}\right)$.

a Find the volume in the reservoir at time $t = 10$.

b Find the rate of change of the volume of water in the reservoir at time $t = 10$.

12 A manufacturing company has a daily output on day t of a production run given by

$$y = 600(1 - e^{-0.5t})$$

where the first day of the production run is $t = 0$.

a Sketch the graph of y against t. (Assume a continuous model.)

b Find the instantaneous rate of change of output y with respect to t on the 10th day.

13 For each of the following, find $\frac{dy}{dx}$ in terms of y:

a $y = e^{-2x}$

b $y = Ae^{kx}$

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CF

14 The mass, m kg, of radioactive lead remaining in a sample t hours after observations began is given by $m = 2e^{-0.2t}$.

a Find the mass left after 12 hours.

b Find how long it takes for the mass to fall to half of its value at $t = 0$.

c Find how long it takes for the mass to fall to **i** one-quarter and **ii** one-eighth of its value at $t = 0$.

d Express the rate of decay as a function of m.

Example 29 **15** The spread of a disease can be modelled by a rule of the form $N = Ae^{kt}$, where N is the number of people who have been infected after t days. When the disease is first identified ($t = 0$), there are 20 infected people. The relative growth rate is 10%.

a State the values of the constants A and k.

b What is the initial rate of increase of the disease?

c When does the rate of increase reach 100 people per day?

8J Motion in a straight line

We continue our study of motion in a straight line from Mathematical Methods Units 1 & 2.

▶ Position

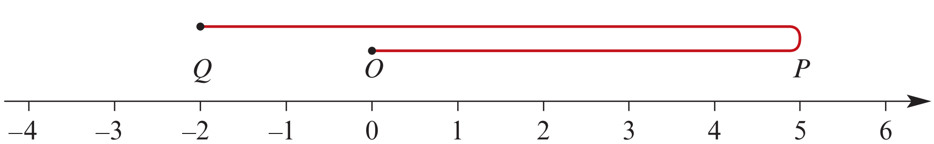
The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the **origin**, and whether it is to the right or left of O. By convention, the direction to the right of the origin is considered to be positive.

Consider a particle which starts at O and begins to move. The position of the particle at any instant can be specified by a real number x. For example, if the unit is metres and if $x = -3$, the position is 3 m to the left of O; while if $x = 3$, the position is 3 m to the right of O.

▶ Displacement and distance

The **displacement** of a particle is defined as the change in position of the particle.

It is important to distinguish between the scalar quantity **distance** and the vector quantity displacement (which has a direction). For example, consider a particle that starts at O and moves first 5 units to the right to point P, and then 7 units to the left to point Q.



The difference between its final position and its initial position is −2. So the displacement of the particle is −2 units. However, the distance it has travelled is 12 units.

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► Velocity and speed

Average velocity

The average rate of change of position with respect to time is **average velocity**.

A particle's average velocity for a time interval $[t_1, t_2]$ is given by

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

where x_1 is the position at time t_1 and x_2 is the position at time t_2.

Instantaneous velocity

The instantaneous rate of change of position with respect to time is **instantaneous velocity**. We will refer to the instantaneous velocity as simply the **velocity**.

If a particle's position, x, at time t is given as a function of t, then the velocity of the particle at time t is determined by differentiating the rule for position with respect to time.

If x is the position of a particle at time t, then

$$\text{velocity } v = \frac{dx}{dt}$$

Velocity may be positive, negative or zero. If the velocity is positive, the particle is moving to the right, and if it is negative, the particle is moving to the left. A velocity of zero means the particle is instantaneously at rest.

Speed and average speed

- **Speed** is the magnitude of the velocity.
- **Average speed** for a time interval $[t_1, t_2]$ is given by $\dfrac{\text{distance travelled}}{t_2 - t_1}$

Units of measurement

Common units for velocity (and speed) are:

$$\begin{aligned} &\text{1 metre per second} &&= 1 \text{ m/s} &&= 1 \text{ m s}^{-1} \\ &\text{1 centimetre per second} &&= 1 \text{ cm/s} &&= 1 \text{ cm s}^{-1} \\ &\text{1 kilometre per hour} &&= 1 \text{ km/h} &&= 1 \text{ km h}^{-1} \end{aligned}$$

The first and third units are connected in the following way:

$$1 \text{ km/h} = 1000 \text{ m/h} = \frac{1000}{60 \times 60} \text{ m/s} = \frac{5}{18} \text{ m/s}$$

$$\therefore \quad 1 \text{ m/s} = \frac{18}{5} \text{ km/h}$$

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Example 30

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 7t + 6$, $t \geq 0$.

a Find its initial velocity.

b When does its velocity equal zero, and what is its position at this time?

c What is its average velocity for the first 4 seconds?

d Determine its average speed for the first 4 seconds.

Solution

a $x = t^2 - 7t + 6$

$$v = \frac{dx}{dt} = 2t - 7$$

At $t = 0$, $v = -7$. The particle is initially moving to the left at 7 cm/s.

b $\frac{dx}{dt} = 0$ implies $2t - 7 = 0$, i.e. $t = 3.5$

When $t = 3.5$, $x = (3.5)^2 - 7(3.5) + 6$

$= -6.25$

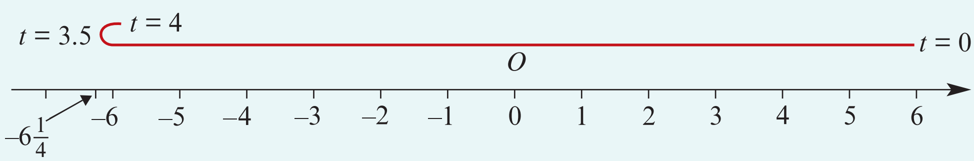
So, at $t = 3.5$ seconds, the particle is at rest 6.25 cm to the left of O.

c $$\text{Average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

Position is given by $x = t^2 - 7t + 6$. So at $t = 4$, $x = -6$, and at $t = 0$, $x = 6$.

$$\therefore \text{Average velocity} = \frac{-6 - 6}{4} = -3 \text{ cm/s}$$

d $$\text{Average speed} = \frac{\text{distance travelled}}{\text{change in time}}$$



The particle stopped at $t = 3.5$ and began to move in the opposite direction. So we must consider the distance travelled in the first 3.5 seconds (from $x = 6$ to $x = -6.25$) and then the distance travelled in the final 0.5 seconds (from $x = -6.25$ to $x = -6$).

Total distance travelled = 12.25 + 0.25 = 12.5

$$\therefore \text{Average speed} = \frac{12.5}{4} = 3.125 \text{ cm/s}$$

Note: Remember that speed is the magnitude of the velocity. However, we can see from this example that average speed is *not* the magnitude of the average velocity.

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Acceleration

The acceleration of a particle is the rate of change of its velocity with respect to time.

- **Average acceleration** for the time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.
- **Instantaneous acceleration** $a = \dfrac{dv}{dt}$

Acceleration may be positive, negative or zero. Zero acceleration means the particle is moving at a constant velocity.

The direction of motion and the acceleration need not coincide. For example, a particle may have a positive velocity, indicating it is moving to the right, but a negative acceleration, indicating it is slowing down.

Also, although a particle may be instantaneously at rest, its acceleration at that instant need not be zero. If acceleration has the same sign as velocity, then the particle is 'speeding up'. If the sign is opposite, the particle is 'slowing down'.

The most commonly used units for acceleration are cm/s^2 and m/s^2.

Example 31

A particle moves along a straight line. Its position, x m, relative to a fixed point O on the line is given by $x = 3 + 2\sin(3t)$, where t is the time in seconds $(0 \le t \le 2\pi)$.

a **i** Find the particle's greatest distance from O.
ii Find the particle's least distance from O.

b Find the times at which the particle is 5 m from O.

c Find the velocity, v m/s, and acceleration, a m/s^2, of the particle at time t seconds.

d Find the maximum speed, and find the times and locations at which this occurs.

e Find the maximum magnitude of the acceleration, and find the times and locations at which this occurs.

f Describe the motion of the particle.

Solution

a **i** Greatest distance from O occurs when $\sin(3t) = 1$. Greatest distance is 5 m.
ii Least distance from O occurs when $\sin(3t) = -1$. Least distance is 1 m.

b
$$3 + 2\sin(3t) = 5$$
$$\sin(3t) = 1$$
$$3t = \frac{\pi}{2}, \frac{5\pi}{2} \text{ or } \frac{9\pi}{2}$$
$$\therefore \quad t = \frac{\pi}{6}, \frac{5\pi}{6} \text{ or } \frac{3\pi}{2}$$

c
$$x = 3 + 2\sin(3t)$$
$$v = \frac{dx}{dt} = 6\cos(3t)$$
$$a = \frac{dv}{dt} = -18\sin(3t)$$

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d Maximum speed is 6 m/s and occurs when:

$\cos(3t) = \pm 1$

$3t = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$ or 6π

$\therefore \quad t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ or 2π

Locations are all $x = 3$.

e Maximum magnitude of acceleration is 18 m/s^2 and occurs when:

$\sin(3t) = \pm 1$

$3t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$ or $\frac{11\pi}{2}$

$\therefore \quad t = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}$ or $\frac{11\pi}{6}$

Locations are $x = 5$ and $x = 1$ alternating.

f The particle starts at $x = 3$ and oscillates between $x = 5$ and $x = 1$ three times.

Section summary

- Position (x m) is specified with respect to a reference point O on the line.
- Velocity (v m/s) and acceleration (a m/s^2) are given by:
  - velocity $v = \frac{dx}{dt}$
  - acceleration $a = \frac{dv}{dt}$

Exercise 8J

Example 30

1 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 6t + 8$, $t \geq 0$.

a Find its initial velocity.

b When does its velocity equal zero, and what is its position at this time?

c What is its average velocity for the first 4 seconds?

d Determine its average speed for the first 4 seconds.

SF

2 A particle moving in a straight line has position x cm relative to the point O at time t seconds ($t \geq 0$), where $x = t^3 - 11t^2 + 24t - 3$. Find:

a its initial position and velocity

b its velocity at time t

c at what times the particle is stationary

d where the particle is stationary

e how far it travels in the first 2 seconds

f how far it travels in the first 7 seconds.

Example 31

3 A particle moves along a straight line. Its position, x m, relative to a fixed point O on the line is given by $x = 30 + 20\sin\left(\frac{\pi t}{6}\right)$, where t is the time in seconds ($0 \leq t \leq 24$).

CF

a **i** Find the particle's greatest distance from O.

ii Find the particle's least distance from O.

b Find the times at which the particle is 10 m from O.

c Find the velocity, v m/s, and acceleration, a m/s^2, of the particle at time t seconds.

d Find the maximum speed, and when and where this occurs.

e Find the maximum magnitude of the acceleration, and when and where this occurs.

f Find the distance travelled by the particle during the 24 seconds of motion.

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SF

4 A particle moves along a straight line such that its position, x cm, relative to a point O at time t seconds is given by $x = 2t^3 - 9t^2 + 12t$ for $t \geq 0$.

a Find the velocity, v cm/s, as a function of t.

b Find the acceleration, a cm/s^2, as a function of t.

c At what times and in what positions will the particle have zero velocity?

d At what times and in what positions will the particle have zero acceleration?

5 A particle moves in a straight line such that its position, x cm, relative to a point O at time t seconds is given by $x = 8 + 2t - t^2$ for $t \geq 0$. Find:

a its initial position

b its initial velocity

c when and where the velocity is zero

d its acceleration at time t.

CF

6 A particle is moving in a straight line such that its position, x cm, relative to a point O at time t seconds is given by $x = \sqrt{2t^2 + 2}$. Find:

a the velocity as a function of t

b the velocity when $t = 1$

c the acceleration as a function of t

d the acceleration when $t = 1$.

7 A vehicle is travelling in a straight line away from point O. Its distance from O after t seconds is $0.4e^t$ metres. Find the velocity and acceleration of the vehicle when $t = 0$, $t = 1$ and $t = 2$.

8 A particle moves in a straight line such that its velocity, v m/s, at time t seconds is given by $v = e^{-2t} \sin(3t)$ for $t \geq 0$.

a Find the first three times at which the particle's velocity is zero.

b Find the particle's acceleration at time t seconds.

c Use a graphics calculator to find the first time at which the particle's acceleration is zero. Give your answer correct to three decimal places.

9 A particle moves in a straight line such that its velocity, v m/s, at time t seconds is given by $v = c - \ln(t + d)$ for $t \geq 0$, where c and d are positive constants.

a Find the particle's acceleration, a m/s^2, at time t seconds.

b Given that the acceleration is $-\frac{1}{20}$ m/s^2 when $t = 10$, find the value of d.

c Find the time at which the particle is stationary in terms of c.

10 A block of stone is falling through a layer of mud. The depth, x metres, of the block below ground level at time t minutes is given by $x = 10 - 10e^{-0.5t}$.

a Find the velocity of the block at time t.

b Find the acceleration of the block at time t.

c When is the block 5 m below ground level and what is its velocity at this time?

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8K Stationary points

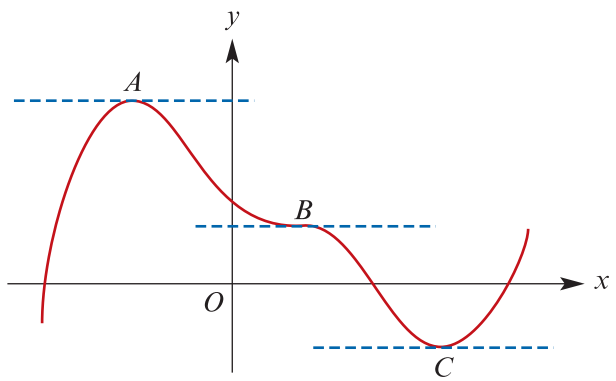
In the previous chapter, we have seen that the gradient of the tangent at a point $(a, f(a))$ on the curve with rule $y = f(x)$ is given by $f'(a)$.

A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.

Equivalently, a point $(a, f(a))$ on $y = f(x)$ is a stationary point if $\frac{dy}{dx} = 0$ when $x = a$.

In the graph shown, there are stationary points at A, B and C.

At such points, the tangents are parallel to the x-axis (illustrated as dashed lines).



The reason for the name *stationary point* becomes clear if we look at an application to the motion of a particle.

Example 32

A particle is moving in a straight line. Its position, x metres, relative to a point O on the line at time t seconds is given by

$$x = 9t - \frac{1}{3}t^3, \quad 0 \le t \le 4$$

Find the particle's maximum distance from O. (Here the particle is always on the right of O and so its distance from O is its position.)

Solution

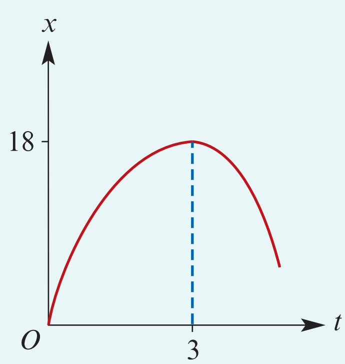
$$\frac{dx}{dt} = 9 - t^2$$

The maximum distance from O occurs when $\frac{dx}{dt} = 0$.

So $t = 3$ or $t = -3$. But $t = -3$ lies outside the domain.

At $t = 3$, $x = 18$.

Thus the stationary point is $(3, 18)$ and the maximum distance from O is 18 metres.



Note: The stationary point occurs when the rate of change of position with respect to time (the velocity) is zero. At this moment, the particle is stationary.

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Example 33

Find the stationary points of the following functions:

a $y = 9 + 12x - 2x^2$ **b** $y = 4 + 3x - x^3$ **c** $p = 2t^3 - 5t^2 - 4t + 13,\ t > 0$

Solution

a $y = 9 + 12x - 2x^2$

$$\frac{dy}{dx} = 12 - 4x$$

A stationary point occurs when $\frac{dy}{dx} = 0$, i.e. when $12 - 4x = 0$.

Hence $x = 3$ and
$$y = 9 + 12 \times 3 - 2 \times 3^2 = 27$$

The stationary point is $(3, 27)$.

b $y = 4 + 3x - x^3$

$$\frac{dy}{dx} = 3 - 3x^2$$

$\frac{dy}{dx} = 0$ implies $3(1 - x^2) = 0$

$$\therefore \quad x = \pm 1$$

The stationary points are $(1, 6)$ and $(-1, 2)$.

c $p = 2t^3 - 5t^2 - 4t + 13$

$$\frac{dp}{dt} = 6t^2 - 10t - 4, \quad t > 0$$

$\frac{dp}{dt} = 0$ implies $2(3t^2 - 5t - 2) = 0$

$$(3t + 1)(t - 2) = 0$$

$$\therefore \quad t = -\frac{1}{3} \text{ or } t = 2$$

But $t > 0$, and so the only acceptable solution is $t = 2$. The corresponding stationary point is $(2, 1)$.

Example 34

Find the stationary points of the following functions:

a $y = \sin(2x),\ x \in [0, 2\pi]$ **b** $y = e^{2x} - x$ **c** $y = x\ln(2x),\ x \in (0, \infty)$

Solution

a $y = \sin(2x)$

$$\frac{dy}{dx} = 2\cos(2x)$$

So $\frac{dy}{dx} = 0$ implies $2\cos(2x) = 0$

$$\cos(2x) = 0$$

$$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \text{ or } \frac{7\pi}{2}$$

$$\therefore \quad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

The stationary points are $\left(\frac{\pi}{4}, 1\right), \left(\frac{3\pi}{4}, -1\right), \left(\frac{5\pi}{4}, 1\right)$ and $\left(\frac{7\pi}{4}, -1\right)$.

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b $y = e^{2x} - x$

$\frac{dy}{dx} = 2e^{2x} - 1$

So $\frac{dy}{dx} = 0$ implies

$$2e^{2x} - 1 = 0$$
$$e^{2x} = \frac{1}{2}$$
$$\therefore \quad x = \frac{1}{2}\ln\left(\frac{1}{2}\right)$$
$$= -\frac{1}{2}\ln 2$$

When $x = -\frac{1}{2}\ln 2$,

$$y = e^{2\times\frac{1}{2}\ln\left(\frac{1}{2}\right)} + \frac{1}{2}\ln 2$$
$$= \frac{1}{2} + \frac{1}{2}\ln 2$$

The coordinates of the stationary point are $(-\frac{1}{2}\ln 2, \frac{1}{2} + \frac{1}{2}\ln 2)$.

c $y = x\ln(2x)$

$\frac{dy}{dx} = \ln(2x) + 1$

So $\frac{dy}{dx} = 0$ implies

$$\ln(2x) + 1 = 0$$
$$\ln(2x) = -1$$
$$2x = e^{-1}$$
$$\therefore \quad x = \frac{1}{2e}$$

When $x = \frac{1}{2e}$, $y = \frac{1}{2e}\ln\left(\frac{2}{2e}\right)$

$$= \frac{-1}{2e}$$

The coordinates of the stationary point are $\left(\frac{1}{2e}, \frac{-1}{2e}\right)$.

Example 35

The curve with equation $y = x^3 + ax^2 + bx + c$ passes through the point $(0, 5)$ and has a stationary point at $(2, 7)$. Find a, b and c.

Solution

When $x = 0$, $y = 5$. Thus $5 = c$.

$\frac{dy}{dx} = 3x^2 + 2ax + b$ and at $x = 2$, $\frac{dy}{dx} = 0$. Therefore

$$12 + 4a + b = 0 \qquad (1)$$

The point $(2, 7)$ is on the curve and so

$$8 + 4a + 2b + 5 = 7$$
$$\therefore \quad 6 + 4a + 2b = 0 \qquad (2)$$

Subtracting (1) from (2) gives $-6 + b = 0$. Thus $b = 6$. Substitute in (1):

$$12 + 4a + 6 = 0$$
$$4a = -18$$

Hence $a = -\frac{9}{2}$, $b = 6$ and $c = 5$.

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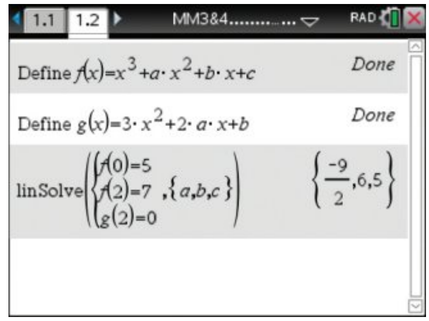
TI-84

Using the TI-Nspire CX non-CAS

- Define the function $f(x) = x^3 + ax^2 + bx + c$ and derivative function $g(x) = 3x^2 + 2ax + b$ as shown.
- Use (menu) > **Algebra** > **Solve System of Linear Equations** to solve for a, b and c given that $f(0) = 5$, $f(2) = 7$ and $g(2) = 0$.
- Hence $a = -\frac{9}{2}$, $b = 6$ and $c = 5$.

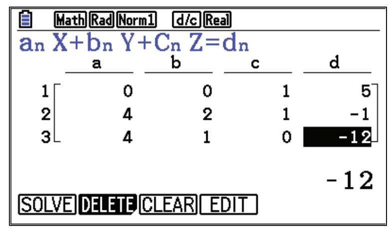


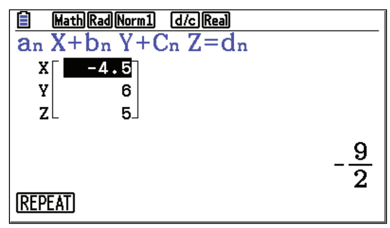
Using the Casio

- Select **Equation** mode (MENU) (ALPHA) (X,θ,T).
- Select **Simultaneous** (F1), then select three unknowns (F2).
- For $f(x) = x^3 + ax^2 + bx + c$, we obtain the following three equations:

$$c = 5 \qquad (\text{as } f(0) = 5)$$
$$4a + 2b + c = -1 \qquad (\text{as } f(2) = 7)$$
$$4a + b = -12 \qquad (\text{as } f'(2) = 0)$$

- Enter the coefficients of these equations in the table as shown. Select **Solve** (F1).
- Hence $a = -\frac{9}{2}$, $b = 6$ and $c = 5$.





Section summary

- A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.
- Equivalently, a point $(a, f(a))$ on $y = f(x)$ is a stationary point if $\frac{dy}{dx} = 0$ when $x = a$.

Exercise 8K

Example 33

SF

1 Find the stationary points for each of the following:

a $f(x) = x^3 - 12x$

b $g(x) = 2x^2 - 4x$

c $h(x) = 5x^4 - 4x^5$

d $f(t) = 8t + 5t^2 - t^3$ for $t > 0$

e $g(z) = 8z^2 - 3z^4$

f $f(x) = 5 - 2x + 3x^2$

g $h(x) = x^3 - 4x^2 - 3x + 20$, $x > 0$

h $f(x) = 3x^4 - 16x^3 + 24x^2 - 10$

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Example 34

2 Find the stationary points of the following functions:

a $y = e^{2x} - 2x$

b $y = x\ln(3x),\ x > 0$

c $y = \cos(2x),\ x \in [-\pi, \pi]$

d $y = xe^{x}$

e $y = x^2e^{-x}$

f $y = 2x\ln x,\ x > 0$

CF

3 Find the coordinates of the stationary point on the graph of $y = \ln x - 4x$.

SF

4 Find the coordinates of the stationary point on the graph of $y = \ln x + \dfrac{2}{x}$.

5 Find the coordinates of the stationary points on the graph of $y = x^2e^{-ax}$, where a is a non-zero constant.

6 **a** The curve with rule $f(x) = x^2 - ax + 9$ has a stationary point when $x = 3$. Find a.

b The curve with rule $h(x) = x^3 - bx^2 - 9x + 7$ has a stationary point when $x = -1$. Find b.

CF

Example 35

7 The curve with equation $y = x^3 + bx^2 + cx + d$ passes through the point $(0, 3)$ and has a stationary point at $(1, 3)$. Find b, c and d.

8 The tangent to the curve of $y = ax^2 + bx + c$ at the point where $x = 2$ is parallel to the line $y = 4x$. There is a stationary point at $(1, -3)$. Find the values of a, b and c.

9 The graph of $y = ax^3 + bx^2 + cx + d$ touches the line $2y + 6x = 15$ at the point $A(0, 7\frac{1}{2})$ and has a stationary point at $B(3, -6)$. Find the values of a, b, c and d.

10 The curve with equation $y = ax + \dfrac{b}{2x-1}$ has a stationary point at $(2, 7)$. Find:

a the values of a and b

b the coordinates of the other stationary point.

11 Find the x-coordinates, in terms of n, of the stationary points of the curve with equation $y = (2x - 1)^n(x + 2)$, where n is a natural number.

12 Find the x-coordinates of the stationary points of the curve with equation $y = (x^2 - 1)^n$ where n is an integer greater than 1.

13 Find the coordinates of the stationary points of the curve with equation $y = \dfrac{x}{x^2+1}$.

SF

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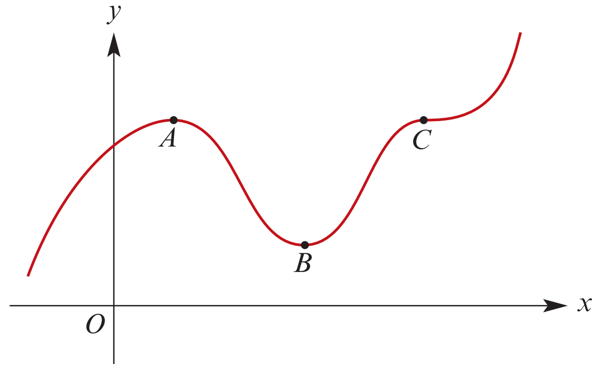
8L Types of stationary points

The graph of $y = f(x)$ shown has three stationary points A, B, C.



A Point A is called a **local maximum** point.
Notice that immediately to the left of A the gradient is positive, and immediately to right the gradient is negative.

gradient	+	0	−
shape of f	/	—	\

B Point B is called a **local minimum** point.
Notice that immediately to the left of B the gradient is negative, and immediately to the right the gradient is positive.

gradient	−	0	+
shape of f	\	—	/

C Point C is called a **stationary point of inflection**.
The gradient is positive immediately to the left and right of C.

gradient	+	0	+
shape of f	/	—	/

Clearly it is also possible to have stationary points of inflection such that the gradient is negative immediately to the left and right.

gradient	−	0	−
shape of f	\	—	\

Stationary points of types A and B are referred to as **turning points**.

Example 36

For the function $f(x) = 3x^3 - 4x + 1$:

a Find the stationary points and state their nature.

b Sketch the graph.

Solution

a The derivative is $f'(x) = 9x^2 - 4$.

The stationary points occur where $f'(x) = 0$:

$$9x^2 - 4 = 0$$

$$\therefore \quad x = \pm\frac{2}{3}$$

There are stationary points at $(-\frac{2}{3}, f(-\frac{2}{3}))$ and $(\frac{2}{3}, f(\frac{2}{3}))$, that is, at $(-\frac{2}{3}, 2\frac{7}{9})$ and $(\frac{2}{3}, -\frac{7}{9})$.
So $f'(x)$ is of constant sign for each of

$$x < -\tfrac{2}{3}, \quad -\tfrac{2}{3} < x < \tfrac{2}{3} \quad \text{and} \quad x > \tfrac{2}{3}$$

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To calculate the sign of $f'(x) = 9x^2 - 4$ for each of these intervals, simply choose a representative number in the interval.

Thus $f'(-1) = 9 - 4 = 5 > 0$

$f'(0) = 0 - 4 = -4 < 0$

$f'(1) = 9 - 4 = 5 > 0$

We can now put together the table shown on the right.

x		$-\frac{2}{3}$		$\frac{2}{3}$	
$f'(x)$	+	0	−	0	+
shape of f	／	—	＼	—	／

There is a local maximum at $(-\frac{2}{3}, 2\frac{7}{9})$ and a local minimum at $(\frac{2}{3}, -\frac{7}{9})$.

b To sketch the graph of $f(x) = 3x^3 - 4x + 1$, we now need to find the axis intercepts and investigate the behaviour of the graph for $x > \frac{2}{3}$ and $x < -\frac{2}{3}$.

The y-axis intercept is $f(0) = 1$.

To find the x-axis intercepts, consider $f(x) = 0$, which implies $3x^3 - 4x + 1 = 0$. Using the factor theorem, we find that $x - 1$ is a factor of $3x^3 - 4x + 1$. By division:

$$3x^3 - 4x + 1 = (x - 1)(3x^2 + 3x - 1)$$

Now $f(x) = (x - 1)(3x^2 + 3x - 1) = 0$ implies that $x = 1$ or $3x^2 + 3x - 1 = 0$. We have

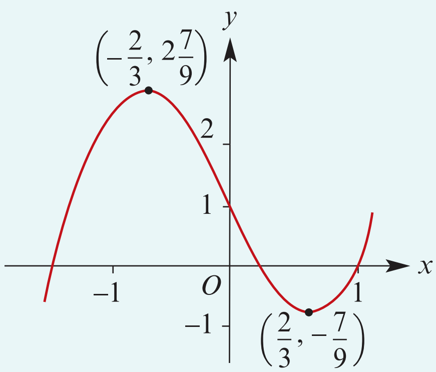
$$\begin{aligned} 3x^2 + 3x - 1 &= 3\left[\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{1}{3}\right] \\ &= 3\left[\left(x + \frac{1}{2}\right)^2 - \frac{21}{36}\right] \\ &= 3\left(x + \frac{1}{2} - \frac{\sqrt{21}}{6}\right)\left(x + \frac{1}{2} + \frac{\sqrt{21}}{6}\right) \end{aligned}$$

Thus the x-axis intercepts are at

$$x = -\frac{1}{2} + \frac{\sqrt{21}}{6}, \quad x = -\frac{1}{2} - \frac{\sqrt{21}}{6}, \quad x = 1$$

For $x > \frac{2}{3}$, $f(x)$ becomes larger.

For $x < \frac{2}{3}$, $f(x)$ becomes smaller.



A graphics calculator can be used to plot the graph of a function and determine its key features, including:

- the value of the function at any point
- the axis intercepts
- the value of its derivative at any point
- the turning points.

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Example 37

Plot the graph of $y = x^3 - 19x + 20$ and determine:

a the value of y when $x = -4$

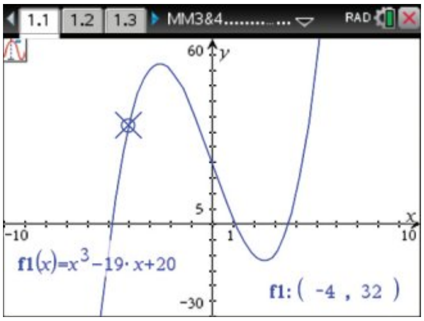
b the values of x when $y = 0$

c the value of $\frac{dy}{dx}$ when $x = -1$

d the coordinates of the local maximum.

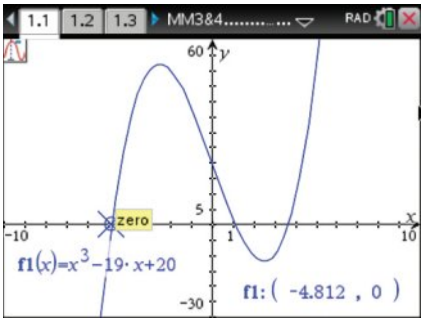
Using the TI-Nspire CX non-CAS

Plot the graph of $f1(x) = x^3 - 19x + 20$ and select an appropriate window (menu > **Window/Zoom** > **Window Settings**).



a Use menu > **Trace** > **Graph Trace** and type −4. The cursor will jump to the point where $x = -4$.

Hence $y = 32$ when $x = -4$.

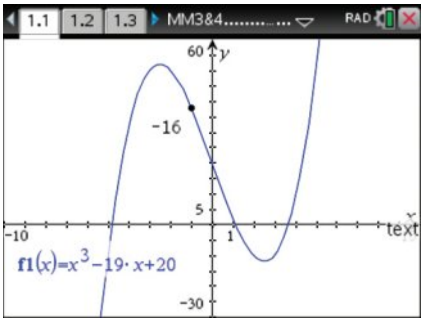


b Use menu > **Trace** > **Graph Trace** and drag the trace cursor (ctrl) to the x-axis. Repeat to find the other zeroes.

The zeroes are −4.812, 1.128 and 3.684.

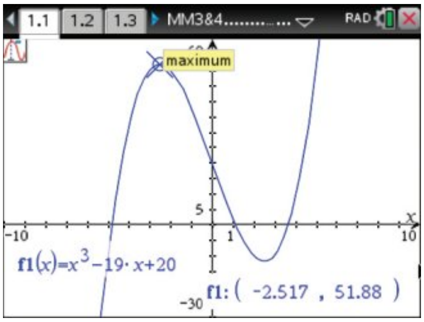
Note: Change the display precision if necessary using menu > **Settings**.

Alternatively, the zeroes can be found using menu > **Analyze Graph** > **Zeros**.



c Use menu > **Analyze Graph** > **dy/dx** and type −1. The cursor will jump to $x = -1$ and the derivative will be displayed.

Hence $\frac{dy}{dx} = -16$ when $x = -1$.



d Use menu > **Trace** > **Graph Trace** and drag the trace cursor (ctrl) to the local maximum.

The local maximum is at $(-2.517, 51.88)$.

Note: Alternatively, the local maximum can be found using menu > **Analyze Graph** > **Maximum**.

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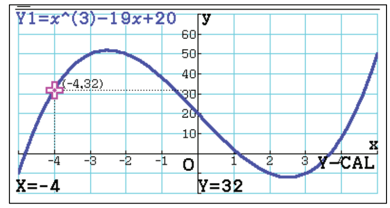
Using the Casio

Plot the graph of $y = x^3 - 19x + 20$ and select an appropriate View Window.

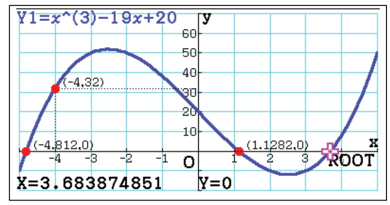
a To find the y-value when $x = -4$:

- Go to the **G-Solve** menu (SHIFT) (F5) and select **y-Cal** (F6) (F1).
- Enter the x-value -4 and press (EXE).
- Hence $y = 32$ when $x = -4$.



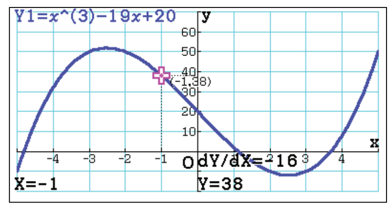
b To find the x-axis intercepts:

- Go to the **G-Solve** menu (SHIFT) (F5) and select **Root** (F1).
- Hence the x-axis intercepts are -4.812, 1.128 and 3.684.



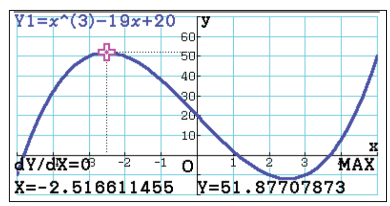
c To find the derivative at $x = -1$:

- Ensure that the derivative setting is On. (Go to the set-up screen (SHIFT) (MENU).)
- On the graph screen, go to **Trace** (SHIFT) (F1) and enter the x-value -1.
- Hence the derivative at $x = -1$ is -16.



d To find the local maximum:

- Go to the **G-Solve** menu (SHIFT) (F5) and select **Maximum** (F2).
- Hence the local maximum has coordinates $(-2.517, 51.877)$.



Example 38

Sketch the graph of $f(x) = e^{x^3}$.

Solution

As $x \to -\infty$, $f(x) \to 0$.

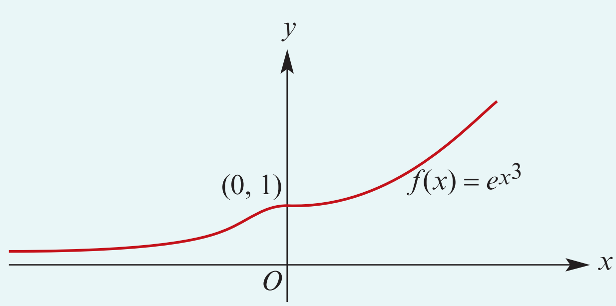
Axis intercepts

When $x = 0$, $f(x) = 1$.

Stationary points

$f'(x) = 3x^2 e^{x^3}$

So $f'(x) = 0$ implies $x = 0$.



The gradient of f is always greater than or equal to 0, which means that $(0, 1)$ is a stationary point of inflection.

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Example 39

Consider the function f given by $f(x) = x \ln x$ for $x > 0$.

a Find $f'(x)$.

b Solve the equation $f(x) = 0$.

c Solve the equation $f'(x) = 0$.

d Sketch the graph of $y = f(x)$.

Solution

a $f'(x) = x \times \frac{1}{x} + \ln x$ (product rule)

$= 1 + \ln x$

b $f(x) = x \ln x$

Thus $f(x) = 0$ implies $x = 0$ or $\ln x = 0$.

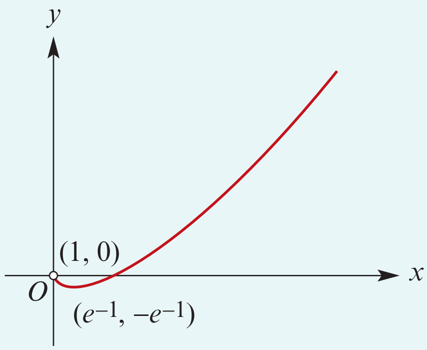
Since $x > 0$, the only solution is $x = 1$.

c $f'(x) = 0$ implies $1 + \ln x = 0$.

Therefore $\ln x = -1$ and so $x = e^{-1}$.

d When $x = e^{-1}$, $y = e^{-1}\ln(e^{-1})$

$= e^{-1} \times (-1) = -e^{-1}$



Example 40

Find the local maximum and local minimum points of $f(x) = 2\sin x + 1 - 2\sin^2 x$, where $0 < x < 2\pi$.

Solution

Find $f'(x)$ and solve $f'(x) = 0$:

$$f(x) = 2\sin x + 1 - 2\sin^2 x$$

$$\therefore \quad f'(x) = 2\cos x - 4\sin x \cos x$$

$$= 2\cos x \cdot (1 - 2\sin x)$$

Thus $f'(x) = 0$ implies

$\cos x = 0$ or $1 - 2\sin x = 0$

i.e. $\cos x = 0$ or $\sin x = \frac{1}{2}$

i.e. $x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $x = \frac{\pi}{6}, \frac{5\pi}{6}$

We have $f\left(\frac{\pi}{2}\right) = 1$, $f\left(\frac{3\pi}{2}\right) = -3$, $f\left(\frac{\pi}{6}\right) = \frac{3}{2}$ and $f\left(\frac{5\pi}{6}\right) = \frac{3}{2}$

x		$\frac{\pi}{6}$		$\frac{\pi}{2}$		$\frac{5\pi}{6}$		$\frac{3\pi}{2}$	
$f'(x)$	+	0	−	0	+	0	−	0	+
shape of f	/	—	\	—	/	—	\	—	/

Local maximums at $\left(\frac{\pi}{6}, \frac{3}{2}\right)$ and $\left(\frac{5\pi}{6}, \frac{3}{2}\right)$. Local minimums at $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{3\pi}{2}, -3\right)$.

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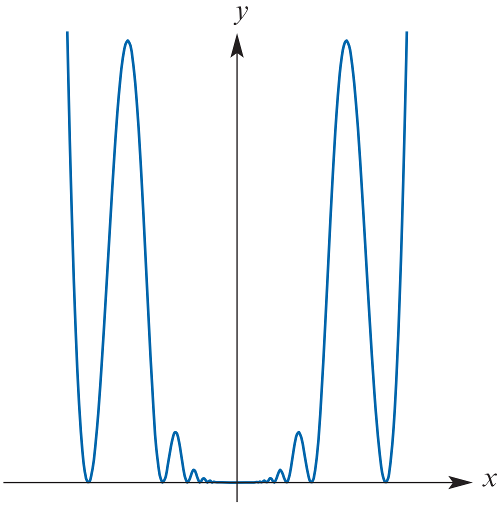
Bad behaviour? In this course, and in school courses around the world, we deal with functions that are 'conveniently behaved'. This avoids some complications.

For an example of a function which is not in this category, consider

$$f(x) = \begin{cases} x^4 \sin^2\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

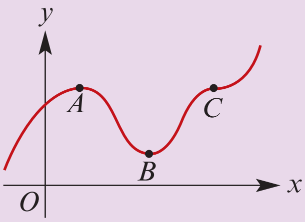
The derivative of this function is defined for all $x \in \mathbb{R}$. In any open interval around $x = 0$, the graph of this function has infinitely many stationary points, no matter how small the interval.



Section summary

A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.

Types of stationary points



A Point A is a **local maximum**:
- $f'(x) > 0$ immediately to the left of A
- $f'(x) < 0$ immediately to the right of A.

B Point B is a **local minimum**:
- $f'(x) < 0$ immediately to the left of B
- $f'(x) > 0$ immediately to the right of B.

C Point C is a **stationary point of inflection**.

Stationary points of types A and B are called **turning points**.

Exercise 8L

Skillsheet

Example 36

SF

1 For each of the following derivative functions, write down the values of x at which the derivative is zero and prepare a gradient table (as in Example 36) showing whether the corresponding points on the graph of $y = f(x)$ are local maximums, local minimums or stationary points of inflection:

a $f'(x) = 4x^2$

b $f'(x) = (x - 2)(x + 5)$

c $f'(x) = (x + 1)(2x - 1)$

d $f'(x) = -x^2 + x + 12$

e $f'(x) = x^2 - x - 12$

f $f'(x) = 5x^4 - 27x^3$

g $f'(x) = (x - 1)(x - 3)$

h $f'(x) = -(x - 1)(x - 3)$

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2 Find the x-coordinates of the stationary points on each of the following curves and state their nature: SF

a $y = x(x^2 - 12)$ **b** $y = x^2(3 - x)$

c $y = x^3 - 5x^2 + 3x + 2$ **d** $y = 3 - x^3$

e $y = 3x^4 + 16x^3 + 24x^2 + 3$ **f** $y = x(x^2 - 1)$

3 Sketch the graph of each of the following, finding **i** axis intercepts and **ii** stationary points:

a $y = 4x^3 - 3x^4$ **b** $y = x^3 - 6x^2$ **c** $y = 3x^2 - x^3$

d $y = x^3 + 6x^2 + 9x + 4$ **e** $y = (x^2 - 1)^5$ **f** $y = (x^2 - 1)^4$

4 **a** Find the stationary points of the graph of $y = 2x^3 + 3x^2 - 12x + 7$, stating the nature of each.

b Show that the graph passes through $(1, 0)$.

c Find the other axis intercepts.

d Sketch the graph.

5 **a** Show that the polynomial $P(x) = x^3 + ax^2 + b$ has a stationary point at $x = 0$ for all a and b. CF

b Given that $P(x)$ has a second stationary point at $(-2, 6)$, find the values of a and b and the nature of both stationary points.

6 Sketch the graph of $f(x) = (2x - 1)^5(2x - 4)^4$.

a State the coordinates of the axis intercepts.

b State the coordinates and nature of each stationary point.

7 **a** Sketch the graphs of $f(x) = (4x^2 - 1)^6$ and $g(x) = (4x^2 - 1)^5$ on the one set of axes.

b **i** Find the values of x for which $f(x) > g(x)$.

ii Find the values of x for which $f'(x) > g'(x)$.

8 Sketch the graph of each of the following. State the axis intercepts and the coordinates of stationary points. SF

a $y = x^3 + x^2 - 8x - 12$ **b** $y = 4x^3 - 18x^2 + 48x - 290$

9 For each of the following, find the coordinates of the stationary points and determine their nature:

a $f(x) = 3x^4 + 4x^3$ **b** $f(x) = x^4 + 2x^3 - 1$

c $f(x) = 3x^3 - 3x^2 + 12x + 9$

10 Consider the function f defined by $f(x) = \frac{1}{8}(x - 1)^3(8 - 3x) + 1$. CF

a Show that $f(0) = 0$ and $f(3) = 0$.

b Show that $f'(x) = \frac{3}{8}(x - 1)^2(9 - 4x)$ and specify the values of x for which $f'(x) \geq 0$.

c Sketch the graph of $y = f(x)$.

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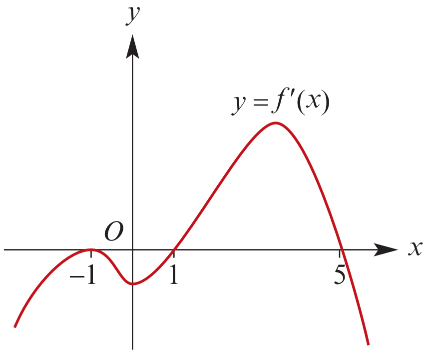
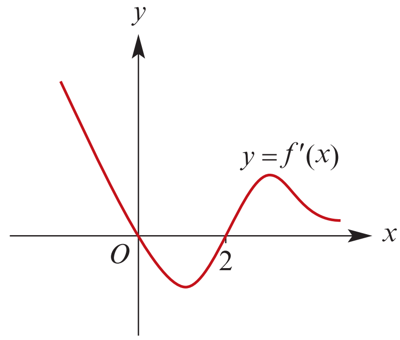
11 Each graph below shows the graph of f' for a function f. Find the values of x for which the graph of $y = f(x)$ has a stationary point and state the nature of each stationary point.

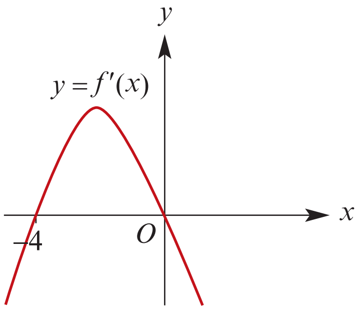
a



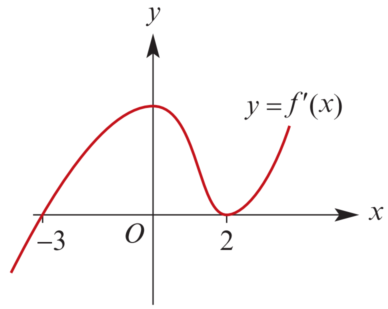
b



c



d



12 Find the coordinates of the stationary points, and state the nature of each, for the curve with equation:

a $y = x^4 - 16x^2$

b $y = x^{2m} - 16x^{2m-2}$, where m is a natural number greater than or equal to 2.

Example 38 **13** Sketch the graph of $f(x) = e^{-\frac{x^2}{2}}$.

14 Let $f(x) = x^2e^x$. Find the values of x for which $f'(x) < 0$.

CF

15 Find the values of x for which $100e^{-x^2+2x-5}$ increases as x increases and hence find the maximum value of $100e^{-x^2+2x-5}$.

16 Let $f(x) = e^x - 1 - x$.

a Find the minimum value of $f(x)$.

b Hence show $e^x \geq 1 + x$ for all real x.

17 For $f(x) = x + e^{-x}$:

a Find the position and nature of any stationary points.

b Find, if they exist, the equations of any asymptotes.

c Sketch the graph of $y = f(x)$.

18 The curve $y = e^x(px^2 + qx + r)$ is such that the tangents at $x = 1$ and $x = 3$ are parallel to the x-axis. The point with coordinates $(0, 9)$ is on the curve. Find p, q and r.

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CF

19 **a** Let $y = e^{4x^2-8x}$. Find $\frac{dy}{dx}$.

b Find the coordinates of the stationary point on the curve of $y = e^{4x^2-8x}$ and state its nature.

c Sketch the graph of $y = e^{4x^2-8x}$.

d Find the equation of the normal to the curve of $y = e^{4x^2-8x}$ at the point where $x = 2$.

20 On the same set of axes, sketch the graphs of $y = \ln x$ and $y = \ln(5x)$, and use them to explain why $\frac{d}{dx}\left(\ln x\right) = \frac{d}{dx}\left(\ln(5x)\right)$.

Example 39 **21** Consider the function f given by $f(x) = x^2 \ln x$ for $x > 0$.

a Find $f'(x)$.

b Solve the equation $f(x) = 0$.

c Solve the equation $f'(x) = 0$.

d Sketch the graph of $y = f(x)$.

SF

22 Find the stationary point on the graph of $y = \frac{x}{\ln x}$ and state its nature.

CF

23 Let $f(x) = x^3 - 3x^2 - 9x + 11$. Sketch the graph of:

a $y = f(x)$ **b** $y = 2f(x)$ **c** $y = f(x + 2)$ **d** $y = f(x - 2)$ **e** $y = -f(x)$

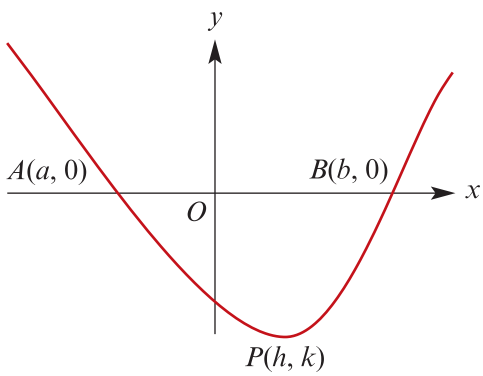
24 Let $f(x) = 2 + 3x - x^3$. Sketch the graph of:

a $y = f(x)$ **b** $y = -2f(x)$ **c** $y = 2f(x - 1)$ **d** $y = f(x) - 3$ **e** $y = 3f(x + 1)$

25 The graph shown opposite has equation $y = f(x)$. Suppose a dilation of factor p from the x-axis followed by a translation of ℓ units in the positive direction of the x-axis is applied to the graph.
For the graph of the image, state:

a the axis intercepts

b the coordinates of the turning point.



Example 40 **26** Find the values of x for which the graph of $y = f(x)$ has a stationary point and state the nature of each stationary point. Consider $0 \le x \le 2\pi$ only.

a $f(x) = 2\cos x - (2\cos^2 x - 1)$

b $f(x) = 2\cos x + 2\sin x \cos x$

c $f(x) = 2\sin x - (2\cos^2 x - 1)$

d $f(x) = 2\sin x + 2\sin x \cos x$

27 The graph of a quartic function passes through the points with coordinates $(1, 21)$, $(2, 96)$, $(5, 645)$, $(6, 816)$ and $(7, 861)$.

a Find the rule of the quartic and plot the graph. Determine the turning points and axis intercepts.

b Plot the graph of the derivative on the same screen.

c Find the value of the function when $x = 10$.

d For what value(s) of x is the value of the function 500?

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Review

AS
Nrich

Chapter summary

Basic derivatives

$f(x)$	$f'(x)$	
c	0	where c is a constant
x^a	ax^{a-1}	where $a \in \mathbb{R} \setminus \{0\}$
e^{ax}	ae^{ax}	
$\ln(ax)$	$\dfrac{1}{x}$	
$\sin(ax+b)$	$a\cos(ax+b)$	
$\cos(ax+b)$	$-a\sin(ax+b)$	

Rules for differentiation

- Chain rule
  - If $h(x) = f(g(x))$, then $h'(x) = f'(g(x))\,g'(x)$
  - $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
- Product rule
  - If $h(x) = f(x)g(x)$, then $h'(x) = f(x)g'(x) + f'(x)g(x)$.
  - If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.
- Quotient rule
  - If $h(x) = \dfrac{f(x)}{g(x)}$, then $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.
  - If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

Tangents and normals

Let (x_1, y_1) be a point on the curve $y = f(x)$. If f is differentiable at $x = x_1$, then

- the equation of the **tangent** to the curve at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$
- the equation of the **normal** to the curve at (x_1, y_1) is given by $y - y_1 = \dfrac{-1}{f'(x_1)}(x - x_1)$.

Motion in a straight line

For an object moving in a straight line with position x at time t:

- velocity $v = \dfrac{dx}{dt}$
- acceleration $a = \dfrac{dv}{dt}$

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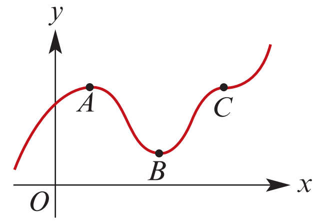
Stationary points

- A point with coordinates $(a, f(a))$ on a curve $y = f(x)$ is a **stationary point** if $f'(a) = 0$.
- The graph shown has three stationary points: A, B and C.



A Point A is a **local maximum** point. Notice that immediately to the left of A the gradient is positive, and immediately to the right the gradient is negative.

B Point B is a **local minimum** point. Notice that immediately to the left of B the gradient is negative, and immediately to the right the gradient is positive.

C Point C is a **stationary point of inflection**.

Stationary points of types A and B are referred to as **turning points**.

Technology-free questions

SF

1 Differentiate each of the following with respect to x:

a $x + \sqrt{1 - x^2}$ **b** $\dfrac{4x+1}{x^2+3}$ **c** $\sqrt{1+3x}$ **d** $\dfrac{2+\sqrt{x}}{x}$

e $(x-9)\sqrt{x-3}$ **f** $x\sqrt{1+x^2}$ **g** $\dfrac{x^2-1}{x^2+1}$ **h** $\dfrac{x}{x^2+1}$

i $(2+5x^2)^{\frac{1}{3}}$ **j** $\dfrac{2x+1}{x^2+2}$ **k** $(3x^2+2)^{\frac{2}{3}}$

2 For each of the following functions, find the gradient of the tangent to the curve at the point corresponding to the given x-value:

a $y = 3x^2 - 4$ at $x = -1$ **b** $y = \dfrac{x-1}{x^2+1}$ at $x = 0$

c $y = (x-2)^5$ at $x = 1$ **d** $y = (2x+2)^{\frac{1}{3}}$ at $x = 3$

3 Differentiate each of the following with respect to x:

a $\ln(x+2)$ **b** $\sin(3x+2)$ **c** $\cos\left(\dfrac{x}{2}\right)$ **d** e^{x^2-2x}

e $\ln(3-x)$ **f** $\sin(2\pi x)$ **g** $\sin^2(3x+1)$ **h** $\sqrt{\ln x},\ x > 1$

i $\dfrac{2\ln(2x)}{x}$ **j** $x^2\sin(2\pi x)$

4 Differentiate each of the following with respect to x:

a $e^x\sin(2x)$ **b** $2x^2\ln x$ **c** $\dfrac{\ln x}{x^3}$ **d** $\sin(2x)\cos(3x)$

e $\dfrac{\sin(2x)}{\cos(2x)}$ **f** $\cos^3(3x+2)$ **g** $x^2\sin^2(3x)$

5 Find the gradient of each of the following curves at the stated value of x:

a $y = e^{2x} + 1,\ x = 1$ **b** $y = e^{x^2+1},\ x = 0$

c $y = 5e^{3x} + x^2,\ x = 1$ **d** $y = 5 - e^{-x},\ x = 0$

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Review

SF

6 Differentiate each of the following with respect to x:

a e^{ax} **b** e^{ax+b} **c** e^{a-bx} **d** $be^{ax} - ae^{bx}$ **e** $\dfrac{e^{ax}}{e^{bx}}$

CF

7 **a** For $y = \dfrac{2x - 3}{x^2 + 4}$, show that $\dfrac{dy}{dx} = \dfrac{8 + 6x - 2x^2}{(x^2 + 4)^2}$.

b Find the values of x for which both y and $\dfrac{dy}{dx}$ are positive.

8 Find the derivative of each of the following, given that the function f is differentiable for all real numbers:

a $xf(x)$ **b** $\dfrac{1}{f(x)}$ **c** $\dfrac{x}{f(x)}$ **d** $\dfrac{x^2}{[f(x)]^2}$

9 **a** Find the equation of the tangent to the curve $y = x^3 - 8x^2 + 15x$ at the point with coordinates $(4, -4)$.

b Find the coordinates of the point where the tangent meets the curve again.

10 Find the equation of the tangent to the curve $y = 3x^2$ at the point where $x = a$. If this tangent meets the y-axis at P, find the y-coordinate of P in terms of a.

11 **a** Find the equation of the tangent to the curve with equation $y = x^3 - 7x^2 + 14x - 8$ at the point where $x = 1$.

b Find the x-coordinate of a second point on this curve at which the tangent is parallel to the tangent at $x = 1$.

12 Use the formula $A = \pi r^2$ for the area of a circle to find:

a the average rate at which the area of a circle changes with respect to the radius as the radius increases from $r = 2$ to $r = 3$

b the instantaneous rate at which the area changes with respect to r when $r = 3$.

13 For the function with rule $f(x) = (x - 1)^{\frac{4}{5}}$:

a State the values of x for which the function is differentiable, and find the rule for f'.

b Find the equations of the tangents at the points $(2, 1)$ and $(0, 1)$.

c Find the coordinates of the point of intersection of the two tangents.

14 A vehicle is travelling in a straight line away from a point O. Its distance from O after t seconds is $0.25e^t$ metres. Find the velocity and acceleration of the vehicle at $t = 0$, $t = 1$ and $t = 4$.

15 The temperature, θ°C, of material inside a nuclear power station at time t seconds after a reaction begins is given by $\theta = \dfrac{1}{4}e^{100t}$.

a Find the rate of increase of temperature at time t.

b Find the rate of increase of temperature when $t = \dfrac{1}{20}$.

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16 Find the equation of the tangent to $y = e^x$ at $(1, e)$.

17 The diameter of a tree (D cm) t years after 1 January 2010 is given by $D = 50e^{kt}$.

a Prove that $\frac{dD}{dt} = cD$ for some constant c.

b If $k = 0.2$, find the rate of increase of D when $D = 100$.

18 **a** Find the equation of the tangent to $y = \ln x$ at the point $(e, 1)$.

b Find the equation of the tangent to $y = 2\sin\left(\frac{x}{2}\right)$ at the point $\left(\frac{\pi}{2}, \sqrt{2}\right)$.

c Find the equation of the tangent to $y = \cos x$ at the point $\left(\frac{3\pi}{2}, 0\right)$.

d Find the equation of the tangent to $y = \ln(x^2)$ at the point $(-\sqrt{e}, 1)$.

19 For each of the following, find the stationary points of the graph and state their nature:

a $f(x) = 4x^3 - 3x^4$ **b** $g(x) = x^3 - 3x - 2$ **c** $h(x) = x^3 - 9x + 1$

20 Sketch the graph of $y = x^3 - 6x^2 + 9x$.

21 The derivative of the function $y = f(x)$ is $\frac{dy}{dx} = (x-1)^2(x-2)$. Find the x-coordinate and state the nature of each stationary point.

22 Find the minimum value of $e^{3x} + e^{-3x}$.

Multiple-choice questions

1 The average rate of change of the function with rule $f(x) = e^x + x^3$ for $x \in [0, 1]$ is

A e **B** $e^3 + 1$ **C** $\frac{e^3 + 1}{2}$ **D** $e + 1$ **E** $e^x + 3x^2$

2 Let $f(x) = 5 + \frac{5}{(7-x)^2}$ for $x \neq 7$. Then $f'(x) > 0$ for

A $x \neq 7$ **B** $x \in \mathbb{R}$ **C** $x < 7$ **D** $x > 7$ **E** $x > 5$

3 Let $y = f(g(x))$ where $g(x) = 2x^4$. Then $\frac{dy}{dx}$ is equal to

A $8x^3 f'(2x^4)$ **B** $8x^2 f(4x^3)$ **C** $8x^4 f(x) f'(x^3)$

D $2f(x) f'(x^3)$ **E** $8x^3$

4 Which of the following is *not true* for the curve of $y = f(x)$ where $f(x) = x^{\frac{1}{3}}$?

A The gradient is defined for all real numbers.

B The curve passes through the origin.

C The curve passes through the points with coordinates $(1, 1)$ and $(-1, -1)$.

D For $x > 0$, the gradient is positive.

E For $x > 0$, the gradient is decreasing.

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Review

5 The graph of the function $y = \dfrac{k}{2(x^3+1)}$ has gradient 1 when $x = 1$. The value of k is

A 1 **B** $\dfrac{-8}{3}$ **C** $\dfrac{-1}{2}$ **D** -4 **E** $\dfrac{-1}{4}$

6 If $y = \sqrt{3 - 2f(x)}$, then $\dfrac{dy}{dx}$ is equal to

A $\dfrac{2f'(x)}{\sqrt{3-2f(x)}}$ **B** $\dfrac{-1}{2\sqrt{3-2f(x)}}$ **C** $\dfrac{1}{2}\sqrt{3-2f'(x)}$

D $\dfrac{3}{2(3-2f'(x))}$ **E** $\dfrac{-f'(x)}{\sqrt{3-2f(x)}}$

7 The derivative of $e^{-2ax}\cos(ax)$ with respect to x is

A $-ae^{-2ax}\cos(ax) - 2ae^{-2ax}\sin(ax)$ **B** $ae^{-2ax}\cos(ax) - 2ae^{-2ax}\sin(ax)$

C $-2ae^{-2ax}\cos(ax) - ae^{-2ax}\sin(ax)$ **D** $2ae^{-2ax}\cos(ax) + 2ae^{-2ax}\sin(ax)$

E $-ae^{-2ax}\cos(ax) - 2ae^{2ax}\sin(ax)$

8 If $f(x) = \dfrac{\cos x}{x-a}$, where a is a constant, then $f'(x)$ is

A $\dfrac{\sin x}{x-a} + \dfrac{\cos x}{(x-a)^2}$ **B** $-\dfrac{\sin x}{x-a} - \dfrac{\cos x}{(x-a)^2}$ **C** $\dfrac{\sin x}{x-a} - \dfrac{\cos x}{(x-a)^2}$

D $\dfrac{x\sin x}{x-a} + \dfrac{x\cos x}{(x-a)^2}$ **E** $\dfrac{\sin x}{x} - \dfrac{\cos x}{x}$

9 The line with equation $y = 4x + c$ is a tangent to the curve with equation $y = x^2 - x - 5$. The value of c is

A $-\dfrac{45}{4}$ **B** $-1 + 2\sqrt{2}$ **C** 2 **D** $\dfrac{5}{2}$ **E** $-\dfrac{2}{5}$

10 The equation of the tangent to the curve with equation $y = x^4$ at the point where $x = 1$ is

A $y = -4x - 3$ **B** $y = \dfrac{1}{4}x - 3$ **C** $y = -4x$

D $y = \dfrac{1}{4}x + \dfrac{5}{4}$ **E** $y = 4x - 3$

11 The equation of the normal to the curve with equation $y = x^2$ at the point where $x = a$ is

A $y = \dfrac{-1}{2a}x + 2 + a^2$ **B** $y = \dfrac{-1}{2a}x + \dfrac{1}{2} + a^2$ **C** $y = 2ax - a^2$

D $y = 2ax + 3a^2$ **E** $y = \dfrac{1}{2a}x + 2 + a^2$

12 The equation of the tangent to $y = e^{ax}$ at the point $\left(\dfrac{1}{a}, e\right)$ is

A $y = e^{ax-1} + 1$ **B** $y = ae^{ax}x$ **C** $y = 1 - ae^{ax}$ **D** $y = \dfrac{e^2x}{a}$ **E** $y = aex$

13 Under certain conditions, the number of bacteria, N, in a sample increases with time, t hours, according to the rule $N = 4000e^{0.2t}$. The rate, to the nearest whole number of bacteria per hour, that the bacteria are growing 3 hours from the start is

A 1458 **B** 7288 **C** 16 068 **D** 80 342 **E** 109 731

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14 The gradient of the tangent to the curve $y = x^2 \cos(5x)$ at the point where $x = \pi$ is

A $5\pi^2$ **B** $-5\pi^2$ **C** 5π **D** -5π **E** -2π

15 The equation of the tangent to the curve with equation $y = e^{-x} - 1$ at the point where the curve crosses the y-axis is

A $y = x$ **B** $y = -x$ **C** $y = \frac{1}{2}x$ **D** $y = -\frac{1}{2}x$ **E** $y = -2x$

16 For $f(x) = x^3 - x^2 - 1$, the values of x for which the graph of $y = f(x)$ has stationary points are

A $\frac{2}{3}$ only **B** 0 and $\frac{2}{3}$ **C** 0 and $-\frac{2}{3}$ **D** $-\frac{1}{3}$ and 1 **E** $\frac{1}{3}$ and -1

17 For $f(x) = e^x - ex$, the coordinates of the turning point of the graph of $y = f(x)$ are

A $\left(1, \frac{1}{e}\right)$ **B** $(1, e)$ **C** $(0, 1)$ **D** $(1, 0)$ **E** $(e, 1)$

18 For $f(x) = e^{ax} - \frac{ax}{e}$, the coordinates of the turning point of the graph of $y = f(x)$ are

A $\left(-\frac{1}{a}, 0\right)$ **B** $\left(\frac{1}{a}, \frac{1}{e}\right)$ **C** $\left(-\frac{1}{a}, \frac{2}{e}\right)$ **D** $\left(-1, \frac{1}{e}\right)$ **E** $(1, 0)$

Extended-response questions

1 The amount of salt (s grams) in 100 litres of salt solution at time t minutes is given by CF

$$s = 50 + 30e^{-\frac{1}{5}t}$$

a Find the amount of salt in the mixture after 10 minutes.

b Sketch the graph of s against t for $t \geq 0$.

c Find the rate of change of the amount of salt at time t (in terms of t).

d Find the rate of change of the amount of salt at time t (in terms of s).

e Find the concentration (grams per litre) of salt at time $t = 0$.

f Find the value of t for which the salt solution first reaches a concentration of 0.51 grams per litre.

2 A medium is kept at a constant temperature of 20°C. An object is placed in this medium. The temperature, T°C, of the object at time t minutes is given by

$$T = 40e^{-0.36t} + 20, \quad t \geq 0$$

a Find the initial temperature of the object.

b Sketch the graph of T against t for $t \geq 0$.

c Find the rate of change of temperature with respect to time (in terms of t).

d Find the rate of change of temperature with respect to time (in terms of T).

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Review

CU

3 A certain food is susceptible to contamination from bacterial spores of two types, F and G. In order to kill the spores, the food is heated to a temperature of 120°C. The number of live spores after t minutes can be approximated by $f(t) = 1000e^{-0.5t}$ for F-type spores and by $g(t) = 1200e^{-0.7t}$ for G-type spores.

a Find the time required to kill 50% of the F-type spores.

b Find the total number of live spores of both types when $t = 0$, and find the percentage of these that are still alive when $t = 5$.

c Find the rate at which the total number of live spores is decreasing when $t = 5$.

d Find the value of t for which the number of live F-type spores and the number of live G-type spores are equal.

e On the same set of axes, sketch the graphs of $y = f(t)$ and $y = g(t)$ for $t \geq 0$.

4 An object falls from rest in a medium and its velocity, V m/s, after t seconds is given by $V = 100(1 - e^{-0.2t})$.

a Sketch the graph of V against t for $t \geq 0$.

b Express the acceleration at any instant:

i in terms of t **ii** in terms of V.

c Find the value of t for which the velocity of the object is 80 m/s.

5 The depth of water at the entrance to a harbour t hours after high tide is D m, where $D = p + q\cos(rt)°$ for suitable constants p, q, r. At high tide the depth is 7 m; at low tide, 6 hours later, the depth is 3 m.

a Show that $r = 30$ and find the values of p and q.

b Sketch the graph of D against t for $0 \leq t \leq 12$.

c Find the average rate of change of depth with respect to time over the period from high tide to low tide.

d Find the instantaneous rate of change of the depth of the water at time t hours.

e Find how soon after low tide a ship that requires a depth of at least 4 m of water will be able to enter the harbour.

6 Let $f(x) = x^3 - 3x^2 + 6x - 10$.

a Find the coordinates of the point on the graph of f for which $f'(x) = 3$.

b Express $f'(x)$ in the form $a(x + p)^2 + q$.

c Hence show that the gradient of f is greater than 3 for all points on the curve of f other than the point found in **a**.

7 The kangaroo population in a certain confined region is given by $f(x) = \dfrac{100\,000}{1 + 100e^{-0.3x}}$, where x is the time in years.

a Find $f'(x)$.

b Find the rate of growth of the kangaroo population when:

i $x = 0$ **ii** $x = 4$

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8 Consider the function with rule $f(x) = 8\ln(6 - 0.2x)$ for $x < a$, where a is the largest possible value such that f can be defined in this way.

a What is the value of a?

b Find the exact values for the coordinates of the points where the graph of $y = f(x)$ crosses each axis.

c Find the gradient of the tangent to the graph of $y = f(x)$ at the point where $x = 20$.

d Sketch the graph of $y = f(x)$.

9 **a** Show that the tangent to the graph of $y = e^x$ for $x = 0$ has equation $y = x + 1$.

b Plot the graphs of $y = e^x$ and $y = x + 1$ on a calculator.

c Let $f(x) = e^x$ and $g(x) = x + 1$. Use a calculator to investigate functions of the form

$$h(x) = af(x - b) + c \quad \text{and} \quad k(x) = ag(x - b) + c$$

Comment on your observations.

d Use the chain rule and properties of transformations to prove that, if the tangent to the curve $y = f(x)$ at the point (x_1, y_1) has equation $y = mx + c$, then the tangent to the curve $y = af(bx)$ at the point $\left(\frac{x_1}{b}, y_1 a\right)$ has equation $y = a(mbx + c)$.

10 A culture contains 1000 bacteria and 5 hours later the number has increased to 10 000. The number, N, of bacteria present at any time, t hours, is given by $N = Ae^{kt}$.

a Find the values of A and k.

b Find the rate of growth at time t.

c Show that, at time t, the rate of growth is proportional to the number of bacteria present.

d Find this rate of growth when:

i $t = 4$ **ii** $t = 50$

11 The populations of two ant colonies, A and B, are increasing according to the rules:

A population $= 2 \times 10^4 e^{0.03t}$

B population $= 10^4 e^{0.05t}$

After how many years will their populations:

a be equal **b** be increasing at the same rate?

12 A particle on the end of a spring, which is hanging vertically, is oscillating such that its height, h metres, above the floor after t seconds is given by

$$y = 0.5 + 0.2\sin(3\pi t), \quad t \geq 0$$

a Find the greatest height above the floor and the time at which this height is first reached.

b Find the period of oscillation.

c Find the speed of the particle when $t = \frac{1}{3}, \frac{2}{3}, \frac{1}{6}$.

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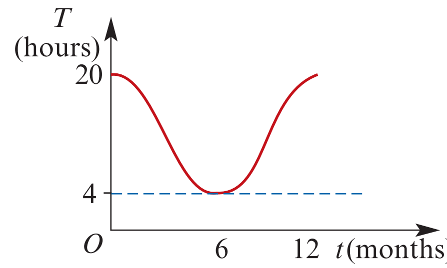
Review

CU

13 The length of night on Seal Island varies between 20 hours in midwinter and 4 hours in midsummer. The relationship between T, the number of hours of night, and t, the number of months past the longest night in 2010, is given by

$$T(t) = p + q\cos(\pi r t)$$

where p, q and r are constants.
Assume that the year consists of 12 months of equal length.
The graph of T against t is illustrated.



a Find the value of:

i r **ii** p and q

b Find $T'(3)$ and $T'(9)$ and find the rate of change of hours of night with respect to the number of months.

c Find the average rate of change of hours of night from $t = 0$ to $t = 6$.

d After how many months is the rate of change of hours of night a maximum?

14 A curve with equation of the form $y = ax^3 + bx^2 + cx + d$ has zero gradient at the point $\left(\frac{1}{3}, \frac{4}{27}\right)$ and also touches, but does not cross, the x-axis at the point $(1, 0)$.

a Find a, b, c and d.

b Find the values of x for which the curve has a negative gradient.

c Sketch the curve.

15 **a** For the function with rule $f(x) = x^3 + ax^2 + bx$, plot the graph of each of the following using a calculator. (Give axis intercepts, coordinates of stationary points and the nature of stationary points.)

i $a = 1,\ b = 1$ **ii** $a = -1,\ b = -1$

iii $a = 1,\ b = -1$ **iv** $a = -1,\ b = 1$

b **i** Find $f'(x)$.

ii Solve the equation $f'(x) = 0$ for x, giving your answer in terms of a and b.

c **i** Show that the graph of $y = f(x)$ has exactly one stationary point if $a^2 - 3b = 0$.

ii If $b = 3$, find the corresponding value(s) of a which satisfy $a^2 - 3b = 0$. Find the coordinates of the stationary points and state the nature of each.

iii Using a calculator, plot the graph(s) of $y = f(x)$ for these values of a and b.

iv Plot the graphs of the corresponding derivative functions on the same set of axes.

d State the relationship between a and b if no stationary points exist for the graph of $y = f(x)$.

16 For what value of x is $\dfrac{\ln x}{x}$ a maximum? That is, when is the ratio of the logarithm of a number to the number a maximum?

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17 For the quartic function f with rule $f(x) = (x - a)^2(x - b)^2$, where $a > 0$ and $b > 0$:

a Show that $f'(x) = 2(x - a)(x - b)[2x - (b + a)]$.

b **i** Solve the equation $f'(x) = 0$ for x.

ii Solve the equation $f(x) = 0$ for x.

c Hence find the coordinates of the stationary points of the graph of $y = f(x)$.

d Using a calculator, plot the graph of $y = f(x)$ for several values of a and b.

e **i** If $a = b$, then $f(x) = (x - a)^4$. Sketch the graph of $y = f(x)$.

ii If $a = -b$, find the coordinates of the stationary points.

iii Plot the graph of $y = f(x)$ for several values of a, given that $a = -b$.

18 For the quartic function f with rule $f(x) = (x - a)^3(x - b)$, where $a > 0$ and $b > 0$:

a Show that $f'(x) = (x - a)^2[4x - (3b + a)]$.

b **i** Solve the equation $f'(x) = 0$ for x.

ii Solve the equation $f(x) = 0$ for x.

c Find the coordinates of the stationary points of the graph of $y = f(x)$ and state the nature of the stationary points.

d Using a calculator, plot the graph of $y = f(x)$ for several values of a and b.

e If $a = -b$, state the coordinates of the stationary points in terms of a.

f **i** State the relationship between b and a if there is a local minimum for $x = 0$.

ii Illustrate this for $b = 1$ and $a = -3$ on a calculator.

g Show that, if there is a turning point for $x = \frac{a + b}{2}$, then $b = a$ and $f(x) = (x - a)^4$.

19 **a** Find the equation of the tangent to the curve $y = e^x$ at the point $(1, e)$.

b Find the equation of the tangent to the curve $y = e^{2x}$ at the point $(\frac{1}{2}, e)$.

c Find the equation of the tangent to the curve $y = e^{kx}$ at the point (k^{-1}, e).

d Show that $y = xke$ is the only tangent to the curve $y = e^{kx}$ which passes through the origin.

e Hence determine for what values of k the equation $e^{kx} = x$ has:

i a unique real solution **ii** no real solution.

20 A certain chemical starts to dissolve in water at time $t = 0$. It is known that, if x is the number of grams not dissolved after t hours, then

$$x = \frac{60}{5e^{\lambda t} - 3}, \quad \text{where } \lambda = \frac{1}{2}\ln\left(\frac{6}{5}\right)$$

a Find the amount of chemical present when:

i $t = 0$ **ii** $t = 5$

b Find $\frac{dx}{dt}$ in terms of t.

c **i** Show that $\frac{dx}{dt} = -\lambda x - \frac{\lambda x^2}{20}$.

ii Sketch the graph of $\frac{dx}{dt}$ against x for $x > 0$.

iii Write a short explanation of your result.

ISBN 978-1-108-45164-2

Chapter 9

9 Anti-differentiation

Objectives

- To find **anti-derivatives** of polynomial functions, power functions, exponential functions and trigonometric functions.
- To understand the properties of anti-differentiation applied to sums, differences and multiples of functions.
- To find the rule for a function given one point on its graph and the rule for its derivative function.
- To use anti-differentiation to **solve problems**.
- To apply anti-differentiation to **motion in a straight line**.

In the previous two chapters, we have been given the rule for a function and found a rule for its derivative function. In this chapter, we try to go in the opposite direction:

- If we are given the rule for the derivative of a function, can we find a rule for the original function? What extra information do we need?
- For example, if we are given the rule for the velocity of an object moving in a straight line, can we find a rule for its position? What extra information do we need?

The process of 'undoing the derivative' is called anti-differentiation.

In the first case, the required extra information is a point on the graph of the function, and in the second case it is the position of the object at a particular time.

In the next chapter, we will see a more surprising application of anti-differentiation to finding the areas of regions bounded by curves.

Chapters 9 and 10 cover Unit 3 Topic 3: Integrals.

ISBN 978-1-108-45164-2

9A Anti-differentiation of polynomial functions

The derivative of x^2 with respect to x is $2x$. Conversely, given that an unknown expression has derivative $2x$, it is clear that the unknown expression could be x^2. The process of finding a function from its derivative is called **anti-differentiation**.

Now consider the functions $f(x) = x^2 + 1$ and $g(x) = x^2 - 7$.

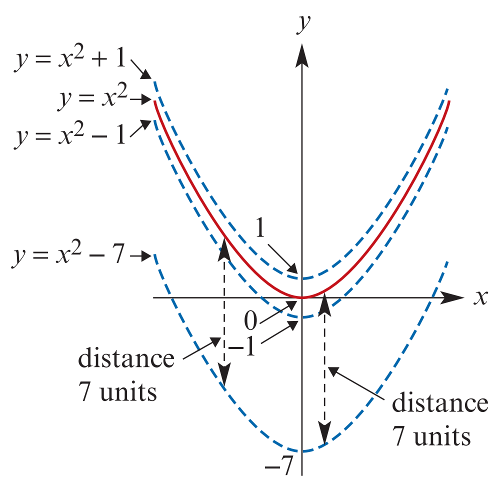
We have $f'(x) = 2x$ and $g'(x) = 2x$. So the two different functions have the same derivative function.

Both $x^2 + 1$ and $x^2 - 7$ are said to be **anti-derivatives** of $2x$.

If two functions have the same derivative function, then they differ by a constant. So the graphs of the two functions can be obtained from each other by translation parallel to the y-axis.

The diagram shows several anti-derivatives of $2x$.

Each of the graphs is a translation of $y = x^2$ parallel to the y-axis.



Notation

The general anti-derivative of $2x$ is $x^2 + c$, where c is an arbitrary real number. We use the notation of Leibniz to state this with symbols:

$$\int 2x\,dx = x^2 + c$$

This is read as 'the **general anti-derivative** of $2x$ with respect to x is equal to $x^2 + c$' or as 'the **indefinite integral** of $2x$ with respect to x is $x^2 + c$'.

To be more precise, the indefinite integral is the set of all anti-derivatives and to emphasise this we could write:

$$\begin{aligned}\int 2x\,dx &= \{\, f(x) : f'(x) = 2x \,\}\\ &= \{\, x^2 + c : c \in \mathbb{R} \,\}\end{aligned}$$

This set notation is not commonly used, but it should be clearly understood that there is not a unique anti-derivative for a given function. We will not use this set notation, but it is advisable to keep it in mind when considering further results.

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In general:

> If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + c$, where c is an arbitrary real number.

Rules for anti-differentiation

We know that:

$f(x) = x^3$ implies $f'(x) = 3x^2$

$f(x) = x^8$ implies $f'(x) = 8x^7$

$f(x) = x$ implies $f'(x) = 1$

$f(x) = x^n$ implies $f'(x) = nx^{n-1}$

Reversing this process we have:

$\int 3x^2\,dx = x^3 + c$ where c is an arbitrary constant

$\int 8x^7\,dx = x^8 + c$ where c is an arbitrary constant

$\int 1\,dx = x + c$ where c is an arbitrary constant

$\int nx^{n-1}\,dx = x^n + c$ where c is an arbitrary constant

We also have:

$$\int x^2\,dx = \frac{1}{3}x^3 + c$$

$$\int x^7\,dx = \frac{1}{8}x^8 + c$$

$$\int 1\,dx = x + c$$

$$\int x^{n-1}\,dx = \frac{1}{n}x^n + c$$

From this we see that:

> $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad \text{where } n \in \mathbb{N} \cup \{0\}$$

We also record the following results, which follow immediately from the corresponding results for differentiation:

Sum $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

Difference $\int f(x) - g(x)\,dx = \int f(x)\,dx - \int g(x)\,dx$

Multiple $\int kf(x)\,dx = k\int f(x)\,dx$, where k is a real number

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Example 1

Find the general anti-derivative (indefinite integral) of each of the following:

a $3x^5$

b $3x^2 + 4x^3 + 3$

Solution

a
$$\int 3x^5 \, dx = 3\int x^5 \, dx$$
$$= 3 \times \frac{x^6}{6} + c$$
$$= \frac{x^6}{2} + c$$

b
$$\int 3x^2 + 4x^3 + 3 \, dx$$
$$= 3\int x^2 \, dx + 4\int x^3 \, dx + 3\int 1 \, dx$$
$$= \frac{3x^3}{3} + \frac{4x^4}{4} + \frac{3x}{1} + c$$
$$= x^3 + x^4 + 3x + c$$

Given extra information, we can find a unique anti-derivative.

Example 2

It is known that $f'(x) = x^3 + 4x^2$ and $f(0) = 0$. Find $f(x)$.

Solution

$$\int x^3 + 4x^2 \, dx = \frac{x^4}{4} + \frac{4x^3}{3} + c$$

Thus $f(x) = \dfrac{x^4}{4} + \dfrac{4x^3}{3} + c$ for some real number c.

Since $f(0) = 0$, we have $c = 0$.

$$\therefore \quad f(x) = \frac{x^4}{4} + \frac{4x^3}{3}$$

Example 3

If the gradient of the tangent at a point (x, y) on a curve is given by $5x$ and the curve passes through the point $(0, 6)$, find the equation of the curve.

Solution

Let the curve have equation $y = f(x)$. Then $f'(x) = 5x$.

$$\int 5x \, dx = \frac{5x^2}{2} + c$$

$$\therefore \quad f(x) = \frac{5x^2}{2} + c$$

This describes the family of curves for which $f'(x) = 5x$. Here we are given the additional information that the curve passes through the point $(0, 6)$, i.e. $f(0) = 6$.

Hence $c = 6$ and so $f(x) = \dfrac{5x^2}{2} + 6$.

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Example 4

Find y in terms of x if:

a $\dfrac{dy}{dx} = x^2 + 2x$, and $y = 1$ when $x = 1$

b $\dfrac{dy}{dx} = 3 - x$, and $y = 2$ when $x = 4$

Solution

a $\int x^2 + 2x\,dx = \dfrac{x^3}{3} + x^2 + c$

$\therefore \quad y = \dfrac{x^3}{3} + x^2 + c$

As $y = 1$ when $x = 1$,

$1 = \dfrac{1}{3} + 1 + c$

$c = -\dfrac{1}{3}$

Hence $y = \dfrac{x^3}{3} + x^2 - \dfrac{1}{3}$

b $\int 3 - x\,dx = 3x - \dfrac{x^2}{2} + c$

$\therefore \quad y = 3x - \dfrac{x^2}{2} + c$

As $y = 2$ when $x = 4$,

$2 = 3 \times 4 - \dfrac{4^2}{2} + c$

$c = -2$

Hence $y = 3x - \dfrac{x^2}{2} - 2$

Section summary

- Anti-derivative of x^n, where $n \in \mathbb{N} \cup \{0\}$:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

- Properties of anti-differentiation:
  - $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
  - $\int f(x) - g(x)\,dx = \int f(x)\,dx - \int g(x)\,dx$
  - $\int kf(x)\,dx = k\int f(x)\,dx$, where k is a real number

Exercise 9A

SF

Example 1

1 Find:

a $\int \frac{1}{2}x^3\,dx$ **b** $\int 3x^2 - 2\,dx$ **c** $\int 5x^3 - 2x\,dx$

d $\int \frac{4}{5}x^3 - 2x^2\,dx$ **e** $\int (x-1)^2\,dx$ **f** $\int x(x + \frac{1}{x})\,dx,\ x \neq 0$

g $\int 2z^2(z-1)\,dz$ **h** $\int (2t-3)^2\,dt$ **i** $\int (t-1)^3\,dt$

Example 2

2 It is known that $f'(x) = 4x^3 + 6x^2 + 2$ and $f(0) = 0$. Find $f(x)$.

Example 3

3 If the gradient at a point (x, y) on a curve is given by $6x^2$ and the curve passes through the point $(0, 12)$, find the equation of the curve.

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SF

Example 4

4 Find y in terms of x in each of the following:

a $\dfrac{dy}{dx} = 2x - 1$, and $y = 0$ when $x = 1$

b $\dfrac{dy}{dx} = 3 - x$, and $y = 1$ when $x = 0$

c $\dfrac{dy}{dx} = x^2 + 2x$, and $y = 2$ when $x = 0$

d $\dfrac{dy}{dx} = 3 - x^2$, and $y = 2$ when $x = 3$

e $\dfrac{dy}{dx} = 2x^4 + x$, and $y = 0$ when $x = 0$

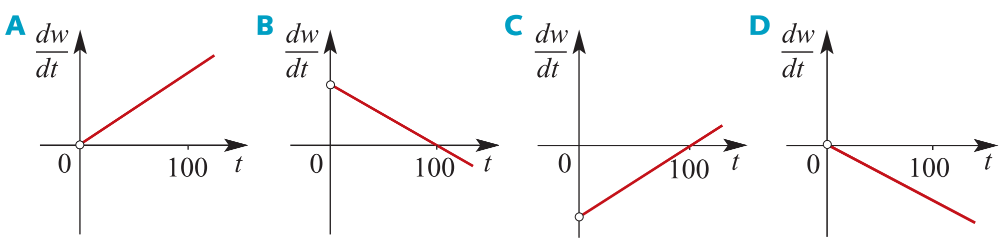
5 Assume that $\dfrac{dV}{dt} = t^2 - t$ for $t > 1$, and that $V = 9$ when $t = 3$.

a Find V in terms of t.

b Calculate the value of V when $t = 10$.

6 The gradient of the tangent at any point $(x, f(x))$ on the curve with equation $y = f(x)$ is given by $3x^2 - 1$. Find $f(x)$ if the curve passes through the point $(1, 2)$, i.e. $f(1) = 2$.

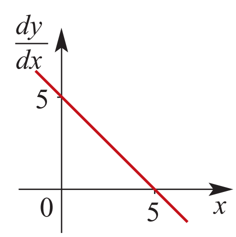
7 **a** Which one of the following graphs represents $\dfrac{dw}{dt} = 2000 - 20t$, $t > 0$?



b Find w in terms of t if when $t = 0$, $w = 100\,000$.

8 The graph shows $\dfrac{dy}{dx}$ against x for a certain curve with equation $y = f(x)$.

Find $f(x)$, given that the point $(0, 4)$ lies on the curve.



9 Find the equation of the curve $y = f(x)$ which passes through the point $(2, -6)$ and for which $f'(x) = x^2(x - 3)$.

CF

10 The curve $y = f(x)$ for which $f'(x) = 4x + k$, where k is a constant, has a turning point at $(-2, -1)$.

a Find the value of k.

b Find the coordinates of the point at which the curve meets the y-axis.

11 Given that $\dfrac{dy}{dx} = ax^2 + 1$ and that when $x = 1$, $\dfrac{dy}{dx} = 3$ and $y = 3$, find the value of y when $x = 2$.

ISBN 978-1-108-45164-2

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12 The curve for which $\frac{dy}{dx} = 2x + k$, where k is a constant, is such that the tangent at $(3, 6)$ passes through the origin. Find the gradient of this tangent and hence determine:

a the value of k

b the equation of the curve.

13 The curve $y = f(x)$ for which $f'(x) = 16x + k$, where k is a constant, has a turning point at $(2, 1)$. Find:

a the value of k

b the value of $f(x)$ when $x = 7$.

14 Suppose that a point moves along some unknown curve $y = f(x)$ in such a way that, at each point (x, y) on the curve, the tangent line has slope x^2. Find an equation for the curve, given that it passes through $(2, 1)$.

9B Anti-differentiation of power functions

A power function has a rule of the form $f(x) = x^r$, where r is a rational number. In this section, we consider linear combinations of power functions:

e.g. $f(x) = 5x + 7x^{\frac{1}{2}}$ for $x > 0$

$f(x) = 2x^{-2} - 3x^{-3}$ for $x \neq 0$

We restrict to power functions $f(x) = x^r$ where $r \neq -1$. We will consider the case $r = -1$ in the next section.

▶ The anti-derivative of x^r where $r \neq -1$

We know that:

$f(x) = x^{\frac{3}{2}}$ implies $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$

$f(x) = x^{-4}$ implies $f'(x) = -4x^{-5}$

Reversing this process gives:

$\int \frac{3}{2}x^{\frac{1}{2}}\,dx = x^{\frac{3}{2}} + c$ where c is an arbitrary constant

$\int -4x^{-5}\,dx = x^{-4} + c$ where c is an arbitrary constant

We can also show that:

$\int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}} + c$ $\qquad$ $\int x^{-5}\,dx = -\frac{1}{4}x^{-4} + c$

Generalising, it is seen that:

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c \qquad \text{where } r \in \mathbb{Q} \setminus \{-1\}$$

Note: This result can only be applied for suitable values of x for a given value of r. For example, if $r = \frac{1}{2}$, then $x \in \mathbb{R}^+$ is a suitable restriction. If $r = -2$, we can take $x \in \mathbb{R} \setminus \{0\}$, and if $r = 3$, we can take $x \in \mathbb{R}$.

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Example 5

Find y in terms of x if:

a $\dfrac{dy}{dx} = \dfrac{1}{x^2}$ **b** $\dfrac{dy}{dx} = 3\sqrt{x}$ **c** $\dfrac{dy}{dx} = x^{\frac{3}{4}} + x^{-\frac{3}{4}}$

Solution

a
$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + c$$
$$\therefore \quad y = \frac{-1}{x} + c$$

b
$$\int 3\sqrt{x}\,dx = 3\int x^{\frac{1}{2}}\,dx = 3 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$\therefore \quad y = 2x^{\frac{3}{2}} + c$$

c
$$y = \frac{4}{7}x^{\frac{7}{4}} + 4x^{\frac{1}{4}} + c$$

Example 6

It is known that $f'(x) = x - 3x^{\frac{1}{2}}$ and $f(1) = 3$. Find $f(x)$.

Solution

$$\int x - 3x^{\frac{1}{2}}\,dx = \frac{1}{2}x^2 - 3 \times \frac{2}{3}x^{\frac{3}{2}} + c$$

$$\therefore \quad f(x) = \frac{1}{2}x^2 - 2x^{\frac{3}{2}} + c$$

As $f(1) = 3$, we have

$$3 = \frac{1}{2} - 2 + c$$

$$\therefore \quad c = \frac{9}{2}$$

Hence $\quad f(x) = \dfrac{1}{2}x^2 - 2x^{\frac{3}{2}} + \dfrac{9}{2}$

Section summary

Anti-derivative of x^r, where $r \in \mathbb{Q} \setminus \{-1\}$:

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c$$

Exercise 9B

Skillsheet

Example 5

1 Find y in terms of x if:

a $\dfrac{dy}{dx} = \dfrac{1}{x^3}$ **b** $\dfrac{dy}{dx} = 4\sqrt[3]{x}$ **c** $\dfrac{dy}{dx} = x^{\frac{1}{4}} + x^{-\frac{3}{5}}$

SF

2 Find:

a $\int 3x^{-2}\,dx$ **b** $\int 2x^{-4} + 6x\,dx$ **c** $\int 2x^{-2} + 6x^{-3}\,dx$

d $\int 3x^{\frac{1}{3}} - 5x^{\frac{5}{4}}\,dx$ **e** $\int 3x^{\frac{3}{4}} - 7x^{\frac{1}{2}}\,dx$ **f** $\int 4x^{\frac{3}{5}} + 12x^{\frac{5}{3}}\,dx$

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Example 6

3 Find $f(x)$ for each of the following:

a $f'(x) = x^{\frac{5}{2}}$ and $f(1) = 1$ **b** $f'(x) = x^{\frac{1}{3}}$ and $f(0) = 5$

c $f'(x) = x^{\frac{1}{2}} + x$ and $f(4) = 6$

SF

4 Find:

a $\int \sqrt{x}\,(2 + x)\,dx$ **b** $\int \frac{3z^4 + 2z}{z^3}\,dz$ **c** $\int \frac{5x^3 + 2x^2}{x}\,dx$

d $\int \sqrt{x}\,(2x + x^2)\,dx$ **e** $\int x^2(2 + 3x^2)\,dx$ **f** $\int \sqrt[3]{x}\,(x + x^4)\,dx$

CF

5 A curve with equation $y = f(x)$ passes through the point $(2, 0)$ and $f'(x) = 3x^2 - \frac{1}{x^2}$. Find $f(x)$.

SF

6 Find s in terms of t if $\frac{ds}{dt} = 3t - \frac{8}{t^2}$ and $s = 1\frac{1}{2}$ when $t = 1$.

7 A curve $y = f(x)$ for which $f'(x) = x - \frac{2}{\sqrt{x}} + k$, where k is a constant, has a stationary point at $(1, 4)$. Find:

a the value of k

b the value of $f(x)$ when $x = 4$.

CF

9C The anti-derivative of $(ax + b)^r$

Case 1: $r \neq -1$

For $f(x) = (ax + b)^{r+1}$, where $r \neq -1$, we can use the chain rule to find

$$f'(x) = a(r + 1)(ax + b)^r$$

Thus it follows that:

$$\int (ax + b)^r\,dx = \frac{1}{a(r + 1)}(ax + b)^{r+1} + c \qquad \text{where } r \neq -1$$

This result does not hold for $r = -1$.

Example 7

Find the general anti-derivative of:

a $(3x + 1)^5$ **b** $(2x - 1)^{-2}$

Solution

a
$$\int (3x + 1)^5\,dx = \frac{1}{3 \times 6}(3x + 1)^6 + c = \frac{1}{18}(3x + 1)^6 + c$$

b
$$\int (2x - 1)^{-2}\,dx = \frac{1}{2 \times (-1)}(2x - 1)^{-1} + c = -\frac{1}{2}(2x - 1)^{-1} + c$$

ISBN 978-1-108-45164-2

Case 2: $r = -1$

But what happens when $r = -1$? In other words, what is $\int \frac{1}{ax+b}\,dx$?

Remember that $\frac{d}{dx}(\ln x) = \frac{1}{x}$. Thus $\int \frac{1}{x}\,dx = \ln x + c$ provided that $x > 0$.

More generally:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + c \qquad \text{for } ax+b>0$$

Example 8

a Find the general anti-derivative of $\frac{2}{3x-2}$ for $x > \frac{2}{3}$.

b Find the general anti-derivative of $\frac{2}{2-3x}$ for $x < \frac{2}{3}$.

Solution

a $\int \frac{2}{3x-2}\,dx = \frac{1}{3} \times 2\ln(3x-2) + c$

$= \frac{2}{3}\ln(3x-2) + c$

b $\int \frac{2}{2-3x}\,dx = -\frac{1}{3} \times 2\ln(2-3x) + c$

$= -\frac{2}{3}\ln(2-3x) + c$

Example 9

a Given that $\frac{dy}{dx} = \frac{3}{x}$ and $y = 10$ when $x = 1$, find y in terms of x for $x > 0$.

b Given that $\frac{dy}{dx} = \frac{2}{1-x}$ and $y = 5$ when $x = 1-e$, find y in terms of x for $x < 1$.

Solution

a $y = \int \frac{3}{x}\,dx = 3\ln x + c$

When $x = 1$, $y = 10$ and so

$10 = 3\ln 1 + c$

$10 = 0 + c$

$\therefore \quad c = 10$

Hence $y = 3\ln x + 10$.

b $y = \int \frac{2}{1-x}\,dx = -2\ln(1-x) + c$

When $x = 1-e$, $y = 5$ and so

$5 = -2\ln e + c$

$5 = -2 + c$

$\therefore \quad c = 7$

Hence $y = -2\ln(1-x) + 7$.

Section summary

- $\int (ax+b)^r\,dx = \frac{1}{a(r+1)}(ax+b)^{r+1} + c \qquad$ where $r \in \mathbb{Q} \setminus \{-1\}$
- $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + c \qquad$ for $ax+b>0$

ISBN 978-1-108-45164-2

Exercise 9C

Skillsheet

Example 7

1 Find:

a $\int (2x-1)^2\,dx$ **b** $\int (2-t)^3\,dt$ **c** $\int (5x-2)^3\,dx$

d $\int (4x-6)^{-2}\,dx$ **e** $\int (6-4x)^{-3}\,dx$ **f** $\int (4x+3)^{-3}\,dx$

g $\int (3x+6)^{\frac{1}{2}}\,dx$ **h** $\int (3x+6)^{-\frac{1}{2}}\,dx$ **i** $\int (2x-4)^{\frac{7}{2}}\,dx$

j $\int (3x+11)^{\frac{4}{3}}\,dx$ **k** $\int \sqrt{2-3x}\,dx$ **l** $\int (5-2x)^4\,dx$

2 Find the function f such that $f'(x) = \sqrt{1-x}$ and $f(0) = 1$.

Example 8

3 Find an anti-derivative of each of the following:

a $\dfrac{1}{2x},\ x > 0$ **b** $\dfrac{1}{3x+2},\ x > -\dfrac{2}{3}$ **c** $\dfrac{4}{1+4x},\ x > -\dfrac{1}{4}$

d $\dfrac{5}{3x-2},\ x > \dfrac{2}{3}$ **e** $\dfrac{3}{1-4x},\ x < \dfrac{1}{4}$ **f** $\dfrac{3}{2-\frac{x}{2}},\ x < 4$

4 Find:

a $\int \dfrac{5}{x}\,dx$, for $x > 0$ **b** $\int \dfrac{3}{x-4}\,dx$, for $x > 4$

c $\int \dfrac{10}{2x+1}\,dx$, for $x > -\dfrac{1}{2}$ **d** $\int \dfrac{6}{5-2x}\,dx$, for $x < \dfrac{5}{2}$

e $\int 6(1-2x)^{-1}\,dx$, for $x < \dfrac{1}{2}$ **f** $\int (4-3x)^{-1}\,dx$, for $x < \dfrac{4}{3}$

5 Find an anti-derivative of each of the following:

a $\dfrac{3x+1}{x},\ x > 0$ **b** $\dfrac{x+1}{x},\ x > 0$ **c** $\dfrac{1}{(x+1)^2},\ x \neq -1$

d $\dfrac{(x+1)^2}{x},\ x > 0$ **e** $\dfrac{3}{(x-1)^3},\ x \neq 1$ **f** $\dfrac{1-2x}{x},\ x > 0$

Example 9

6 **a** Given that $\dfrac{dy}{dx} = \dfrac{1}{2x}$ and $y = 2$ when $x = e^2$, find y in terms of x for $x > 0$.

b Given that $\dfrac{dy}{dx} = \dfrac{2}{5-2x}$ and $y = 10$ when $x = 2$, find y in terms of x for $x < \dfrac{5}{2}$.

7 A curve with equation $y = f(x)$ passes through the point $(5+e, 10)$ and $f'(x) = \dfrac{10}{x-5}$. Find the equation of the curve for $x > 5$.

8 Find an anti-derivative of each of the following:

a $\dfrac{x}{x+1},\ x > -1$ **b** $\dfrac{1-2x}{x+1},\ x > -1$ **c** $\dfrac{2x+1}{x+1},\ x > -1$

9 Given that $\dfrac{dy}{dx} = \dfrac{3}{(x-2)^2}$ and $y = -1$ when $x = 0$, find y in terms of x for $x < 2$.

10 Given that $\dfrac{dy}{dx} = \dfrac{5}{2-4x}$ and $y = 10$ when $x = -2$, find y in terms of x for $x < \dfrac{1}{2}$.

SF

CF

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ISBN 978-1-108-45164-2

9D The anti-derivative of e^{kx}

In Chapter 8 we found that, if $f(x) = e^{kx}$, then $f'(x) = ke^{kx}$. Thus:

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c \qquad \text{where } k \neq 0$$

Example 10

Find the general anti-derivative of each of the following:

a e^{4x} **b** $e^{5x} + 6x$ **c** $e^{3x} + 2$ **d** $e^{-x} + e^{x}$

Solution

a $\int e^{4x}\,dx = \frac{1}{4}e^{4x} + c$

b $\int e^{5x} + 6x\,dx = \frac{1}{5}e^{5x} + 3x^2 + c$

c $\int e^{3x} + 2\,dx = \frac{1}{3}e^{3x} + 2x + c$

d $\int e^{-x} + e^{x}\,dx = -e^{-x} + e^{x} + c$

Example 11

If the gradient of the tangent at a point (x, y) on a curve is given by $5e^{2x}$ and the curve passes through the point $(0, 7.5)$, find the equation of the curve.

Solution

Let the curve have equation $y = f(x)$. Then $f'(x) = 5e^{2x}$.

$$\int 5e^{2x}\,dx = \frac{5}{2}e^{2x} + c$$

$$\therefore \quad f(x) = \frac{5}{2}e^{2x} + c$$

But $f(0) = 7.5$ and therefore

$$7.5 = \frac{5}{2}e^{0} + c$$

$$= 2.5 + c$$

$$\therefore \quad c = 5$$

Hence $f(x) = \frac{5}{2}e^{2x} + 5$.

Section summary

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c$$

Exercise 9D

Skillsheet

Example 10

1 Find the general anti-derivative of each of the following: SF

a e^{6x} **b** $e^{2x} + 3x$ **c** $e^{-3x} + 2x$ **d** $e^{-2x} + e^{2x}$

ISBN 978-1-108-45164-2

SF

2 Find:

a $\int e^{2x} - e^{\frac{x}{2}}\, dx$ **b** $\int \frac{e^{2x}+1}{e^x}\, dx$ **c** $\int 2e^{3x} - e^{-x}\, dx$

d $\int 5e^{\frac{x}{3}} - 2e^{\frac{x}{5}}\, dx$ **e** $\int 3e^{\frac{2x}{3}} - 3e^{\frac{7x}{5}}\, dx$ **f** $\int 5e^{\frac{4x}{3}} - 3e^{\frac{2x}{3}}\, dx$

Example 11

3 Find y in terms of x for each of the following:

a $\frac{dy}{dx} = e^{2x} - x$ and $y = 5$ when $x = 0$ **b** $\frac{dy}{dx} = \frac{3 - e^{2x}}{e^x}$ and $y = 4$ when $x = 0$

4 For the function f it is known that $f'(x) = e^{2x}$ and $f(1) = 4$. Find $f(x)$.

CF

5 Given that $\frac{dy}{dx} = ae^{-x} + 1$ and that when $x = 0$, $\frac{dy}{dx} = 3$ and $y = 5$, find the value of y when $x = 2$.

6 A curve for which $\frac{dy}{dx} = e^{kx}$, where k is a constant, is such that the tangent at $(1, e^2)$ passes through the origin. Find the gradient of this tangent and hence determine:

a the value of k **b** the equation of the curve.

7 A curve for which $\frac{dy}{dx} = -e^{kx}$, where k is a constant, is such that the tangent at $(1, -e^3)$ passes through the origin. Find the gradient of this tangent and hence determine:

a the value of k **b** the equation of the curve.

9E Anti-differentiation of trigonometric functions

Recall the following results from Chapter 8:

- If $f(x) = \sin(ax + b)$, then $f'(x) = a\cos(ax + b)$.
- If $g(x) = \cos(ax + b)$, then $g'(x) = -a\sin(ax + b)$.

Thus:

$$\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + c$$

Example 12

Find an anti-derivative of each of the following:

a $\sin\left(3x + \frac{\pi}{4}\right)$ **b** $\frac{1}{4}\sin(4x)$

Solution

a $-\frac{1}{3}\cos\left(3x + \frac{\pi}{4}\right) + c$ **b** $-\frac{1}{16}\cos(4x) + c$

ISBN 978-1-108-45164-2

Section summary

$$\int \sin(ax + b)\,dx = -\frac{1}{a}\cos(ax + b) + c$$

$$\int \cos(ax + b)\,dx = \frac{1}{a}\sin(ax + b) + c$$

Exercise 9E

SF

Example 12

1 Find an anti-derivative of each of the following:

a $\cos(3x)$ **b** $\sin\left(\frac{1}{2}x\right)$ **c** $3\cos(3x)$

d $2\sin\left(\frac{1}{2}x\right)$ **e** $\sin\left(2x - \frac{\pi}{3}\right)$ **f** $\cos(3x) + \sin(2x)$

g $\cos(4x) - \sin(4x)$ **h** $-\frac{1}{2}\sin(2x) + \cos(3x)$ **i** $-\frac{1}{2}\cos\left(2x + \frac{\pi}{3}\right)$

j $\sin(\pi x)$ **k** $2\pi\cos(2\pi x)$ **l** $-2\cos(\pi - 2x)$

2 For each of the following, find $f(x)$:

a $f'(x) = \cos(2x)$ and $f\left(\frac{\pi}{4}\right) = 1$

b $f'(x) = \sin\left(\frac{x}{2}\right)$ and $f\left(\frac{4\pi}{3}\right) = 2$

3 The gradient of a curve is given by $\frac{dy}{dx} = x + \sin(2x)$. Find the equation of the curve, given that it passes through the point $(0, 1)$.

4 Find the function g such that $g'(x) = 4\sin(2x)$ and $g\left(\frac{\pi}{4}\right) = 2$.

5 A curve has gradient function $\frac{dy}{dx} = \cos(2x) + \sin(3x)$ and the point $\left(\frac{\pi}{2}, 0\right)$ lies on the curve.

a Find the equation of the curve.

b Find the equation of the tangent to the curve at the point $\left(\frac{\pi}{2}, 0\right)$.

CF

6 Given that $f'(x) = \cos x - \cos(2x)$ and $f(0) = 8$, find $f(x)$.

7 A curve with equation $y = f(x)$ passes through the point $(\pi, 0)$ and $f'(x) = 1 - 3\sin(3x)$.

a Find $f(x)$.

b Find the equation of the tangent to the curve $y = f(x)$ at the point $(\pi, 0)$.

ISBN 978-1-108-45164-2

9F Further anti-differentiation techniques

In this section, we introduce further methods for finding an anti-derivative.

Example 13

Let $f(x) = \ln(x^2 + 1)$.

a Show that $f'(x) = \dfrac{2x}{x^2 + 1}$.

b Hence find an anti-derivative of $\dfrac{x}{x^2 + 1}$.

Solution

a Let $y = \ln(x^2 + 1)$ and $u = x^2 + 1$.
Then $y = \ln u$. By the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{u} \cdot 2x$$

$$\therefore \quad f'(x) = \frac{2x}{x^2 + 1}$$

b

$$\int \frac{x}{x^2 + 1}\, dx = \frac{1}{2} \int \frac{2x}{x^2 + 1}\, dx$$

$$= \frac{1}{2} \ln(x^2 + 1) + c$$

We can generalise the previous example as follows.

Let $g(x) = \ln(f(x))$, for $f(x) > 0$. Then the chain rule gives $g'(x) = \dfrac{f'(x)}{f(x)}$, and therefore we have

$$\int \frac{f'(x)}{f(x)}\, dx = \ln(f(x)) + c \qquad \text{for } f(x) > 0$$

This result is used in the following example.

Example 14

Find an anti-derivative for each of the following:

a $\dfrac{6x}{x^2 + 1}$

b $\dfrac{\sin x}{\cos x}$, for $\cos x > 0$

c $\dfrac{e^x}{e^x + 1}$

Solution

a

$$\int \frac{6x}{x^2 + 1}\, dx$$

$$= 3 \int \frac{2x}{x^2 + 1}\, dx$$

$$= 3 \ln(x^2 + 1) + c$$

b

$$\int \frac{\sin x}{\cos x}\, dx$$

$$= -\int \frac{-\sin x}{\cos x}\, dx$$

$$= -\ln(\cos x) + c$$

c

$$\int \frac{e^x}{e^x + 1}\, dx$$

$$= \ln(e^x + 1) + c$$

ISBN 978-1-108-45164-2

Example 15

Let $f(x) = \dfrac{\cos x}{\sin x}$.

a Show that $f'(x) = \dfrac{-1}{\sin^2 x}$.

b Hence find $\displaystyle\int \frac{1}{\sin^2 x}\, dx$.

Solution

a Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\ &= \frac{-1}{\sin^2 x} \end{aligned}$$

b

$$\begin{aligned} \int \frac{1}{\sin^2 x}\, dx &= -\int \frac{-1}{\sin^2 x}\, dx \\ &= -\frac{\cos x}{\sin x} + c \end{aligned}$$

Example 16

a If $f(x) = x\ln(kx)$, find $f'(x)$ and hence find $\int \ln(kx)\, dx$, where k is a positive constant.

b If $f(x) = x^2\ln(kx)$, find $f'(x)$ and hence find $\int x\ln(kx)\, dx$, where k is a positive constant.

Solution

a

$$\begin{aligned} f'(x) &= \ln(kx) + x \times \frac{1}{x} \\ &= \ln(kx) + 1 \end{aligned}$$

Anti-differentiate both sides of the equation with respect to x:

$$\begin{aligned} \int f'(x)\, dx &= \int \ln(kx)\, dx + \int 1\, dx \\ x\ln(kx) + c_1 &= \int \ln(kx)\, dx + x + c_2 \end{aligned}$$

Thus

$$\begin{aligned} \int \ln(kx)\, dx &= x\ln(kx) - x + c_1 - c_2 \\ &= x\ln(kx) - x + c \end{aligned}$$

b

$$\begin{aligned} f'(x) &= 2x\ln(kx) + x^2 \times \frac{1}{x} \\ &= 2x\ln(kx) + x \end{aligned}$$

Anti-differentiate both sides of the equation with respect to x:

$$\begin{aligned} \int f'(x)\, dx &= \int 2x\ln(kx)\, dx + \int x\, dx \\ x^2\ln(kx) + c_1 &= \int 2x\ln(kx)\, dx + \frac{x^2}{2} + c_2 \end{aligned}$$

Thus $\displaystyle\int x\ln(kx)\, dx = \frac{1}{2}x^2\ln(kx) - \frac{x^2}{4} + c$

ISBN 978-1-108-45164-2

Exercise 9F

CF

Example 13 **1** Differentiate $\ln(3x^2 + 7)$ and hence determine $\int \frac{x}{3x^2 + 7}\, dx$.

Example 14 **2** Find an anti-derivative for each of the following:

a $\frac{x^2}{x^3 + 3}$, for $x^3 + 3 > 0$ **b** $\frac{x + 2}{x^2 + 4x}$, for $x^2 + 4x > 0$ **c** $\frac{e^{2x}}{3 + e^{2x}}$

d $\frac{x^2 + 1}{x^3 + 3x}$, for $x^3 + 3x > 0$ **e** $\frac{5}{3x - 2}$, for $x > \frac{2}{3}$

Example 15 **3** **a** Differentiate $\frac{\sin x}{\cos x}$ and hence find an anti-derivative of $\frac{1}{\cos^2 x}$.

b Differentiate $\frac{\cos(2x)}{\sin(2x)}$ and hence find an anti-derivative of $\frac{1}{\sin^2(2x)}$.

4 Differentiate $e^{2\sqrt{x}}$ with respect to x and hence determine $\int \frac{e^{2\sqrt{x}}}{\sqrt{x}}\, dx$.

5 Differentiate $\cos^3 x$ with respect to x and hence find an anti-derivative of $\cos^2 x \sin x$.

Example 16 **6** **a** If $f(x) = x\ln(2x)$, find $f'(x)$ and hence find $\int \ln(2x)\, dx$.

b If $f(x) = x^2 \ln(2x)$, find $f'(x)$ and hence find $\int x\ln(2x)\, dx$.

7 Differentiate $x \sin x$ with respect to x and hence determine $\int x \cos x\, dx$.

8 Differentiate xe^{2x} with respect to x and hence determine $\int xe^{2x}\, dx$.

9 Let $f(x) = (2x + a)\sqrt{4x - a}$, where a is a positive constant.

a Find $f'(x)$. Simplify your answer.

b Hence find an anti-derivative of $\frac{x}{\sqrt{4x - a}}$.

10 Differentiate $\sqrt{1 + 4x^2}$ and hence find an anti-derivative of $\frac{x}{\sqrt{1 + 4x^2}}$.

11 Show that $\frac{4}{x^2 - 4} = \frac{1}{x - 2} - \frac{1}{x + 2}$. Hence find an anti-derivative of $\frac{1}{x^2 - 4}$.

12 Let $f(x) = g'(x)$ and $h(x) = k'(x)$, where $g(x) = (x^2 + 1)^3$ and $k(x) = \sin(x^2)$. Find:

a $\int f(x)\, dx$ **b** $\int h(x)\, dx$ **c** $\int f(x) + h(x)\, dx$

d $\int -f(x)\, dx$ **e** $\int f(x) - 4\, dx$ **f** $\int 3h(x)\, dx$

13 Find the derivatives of $x + \sqrt{1 + x^2}$ and $\ln(x + \sqrt{1 + x^2})$.

By simplifying your last result if necessary, determine $\int \frac{1}{\sqrt{1 + x^2}}\, dx$.

ISBN 978-1-108-45164-2

9G Applications of anti-differentiation to motion in a straight line

In the previous chapter, we considered examples in which we were given a rule for the position of a particle in terms of time, and from it we derived rules for the velocity and the acceleration by using differentiation.

We may be given a rule for acceleration and, by using anti-differentiation and some additional information, we can deduce rules for both velocity and position.

Example 17

A particle starts from a point O and moves in a straight line. After t seconds ($t \geq 0$) its velocity, v m/s, is given by $v = 2t - 4$. Find:

a its position, x m, relative to O after t seconds

b its position after 3 seconds

c its average velocity in the first 3 seconds

d the distance travelled in the first 3 seconds

e its average speed in the first 3 seconds.

Solution

a Anti-differentiate velocity to find the expression for position:

$$x = t^2 - 4t + c$$

When $t = 0$, $x = 0$, and so $c = 0$.

$$\therefore \quad x = t^2 - 4t$$

b When $t = 3$, $x = -3$. The particle is 3 m to the left of O.

c
$$\text{Average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{-3 - 0}{3} = -1 \text{ m/s}$$

d First find when the particle is at rest: $v = 0$ implies $2t - 4 = 0$, i.e. $t = 2$.

When $t = 2$, $x = -4$. Therefore the particle goes from $x = 0$ to $x = -4$ in the first 2 seconds, and then back to $x = -3$ in the next second.

Thus it has travelled 5 m in the first 3 seconds.

e
$$\text{Average speed} = \frac{\text{distance travelled}}{\text{change in time}} = \frac{5}{3} \text{ m/s}$$

ISBN 978-1-108-45164-2

Example 18

A particle moves in a straight line. It starts from rest 3 metres to the right of a point O. Its acceleration, a m/s^2, at time t seconds is given by $a = 6t + 8$. Find its position, x m, and velocity, v m/s, at time t seconds.

Solution

We are given the acceleration:

$$a = \frac{dv}{dt} = 6t + 8$$

Find the velocity by anti-differentiating:

$$v = 3t^2 + 8t + c$$

At $t = 0$, $v = 0$, and so $c = 0$.

$$\therefore \quad v = 3t^2 + 8t$$

Find the position by anti-differentiating again:

$$x = t^3 + 4t^2 + d$$

At $t = 0$, $x = 3$, and so $d = 3$.

$$\therefore \quad x = t^3 + 4t^2 + 3$$

Example 19

A stone is projected vertically upwards from the top of a building 20 m high with an initial velocity of 15 m/s. Find:

a the time taken for the stone to reach its maximum height

b the maximum height reached by the stone

c the time taken for the stone to reach the ground

d the velocity of the stone as it hits the ground.

In this case we only consider the stone's motion in a vertical direction, so we can treat it as motion in a straight line. Also we will assume that the acceleration due to gravity is approximately -10 m/s^2. (Note that downwards is considered the negative direction.)

Solution

Anti-differentiating $a = -10$ gives $v = -10t + c$.

At $t = 0$, $v = 15$ and therefore $v = -10t + 15$.

Anti-differentiating v gives $x = -5t^2 + 15t + d$.

At $t = 0$, $x = 20$ and so $x = -5t^2 + 15t + 20$.

a The stone will reach its maximum height when $v = 0$:

$$-10t + 15 = 0$$

$$\therefore \quad t = 1.5$$

The stone takes 1.5 seconds to reach its maximum height.

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b At $t = 1.5$, $x = -5(1.5)^2 + 15(1.5) + 20$
$= 31.25$

The maximum height reached by the stone is 31.25 metres.

c The stone reaches the ground when $x = 0$:

$$-5t^2 + 15t + 20 = 0$$
$$-5(t^2 - 3t - 4) = 0$$
$$-5(t - 4)(t + 1) = 0$$
$$\therefore \quad t = 4$$

(The solution $t = -1$ is rejected as $t \geq 0$.)
The stone takes 4 seconds to reach the ground.

d At $t = 4$, $v = -10(4) + 15$
$= -25$

The velocity on impact is -25 m/s.

Section summary

Anti-differentiation may be used to go from acceleration to velocity, and from velocity to position.

Exercise 9G

SF

Example 17

1 An object starts from a point O and moves in a straight line. After t seconds ($t \geq 0$) its velocity, v m/s, is given by $v = 2t - 3$. Find:

- **a** its position, x m, relative to O after t seconds
- **b** its position after 3 seconds
- **c** its average velocity in the first 3 seconds
- **d** the distance travelled in the first 3 seconds
- **e** its average speed in the first 3 seconds.

2 A particle is moving in a straight line. Its velocity, v m/s, at time t seconds is given by $v = 3t^2 - 2$. Find an expression for its position, x m, at time t seconds, given that $x = 4$ when $t = 1$.

3 The velocity of a particle, v m/s, at time t seconds ($t \geq 0$) is given by $v = 2t^2 - 8t + 6$. It is initially 4 m to the right of a point O. Find:

- **a** its position and acceleration at time t
- **b** its position when the velocity is zero
- **c** its acceleration when the velocity is zero.

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Example 18 **4** An object moves in a straight line so that its acceleration, a m/s^2, after t seconds ($t \geq 0$) is given by $a = 2t - 3$. If the initial position of the object is 2 m to the right of a point O and its initial velocity is 3 m/s, find the object's position and velocity after 10 seconds.

SF

5 An object moves in a straight line with an acceleration of 8 m/s^2. If after 1 second it passes through point O and after 3 seconds it is 30 metres from O, find its initial position relative to O.

Example 19 **6** An object is projected vertically upwards with a velocity of 25 m/s. (Its acceleration due to gravity is -10 m/s^2.) Find:

a the object's velocity at time t

b its height above the point of projection at time t

c the time it takes to reach its maximum height

d the maximum height reached

e the time taken to return to the point of projection.

CF

7 A particle moves in a straight line. At time t seconds after leaving a fixed point O, the velocity of the particle, v m/s, is given by $v = 9(1 - e^{-\frac{1}{3}t})$ for $t \geq 0$.

a Sketch the graph of v against t for $t \geq 0$.

b Find an expression for the particle's position, x m, relative to O at time t seconds.

c Find the position of the particle after 3 seconds.

8 A particle starts at a point O and moves in a straight line. Its velocity, v m/s, at time t seconds is given by $v = 4\sin\left(\frac{\pi t}{4}\right)$ for $0 \leq t \leq 8$.

a Sketch the graph of v against t for $0 \leq t \leq 8$.

b Find an expression for the particle's position, x m, relative to O at time t seconds.

c Sketch the graph of x against t for $0 \leq t \leq 8$.

d State the maximum speed of the particle, and state its position(s) when it has this speed.

CU

9 The acceleration of a lift in a tall building is $\frac{1}{9}(t - 5)$ m/s^2 at time t seconds. At time $t = 0$, the lift is passing the 50th floor with a velocity of -8 m/s. If each floor spans a height of 6 metres, find the floor at which the lift will stop.

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Review

Chapter summary

AS

- To find the general anti-derivative:
  If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + c$, where c is an arbitrary real number.
- Basic anti-derivatives

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c \qquad \text{where } r \in \mathbb{Q} \setminus \{-1\}$$

$$\int (ax+b)^r\,dx = \frac{1}{a(r+1)}(ax+b)^{r+1} + c \qquad \text{where } r \in \mathbb{Q} \setminus \{-1\}$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + c \qquad \text{for } ax+b > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$$

- Properties of anti-differentiation
  - $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
  - $\int f(x) - g(x)\,dx = \int f(x)\,dx - \int g(x)\,dx$
  - $\int k\,f(x)\,dx = k\int f(x)\,dx$, where k is a real number

Technology-free questions

SF

1 Find:

a $\int \frac{1}{2}\,dx$ **b** $\int \frac{1}{2}x^2\,dx$ **c** $\int x^2 + 3x\,dx$

d $\int (2x+3)^2\,dx$ **e** $\int at\,dt$ **f** $\int \frac{1}{3}t^3\,dt$

g $\int (t+1)(t-2)\,dt$ **h** $\int (2-t)(t+1)\,dt$

2 Suppose that $\frac{dy}{dx} = k\sqrt{x}$, where k is a positive constant. When $x = 1$, $\frac{dy}{dx} = 6$ and $y = 6$. Find y in terms of x.

3 Suppose that $\frac{dy}{dx} = \frac{k}{x^2}$, where k is a positive constant. When $x = 2$, $\frac{dy}{dx} = 6$ and $y = 6$. Find y in terms of x.

4 A curve with equation $y = f(x)$ passes through the point $(3, -1)$ and its gradient is given by $f'(x) = 2x + 5$. Find the equation of the curve.

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Review

SF

5 For each of the following, find $f(x)$:

a $f'(x) = \cos(2x)$ and $f(\pi) = 1$ **b** $f'(x) = \frac{3}{x}$ and $f(1) = 6$ (for $x > 0$)

c $f'(x) = e^{\frac{x}{2}}$ and $f(0) = 1$

6 A curve with equation $y = f(x)$ passes through the origin and its gradient is given by $f'(x) = 3x^2 - 8x + 3$.

a Find the equation of the curve.

b Find the x-axis intercepts of the curve.

7 A curve with equation $y = f(x)$ passes through the point $(2, 0)$ and its gradient is given by $f'(x) = 3x^2 - 12x + 8$.

a Find the equation of the curve.

b Find the x-axis intercepts of the curve.

CF

8 The gradient of a curve is given by $\frac{dy}{dx} = 4 - kx$, where k is a positive constant.

The tangent to the curve at $x = 1$ is perpendicular to the tangent at $x = -1$.

a Find the value of k.

b Find the equation of the curve, given that it passes through the point $(6, 0)$.

9 A particle moves in a straight line. It starts from rest at an origin O and its acceleration, a m/s^2, at time t seconds is given by $a = 4 - t$. Calculate:

a the velocity of the particle when $t = 3$

b the displacement from O when it next comes to rest

c the displacement from O when $t = 12$

d the average speed during the first 12 seconds.

Multiple-choice questions

1 An anti-derivative of $x^3 + 3x$ is

A $x(x^2 + 3)$ **B** $3(x^2 + 1)$ **C** $\frac{x^4}{4} + \frac{3x^2}{2} + c$

D $x^3 + 3 + c$ **E** $3x^3 + x^2 + c$

2 $\int \sqrt{x} + x \, dx$ equals

A $\frac{\sqrt{x^2} + x^2}{2} + c$ **B** $x^{\frac{3}{2}} + x^2 + c$ **C** $\frac{1}{2}x + 1$

D $\frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 + c$ **E** 2

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3 $3\int x^{-4}\,dx$ equals

A $\dfrac{-1}{x^3}+c$ **B** $-x^3+c$ **C** $-12x^{-3}+c$ **D** $-12x^{-5}+c$ **E** $\dfrac{-3x^{-4}}{5}+c$

4 An expression for y, if $\dfrac{dy}{dx}=2x+5$ and if $y=1$ when $x=0$, is

A $y=x^2+5x$ **B** $y=2$ **C** $y=x^2+5x-1$
D $y=x^2+5x+1$ **E** $y=x^2+5x-5$

5 If $f'(x)=5x^4-9x^2$ and $f(1)=2$, then $f(x)=$

A x^5-3x^2+2 **B** x^5-3x^3+4 **C** x^5-9x^2+5
D x^5-3x^3 **E** x^5-3x^2+2

6 An expression for y, if $\dfrac{dy}{dx}=\dfrac{4}{x^3}$ and if $y=0$ when $x=1$, is

A $y=\dfrac{-2}{x^2}$ **B** $y=\dfrac{-2}{x^2}+2$ **C** $y=\dfrac{-2}{3x^2}-\dfrac{2}{3}$
D $y=\dfrac{-4}{x^2}-2$ **E** $y=2x$

7 $\int 3x^2+6\,dx=$

A x^3+6x+c **B** $6x$ **C** $6x+c$ **D** $\dfrac{x^3}{3}+\dfrac{6}{x}+c$ **E** x^3+3x+c

8 Let a and b be positive constants. The general anti-derivative of $\sqrt{(ax-b)^3}$ for $x>\dfrac{b}{a}$ is

A $\dfrac{3}{2}(ax-b)^{\frac{1}{2}}+c$ **B** $\dfrac{1}{5\sqrt{ax-b}}+c$ **C** $\dfrac{2}{5a}(ax-b)^{\frac{5}{2}}+c$
D $\dfrac{1}{5a}\sqrt{(ax-b)^5}+c$ **E** $\dfrac{2}{5\sqrt{(ax-b)^5}}+c$

9 An expression for y, if $\dfrac{dy}{dx}=\dfrac{1}{2}ax+1$ and if $y=1$ when $x=0$, is

A $y=\dfrac{1}{4}ax^2+x+1$ **B** $y=a$ **C** $y=ax^2+x-1$
D $y=ax^2+x+a$ **E** $y=ax^2+ax+a$

10 If $f'(x)=-6\sin(3x)$ and $f\left(\dfrac{2\pi}{3}\right)=3$, then $f(x)=$

A $-18\cos(3x)+21$ **B** $-2\cos(2x)+5$ **C** $-2\sin(3x)+1$
D $2\cos(3x)+1$ **E** $2\sin(4x)+3$

11 Suppose that $\dfrac{dy}{dx}=ae^{-x}+2$ and that, when $x=0$, $\dfrac{dy}{dx}=5$ and $y=1$. When $x=2$, $y=$

A $-\dfrac{3}{e^2}+2$ **B** $-\dfrac{3}{e^2}+4$ **C** $-\dfrac{3}{e^2}+8$ **D** $3e^2+4$ **E** $3e^2+8$

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Review

12 An anti-derivative of $\sin(3x) + \cos(3x)$ is

A $\frac{1}{3}\cos(3x) + \frac{1}{3}\sin(3x)$ **B** $-\frac{1}{3}\cos(3x) + \frac{1}{3}\sin(3x)$ **C** $3\cos(3x) + 3\sin(3x)$

D $3\cos(3x) - 3\sin(3x)$ **E** $-3\cos(3x) + 3\sin(3x)$

13 If $g'(x) = \frac{1}{x+1}$ and $g(0) = 2$, then $g(x) =$

A $\ln(x+1) + 2$ **B** $-\frac{1}{(x+1)^2} + 3$ **C** $-\ln(x+1) + 3$

D $\ln(x+1)$ **E** $\frac{1}{(x+1)^2} + 1$

14 The derivative of xe^{3x} is $(3x+1)e^{3x}$. It follows that $\int xe^{3x}\,dx =$

A $(3x+1)e^{3x} + c$ **B** $(3x-1)e^{3x} + c$ **C** $3xe^{3x} + \int e^{3x}\,dx$

D $(3x+1)e^{x} + c$ **E** $\frac{1}{3}e^{3x}\left(x - \frac{1}{3}\right) + c$

15 A particle moves along the x-axis with velocity given by $v = 3t^2 + 6t$ at time $t \geq 0$. The particle's position at time $t = 0$ is $x = 2$. What is the particle's position at time $t = 1$?

A $x = 4$ **B** $x = 6$ **C** $x = 9$ **D** $x = 11$ **E** $x = 12$

Extended-response questions

1 The slope of a children's slide is given by

$$\frac{dy}{dx} = \frac{9}{32}(x^2 - 4x) \quad \text{for } x \in [0, 4]$$

where the origin is taken to be at ground level beneath the highest point of the slide, which is 3 m above the ground. All units are in metres.

a Find the equation of the curve which describes the slide.

b Sketch the curve of the slide, labelling stationary points.

c Does the slope of the slide ever exceed 45°?

CU

2 **a** Differentiate $e^{-3x}\sin(2x)$ and $e^{-3x}\cos(2x)$ with respect to x.

b Hence show that

$$e^{-3x}\sin(2x) + c_1 = -3\int e^{-3x}\sin(2x)\,dx + 2\int e^{-3x}\cos(2x)\,dx$$

and $$e^{-3x}\cos(2x) + c_2 = -3\int e^{-3x}\cos(2x)\,dx - 2\int e^{-3x}\sin(2x)\,dx$$

c Use the two equations from **b** to determine $\int e^{-3x}\sin(2x)\,dx$.

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3 A particle moves along a straight line. Let x m be its position relative to a fixed point O at time t seconds ($t \geq 0$). The particle's initial position is $x = 0$ and its velocity, v m/s, at time t seconds is given by

$$v = 5 - \ln(t + k)$$

where k is a positive constant.

a Find $\frac{d}{dt}\big((t + k)\ln(t + k)\big)$ and hence find $\int \ln(t + k)\, dt$.

b Find an expression for the position, x m, of the particle in terms of t.

c Given that the particle is at rest when $t = 100$, find the value of k correct to two decimal places.

d Find the position of the particle when it is at rest.

e Sketch the graph of v against t. (Use the value of k from part **c**.)

4 A particle moves along a straight line. Let x_1 m be the particle's position relative to a fixed point O at time t seconds ($0 \leq t \leq 24$). Its initial position is $x_1 = 0$ and its velocity, v_1 m/s, is given by

$$v_1 = 6\sin\left(\frac{\pi t}{6}\right)$$

a Sketch the graph of v_1 against t for $0 \leq t \leq 24$.

b Find an expression for x_1 in terms of t.

c Sketch the graph of x_1 against t for $0 \leq t \leq 24$.

d State the maximum speed of the particle and its position when it has this speed.

A second particle moves along the same straight line. Let x_2 m be the particle's position relative to O at time t. Its initial position is $x_2 = 12$ and its velocity, v_2 m/s, is given by

$$v_2 = 6\sin\left(\frac{\pi t}{6}\right)$$

e Find an expression for x_2 in terms of t.

f Sketch the graph of x_2 against t for $0 \leq t \leq 24$.

A third particle moves along the same straight line. Let x_3 m be the particle's position relative to O at time t. Its velocity, v_3 m/s, is given by

$$v_3 = 6\sin\left(\frac{\pi t}{3}\right)$$

g Find the initial position of the third particle given that it meets the first particle twice during the 24 seconds of motion.

ISBN 978-1-108-45164-2

Chapter 10

10 Integration

Objectives

- ▶ To estimate the area under the graph of a function using numerical methods, including the **trapezoidal rule**.
- ▶ To be able to calculate **definite integrals**.
- ▶ To use the definite integral to find the **exact area** under the graph of a function.
- ▶ To integrate **polynomial functions**, **power functions**, **exponential functions** and **trigonometric functions**.
- ▶ To use integration to determine **areas under curves** and **areas between curves**.
- ▶ To apply integration to find the **total change** in a quantity given its rate of change.

We have used the derivative to find the gradients of tangents to curves, and in turn this has been used in graph sketching. The derivative has also been used to define instantaneous rate of change and to solve problems involving motion in a straight line. It comes as a surprise that the reverse process – anti-differentiation – can be used to determine areas.

In this chapter, we define an area function A for a given function f on an interval $[a, b]$, and show that the derivative of the area function is the original function f. Hence, you can go from the function f to its area function by anti-differentiation. This result is so important that it carries the title **fundamental theorem of calculus**.

The result was developed over many centuries, and some of the methods discussed in this chapter date back to Archimedes. The final result was brought together by both Leibniz and Newton in the seventeenth century. The wonder of it is that the two seemingly distinct ideas – calculation of areas and calculation of gradients – were shown to be so closely related.

Chapters 9 and 10 cover Unit 3 Topic 3: Integrals.

ISBN 978-1-108-45164-2

10A Estimating the area under a graph

Consider the graph of a function f. We want to find the area under the graph. For now we'll assume that the graph $y = f(x)$ is always *above the x-axis*, and we will estimate the area between the graph $y = f(x)$ and the x-axis. We set left and right endpoints and estimate the area between those endpoints.

Below is the graph of $f(x) = 9 - 0.1x^2$. We consider three methods for determining the area under this graph between $x = 2$ and $x = 5$.

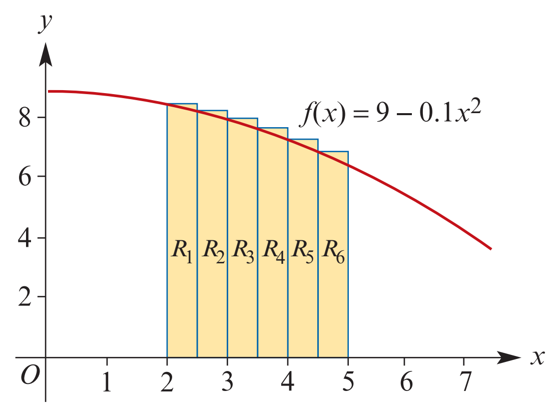
▶ The left-endpoint method

We first find an approximation for the area under the graph between $x = 2$ and $x = 5$ by dividing the region into rectangles as illustrated. The width of each rectangle is 0.5.

Areas of rectangles:

- $R_1 = 0.5 \times f(2.0) = 0.5 \times 8.60 = 4.30$
- $R_2 = 0.5 \times f(2.5) = 0.5 \times 8.38 = 4.19$
- $R_3 = 0.5 \times f(3.0) = 0.5 \times 8.10 = 4.05$
- $R_4 = 0.5 \times f(3.5) = 0.5 \times 7.78 = 3.89$
- $R_5 = 0.5 \times f(4.0) = 0.5 \times 7.40 = 3.70$
- $R_6 = 0.5 \times f(4.5) = 0.5 \times 6.98 = 3.49$

The sum of the areas of the rectangles is 23.62 square units.



This is called the **left-endpoint estimate** for the area under the graph.

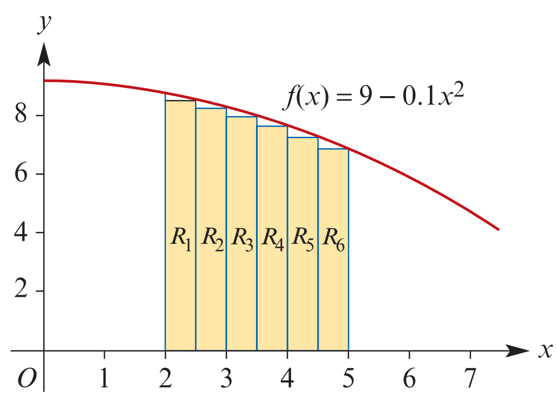
The left-endpoint estimate will be larger than the actual area for a graph that is decreasing over the interval, and smaller than the actual area for a graph that is increasing.

▶ The right-endpoint method

Areas of rectangles:

- $R_1 = 0.5 \times f(2.5) = 0.5 \times 8.38 = 4.19$
- $R_2 = 0.5 \times f(3.0) = 0.5 \times 8.10 = 4.05$
- $R_3 = 0.5 \times f(3.5) = 0.5 \times 7.78 = 3.89$
- $R_4 = 0.5 \times f(4.0) = 0.5 \times 7.40 = 3.70$
- $R_5 = 0.5 \times f(4.5) = 0.5 \times 6.98 = 3.49$
- $R_6 = 0.5 \times f(5.0) = 0.5 \times 6.50 = 3.25$

The sum of the areas of the rectangles is 22.67 square units.



This is called the **right-endpoint estimate** for the area under the graph.

ISBN 978-1-108-45164-2

For f decreasing over $[a, b]$: left-endpoint estimate $\geq$ true area $\geq$ right-endpoint estimate

For f increasing over $[a, b]$: left-endpoint estimate $\leq$ true area $\leq$ right-endpoint estimate

It is clear that, if narrower strips are chosen, we obtain an estimate that is closer to the true value. This is time-consuming to do by hand, but a computer program or spreadsheet makes the process quite manageable. In the following spreadsheet, the right-endpoint estimate is calculated using rectangles of width 0.1 units.

x	$f(x)$	$0.1 \times f(x)$
2.1	8.559	0.8559
2.2	8.516	0.8516
2.3	8.471	0.8471
2.4	8.424	0.8424
2.5	8.375	0.8375
2.6	8.324	0.8324
2.7	8.271	0.8271
2.8	8.216	0.8216
2.9	8.159	0.8159
3	8.1	0.81

x	$f(x)$	$0.1 \times f(x)$
3.1	8.039	0.8039
3.2	7.976	0.7976
3.3	7.911	0.7911
3.4	7.844	0.7844
3.5	7.775	0.7775
3.6	7.704	0.7704
3.7	7.631	0.7631
3.8	7.556	0.7556
3.9	7.479	0.7479
4	7.4	0.74

x	$f(x)$	$0.1 \times f(x)$
4.1	7.319	0.7319
4.2	7.236	0.7236
4.3	7.151	0.7151
4.4	7.064	0.7064
4.5	6.975	0.6975
4.6	6.884	0.6884
4.7	6.791	0.6791
4.8	6.696	0.6696
4.9	6.599	0.6599
5	6.5	0.65

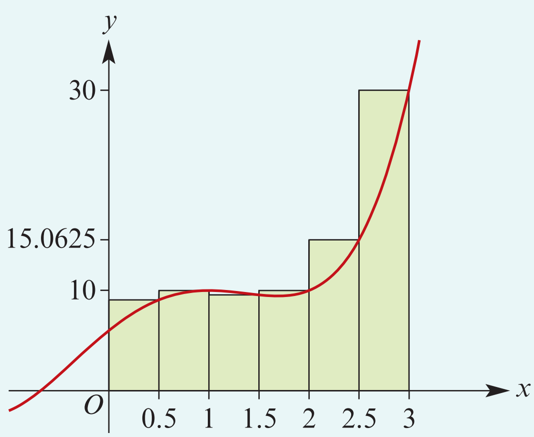
The sum of the areas of the rectangles is 22.9945 square units.

Example 1

Find the sum of the areas of the shaded rectangles to approximate the area under the graph of

$$f(x) = (x-2)(x+2)(x-1)^2 + 10$$

between $x = 0$ and $x = 3$.



Solution

We use the right-endpoint method with rectangles of width 0.5:

$$\begin{aligned}\text{Area} &= f(0.5)\times 0.5 + f(1)\times 0.5 + f(1.5)\times 0.5 + f(2)\times 0.5 + f(2.5)\times 0.5 + f(3)\times 0.5\\ &= \big(f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)\big)\times 0.5\\ &= 83.6875 \times 0.5\\ &= 41.84375\end{aligned}$$

Note: The left-endpoint estimate with the same intervals is 29.84375.
The actual area under the graph is 35.1.

ISBN 978-1-108-45164-2

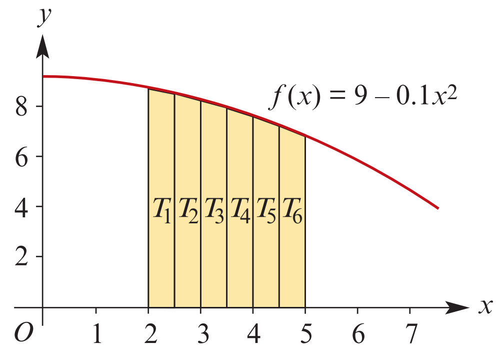
▶ The trapezoidal rule

For this method we work with trapeziums instead of rectangles.

The area of a trapezium is $\frac{1}{2}(a+b)h$, where a and b are the lengths of the two parallel sides and h is their distance apart.

Areas of trapeziums:

- $T_1 = \frac{1}{2}[f(2.0) + f(2.5)] \times 0.5 = 4.24375$
- $T_2 = \frac{1}{2}[f(2.5) + f(3.0)] \times 0.5 = 4.11875$
- $T_3 = \frac{1}{2}[f(3.0) + f(3.5)] \times 0.5 = 3.96875$
- $T_4 = \frac{1}{2}[f(3.5) + f(4.0)] \times 0.5 = 3.79375$
- $T_5 = \frac{1}{2}[f(4.0) + f(4.5)] \times 0.5 = 3.59375$
- $T_6 = \frac{1}{2}[f(4.5) + f(5.0)] \times 0.5 = 3.36875$



The sum of the areas is 23.0875 square units.

This is called the **trapezoidal estimate** for the area under the graph.

We can see that the trapezoidal estimate for this example can also be calculated as

$$\frac{1}{2}\Big[f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + 2f(4) + 2f(4.5) + f(5)\Big] \times 0.5$$

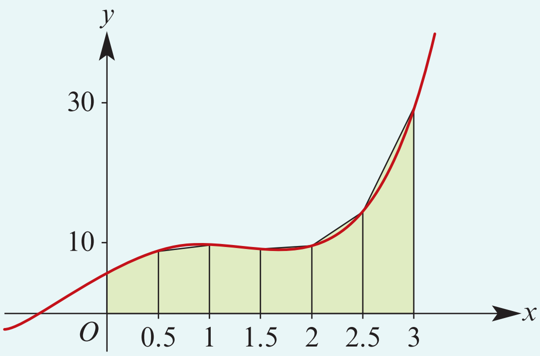
The trapezoidal estimate is the average of the left-endpoint and right-endpoint estimates.

Example 2

Find the sum of the areas of the shaded trapeziums to approximate the area under the graph of

$$f(x) = (x-2)(x+2)(x-1)^2 + 10$$

between $x = 0$ and $x = 3$.



Solution

We use the trapezoidal rule with trapeziums of width 0.5:

$$\text{Area} = \frac{1}{2}\Big[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)\Big] \times 0.5$$

$$= \frac{1}{4}\Big(6 + 2 \times 9.0625 + 2 \times 10 + 2 \times 9.5625 + 2 \times 10 + 2 \times 15.0625 + 30\Big)$$

$$= 35.84375$$

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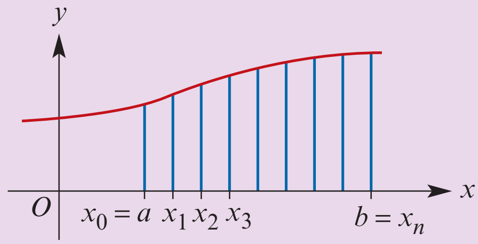
Section summary

Divide the interval $[a, b]$ on the x-axis into n equal subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, $\dots$, $[x_{n-1}, x_n]$ as illustrated.



Estimates for the area under the graph of $y = f(x)$ between $x = a$ and $x = b$:

- Left-endpoint method

$$L_n = \frac{b-a}{n}\Big[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\Big]$$

- Right-endpoint method

$$R_n = \frac{b-a}{n}\Big[f(x_1) + f(x_2) + \cdots + f(x_n)\Big]$$

- Trapezoidal rule

$$T_n = \frac{b-a}{2n}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big]$$

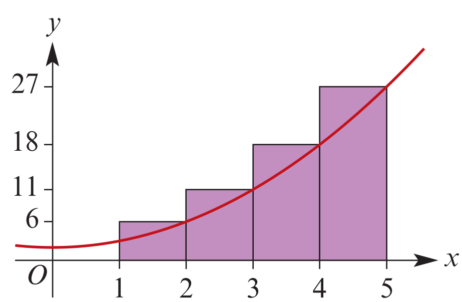
These methods are not limited to situations in which the graph is either increasing or decreasing for the whole interval. They may be used to determine the area under the curve for any continuous function on an interval $[a, b]$.

Exercise 10A

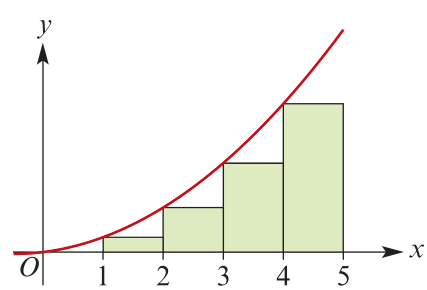
SF

Example 1

1 Find the sum of the areas of the shaded rectangles to approximate the area under the curve $y = x^2 + 2$ between $x = 1$ and $x = 5$.



2 Find the sum of the areas of the shaded rectangles to approximate the area under the curve $y = 2x^2 + 2x$ between $x = 1$ and $x = 5$.



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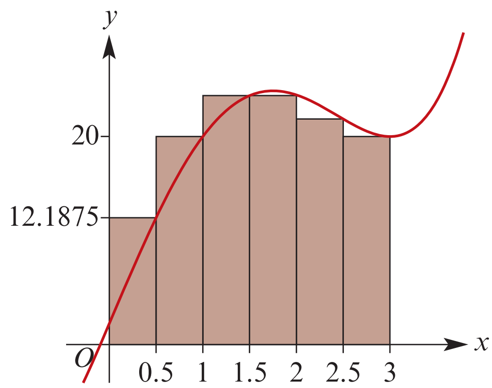
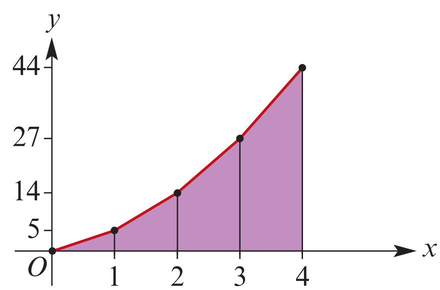
SF

3 Find the sum of the areas of the shaded rectangles to approximate the area under the curve $y = (x - 1)(x + 2)(x - 3)^2 + 20$ between $x = 0$ and $x = 3$.

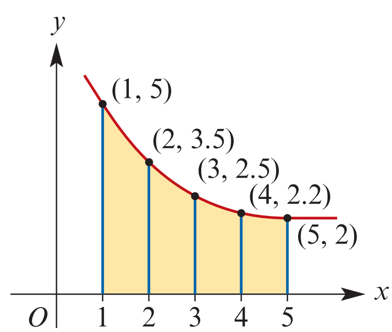


Example 2 **4** Find the sum of the areas of the shaded trapeziums to approximate the area under the curve $y = 2x^2 + 3x$ between $x = 0$ and $x = 4$.



5 To approximate the area of the shaded region, use the subintervals shown to calculate:

a the left-endpoint estimate
b the right-endpoint estimate
c the trapezoidal estimate.



6 Calculate an approximation to the area under the graph of $y = x(3 - x)$ between $x = 0$ and $x = 3$ using strips of width:

a 0.5
b 0.2

7 A table of values is given for the rule $y = f(x)$.

x	0	1	2	3	4	5	6	7	8	9	10
y	3	3.5	3.7	3.8	3.9	3.9	4.0	4.0	3.7	3.3	2.9

Estimate the area enclosed by the graph of $y = f(x)$, the lines $x = 0$ and $x = 10$, and the x-axis by using:

a the left-endpoint method
b the trapezoidal rule.

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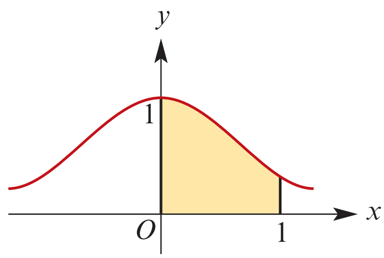
8 The graph is that of $y = \dfrac{1}{1+x^2}$. It is known that the area of the shaded region is $\dfrac{\pi}{4}$.

Apply the trapezoidal rule with strips of width 0.25, and hence find an approximate value for π.



CF

9 Use the trapezoidal rule to find an approximate value for the area under the graph of:

a $y = 2^x$ between $x = 0$ and $x = 2$, using intervals of width 0.5

b $y = \dfrac{1}{\sqrt{1-x^2}}$ between $x = 0$ and $x = 0.9$, using intervals of width 0.1.

SF

10 An engineer takes soundings at intervals of 3 metres across a river 30 metres wide to obtain the data in the following table. Use the trapezoidal rule to find an approximate value for the area of the cross-section of the river's channel.

CF

Distance from bank in metres	0	3	6	9	12	15	18	21	24	27	30
Depth of sounding in metres	1	2	3	4	5	5	6	4	4	2	2

10B Finding the exact area: the definite integral

▶ Definition of the integral

Assume that we want to find the area of the region under the graph of a continuous function f between $x = a$ and $x = b$. In this section, we again assume that the region is *above the x-axis*. (That is, we assume that $f(x) \geq 0$ for all $x \in [a, b]$.)

Using one of the three methods from the previous section, we can make better and better estimates of the area by taking narrower and narrower strips. In the limit (i.e. as $n \to \infty$), these estimates will converge to an answer for the exact area under the graph. Furthermore, it does not matter which of the three methods we choose, we will obtain the same answer in each case.

> The limit we obtain for the exact area under the graph of $y = f(x)$ between $x = a$ and $x = b$ is denoted by
>
> $$\int_a^b f(x)\, dx$$
>
> This is called the **definite integral** of $f(x)$ with respect to x from $x = a$ to $x = b$.

Note: The symbol $\int$ is called the **integral sign**, the numbers a and b are called the **limits** or **endpoints** of the integral, and the function f is called the **integrand**.

You will recognise the integral sign from Chapter 9, where it was used (without endpoints) to denote the general anti-derivative. We now see why the symbol for the area under a graph is also used for anti-derivatives.

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▶ The exact area

In the previous section we looked at ways of approximating the area under the graph of $f(x) = 9 - 0.1x^2$ between $x = 2$ and $x = 5$. We can now denote the exact area by

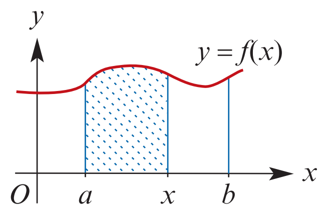
$$\int_2^5 f(x)\,dx$$

We will show that this area can be found by an anti-derivative of f.

The derivative of the area function

Let f be a continuous function on an interval $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$.

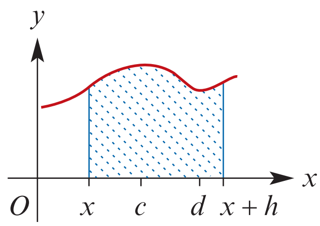
We define the function A geometrically by saying that $A(x)$ is the measure of the area under the curve between a and x. We thus have $A(a) = 0$. We will see that $A'(x) = f(x)$, and thus that A is an anti-derivative of f.



First consider the quotient $\dfrac{A(x+h) - A(x)}{h}$ for $h > 0$.

By our definition of $A(x)$, it follows that $A(x + h) - A(x)$ is the area between x and $x + h$.

Let c be the point in the interval $[x, x+h]$ such that $f(c) \geq f(z)$ for all $z \in [x, x + h]$, and let d be the point in the same interval such that $f(d) \leq f(z)$ for all $z \in [x, x + h]$.



Thus $f(d) \leq f(z) \leq f(c)$ for all $z \in [x, x + h]$.

Therefore $hf(d) \leq A(x + h) - A(x) \leq hf(c)$.

That is, the shaded region has an area less than the area of the rectangle with base h and height $f(c)$ and an area greater than the area of the rectangle with base h and height $f(d)$.

Dividing by h gives

$$f(d) \leq \frac{A(x+h) - A(x)}{h} \leq f(c)$$

As $h \to 0$, both $f(c)$ and $f(d)$ approach $f(x)$.

Thus we have shown that $A'(x) = f(x)$, and therefore A is an anti-derivative of f.

The example again

Now we return to the example $f(x) = 9 - 0.1x^2$ from Section 10A.

Define $A(x)$ to be the area under this graph between 2 and x. Then A is an anti-derivative of f such that $A(2) = 0$. The function defined by

$$A(x) = 9x - \frac{x^3}{30} - \frac{266}{15} = \frac{1}{30}\left(-x^3 + 270x - 532\right)$$

is an anti-derivative of f such that $A(2) = 0$. (Check this for yourself.) So the area under the graph between $x = 2$ and $x = 5$ is given by $A(5) = 23.1$ square units.

We can write $\int_2^5 9 - 0.1x^2\,dx = 23.1$.

ISBN 978-1-108-45164-2

▶ The fundamental theorem of calculus

A general method to find the area under a graph $y = f(x)$ between $x = a$ and $x = b$ is given by the following important theorem. (In the next section, we will see how to interpret this theorem when the graph is not above the x-axis.)

Fundamental theorem of calculus

If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\, dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

where F is any anti-derivative of f.

Proof Let A be the area function we defined above. Since both A and F are anti-derivatives of f, they must differ by a constant. That is,

$$A(x) = F(x) + k$$

where k is a constant. First let $x = a$. We then have

$$0 = A(a) = F(a) + k$$

and so $k = -F(a)$.

Thus $A(x) = F(x) - F(a)$, and letting $x = b$ yields

$$A(b) = F(b) - F(a)$$

Therefore the area under the curve $y = f(x)$ between a and b is equal to $F(b) - F(a)$, where F is an anti-derivative of f.

The notation $[F(x)]_a^b$ in the statement of this theorem helps with setting out integral calculations. It is just another way of writing $F(b) - F(a)$. The use of this notation is illustrated in the following example.

Example 3

Evaluate each of the following definite integrals:

a $\int_2^3 x^2\, dx$ **b** $\int_3^2 x^2\, dx$ **c** $\int_0^1 3x^3 + 2\, dx$

Solution

a $\int_2^3 x^2\, dx$

$= \left[\dfrac{x^3}{3}\right]_2^3$

$= \dfrac{27}{3} - \dfrac{8}{3}$

$= 9 - 2\tfrac{2}{3}$

$= 6\tfrac{1}{3}$

b $\int_3^2 x^2\, dx$

$= \left[\dfrac{x^3}{3}\right]_3^2$

$= \dfrac{8}{3} - \dfrac{27}{3}$

$= -6\tfrac{1}{3}$

c $\int_0^1 3x^3 + 2\, dx$

$= \left[\dfrac{3x^4}{4} + 2x\right]_0^1$

$= \dfrac{3}{4} + 2$

$= \dfrac{11}{4}$

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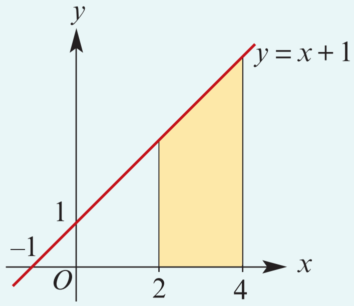
Example 4

Find the area of the shaded region.



Solution

Area $= \int_2^4 x + 1 \, dx$

An anti-derivative of $x + 1$ is $\frac{x^2}{2} + x$.

We write $\int_2^4 x + 1 \, dx = \left[\frac{x^2}{2} + x\right]_2^4$

$$= \left(\frac{4^2}{2} + 4\right) - \left(\frac{2^2}{2} + 2\right)$$

$$= 12 - 4$$

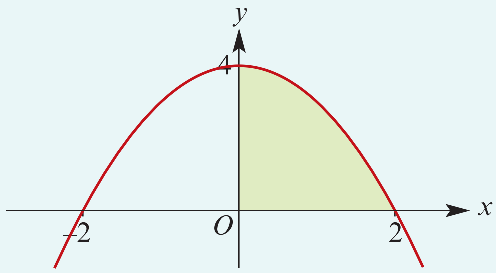
$$= 8$$

The shaded region has area 8 square units. (You do not have to write 'square units'. This is understood from the context.)

Note: This area can also be found without using calculus, since the region is a trapezium with area $\frac{1}{2}(3 + 5) \times 2 = 8$.

Example 5

Part of the graph of $y = 4 - x^2$ is shown on the right. Find the area of the shaded region.



Solution

By the fundamental theorem of calculus, the shaded area is given by

$$\int_0^2 4 - x^2 \, dx = \left[4x - \frac{x^3}{3}\right]_0^2$$

$$= 8 - \frac{8}{3}$$

$$= \frac{16}{3}$$

ISBN 978-1-108-45164-2

Section summary

- Let f be a continuous function on an interval $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$. Then the **definite integral** $\int_a^b f(x)\,dx$ denotes the area under the graph of $y = f(x)$ between $x = a$ and $x = b$.
- Fundamental theorem of calculus
  If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

where F is any anti-derivative of f.

Exercise 10B

SF

1 Evaluate each of the following definite integrals by using geometry to find the exact area under the graph:

a $\int_2^5 x - 2\,dx$ **b** $\int_{-1}^2 (2 - x)\,dx + \int_2^5 (x - 2)\,dx$ **c** $\int_1^2 2x + 1\,dx$

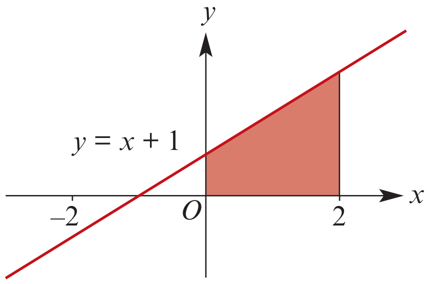
Example 3 **2** Evaluate each of the following definite integrals by using the fundamental theorem of calculus:

a $\int_1^2 x^2\,dx$ **b** $\int_2^3 x^3\,dx$ **c** $\int_1^2 x^3 - x\,dx$

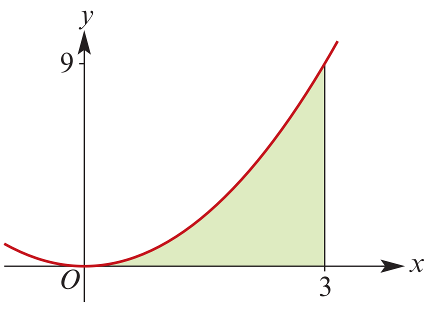
d $\int_{-1}^2 (x + 1)^2\,dx$ **e** $\int_1^2 x^3\,dx$ **f** $\int_1^4 x + 2x^2\,dx$

g $\int_0^2 x^3 + 2x^2 + x + 2\,dx$ **h** $\int_1^4 2x + 5\,dx$

Example 4 **3** Part of the graph of $y = x + 1$ is shown to the right. Find the area of the shaded region.



Example 5 **4** Part of the graph of $y = x^2$ is shown to the right. Find the area of the shaded region.



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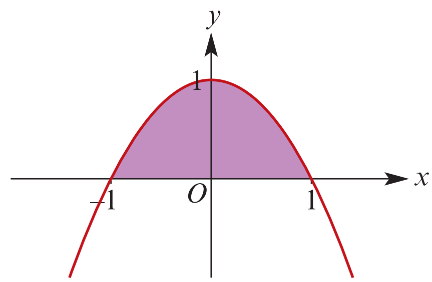
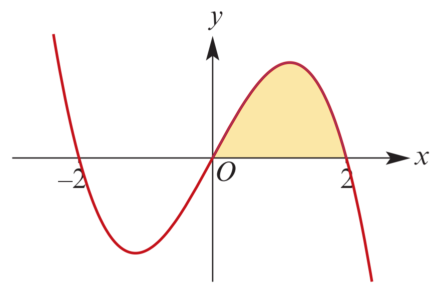
5 Part of the graph of $y = 1 - x^2$ is shown to the right. Find the area of the shaded region.



6 Part of the graph of $y = 4x - x^3$ is shown to the right. Find the area of the shaded region.



7 For each of the following, sketch a graph to illustrate the region for which the definite integral gives the area:

a $\int_0^3 3 - x\,dx$ **b** $\int_{-1}^1 4 - 2x^2\,dx$ **c** $\int_0^1 (1 - x)(1 + x)^2\,dx$

8 Sketch the graph of $f(x) = 1 + x^3$ and find the exact area of the region bounded by the curve and the axes.

10C Signed area

We now look at regions below the x-axis as well as those above the x-axis.

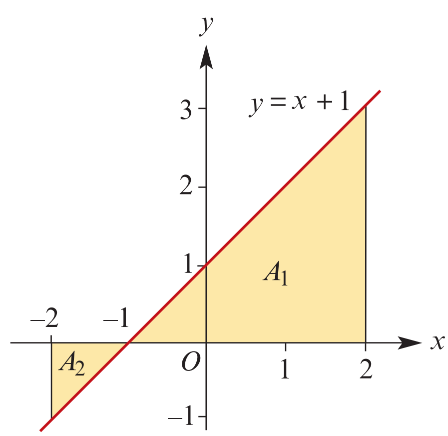
Consider the graph of $y = x + 1$ shown to the right.

$A_1 = \frac{1}{2} \times 3 \times 3 = 4\frac{1}{2}$ (area of a triangle)

$A_2 = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

The total area is $A_1 + A_2 = 5$.

The **signed area** is $A_1 - A_2 = 4$.



- Regions above the x-axis have *positive* signed area.
- Regions below the x-axis have *negative* signed area.

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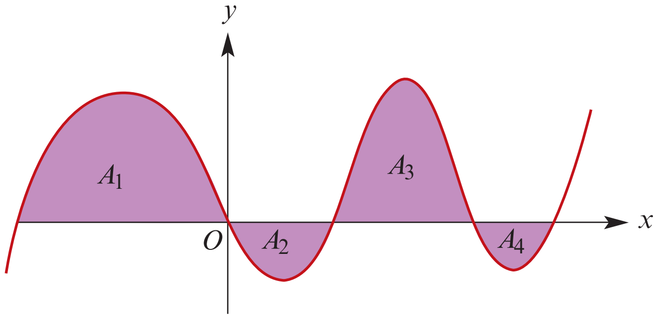
The total area of the shaded region is $A_1 + A_2 + A_3 + A_4$.

The signed area of the shaded region is $A_1 - A_2 + A_3 - A_4$.

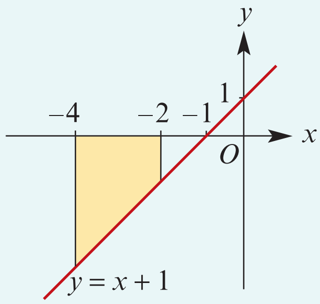


The left-endpoint, right-endpoint and trapezoidal estimates from Section 10A clearly give negative values when the graph lies below the x-axis, and so our definition of the integral is satisfactory in this situation.

The definite integral $\int_a^b f(x)\,dx$ gives the **signed area** enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$.

Example 6

Find the area of the shaded region.



Solution

$$\text{Area} = -\int_{-4}^{-2} x + 1\,dx$$

$$= -\left[\frac{x^2}{2} + x\right]_{-4}^{-2}$$

$$= -(0 - 4)$$

$$= 4$$

The area of the shaded region is 4 square units.

Explanation

The integral gives the signed area. Since the region is below the x-axis, its signed area is negative, and so its area is the negative of the integral from −4 to −2.

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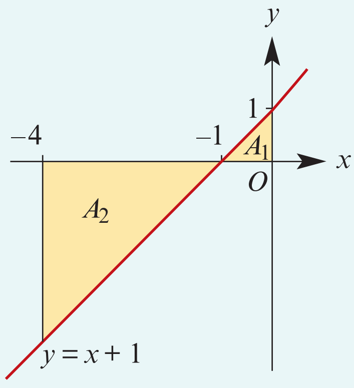
Example 7

a Find the total area of the shaded regions.

b Find the signed area of the shaded regions.



Solution

Area A_1 is above the x-axis and area A_2 is below the x-axis. Therefore

$$A_1 = \int_{-1}^{0} x + 1\,dx = \left[\frac{x^2}{2} + x\right]_{-1}^{0} = 0 - \left(\frac{1}{2} - 1\right) = \frac{1}{2}$$

$$A_2 = -\int_{-4}^{-1} x + 1\,dx = -\left[\frac{x^2}{2} + x\right]_{-4}^{-1} = -\left(\left(\frac{1}{2} - 1\right) - \left(\frac{16}{2} - 4\right)\right) = -\left(-\frac{1}{2} - 4\right) = 4\tfrac{1}{2}$$

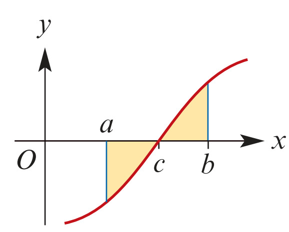
a Total area $= A_1 + A_2 = \frac{1}{2} + 4\frac{1}{2} = 5$

b Signed area $= A_1 - A_2 = \frac{1}{2} - 4\frac{1}{2} = -4$

Note: You can also find the signed area directly by evaluating $\int_{-4}^{0} x + 1\,dx$.

Finding areas

- If $f(x) \geq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.
- If $f(x) \leq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $-\int_a^b f(x)\,dx$.
- If $c \in (a, b)$ with $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area of the shaded region is given by $\int_c^b f(x)\,dx + \left(-\int_a^c f(x)\,dx\right)$.



Note: In determining the area 'under' a curve $y = f(x)$, the sign of $f(x)$ in the given interval is the critical factor.

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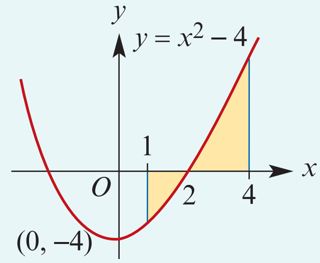
Example 8

Find the area of the shaded region.



Solution

$$\text{Area} = \int_2^4 (x^2 - 4)\,dx + \left(-\int_1^2 (x^2 - 4)\,dx\right)$$

$$= \left[\frac{x^3}{3} - 4x\right]_2^4 - \left[\frac{x^3}{3} - 4x\right]_1^2$$

$$= \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right) - \left(\left(\frac{8}{3} - 8\right) - \left(\frac{1}{3} - 4\right)\right)$$

$$= \frac{56}{3} - 8 - \left(\frac{7}{3} - 4\right) = \frac{37}{3}$$

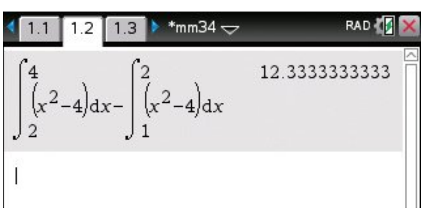
The area is $\frac{37}{3}$ square units.

TI-84

Using the TI-Nspire CX non-CAS

Use (menu) > **Calculus** > **Numerical Integral** and complete as shown.

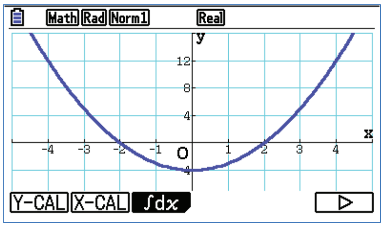
Note: The definite-integral template can also be accessed from the 2D-template palette or by using (shift) (+).

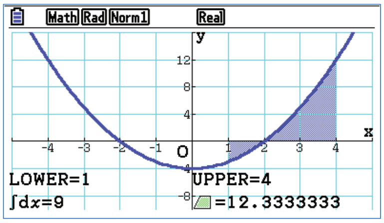


Using the Casio

Method 1: Using Graph mode

- In **Graph** mode, plot the graph of $y = x^2 - 4$.
- Adjust the View Window (SHIFT) (F3) for $-5 \le x \le 5$ and $-10 \le y \le 16$ with y-scale 4.
- Go to the **G-Solve** menu (SHIFT) (F5) and select **∫dx** (F6) (F3), then **Mixed** (F4).
- Enter 1 for the lower bound and 4 for the upper bound.
- The signed area is 9 (shown bottom left). The area is 12.33 (shown bottom right).





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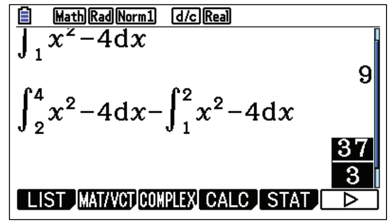
Method 2: Using Run-Matrix mode



To find the signed area:

- In **Run-Matrix** mode, go to the **Calculation** menu (OPTN) (F4) and select **∫dx** (F4).
- Enter the integrand $x^2 - 4$ and endpoints 1 and 4:

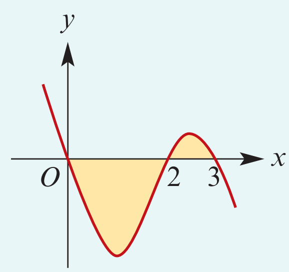
  (X,θ,T) (x^2) (−) (4) ► (1) ► (4) (EXE)
- The signed area is 9.

To find the area:

- Enter the expression $\int_2^4 x^2 - 4\,dx - \int_1^2 x^2 - 4\,dx$ as shown, using **∫dx** from the **Calculation** menu.
- The exact shaded area is $\dfrac{37}{3}$.

Example 9

Find the area enclosed by the graph of $y = x(2 - x)(x - 3)$ and the x-axis.



Solution

$$y = x(-x^2 + 5x - 6)$$
$$= -x^3 + 5x^2 - 6x$$

$$\text{Area} = \int_2^3 (-x^3 + 5x^2 - 6x)\,dx + \left(-\int_0^2 (-x^3 + 5x^2 - 6x)\,dx\right)$$

$$= \left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_2^3 - \left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_0^2$$

$$= \left(\frac{-81}{4} + 45 - 27\right) - \left(-4 + \frac{40}{3} - 12\right) - \left(-4 + \frac{40}{3} - 12\right)$$

$$= \frac{-81}{4} + 18 + 32 - \frac{80}{3}$$

$$= 50 - \frac{243 + 320}{12}$$

$$= \frac{37}{12}$$

The area is $\dfrac{37}{12}$ square units.

Note: There is no need to find the coordinates of stationary points.

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Properties of the definite integral

We present these properties without proof. The proofs are straightforward.

- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
- $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Example 10

Given that $\int_1^3 f(x)\,dx = 6$, evaluate:

a $\int_1^3 2f(x)\,dx$ **b** $\int_1^3 f(x) + 3\,dx$ **c** $\int_3^1 f(x)\,dx$

Solution

a $\int_1^3 2f(x)\,dx$

$= 2\int_1^3 f(x)\,dx$

$= 2 \times 6$

$= 12$

b $\int_1^3 f(x) + 3\,dx$

$= \int_1^3 f(x)\,dx + \int_1^3 3\,dx$

$= 6 + \big[3x\big]_1^3$

$= 6 + (9 - 3)$

$= 12$

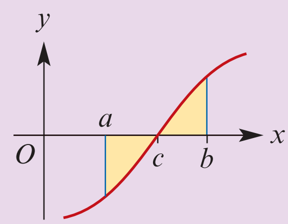
c $\int_3^1 f(x)\,dx$

$= -\int_1^3 f(x)\,dx$

$= -6$

Section summary

- For any continuous function f on an interval $[a, b]$, the **definite integral** $\int_a^b f(x)\,dx$ gives the **signed area** enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$.
- Finding areas
  - If $f(x) \geq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.
  - If $f(x) \leq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $-\int_a^b f(x)\,dx$.
  - If $c \in (a, b)$ with $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area of the shaded region is given by $\int_c^b f(x)\,dx + \left(-\int_a^c f(x)\,dx\right)$.



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Exercise 10C

Example 6

SF

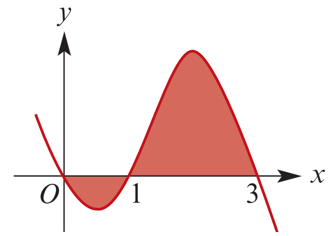
1 On a graph of $y = x^2 - 4x$, shade the region corresponding to $\int_0^4 x^2 - 4x\,dx$ and calculate its value.

Example 7, 8

2 On a graph of $y = x^2 - 9$, shade the region corresponding to $\int_{-3}^{3} x^2 - 9\,dx$ and calculate its value.

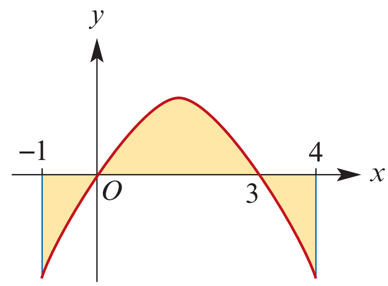
Example 9

3 The figure shows part of the graph of the curve with equation $y = x(x - 1)(3 - x)$. Calculate the area of the shaded region.



4 **a** Evaluate $\int_{-1}^{4} x(3 - x)\,dx$.

b Find the exact area of the shaded region in the figure.



Example 10

CF

5 Given that $\int_1^5 h(x)\,dx = 4$, evaluate:

a $\int_1^5 2h(x)\,dx$ **b** $\int_1^5 h(x) + 3\,dx$ **c** $\int_5^1 h(x)\,dx$ **d** $\int_1^5 h(x) - x\,dx$

6 Given that $\int_2^5 f(x)\,dx = 12$, evaluate:

a $\int_5^2 f(x)\,dx$ **b** $\int_2^5 3f(x)\,dx$ **c** $\int_2^4 (f(x) + 4)\,dx + \int_4^5 f(x)\,dx$

SF

7 Calculate the values of $\int_1^3 f(x)\,dx$, $\int_3^4 f(x)\,dx$ and $\int_1^4 f(x)\,dx$ for:

a $f(x) = 6x$

b $f(x) = 6 - 2x$

What is the relationship between your three answers in each case?

8 Sketch the graph of $y = 5x - x^2 - 4$ and find the area enclosed by the x-axis and the portion of the curve above the x-axis.

9 Sketch the graph of $y = x(10 - x)$ and hence find the area enclosed between the x-axis and the portion of the curve above the x-axis.

CF

10 Sketch the graph of $y = x(x - 2)(x + 1)$ and find the area of the region contained between the graph and the x-axis. (Do not attempt to find the coordinates of the turning points.)

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CF

11 Find the area bounded by the x-axis and the graph of each of the following functions:

a $f(x) = x^2 - 2x$

b $f(x) = (4 - x)(3 - x)$

c $f(x) = (x + 2)(7 - x)$

d $f(x) = x^2 - 5x + 6$

e $f(x) = 3 - x^2$

f $f(x) = x^3 - 6x^2$

10D Integration of more families of functions

So far in this chapter, we have been integrating polynomial functions. We now broaden our scope to include power functions, exponential functions and trigonometric functions. We use the following basic anti-derivatives, which were established in Chapter 9.

Basic anti-derivatives

$f(x)$	$\int f(x)\,dx$	
x^r	$\dfrac{x^{r+1}}{r+1} + c$	where $r \in \mathbb{Q} \setminus \{-1\}$
$(ax+b)^r$	$\dfrac{1}{a(r+1)}(ax+b)^{r+1} + c$	where $r \in \mathbb{Q} \setminus \{-1\}$
$\dfrac{1}{ax+b}$	$\dfrac{1}{a}\ln(ax+b) + c$	for $ax + b > 0$
e^{ax}	$\dfrac{1}{a}e^{ax} + c$	
$\sin(ax+b)$	$-\dfrac{1}{a}\cos(ax+b) + c$	
$\cos(ax+b)$	$\dfrac{1}{a}\sin(ax+b) + c$	

Example 11

a Evaluate $\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx$.

b Evaluate $\int_1^2 \frac{1}{x^3}\,dx$.

Solution

a
$$\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx = \left[\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}\right]_0^1$$
$$= \frac{2}{3} + \frac{2}{5}$$
$$= \frac{16}{15}$$

b
$$\int_1^2 x^{-3}\,dx = \left[-\frac{1}{2}x^{-2}\right]_1^2$$
$$= -\frac{1}{2}(2^{-2} - 1^{-2})$$
$$= -\frac{1}{2}\left(\frac{1}{4} - 1\right) = \frac{3}{8}$$

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Example 12

Evaluate each of the following definite integrals:

a $\int_0^1 2e^{-2x}\,dx$ **b** $\int_1^4 2x^{\frac{1}{2}} + e^{\frac{x}{2}}\,dx$

Solution

a $\int_0^1 2e^{-2x}\,dx = \left[\frac{2}{-2}e^{-2x}\right]_0^1$

$= -1(e^{-2\times 1} - e^{-2\times 0})$

$= -1(e^{-2} - 1)$

$= 1 - e^{-2}$

b $\int_1^4 2x^{\frac{1}{2}} + e^{\frac{x}{2}}\,dx = \left[\frac{4}{3}x^{\frac{3}{2}} + 2e^{\frac{x}{2}}\right]_1^4$

$= \frac{4}{3}\times 8 + 2e^2 - \left(\frac{4}{3} + 2e^{\frac{1}{2}}\right)$

$= \frac{28}{3} + 2e^2 - 2e^{\frac{1}{2}}$

$= 2\left(\frac{14}{3} + e^2 - e^{\frac{1}{2}}\right)$

Example 13

Evaluate each of the following definite integrals:

a $\int_6^8 \frac{1}{x-5}\,dx$ **b** $\int_4^5 \frac{1}{2x-5}\,dx$

Solution

a $\int_6^8 \frac{1}{x-5}\,dx = \left[\ln(x-5)\right]_6^8$

$= \ln 3 - \ln 1$

$= \ln 3$

b $\int_4^5 \frac{1}{2x-5}\,dx = \frac{1}{2}\left[\ln(2x-5)\right]_4^5$

$= \frac{1}{2}(\ln 5 - \ln 3)$

$= \frac{1}{2}\ln\left(\frac{5}{3}\right)$

Example 14

Find the exact value of each of the following definite integrals:

a $\int_0^{\frac{\pi}{4}} \sin(2x)\,dx$ **b** $\int_0^{\frac{\pi}{2}} 2\cos x + 1\,dx$

Solution

a $\int_0^{\frac{\pi}{4}} \sin(2x)\,dx$

$= \left[\frac{-1}{2}\cos(2x)\right]_0^{\frac{\pi}{4}}$

$= \frac{-1}{2}\cos\left(\frac{\pi}{2}\right) - \left(\frac{-1}{2}\cos 0\right)$

$= 0 + \frac{1}{2} = \frac{1}{2}$

b $\int_0^{\frac{\pi}{2}} 2\cos x + 1\,dx$

$= \left[2\sin x + x\right]_0^{\frac{\pi}{2}}$

$= 2\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2} - (2\sin 0 + 0)$

$= 2 + \frac{\pi}{2}$

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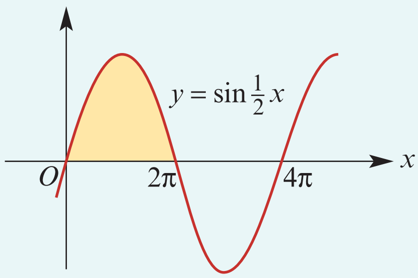
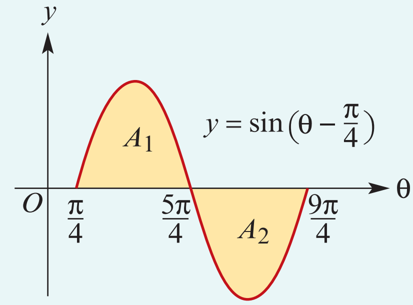
Example 15

Find the exact area of the shaded region for each graph:

a



b



Solution

a Area $= \int_0^{2\pi} \sin\left(\frac{1}{2}x\right) dx$

$= \left[-2\cos\left(\frac{1}{2}x\right)\right]_0^{2\pi}$

$= -2\cos\pi - (-2\cos 0)$

$= 4$

∴ Area of shaded region is 4 square units.

b Regions A_1 and A_2 must be considered separately:

Area $A_1 = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin\left(\theta - \frac{\pi}{4}\right) d\theta$

$= \left[-\cos\left(\theta - \frac{\pi}{4}\right)\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$

$= -(\cos\pi - \cos 0)$

$= 2$

Area $A_2 = -\int_{\frac{5\pi}{4}}^{\frac{9\pi}{4}} \sin\left(\theta - \frac{\pi}{4}\right) d\theta$

$= -\left[-\cos\left(\theta - \frac{\pi}{4}\right)\right]_{\frac{5\pi}{4}}^{\frac{9\pi}{4}}$

$= \cos(2\pi) - \cos\pi$

$= 2$

∴ Total area of shaded region is 4 square units.

Exercise 10D

Skillsheet

Example 11

1 Evaluate each of the following:

a $\int_1^2 \frac{1}{x^2}\, dx$ **b** $\int_1^4 x^{\frac{1}{2}} + 2x^2\, dx$ **c** $\int_1^4 2x^{\frac{3}{2}} + 5x^3\, dx$

SF

2 Evaluate each of the following:

a $\int_1^2 \frac{(2-x)(2+x)}{x^2}\, dx$ **b** $\int_1^4 2x - 3\sqrt{x}\, dx$ **c** $\int_1^3 \frac{4x^2+9}{x^2}\, dx$

d $\int_1^4 6x - 3\sqrt{x}\, dx$ **e** $\int_1^4 \frac{x^2-1}{x^2}\, dx$ **f** $\int_1^4 \frac{2x-3\sqrt{x}}{x}\, dx$

CF

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SF

3 Evaluate each of the following:

a $\int_0^1 (2x+1)^3\,dx$ **b** $\int_0^2 (4x+1)^{-\frac{1}{2}}\,dx$ **c** $\int_1^2 (1-2x)^2\,dx$

d $\int_0^1 (3-2x)^{-2}\,dx$ **e** $\int_0^2 (3+2x)^{-3}\,dx$ **f** $\int_{-1}^1 (4x+1)^3\,dx$

g $\int_0^1 \sqrt{2-x}\,dx$ **h** $\int_3^4 \frac{1}{\sqrt{2x-4}}\,dx$ **i** $\int_0^1 \frac{1}{(3+2x)^2}\,dx$

Example 12 **4** Evaluate each of the following:

a $\int_0^1 e^{2x}\,dx$ **b** $\int_0^1 e^{-2x}+1\,dx$ **c** $\int_0^1 2e^{\frac{x}{3}}+2\,dx$ **d** $\int_{-2}^2 \frac{e^x+e^{-x}}{2}\,dx$

Example 13 **5** Find each of the following definite integrals:

a $\int_7^8 \frac{1}{x-6}\,dx$ **b** $\int_2^4 \frac{1}{2x-3}\,dx$ **c** $\int_5^6 \frac{3}{2x+7}\,dx$

d $\int_2^5 \frac{3}{x+3}\,dx$ **e** $\int_{-1}^2 \frac{3}{x+5}\,dx$ **f** $\int_{-2}^{-1} \frac{3}{2x+5}\,dx$

6 Find the exact area of the region bounded by the curve $y = 3x + 2x^{-2}$, the lines $x = 2$ and $x = 5$ and the x-axis.

7 Sketch the graph of $f(x) = 4e^{2x} + 3$. Find the exact area of the region enclosed by the curve, the axes and the line $x = 1$.

8 Sketch the graph of $y = \frac{1}{3x-2}$. Find the exact area of the region enclosed by the curve, the x-axis and the lines with equations $x = 2$ and $x = 3$.

9 Sketch the graph of $y = \frac{1}{x+4} - 2$ for $x > -4$. Find the exact area of the region enclosed by the curve, the x-axis and the line $x = 2$.

Example 14 **10** Find the exact value of each of the following definite integrals:

a $\int_0^{\frac{\pi}{4}} \sin x\,dx$ **b** $\int_0^{\frac{\pi}{4}} \cos(2x)\,dx$ **c** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos\theta\,d\theta$

d $\int_0^{\frac{\pi}{2}} \sin\theta+\cos\theta\,d\theta$ **e** $\int_0^{\frac{\pi}{2}} \sin(2\theta)\,d\theta$ **f** $\int_0^{\frac{\pi}{3}} \cos(3\theta)+\sin(3\theta)\,d\theta$

g $\int_0^{\pi} \sin\left(\frac{x}{4}\right)+\cos\left(\frac{x}{4}\right)dx$ **h** $\int_0^{\frac{\pi}{4}} \sin\left(2x-\frac{\pi}{3}\right)dx$ **i** $\int_0^{\pi} \cos(2x)-\sin\left(\frac{x}{2}\right)dx$

Example 15 **11** Calculate the exact area of the region bounded by the curve $y = \sin\left(\frac{1}{2}x\right)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$.

12 For each of the following, draw a graph to illustrate the area given by the definite integral and evaluate the integral:

a $\int_0^{\frac{\pi}{4}} \cos x\,dx$ **b** $\int_0^{\frac{\pi}{3}} \sin(2x)\,dx$ **c** $\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \cos(2x)\,dx$

d $\int_0^{\frac{\pi}{2}} \cos\theta+\sin\theta\,d\theta$ **e** $\int_0^{\frac{\pi}{2}} \sin(2\theta)+1\,d\theta$ **f** $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} 1-\cos(2\theta)\,d\theta$

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13 Find the exact value of each of the following definite integrals:

a $\int_0^{\frac{\pi}{2}} \sin\left(2x + \frac{\pi}{4}\right) dx$ **b** $\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{6}\right) dx$

c $\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{3}\right) dx$ **d** $\int_0^{\frac{\pi}{4}} \cos(3\pi - x)\, dx$

SF

14 Sketch the curve $y = 2 + \sin(3x)$ for the interval $0 \le x \le \frac{2\pi}{3}$ and calculate the exact area enclosed by the curve, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{3}$.

CF

15 Find the exact value of each of the following:

a $\int_1^4 \sqrt{x}\, dx$ **b** $\int_{-1}^1 (1 + x)^2\, dx$ **c** $\int_0^8 \sqrt[3]{x}\, dx$

d $\int_0^{\frac{\pi}{3}} \cos(2x) - \sin\left(\frac{x}{2}\right) dx$ **e** $\int_1^2 e^{2x} + \frac{4}{x}\, dx$ **f** $\int_0^{\pi} \sin\left(\frac{x}{4}\right) + \cos\left(\frac{x}{4}\right) dx$

g $\int_0^{\frac{\pi}{2}} 5x + \sin(2x)\, dx$ **h** $\int_1^4 \left(2 + \frac{1}{x}\right)^2 dx$ **i** $\int_0^1 x^2(1 - x)\, dx$

SF

10E Further integration techniques

The techniques used for anti-differentiation in Section 9F can also be used for finding definite integrals. In the examples and exercise questions for this section, we revisit these techniques and introduce some new techniques.

Example 16

Let $f(x) = \ln(2x^2 - 1)$.

a Show that $f'(x) = \dfrac{4x}{2x^2 - 1}$.

b Hence evaluate $\int_1^2 \dfrac{x}{2x^2 - 1}\, dx$.

Solution

a Let $y = \ln(2x^2 - 1)$ and $u = 2x^2 - 1$.
Then $y = \ln u$. By the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \frac{1}{u} \cdot 4x$$
$$\therefore \quad f'(x) = \frac{4x}{2x^2 - 1}$$

b
$$\int_1^2 \frac{x}{2x^2 - 1}\, dx = \frac{1}{4} \int_1^2 \frac{4x}{2x^2 - 1}\, dx$$
$$= \frac{1}{4}\Big[\ln(2x^2 - 1)\Big]_1^2$$
$$= \frac{1}{4}(\ln 7 - \ln 1)$$
$$= \frac{1}{4} \ln 7$$

As seen in Section 9F, we can generalise this example. Let $g(x) = \ln(f(x))$, for $f(x) > 0$. Then the chain rule gives $g'(x) = \dfrac{f'(x)}{f(x)}$, and therefore we have

$$\int \frac{f'(x)}{f(x)}\, dx = \ln(f(x)) + c \qquad \text{for } f(x) > 0$$

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This result is used in the following example.

Example 17

Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x}\, dx$.

Solution

We know that the derivative of $\sin x$ is $\cos x$.

Hence

$$\begin{aligned}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x}\, dx &= \left[\ln(\sin x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \ln\left(\sin\left(\frac{\pi}{2}\right)\right) - \ln\left(\sin\left(\frac{\pi}{6}\right)\right) \\ &= \ln 1 - \ln\left(\frac{1}{2}\right) \\ &= 0 + \ln 2 \\ &= \ln 2\end{aligned}$$

It is not possible to find rules for anti-derivatives of all continuous functions: for example, for e^{-x^2}. However, for these functions we can find approximations of definite integrals.

Exercise 10E

Example 16, 17

1 Evaluate each of the following definite integrals: CF

a $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin x}{\cos x}\, dx$ **b** $\int_{-1}^{1} \frac{2x}{x^2+1}\, dx$ **c** $\int_{0}^{1} \frac{3x^2}{x^3+6}\, dx$

d $\int_{-1}^{2} \frac{20x}{10x^2+11}\, dx$ **e** $\int_{-1}^{1} \frac{5x}{5x^2+11}\, dx$ **f** $\int_{1}^{2} \frac{3x^2+2x}{x^3+x^2}\, dx$

g $\int_{1}^{2} \frac{5x}{5x^2+11}\, dx$ **h** $\int_{1}^{e} \frac{e^x}{e^x+1}\, dx$ **i** $\int_{2}^{e} \frac{1}{x\ln x}\, dx$

2 Find $\frac{d}{dx}(e^{\sqrt{x}})$ and hence evaluate $\int_{1}^{2} \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$.

3 Find $\frac{d}{dx}\left(\sin^3(2x)\right)$ and hence evaluate $\int_{0}^{\frac{\pi}{4}} \sin^2(2x)\cos(2x)\, dx$.

4 Using a graphics calculator, find the value of each of the following definite integrals correct to two decimal places: SF

a $\int_{0}^{20} 10\cos\left(\frac{\pi x}{40}\right) e^{\frac{x}{80}}\, dx$ **b** $\int_{2}^{5} \frac{e^x}{(x-1)^2}\, dx$ **c** $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos x}{(x-1)^2}\, dx$

5 Show that $\frac{2x+3}{x-1} = 2 + \frac{5}{x-1}$. Hence evaluate $\int_{2}^{4} \frac{2x+3}{x-1}\, dx$. CF

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6 Show that $\dfrac{5x-4}{x-2} = 5 + \dfrac{6}{x-2}$. Hence evaluate $\displaystyle\int_3^4 \frac{5x-4}{x-2}\,dx$.

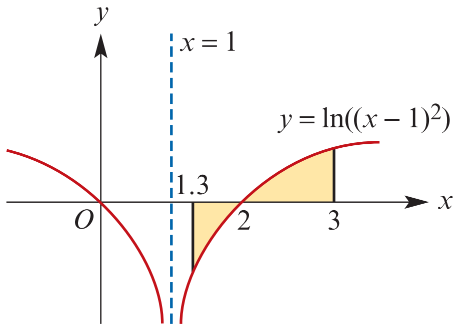
7 **a** Find $\dfrac{d}{dx}\left(xe^x\right)$ and hence evaluate $\displaystyle\int_0^1 xe^x\,dx$.

b Find $\dfrac{d}{dx}\left(x\sin(3x)\right)$ and hence evaluate $\displaystyle\int_0^{\frac{\pi}{6}} x\cos(3x)\,dx$.

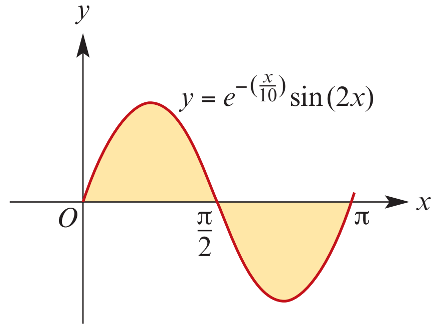
8 The curve with equation $y = a + b\sin\left(\dfrac{\pi x}{2}\right)$ passes through the points $(0, 1)$ and $(3, 3)$. Find a and b. Find the area of the region enclosed by this curve, the x-axis and the lines $x = 0$ and $x = 1$.

9 For each of the following, find the area of the shaded region correct to three decimal places:

a



b



10 Evaluate $\displaystyle\int_0^{\pi} e^{-\left(\frac{x}{10}\right)}\sin(2x)\,dx$, correct to four decimal places.

11 Sketch the graph of $y = \dfrac{2}{x-1} + 4$ and evaluate $\displaystyle\int_2^3 \frac{2}{x-1} + 4\,dx$.
Indicate on your graph the region for which you have determined the area.

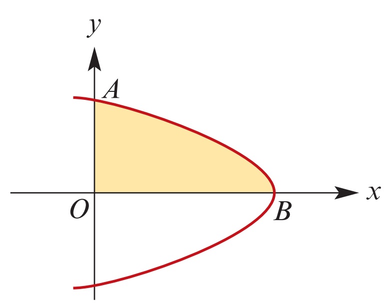
12 Sketch the graph of $y = \sqrt{2x-4} + 1$ and evaluate $\displaystyle\int_2^3 \sqrt{2x-4} + 1\,dx$.
Indicate on your graph the region for which you have determined the area.

13 **a** In the figure, the graph of $y^2 = 9(1 - x)$ is shown. Find the coordinates of A and B.

b Find the exact area of the shaded region by evaluating

$$\int_0^b 1 - \frac{y^2}{9}\,dy$$

for a suitable choice of b.



14 Let $a > 0$ with $a \neq 1$.

a Show that $a^x = e^{x(\ln a)}$.

b Hence find the derivative and an anti-derivative of a^x.

c Hence, or otherwise, show that the area under the curve $y = a^x$ between the lines $x = 0$ and $x = b$ is $\dfrac{1}{\ln a}(a^b - 1)$.

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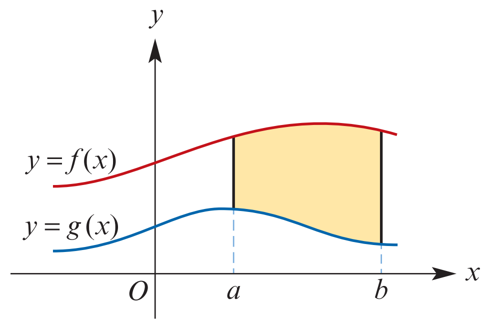
10F The area of a region between two curves

Let f and g be continuous functions on the interval $[a, b]$ such that

$$f(x) \geq g(x) \quad \text{for all } x \in [a, b]$$

Then the area of the region bounded by the two curves and the lines $x = a$ and $x = b$ can be found by evaluating

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$



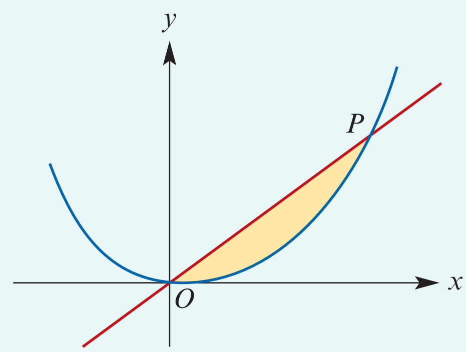
Example 18

Find the area of the region bounded by the parabola $y = x^2$ and the line $y = 2x$.

Solution

We first find the coordinates of the point P:

$$x^2 = 2x$$
$$x(x - 2) = 0$$
$$\therefore \quad x = 0 \text{ or } x = 2$$



Therefore the coordinates of P are $(2, 4)$.

$$\text{Required area} = \int_0^2 2x - x^2\,dx$$
$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$
$$= 4 - \frac{8}{3} = \frac{4}{3}$$

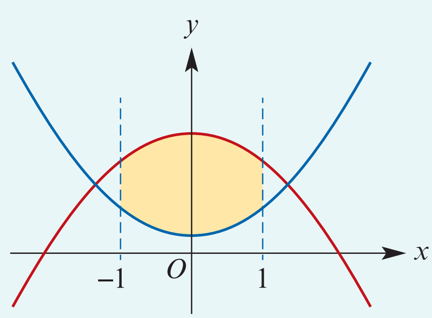
The area is $\frac{4}{3}$ square units.

Example 19

Calculate the area of the region enclosed by the curves with equations $y = x^2 + 1$ and $y = 4 - x^2$ and the lines $x = -1$ and $x = 1$.

Solution

$$\text{Required area} = \int_{-1}^1 4 - x^2 - (x^2 + 1)\,dx$$
$$= \int_{-1}^1 3 - 2x^2\,dx$$
$$= \left[3x - \frac{2x^3}{3}\right]_{-1}^1$$
$$= 3 - \frac{2}{3} - \left(-3 + \frac{2}{3}\right)$$
$$= \frac{14}{3}$$



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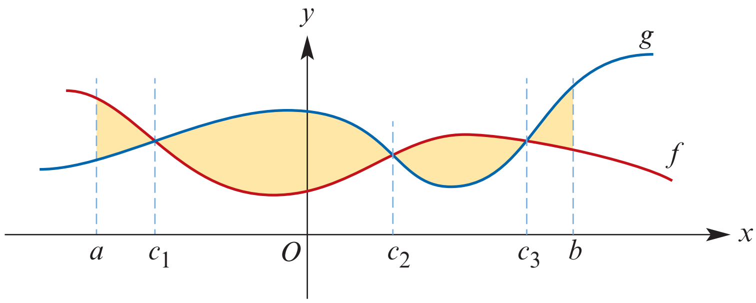
In Examples 18 and 19, the graph of one function is always 'above' the graph of the other for the intervals considered. What happens if the graphs cross?



To find the area of the shaded region, we must consider the intervals $[a, c_1]$, $[c_1, c_2]$, $[c_2, c_3]$ and $[c_3, b]$ separately. Thus, the shaded area is given by

$$\int_a^{c_1} f(x) - g(x)\,dx + \int_{c_1}^{c_2} g(x) - f(x)\,dx + \int_{c_2}^{c_3} f(x) - g(x)\,dx + \int_{c_3}^{b} g(x) - f(x)\,dx$$

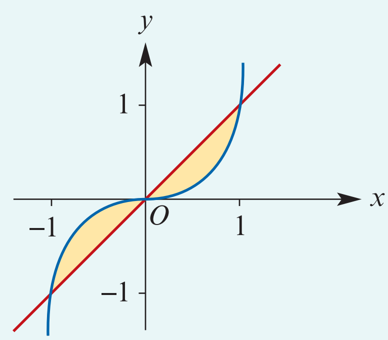
Example 20

Find the area of the region enclosed by the graphs of $f(x) = x^3$ and $g(x) = x$.

Solution

The graphs intersect where $f(x) = g(x)$:

$$x^3 = x$$
$$x^3 - x = 0$$
$$x(x^2 - 1) = 0$$
$$\therefore \quad x = 0 \text{ or } x = \pm 1$$



We see that:

- $f(x) \geq g(x)$ for $-1 \leq x \leq 0$
- $f(x) \leq g(x)$ for $0 \leq x \leq 1$

Thus the area is given by

$$\int_{-1}^{0} f(x) - g(x)\,dx + \int_0^1 g(x) - f(x)\,dx = \int_{-1}^{0} x^3 - x\,dx + \int_0^1 x - x^3\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= -\left(-\frac{1}{4}\right) + \frac{1}{4}$$

$$= \frac{1}{2}$$

The area is $\frac{1}{2}$ square unit.

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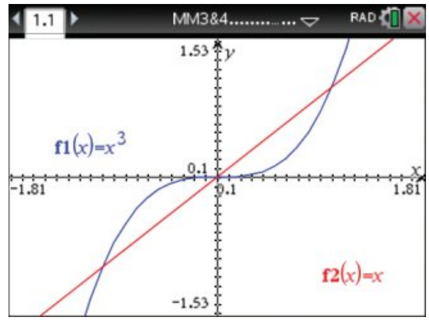
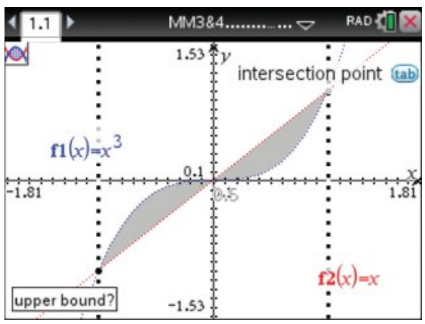
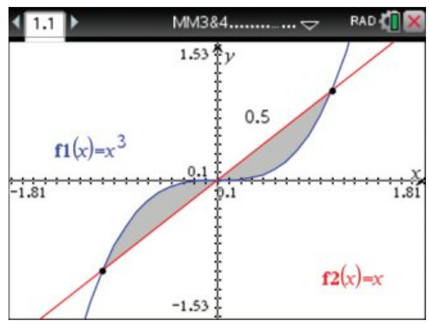
TI-84

Using the TI-Nspire CX non-CAS

- Plot the graphs of $f1(x) = x^3$ and $f2(x) = x$ as shown.
- Use (menu) > **Analyze Graph** > **Bounded Area** and select the left intersection point. Then move the cursor to select the right intersection point. Press (enter).
- Hence the enclosed area is 0.5 square units.

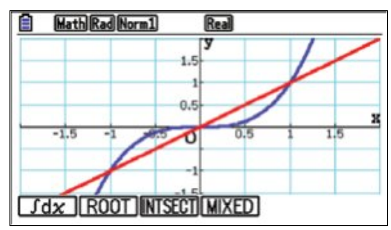


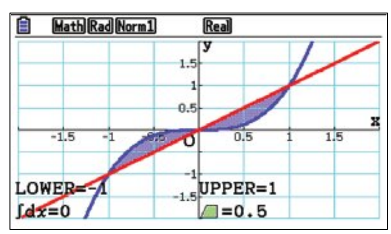




Using the Casio

- Plot the graphs of $y = x^3$ and $y = x$.
- Adjust the View Window (SHIFT) (F3) as shown.
- Go to **G-Solve** (SHIFT) (F5) and select **∫dx** (F6) (F3), then **Intersection** (F3).
- Select the first and third intersection points by pressing (EXE) ► ► (EXE).
- Hence the enclosed area is 0.5 square units.





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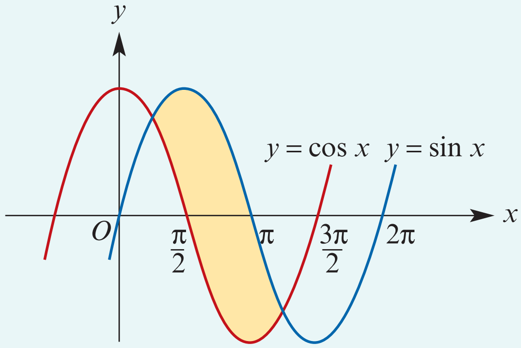
Example 21

Find the area of the shaded region.



Solution

First find the x-coordinates of the two points of intersection.

If $\sin x = \cos x$, then $\tan x = 1$ and so $x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$.

$$\begin{aligned}
\text{Area} &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x - \cos x \, dx \\
&= \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\
&= -\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right) - \left[-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right] \\
&= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\
&= \frac{4}{\sqrt{2}} = 2\sqrt{2}
\end{aligned}$$

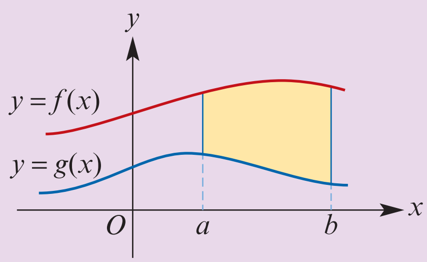
The area is $2\sqrt{2}$ square units.

Section summary

To find the area of the shaded region bounded by the two curves and the lines $x = a$ and $x = b$, use

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

where f and g are continuous functions on $[a, b]$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$.



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Exercise 10F

CF

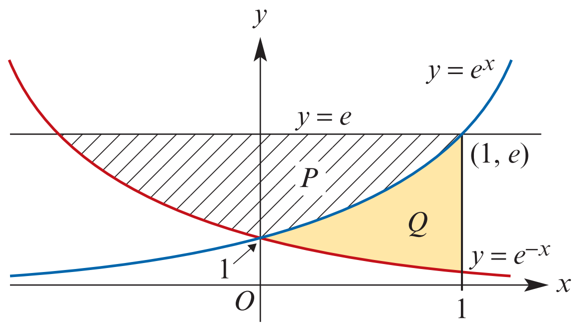
Example 18 **1** Find the exact area of the region bounded by the graphs of $y = 12 - x - x^2$ and $y = x + 4$.

Example 19 **2** Find the exact area of the region bounded by the graphs of $f(x) = 5 - x^2$ and $g(x) = (x - 1)^2$.

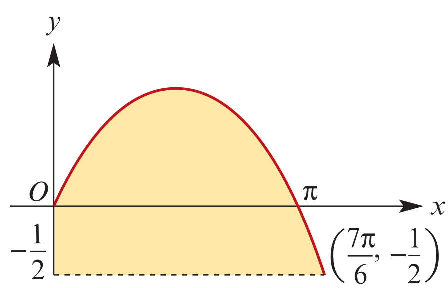
Example 20 **3** Find the exact area of the region bounded by the graphs with equations:

a $y = x + 3$ and $y = 12 + x - x^2$
b $y = 3x + 5$ and $y = x^2 + 1$
c $y = 3 - x^2$ and $y = 2x^2$
d $y = x^2$ and $y = 3x$
e $y^2 = x$ and $x - y = 2$

4 **a** Find the area of region P.
b Find the area of region Q.



5 The figure shows part of the curve $y = \sin x$. Calculate the area of the shaded region, correct to three decimal places.



Example 21 **6** Using the same axes, sketch the curves $y = \sin x$ and $y = \sin(2x)$ for $0 \le x \le \pi$. Calculate the smaller of the two areas enclosed by the curves.

7 Find the coordinates of P, the point of intersection of the curves $y = e^x$ and $y = 2 + 3e^{-x}$. If these curves cut the y-axis at points A and B respectively, calculate the area bounded by AB and the arcs AP and BP. Give your answer correct to three decimal places.

8 **a** Find the point of intersection of the graphs of $y = 8e^{-x}$ and $y = e^x - 2$.
b Sketch the graphs of $y = 8e^{-x}$ and $y = e^x - 2$ and find the area of the region bounded by the two curves and the y-axis.

9 Find the area of the region bounded by the graphs of $y = \frac{3}{x}$ and $y = 4 - x$.

10 Find the area of the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$.

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10G Applications of integration

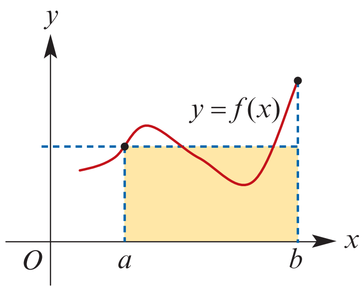
In this section we look at three applications of integration.

▶ Average value of a function

The average value of a function f for an interval $[a, b]$ is defined as:

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

In terms of the graph of $y = f(x)$, the average value is the height of a rectangle having the same area as the area under the graph for the interval $[a, b]$.



Example 22

Find the average value of $f(x) = x^2$ for the interval $[0, 2]$. Illustrate with a horizontal line determined by this value.

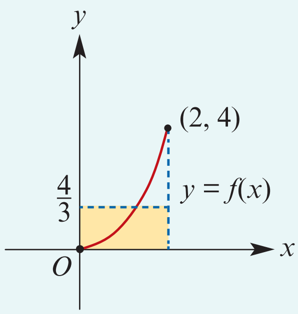
Solution

$$\text{Average} = \frac{1}{2-0}\int_0^2 x^2\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{2}\times\frac{8}{3} = \frac{4}{3}$$

Note: Area of rectangle $= \int_0^2 f(x)\,dx$



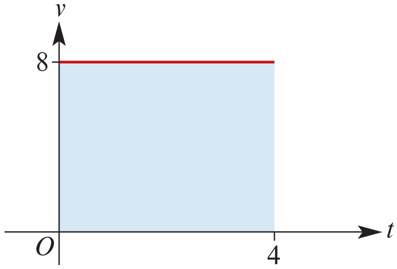
▶ Motion in a straight line

We have applied anti-differentiation to motion in a straight line in Section 9G. We now use the connection between anti-differentiation and the area under a graph.

When a particle is moving with constant velocity, the corresponding velocity–time graph (v against t) is a straight line parallel to the t-axis.

The velocity–time graph for a particle moving at 8 m/s for 4 seconds is shown on the right.

The shaded region is a rectangle of area $8 \times 4 = 32$, which is the product of the velocity and the time taken. Therefore this area is equal to the particle's displacement, 32 m, over the 4 seconds.



In the next example, we look at a situation where the velocity is not constant.

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Example 23

A particle starts moving along a straight line. Its velocity, $v(t)$ m/s, after t seconds is given by $v(t) = 2t - 6$.

a Find the particle's displacement after 4 seconds.

b Find the distance travelled by the particle in the first 4 seconds.

c Sketch the velocity–time graph and find:

  i the signed area enclosed by the graph and the t-axis between $t = 0$ and $t = 4$

  ii the area enclosed by the graph and the t-axis between $t = 0$ and $t = 4$.

Solution

a Anti-differentiate $v(t)$ to find the position, $x(t)$ m, after t seconds:

$$x(t) = t^2 - 6t + c$$

The particle's displacement is its change in position:

$$x(4) - x(0) = (-8 + c) - c = -8$$

The displacement after 4 seconds is -8 m.

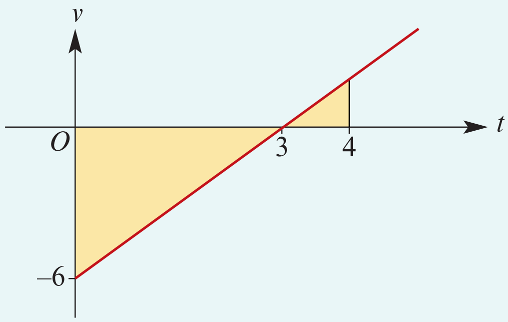
b First find when the particle is at rest: $v(t) = 0$ implies $2t - 6 = 0$, i.e. $t = 3$.

Now consider the displacement over the two time intervals $[0, 3]$ and $[3, 4]$:

- Displacement over $[0, 3]$: $x(3) - x(0) = (-9 + c) - c = -9$ m
- Displacement over $[3, 4]$: $x(4) - x(3) = (-8 + c) - (-9 + c) = 1$ m

The particle travelled 10 m in the first 4 seconds.

c



i Signed area $= \int_0^4 v(t)\,dt$

$= \int_0^4 2t - 6\,dt$

$= \left[t^2 - 6t\right]_0^4$

$= -8 - 0$

$= -8$

ii Area $= -\int_0^3 v(t)\,dt + \int_3^4 v(t)\,dt$

$= -\int_0^3 2t - 6\,dt + \int_3^4 2t - 6\,dt$

$= -\left[t^2 - 6t\right]_0^3 + \left[t^2 - 6t\right]_3^4$

$= -(-9 - 0) + (-8 - (-9))$

$= 9 + 1$

$= 10$

Note: The displacement is equal to the signed area.
The distance travelled is equal to the area.

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Since a particle's position, $x(t)$, is an anti-derivative of its velocity, $v(t)$, the results below follow directly from the fundamental theorem of calculus.

Using definite integrals for motion in a straight line

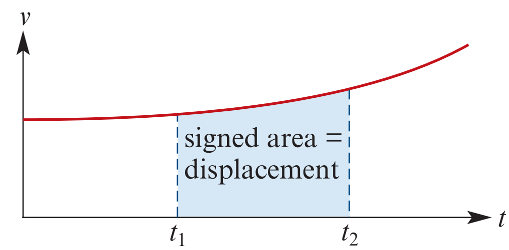
- The displacement of a particle over a time interval $[t_1, t_2]$ is given by

$$\text{displacement} = x(t_2) - x(t_1) = \int_{t_1}^{t_2} v(t)\, dt$$

- The average velocity of a particle over a time interval $[t_1, t_2]$ is given by

$$\text{average velocity} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v(t)\, dt$$

Therefore the signed area of the region(s) between the velocity–time graph and the t-axis corresponds to the displacement of the particle between times t_1 and t_2.



Example 24

A particle starts from rest and moves in a straight line with acceleration $a = 6t + 8$ m/s^2, where t is the time in seconds ($t \geq 0$).

a Find the displacement (change in position) over the first 4 seconds of motion.

b Find the average velocity over the first 4 seconds of motion.

Solution

We are given the acceleration:

$$a = \frac{dv}{dt} = 6t + 8$$

Find the velocity by anti-differentiating:

$$v = 3t^2 + 8t + c$$

At $t = 0$, $v = 0$, and so $c = 0$.

$$\therefore \quad v = 3t^2 + 8t$$

a
$$\begin{aligned} \text{Displacement} &= \int_0^4 v(t)\, dt \\ &= \int_0^4 3t^2 + 8t\, dt \\ &= \left[t^3 + 4t^2 \right]_0^4 \\ &= 128 \text{ m} \end{aligned}$$

b
$$\begin{aligned} \text{Average velocity} &= \frac{\text{change in position}}{\text{change in time}} \\ &= \frac{128}{4} \\ &= 32 \text{ m/s} \end{aligned}$$

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▶ Rates of change

Given the rate of change of a quantity, we can obtain information about how the quantity varies. For example, we have seen that if the velocity of an object travelling in a straight line is given at time t, then the position of the object at time t can be determined using information about the initial position of the object.

Example 25

The rate of change of temperature with respect to time of a liquid which has been boiled and then allowed to cool is given by

$$\frac{dT}{dt} = -0.5(T - 30)$$

where T is the temperature (°C) at time t (minutes).

a Sketch the graph of $\frac{dT}{dt}$ against T for $T > 30$.

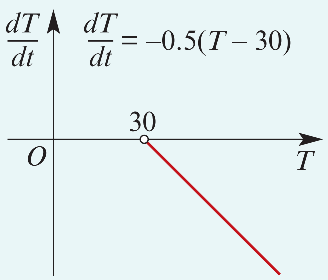
b Sketch the graph of $\frac{dt}{dT}$ against T for $T > 30$.

c **i** Find the area of the region enclosed by the graph of **b**, the T-axis and the lines $T = 35$ and $T = 120$. Give your answer correct to two decimal places.

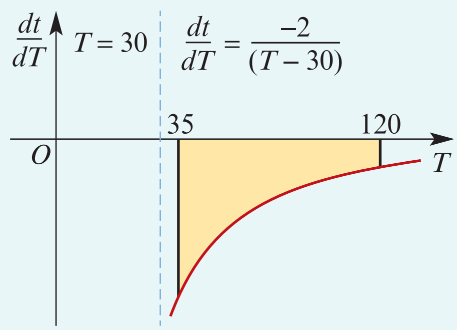
ii What does this area represent?

Solution

a



b



c **i** Area $= -\int_{35}^{120} \frac{-2}{T-30}\, dT = 5.78$

ii The area represents the time taken for the liquid to cool from 120°C to 35°C.

We can use integration to find the total change in a quantity from the rate of change of the quantity. The result below follows directly from the fundamental theorem of calculus.

Given the rule for $f'(x)$, the **total change** in the value of $f(x)$ between $x = a$ and $x = b$ can be found using

$$f(b) - f(a) = \int_a^b f'(x)\, dx$$

Note: In Example 25, the definite integral $\int_{120}^{35} \frac{dt}{dT}\, dT = \int_{120}^{35} \frac{-2}{T-30}\, dT = 5.78$ gives the change in t between $T = 120$ and $T = 35$.

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Exercise 10G

SF

Example 22

1 Find the average value of each of the following functions for the stated interval:

a $f(x) = x(2 - x),\ x \in [0, 2]$

b $f(x) = \sin x,\ x \in [0, \pi]$

c $f(x) = \sin x,\ x \in \left[0, \frac{\pi}{2}\right]$

d $f(x) = \sin(nx),\ x \in \left[0, \frac{2\pi}{n}\right]$

e $f(x) = e^x + e^{-x},\ x \in [-2, 2]$

CF

2 An object is cooling and its temperature, T°C, after t minutes is given by $T = 50e^{\frac{-t}{2}}$. What is its average temperature over the first 10 minutes of cooling?

3 Find the average speed over the given interval for each of the following speed functions. For each of them, sketch a graph and mark in the average as a horizontal line. Time is in seconds and speed in metres per second.

a $v = 20t,\ t \in [0, 5]$

b $v = 24\sin\left(\frac{1}{4}\pi t\right),\ t \in [0, 4]$

c $v = 5(1 - e^{-t}),\ t \in [0, 5]$

4 An object falls from rest. Its velocity, v m/s, at time t seconds is given by $v = 9.8t$. Find the average velocity of the object over the first 3 seconds of its motion.

5 Find the mean value of $x(a - x)$ from $x = 0$ to $x = a$.

6 A quantity of gas expands according to the law $pv^{0.9} = 300$, where v m^3 is the volume of the gas and p N/m^2 is the pressure.

a What is the average pressure as the volume changes from $\frac{1}{2}$ m^3 to 1 m^3?

b If the change in volume in terms of t is given by $v = 3t + 1$, what is the average pressure over the time interval from $t = 0$ to $t = 1$?

Example 23

7 A particle moves in a straight line. The velocity of the particle, v m/s, at time t seconds is given by $v = 2t - 3$ for $t \geq 0$.

a Sketch the velocity–time graph for the motion of the particle from $t = 0$ to $t = 4$.

b By calculating a total signed area, find the particle's displacement after 4 seconds.

c By calculating a total area, find the distance travelled by the particle in the first 4 seconds.

8 A particle is oscillating in a straight line. Its velocity, v m/s, at time t seconds is given by $v = \frac{3\pi}{2}\sin\left(\frac{\pi t}{2}\right)$. Find the particle's displacement between $t = 1$ and $t = 4$.

9 A particle's velocity, v cm/s, at time t seconds is given by a function of the form $v = 3 + at$, where a is a constant. Given that the particle's displacement between $t = 1$ and $t = 2$ is -7 cm, find the value of a.

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Example 24 **10** A particle moves along a straight line. Its acceleration, a m/s^2, at time t seconds ($t \geq 0$) is given by $a = 3t - 5$. Its initial velocity is 3 m/s.

a Find the displacement of the particle in the period from $t = 1$ to $t = 5$.

b Find the average velocity of the particle over this period.

CF

11 After t seconds ($t \geq 0$), a particle has acceleration $a = kt$ m/s^2, where k is a constant. Given that, after 2 seconds, the particle's displacement is 7 m and its velocity is 4 m/s, find the value of k.

CU

Example 25 **12** Heat escapes from a storage tank such that the rate of heat loss, in kilojoules per day, is given by

$$\frac{dH}{dt} = 1 + \frac{3}{4}\sin\left(\frac{\pi t}{60}\right), \qquad 0 \leq t \leq 200$$

where $H(t)$ is the total accumulated heat loss at time t days after noon on 1 April.

a Sketch the graph of $\frac{dH}{dt}$ against t for $0 \leq t \leq 200$.

b Find the values of t for which the rate of heat loss, i.e. $\frac{dH}{dt}$, is greater than 1.375.

c Find the values of t for which the rate of heat loss reaches its maximum.

d Find the heat lost between:

i $t = 0$ and $t = 120$ **ii** $t = 0$ and $t = 200$

13 The rate of flow of water from a reservoir is given by $\frac{dV}{dt} = 1000 - 30t^2 + 2t^3$ for $0 \leq t \leq 15$, where V is measured in millions of litres and t is the number of hours after the sluice gates are opened.

a Find the rate of flow (in million litres per hour) when $t = 0$ and $t = 2$.

b **i** Find the times when the rate of flow is a maximum.

ii Find the maximum flow.

c Sketch the graph of $\frac{dV}{dt}$ against t for $0 \leq t \leq 15$.

d **i** Find the area beneath the graph between $t = 0$ and $t = 10$.

ii What does this area represent?

14 The population of penguins on an island off the coast of Tasmania is increasing steadily. The rate of growth is given by the function $R(t) = 10\ln(t + 1)$ for $t \geq 0$. The rate is measured in number of penguins per year. The date 1 January 1875 coincides with $t = 0$.

a Find the rate of growth of penguins when $t = 5$, $t = 10$ and $t = 100$.

b Sketch the graph of $y = R(t)$.

c Using a graphics calculator, find the area under the graph of $y = R(t)$ between $t = 0$ and $t = 100$. What does this area represent?

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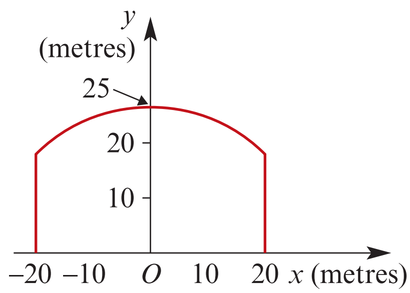
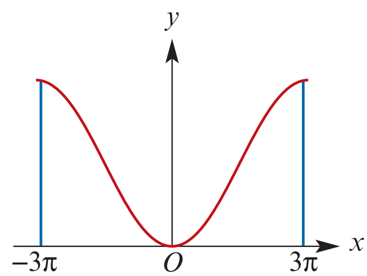
CU

15 The roof of an exhibition hall has the shape of the function $f(x) = 25 - 0.02x^2$, $x \in [-20, 20]$. The hall is 80 metres long. A cross-section of the hall is shown in the figure. An air-conditioning company wishes to find the volume of the hall so that a suitable system may be installed. Find this volume.



16 A long trough with a parabolic cross-section is $1\frac{1}{2}$ metres wide at the top and 2 metres deep. Find the depth of water when the trough is half full.

17 A sculpture has cross-section as shown. The equation of the curve is $y = 3 - 3\cos\left(\frac{x}{3}\right)$ for $x \in [-3\pi, 3\pi]$. All measurements are in metres.



a Find the maximum value of the function and hence the height of the sculpture.

b The sculpture has a flat metal finish on one face, which in the diagram is represented by the region between the curve and the x-axis. Find the area of this region.

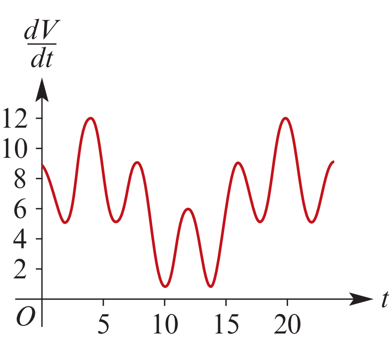
c There is a strut that meets the right side of the curve at right angles and passes through the point $(9, 0)$.

i Find the equation of the normal to the curve where $x = a$.

ii Find, correct to three decimal places, the value of a if the normal passes through $(9, 0)$.

18 The graph shows the number of litres per minute of water flowing through a pipe against the number of minutes since the machine started. The pipe is attached to the machine, which requires the water for cooling.
The curve has equation

$$\frac{dV}{dt} = 3\left[\cos\left(\frac{\pi t}{2}\right) + \sin\left(\frac{\pi t}{8}\right) + 2\right]$$



a What is the rate of flow of water when:

i $t = 0$ **ii** $t = 2$ **iii** $t = 4$?

b Find, correct to three decimal places, the maximum and minimum flow through the pipe.

c Find the volume of water which flows through the pipe in the first 8 minutes.

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10H The area under a graph as the limit of a sum

In this final section, we consider a special case of finding the exact area under a graph as the limit of a sum. This discussion gives an indication of how the limiting process can be undertaken in general.

Notation

We first introduce a notation to help us express sums. We do this through examples:

$$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2$$

$$\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5$$

$$\sum_{i=1}^{n} x_i f(x_i) = x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + \cdots + x_n f(x_n)$$

The symbol $\sum$ is the uppercase Greek letter 'sigma', which is used to denote *sum*.

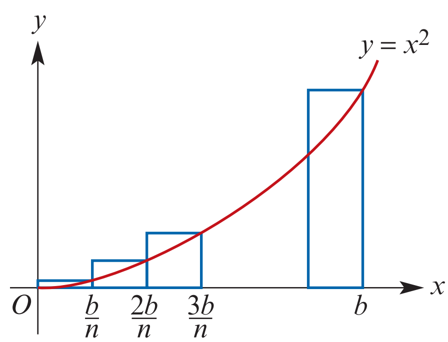
The area under a parabola

Consider the graph of $y = x^2$. We will find the area under the curve from $x = 0$ to $x = b$ using a technique due to Archimedes.

Divide the interval $[0, b]$ into n equal subintervals:

$$\left[0, \frac{b}{n}\right], \left[\frac{b}{n}, \frac{2b}{n}\right], \left[\frac{2b}{n}, \frac{3b}{n}\right], \ldots, \left[\frac{(n-1)b}{n}, b\right]$$

Each subinterval is the base of a rectangle with height determined by the right endpoint of the subinterval.



We obtain:

$$\begin{aligned} \text{area of rectangles} &= \frac{b}{n}\left[\left(\frac{b}{n}\right)^2 + \left(\frac{2b}{n}\right)^2 + \left(\frac{3b}{n}\right)^2 + \cdots + \left(\frac{nb}{n}\right)^2\right] \\ &= \frac{b}{n}\left(\frac{b^2}{n^2} + \frac{4b^2}{n^2} + \frac{9b^2}{n^2} + \cdots + \frac{n^2 b^2}{n^2}\right) \\ &= \frac{b^3}{n^3}(1 + 4 + 9 + \cdots + n^2) \end{aligned}$$

There is a formula for working out the sum of the first n square numbers:

$$\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$$

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Using this formula gives:

$$\begin{aligned}\text{area of rectangles} &= \frac{b^3}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{b^3}{n^3} \times \frac{n}{6}(n+1)(2n+1) \\ &= \frac{b^3}{6n^2}(2n^2+3n+1) \\ &= \frac{b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\end{aligned}$$

As n becomes very large, the terms $\frac{3}{n}$ and $\frac{1}{n^2}$ become very small. We write:

$$\lim_{n\to\infty} \frac{b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) = \frac{b^3}{3}$$

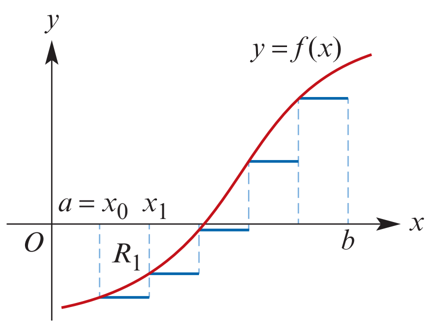
We read this as: the limit of the sum as n approaches infinity is $\frac{b^3}{3}$.

Using n left-endpoint rectangles, and considering the limit as $n \to \infty$, also gives the area $\frac{b^3}{3}$.

The signed area enclosed by a curve

This technique may be applied in general to a continuous function f on an interval $[a, b]$. For convenience, we will consider an increasing function.

Divide the interval $[a, b]$ into n equal subintervals. Each subinterval is the base of a rectangle with its 'height' determined by the left endpoint of the subinterval.



The contribution of rectangle R_1 is $(x_1 - x_0)f(x_0)$. Since $f(x_0) < 0$, the result is negative and so we have found the *signed area* of R_1.

The sum of the signed areas of the rectangles is

$$\frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i)$$

If the limit as $n \to \infty$ exists, then we can make the following definition:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i)\right)$$

We could also have used the right-endpoint estimate or the trapezoidal estimate. All three estimates will converge to the same limit as n approaches infinity. Definite integrals may be defined as the limit of suitable sums, and the fundamental theorem of calculus holds true under this definition.

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Review

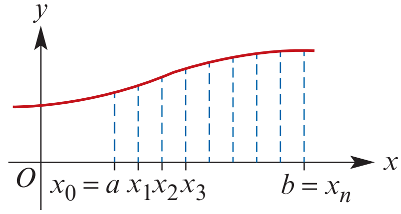
Chapter summary

AS

Nrich

- Numerical methods for approximating the area under a graph:
  The interval $[a, b]$ on the x-axis is divided into n equal subintervals $[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$.
  - Left-endpoint method

    $$L_n = \frac{b-a}{n}\Big[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\Big]$$



  - Right-endpoint method

    $$R_n = \frac{b-a}{n}\Big[f(x_1) + f(x_2) + \cdots + f(x_n)\Big]$$

  - Trapezoidal rule

    $$T_n = \frac{b-a}{2n}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big]$$

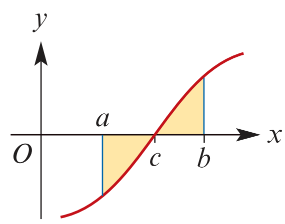
- Definite integral The signed area enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$ is denoted by $\int_a^b f(x)\,dx$.

- Fundamental theorem of calculus
  If f is a continuous function on an interval $[a, b]$, then

  $$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

  where F is any anti-derivative of f.

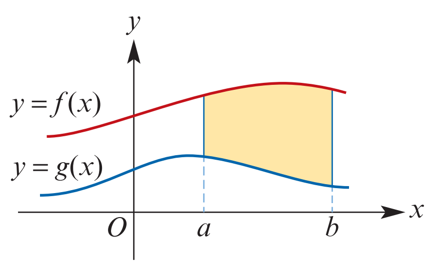
- Finding areas:
  - If $f(x) \geq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.
  - If $f(x) \leq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $-\int_a^b f(x)\,dx$.
  - If $c \in (a, b)$ with $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area of the shaded region is given by $\int_c^b f(x)\,dx + \Big(-\int_a^c f(x)\,dx\Big)$.



  - To find the area of the shaded region bounded by the two curves and the lines $x = a$ and $x = b$, use

    $$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

    where f and g are continuous functions on $[a, b]$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$.



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Review

- Properties of the definite integral:
  - $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
  - $\int_a^a f(x)\,dx = 0$
  - $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
  - $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
  - $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- The **average value** of a continuous function f for an interval $[a, b]$ is $\dfrac{1}{b-a}\int_a^b f(x)\,dx$.
- The **total change** in the value of f for an interval $[a, b]$ is $f(b) - f(a) = \int_a^b f'(x)\,dx$.

Technology-free questions

SF

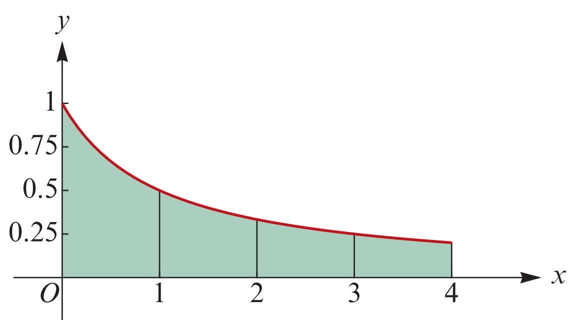
1 The graph of $y = \dfrac{1}{x+1}$ is shown.

Estimate the area under the graph between $x = 0$ and $x = 4$ using:

a the left-endpoint method

b the right-endpoint method

c the trapezoidal rule.



2 Evaluate each of the following definite integrals:

a $\int_1^4 \sqrt{x}\,dx$ **b** $\int_1^4 x^3 - 2x\,dx$ **c** $\int_1^2 \dfrac{1}{x^2}\,dx$

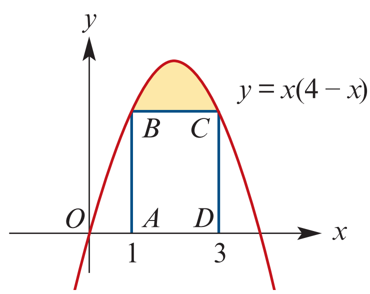
d $\int_1^4 \dfrac{2}{x^3}\,dx$ **e** $\int_0^1 \sqrt{x}\,(x+1)\,dx$

CF

3 **a** Find the coordinates of points B and C.

b Find the area of rectangle $ABCD$.

c Find the area of the shaded region.



4 Let $f(x) = x(x-3)(x+2)$. Calculate the exact area of the region enclosed between the graph of $y = f(x)$ and the x-axis.

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Review

SF

5 Evaluate each of the following definite integrals:

a $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta \, d\theta$ **b** $\int_{a}^{4a} (a^{\frac{1}{2}} - x^{\frac{1}{2}}) \, dx$, where a is a positive constant

c $\int_{1}^{4} \frac{3}{\sqrt{x}} - 5\sqrt{x} - x^{-\frac{3}{2}} \, dx$ **d** $\int_{0}^{\frac{\pi}{4}} \cos(2\theta) \, d\theta$ **e** $\int_{1}^{e} \frac{1}{x} \, dx$

f $\int_{0}^{\frac{\pi}{2}} \sin 2\left(\theta + \frac{\pi}{4}\right) d\theta$ **g** $\int_{0}^{\pi} \sin(4\theta) \, d\theta$

CF

6 Find $\int_{-1}^{2} x + 2f(x) \, dx$ if $\int_{-1}^{2} f(x) \, dx = 5$.

7 Find $\int_{1}^{5} f(x) \, dx$ if $\int_{0}^{1} f(x) \, dx = -2$ and $\int_{0}^{5} f(x) \, dx = 1$.

8 Find $\int_{3}^{-2} f(x) \, dx$ if $\int_{-2}^{1} f(x) \, dx = 2$ and $\int_{1}^{3} f(x) \, dx = -6$.

SF

9 Evaluate $\int_{0}^{2} (x + 1)^7 \, dx$.

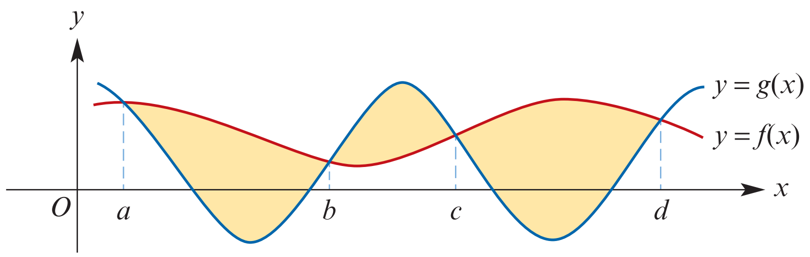
10 Evaluate $\int_{0}^{1} (3x + 1)^3 \, dx$.

CF

11 Find $\int_{0}^{3} f(3x) \, dx$ if $\int_{0}^{9} f(x) \, dx = 5$.

12 Find $\int_{0}^{1} f(3x + 1) \, dx$ if $\int_{1}^{4} f(x) \, dx = 5$.

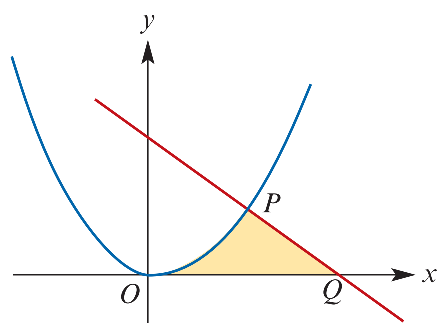
13 Set up a sum of definite integrals that represents the total shaded area between the curves $y = f(x)$ and $y = g(x)$.



14 The figure shows the curve $y = x^2$ and the straight line $2x + y = 15$. Find:

a the coordinates of P and Q

b the area of the shaded region.



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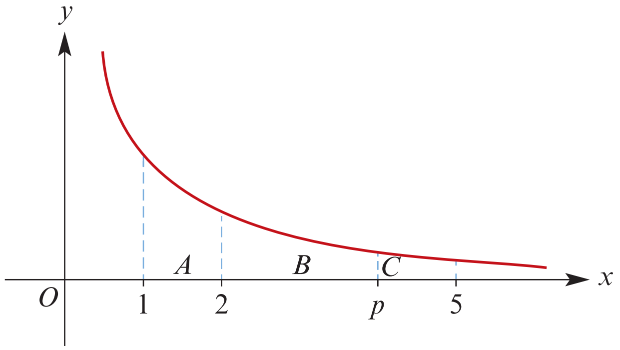
Review

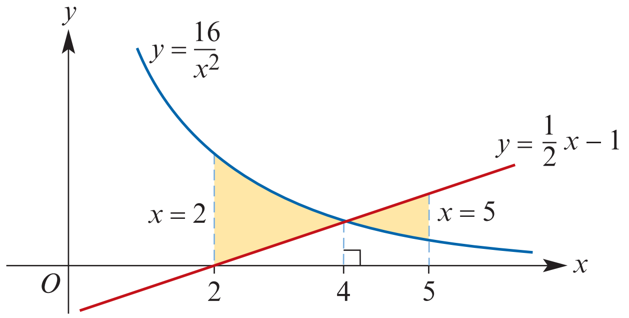
15 The figure shows part of the curve $y = \dfrac{10}{x^2}$. Find:

a the area of region A

b the value of p for which the regions B and C are of equal area.



16 Find the area of the shaded region.

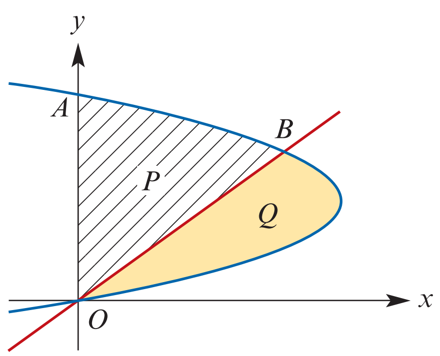


17 The figure shows part of the curve $x = 6y - y^2$ and part of the line $y = x$.

a Find the coordinates of A and B.

b Find the area of region P.

c Find the area of region Q.



18 **a** Sketch the graph of $y = e^x + 1$ and clearly indicate, by shading the region, the area given by the definite integral $\int_0^2 e^x + 1\,dx$.

b Evaluate $\int_0^2 e^x + 1\,dx$.

19 **a** Sketch the graphs of $y = e^{-x}$ and $y = e^x$ on the one set of axes and clearly indicate, by shading the region, the area given by $\int_0^2 e^{-x}\,dx + \int_{-2}^0 e^x\,dx$.

b Evaluate $\int_0^2 e^{-x}\,dx + \int_{-2}^0 e^x\,dx$.

CF

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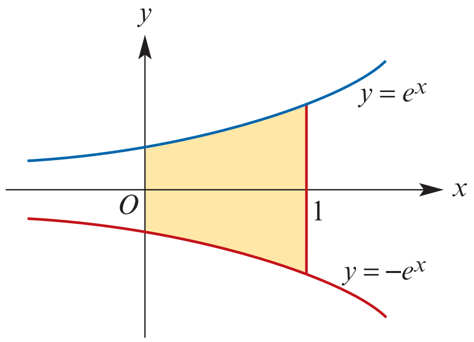
Review

CF

20 **a** Evaluate $\int_0^1 e^x \, dx$.

b By symmetry, find the area of the region shaded in the figure.



21 Sketch the graph of $f(x) = 2e^{2x} + 3$ and find the area of the region enclosed between the curve, the axes and the line $x = 1$.

SF

22 Evaluate each of the following definite integrals:

a $\int_0^2 e^{-x} + x \, dx$

b $\int_2^3 x + \frac{1}{x-1} \, dx$

c $\int_0^{\frac{\pi}{2}} \sin x + x \, dx$

d $\int_{-4}^{-5} e^x + \frac{1}{2-2x} \, dx$

Multiple-choice questions

1 If $F'(x) = f(x)$, then $\int_3^5 f(x) \, dx$ is

A $f(5) - f(3)$ **B** $f(5) + c$ **C** $f(5) - f(3) + c$
D $F(5) - F(3)$ **E** $F(5) + F(3)$

2 $\int_0^2 3x^2 - 2x \, dx =$

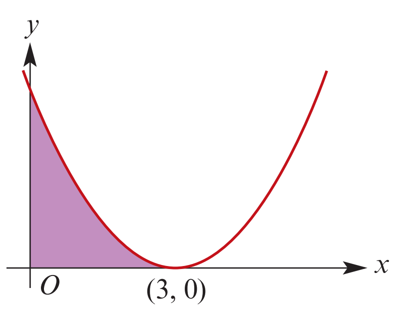
A 12 **B** 4 **C** 8 **D** 32 **E** 2

3 An equivalent expression for $\int_0^2 3f(x) + 2 \, dx$ is

A $3\int_0^2 f(x) \, dx + 2x$ **B** $3\int_0^2 f(x) \, dx + 2x$ **C** $3\int_0^2 f(x) \, dx + 4$
D $2f'(x) + 3$ **E** $\int_0^2 f(x) + 6$

4 The graph with the equation $y = k(x-3)^2$ is shown. If the area of the shaded region is 36 square units, the value of k is

A 4 **B** −4 **C** 9 **D** $\frac{4}{7}$ **E** 32



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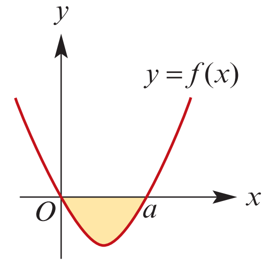
5 The area of the shaded region is given by



A $\int_0^a -f(x)\,dx$

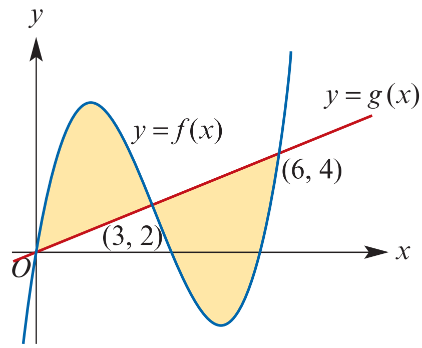
B $\int_0^a f(x)\,dx$

C $\int_0^a x - f(x)\,dx$

D $\int_0^a f(x) - x\,dx$

E none of these

6 An expression using integral notation for the area of the shaded region shown is



A $\int_0^6 f(x) - g(x)\,dx$

B $\int_0^3 f(x) - g(x)\,dx + \int_3^6 g(x) - f(x)\,dx$

C $\int_0^4 f(x) - g(x)\,dx$

D $\int_0^2 f(x) - g(x)\,dx + \int_2^4 f(x) - g(x)\,dx$

E $\int_0^2 f(x) - g(x)\,dx + \int_2^4 g(x) - f(x)\,dx$

7 $\int_a^b c\,dx$, where a, b and c are distinct real number constants, is equal to

A ca **B** $cb - ca$ **C** $ca - b$ **D** $cb - a$ **E** $c(a + b)$

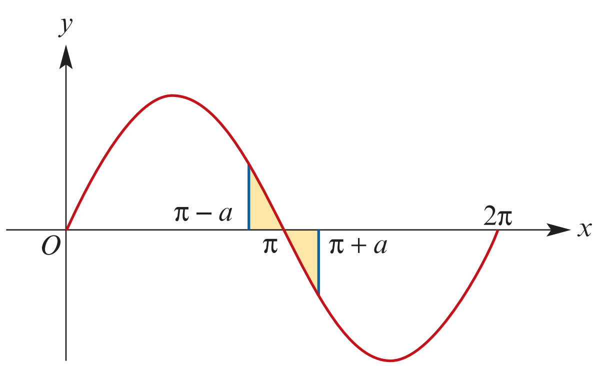
8 The area of the region enclosed by the curve $y = e^{5x} - 2\sin(4x)$, the x-axis and the lines $x = -1$ and $x = 1$, correct to two decimal places, is

A 0.17 **B** 29.55 **C** 29.68 **D** 29.85 **E** 30.02

9 The rate of flow of water from a tap follows the rule $R(t) = 5e^{-0.1t}$, where $R(t)$ litres per minute is the rate of flow after t minutes. The number of litres, to the nearest litre, which flowed out in the first 3 minutes is

A 0 **B** 5 **C** 13 **D** 50 **E** 153

10 The graph represents the function $y = \sin x$, $0 \le x \le 2\pi$. The total area of the shaded regions is



A $\frac{1}{2}\cos a$

B $2\cos a$

C $\frac{1}{2}(1 - \cos a)$

D $2(1 - \cos a)$

E $2\sin^2 a$

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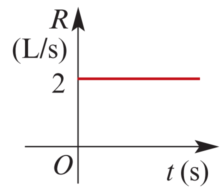
Extended-response questions

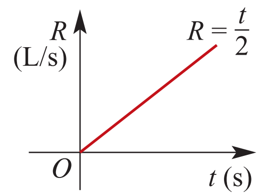
1 **a** Water flows into a container at the constant rate (R) of 2 litres per second. A graph of the rate of flow is as shown.



i How much water has flowed into the container after 1 minute?

ii Illustrate this quantity as an area under the graph of $R = 2$.

b Water flows into a container at a rate R L/s, where $R = \frac{t}{2}$ and t is the time measured in seconds. The graph is as shown.

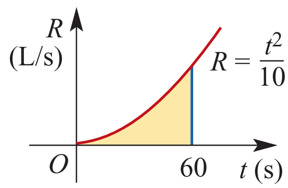


i How much water has flowed into the container after 1 minute?

ii Illustrate this quantity as an area under the graph of $R = \frac{t}{2}$.

iii How much water has flowed into the container after a minutes?

c Water flows into a container at a rate R L/s, where $R = \frac{t^2}{10}$ and t is the time measured in seconds.



i Find the area of the shaded region.

ii What does the area represent?

iii After how many seconds will 10 000 litres have flowed into the container?

2 **a** **i** A car travels on a straight road at 60 km/h for 2 hours. Sketch the speed–time graph illustrating this.

ii Shade the region which indicates the total distance travelled by the car after 2 hours.

b **i** Sketch the speed–time graph of a car travelling for 5 minutes, if the car starts from rest and accelerates at a rate of 0.3 km/min^2.

ii How far has the car travelled at the end of 5 minutes?

c A particle starts from a point O and travels at a velocity V m/s given by $V = 20t - 3t^2$, where t seconds is the time the particle has been travelling.

i Find the acceleration of the particle at time t.

ii Sketch the graph of V against t for $0 \le t \le \frac{20}{3}$.

iii How far has the particle travelled after 6 seconds? Illustrate this quantity by shading a suitable region under the graph.

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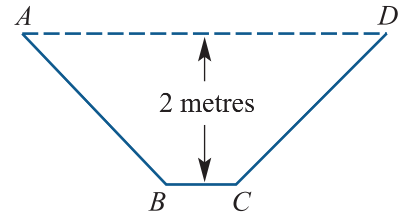
Review

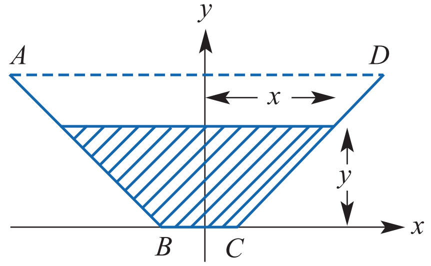
CU

3 An irrigation channel 2 metres deep is constructed. Its cross-section, trapezium $ABCD$, is shown in the diagram. The length AD is 5 metres and the length BC is 1 metre.



a Calculate the area of this cross-section.

b The cross-section is placed symmetrically on coordinate axes as shown.



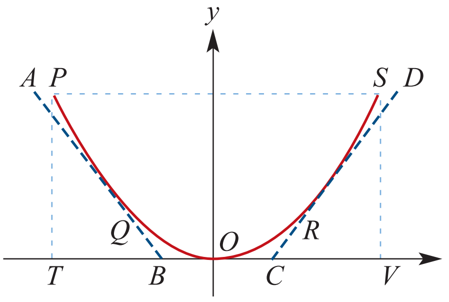
i Find the equation of the line CD.

ii Calculate the area of the cross-section of the water when its depth is y m. Give your answer in terms of x.

c In an attempt to improve water flow, a metal chute is added. Its cross-section has the shape of a parabola $PQORS$ and it just touches the original channel at Q, O and R.

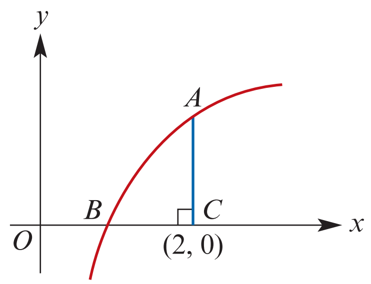
i Given that the x-coordinate of R is 1, find the equation of the parabola and find the coordinates of P and S.

ii Calculate the area of the parabolic cross-section $PQORS$.



4 The diagram shows part of the curve with equation

$$y = x - \frac{1}{x^2}$$



The point C has coordinates $(2, 0)$. Find:

a the equation of the tangent to the curve at point A

b the coordinates of the point T where this tangent meets the x-axis

c the coordinates of the point B where the curve meets the x-axis

d the area of the region enclosed by the curve and the lines AT and BT

e the ratio of the area found in part **d** to the area of the triangle ATC.

5 It is thought that the temperature, θ, of a piece of charcoal in a barbecue will increase at a rate $\frac{d\theta}{dt}$ given by $\frac{d\theta}{dt} = e^{2.6t}$, where θ is in degrees and t is in minutes.

a If the charcoal starts at a temperature of 30°C, find the expected temperature of the charcoal after 3 minutes.

b Sketch the graph of θ against t.

c At what time does the temperature of the charcoal reach 500°C?

d Find the average rate of increase of temperature from $t = 1$ to $t = 2$.

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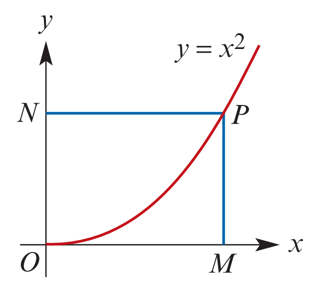
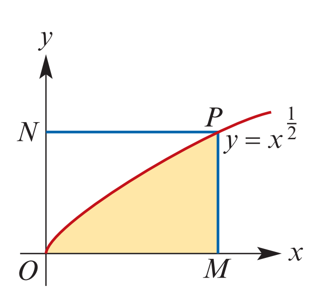
6 **a** In the figure, the point P is on the curve $y = x^2$.
Prove that the curve divides the rectangle $OMPN$ into two regions whose areas are in the ratio 2 : 1.



b In the figure, the point P is on the curve $y = x^{\frac{1}{2}}$.
Prove that the area of the shaded region is two-thirds the area of the rectangle $OMPN$.



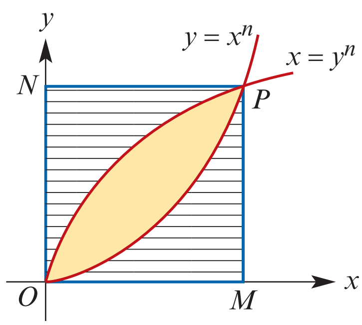
c Consider a point P on the curve $y = x^n$, with PM and PN the perpendiculars from P to the x-axis and the y-axis respectively. Prove that the area of the region enclosed between PM, the x-axis and the curve is equal to $\frac{1}{n+1}$ of the area of the rectangle $OMPN$.

7 **a** Find the area enclosed between the parabolas $y = x^2$ and $y^2 = x$.

b Show that the curves with equations $y = x^n$ and $y^n = x$ intersect at (1, 1), where $n = 1, 2, 3, \ldots$.

c Show that the area of the region contained between the curves $y = x^n$ and $y^n = x$ is $\frac{n-1}{n+1}$.

d Find the area of the region indicated by horizontal shading in the diagram.



e Use your result from **c** to find the area of the region between the curves for $n = 10$, $n = 100$ and $n = 1000$.

f Describe the result for n very large.

8 It is believed that the velocity of a certain subatomic particle t seconds after a collision will be given by the expression

$$\frac{dx}{dt} = ve^{-t}, \quad v = 5 \times 10^4 \text{ m/s}$$

where x is the distance travelled in metres.

a What is the initial velocity of the particle?

b What happens to the velocity as $t \to \infty$ (i.e. as t becomes very large)?

c How far will the particle travel between $t = 0$ and $t = 20$?

d Find an expression for x in terms of t.

e Sketch the graph of x against t.

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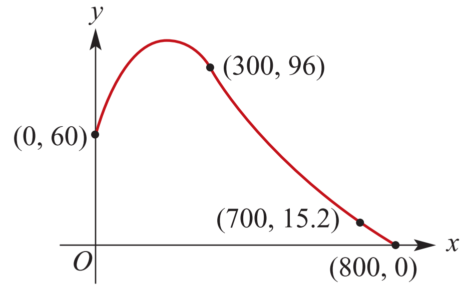
Review

CU

9 A small hill has a cross-section as shown. The coordinates of four points are given. Measurements are in metres.



a Find the equation of the cubic.

b Find the maximum height of the hill correct to the nearest metre.

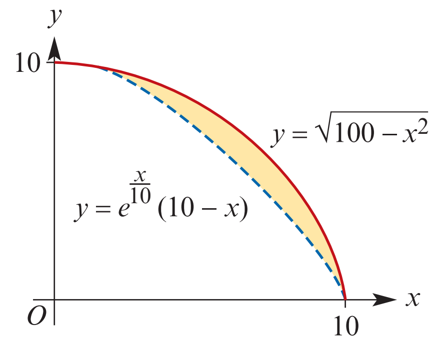
c **i** Using a graphics calculator, plot the graph of the gradient function.

ii State the coordinates of the point on the curve where the magnitude of the gradient is a maximum.

d Use a graphics calculator to find the cross-sectional area correct to the nearest square metre.

10 A teacher attempts to draw a quarter circle of radius 10 on the white board. However, the first attempt results in a curve with equation $y = e^{\frac{x}{10}}(10 - x)$.

The quarter circle has equation $y = \sqrt{100 - x^2}$.



a Find $\dfrac{dy}{dx}$ for both functions.

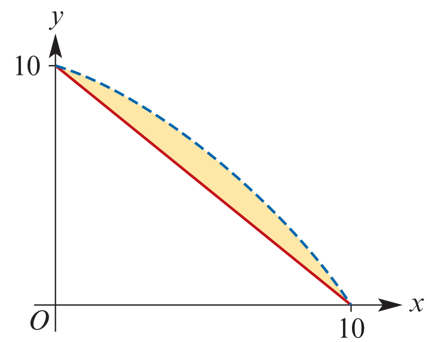
b Find the gradient of each of the functions when $x = 0$.

c Find the gradient of $y = e^{\frac{x}{10}}(10 - x)$ when $x = 10$.

d Find the area of the shaded region correct to two decimal places using a calculator.

e Find the percentage error for the calculation of the area of the quarter circle.

f The teacher draws in a chord from (0, 10) to (10, 0). Find the area of the shaded region using a calculator.



g **i** Use the result that the derivative of $e^{\frac{x}{10}}(10 - x)$ is $-e^{\frac{x}{10}} + \frac{1}{10}e^{\frac{x}{10}}(10 - x)$ to find $\int_0^{10} e^{\frac{x}{10}}(10 - x)\,dx$ by analytic techniques.

ii Find the exact area of the original shaded region and compare it to the answer of **d**.

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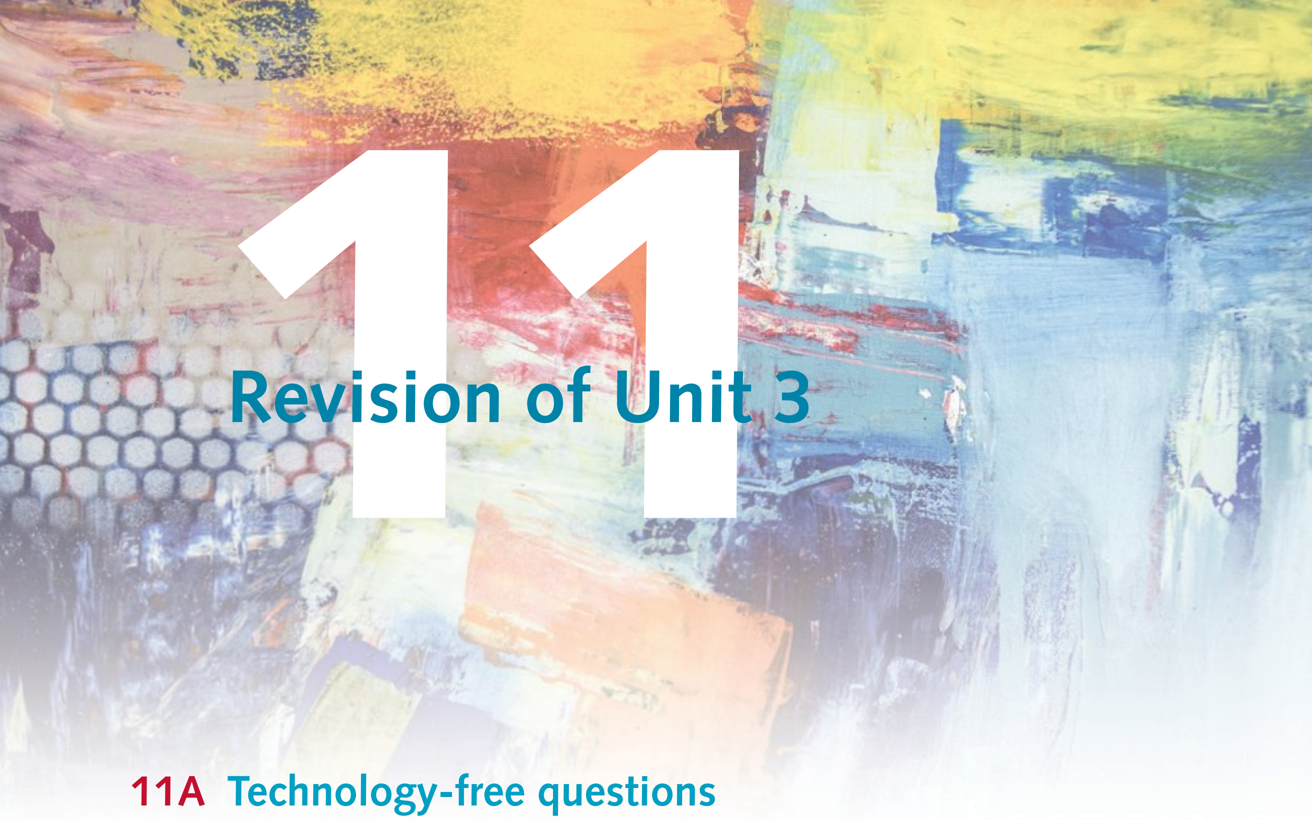

11 Revision of Unit 3

11A Technology-free questions

Exponential and logarithmic functions

1 Let $g(x) = x^2$. For each of the following functions f:

i Find $f(g(x))$ and find the maximal domain and range of the function $y = f(g(x))$.

ii Find $g(f(x))$ and find the maximal domain and range of the function $y = g(f(x))$.

a $f(x) = \ln(3x)$ **b** $f(x) = \ln(2 - x)$ **c** $f(x) = -\ln(2x)$

2 Simplify $2\log_{10} 5 + 3\log_{10} 2 - \log_{10} 20$.

3 Find x in terms of a if $3\log_a x = 3 + \log_a 12$.

4 Solve $2 \times 2^{-x} = 1024$.

5 Solve the equation $4e^{2x} = 9$ for x.

6 Solve the equation $\ln(x + 12) = 1 + \ln(2 - x)$.

7 Evaluate $\log_a 4 \times \log_{16} a$.

8 **a** The graph of the function f with rule $f(x) = 2\ln(x + 2)$ intersects the axes at the points $(a, 0)$ and $(0, b)$. Find the exact values of a and b.

b Hence sketch the graph of $y = f(x)$.

9 Solve the equation $2^{4x} - 5 \times 2^{2x} + 4 = 0$ for x.

10 A function has rule $y = Ae^{kt}$. Given that $y = 4$ when $t = 1$ and that $y = 10$ when $t = 2$, find the values of A and k.

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Differentiation

11 Let $y = \dfrac{x^2 - 1}{x^4 - 1}$.

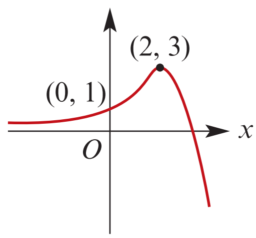
a Find $\dfrac{dy}{dx}$.

b Find the value(s) of x for which $\dfrac{dy}{dx} = 0$.

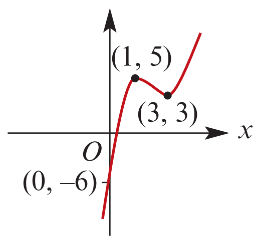
12 Let $y = (3x^2 - 4x)^4$. Find $\dfrac{dy}{dx}$.

13 Sketch the graph of the derivative function for each of the following functions:

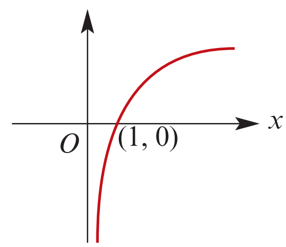
a



b



c



14 Find the derivative of $\left(4x + \dfrac{9}{x}\right)^2$ and find the values of x at which the derivative is zero.

15 Let $f(x) = x^2 \ln(2x)$. Find $f'(x)$.

16 **a** Let $f(x) = e^{2x+1}$. The tangent to the graph of f at the point where $x = b$ passes through the point $(0, 0)$. Find b.

b Let $f(x) = e^{2x+1} + k$, where k is a real number. The tangent to the graph of f at the point where $x = b$ passes through the point $(0, 0)$. Find k in terms of b.

17 The line $y = mx - 8$ is tangent to the curve $y = x^{\frac{1}{3}} + c$ at the point $(8, a)$. Find the values of a, c and m.

18 If $f(3) = -2$ and $f'(3) = 5$, find $g'(3)$ where:

a $g(x) = 3x^2 - 5f(x)$

b $g(x) = \dfrac{3x + 1}{f(x)}$

c $g(x) = [f(x)]^2$

19 A particle starts at a point O and moves along a straight line. The position, x m, of the particle relative to O after t seconds is given by $x = 6(1 - e^{-0.5t})$.

a Find the velocity, v m/s, of the particle at time t.

b Find the acceleration, a m/s^2, of the particle at time t.

c What is the velocity of the particle when it is 3 m from O?

20 Find the equation of the tangent to the curve $y = 8 + \ln(x + 2)$ at the point where $x = 0$.

21 Let $f(x) = 2x^2 - \ln x$ for $x > 0$. Find the coordinates of the stationary point on the graph of f, and state whether this point is a local maximum or a local minimum.

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22 Find the equation of the tangent to the curve $y = x \sin x$ at the point where $x = \frac{\pi}{2}$. SF

23 Find the coordinates of the three stationary points on the graph of $y = e^{x^4 - 18x^2}$. CF

Anti-differentiation and integration

24 Find an anti-derivative of: SF

a $\frac{3}{5x-2}$, $x > \frac{2}{5}$ **b** $\frac{3}{(5x-2)^2}$, $x \neq \frac{2}{5}$

25 Given that $\frac{dy}{dx} = e^{2x} + \sin(2x)$ and that $y = 0$ when $x = \frac{\pi}{2}$, find y in terms of x.

26 **a** Differentiate $\sin^3(2x)$ with respect to x. CF

b Write $\cos^3(2x)$ as $(1 - \sin^2(2x))\cos(2x)$ and hence find an anti-derivative of $\cos^3(2x)$.

27 Given that $\frac{dy}{dx} = \frac{2}{x^2} + \cos(2x)$ and that $y = 0$ when $x = \frac{\pi}{2}$, find y in terms of x. SF

28 A particle moves along a straight line. It starts at rest at a point O on the line. Its acceleration, a m/s^2, after t seconds is given by $a = 18\sin(3t)$. CF

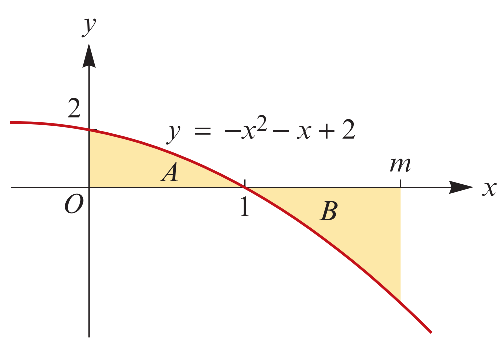
a Find an expression for the particle's velocity, v m/s, after t seconds.

b Find an expression for the particle's position, x m, relative to O after t seconds.

c Find the velocity and position of the particle when $t = \frac{\pi}{2}$.

29 If $f'(x) = 5e^x$ and $f(\ln 3) = 11$, find $f(x)$. SF

30 The graph of $y = -x^2 - x + 2$ is shown. Find the value of m such that regions A and B have the same area. CF



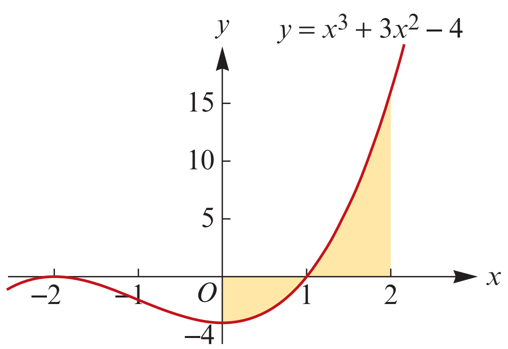
31 Let $f(x) = x^3 + 3x^2 - 4$. The graph of $y = f(x)$ is as shown. Find:

a the coordinates of the stationary points

b $\int_{-2}^{2} f(x)\,dx$

c $\int_{0}^{2} f(x)\,dx$

d the area of the shaded region.



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Revision

SF

32 Evaluate each of the following integrals:

a $\int_0^{\frac{\pi}{2}} 2\sin\left(\frac{x}{2}\right) dx$ **b** $\int_0^{\frac{3}{2}} e^{\frac{x}{2}}\, dx$ **c** $\int_{\frac{1}{2}}^{1} \frac{1}{2x}\, dx$

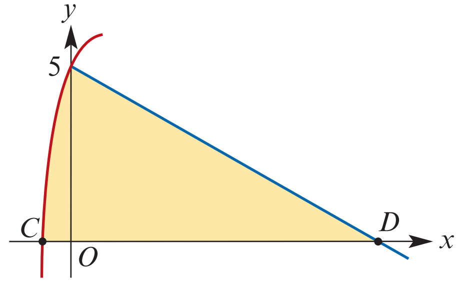
d $\int_{-2}^{-1} \frac{1}{1-x}\, dx$ **e** $\int_3^4 \frac{1}{2(x-2)^2}\, dx$ **f** $\int_2^4 \frac{1}{(3x-2)^2}\, dx$

CF

33 Let $f(x) = 6 - e^{-2x}$. The diagram shows part of the graph of f and also shows the normal to the graph of f at the point $(0, 5)$.

a Find the coordinates of points C and D.

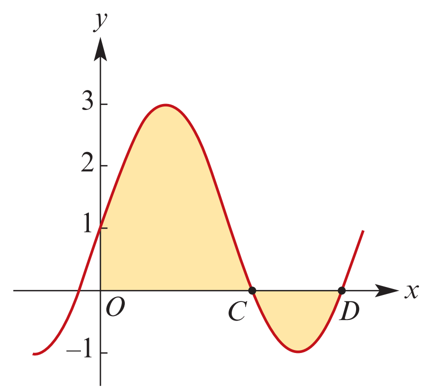
b Find the area of the shaded region.



34 Part of the graph of $y = 2\sin(\pi x) + 1$ is shown.

a Find the coordinates of points C and D.

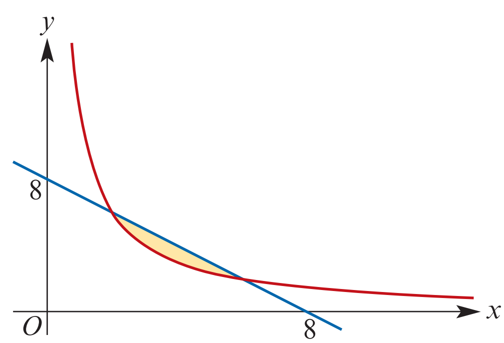
b Find the total area of the shaded regions.



35 The diagram shows the graphs of $f(x) = 8 - x$ and $g(x) = \frac{12}{x}$.

a Find the coordinates of the points of intersection of the two graphs.

b Find the area of the shaded region.



36 **a** Show that the derivative of $\frac{x}{3x+1}$ with respect to x is $\frac{1}{(3x+1)^2}$.

b Hence evaluate $\int_1^3 \frac{4}{(3x+1)^2}\, dx$.

37 **a** Find the derivative of $2x\sin(4x)$ with respect to x.

b Hence evaluate $\int_0^{\frac{\pi}{2}} 8x\cos(4x)\, dx$.

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11B Multiple-choice questions

1 Let $f(x) = e^{-x} - 1$ for $x \in \mathbb{R}$. The range of the function f is

A $(1, \infty)$ **B** $\mathbb{R}$ **C** $[-1, \infty)$ **D** $(-1, \infty)$ **E** $[1, \infty)$

2 If $y = 2 \ln x + 1$, then x can be expressed as a function of y using the rule

A $x = 2e^{y-1}$ **B** $x = e^{\frac{1}{2}(y-1)}$ **C** $x = e^{\frac{y}{2}-1}$ **D** $x = 2e^{y+1}$ **E** $x = \frac{1}{2}e^{y-1}$

3 Rearranging the equation $y = e^x - 1$ to make x the subject yields the equation

A $x = \dfrac{1}{e^y - 1}$ **B** $x = -\ln(y + 1)$ **C** $x = \ln(y - 1)$

D $x = \ln(1 - y)$ **E** $x = \ln(y + 1)$

4 The equation $A = e^{b-1}$ is equivalent to

A $b = e^{-(A-1)}$ **B** $b = -\ln A$ **C** $b = 1 + \ln A$

D $b = \ln(A + 1)$ **E** $b = \ln(A - 1)$

5 When rewritten with x as the subject, the equation $y = \ln\left(\dfrac{x}{2}\right)$ becomes

A $x = e^{\frac{1}{2}y}$ **B** $x = \ln\left(\dfrac{2}{y}\right)$ **C** $x = \dfrac{1}{2}e^{\frac{y}{2}}$ **D** $x = 2e^y$ **E** $x = \dfrac{1}{\ln\left(\dfrac{2}{y}\right)}$

6 For which values of x is the function f with the rule $f(x) = -2 + \ln(3x - 2)$ defined?

A $x > -2$ **B** $x > \dfrac{2}{3}$ **C** $x \geq -2$ **D** $x \geq \dfrac{2}{3}$ **E** $x > 2$

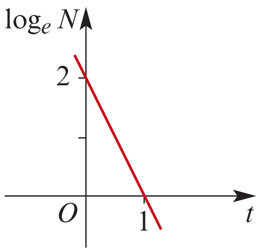
7 If $\log_2(8x) + \log_2(2x) = 6$, then $x =$

A 1.5 **B** ± 1.5 **C** 2 **D** ± 2 **E** 6.4

8 The equation $\log_{10} x = y(\log_{10} 3) + 1$ is equivalent to the equation

A $x = 10(3^y)$ **B** $x = 30^y$ **C** $x = 3^y + 10$ **D** $x = y^3 + 10$ **E** $x = 10y^3$

9 The graph indicates that the relationship between N and t is

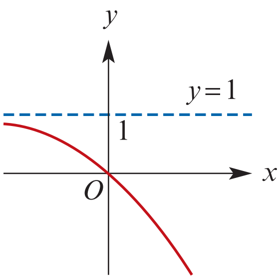


A $N = 2 - e^{-2t}$ **B** $N = e^{2-2t}$

C $N = e^{2t} + 2$ **D** $N = \dfrac{e^{-2t}}{100}$

E $N = -2e^{2t}$

10 A possible equation for the graph is



A $y = 1 - e^x$

B $y = 1 - e^{-x}$

C $y = 1 + e^x$

D $y = 1 + e^{-x}$

E $y = e^{-x} - 1$

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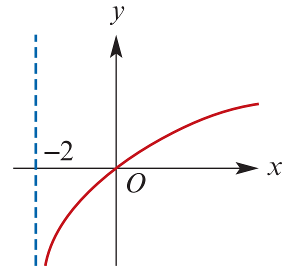
Revision

11 A possible equation for the graph is



A $y = \ln(x - 2)$
B $y = \ln(\frac{1}{2}x + 1)$
C $y = \ln(2x + 2)$
D $y = 2\ln(x + 1)$
E $y = \frac{1}{2}\ln(x + 2)$

12 If $y = \dfrac{x^4 + x}{x^2}$, then $\dfrac{dy}{dx}$ equals

A $\dfrac{4x^3 + 2x}{2x}$ **B** $x^2 + 1$ **C** $2x$ **D** $\dfrac{2x^3 + 1}{x^2}$ **E** $2x - \dfrac{1}{x^2}$

13 If $y = (4 - 9x^4)^{\frac{1}{2}}$, then $\dfrac{dy}{dx}$ equals

A $-\dfrac{9}{2}(4 - 9x^4)^{-\frac{1}{2}}$ **B** $\dfrac{1}{2}(4 - 9x^4)^{-\frac{1}{2}}$ **C** $2(4 - 9x^4)^{-\frac{1}{2}}$
D $-3x(4 - 9x^4)^{-\frac{1}{2}}$ **E** $-18x^3(4 - 9x^4)^{-\frac{1}{2}}$

14 The gradient of the curve with equation $y = \sin(2x) + 1$ at $(0, 1)$ is

A 1 **B** −1 **C** 0 **D** 2 **E** −2

15 $\dfrac{d}{dx}(e^{x^2+1})$ is

A $2x$ **B** $2xe^{x^2+1}$ **C** $2xe^{2x}$ **D** $(x^2 + 1)e^{x^2}$ **E** $(x^2 + 1)e^{x^2+1}$

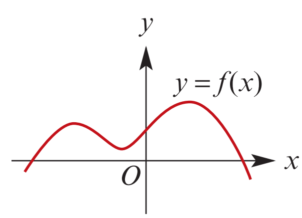
16 The derivative of $\dfrac{1}{1 + x}$ is

A $\dfrac{1}{(1 + x)^2}$ **B** $\dfrac{1}{1 - x}$ **C** $\dfrac{-1}{(1 + x)^2}$ **D** 1 **E** $\dfrac{1}{2}$

17 The gradient of $y = ce^{2x}$ is equal to 11 when $x = 0$. The value of c is

A 0 **B** 1 **C** 5 **D** 5.5 **E** $5e^{-2}$

18 For the graph of $y = f(x)$ shown, $f'(x) = 0$ at



A 3 points **B** 2 points
C 5 points **D** 0 points
E none of these

19 If $y = (3x^4 - 2)^4$, then $\dfrac{dy}{dx}$ equals

A $x^4(3x^4 - 2)^3$ **B** $4(3x^4 - 2)^3$ **C** $12x^{12}$
D $(12x^2 - 2)^4$ **E** $48x^3(3x^4 - 2)^3$

20 Let $f(x) = 3x^2 + 2$. If $g'(x) = f'(x)$ and $g(2) = 29$, then $g(x) =$

A $3x^3 + 5$ **B** $3x^2 - 3$ **C** $\dfrac{x^3}{3} + 2x$ **D** $3x^2 + 17$ **E** $6x + 17$

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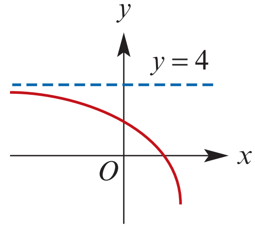
21 Let $f(x) = 4 - e^{-2x}$. The graph of $y = f'(x)$ is best represented by

A



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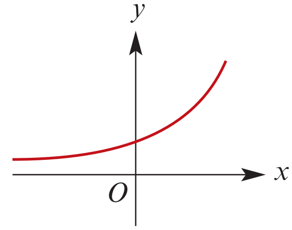

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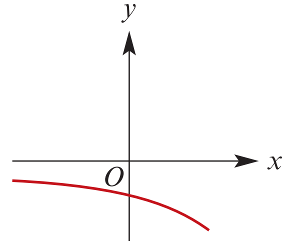
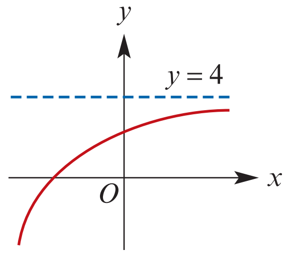


D



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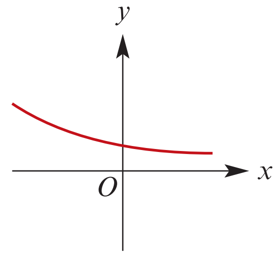
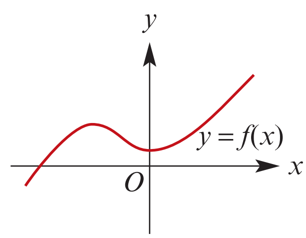


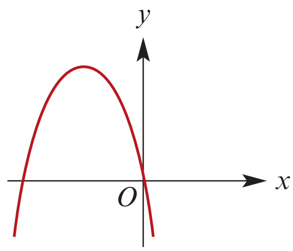
22 If $f(x) = e^{kx} + e^{-kx}$, then $f'(x) > 0$ for

A $x \in \mathbb{R}$ **B** $x \geq 0$ **C** $x < 0$ **D** $x \leq 0$ **E** $x > 0$

23 The graph of $y = f(x)$ is shown on the right. The graph that best represents the graph of $y = f'(x)$ is



A



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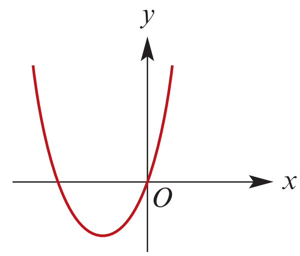

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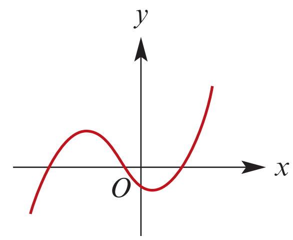

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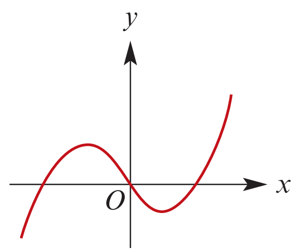

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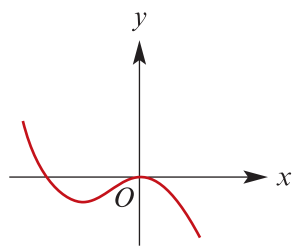

24 Rainwater is being collected in a water tank. The volume, V m^3, of water in the tank after time t minutes is given by $V = 2t^2 + 3t + 1$. The average rate of change of volume of water between times $t = 2$ and $t = 4$, in m^3/min, is

A 11 **B** 13 **C** 15 **D** 17 **E** 19

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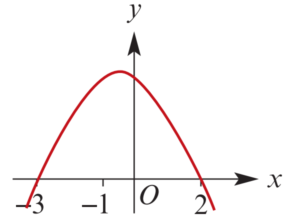
25 The graph of the derivative function f' given by $y = f'(x)$ is shown. The function f is increasing for



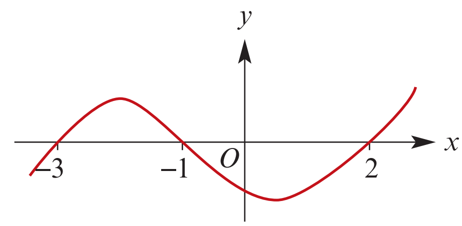
A $x \geq 0$ **B** $-3 \leq x \leq 2$
C $x \geq 2$ **D** $x \leq -3$ and $x \geq 2$
E $x \leq 0$

26 Which one of the following gives the gradient of the tangent to a curve with the equation $y = f(x)$ at the point $x = 2$?

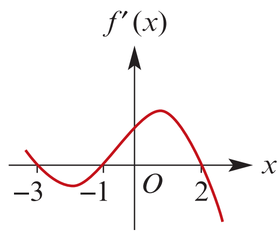
A $\dfrac{f(x+h) - f(x)}{h}$ **B** $f(2+h) - f(2)$ **C** $\dfrac{f(2+h) - f(2)}{h}$

D $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ **E** $\lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$

27 The graph of $y = f(x)$ is shown.
A possible graph of the gradient function f' with rule given by $f'(x)$ is



A



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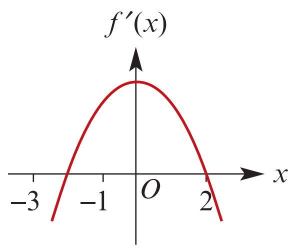
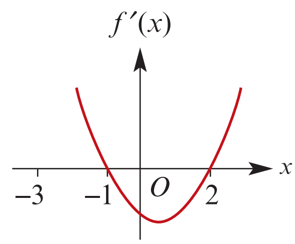
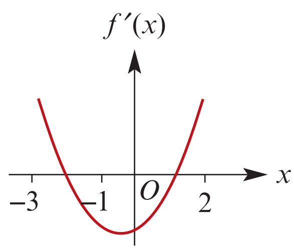


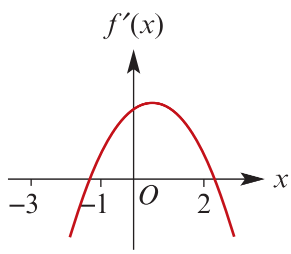
C



D



E



28 The derivative of $\dfrac{e^{2x} + e^{-2x}}{e^x}$ is

A $e^x + e^{-3x}$ **B** $e^x - 3e^{-3x}$ **C** $xe^x - 3xe^{-3x}$ **D** $\dfrac{2e^{2x} - e^{-2x}}{e^x}$ **E** $\dfrac{e^{3x} - 3e^{-x}}{e^{x^2}}$

29 The equation of the tangent to the curve $y = 1 + e^{2x}$ at the point $(0, 2)$ is

A $y = 2e^{2x}$ **B** $y = 2x + 2$ **C** $y = \dfrac{-1}{2}x + 2$ **D** $y = 2$ **E** $y = 2e^{2x} + 2$

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30 The graph of $y = x^2 - x^3$ has stationary points where x is equal to

A 0 and $\frac{2}{3}$ **B** 0 and 1 **C** -1 and 0 **D** 0 and $\frac{3}{2}$ **E** 2 and -3

31 The derivative of $\frac{4x^2 + 6}{x}$ is

A $8x + 6$ **B** $8x^3 - 6$ **C** $\frac{4x^2 + 6}{x^2}$ **D** $\frac{4x^2 - 6}{x^2}$ **E** $\frac{8x^3 - 6}{x^2}$

32 If $f(x) = 4x^3 - 3x + 7 - \frac{2}{x}$, then $f'(1)$ is equal to

A -2 **B** 18 **C** 7 **D** 11 **E** 14

33 If $f'(x) = x^2 + \frac{1}{x}$ and $f(1) = \frac{1}{3}$, then $f(x)$ is equal to

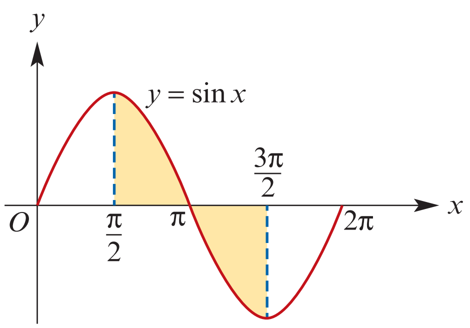
A $\frac{x^3}{3} + \ln x$ **B** $\frac{x^3}{3} + \ln x + \frac{2}{3}$ **C** $\frac{x^3}{3} - \ln x - \frac{1}{3}$

D $\frac{-x^3}{3} + \ln x + \frac{2}{3}$ **E** $\frac{x^3}{3} - \ln x + \frac{1}{3}$

34 If $y = F(x)$ and $\frac{dy}{dx} = f(x)$, then $\int_2^3 f(x)\,dx$ is equal to

A $f(3) - f(2)$ **B** $F'(3) - F'(2)$ **C** $F(3) - F(2)$ **D** $f(x) + c$ **E** $F(3) - f(2)$

35 The area of the shaded region is given by



A $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin x\,dx$

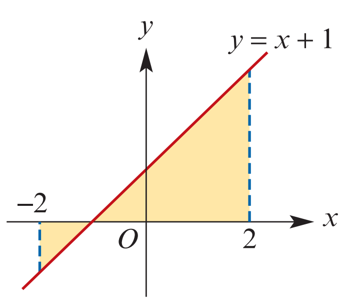
B $\int_{\pi}^{\frac{3\pi}{2}} \sin x\,dx + \int_{\frac{\pi}{2}}^{\pi} \sin x\,dx$

C $\int_{\frac{3\pi}{2}}^{\pi} \sin x\,dx + \int_{\frac{\pi}{2}}^{\pi} \sin x\,dx$

D $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin x\,dx + \int_{\pi}^{\frac{\pi}{2}} \sin x\,dx$

E $\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin^2 x\,dx$

36 The area of the shaded region is given by



A $\int_0^2 (x + 1)\,dx - \int_2^0 (x + 1)\,dx$

B $\int_{-2}^2 (x + 1)\,dx$

C $\int_0^2 (x + 1)\,dx + \int_{-2}^0 (x + 1)\,dx$

D $\int_{-1}^2 (x + 1)\,dx - \int_{-2}^{-1} (x + 1)\,dx$

E $\int_{-1}^2 (x + 1)\,dx + \int_{-2}^{-1} (x + 1)\,dx$

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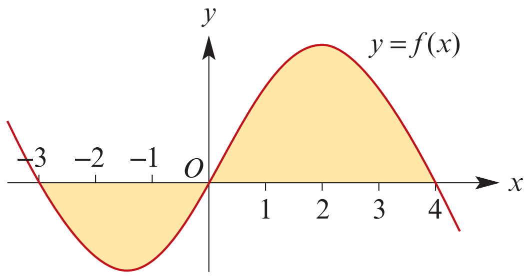
37 If $\frac{dy}{dx} = \frac{1}{x^2}$ and $y = 2$ when $x = 1$, then

A $y = \frac{-1}{x}$ **B** $y = \frac{-1}{x} + 3$ **C** $y = \frac{-2}{x^3}$ **D** $y = \frac{2}{x^3}$ **E** $y = \frac{1}{x} + 1$

38 If $\int_0^{36} \frac{1}{2x+9}\, dx = \ln k$, then k is

A 3 **B** $\frac{9}{2}$ **C** $6\sqrt{2}$ **D** 9 **E** 81

39 The area of the shaded region is given by



A $\int_{-3}^{4} f(x)\, dx$

B $\int_{-3}^{0} f(x)\, dx + \int_{0}^{4} f(x)\, dx$

C $\int_{-3}^{1} f(x)\, dx + \int_{1}^{4} f(x)\, dx$

D $\int_{4}^{0} f(x)\, dx + \int_{-3}^{0} f(x)\, dx$

E none of these

40 $\int x^2 - \frac{1}{x^2} + \sin x\, dx$ is

A $\frac{x^3}{3} + \frac{1}{x} + \cos x + c$ **B** $\frac{x^3}{3} - \frac{2}{x} - \cos x + c$ **C** $2x - \frac{3}{x^2} + \cos x + c$

D $\frac{x^3}{3} + \frac{1}{x} - \cos x + c$ **E** $2x + \frac{2}{x^2} + \cos x + c$

41 The area bounded by the curve $y = \frac{1}{3-x}$, the x-axis, the y-axis and the line $x = 2$ is

A $\ln 3$ **B** $\ln\left(\frac{1}{3}\right)$ **C** $-\ln(3 - x)$ **D** $\ln 2$ **E** $\ln\left(\frac{1}{2}\right)$

42 If $\int_a^b \sin(2x)\, dx = 0$, then possible values for a and b are

A $b = \frac{3\pi}{4},\ a = \frac{\pi}{4}$ **B** $b = \frac{\pi}{2},\ a = 0$ **C** $b = \pi,\ a = \frac{\pi}{2}$

D $b = \frac{\pi}{6},\ a = \frac{\pi}{3}$ **E** $b = \pi,\ a = \frac{\pi}{4}$

43 An anti-derivative of $x^2 - \frac{1}{x}$ is

A $2x - \frac{2}{x^2}$ **B** $\frac{x^3}{3} - \ln x$ **C** $x^3 + \frac{1}{x^2}$ **D** $x^3 - \ln x$ **E** none of these

44 For $f(x) = \frac{\sin x}{x}$, $f'(x) =$

A $\cos x$ **B** $\frac{x\cos x - \sin x}{x^2}$ **C** $\frac{x\cos x - \sin x}{x}$

D $\frac{\sin x - x\cos x}{x^2}$ **E** $\frac{x\cos x - \sin x}{\sin^2 x}$

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45 If $y = \ln(\cos(2x))$ for $0 < x < \frac{\pi}{4}$, then $\frac{dy}{dx}$ is equal to

A $\frac{2}{\cos(2x)}$ **B** $-\frac{2\sin(2x)}{\cos(2x)}$ **C** $\frac{1}{x}\cos(2x) - 2\sin(2x)\ln x$

D $\frac{\sin(2x)}{2\cos(2x)}$ **E** $\frac{1}{\cos(2x)}$

46 If $f'(x) = \sin(2x)$ and $f(0) = 3$, then

A $f(x) = -\frac{1}{2}\cos(2x) + 3$ **B** $f(x) = \frac{1}{2}\cos(2x) + 3$ **C** $f(x) = -\frac{1}{2}\cos(2x) + 3\frac{1}{2}$

D $f(x) = -\frac{1}{2}\cos(2x) + 2\frac{1}{2}$ **E** $f(x) = \frac{1}{2}\cos(2x) + 2\frac{1}{2}$

47 The equation of the tangent to the curve $y = 4e^{3x} - x$ at the point $(0, 4)$ is

A $y = 12x + 4$ **B** $y = -4x + 4$ **C** $y = 4$ **D** $y = 11x + 4$ **E** $y = 4x + 4$

48 The function $f(x) = x^3 - x^2 - x + 2$ has a local minimum at the point

A $(-1, 0)$ **B** $(1, 1)$ **C** $(2, 0)$ **D** $(-1, 1)$ **E** $(1, 0)$

49 $\frac{d}{dx}\left(\frac{x-1}{\sqrt{x}}\right)$ equals

A $2\sqrt{x}$ **B** $\frac{x+1}{x\sqrt{x}}$ **C** $\frac{3x-1}{2\sqrt{x}}$ **D** $\frac{x+1}{2x\sqrt{x}}$ **E** $\frac{3x-1}{2x\sqrt{x}}$

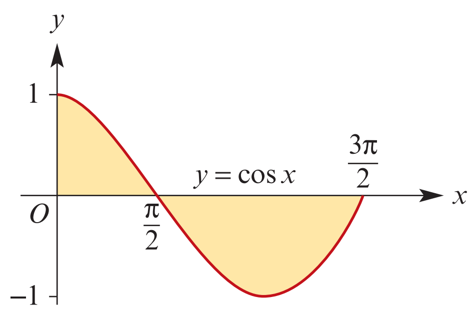
50 $\frac{d}{dx}(e^{\cos x}) =$

A $e^{\cos x}$ **B** $e^{\cos x} \cdot \sin x$ **C** $-e^{\cos x} \cdot \sin x$ **D** $e^{\sin x}$ **E** $e^{\sin x} \cdot \cos x$

51 The total area, in square units, of the shaded regions is

A 3 **B** −1 **C** 1

D 2 **E** −2



52 The gradient of the normal to the curve $y = e^{-\cos x}$ at the point where $x = \frac{\pi}{3}$ is

A $\frac{\sqrt{3}}{2e^{\frac{1}{2}}}$ **B** $\frac{-2e^{\frac{1}{2}}}{\sqrt{3}}$ **C** $\frac{1}{2e^{\frac{\sqrt{3}}{2}}}$ **D** $\frac{2e^{\frac{1}{2}}}{\sqrt{3}}$ **E** $-e^{-\frac{1}{2}}$

53 Rainwater is being collected in a water tank. The rate of change of volume, V L, with respect to time, t seconds, is given by

$$\frac{dV}{dt} = 5t + 2$$

The volume of water that is collected in the tank between times $t = 2$ and $t = 6$ is

A 5 L **B** 20 L **C** 22 L **D** 88 L **E** 168 L

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54 $\int_0^{\frac{\pi}{2}} (\cos x + \sin x)\, dx$ equals

A -2 **B** -1 **C** 1 **D** $\frac{\pi}{2}$ **E** 2

55 If $f(x) = \ln(3x)$, then $f'(1)$ is

A $\frac{1}{3}$ **B** $\ln 3$ **C** 1 **D** $3\ln 3$ **E** 3

56 Let $f(x) = a\sin(3x)$, where a is constant. If $f'(\pi) = 2$, then a is equal to

A -3 **B** $-\frac{3}{2}$ **C** $\frac{3}{2}$ **D** $\frac{2}{3}$ **E** $-\frac{2}{3}$

57 An anti-derivative of $\dfrac{1}{(2x-5)^{\frac{5}{2}}}$ is equal to

A $\dfrac{-3}{(2x-5)^{\frac{5}{2}}}$ **B** $\dfrac{-1}{3(2x-5)^{\frac{3}{2}}}$ **C** $\dfrac{5}{(2x-5)^{\frac{5}{2}}}$ **D** $\dfrac{7}{2(2x-5)^{\frac{7}{2}}}$ **E** $\dfrac{1}{3(2x-5)^{\frac{3}{2}}}$

11C Extended-response questions

1 The population of a country increases by 2.96% each year. The population t years after 1 January 1950 is given by the formula

$$p(t) = (150 \times 10^6)e^{kt}$$

a Find the value of k.

b Find the population on 1 January 1950.

c Find the population on 1 January 2000.

d After how many years would the population be 300×10^6?

2 A large urn was filled with water. It was turned on, and the water was heated until its temperature reached 95°C. This occurred at exactly 2 p.m., at which time the urn was turned off and the water began to cool. The temperature of the room where the urn was located remained constant at 15°C.

Commencing at 2 p.m. and finishing at midnight, Jenny measured the temperature of the water every hour on the hour for the next 10 hours and recorded the results. At 4 p.m., Jenny recorded the temperature of the water as 55°C. She found that the temperature, T°C, of the water could be described by the equation

$$T = Ae^{-kt} + 15 \quad \text{for } 0 \le t \le 10$$

where t is the number of hours after 2 p.m.

a Find the values of A and k.

b Find the temperature of the water at midnight.

c At what time did Jenny first record a temperature less than 24°C?

d Sketch the graph of T against t.

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3 A machine in a factory has 20 different power settings. The noise produced by the machine, N dB, depends on the power setting, P, according to a rule of the form

$$N = a\log_{10}(bP) \quad \text{for } P = 1, 2, 3, \dots, 20$$

where a and b are constants.

a Find the values of a and b, given that the machine produces a noise of 45 dB on power setting 1 and a noise of 90 dB on power setting 10.

b Find the maximum noise level produced by the machine (to the nearest decibel).

c On weekends, the local council imposes a noise-level restriction of 75 dB on the factory. What is the maximum power setting that can be used on the machine if it is being run on the weekend?

4 **a** Find all values of x for which $(\ln x)^2 = 2\ln x$.

b Find the gradient of each of the curves $y = 2\ln x$ and $y = (\ln x)^2$ at the point $(1, 0)$.

c Use these results to sketch, on one set of axes, the graphs of $y = 2\ln x$ and $y = (\ln x)^2$.

d Find the values of x for which $2\ln x > (\ln x)^2$.

5 An object that is at a higher temperature than its surroundings cools according to Newton's law of cooling: $T = T_0\, e^{-kt}$, where T_0 is the original excess of temperature and T is the excess of temperature after time t minutes.

a Prove that $\frac{dT}{dt}$ is proportional to T.

b If the original temperature of the object is 100°C, the temperature of its surroundings is 30°C and the object cools to 70°C in 20 minutes, find the value of k correct to three decimal places.

c At what rate is the temperature decreasing after 30 minutes?

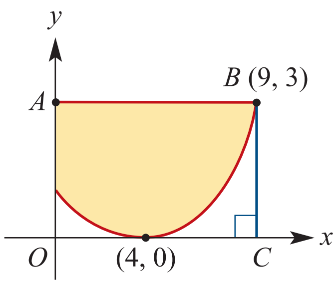
6 A swimming pool has a cross-sectional area as shown.

a Find the area of the rectangle $OABC$.

b Find the equation of the curve given that it is of the form $y = k(x - 4)^2$.

c Find the total area of the region enclosed between the curve and the x-axis for $x \in [0, 9]$.

d Find the area of the cross-section of the pool (i.e. the shaded region).



7 **a** Calculate $\int_{-3}^{1} 1 - t^2\, dt$ and illustrate the region of the Cartesian plane for which this integral gives the signed area.

b Show that $\int_{a}^{1} 1 - t^2\, dt = 0$ implies $a^3 - 3a + 2 = 0$.

c Find the values of a for which $\int_{a}^{1} 1 - t^2\, dt = 0$.

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Revision

CU

8 The rate of flow of water into a tank is given by $\frac{dV}{dt} = 10e^{-(t+1)}(5 - t)$ for $0 \le t \le 5$, where V litres is the amount of water in the tank at time t minutes. Initially the tank is empty.

a **i** Find the initial rate of flow of water into the tank.

ii Find the value of t for which $\frac{dV}{dt} = 0$.

iii Find the time, to the nearest second, when the rate is 1 litre per minute.

iv Find the first time, to the nearest second, when $\frac{dV}{dt} < 0.1$.

b Find the amount of water in the tank when $t = 5$.

c Find the time, to the nearest second, when there are 10 litres of water in the tank.

9 A large clock is hanging on a wall. The height (h cm) of the tip of the second hand above the ground varies as a function of time (t seconds). The second hand is 25 cm long and the centre of the clock face is 250 cm above the ground.

a Find a function to model the height of the tip of the second hand above the ground as a function of time, assuming that the second hand starts at the 9 o'clock position.

b How far above the ground is the tip of the second hand after 15 seconds?

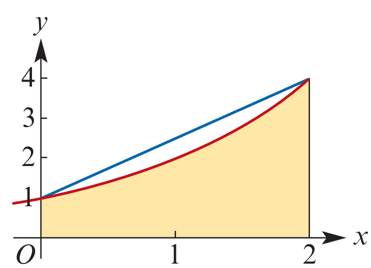
c How far above the ground is the tip of the second hand when it reaches the 8 o'clock position?

d Find the first time that the tip of the second hand is 262.5 cm above the ground.

e Find the average rate of change of h with respect to t as the second hand moves from the 9 o'clock position to the 12 o'clock position.

f Find the instantaneous rate of change of h with respect to t when the second hand is at the 10 o'clock position.

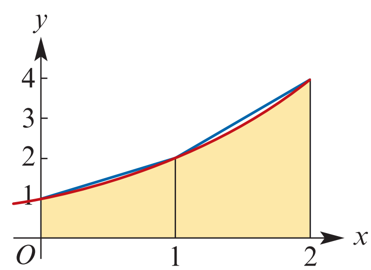
10 It can be shown that $\int 2^x \, dx = \frac{2^x}{\ln 2} + c$.



a Evaluate the definite integral $\int_0^2 2^x \, dx$.

b **i** Find an approximation, A_1, to the definite integral using one trapezium as shown.

ii Find the error $E_1 = A_1 - \int_0^2 2^x \, dx$.



c **i** Find an approximation, A_2, to the definite integral using two trapeziums as shown.

ii Find the error $E_2 = A_2 - \int_0^2 2^x \, dx$.

d Continuing in this way, find A_4 and E_4, then find A_8 and E_8. (You will notice that doubling the number of trapeziums decreases the error by about a factor of 4.)

e Repeat this procedure for the definite integral $\int_0^2 x^2 \, dx$. Find the approximations and errors using one, two, four and eight trapeziums. How many trapeziums would be needed for an approximation to be within 10^{-6} of the definite integral?

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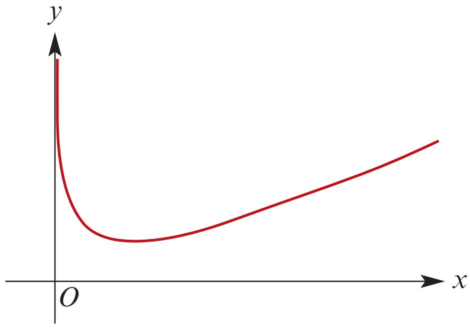
11 The graph of the function

$$f(x) = x - \ln x, \quad x > 0$$

is shown on the right.



a Determine $f'(x)$ and show that:

i $f'(x) < 0$ for $0 < x < 1$

ii $f'(x) = 0$ for $x = 1$

iii $0 < f'(x) < 1$ for $x > 1$.

b Hence state the coordinates of the local minimum on the graph of $y = f(x)$.

c Let n be an integer with $n \geq 2$. Find the value of x such that $f'(x) = \frac{1}{n}$.

d Find the value of a such that the tangent to the graph of $y = f(x)$ at point $P(a, f(a))$ passes through the origin.

e Determine the equation of the tangent to the graph of $y = f(x)$ at $x = e^{-1}$.

f Determine the equation of the tangent to the graph of $y = f(x)$ at $x = e^n$, where n is a positive integer, and state the y-axis intercept of this tangent.

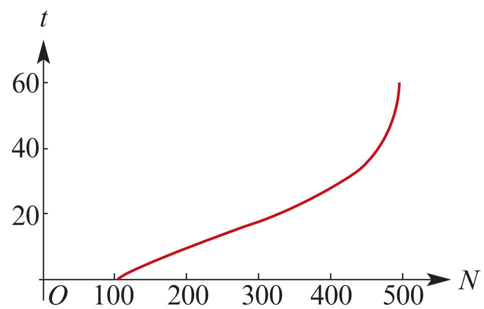
g Differentiate $x \ln x$ and hence find an anti-derivative of $x - \ln x$.

h Evaluate $\int_1^e f(x)\,dx$.

12 A population of single-celled fresh-water organisms grows according to the model

$$t = 10\ln\left(\frac{4N}{500 - N}\right), \quad 100 \leq N < 500$$

where t is the number of days that it takes for the population to reach size N. The graph of t against N is shown.



a Determine $\frac{dt}{dN}$.

b For each of the following values of N, find the corresponding value of t. (Give answers correct to two decimal places.)

i $N = 110$ **ii** $N = 120$ **iii** $N = 250$ **iv** $N = 450$

c For each of the following values of N, find the corresponding value of $\frac{dt}{dN}$. (Give answers correct to two decimal places.)

i $N = 110$ **ii** $N = 120$ **iii** $N = 250$ **iv** $N = 450$

d Find the equation of the tangent to the graph of t against N where:

i $N = 250$ **ii** $N = 100$

e Rearrange the equation $t = 10\ln\left(\frac{4N}{500 - N}\right)$ to make N the subject.

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Revision

11D Degree-of-difficulty classified questions

▶ Simple familiar questions

1 Solve each of the following equations for x:

a $\ln x + \ln 25 = \ln(x^3)$ **b** $3 \times 3^{2x} + 3^x - 10 = 0$

c $\log_{16}(3x - 1) = \log_4(3x) + \log_4\left(\frac{1}{2}\right)$

2 Differentiate each of the following with respect to x:

a $\frac{\cos(2x)}{x}$ **b** $x^2 e^{4x}$ **c** $\sin(3x^2)$

d $\ln(\sin x)$ **e** $x^3 \ln x$ **f** $\frac{e^{2x}}{x+2}$

3 Given that $\int_3^7 f(x)\,dx = 12$, evaluate:

a $\int_7^3 f(x)\,dx$ **b** $\int_3^7 5f(x)\,dx$ **c** $\int_3^5 (f(x)+1)\,dx + \int_5^7 f(x)\,dx$

4 Find:

a $\int \sin(3x+5)\,dx$ **b** $\int e^{-4x}\,dx$ **c** $\int \cos(4-2x)\,dx$

5 Evaluate:

a $\int_0^1 \frac{2}{5x+5}\,dx$ **b** $\int_1^4 2x + \frac{2}{x}\,dx$ **c** $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 5\,dx$

6 The graphs of $y = 2x$ and $y = 8x - x^2$ intersect at the origin and at the point B.

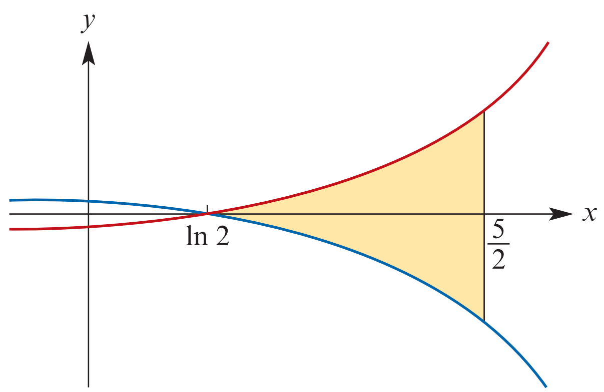
a Determine the coordinates of B.

b Calculate the area of the region bounded by the graphs of $y = 2x$ and $y = 8x - x^2$.

7 Consider the functions

$$f(x) = e^x - 2$$
$$g(x) = -e^x + 2$$

Find the area of the shaded region bounded by the graphs of f and g and the line $x = \frac{5}{2}$.



8 Find an anti-derivative of each of the following:

a $x^2 - 4x + 6$ **b** $x^2 - 3x + \frac{3}{x},\ x > 0$ **c** $4x^3 - 2x - \frac{4}{x^2}$

d $\frac{x+1}{\sqrt{x}}$ **e** $\sin(3x) + \cos(4x)$ **f** e^{2x-3}

9 A curve with equation $y = f(x)$ passes through the point $(0, 12)$ and its gradient is given by $f'(x) = 4(x+2)(x-3)$. Find $f(x)$.

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10 Let $y = 5 \times 3^{2x}$ for $x \geq 0$.

a Determine the values of m and c such that $\log_3 y = mx + c$.

b Sketch the graph of $\log_3 y$ against x.

► Complex familiar questions

1 Let $y = Ae^{bx}$, where A and b are constants with $A > 0$. The graph of $\ln y$ against x is a straight line with ($\ln y$)-axis intercept 2 and gradient −0.25. Find the values of A and b.

2 Alex starts at point A and cycles along a straight path until coming to rest at point B. His velocity, v m/s, at time t seconds after leaving point A is given by $v = 6t - \frac{1}{2}t^2$.

a Find the time taken for Alex to travel from A to B.

b Find the distance AB.

c Find Alex's acceleration 8 seconds after leaving point A.

d Find Alex's average velocity over his journey from A to B.

3 For each of the following functions, find the coordinates of the points on the graph at which the tangent passes through the origin:

a $y = x \sin x, \quad -\pi \leq x \leq \pi$ **b** $y = x\cos(2x), \quad -\pi \leq x \leq \pi$

4 Solve the equation $\log_2(7x^2 + 8x + 3) = \log_2(x^2) + 1$.

5 Recall that the pH of a solution can be found using

$$pH = -\log_{10}([H_3O^+])$$

where $[H_3O^+]$ is the concentration of hydronium ions in moles per litre.

a In a glass of tomato juice, the concentration of hydronium ions is $10^{-4.1}$ moles per litre. Find the pH.

b In the gastric juices in your stomach, the concentration of hydronium ions is 10^{-1} moles per litre. Find the pH.

6 An object is dropped from a great height. Its velocity, v m/s, at time t seconds after being dropped is given by

$$v = 48 - 48e^{-0.2t}, \quad t \geq 0$$

a Find the initial velocity of the object.

b Find the acceleration, a m/s^2, of the object at time t.

c As t becomes very large, what value does v approach?

d As t becomes very large, what value does a approach?

e Explain in words what is happening as t becomes very large.

f Find an expression for the distance, x m, that the object has fallen at time t seconds after being dropped.

g Find, correct to two decimal places, the time that it takes for the object to fall 240 m.

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7 Let $f(x) = 3\sin(\pi x)$ for $-2 \le x \le 2$.

a Sketch the graph of $y = f(x)$ for $-2 \le x \le 2$.

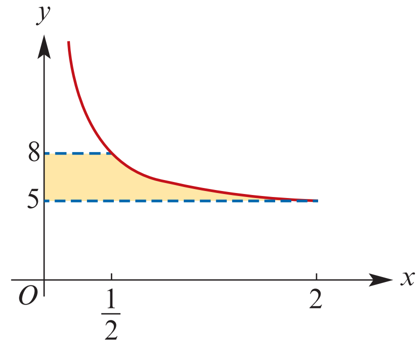
b Find the equation of the tangent to the graph where $x = \frac{1}{2}$.

c Evaluate $\int_0^{\frac{1}{2}} f(x) - \frac{x}{4}\,dx$.

8 The diagram shows the graph of the function

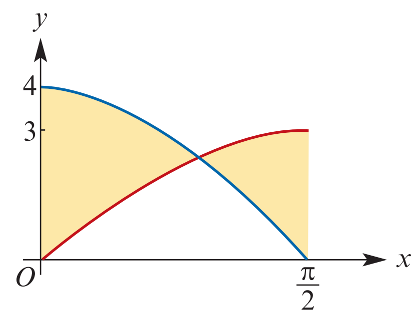
$$f(x) = 4 + \frac{2}{x}, \quad 0 < x \le 2$$

and the lines $y = 5$ and $y = 8$. Find the area of the shaded region.



▶ Complex unfamiliar questions

1 The diagram shows the graphs of $y = 3\sin x$ and $y = 4\cos x$ for $0 \le x \le \frac{\pi}{2}$.



a The graphs intersect at the point $P(a, b)$. Determine the value of:

i $\tan a$ **ii** $\sin a$ **iii** $\cos a$

b Determine the total area of the shaded regions.

2 Let $f(x) = e^{\pi - x}\sin x$ for $x \ge 0$.

a Describe the transformation of the plane that takes the graph of $y = e^x \sin x$ to the graph of $y = f(x)$.

b Solve the equation $f(x) = 0$ for $x \ge 0$.

c Find the equation of the tangent to the graph of $y = f(x)$ at the point where:

i $x = 0$ **ii** $x = \pi$

d Find the coordinates of the point of intersection of the two tangents from part **c**.

e Find the coordinates of the local maximums of the graph of $y = f(x)$.

f Find the coordinates of the local minimums of the graph of $y = f(x)$.

g Solve the equation $f(x) = e^{\pi - x}$ for $x \ge 0$. Comment.

h Show that the x-coordinates of the local minimums form an arithmetic sequence.

i Show that the y-coordinates of the local minimums form an infinite geometric sequence.

j Differentiate $-(\cos x + \sin x)e^{\pi - x}$ with respect to x. Hence evaluate $\int_0^{\pi} f(x)\,dx$ and $\int_{\pi}^{2\pi} f(x)\,dx$. Investigate further.

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3 The graph of f is shown, where $f(x) = x - (x - 2)\ln(x - 2) - 2$ for $x > 2$.

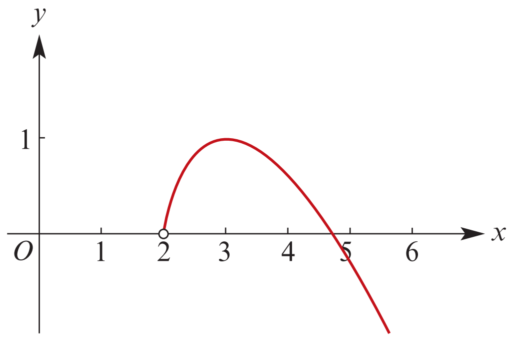
a Find the coordinates of the local maximum of the graph of f.

b Find the values of x for which $\frac{1}{2} < f'(x) < 1$.

c Find the equation of the tangent to the graph of f where:

i $x = e + 2$

ii $x = \frac{1}{e} + 2$



d Find the coordinates of the point of intersection of the two tangents from part **c**.

e For $a > 3$, find the x-axis intercept, b, of the tangent to the graph of f at $(a, f(a))$. Find the minimum possible value of b.

4 A lake is stocked with 2000 fish. The fish population, P, can be modelled by

$$P = \frac{20\,000}{1 + 9e^{-\frac{t}{5}}}$$

where t is the time in months since the lake was initially stocked.

a Find the fish population after 8 months.

b After how many months will the fish population be 10 000?

c What happens to the fish population as t gets very large?

d Find t in terms of P.

e Find the rate of change, $\frac{dP}{dt}$, of the fish population with respect to time.

f Evaluate $\frac{dP}{dt}$ for:

i $t = 5$ **ii** $t = 20$ **iii** $t = 30$ **iv** $t = 50$

5 The curves $y^2 = ax$ and $x^2 = by$, where a and b are both positive, intersect at the origin and at the point (r, s). Find r and s in terms of a and b. Prove that the two curves divide the rectangle with corners $(0, 0)$, $(0, s)$, (r, s), $(r, 0)$ into three regions of equal area.

6 **a** If $y = x\ln x$, find $\frac{dy}{dx}$. Hence find the value of $\int_1^e \ln x\, dx$.

b If $y = x(\ln x)^n$, where n is a positive integer, find $\frac{dy}{dx}$.

c Let $I_n = \int_1^e (\ln x)^n\, dx$. For $n > 1$, show that $I_n + nI_{n-1} = e$.

d Hence find the value of $\int_1^e (\ln x)^3\, dx$.

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Chapter 12

12 The second derivative and applications

Objectives

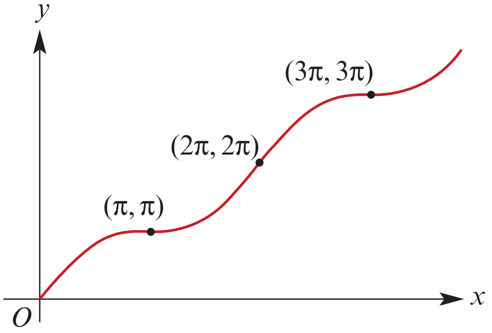
- To use the notation for the **second derivative** of a function.
- To recognise **acceleration** as the second derivative of position with respect to time.
- To use the second derivative in graph sketching:
  - investigate **concavity** and **points of inflection**
  - apply the **second derivative test** to determine the nature of stationary points.
- To solve **optimisation problems**.

In this chapter, we introduce the **second derivative** of a function. This is simply the derivative of the derivative. We will see that the second derivative can provide extra information about the shape of the graph of a function.

For example, part of the graph of $y = x + \sin x$ is shown for $x \geq 0$. You can clearly see how the gradient is changing.

From the point $(0, 0)$ to the point (π, π), the gradient is decreasing. At (π, π), this changes and the gradient starts increasing. At $(2\pi, 2\pi)$, the gradient starts decreasing again, and so on.

These points are called **points of inflection**. They can be identified through the second derivative.



This chapter covers Unit 4 Topic 1: Further differentiation and applications 3.

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12A The second derivative and acceleration

For the function f with rule $f(x)$, the derivative is denoted by f' and has rule $f'(x)$. This notation is extended to taking the derivative of the derivative: the new function is denoted by f'' and has rule $f''(x)$. This new function is known as the **second derivative**.

For example, consider the function g with rule $g(x) = 2x^3 - 4x^2$.

- The derivative has rule $g'(x) = 6x^2 - 8x$.
- The second derivative has rule $g''(x) = 12x - 8$.

Note: The second derivative might not exist at a point even if the first derivative does.
For example, let $f(x) = x^{\frac{4}{3}}$. Then $f'(x) = \frac{4}{3}x^{\frac{1}{3}}$ and $f''(x) = \frac{4}{9}x^{-\frac{2}{3}}$.
We see that $f'(0) = 0$, but the second derivative $f''(x)$ is not defined at $x = 0$.

In Leibniz notation, the second derivative of y with respect to x is denoted by $\frac{d^2y}{dx^2}$.

Example 1

Find the second derivative of each of the following with respect to x:

a $f(x) = 6x^4 - 4x^3 + 4x$ **b** $y = e^x \sin x$

Solution

a
$$f(x) = 6x^4 - 4x^3 + 4x$$
$$f'(x) = 24x^3 - 12x^2 + 4$$
$$f''(x) = 72x^2 - 24x$$

b
$$y = e^x \sin x$$
$$\frac{dy}{dx} = e^x \sin x + e^x \cos x \quad \text{(by the product rule)}$$
$$\frac{d^2y}{dx^2} = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$$
$$= 2e^x \cos x$$

Example 2

If $f(x) = e^{2x}$, find $f''(0)$.

Solution

$$f(x) = e^{2x}$$
$$f'(x) = 2e^{2x}$$
$$f''(x) = 4e^{2x}$$

Therefore $f''(0) = 4e^0 = 4$.

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Example 3

Consider $f(x) = x^3 - 2x^2 + 4x - 6$.

a Find $f''(x)$.

b Solve the equation $f''(x) = 0$ for x.

Solution

a
$$f(x) = x^3 - 2x^2 + 4x - 6$$
$$f'(x) = 3x^2 - 4x + 4$$
$$f''(x) = 6x - 4$$

b
$$f''(x) = 0$$
$$6x - 4 = 0$$
$$\therefore \quad x = \frac{2}{3}$$

Example 4

Consider $y = x^2e^x$.

a Find $\dfrac{d^2y}{dx^2}$.

b Solve the equation $\dfrac{d^2y}{dx^2} = 0$ for x.

Solution

a
$$y = x^2e^x$$
$$\frac{dy}{dx} = 2xe^x + x^2e^x$$
$$\frac{d^2y}{dx^2} = 2e^x + 2xe^x + 2xe^x + x^2e^x$$
$$= 2e^x + 4xe^x + x^2e^x$$

b $\dfrac{d^2y}{dx^2} = 0$ implies
$$2e^x + 4xe^x + x^2e^x = 0$$
$$e^x(2 + 4x + x^2) = 0$$
$$x^2 + 4x + 2 = 0$$

Therefore $x = -2 + \sqrt{2}$ or $x = -2 - \sqrt{2}$.

► Motion in a straight line

We have studied motion in a straight line in Sections 8J, 9G and 10G.

Recall that an object's position, x m, at time t seconds is specified with respect to a reference point O on the line. Velocity, v m/s, and acceleration, a m/s^2, are given by:

$$\text{velocity} \quad v = \frac{dx}{dt} \qquad \text{acceleration} \quad a = \frac{dv}{dt}$$

We can now recognise that acceleration is the second derivative of position with respect to time:

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

That is, the object's acceleration is the rate of change of the rate of change of its position.

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Example 5

A particle moves along a straight line such that its position, x m, relative to a point O at time t seconds is given by $x = 5 + \sin(2\pi t)$ for $0 \le t \le 2$. Find:

a at what times and in what positions the particle will have zero velocity

b its acceleration at those instants.

Solution

a Velocity $v = \frac{dx}{dt} = 2\pi\cos(2\pi t)$

Solve the equation $v = 0$ for $0 \le t \le 2$:

$$2\pi\cos(2\pi t) = 0$$
$$\cos(2\pi t) = 0$$
$$2\pi t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \text{ or } \frac{7\pi}{2}$$
$$\therefore \quad t = \frac{1}{4}, \frac{3}{4}, \frac{5}{4} \text{ or } \frac{7}{4}$$

The times and positions at which the velocity is zero:

- $t = \frac{1}{4}$, $x = 5 + \sin\left(\frac{\pi}{2}\right) = 6$ m
- $t = \frac{3}{4}$, $x = 5 + \sin\left(\frac{3\pi}{2}\right) = 4$ m
- $t = \frac{5}{4}$, $x = 5 + \sin\left(\frac{5\pi}{2}\right) = 6$ m
- $t = \frac{7}{4}$, $x = 5 + \sin\left(\frac{7\pi}{2}\right) = 4$ m

b Acceleration $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -4\pi^2\sin(2\pi t)$

The acceleration when the velocity is zero:

- $t = \frac{1}{4}$, $a = -4\pi^2\sin\left(\frac{\pi}{2}\right) = -4\pi^2$ m/s^2
- $t = \frac{3}{4}$, $a = -4\pi^2\sin\left(\frac{3\pi}{2}\right) = 4\pi^2$ m/s^2
- $t = \frac{5}{4}$, $a = -4\pi^2\sin\left(\frac{5\pi}{2}\right) = -4\pi^2$ m/s^2
- $t = \frac{7}{4}$, $a = -4\pi^2\sin\left(\frac{7\pi}{2}\right) = 4\pi^2$ m/s^2

Section summary

- The **second derivative** of a function f is the derivative of the derivative of f.
- For a function f with rule $f(x)$, the second derivative of f is denoted by f'' and has rule $f''(x)$.
- In Leibniz notation, the second derivative of y with respect to x is denoted by $\frac{d^2y}{dx^2}$.
- For an object moving in a straight line with position x at time t:
  - velocity $v = \frac{dx}{dt}$
  - acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

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Exercise 12A

SF

Example 1 **1** Find the second derivative of each of the following:

a $2x + 5$ **b** x^8 **c** $\sqrt{x}$ **d** $(2x+1)^4$ **e** $\sin x$

f $\cos x$ **g** e^x **h** $\ln x$ **i** $\dfrac{1}{x+1}$ **j** $\sin\left(2x + \dfrac{\pi}{4}\right)$

2 Find the second derivative of each of the following:

a $\sqrt{x^5}$ **b** $(x^2+3)^4$ **c** $\sin\left(\dfrac{x}{2}\right)$

d $3\cos(4x+1)$ **e** $\dfrac{1}{2}e^{2x+1}$ **f** $\ln(2x+1)$

g $x^4 + 3x^2 - 7x + 2$ **h** x^3e^x **i** $x \ln x$

3 For each of the following, find $f''(x)$:

a $f(x) = 6e^{3-2x}$ **b** $f(x) = -8e^{-0.5x^2}$ **c** $f(x) = e^{\ln x}$

d $f(x) = \ln(x^2 + 2x)$ **e** $f(x) = 2(1-3x)^5$ **f** $f(x) = e^{x^2}$

g $f(x) = \dfrac{x-1}{x+1}$ **h** $f(x) = \dfrac{1}{\sqrt{1-x}}$ **i** $f(x) = 5\sin(3-x)$

j $f(x) = \cos(1-3x)$ **k** $f(x) = \sin\left(\dfrac{x}{3}\right)$ **l** $f(x) = \cos\left(\dfrac{x}{4}\right)$

Example 2 **4** For each of the following, find $f''(0)$:

a $f(x) = e^{\sin x}$ **b** $f(x) = e^{-\frac{1}{2}x^2}$ **c** $f(x) = \sqrt{1-x^2}$ **d** $f(x) = \cos(x^2)$

Example 3 **5** For each of the following, solve the equation $f''(x) = 0$ for x:

a $f(x) = 2x^3 + 4x^2$ **b** $f(x) = 5 - x - x^2 + 5x^3$ **c** $f(x) = x^4 - 3x^2 - 4x$

Example 4 **6** For each of the following, solve the equation $\dfrac{d^2y}{dx^2} = 0$ for x:

a $y = 2xe^x$ **b** $y = x^2e^x - xe^x$ **c** $y = x^3e^x$

d $y = \dfrac{x}{\ln x}$ **e** $y = \dfrac{\ln x}{x}$

CF

Example 5 **7** A particle moves along a straight line such that its position, x m, relative to a point O at time t seconds is given by $x = 7 + 2\cos\left(\dfrac{\pi t}{4}\right)$ for $0 \le t \le 8$. Find:

a at what times and in what positions the particle has zero velocity

b its acceleration at those instants.

8 A particle moves along the x-axis so that at time t seconds its distance, x m, from the origin O is given by $x = e^{-t} - e^{-2t}$ for $t \ge 0$.

a Find the particle's velocity and acceleration at time t seconds.

b Find its maximum distance from the origin and the time at which this occurs.

c Find the acceleration of the particle when it reaches this maximum distance.

d Find the maximum speed of the particle as it returns towards O.

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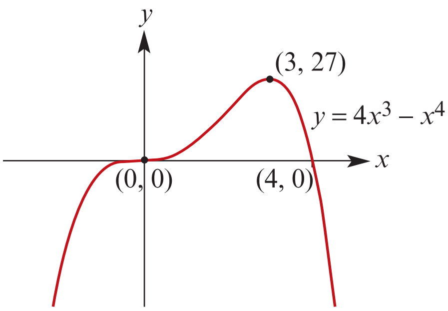
12B Using the second derivative in graph sketching

In Section 8L, you have used the first derivative when sketching the graphs of polynomial and other functions. The second derivative enables us to find out more information about these graphs. We start this section by considering the graph of $y = 4x^3 - x^4$.

▶ The graph of $y = 4x^3 - x^4$

The graph of this function is shown in the diagram on the right.

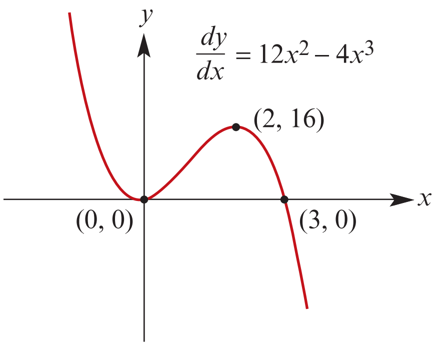
There is a local maximum at $(3, 27)$ and a stationary point of inflection at $(0, 0)$. These points have been determined by considering the derivative function $\frac{dy}{dx} = 12x^2 - 4x^3$.



The graph of the derivative function

Note that the local maximum and the stationary point of inflection of the original graph correspond to the x-axis intercepts of the graph of the derivative.

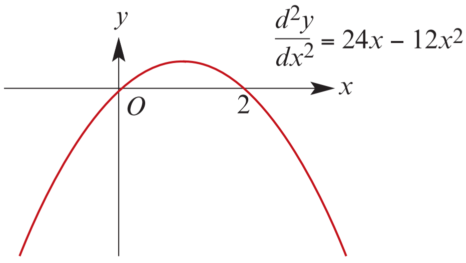
Also, it can be seen that the gradient of the original graph is positive for $x < 0$ and $0 < x < 3$ and negative for $x > 3$.



The graph of the second derivative function

Further information can be obtained by considering the graph of the second derivative.

The graph of the second derivative reveals that, at the points on the original graph where $x = 0$ and $x = 2$, there are important changes in the gradient.



- At the point where $x = 0$, the gradient of $y = 4x^3 - x^4$ changes from decreasing (positive) to increasing (positive). This point is also a stationary point, but it is neither a local maximum nor a local minimum. It is known as a **stationary point of inflection**.
- At the point where $x = 2$, the gradient of $y = 4x^3 - x^4$ changes from increasing (positive) to decreasing (positive). This point is called a **point of inflection**. In this case, the point corresponds to a local maximum of the derivative graph.

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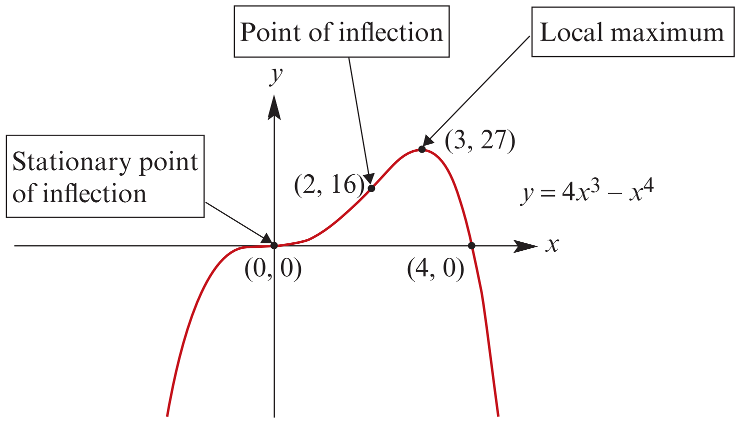
The gradient of $y = 4x^3 - x^4$ increases on the interval $(0, 2)$ and then decreases on the interval $(2, 3)$. The point $(2, 16)$ is the point of maximum gradient of $y = 4x^3 - x^4$ for the interval $(0, 3)$.



Behaviour of tangents

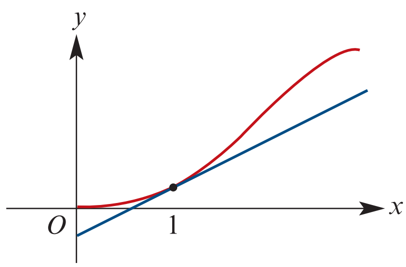
A closer look at the graph of $y = 4x^3 - x^4$ for the interval $(0, 3)$ and, in particular, the behaviour of the tangents to the graph in this interval will reveal more.

The tangents at $x = 1$, 2 and 2.5 have equations $y = 8x - 5$, $y = 16x - 16$ and $y = \frac{25}{2}x - \frac{125}{16}$ respectively. The following graphs illustrate the behaviour.

- The first diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 1$.

  The tangent lies *below* the graph in the immediate neighbourhood of where $x = 1$.

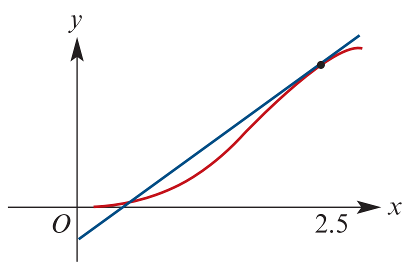
  For the interval $(0, 2)$, the gradient of the graph is increasing; the graph is said to be *concave up*.



- The second diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 2.5$.

  The tangent lies *above* the graph in the immediate neighbourhood of where $x = 2.5$.

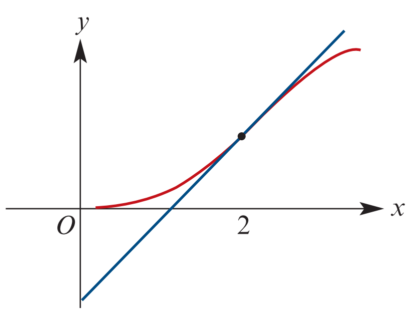
  For the interval $(2, 3)$, the gradient of the graph is decreasing; the graph is said to be *concave down*.



- The third diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 2$.

  The tangent *crosses* the graph at the point $(2, 16)$.

  At $x = 2$, the gradient of the graph changes from increasing to decreasing; the point $(2, 16)$ is said to be a *point of inflection*.



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Concavity and points of inflection

For a curve $y = f(x)$:

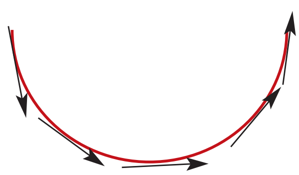
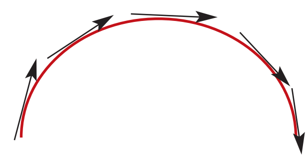
- the derivative $f'(a)$ gives the gradient of the curve at $x = a$
- the second derivative $f''(a)$ gives the rate of change of the gradient of the curve at $x = a$.

We have met the ideas of concave up and concave down in the example at the beginning of this section. We now give the definitions of these ideas.

Concave up and concave down

For a curve $y = f(x)$:

- If $f''(x) > 0$ for all $x \in (a, b)$, then the gradient of the curve is increasing over the interval (a, b). The curve is said to be **concave up**.
- If $f''(x) < 0$ for all $x \in (a, b)$, then the gradient of the curve is decreasing over the interval (a, b). The curve is said to be **concave down**.

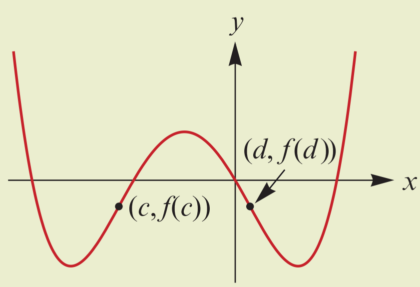
Concave up for an interval	Concave down for an interval
	
The tangent is below the curve at each point and the gradient is increasing i.e. $f''(x) > 0$	The tangent is above the curve at each point and the gradient is decreasing i.e. $f''(x) < 0$

Point of inflection

A point where a curve changes from concave up to concave down or from concave down to concave up is called a **point of inflection**.

That is, a point of inflection occurs where the sign of the second derivative changes.

In the graph on the right, there are points of inflection at $x = c$ and $x = d$.



Note: At a point of inflection, the tangent will pass through the curve.

At a point of inflection of a twice differentiable function f, we must have $f''(x) = 0$. However, this condition does not necessarily guarantee a point of inflection. At a point of inflection, there must also be a change of concavity.

For example, consider $f(x) = x^4$. Then $f''(x) = 12x^2$ and so $f''(0) = 0$. But the graph of $y = x^4$ has a local minimum at $x = 0$.

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From now on, we can use these new ideas in our graphing.

Example 6

For each of the following functions, find the coordinates of the points of inflection of the curve and state the intervals where the curve is concave up:

a $f(x) = x^3$

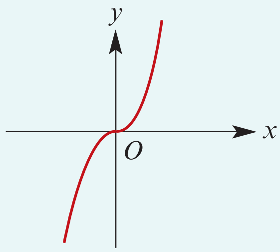
b $f(x) = -x^3$

c $f(x) = x^3 - 3x^2 + 1$

Solution

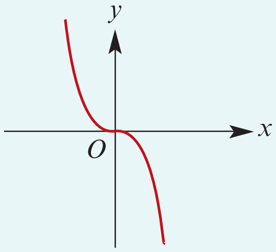
a
- There is a stationary point of inflection at $(0, 0)$.
  At $x = 0$, the gradient is zero and the curve changes from concave down to concave up.
- The curve is concave up on the interval $(0, \infty)$.
  The second derivative is positive on this interval.

Note: The tangent at $x = 0$ is the line $y = 0$.



b
- There is a stationary point of inflection at $(0, 0)$.
  At $x = 0$, the gradient is zero and the curve changes from concave up to concave down.
- The curve is concave up on the interval $(-\infty, 0)$.
  The second derivative is positive on this interval.

Note: The tangent at $x = 0$ is the line $y = 0$.

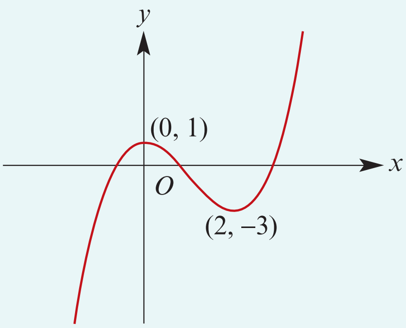


c
$$f(x) = x^3 - 3x^2 + 1$$
$$f'(x) = 3x^2 - 6x$$
$$f''(x) = 6x - 6$$

There is a local maximum at $(0, 1)$ and a local minimum at $(2, -3)$.

The second derivative is zero at $x = 1$, it is positive for $x > 1$, and it is negative for $x < 1$.

- There is a point of inflection at $(1, -1)$.
- The curve is concave up on the interval $(1, \infty)$.



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Test for local maximum or minimum

The following test provides a useful method for identifying local maximums and minimums.

Second derivative test

For the graph of $y = f(x)$:

- If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum, as the curve is concave up.
- If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum, as the curve is concave down.
- If $f''(a) = 0$, then further investigation is necessary.

Example 7

Consider the graph of $y = f(x)$, where $f(x) = x^2(10 - x)$.

a Find the coordinates of the stationary points and determine their nature using the second derivative test.

b Find the coordinates of the point of inflection and find the gradient at this point.

c On the one set of axes, sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ for $x \in [0, 10]$.

Solution

We have
$$f(x) = x^2(10 - x) = 10x^2 - x^3$$
$$f'(x) = 20x - 3x^2$$
$$f''(x) = 20 - 6x$$

a $f'(x) = 0$ implies $x(20 - 3x) = 0$, and therefore $x = 0$ or $x = \frac{20}{3}$.

Since $f''(0) = 20 > 0$, there is a local minimum at $(0, 0)$.

Since $f''\left(\frac{20}{3}\right) = -20 < 0$, there is a local maximum at $\left(\frac{20}{3}, \frac{4000}{27}\right)$.

b $f''(x) = 0$ implies $x = \frac{10}{3}$.

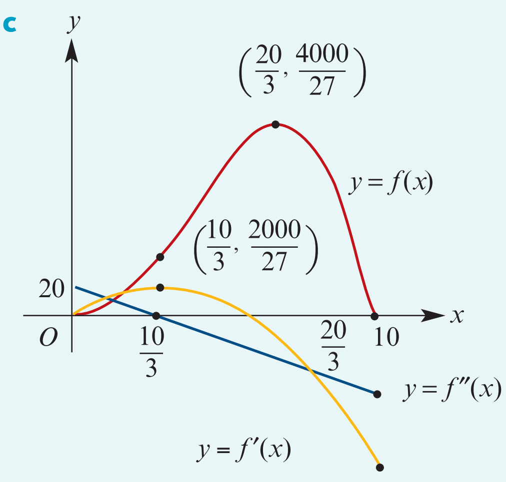
We have $f''(x) > 0$ for $x < \frac{10}{3}$ and $f''(x) < 0$ for $x > \frac{10}{3}$.

Hence there is a point of inflection at $\left(\frac{10}{3}, \frac{2000}{27}\right)$.

The gradient at this point is $\frac{100}{3}$.

Note: The maximum gradient of $y = f(x)$ is at the point of inflection.

c



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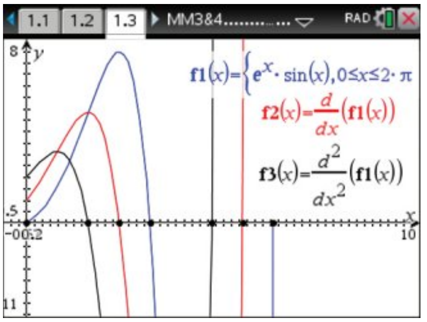
Example 8

Using a graphics calculator, find approximate coordinates for the stationary points and the points of inflection on the graph of the function

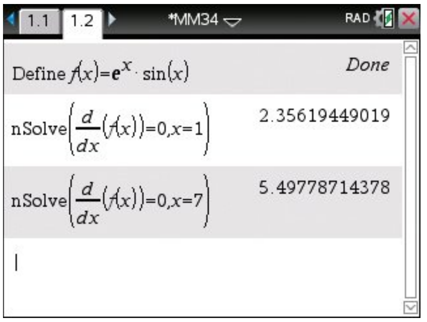
$$f(x) = e^x \sin x, \quad x \in [0, 2\pi]$$

Using the TI-Nspire CX non-CAS

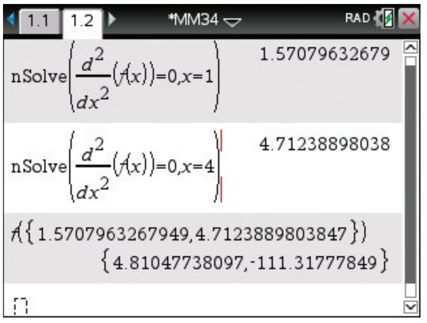
- Plot the graphs of the original function ($f1$), its derivative ($f2$) and its second derivative ($f3$) as shown. Note that:
  - The graph of the derivative ($f2$) has two x-axis intercepts, so there are two stationary points.
  - The graph of the second derivative ($f3$) crosses the x-axis twice, so there are two points of inflection.



- In a **Calculator** application, define the function $f(x) = e^x \sin(x)$.
- Use **nSolve** to find the x-values such that the derivative equals zero, for $x \in [0, 2\pi]$.



- Use **nSolve** to find the x-values such that the second derivative equals zero, for $x \in [0, 2\pi]$.
- Substitute the x-values in $f(x)$ to find the corresponding y-values:
  - The stationary points are at $(2.36, 7.46)$ and $(5.50, -172.64)$.
  - The points of inflection are at $(1.57, 4.81)$ and $(4.71, -111.32)$.



Notes:

- The derivative templates can be accessed from the 2D-template palette ⌸.
- In the **Graphs** application, you can use menu > **Trace** > **Graph Trace** to find the x-axis intercepts of the derivative and the second derivative.
- When using **nSolve**, you will need to try several guess values in order to find all the solutions.

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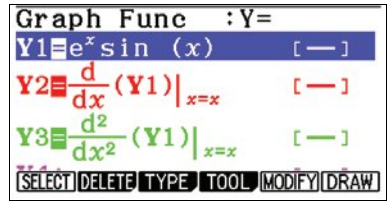
Using the Casio

- Press (MENU) (5) to select **Graph** mode.
- Enter the rule $y = e^x \sin(x)$ in $Y1$.
- Enter the derivative of $Y1$ in $Y2$:

  (OPTN) (F2) (F1) (F1) (1) ► (X,θ,T) (EXE)
- Enter the second derivative of $Y1$ in $Y3$:

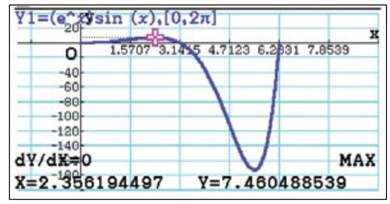
  (OPTN) (F2) (F2) (F1) (1) ► (X,θ,T) (EXE)

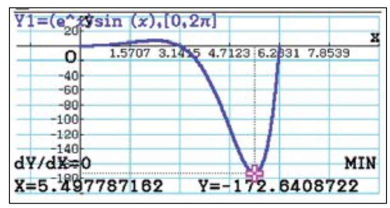


Turning points

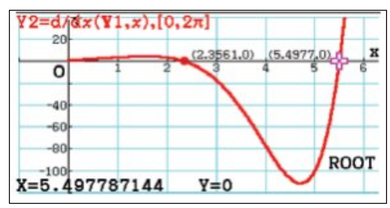
- Plot the graph of the original function $Y1$:

  ▲ (F1) ▲ (F1) (F6)
- Go to **G-Solve** (SHIFT) (F5); select **Maximum** (F2). The local maximum is at $(2.36, 7.46)$.
- Go to **G-Solve** (SHIFT) (F5); select **Minimum** (F3). The local minimum is at $(5.50, -172.64)$.





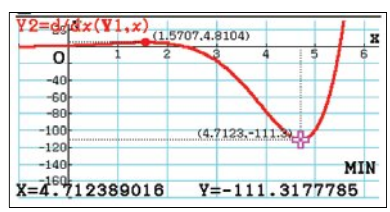
Note: The x-coordinates of the stationary points are the x-axis intercepts of the graph of the derivative function $Y2$.



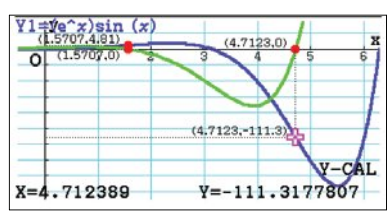
Points of inflection

- Press (EXIT) to return to the function list.
- Plot the graph of the derivative function $Y2$:

  ▲ (F1) ▲ (F1) (F6)
- Find the turning points using **Maximum** and **Minimum** from the **G-Solve** menu.
- The points of inflection occur when $x = 1.571$ and $x = 4.712$.



- On the graph of the original function $Y1$, find the y-coordinate of each inflection point using **y-Cal** from the **G-Solve** menu.
- The points of inflection are at $(1.57, 4.81)$ and $(4.71, -111.32)$.



Note: The x-coordinates of the points of inflection can also be found by determining where the graph of the second derivative function $Y3$ crosses the x-axis.

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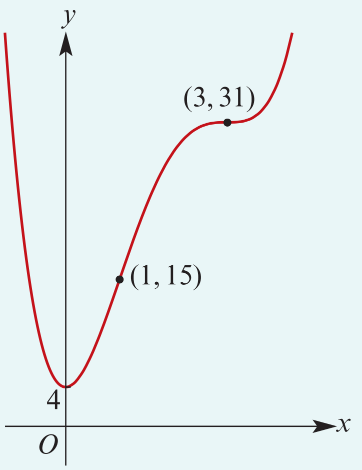
Example 9

Sketch the graph of $f(x) = x^4 - 8x^3 + 18x^2 + 4$, locating the stationary points and the points of inflection.

Solution

$$f(x) = x^4 - 8x^3 + 18x^2 + 4$$
$$f'(x) = 4x^3 - 24x^2 + 36x = 4x(x-3)^2$$
$$f''(x) = 12x^2 - 48x + 36 = 12(x-1)(x-3)$$



Stationary points

$f'(x) = 0$ implies $x = 0$ or $x = 3$

- Since $f''(0) = 36 > 0$, there is a local minimum at $(0, 4)$.
- Since $f''(3) = 0$, the test is inconclusive; further investigation is required.

Points of inflection

$f''(x) = 0$ implies $x = 1$ or $x = 3$

When $x < 1$, $f''(x) > 0$.
When $1 < x < 3$, $f''(x) < 0$.
When $x > 3$, $f''(x) > 0$.

- There is a point of inflection at $(1, 15)$ and a stationary point of inflection at $(3, 31)$.

Section summary

- For a curve $y = f(x)$:
  - If $f''(x) > 0$ for all $x \in (a, b)$, then the gradient of the curve is increasing over the interval (a, b). The curve is said to be **concave up**.
  - If $f''(x) < 0$ for all $x \in (a, b)$, then the gradient of the curve is decreasing over the interval (a, b). The curve is said to be **concave down**.
- A **point of inflection** is where the curve changes from concave up to concave down or from concave down to concave up.
- At a point of inflection of a twice differentiable function f, we must have $f''(x) = 0$. However, this condition does not necessarily guarantee a point of inflection. At a point of inflection, there must also be a change of concavity.
- Second derivative test

  For the graph of $y = f(x)$:
  - If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum.
  - If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum.
  - If $f''(a) = 0$, then further investigation is necessary.

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Exercise 12B

Skillsheet **1** Sketch a small portion of a continuous curve around a point $x = a$ having the property:

a $\dfrac{dy}{dx} > 0$ when $x = a$ and $\dfrac{d^2y}{dx^2} > 0$ when $x = a$

b $\dfrac{dy}{dx} < 0$ when $x = a$ and $\dfrac{d^2y}{dx^2} < 0$ when $x = a$

c $\dfrac{dy}{dx} > 0$ when $x = a$ and $\dfrac{d^2y}{dx^2} < 0$ when $x = a$

d $\dfrac{dy}{dx} < 0$ when $x = a$ and $\dfrac{d^2y}{dx^2} > 0$ when $x = a$

SF

Example 6 **2** For each of the following functions, find the coordinates of the points of inflection of the curve and state the intervals where the curve is concave up:

a $f(x) = x^3 - x$ **b** $f(x) = x^3 - x^2$ **c** $f(x) = x^2 - x^3$ **d** $f(x) = x^4 - x^3$

Example 7 **3** Let $f(x) = \dfrac{x^2}{10}(20 - x)$ for $x \in [0, 20]$.

CF

a Find the coordinates of the stationary points and determine their nature using the second derivative test.

b Find the coordinates of the point of inflection and find the gradient at this point.

c On the one set of axes, sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ for $x \in [0, 20]$.

4 Let $f(x) = 2x^3 + 6x^2 - 12$ for $x \in \mathbb{R}$.

a **i** Find $f'(x)$. **ii** Find $f''(x)$.

b Find the coordinates of the stationary points and use the second derivative test to establish their nature.

c Use $f''(x)$ to find the point of inflection.

5 Repeat Question 4 for each of the following functions:

a $f(x) = \sin x,\ x \in [0, 2\pi]$ **b** $f(x) = xe^x,\ x \in \mathbb{R}$

Example 9 **6** Sketch the graph of $y = x^3(4 - x)$, locating the stationary points and the points of inflection.

7 Sketch the graph of $y = 3x^4 - 44x^3 + 144x^2$, locating the stationary points and the points of inflection.

8 Let $f(x) = x(10 - x)e^x$ for $x \in [0, 10]$.

a Find $f'(x)$ and $f''(x)$.

b Sketch the graphs of $y = f(x)$ and $y = f''(x)$ on the one set of axes for $x \in [0, 10]$.

c Find the value of x for which the gradient of the graph of $y = f(x)$ is a maximum and indicate this point on the graph of $y = f(x)$.

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9 Find the coordinates of the points of inflection of $y = x - \sin x$ for $x \in [0, 4\pi]$. SF

10 For each of the following functions, find the values of x in the interval $[0, 2\pi]$ for which the graph of the function has a point of inflection:

a $y = \sin x$ **b** $y = \tan x$ **c** $y = \sin(2x)$

11 Show that the parabola with equation $y = ax^2 + bx + c$ has no points of inflection. CF

12 For the curve with equation $y = 2x^3 - 9x^2 + 12x + 8$, find the values of x for which:

a $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$ **b** $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} < 0$

13 For each of the following functions, determine the coordinates of any points of inflection and the gradient of the graph at these points: SF

a $y = x^3 - 6x$ **b** $y = x^4 - 6x^2 + 4$ **c** $y = 3 - 10x^3 + 10x^4 - 3x^5$

d $y = (x^2 - 1)(x^2 + 1)$ **e** $y = x\sqrt{x + 1}$ **f** $y = \frac{2x}{x^2 + 1}$

14 Determine the values of x in $[-2\pi, 2\pi]$ for which the graph of $y = e^{-x} \sin x$ has:

a stationary points **b** points of inflection.

15 Given that $f(x) = x^3 + bx^2 + cx$ and $b^2 > 3c$, prove that: CF

a the graph of f has two stationary points

b the graph of f has one point of inflection

c the point of inflection is the midpoint of the interval joining the stationary points.

16 Consider the function with rule $f(x) = 2x^2 \ln(x)$.

a Find $f'(x)$.

b Find $f''(x)$.

c Find the stationary points and the points of inflection of the graph of $y = f(x)$.

17 The graph of $y = x^3 - ax^2 + bx + c$ passes through the point $(2, 7)$, has a local maximum when $x = 1$ and a point of inflection when $x = 3$.

a Find the values of a, b and c.

b Sketch the graph.

18 The graph of $y = ax^3 + bx + c$ has a point of inflection where $y = 3$, a local maximum where $x = 1$ and passes through the point $(2, 1)$.

a Find the values of a, b and c.

b Give the value of x for which there is a local minimum.

c Sketch the graph.

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12C Absolute maximum and minimum values

Local maximum and minimum values were considered in the previous section. These are often not the actual maximum and minimum values of the function.

For a function defined on an interval:

- the actual maximum value of the function is called the **absolute maximum**
- the actual minimum value of the function is called the **absolute minimum**.

The corresponding points on the graph of the function are not necessarily stationary points.

More precisely, for a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the absolute maximum value of the function
- if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the absolute minimum value of the function.

Example 10

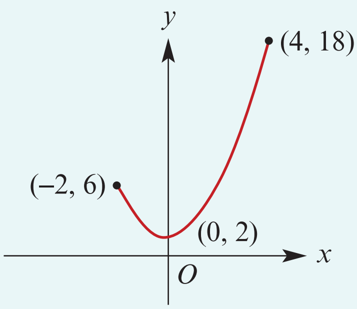
Let $f(x) = x^2 + 2$ for $x \in [-2, 4]$. Find the absolute maximum value and the absolute minimum value of the function.

Solution

The maximum value is 18 and occurs when $x = 4$.

The minimum value is 2 and occurs when $x = 0$.

(Note that the absolute minimum occurs at a stationary point of the graph. The absolute maximum occurs at an endpoint, not at a stationary point.)



Example 11

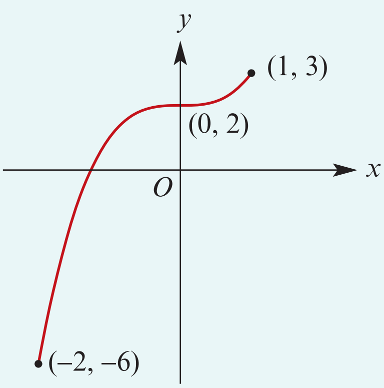
Let $f(x) = x^3 + 2$ for $x \in [-2, 1]$. Find the maximum and minimum values of the function.

Solution

The maximum value is 3 and occurs when $x = 1$.

The minimum value is -6 and occurs when $x = -2$.

(Note that the absolute maximum and minimum values do not occur at stationary points.)



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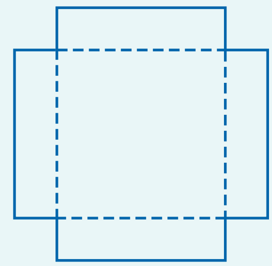
Example 12

From a square piece of metal of side length 2 m, four squares are removed as shown in the diagram. The metal is then folded along the dashed lines to form an open box with height x m.



a Show that the volume of the box, V m^3, is given by $V = 4x^3 - 8x^2 + 4x$.

b Find the value of x that gives the box its maximum volume and show that the volume is a maximum for this value.

c Sketch the graph of V against x for a suitable domain.

d If the height of the box must be less than 0.3 m, i.e. $x \le 0.3$, what will be the maximum volume of the box?

Solution

a The box has length and width $2 - 2x$ metres, and has height x metres. Thus

$$\begin{aligned} V &= (2 - 2x)^2 x \\ &= (4 - 8x + 4x^2)x \\ &= 4x^3 - 8x^2 + 4x \end{aligned}$$

b Let $V(x) = 4x^3 - 8x^2 + 4x$. A local maximum will occur when $V'(x) = 0$.
We have $V'(x) = 12x^2 - 16x + 4$, and so $V'(x) = 0$ implies that

$$\begin{aligned} 12x^2 - 16x + 4 &= 0 \\ 3x^2 - 4x + 1 &= 0 \\ (3x - 1)(x - 1) &= 0 \\ \therefore \quad x &= \frac{1}{3} \text{ or } x = 1 \end{aligned}$$

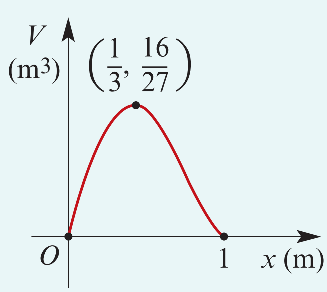
But, when $x = 1$, the length of the box is $2 - 2x = 0$. Therefore the only value to be considered is $x = \frac{1}{3}$. We show the entire chart for completeness.

x		$\frac{1}{3}$		1
$V'(x)$	+	0	−	0
shape of V	/	—	\	—

The maximum occurs when $x = \frac{1}{3}$.

$$\begin{aligned} \therefore \text{ Maximum volume} &= \left(2 - 2 \times \frac{1}{3}\right)^2 \times \frac{1}{3} \\ &= \frac{16}{27} \text{ m}^3 \end{aligned}$$

c



d The local maximum of $V(x)$ defined on $[0, 1]$ is at $\left(\frac{1}{3}, \frac{16}{27}\right)$.

But $\frac{1}{3}$ is not in the interval $[0, 0.3]$.

Since $V'(x) > 0$ for all $x \in [0, 0.3]$, the maximum volume for this situation occurs when $x = 0.3$ and is 0.588 m^3.

ISBN 978-1-108-45164-2

Section summary

For a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the **absolute maximum** value of the function
- if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the **absolute minimum** value of the function.

Exercise 12C

Skillsheet
Example 10

SF

1 Let $f(x) = 2 - 8x^2$ for $x \in [-3, 3]$. Find the absolute maximum value and the absolute minimum value of the function.

Example 11

2 Let $f(x) = x^3 + 2x + 3$ for $x \in [-3, 2]$. Find the absolute maximum value and the absolute minimum value of the function for its domain.

3 Let $f(x) = 2x^3 - 6x^2$ for $x \in [-1.5, 2.5]$. Find the absolute maximum and absolute minimum values of the function.

4 Let $f(x) = 2x^4 - 8x^2$ for $x \in [-2, 6]$. Find the absolute maximum and absolute minimum values of the function.

Example 12

CF

5 A rectangular block is such that the sides of its base are of length x cm and $3x$ cm. The sum of the lengths of all its edges is 20 cm.

a Show that the volume, V cm^3, of the block is given by $V = 15x^2 - 12x^3$.

b Find $\dfrac{dV}{dx}$.

c Find the coordinates of the local maximum of the graph of V against x for $x \in [0, 1.25]$.

d If $x \in [0, 0.8]$, find the absolute maximum value of V and the value of x for which this occurs.

e If $x \in [0, 1]$, find the absolute maximum value of V and the value of x for which this occurs.

6 Variables x, y and z are such that $x + y = 30$ and $z = xy$.

a If $x \in [2, 5]$, find the possible values of y.

b Find the absolute maximum and absolute minimum values of z.

7 Consider the function $f(x) = \dfrac{1}{x-1} + \dfrac{1}{4-x}$, $x \in [2, 3]$.

a Find $f'(x)$.

b Find the coordinates of the stationary point of the graph of $y = f(x)$.

c Find the absolute maximum and absolute minimum of the function.

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CF

8 A piece of string 10 metres long is cut into two pieces to form two squares.

a If one piece of string has length x metres, show that the combined area of the two squares is given by $A = \frac{1}{8}(x^2 - 10x + 50)$.

b Find $\frac{dA}{dx}$.

c Find the value of x that makes A a minimum.

d If two squares are formed but $x \in [0, 1]$, find the maximum possible combined area of the two squares.

9 Find the maximum and minimum values of the function $g(x) = x + \frac{1}{x-2}$, $x \in [2.1, 8]$.

10 Consider the function $f(x) = \frac{1}{x+1} + \frac{1}{4-x}$, $x \in [0, 3]$.

a Find $f'(x)$.

b Find the coordinates of the stationary point of the graph of $y = f(x)$.

c Find the absolute maximum and absolute minimum of the function.

SF

11 Let $f(x) = \sin(2x)$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{8}\right]$. State the absolute maximum and minimum values of the function.

12 Let $f(x) = \cos(2x)$ for $x \in \left[0, \frac{\pi}{8}\right]$. State the absolute maximum and minimum values of the function.

13 Let $f(x) = 2 - x^{\frac{2}{3}}$ for $x \in [-1, 8]$. Sketch the graph and state the absolute maximum and minimum values of the function.

CF

14 Let $f(x) = 2e^x + e^{-x}$ for $x \in [-1, 2]$. Sketch the graph and state the absolute maximum and minimum values of the function.

15 Let $f(x) = (x - 5)\ln\left(\frac{x-5}{10}\right)$ for $x \in [6, 10]$. Sketch the graph and state the absolute maximum and minimum values of the function.

12D Optimisation problems

Many practical problems require that some quantity (for example, cost of manufacture or fuel consumption) be **minimised**, that is, made as small as possible. Other problems require that some quantity (for example, profit on sales or attendance at a concert) be **maximised**, that is, made as large as possible. In both cases, we say that the quantity is to be **optimised**. We can use differential calculus to solve many of these types of problems.

We now have two methods for identifying whether a stationary point corresponds to a local maximum or local minimum value:

Method 1 create a gradient chart

Method 2 use the second derivative test.

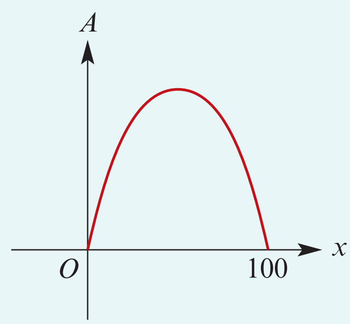
ISBN 978-1-108-45164-2

Example 13

A farmer has sufficient fencing to make a rectangular pen of perimeter 200 metres. What dimensions will give an enclosure of maximum area?

Solution

Let the length of the rectangle be x metres. Then the width is $100 - x$ metres and the area is A m^2, where



$$A = x(100 - x)$$
$$= 100x - x^2$$

The maximum value of A occurs when $\frac{dA}{dx} = 0$.

$$\frac{dA}{dx} = 100 - 2x$$

$$\therefore \quad \frac{dA}{dx} = 0 \text{ implies } x = 50$$

From the gradient chart, the maximum area occurs when $x = 50$.

x		50	
$\frac{dA}{dx}$	+	0	−
shape of A	/	—	\

The pen with maximum area has dimensions 50 m by 50 m, and so has area 2500 m^2.

Example 14

Two variables x and y are such that $x^4y = 8$. A third variable z is defined by $z = x + y$. Find the values of x and y that give z a stationary value. Use the second derivative test to show that this value of z is a minimum.

Solution

Obtain y in terms of x from the equation $x^4y = 8$:

$$y = 8x^{-4}$$

Substitute in the equation $z = x + y$:

$$z = x + 8x^{-4}$$

Now z is expressed in terms of one variable, x. Differentiate with respect to x:

$$\frac{dz}{dx} = 1 - 32x^{-5}$$

A stationary point occurs where $\frac{dz}{dx} = 0$:

$$1 - 32x^{-5} = 0$$
$$32x^{-5} = 1$$
$$x^5 = 32$$
$$\therefore \quad x = 2$$

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There is a stationary point at $x = 2$. The corresponding value of y is $8 \times 2^{-4} = \frac{1}{2}$.

So the corresponding value of z is

$$z = x + y = 2\frac{1}{2}$$

Second derivative test:

When $x = 2$, we have

$$\frac{d^2z}{dx^2} = 160x^{-6} = \frac{160}{2^6} > 0$$

which corresponds to a local minimum.

The minimum value of z is $2\frac{1}{2}$ and occurs when $x = 2$ and $y = \frac{1}{2}$.

Example 15

A cylindrical tin canister closed at both ends has a surface area of 100 cm^2. Find, correct to two decimal places, the greatest volume it can have. If the radius of the canister can be at most 2 cm, find the greatest volume it can have.

Solution

Let the radius of the circular end of the tin be r cm, let the height of the tin be h cm and let the volume of the tin be V cm^3.

Obtain equations for the surface area and the volume.

Surface area: $100 = 2\pi r^2 + 2\pi rh$ (1)

Volume: $V = \pi r^2 h$ (2)

The process we follow now is very similar to Example 14. Obtain h in terms of r from equation (1):

$$h = \frac{1}{2\pi r}(100 - 2\pi r^2)$$

Substitute in equation (2):

$$V = \pi r^2 \times \frac{1}{2\pi r}(100 - 2\pi r^2)$$

$$\therefore \quad V = 50r - \pi r^3 \qquad (3)$$

A stationary point of the graph of $V = 50r - \pi r^3$ occurs when $\frac{dV}{dr} = 0$.

$$\frac{dV}{dr} = 0 \text{ implies } 50 - 3\pi r^2 = 0$$

$$\therefore \quad r = \pm\sqrt{\frac{50}{3\pi}} \approx \pm 2.3$$

But $r = -2.3$ does not fit the practical situation.

Substitute $r = 2.3$ in equation (3) to find $V \approx 76.78$.

So there is a stationary point at $(2.3, 76.8)$.

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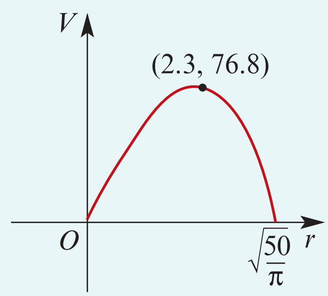
Use a gradient chart to determine the nature of this stationary point.

The maximum volume is 76.78 cm^3 correct to two decimal places.

r		2.3	
$\dfrac{dV}{dr}$	+	0	−
shape of V	/	—	\

It can be observed that the volume is given by a function f with rule $f(r) = 50r - \pi r^3$ and domain $\left[0, \sqrt{\frac{50}{\pi}}\right]$, giving the graph on the right.



If the greatest radius the canister can have is 2 cm, then the function f has domain $[0, 2]$. It has been seen that $f'(r) > 0$ for all $r \in [0, 2]$. The maximum value occurs when $r = 2$. The maximum volume in this case is $f(2) = 100 - 8\pi \approx 74.87 \text{ cm}^3$.

In some situations the variables may not be continuous. For instance, one of them may only take integer values. In such cases, we may be able to model the non-continuous case with a continuous function so that the techniques of differential calculus can be used.

Example 16

A TV cable company has 1000 subscribers who are paying \$5 per month. It can get 100 more subscribers for each \$0.10 decrease in the monthly fee. What monthly fee will yield the maximum revenue and what will this revenue be?

Solution

Let x denote the monthly fee. Then the number of subscribers is $1000 + 100\left(\dfrac{5-x}{0.1}\right)$.

(Note that we are treating a discrete situation with a continuous function.)

Let R denote the revenue. Then

$$\begin{aligned} R &= x(1000 + 1000(5 - x)) \\ &= 1000(6x - x^2) \\ \therefore \quad \frac{dR}{dx} &= 1000(6 - 2x) \end{aligned}$$

Thus $\dfrac{dR}{dx} = 0$ implies $6 - 2x = 0$ and hence $x = 3$.

Second derivative test:

When $x = 3$, we have

$$\frac{d^2R}{dx^2} = -2000 < 0$$

which corresponds to a local maximum.

For maximum revenue, the monthly fee should be \$3, and this gives a total revenue of \$9000.

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Example 17

A manufacturer annually produces and sells 10 000 shirts. Sales are uniformly distributed throughout the year. The production cost of each shirt is \$23 and the carrying costs (storage, insurance, interest) depend on the total number of shirts in a production run. (A production run is the number, x, of shirts which are under production at a given time.)

The set-up costs for a production run are \$40. The annual carrying costs are $\$x^{\frac{3}{2}}$. Find the size of a production run that minimises the total set-up and carrying costs for a year.

Solution

Number of production runs per year $= \dfrac{10\,000}{x}$

Set-up costs for these production runs $= 40\left(\dfrac{10\,000}{x}\right)$

Let C be the total set-up and carrying costs. Then

$$C = x^{\frac{3}{2}} + \frac{400\,000}{x}$$

$$= x^{\frac{3}{2}} + 400\,000x^{-1}, \quad x > 0$$

$$\therefore \quad \frac{dC}{dx} = \frac{3}{2}x^{\frac{1}{2}} - \frac{400\,000}{x^2}$$

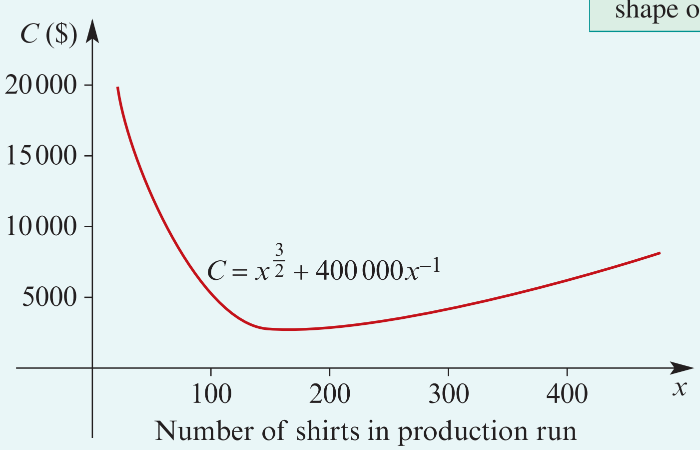
Thus $\dfrac{dC}{dx} = 0$ implies $\dfrac{3}{2}x^{\frac{1}{2}} = \dfrac{400\,000}{x^2}$

$$x^{\frac{5}{2}} = \frac{400\,000 \times 2}{3}$$

$$\therefore \quad x \approx 148.04$$

Each production run should be 148 shirts.

x		148.04	
$\dfrac{dC}{dx}$	−	0	+
shape of C	\	—	/



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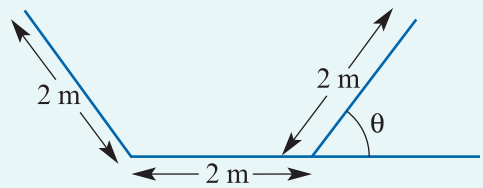
Example 18

The cross-section of a drain is to be an isosceles trapezium, with three sides of length 2 metres, as shown. Find the angle θ that maximises the cross-sectional area, and find this maximum area.



Solution

Let A m^2 be the area of the trapezium. Then

$$A = \frac{1}{2} \times 2\sin\theta \times (2 + 2 + 4 \times \cos\theta)$$
$$= \sin\theta \cdot (4 + 4\cos\theta)$$

and
$$A'(\theta) = \cos\theta \cdot (4 + 4\cos\theta) - 4\sin^2\theta$$
$$= 4\cos\theta + 4\cos^2\theta - 4(1 - \cos^2\theta)$$
$$= 4\cos\theta + 8\cos^2\theta - 4$$

The maximum will occur when $A'(\theta) = 0$:

$$8\cos^2\theta + 4\cos\theta - 4 = 0$$
$$2\cos^2\theta + \cos\theta - 1 = 0$$
$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$
$$\therefore \quad \cos\theta = \frac{1}{2} \text{ or } \cos\theta = -1$$

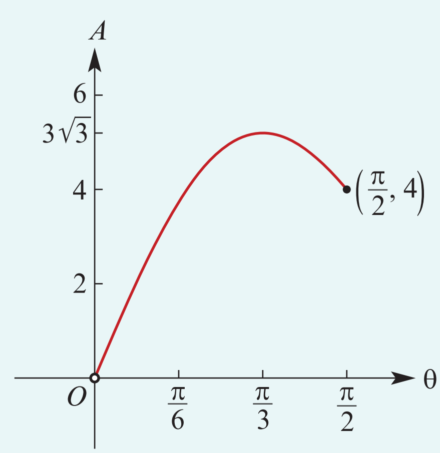
The practical restriction on θ is that $0 < \theta \le \frac{\pi}{2}$.

Therefore the only possible solution is $\theta = \frac{\pi}{3}$, and a gradient chart confirms that $\frac{\pi}{3}$ gives a maximum.

θ		$\frac{\pi}{3}$	
$A'(\theta)$	+	0	−
shape of A	/	—	\

When $\theta = \frac{\pi}{3}$, $A = \frac{\sqrt{3}}{2}(4 + 2) = 3\sqrt{3}$,

i.e. the maximum cross-sectional area is $3\sqrt{3}$ m^2.



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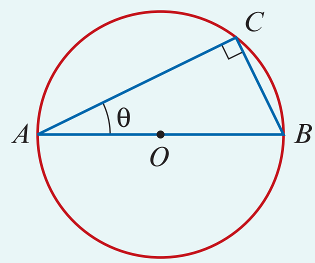
Example 19

The figure shows a circular lake, centre O, of radius 2 km. A man swims across the lake from A to C at 3 km/h and then walks around the edge of the lake from C to B at 4 km/h.



a If $\angle BAC = \theta$ radians and the total time taken is T hours, show that

$$T = \frac{1}{3}(4\cos\theta + 3\theta)$$

b Find the value of θ for which $\frac{dT}{d\theta} = 0$ and determine whether this gives a maximum or minimum value of T $(0° < \theta° < 90°)$.

Solution

a Time taken $= \dfrac{\text{distance travelled}}{\text{speed}}$

Therefore the swim takes $\frac{4\cos\theta}{3}$ hours and the walk takes $\frac{4\theta}{4}$ hours.

Thus the total time taken is given by $T = \frac{1}{3}(4\cos\theta + 3\theta)$.

b $\dfrac{dT}{d\theta} = \dfrac{1}{3}(-4\sin\theta + 3)$

The stationary point occurs where $\frac{dT}{d\theta} = 0$, and $\frac{1}{3}(-4\sin\theta + 3) = 0$ implies $\sin\theta = \frac{3}{4}$.

Therefore $\theta = 48.59°$ to two decimal places.

Second derivative test:

Note that $\frac{d^2T}{d\theta^2} = -\frac{4}{3}\cos\theta$. When $\theta = 48.59°$, we have $\cos\theta > 0$ and so $\frac{d^2T}{d\theta^2} < 0$.

Hence the value of T is a maximum when $\theta = 48.59°$.

Note: The maximum time taken is 1.73 hours. If the man swims straight across the lake, it takes $1\frac{1}{3}$ hours. If he walks all the way, it takes approximately 1.57 hours.

Example 20

Assume that the number of bacteria present in a culture at time t is given by $N(t)$, where $N(t) = 36te^{-0.1t}$. At what time will the population be at a maximum? Find the maximum population.

Solution

$$\begin{aligned} N(t) &= 36te^{-0.1t} \\ \therefore\quad N'(t) &= 36e^{-0.1t} - 3.6te^{-0.1t} \\ &= e^{-0.1t}(36 - 3.6t) \end{aligned}$$

Thus $N'(t) = 0$ implies $t = 10$.

The maximum population is $N(10) = 360e^{-1} \approx 132$.

ISBN 978-1-108-45164-2

▶ Maximum rates of increase and decrease

We know that we can use the derivative of a function to help find the maximum and minimum values of the function. Similarly, we can use the second derivative to help find the maximum rate of increase or decrease of the function.

The second derivative gives the instantaneous rate of change of the derivative.

- If $\frac{d^2y}{dx^2} > 0$, then $\frac{dy}{dx}$ is increasing as x increases.
- If $\frac{d^2y}{dx^2} < 0$, then $\frac{dy}{dx}$ is decreasing as x increases.

Example 21

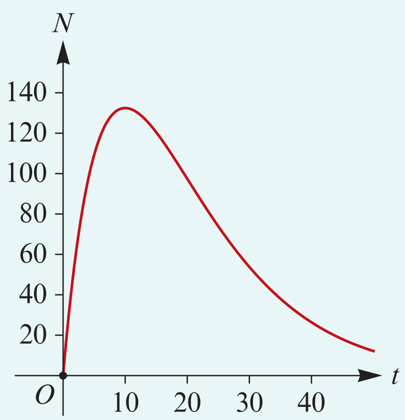
Assume that the number of bacteria present in a culture at time t is given by $N(t)$, where $N(t) = 36te^{-0.1t}$.

a Sketch the graphs of $N(t)$ against t and $N'(t)$ against t.

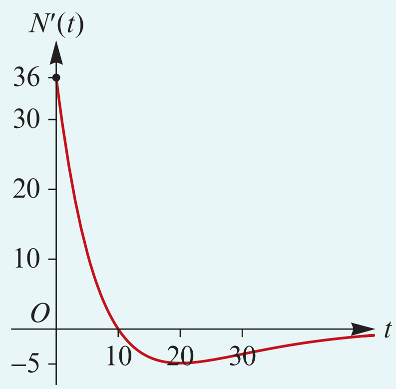
b Find the maximum rates of increase and decrease of the population and the times at which these occur.

Solution

a $N(t) = 36te^{-0.1t}$



$N'(t) = 36e^{-0.1t} - 3.6te^{-0.1t}$



b The rate of change of the population is $N'(t) = e^{-0.1t}(36 - 3.6t)$.
From the graph, the maximum value of $N'(t)$ occurs at $t = 0$. Thus the maximum rate of increase of the population is $N'(0) = 36$ bacteria per unit of time.

We now calculate

$$N''(t) = -7.2e^{-0.1t} + 0.36te^{-0.1t}$$
$$= e^{-0.1t}(-7.2 + 0.36t)$$

Thus $N''(t) = 0$ implies $t = 20$.

The minimum value of $N'(t)$ occurs at $t = 20$. Since $N'(20) = -36e^{-2} \approx -4.9$, the maximum rate of decrease of the population is 4.9 bacteria per unit of time.

Note: The graph of the original function N has a point of inflection at $t = 20$.

ISBN 978-1-108-45164-2

Section summary

Here are some steps for solving optimisation problems:

- Where possible, draw a diagram to illustrate the problem. Label the diagram and designate your variables and constants. Note the values that the variables can take.
- Write an expression for the quantity that is going to be maximised or minimised. Form an equation for this quantity in terms of a single independent variable. This may require some algebraic manipulation.
- If $y = f(x)$ is the quantity to be maximised or minimised, find the values of x for which $f'(x) = 0$.
- Test each point for which $f'(x) = 0$ to determine whether it is a local maximum, a local minimum or neither.
- If the function $y = f(x)$ is defined on an interval, such as $[a, b]$ or $[0, \infty)$, check the values of the function at the endpoints.

Exercise 12D

SF

Example 13 **1** Find the maximum area of a rectangular field that can be enclosed by 100 m of fencing.

Example 14 **2** Find two positive numbers that sum to 4 and such that the sum of the cube of the first and the square of the second is as small as possible.

3 For $x + y = 100$, prove that the product $P = xy$ is a maximum when $x = y$ and find the maximum value of P. (Use the second derivative test.)

4 A farmer has 4 km of fencing wire and wishes to fence a rectangular piece of land through which flows a straight river, which is to be utilised as one side of the enclosure. How can this be done to enclose as much land as possible?

5 Two positive quantities p and q vary in such a way that $p^3q = 9$. Another quantity z is defined by $z = 16p + 3q$. Find values of p and q that make z a minimum. (Use the second derivative test.)

CF

Example 15 **6** A cuboid has a total surface area of 150 cm^2 with a square base of side length x cm.

a Show that the height, h cm, of the cuboid is given by $h = \dfrac{75 - x^2}{2x}$.

b Express the volume of the cuboid in terms of x.

c Hence determine its maximum volume as x varies.

Example 16, 17 **7** A manufacturer finds that the daily profit, \$$P$, from selling n articles is given by $P = 100n - 0.4n^2 - 160$.

a **i** Find the value of n which maximises the daily profit.

ii Find the maximum daily profit.

b Sketch the graph of P against n. (Use a continuous graph.)

c State the allowable values of n for a profit to be made.

d Find the value of n which maximises the profit per article.

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CF

8 The number of salmon swimming upstream in a river to spawn is approximated by $s(x) = -x^3 + 3x^2 + 360x + 5000$ with x representing the temperature of the water in degrees (°C). (This function is valid only if $6 \le x \le 20$.) Find the water temperature that produces the maximum number of salmon swimming upstream.

9 The number of mosquitos, $M(x)$ in millions, in a certain area depends on the average daily rainfall, x mm, during September and is approximated by

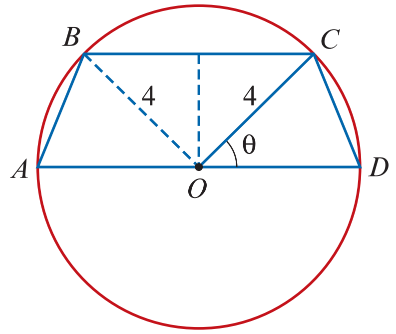
$$M(x) = \frac{1}{30}(50 - 32x + 14x^2 - x^3) \qquad \text{for } 0 \le x \le 10$$

Find the rainfall that will produce the maximum and the minimum number of mosquitos.

Example 18 **10** $ABCD$ is a trapezium with $AB = CD$. The vertices are on a circle with centre O and radius 4 units. The line segment AD is a diameter of the circle.

a Find BC in terms of θ.

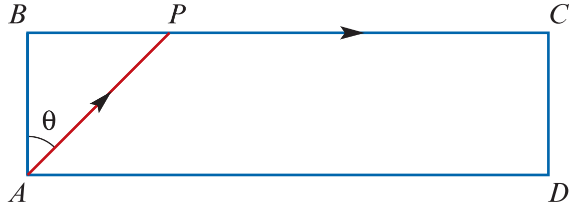
b Find the area of the trapezium in terms of θ and hence find the maximum area.



11 Find the point on the parabola $y = x^2$ that is closest to the point $(3, 0)$.

CU

Example 19 **12** The figure shows a rectangular field in which $AB = 300$ m and $BC = 1100$ m.



a An athlete runs across the field from A to P at 4 m/s. Find the time taken to run from A to P in terms of θ.

b The athlete, on reaching P, immediately runs to C at 5 m/s. Find the time taken to run from P to C in terms of θ.

c Use the results from **a** and **b** to show that the total time taken, T seconds, is given by $T = 220 + \dfrac{75 - 60\sin\theta}{\cos\theta}$.

d Find $\dfrac{dT}{d\theta}$.

e Find the value of θ for which $\dfrac{dT}{d\theta} = 0$ and show that this is the value of θ for which T is a minimum.

f Find the minimum value of T and find the distance of point P from B that will minimise the athlete's running time.

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CF

Example 20 **13** The number, $N(t)$, of insects in a population at time t is given by $N(t) = 50te^{-0.1t}$. At what time will the population be at a maximum? Find the maximum population.

Example 21 **14** The number, $N(t)$, of insects in a population at time t is given by $N(t) = 50te^{-0.1t}$.

a Sketch the graphs of $N(t)$ against t and $N'(t)$ against t.

b Find the maximum rates of increase and decrease of the population and the times at which these occur.

15 Water is being poured into a flask. The volume, V mL, of water in the flask at time t seconds is given by

$$V(t) = \frac{3}{4}\left(10t^2 - \frac{t^3}{3}\right), \quad 0 \le t \le 20$$

a Find the volume of water in the flask when:

i $t = 0$ **ii** $t = 20$

b Find $V'(t)$, the rate of flow of water into the flask.

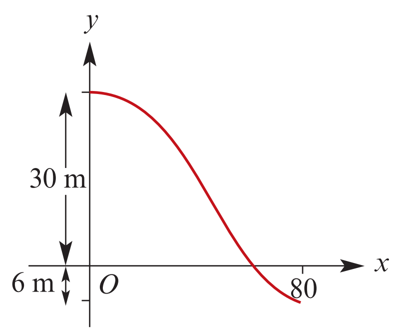
c Sketch the graph of $V(t)$ against t for $0 \le t \le 20$.

d Sketch the graph of $V'(t)$ against t for $0 \le t \le 20$.

e At what time is the flow greatest and what is the flow at this time?

16 A section of a roller coaster can be described by the rule

$$y = 18\cos\left(\frac{\pi x}{80}\right) + 12, \quad 0 \le x \le 80$$



a Find the gradient function, $\frac{dy}{dx}$.

b Sketch the graph of $\frac{dy}{dx}$ against x.

c State the coordinates of the point on the track for which the magnitude of the gradient is maximum.

17 The depth, $D(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by

$$D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right), \quad 0 \le t \le 24$$

a Sketch the graph of $y = D(t)$ for $0 \le t \le 24$.

b Find the values of t for which $D(t) \ge 8.5$.

c Find the rate at which the depth is changing when:

i $t = 3$ **ii** $t = 6$ **iii** $t = 12$

d **i** At what times is the depth increasing most rapidly?

ii At what times is the depth decreasing most rapidly?

ISBN 978-1-108-45164-2

Chapter summary

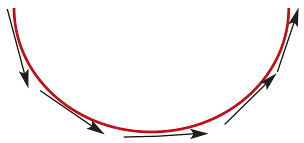
AS

The second derivative

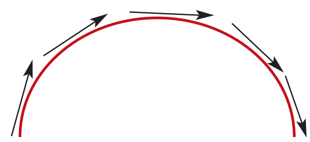
- The **second derivative** of a function f is the derivative of the derivative of f, and it is denoted by f''.
- In Leibniz notation, the second derivative of y with respect to x is denoted by $\dfrac{d^2y}{dx^2}$.

Using the second derivative in graph sketching

- Concave up: $f''(x) > 0$



- Concave down: $f''(x) < 0$



- A **point of inflection** is where the curve changes from concave up to concave down or from concave down to concave up.
- At a point of inflection of a twice differentiable function f, we must have $f''(x) = 0$. However, this condition does not necessarily guarantee a point of inflection. At a point of inflection, there must also be a change of concavity.
- Second derivative test

  For the graph of $y = f(x)$:
  - If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum.
  - If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum.
  - If $f''(a) = 0$, then further investigation is necessary.

Maximum and minimum values

For a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \le M$ for all $x \in [a, b]$, then M is the **absolute maximum** value of the function
- if N is a value of the function such that $f(x) \ge N$ for all $x \in [a, b]$, then N is the **absolute minimum** value of the function.

Motion in a straight line

For an object moving in a straight line with position x at time t:

- velocity $v = \dfrac{dx}{dt}$
- acceleration $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

Technology-free questions

SF

1 Find the second derivative of each of the following:

a $(2x + 5)^8$ **b** $\sin(2x)$ **c** $\cos\left(\dfrac{x}{3}\right)$

d e^{-4x} **e** $\ln(6x)$ **f** $\ln(\sin x)$

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Review

SF

2 For each of the following curves, state the coordinates of the point(s) of inflection:

a $y = x^3 - 8x^2$

b $y = \sin\left(x - \frac{\pi}{6}\right)$, $0 \le x \le 2\pi$

c $y = \ln x + \frac{1}{x}$

d $y = \frac{x}{\ln x}$

3 Consider the graph of $f(x) = x^2 \ln x$.

a Find the value of x for which there is a local minimum.

b Find the value of x for which there is a point of inflection.

4 Consider the graph of $f(x) = x^3 e^x$.

a Find the value of x for which there is a local minimum.

b Find the value of x for which there is a stationary point of inflection.

CF

5 Consider the graph of $f(x) = ce^{-kx^2}$, where c and k are constants.

a Find $f'(x)$ and $f''(x)$ in terms of c and k.

b Find the coordinates of the points of inflection if $c = 1$ and $k = 2$.

c Find the value of k for which there are points of inflection at $x = \frac{1}{4}$ and $x = -\frac{1}{4}$.

6 A particle moves in a straight line. Its position, x m, at time t seconds is given by

$$x = \frac{1}{9}\left(1 - (3t + 1)e^{-3t}\right)$$

a Find the velocity of the particle at time t.

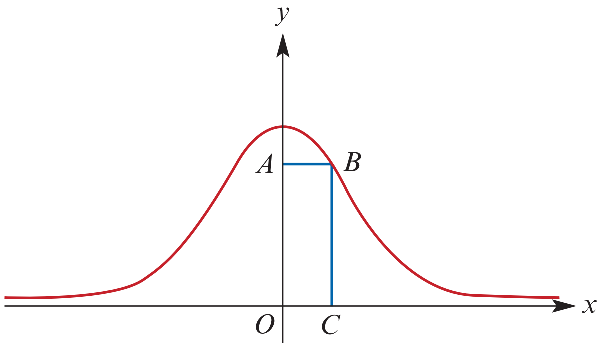
b Find the acceleration of the particle at time t.

c Find the maximum velocity of the particle and the time and location at which this maximum velocity occurs.

7 The graph of $f(x) = e^{-x^2}$ is shown.

Let $B(x, f(x))$ be a point on the graph, where $x > 0$.

Draw the rectangle $OABC$ as shown, where A is on the y-axis and C is on the x-axis.



a Find an expression for the area, W, of the rectangle in terms of x.

b Find $\frac{dW}{dx}$ and $\frac{d^2W}{dx^2}$.

c Find the coordinates of the point of inflection on the graph of W against x.

d Find the maximum value of W and the value of x for which this occurs. Verify that the point is a local maximum by showing that $\frac{d^2W}{dx^2} < 0$.

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Multiple-choice questions

1 For a polynomial function with rule $f(x)$, the derivative satisfies $f'(a) = f'(b) = 0$, $f'(x) > 0$ for $x \in (a, b)$, $f'(x) < 0$ for $x < a$ and $f'(x) > 0$ for $x > b$. The nature of the stationary points of the graph of $y = f(x)$ is

A local maximum at $(a, f(a))$ and local minimum at $(b, f(b))$
B local minimum at $(a, f(a))$ and local maximum at $(b, f(b))$
C stationary point of inflection at $(a, f(a))$ and local minimum at $(b, f(b))$
D stationary point of inflection at $(a, f(a))$ and local maximum at $(b, f(b))$
E local minimum at $(a, f(a))$ and stationary point of inflection at $(b, f(b))$

2 The second derivative of e^{-x^3} is

A e^{-6x} **B** $-3x^2e^{-x^3}$ **C** $-6x^4e^{-x^3}$
D $3x(3x^3 - 2)e^{-x^3}$ **E** $(9x^4 - 2)e^{-x^3}$

3 The graph of $y = 5x^4 - x^5$ has a point of inflection at

A $(0, 0)$ only **B** $(3, 162)$ only **C** $(4, 256)$ only
D $(0, 0)$ and $(3, 162)$ **E** $(0, 0)$ and $(4, 256)$

4 The position of a particle moving along the x-axis is given by $x(t) = \sin(2t) - \cos(3t)$ at time $t \geq 0$. When $t = \pi$, the acceleration of the particle is

A 9 **B** 19 **C** 0 **D** −19 **E** −9

5 The volume, V cm^3, of a solid is given by the formula $V = -10x(2x^2 - 6)$, where x cm is a particular measurement. The value of x for which the volume is a maximum is

A 0 **B** 1 **C** $\sqrt{2}$ **D** $\sqrt{3}$ **E** 2

6 If $f(x) = ax^4 + bx^2$, where $a > 0$ and $b > 0$, then which of the following must be true?

A There are no stationary points. **B** There are two stationary points.
C The graph is concave up for all x. **D** The graph is concave down for all x.
E There is one point of inflection.

7 The coordinates of the points of inflection of $y = \sin x$ for $x \in [0, 2\pi]$ are

A $\left(\frac{\pi}{2}, 1\right)$ and $\left(-\frac{\pi}{2}, -1\right)$ **B** $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right), \left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$
C $(\pi, 0)$ **D** $(1, 0)$ **E** $(0, 0)$, $(\pi, 0)$ and $(2\pi, 0)$

8 Let $g(x) = e^{f(x)}$, where the function f is twice differentiable. Then $g''(x)$ is equal to

A $f'(x)e^{f(x)}$ **B** $f''(x)e^{f(x)} + (f'(x))^2e^{f(x)}$
C $e^{f''(x)}$ **D** $f''(x)e^{f(x)}$ **E** $f''(x)e^{f(x)} + f'(x)e^{f(x)}$

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Review

9 Let $g(x) = e^{-x} f(x)$, where the function f is twice differentiable. There is a point of inflection on the graph of $y = g(x)$ at $(a, g(a))$. An expression for $f''(a)$ in terms of $f'(a)$ and $f(a)$ is

A $f''(a) = f(a) + f'(a)$ **B** $f''(a) = 2f(a)f'(a)$ **C** $f''(a) = 2f(a) + f'(a)$

D $f''(a) = \dfrac{f'(a)}{f(a)}$ **E** $f''(a) = 2f'(a) - f(a)$

10 The graph of a polynomial function with rule $y = f(x)$ has a local maximum at the point with coordinates $(a, f(a))$. The graph also has a local minimum at the origin, but no other stationary points. The graph of the function with rule $y = -2f\left(\dfrac{x}{2}\right) + k$, where k is a positive real number, has

A a local minimum at the point with coordinates $(2a, -2f(a) + k)$

B a local maximum at the point with coordinates $(2a, -2f(a) + k)$

C a local minimum at the point with coordinates $\left(\dfrac{a}{2}, 2f(a) + k\right)$

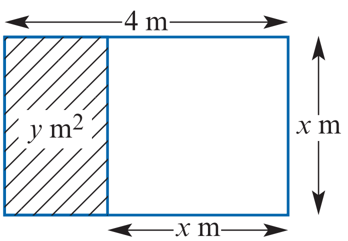
D a local maximum at the point with coordinates $\left(\dfrac{a}{2}, -2f(a) + k\right)$

E a local maximum at the point with coordinates $(2a, -2f(a) - k)$

Extended-response questions

CF

1 The diagram shows a rectangle with sides 4 m and x m and a square with side x m. The area of the shaded region is y m^2.



a Find an expression for y in terms of x.

b Find the possible values for x.

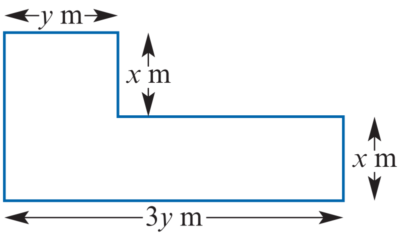
c Find the maximum value of y and the corresponding value of x.

d Explain briefly why this value of y is a maximum.

e Sketch the graph of y against x.

f State the possible values for y.

2 A flower bed is to be L-shaped, as shown in the figure, and its perimeter is 48 m.



a Write down an expression for the area, A m^2, in terms of y and x.

b Find y in terms of x.

c Write down an expression for A in terms of x.

d Find the values of x and y that give the maximum area.

e Find the maximum area.

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3 It costs $(12 + 0.008x)$ dollars per kilometre to operate a truck at x km/h. In addition it costs \$14.40 per hour to pay the driver.

a What is the total cost per kilometre if the truck is driven at:

i 40 km/h

ii 64 km/h?

b Write an expression for C, the total cost per kilometre, in terms of x.

c Sketch the graph of C against x for $0 < x < 120$.

d At what speed should the truck be driven to minimise the total cost per kilometre?

4 A box is to be made from a 10 cm by 16 cm sheet of metal by cutting equal squares out of the corners and bending up the flaps to form the box. Let the lengths of the sides of the squares be x cm and let the volume of the box formed be V cm^3.

a Show that $V = 4(x^3 - 13x^2 + 40x)$.

b State the set of x-values for which the expression for V in terms of x is valid.

c Find the values of x such that $\dfrac{dV}{dx} = 0$.

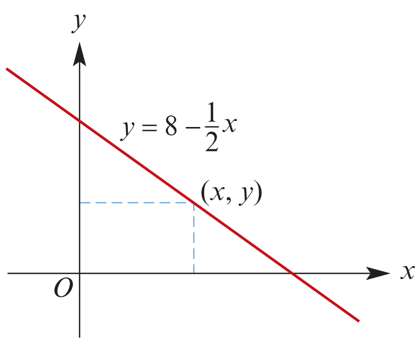
d Find the dimensions of the box if the volume is to be a maximum.

e Find the maximum volume of the box.

f Sketch the graph of V against x for the domain established in **b**.

5 A rectangle has one vertex at the origin, another on the positive x-axis, another on the positive y-axis and a fourth on the line $y = 8 - \dfrac{x}{2}$.

What is the greatest area the rectangle can have?



6 At a factory the time, T seconds, spent in producing a certain size metal component is related to its weight, w kg, by $T = k + 2w^2$, where k is a constant.

a If a 5 kg component takes 75 seconds to produce, find k.

b Sketch the graph of T against w.

c Write down an expression for the average time A (in seconds per kilogram).

d **i** Find the weight that yields the minimum average machining time per kilogram.

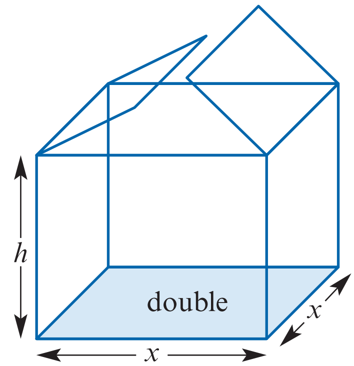
ii State the minimum average machining time.

7 An open tank is to be constructed with a square base and vertical sides to contain 500 m^3 of water. What must be the area of sheet metal used in its construction if this area is to be a minimum?

ISBN 978-1-108-45164-2

Review

8 A manufacturer produces cardboard boxes that have a square base. The top of each box consists of a double flap that opens as shown. The bottom of the box has a double layer of cardboard for strength. Each box must have a volume of 12 cubic metres.



a Show that the area of cardboard required is given by $C = 3x^2 + 4xh$.

b Express C as a function of x only.

c Sketch the graph of C against x for $x > 0$.

d **i** What dimensions of the box will minimise the amount of cardboard used?

ii What is the minimum area of cardboard used?

9 A piece of wire of length 1 m is bent into the shape of a sector of a circle of radius a cm and sector angle θ. Let the area of the sector be A cm^2.

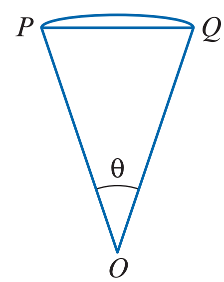
a Find A in terms of a and θ.

b Find A in terms of θ.

c Find the value of θ for which A is a maximum.

d Find the maximum area of the sector.

10 A piece of wire of fixed length, L cm, is bent to form the boundary $OPQO$ of a sector of a circle. The circle has centre O and radius r cm. The angle of the sector is θ radians.



a Show that the area, A cm^2, of the sector is given by

$$A = \frac{1}{2}rL - r^2$$

b **i** Find a relationship between r and L for which $\frac{dA}{dr} = 0$.

ii Find the corresponding value of θ.

iii Determine the nature of the stationary point found in **i**.

c Show that, for the value of θ found in **b ii**, the area of the triangle OPQ is approximately 45.5% of the area of sector OPQ.

11 Assume that the number of bacteria present in a culture at time t is given by $N(t)$ where $N(t) = 24te^{-0.2t}$. At what time will the population be at a maximum? Find the maximum population.

12 At noon the captain of a ship sees two fishing boats approaching. One of them is 10 km due east and travelling west at 8 km/h. The other is 6 km due north and travelling south at 6 km/h. At what time will the fishing boats be closest together and how far apart will they be?

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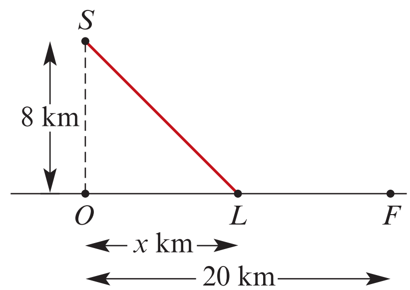
13 The point S is 8 km offshore from the point O, which is located on the straight shore of a lake, as shown in the diagram. The point F is on the shore, 20 km from O. Contestants race from the start, S, to the finish, F, by rowing in a straight line to some point, L, on the shore and then running along the shore to F. A certain contestant rows at 5 km per hour and runs at 15 km per hour.

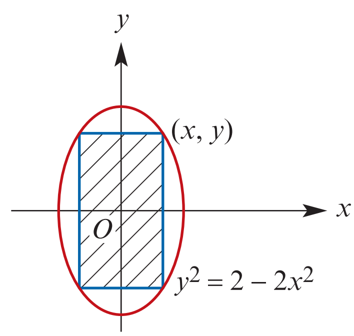


a Show that, if the distance OL is x km, the time taken by this contestant to complete the course is (in hours):

$$T(x) = \frac{\sqrt{64 + x^2}}{5} + \frac{20 - x}{15}$$

b Show that the time taken by this contestant to complete the course has its minimum value when $x = 2\sqrt{2}$. Find this time.

14 A rectangular beam is to be cut from a non-circular tree trunk whose cross-sectional outline can be represented by the equation $y^2 = 2 - 2x^2$.



a Show that the area of the cross-section of the beam is given by

$$A = 4x\sqrt{2 - 2x^2}$$

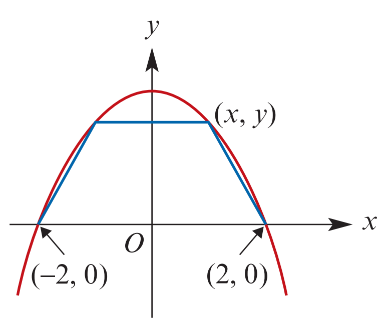
where x is the half-width of the beam.

b State the possible values for x.

c Find the value of x for which the cross-sectional area of the beam is a maximum and find the corresponding value of y.

d Find the maximum cross-sectional area of the beam.

15 An isosceles trapezium is inscribed in the parabola $y = 4 - x^2$ as illustrated.



a Show that the area of the trapezium is given by $\frac{1}{2}(4 - x^2)(2x + 4)$.

b Show that the trapezium has its greatest area when $x = \frac{2}{3}$.

c Repeat with the parabola $y = a^2 - x^2$:

i Show that the area, A, of the trapezium is given by $(a^2 - x^2)(a + x)$.

ii Use the product rule to find $\frac{dA}{dx}$.

iii Show that a maximum occurs when $x = \frac{a}{3}$.

ISBN 978-1-108-45164-2

Review

16 Consider the function with rule $f(x) = 6x^4 - x^3 + ax^2 - 6x + 8$.

CU

a **i** If $x + 1$ is a factor of $f(x)$, find the value of a.

ii Using a calculator, plot the graph of $y = f(x)$ for this value of a.

b Let $g(x) = 6x^4 - x^3 + 21x^2 - 6x + 8$.

i Plot the graph of $y = g(x)$.

ii Find the minimum value of $g(x)$ and the value of x for which this occurs.

iii Find $g'(x)$.

iv Using a calculator, solve the equation $g'(x) = 0$ for x.

v Find $g'(0)$ and $g'(10)$.

vi Find $g''(x)$.

vii Show that the graph of $y = g'(x)$ has no stationary points and thus deduce that $g'(x) = 0$ has only one solution.

17 A psychologist hypothesised that the ability of a mouse to memorise during the first 6 months of its life can be modelled by $f(x) = x \ln x + 1$ for $x \in (0, 6]$, where $f(x)$ is the ability to memorise at age x months.

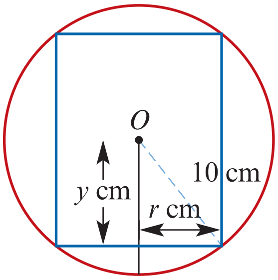
a Find $f'(x)$.

b Find the value of x for which $f'(x) = 0$ and hence find when the mouse's ability to memorise is a minimum.

c Sketch the graph of f.

d When is the mouse's ability to memorise a maximum in this period?

18 A cylinder is to be cut from a sphere. The cross-section through the centre of the sphere is as shown. The radius of the sphere is 10 cm. Let r cm be the radius of the cylinder.



a **i** Find y in terms of r and hence the height, h cm, of the cylinder.

ii The volume of a cylinder is given by $V = \pi r^2 h$. Find V in terms of r.

b **i** Plot the graph of V against r using a calculator.

ii Find the maximum volume of the cylinder and the corresponding values of r and h. (Use a calculator.)

iii Find the two possible values of r if the volume is 2000 cm^3.

c **i** Find $\frac{dV}{dr}$.

ii Hence find the exact value of the maximum volume and the value of r for which this occurs.

d **i** Plot the graph of the derivative function $\frac{dV}{dr}$ against r, using a calculator.

ii From the calculator, find the values of r for which $\frac{dV}{dr}$ is positive.

iii From the calculator, find the values of r for which $\frac{dV}{dr}$ is increasing.

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19 Consider the curve with equation $y = (x^2 - 2x)e^x$.

a Find the x-axis intercepts.

b Find the equation of the tangent to the curve at $x = 1$.

c Find the equation of the tangent to the curve at $x = 2$.

d Find the x-values for which there is a turning point.

e Find the x-values for which there is a point of inflection.

f Sketch the curve.

20 Consider the family of curves with equations $y = (2x^2 - 5x)e^{ax}$, where $a \in \mathbb{R} \setminus \{0\}$.

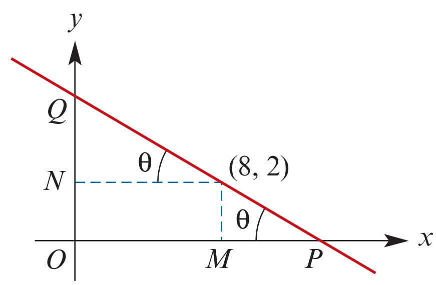
a Find the x-axis intercepts.

b Find the x-values for which there is a turning point, in terms of a.

c Find the x-values for which there is a point of inflection, in terms of a.

d If a curve from this family passes through the point $(3, 10)$, find the value of a.

21 A straight line is drawn through the point $(8, 2)$ to intersect the positive y-axis at Q and the positive x-axis at P. (In this question, we find the minimum value of $OP + OQ$.)



a Show that $\dfrac{d}{d\theta}\left(\dfrac{1}{\tan\theta}\right) = -\dfrac{1}{\sin^2\theta}$.

b Find MP in terms of θ.

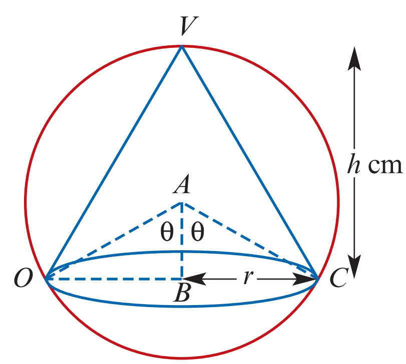
c Find NQ in terms of θ.

d Hence find $OP + OQ$ in terms of θ. Denote $OP + OQ$ by x.

e Find $\dfrac{dx}{d\theta}$.

f Find the minimum value of x and the value of θ for which this occurs.

22 A cone is inscribed inside a sphere as illustrated. The radius of the sphere is a cm, and both the angles $\angle OAB$ and $\angle CAB$ have magnitude θ. The height of the cone is h cm and the radius of the cone is r cm.



a Find h in terms of a and θ.

b Find r in terms of a and θ.

The volume, V cm^3, of the cone is given by $V = \dfrac{1}{3}\pi r^2 h$.

c Use the results from **a** and **b** to show that

$$V = \frac{1}{3}\pi a^3 \sin^2\theta \cdot (1 + \cos\theta)$$

d Find $\dfrac{dV}{d\theta}$ (where a is a constant) and hence find the value of θ for which the volume is a maximum.

e Find the maximum volume of the cone in terms of a.

ISBN 978-1-108-45164-2

Cambridge University Press

Review

CU

23 Some bacteria are introduced into a supply of fresh milk. After t hours there are y grams of bacteria present, where

$$y = \frac{Ae^{bt}}{1 + Ae^{bt}} \qquad (1)$$

and A and b are positive constants.

a Show that $0 < y < 1$ for all values of t.

b Find $\frac{dy}{dt}$ in terms of t.

c From equation (1), show that $Ae^{bt} = \frac{y}{1 - y}$.

d **i** Show that $\frac{dy}{dt} = by(1 - y)$.

ii Hence, or otherwise, show that the maximum value of $\frac{dy}{dt}$ occurs when $y = 0.5$.

e If $A = 0.01$ and $b = 0.7$, find when, to the nearest hour, the bacteria will be increasing at the fastest rate.

24 Let $f(x) = \frac{e^x}{x}$ for $x > 0$.

a Find $f'(x)$.

b Find the value of x such that $f'(x) = 0$.

c Find the coordinates of the stationary point and state its nature.

d **i** Find $\frac{f'(x)}{f(x)}$. **ii** Find $\lim_{x \to \infty} \frac{f'(x)}{f(x)}$ and comment.

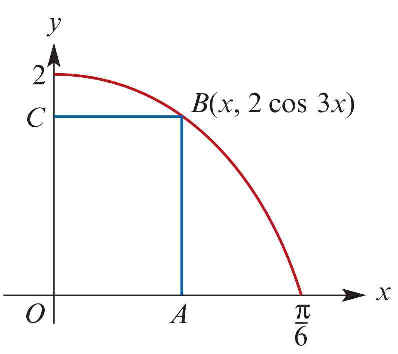
e Sketch the graph of f.

f Over a period of years, the number of birds (n) in an island colony decreased and increased with time (t years) according to the approximate formula

$$n = \frac{ae^{kt}}{t}$$

where t is measured from 1900 and a and k are constant. If during this period the population was the same in 1965 as it was in 1930, when was it least?

25 A section of the graph of $y = 2\cos(3x)$ is shown in the diagram.



a Show that the area, A, of the rectangle $OABC$ in terms of x is $2x\cos(3x)$.

b **i** Find $\frac{dA}{dx}$.

ii Find $\frac{dA}{dx}$ when $x = 0$ and $x = \frac{\pi}{6}$.

c **i** On a calculator, plot the graph of $A = 2x\cos(3x)$ for $x \in \left[0, \frac{\pi}{6}\right]$.

ii Find the two values of x for which the area of the rectangle is 0.2 square units.

iii Find the maximum area of the rectangle and the value of x for which this occurs.

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13 Trigonometry using the sine and cosine rules

Chapter 13

Objectives

- To solve practical problems using the trigonometric ratios.
- To use the **sine rule** and the **cosine rule** to solve problems.
- To find the **area of a triangle** given two sides and the included angle.
- To solve problems involving **angles of elevation** and **angles of depression**.
- To solve problems involving **compass bearings**.
- To solve trigonometric problems in **three dimensions**.
- To identify the **line of greatest slope** of a plane.
- To determine the angle between two planes.

We have revised the trigonometric functions in Chapter 4, where we considered the symmetry properties and the graphs of these functions. In this chapter, we apply the trigonometric ratios to solve problems.

Trigonometry deals with the side lengths and angles of a triangle: the word *trigonometry* comes from the Greek words for triangle and measurement.

We start this chapter by revising the four standard congruence tests for triangles. If you have the information about a triangle given in one of the congruence tests, then the triangle is uniquely determined (up to congruence). You can find the unknown side lengths and angles of the triangle using the **sine rule** or the **cosine rule**. In this chapter, we establish these rules, and apply them in two- and three-dimensional problems.

This chapter covers Unit 4 Topic 2: Trigonometric functions 2.

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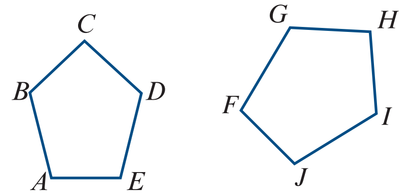
13A Reviewing trigonometry

▶ Congruent triangles

Two plane figures are called **congruent** if one figure can be moved on top of the other figure, by a sequence of translations, rotations and reflections, so that they coincide exactly.

Congruent figures have exactly the same shape and size. For example, the two figures shown are congruent. We can write:

$$\text{pentagon } ABCDE \equiv \text{pentagon } FGHIJ$$

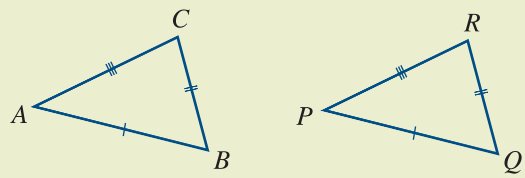


When two figures are congruent, we can find a transformation that pairs up every part of one figure with the corresponding part of the other, so that:

- paired angles have the same size
- paired line segments have the same length
- paired regions have the same area.

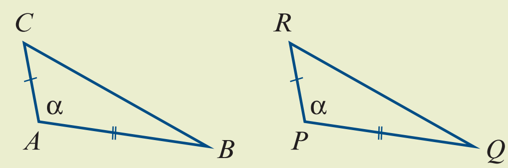
There are four standard tests for two triangles to be congruent.

- **The SSS congruence test**
  If the three sides of one triangle are respectively equal to the three sides of another triangle, then the two triangles are congruent.



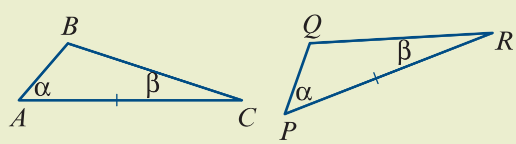
$\triangle ABC \equiv \triangle PQR$

- **The SAS congruence test**
  If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, then the two triangles are congruent.



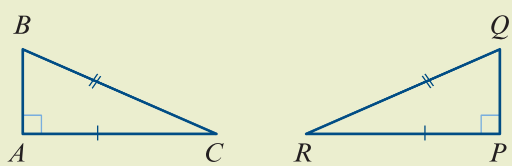
$\triangle ABC \equiv \triangle PQR$

- **The AAS congruence test**
  If two angles and one side of one triangle are respectively equal to two angles and the matching side of another triangle, then the two triangles are congruent.



$\triangle ABC \equiv \triangle PQR$

- **The RHS congruence test**
  If the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle, then the two triangles are congruent.



$\triangle ABC \equiv \triangle PQR$

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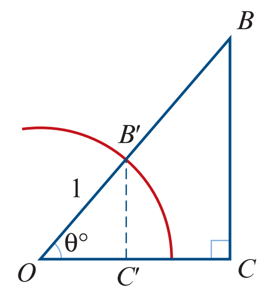
▶ The trigonometric ratios

For acute angles, the unit-circle definition of sine and cosine given in Section 4B is equivalent to the ratio definition.

For a right-angled triangle OBC, we can construct a similar triangle $OB'C'$ that lies in the unit circle. From the diagram:

$$B'C' = \sin(\theta^\circ) \quad \text{and} \quad OC' = \cos(\theta^\circ)$$



As triangles OBC and $OB'C'$ are similar, we have

$$\frac{BC}{OB} = \frac{B'C'}{1} \quad \text{and} \quad \frac{OC}{OB} = \frac{OC'}{1}$$

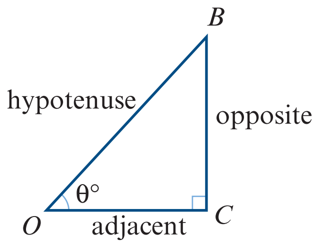
$$\therefore \quad \frac{BC}{OB} = \sin(\theta^\circ) \quad \text{and} \quad \frac{OC}{OB} = \cos(\theta^\circ)$$

This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle θ° is as shown.

$$\sin(\theta^\circ) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta^\circ) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta^\circ) = \frac{\text{opposite}}{\text{adjacent}}$$

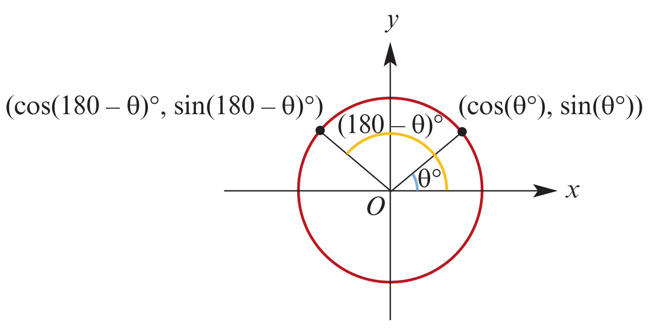


▶ Obtuse angles

From the unit circle, we see that

$$\sin(180 - \theta)^\circ = \sin(\theta^\circ)$$

$$\cos(180 - \theta)^\circ = -\cos(\theta^\circ)$$



For example:

$$\sin 135^\circ = \sin 45^\circ$$

$$\cos 135^\circ = -\cos 45^\circ$$

In this chapter, we will generally use the ratio definition of tangent for acute angles. But we can also find the tangent of an obtuse angle by using

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

We will not consider angles greater than 180° or less than 0° in this chapter, since we are dealing with triangles.

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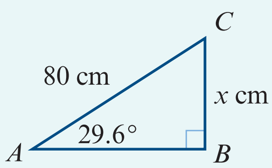
Solving right-angled triangles

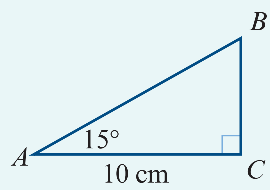
Here we provide some examples of using the trigonometric ratios.

Example 1

a Find the value of x correct to two decimal places.



b Find the length of the hypotenuse correct to two decimal places.



Solution

a

$$\frac{x}{80} = \sin 29.6°$$

$$\therefore \quad x = 80 \sin 29.6°$$

$$= 39.5153\ldots$$

Hence $x = 39.52$, correct to two decimal places.

b

$$\frac{10}{AB} = \cos 15°$$

$$10 = AB \cos 15°$$

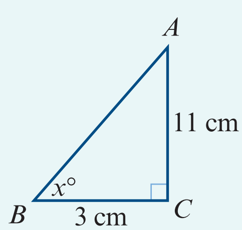
$$\therefore \quad AB = \frac{10}{\cos 15°}$$

$$= 10.3527\ldots$$

The length of the hypotenuse is 10.35 cm, correct to two decimal places.

Example 2

Find the magnitude of $\angle ABC$.



Solution

$$\tan x = \frac{11}{3}$$

$$\therefore \quad x = \tan^{-1}\left(\frac{11}{3}\right)$$

$$= (74.7448\ldots)°$$

Hence $x = 74.74°$, correct to two decimal places.

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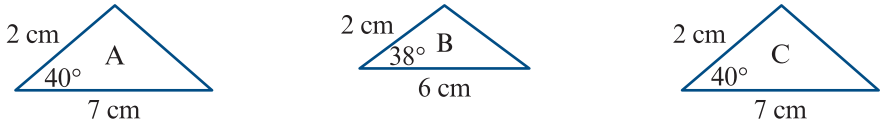
Exercise 13A

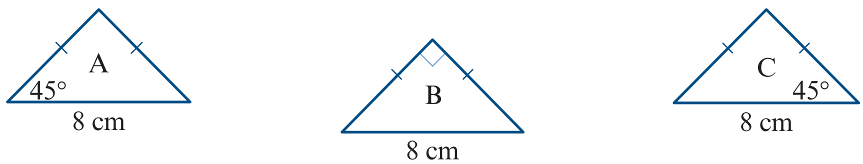
SF

1 In each part, find pairs of congruent triangles. State the congruence tests used.

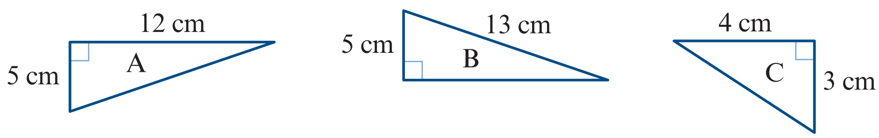
a



b



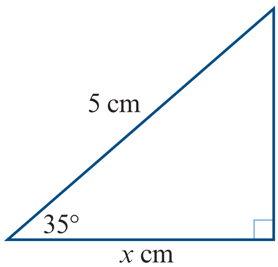
c



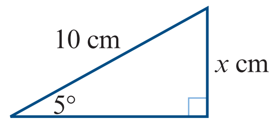
Example 1, 2

2 Find the value of x in each of the following:

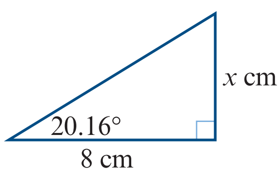
a



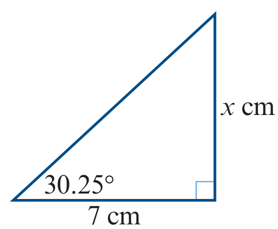
b



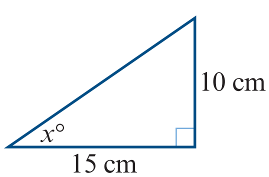
c



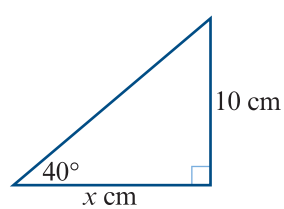
d



e



f



3 An equilateral triangle has altitudes of length 20 cm. Find the length of one side.

4 The base of an isosceles triangle is 12 cm long and the equal sides are 15 cm long. Find the magnitude of each of the three angles of the triangle.

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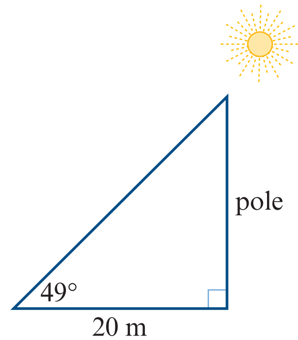
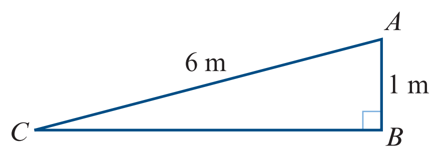
5 A pole casts a shadow 20 m long when the altitude of the sun is 49°. Calculate the height of the pole.



6 This figure represents a ramp.

a Find the magnitude of angle ACB.

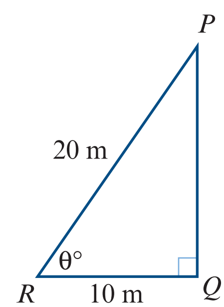
b Find the distance BC.



7 This figure shows a vertical mast PQ, which stands on horizontal ground. A straight wire 20 m long runs from P at the top of the mast to a point R on the ground, which is 10 m from the foot of the mast.

a Calculate the angle of inclination, $\theta°$, of the wire to the ground.

b Calculate the height of the mast.



8 A ladder leaning against a vertical wall makes an angle of 26° with the wall. If the foot of the ladder is 3 m from the wall, calculate:

a the length of the ladder

b the height it reaches above the ground.

9 An engineer is designing a straight concrete entry ramp, 60 m long, for a car park that is 13 m above street level. Calculate the angle of the ramp to the horizontal.

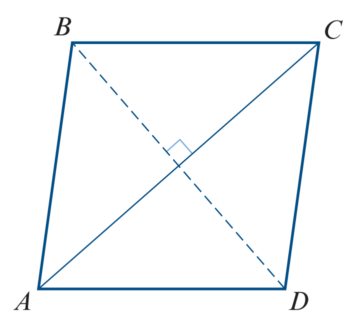
10 A vertical mast is secured from its top by straight cables 200 m long fixed at the ground. The cables make angles of 66° with the ground. What is the height of the mast?

11 A mountain railway rises 400 m at a uniform slope of 16° with the horizontal. What is the distance travelled by a train for this rise?

12 The diagonals of a rhombus bisect each other at right angles. If $BD = AC = 10$ cm, find:

a the length of the sides of the rhombus

b the magnitude of angle ABC.



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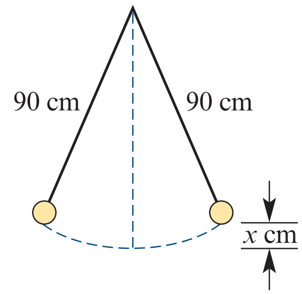
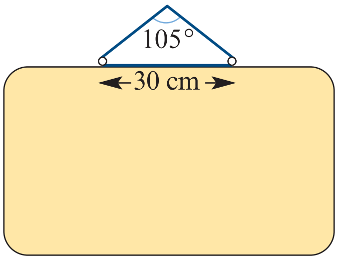
CF

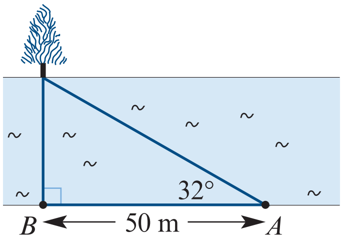
13 A pendulum swings from the vertical through an angle of 15° on each side of the vertical. If the pendulum is 90 cm long, what is the distance, x cm, between its highest and lowest points?



14 A picture is hung symmetrically by means of a string passing over a nail, with the ends of the string attached to two rings on the upper edge of the picture. The distance between the rings is 30 cm, and the string makes an angle of 105° at the nail. Find the length of the string.

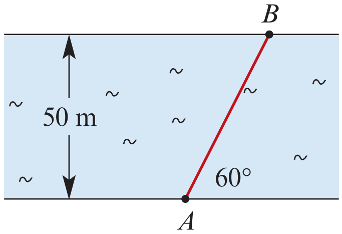


15 The distance AB is 50 m. If the line of sight to the tree of a person standing at A makes an angle of 32° with the bank, how wide is the river?



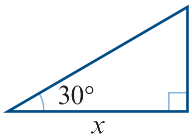
16 A ladder 4.7 m long is placed against a wall. The foot of the ladder must not be placed in a flower bed, which extends a distance of 1.7 m out from the base of the wall. How high up the wall can the ladder reach?

17 A river is known to be 50 m wide. A swimmer sets off from A to cross the river, and the path of the swimmer AB is as shown. How far does the person swim?



18 A rope is tied to the top of a flagpole. When it hangs straight down, it is 2 m longer than the pole. When the rope is pulled tight with the lower end on the ground, it makes an angle of 60° to the horizontal. How tall is the flagpole?

19 The triangle shown has perimeter 10. Find the value of x.



20 Consider the circle with equation $x^2 + y^2 - 4y = 0$ and the point $P(5, 2)$. Draw a diagram to show the circle and the two lines from P that are tangent to the circle. Find the angle between the two tangent lines, $\angle APB$, where A and B are the two points of contact.

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13B The sine rule

In the previous section, we focused on right-angled triangles. In this section and the next, we consider non-right-angled triangles.

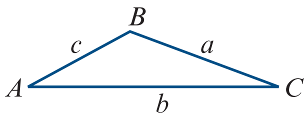
The **sine rule** is used to find unknown side lengths or angles of a triangle in the following two situations:

1. one side and two angles are given
2. two sides and a non-included angle are given (that is, the given angle is not 'between' the two given sides).

In the first case, the triangle is uniquely defined up to congruence. In the second case, there may be two triangles.

Labelling triangles

The following convention is used in the remainder of this chapter:



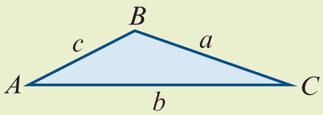
- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.

For example, the magnitude of angle BAC is denoted by A, and the length of side BC is denoted by a.

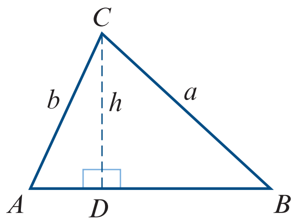
Sine rule

For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Proof We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.



In triangle ACD:

$$\sin A = \frac{h}{b}$$

$$\therefore \quad h = b \sin A$$

In triangle BCD:

$$\sin B = \frac{h}{a}$$

$$\therefore \quad a \sin B = b \sin A$$

$$\text{i.e.} \quad \frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, starting with a perpendicular from A to BC would give

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

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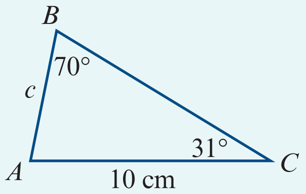
► One side and two angles

When one side and two angles are given, this corresponds to the AAS congruence test. The triangle is uniquely defined up to congruence.

Example 3

Use the sine rule to find the length of AB.



Solution

$$\frac{c}{\sin 31^\circ} = \frac{10}{\sin 70^\circ}$$

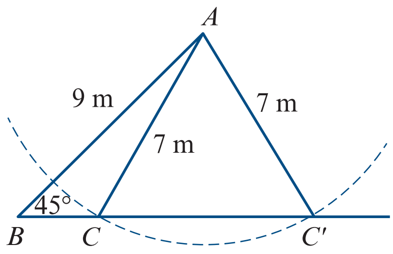
$$\therefore \quad c = \frac{10 \sin 31^\circ}{\sin 70^\circ}$$

$$= 5.4809\ldots$$

The length of AB is 5.48 cm, correct to two decimal places.

► Two sides and a non-included angle

Suppose that we are given the two side lengths 7 m and 9 m and a non-included angle of 45°. There are two triangles that satisfy these conditions, as shown in the diagram.



Warning

- When you are given two sides and a non-included angle, you must consider the possibility that there are two such triangles.
- An angle found using the sine rule is possible if the sum of the given angle and the found angle is less than 180°.

Note: If the given angle is obtuse or a right angle, then there is only one such triangle.

The following example illustrates the case where there are two possible triangles.

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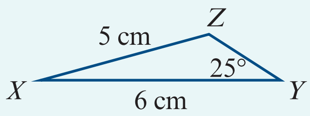
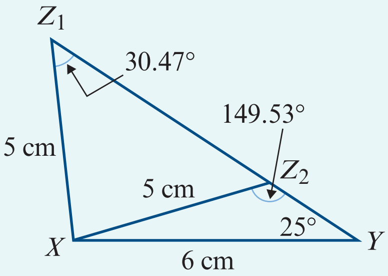
Example 4

Use the sine rule to find the magnitude of angle XZY in the triangle, given that $Y = 25°$, $y = 5$ cm and $z = 6$ cm.



Solution



$$\frac{5}{\sin 25°} = \frac{6}{\sin Z}$$

$$\frac{\sin Z}{6} = \frac{\sin 25°}{5}$$

$$\sin Z = \frac{6 \sin 25°}{5} = 0.5071\ldots$$

$\therefore \quad Z = (30.473\ldots)°$ or $Z = (180 - 30.473\ldots)°$

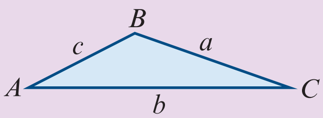
Hence $Z = 30.47°$ or $Z = 149.53°$, correct to two decimal places.

Note: Remember that $\sin(180 - \theta)° = \sin(\theta°)$.

Section summary

- **Sine rule** For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



- When to use the sine rule:
  - one side and two angles are given (AAS)
  - two sides and a non-included angle are given.

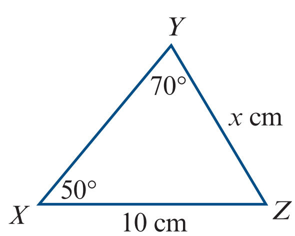
Exercise 13B

Skillsheet
Example 3

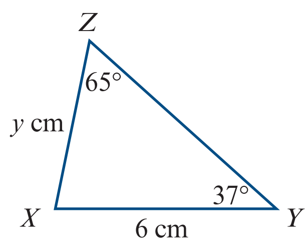
SF

1 Find the value of the pronumeral for each of the following triangles:

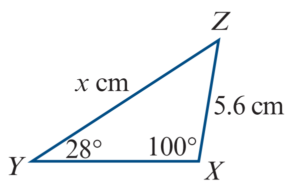
a



b



c



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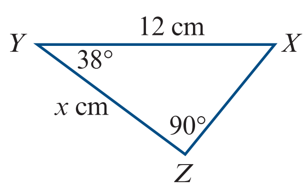

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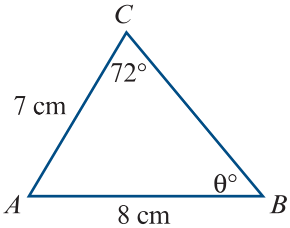
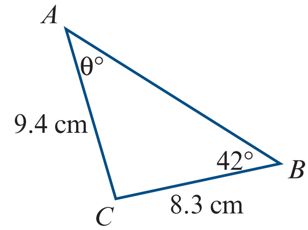
Example 4

2 Find the value of θ for each of the following triangles:

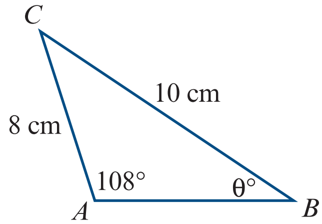
a



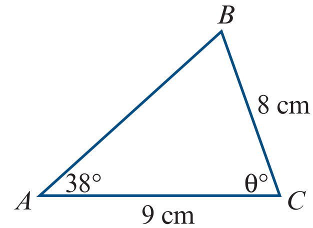
b



c



d



CF

3 Solve the following triangles (i.e. find all sides and angles):

a $a = 12,\ B = 59°,\ C = 73°$ **b** $A = 75.3°,\ b = 5.6,\ B = 48.25°$

c $A = 123.2°,\ a = 11.5,\ C = 37°$ **d** $A = 23°,\ a = 15,\ B = 40°$

e $B = 140°,\ b = 20,\ A = 10°$

SF

4 Solve the following triangles (i.e. find all sides and angles):

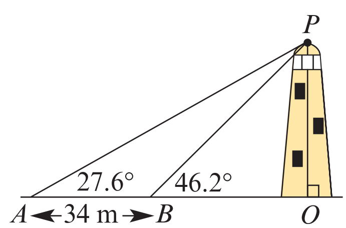
a $b = 17.6,\ C = 48.25°,\ c = 15.3$ **b** $B = 129°,\ b = 7.89,\ c = 4.56$

c $A = 28.35°,\ a = 8.5,\ b = 14.8$

CF

5 A landmark A is observed from two points B and C, which are 400 m apart. The magnitude of angle ABC is measured as 68° and the magnitude of angle ACB as 70°. Find the distance of A from C.

6 P is a point at the top of a lighthouse. Measurements of the length AB and angles PBO and PAO are as shown in the diagram. Find the height of the lighthouse.

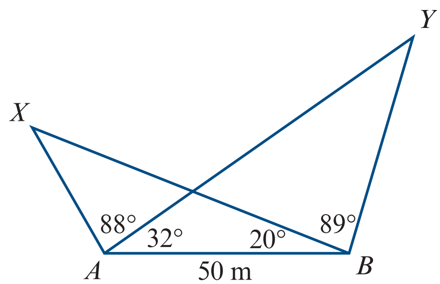


7 A and B are two points on a coastline, and C is a point at sea. The points A and B are 1070 m apart. The angles CAB and CBA have magnitudes of 74° and 69° respectively. Find the distance of C from A.

8 Find:

a AX

b AY



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9 Use the sine rule to establish the following identities for triangles:

a $\dfrac{a+b}{c} = \dfrac{\sin A + \sin B}{\sin C}$ **b** $\dfrac{a-b}{c} = \dfrac{\sin A - \sin B}{\sin C}$

13C The cosine rule

The **cosine rule** is used to find unknown side lengths or angles of a triangle in the following two situations:

1 two sides and the included angle are given

2 three sides are given.

In each case, the triangle is uniquely defined up to congruence.

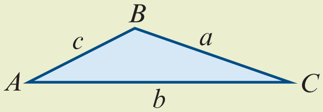
Cosine rule

For triangle ABC:

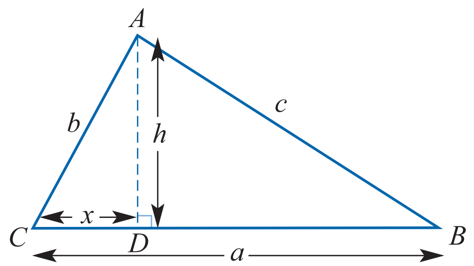
$$c^2 = a^2 + b^2 - 2ab\cos C$$

or equivalently

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



Proof We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.



In triangle ACD:

$$\cos C = \frac{x}{b}$$

$$\therefore \quad x = b\cos C$$

Using Pythagoras' theorem in triangles ACD and ABD:

$$b^2 = x^2 + h^2$$

$$c^2 = (a-x)^2 + h^2$$

Expanding gives

$$\begin{aligned} c^2 &= a^2 - 2ax + x^2 + h^2 \\ &= a^2 - 2ax + b^2 && (\text{as } b^2 = x^2 + h^2) \\ \therefore \quad c^2 &= a^2 + b^2 - 2ab\cos C && (\text{as } x = b\cos C) \end{aligned}$$

Note: By symmetry, the following results also hold:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

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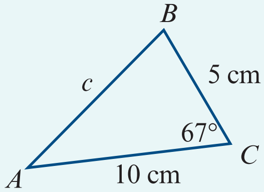
► Two sides and the included angle

When two sides and the included angle are given, this corresponds to the SAS congruence test. The triangle is uniquely defined up to congruence.

Example 5

For triangle ABC, find the length of AB in centimetres correct to two decimal places.



Solution

$$c^2 = 5^2 + 10^2 - 2 \times 5 \times 10 \cos 67^\circ$$
$$= 85.9268\ldots$$
$$\therefore \quad c = 9.2696\ldots$$

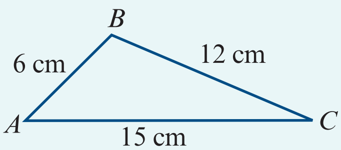
The length of AB is 9.27 cm, correct to two decimal places.

► Three sides

When three sides are given, this corresponds to the SSS congruence test. The triangle is uniquely defined up to congruence.

Example 6

Find the magnitude of angle ABC.



Solution

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{12^2 + 6^2 - 15^2}{2 \times 12 \times 6}$$
$$= -0.3125$$
$$\therefore \quad B = (108.2099\ldots)^\circ$$

The magnitude of angle ABC is 108.21°, correct to two decimal places.

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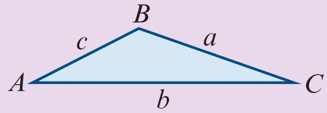
Section summary

- **Cosine rule** For triangle ABC:

$$c^2 = a^2 + b^2 - 2ab\cos C \quad \text{or} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

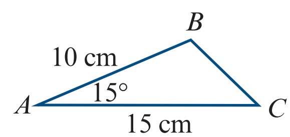
- When to use the cosine rule:
  - two sides and the included angle are given (SAS)
  - three sides are given (SSS).



Exercise 13C

Skillsheet
Example 5

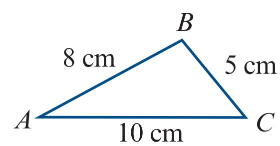
1 Find the length of BC.



SF

Example 6

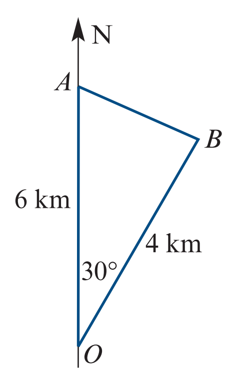
2 Find the magnitudes of angles ABC and ACB.



3 For triangle ABC with:

a $A = 60°$ $b = 16$ $c = 30$, find a
b $a = 14$ $B = 53°$ $c = 12$, find b
c $a = 27$ $b = 35$ $c = 46$, find the magnitude of angle ABC
d $a = 17$ $B = 120°$ $c = 63$, find b
e $a = 31$ $b = 42$ $C = 140°$, find c
f $a = 10$ $b = 12$ $c = 9$, find the magnitude of angle BCA
g $a = 11$ $b = 9$ $C = 43.2°$, find c
h $a = 8$ $b = 10$ $c = 15$, find the magnitude of angle CBA.

4 Two ships sail in different directions from a point O. At a particular time, their positions A and B are as shown. Find the distance between the ships at this time.



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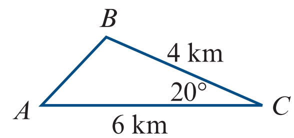
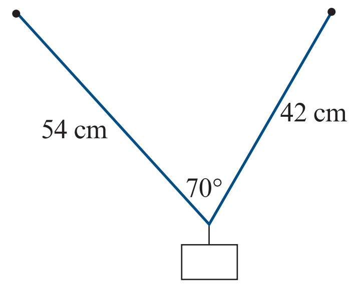
5 A section of an orienteering course is as shown. Find the length of leg AB.



6 A weight is hung from two hooks in a ceiling by strings of length 54 cm and 42 cm, which are inclined at 70° to each other. Find the distance between the hooks.

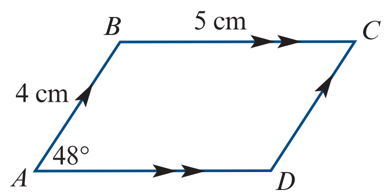


CF

7 $ABCD$ is a parallelogram. Find the lengths of the diagonals:

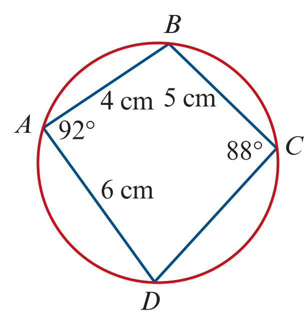
a AC

b BD



8 **a** Find the length of diagonal BD.

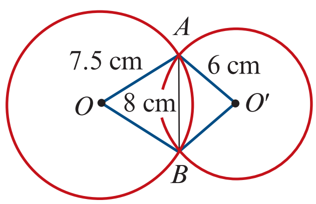
b Use the sine rule to find the length of CD.



9 Two circles of radius 7.5 cm and 6 cm have a common chord of length 8 cm.

a Find the magnitude of angle $AO'B$.

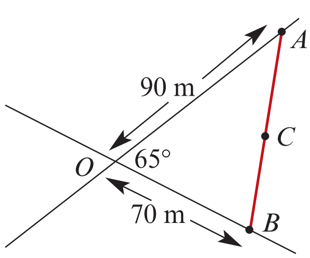
b Find the magnitude of angle AOB.



10 Two straight roads intersect at an angle of 65°. A point A on one road is 90 m from the intersection and a point B on the other road is 70 m from the intersection, as shown.

a Find the distance of A from B.

b If C is the midpoint of AB, find the distance of C from the intersection.



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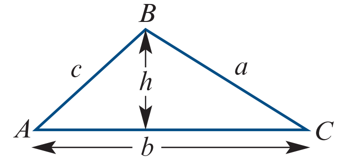
13D The area of a triangle

The area of a triangle is given by

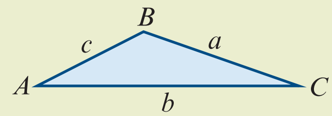
$$\text{Area} = \frac{1}{2} \times \text{base length} \times \text{height}$$
$$= \frac{1}{2}bh$$



By observing that $h = c \sin A$, we obtain the following useful formula.

For triangle ABC:

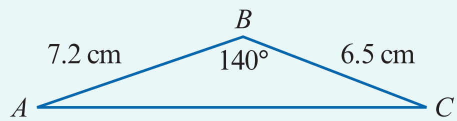
$$\text{Area} = \frac{1}{2}bc \sin A$$



That is, the area is half the product of the lengths of two sides and the sine of the angle included between them.

Example 7

Find the area of triangle ABC shown in the diagram.

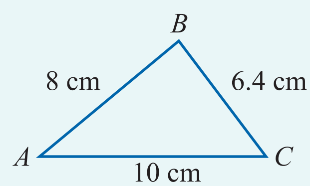


Solution

$$\text{Area} = \frac{1}{2} \times 7.2 \times 6.5 \sin 140^\circ$$
$$= 15.04 \text{ cm}^2 \qquad \text{(correct to two decimal places)}$$

Example 8

Find the area of the triangle, correct to three decimal places.



Solution

Using the cosine rule:

$$8^2 = 6.4^2 + 10^2 - 2 \times 6.4 \times 10 \cos C$$
$$64 = 140.96 - 128 \cos C$$
$$\cos C = 0.60125$$
$$\therefore \quad C^\circ = (53.0405\ldots)^\circ \qquad \text{(store exact value on your calculator)}$$

$$\text{Area } \triangle ABC = \frac{1}{2} \times 6.4 \times 10 \times \sin C$$
$$= 25.570 \text{ cm}^2 \qquad \text{(correct to three decimal places)}$$

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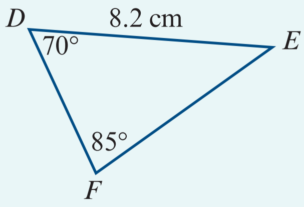
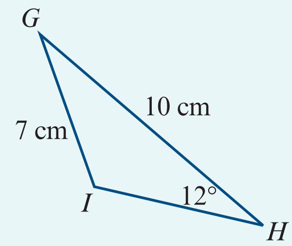
Example 9

Find the area of each of the following triangles, correct to three decimal places:

a



b



Solution

a Note that $E^\circ = (180 - (70 + 85))^\circ = 25^\circ$.

Using the sine rule:

$$\begin{aligned} DF &= \sin 25^\circ \times \frac{8.2}{\sin 85^\circ} \\ &= 3.4787\ldots \qquad \text{(store exact value on your calculator)} \end{aligned}$$

$$\begin{aligned} \text{Area } \triangle DEF &= \frac{1}{2} \times 8.2 \times DF \times \sin 70^\circ \\ &= 13.403 \text{ cm}^2 \qquad \text{(correct to three decimal places)} \end{aligned}$$

b Using the sine rule:

$$\begin{aligned} \sin I &= 10 \times \frac{\sin 12^\circ}{7} \\ &= 0.2970\ldots \end{aligned}$$

$$\begin{aligned} \therefore \quad I^\circ &= (180 - 17.27\ldots)^\circ \qquad \text{(since } I \text{ is an obtuse angle)} \\ &= (162.72\ldots)^\circ \qquad \text{(store exact value on your calculator)} \end{aligned}$$

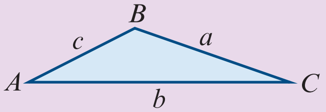
$$\begin{aligned} \therefore \quad G^\circ &= (180 - (12 + I))^\circ \\ &= (5.27\ldots)^\circ \qquad \text{(store exact value on your calculator)} \end{aligned}$$

$$\begin{aligned} \text{Area } \triangle GHI &= \frac{1}{2} \times 10 \times 7 \times \sin G \\ &= 3.220 \text{ cm}^2 \qquad \text{(correct to three decimal places)} \end{aligned}$$

Section summary

For triangle ABC:

$$\text{Area} = \frac{1}{2} bc \sin A$$



That is, the area is half the product of the lengths of two sides and the sine of the angle included between them.

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Exercise 13D

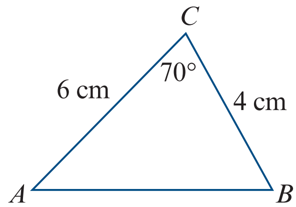
SF

Skillsheet

Example 7

1 Find the area of each of the following triangles:

a



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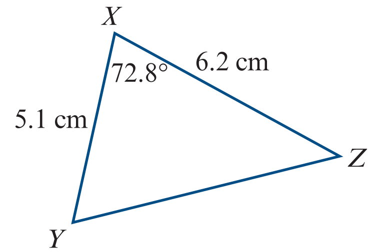
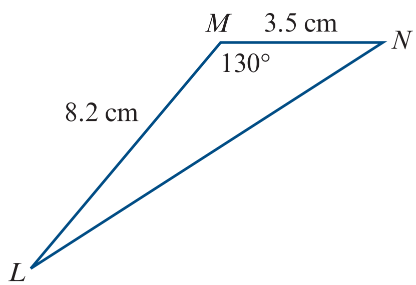
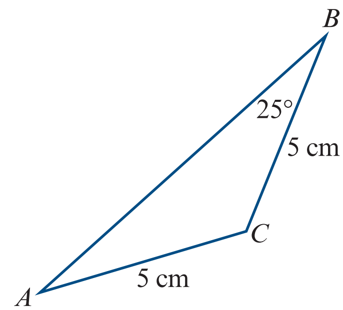


c



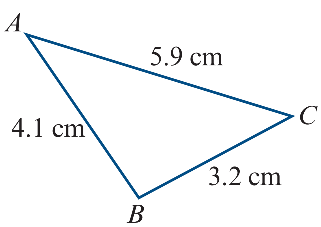
d



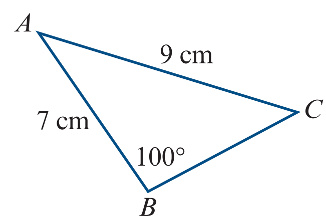
Example 8, 9

2 Find the area of each of the following triangles, correct to three decimal places:

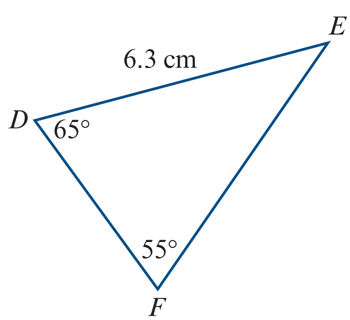
a



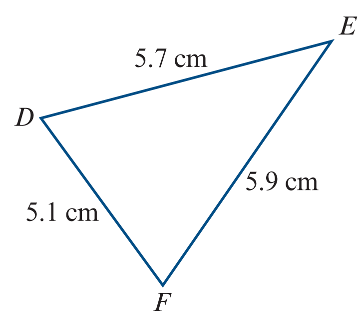
b



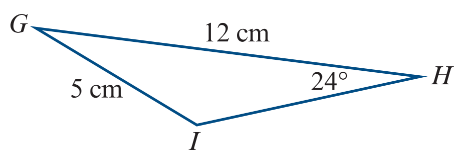
c



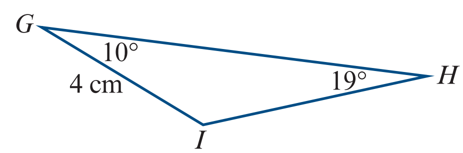
d



e



f



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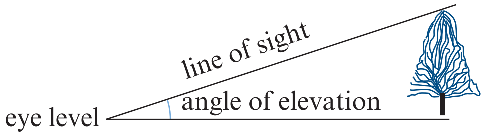
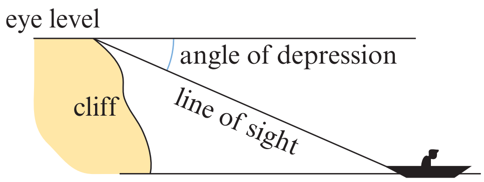
13E Angles of elevation, angles of depression and bearings

The **angle of elevation** is the angle between the horizontal and a direction above the horizontal.



The **angle of depression** is the angle between the horizontal and a direction below the horizontal.



Example 10

The pilot of a helicopter flying at 400 m observes a small boat at an angle of depression of 1.2°. Calculate the horizontal distance of the boat to the helicopter.

Solution

Note that $\angle ABH = 1.2°$, using alternate angles.

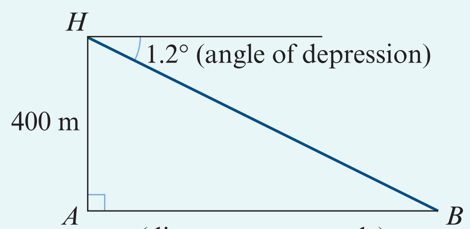
Thus

$$\frac{AH}{AB} = \tan 1.2°$$

$$\frac{400}{AB} = \tan 1.2°$$

$$\therefore \quad AB = \frac{400}{\tan 1.2°}$$

$$= 19\,095.800\ldots$$



(diagram not to scale)

The horizontal distance is 19 100 m, correct to the nearest 10 m.

Example 11

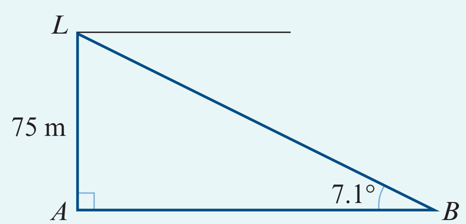
The light on a cliff-top lighthouse, known to be 75 m above sea level, is observed from a boat at an angle of elevation of 7.1°. Calculate the distance of the boat from the lighthouse.

Solution

$$\frac{75}{AB} = \tan 7.1°$$

$$\therefore \quad AB = \frac{75}{\tan 7.1°}$$

$$= 602.135\ldots$$



The distance of the boat from the lighthouse is 602 m, correct to the nearest metre.

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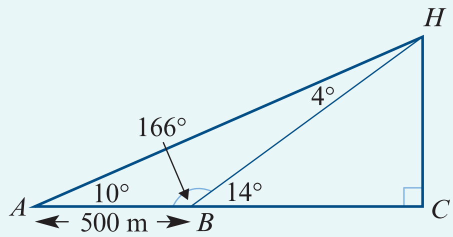
Example 12

From the point A, a man observes that the angle of elevation of the summit of a hill is 10°. He then walks towards the hill for 500 m along flat ground. The summit of the hill is now at an angle of elevation of 14°. Find the height of the hill above the level of A.

Solution



Magnitude of $\angle HBA = (180 - 14)^\circ = 166^\circ$

Magnitude of $\angle AHB = (180 - (166 + 10))^\circ = 4^\circ$

Using the sine rule in triangle ABH:

$$\frac{500}{\sin 4^\circ} = \frac{HB}{\sin 10^\circ}$$

$$\therefore \quad HB = \frac{500 \sin 10^\circ}{\sin 4^\circ}$$

$$= 1244.67\ldots$$

In triangle BCH:

$$\frac{HC}{HB} = \sin 14^\circ$$

$$\therefore \quad HC = HB \sin 14^\circ$$

$$= 301.11\ldots$$

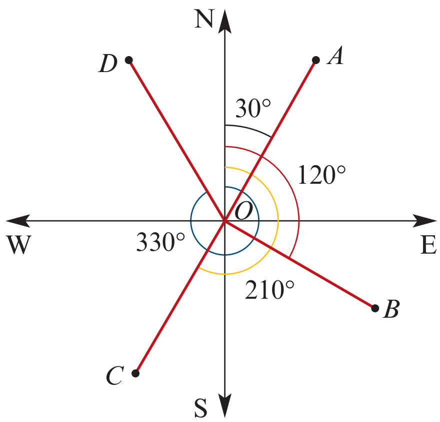
The height of the hill is 301 m, correct to the nearest metre.

Bearings

The **bearing** (or compass bearing) is the direction measured from north clockwise.

For example:

- The bearing of A from O is 030°.
- The bearing of B from O is 120°.
- The bearing of C from O is 210°.
- The bearing of D from O is 330°.



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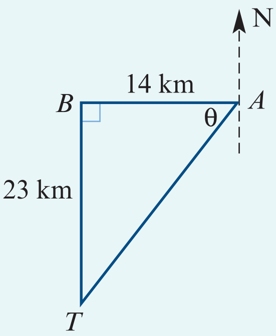
Example 13

The road from town A runs due west for 14 km to town B. A television mast is located due south of B at a distance of 23 km. Calculate the distance and bearing of the mast from the centre of town A.

Solution



$$\tan\theta = \frac{23}{14}$$

$$\therefore \quad \theta = 58.67^\circ \quad \text{(to two decimal places)}$$

Thus the bearing is

$$180^\circ + (90 - 58.67)^\circ = 211.33^\circ$$

To find the distance, use Pythagoras' theorem:

$$\begin{aligned} AT^2 &= AB^2 + BT^2 \\ &= 14^2 + 23^2 \\ &= 725 \end{aligned}$$

$$\therefore \quad AT = 26.925\ldots$$

The mast is 27 km from the centre of town A (to the nearest kilometre) and on a bearing of 211.33°.

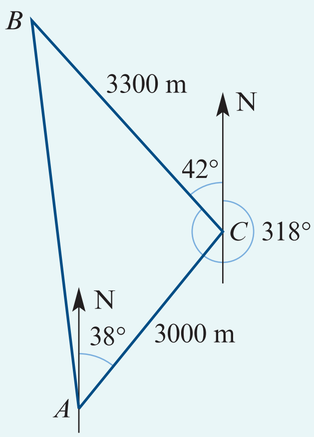
Example 14

A yacht starts from a point A and sails on a bearing of 038° for 3000 m. It then alters its course to a bearing of 318° and after sailing for a further 3300 m reaches a point B. Find:

a the distance AB

b the bearing of B from A.

Solution



a The magnitude of angle ACB needs to be found so that the cosine rule can be applied in triangle ABC:

$$\angle ACB = (180 - (38 + 42))^\circ = 100^\circ$$

In triangle ABC:

$$\begin{aligned} AB^2 &= 3000^2 + 3300^2 - 2 \times 3000 \times 3300 \cos 100^\circ \\ &= 23\,328\,233.917\ldots \end{aligned}$$

$$\therefore \quad AB = 4829.931\ldots$$

The distance of B from A is 4830 m (to the nearest metre).

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b To find the bearing of B from A, the magnitude of angle BAC must first be found. Using the sine rule:

$$\frac{3300}{\sin A} = \frac{AB}{\sin 100^\circ}$$

$$\therefore \quad \sin A = \frac{3300 \sin 100^\circ}{AB}$$

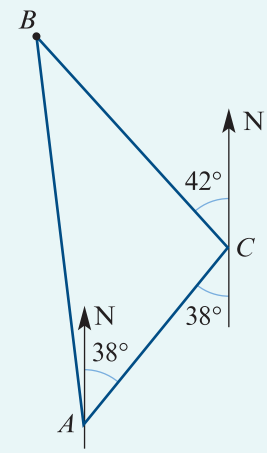
$$= 0.6728\ldots$$

$$\therefore \quad A = (42.288\ldots)^\circ$$

The bearing of B from A $= 360^\circ - (42.29^\circ - 38^\circ)$

$= 355.71^\circ$

The bearing of B from A is 356° to the nearest degree.



Exercise 13E

SF

Example 10 **1** From the top of a vertical cliff 130 m high, the angle of depression of a buoy at sea is 18°. What is the distance of the buoy from the foot of the cliff?

Example 11 **2** The angle of elevation of the top of an old chimney stack at a point 40 m from its base is 41°. Find the height of the chimney.

3 A hiker standing on top of a mountain observes that the angle of depression to the base of a building is 41°. If the height of the hiker above the base of the building is 500 m, find the horizontal distance from the hiker to the building.

4 A person lying down on top of a cliff 40 m high observes the angle of depression to a buoy in the sea below to be 20°. If the person is in line with the buoy, find the distance between the buoy and the base of the cliff, which may be assumed to be vertical.

CF

Example 12 **5** A person standing on top of a cliff 50 m high is in line with two buoys whose angles of depression are 18° and 20°. Calculate the distance between the buoys.

SF

Example 13 **6** A ship sails 10 km north and then sails 15 km east. What is its bearing from the starting point?

7 A ship leaves port A and travels 15 km due east. It then turns and travels 22 km due north.

a What is the bearing of the ship from port A?

b What is the bearing of port A from the ship?

Example 14 **8** A yacht sails from point A on a bearing of 035° for 2000 m. It then alters course to a direction with a bearing of 320° and after sailing for 2500 m it reaches point B.

a Find the distance AB.

b Find the bearing of B from A.

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9 The bearing of a point A from a point B is 207°. What is the bearing of B from A?

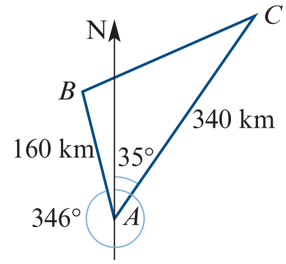
10 The bearing of a ship S from a lighthouse A is 055°. A second lighthouse B is due east of A. The bearing of S from B is 302°. Find the magnitude of angle ASB.

11 A yacht starts from L and sails 12 km due east to M. It then sails 9 km on a bearing of 142° to K. Find the magnitude of angle MLK.

12 The bearing of C from A is 035°. The bearing of B from A is 346°. The distance of C from A is 340 km. The distance of B from A is 160 km.

a Find the magnitude of angle BAC.

b Use the cosine rule to find the distance from B to C.



13 From a ship S, two other ships P and Q are on bearings 320° and 075° respectively. The distance PS is 7.5 km and the distance QS is 5 km. Find the distance PQ.

13F Problems in three dimensions

Some problems in three dimensions can be solved by picking out triangles from a main figure and finding lengths and angles through these triangles.

Example 15

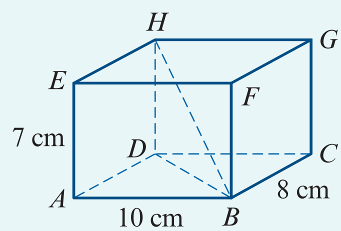
$ABCDEFGH$ is a cuboid. Find:

a the distance DB

b the distance HB

c the magnitude of angle HBD

d the magnitude of angle HBA.



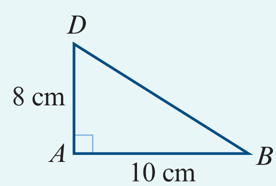
Solution

a $DB^2 = 8^2 + 10^2$

$= 164$

$\therefore\ DB = \sqrt{164}$

$= 12.81$ cm (correct to two decimal places)



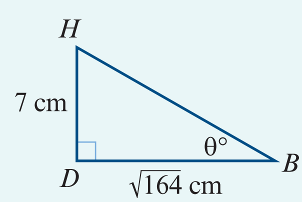
b $HB^2 = HD^2 + DB^2$

$= 7^2 + 164$

$= 213$

$\therefore\ HB = \sqrt{213}$

$= 14.59$ cm (correct to two decimal places)



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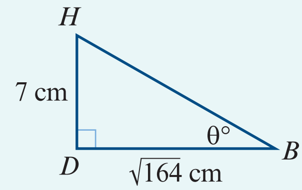
c $\tan\theta = \dfrac{HD}{BD}$

$= \dfrac{7}{\sqrt{164}}$

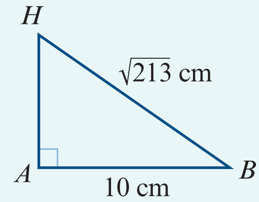
$\therefore\ \theta = 28.66^\circ$ (correct to two decimal places)



d From triangle HBA:

$\cos B = \dfrac{10}{\sqrt{213}}$

$\therefore\ B = 46.75^\circ$ (correct to two decimal places)



Example 16

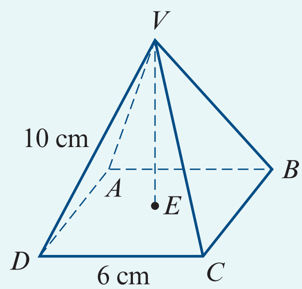
The figure shows a pyramid with a square base. The base has sides 6 cm long and the edges VA, VB, VC and VD are each 10 cm long.

a Find the length of DB.

b Find the length of BE.

c Find the length of VE.

d Find the magnitude of angle VBE.



Solution

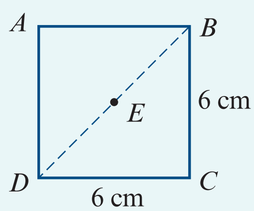
a $DB^2 = 6^2 + 6^2$

$= 72$

$\therefore\ DB = \sqrt{72}$

$= 6\sqrt{2}$

$= 8.4852\ldots$



The length of DB is 8.49 cm, correct to two decimal places.

b $BE = \dfrac{1}{2}DB$

$= 3\sqrt{2}$

$= 4.2426\ldots$

The length of BE is 4.24 cm, correct to two decimal places.

c $VE^2 = VB^2 - BE^2$

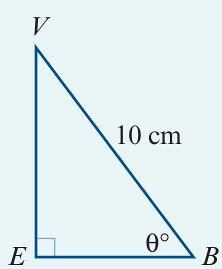
$= 10^2 - (3\sqrt{2})^2$

$= 100 - 18$

$= 82$

$\therefore\ VE = \sqrt{82}$

$= 9.0553\ldots$



The length of VE is 9.06 cm, correct to two decimal places.

d $\sin\theta = \dfrac{VE}{VB}$

$= \dfrac{\sqrt{82}}{10}$

$= 0.9055\ldots$

$\therefore\ \theta = (64.8959\ldots)^\circ$

The magnitude of $\angle VBE$ is 64.90°, correct to two decimal places.

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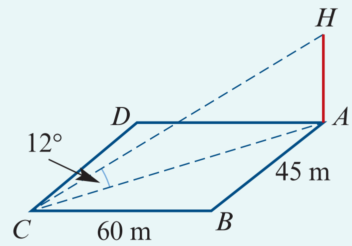
Example 17

A communications mast is erected at corner A of a rectangular courtyard $ABCD$ with side lengths 60 m and 45 m as shown. If the angle of elevation of the top of the mast from C is 12°, find:

a the height of the mast

b the angle of elevation of the top of the mast from B.

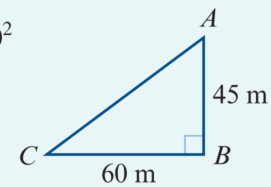


Solution

a $AC^2 = 45^2 + 60^2$

$= 5625$

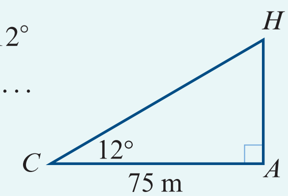
$\therefore \quad AC = 75$



$\dfrac{HA}{75} = \tan 12°$

$\therefore \quad HA = 75 \tan 12°$

$= 15.941\ldots$

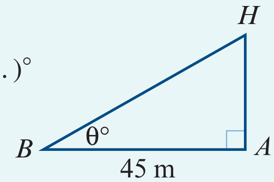


The height of the mast is 15.94 m, correct to two decimal places.

b $\tan\theta = \dfrac{HA}{45}$

$= 0.3542\ldots$

$\therefore \quad \theta = (19.507\ldots)°$



The angle of elevation of the top of the mast, H, from B is 19.51°, correct to two decimal places.

Exercise 13F

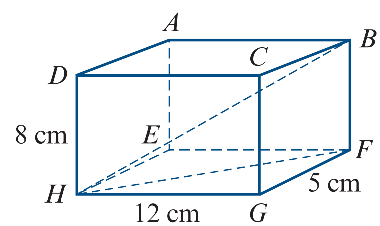
Example 15

1 $ABCDEFGH$ is a cuboid with dimensions as shown. Find:

a the length of FH **b** the length of BH

c the magnitude of angle BHF

d the magnitude of angle BHG.



Example 16

2 $VABCD$ is a right pyramid with a square base. The sides of the base are 8 cm in length. The height, VF, of the pyramid is 12 cm. If E is the midpoint of AD, find:

a the length of EF

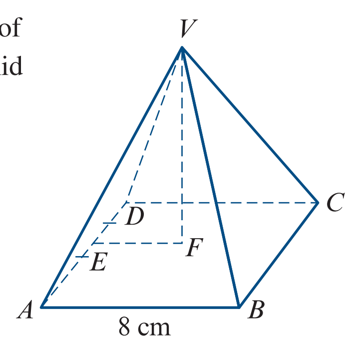
b the magnitude of angle VEF

c the length of VE

d the length of a sloping edge

e the magnitude of angle VAD

f the surface area of the pyramid.



CF

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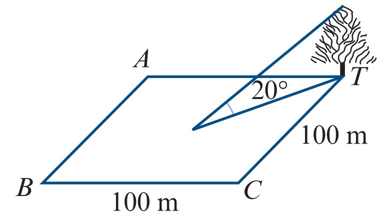
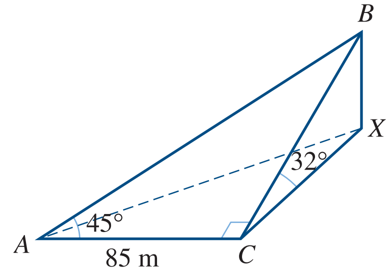
Example 17

3 A tree stands at a corner of a square playing field. Each side of the square is 100 m long. At the centre of the field, the tree subtends an angle of 20°. What angle does it subtend at each of the other three corners of the field?



4 Suppose that A, C and X are three points in a horizontal plane and that B is a point vertically above X. The length of AC is 85 m and the magnitudes of angles BAC, ACB and BCX are 45°, 90° and 32° respectively. Find:

a the distance CB **b** the height XB.

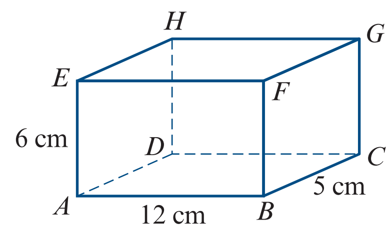


5 Standing due south of a tower 50 m high, the angle of elevation of the top is 26°. What is the angle of elevation after walking a distance 120 m due east?

6 From the top of a cliff 160 m high, two buoys are observed. Their bearings are 337° and 308°. Their respective angles of depression are 3° and 5°. Calculate the distance between the buoys.

7 Find the magnitude of each of the following angles for the cuboid shown:

a ACE **b** HDF **c** ECH



8 From a point A due north of a tower, the angle of elevation to the top of the tower is 45°. From point B, which is 100 m from A on a bearing of 120°, the angle of elevation is 26°. Find the height of the tower.

9 A and B are two positions on level ground. From an advertising balloon at a vertical height of 750 m, point A is observed in an easterly direction and point B at a bearing of 160°. The angles of depression of A and B, as viewed from the balloon, are 40° and 20° respectively. Find the distance between A and B.

10 A right pyramid, height 6 cm, stands on a square base of side length 5 cm. Find:

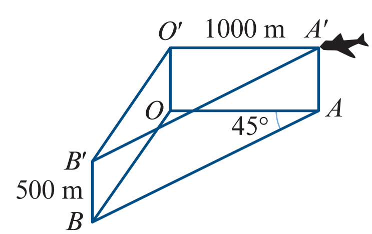
a the length of a sloping edge **b** the area of a triangular face.

11 A light aircraft flying at a height of 500 m above the ground is sighted at a point A' due east of an observer at a point O on the ground, measured horizontally to be 1 km from the plane. The aircraft is flying south-west (along $A'B'$) at 300 km/h.

a How far will it travel in one minute?

b Find its bearing from O (O') at this time.

c What will be its angle of elevation from O at this time?



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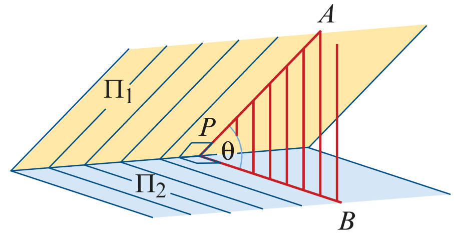
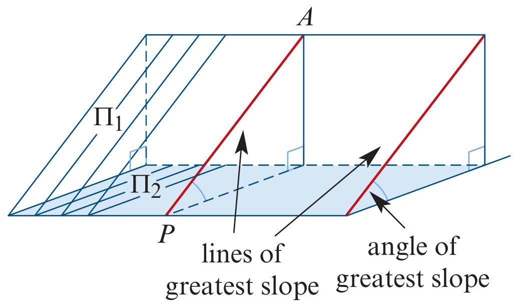
13G Angles between planes and more complex 3D problems

▶ Angles between planes



Consider any point P on the common line of two planes Π_1 and Π_2. If lines PA and PB are drawn at right angles to the common line so that PA is in Π_1 and PB is in Π_2, then $\angle APB$ is the angle between planes Π_1 and Π_2.

Note: If the plane Π_2 is horizontal, then PA is called a **line of greatest slope** in the plane Π_1.

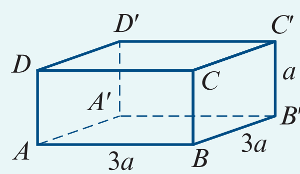


Example 18

For the cuboid shown in the diagram, find:

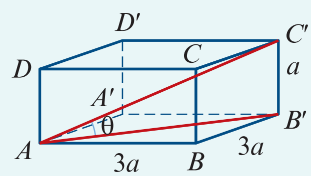
a the angle between AC' and the plane $ABB'A'$

b the angle between the planes ACD' and DCD'.



Solution

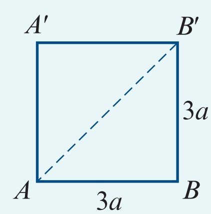
a To find the angle θ between AC' and the plane $ABB'A'$, we need the projection of AC' in the plane.



We drop a perpendicular from C' to the plane (line $C'B'$), and join the foot of the perpendicular to A (line $B'A$).

The required angle, θ, lies between $C'A$ and $B'A$.

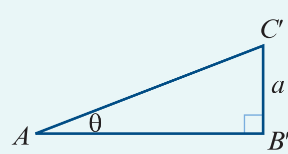
Draw separate diagrams showing the base and the section through A, C' and B'. Then we see that



$$AB' = \sqrt{(3a)^2 + (3a)^2} = 3a\sqrt{2}$$

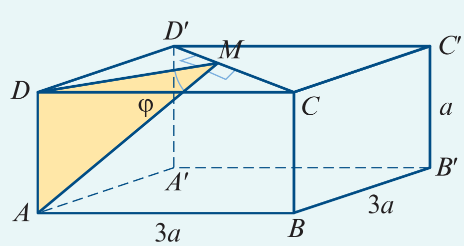
and $\quad \tan\theta = \dfrac{a}{3a\sqrt{2}} = \dfrac{1}{3\sqrt{2}}$

Hence the required angle, θ, is 13.26°.



b The line common to the planes ACD' and DCD' is CD'. Let M be the midpoint of the line segment CD'.

Then MD is perpendicular to CD' in the plane DCD', and MA is perpendicular to CD' in the plane ACD'.



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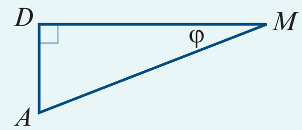
Thus φ is the angle between the planes DCD' and ACD'.

We have

$$DM = \frac{1}{2}DC' = \frac{1}{2}(3a\sqrt{2})$$

$$\therefore \quad \tan\varphi = a \div \left(\frac{3a\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{3}$$

Hence the required angle is $\varphi = 25.24°$.



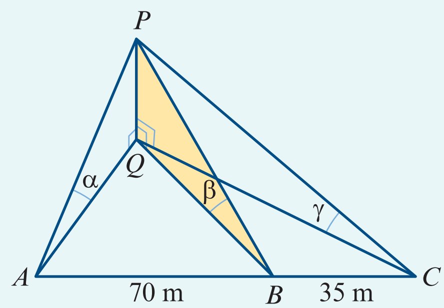
Example 19

Three points A, B and C are on a horizontal line such that $AB = 70$ m and $BC = 35$ m. The angles of elevation of the top of a tower are α, β and γ, where

$$\tan\alpha = \frac{1}{13}, \quad \tan\beta = \frac{1}{15}, \quad \tan\gamma = \frac{1}{20}$$

as shown in the diagram.

The base of the tower is at the same level as A, B and C. Find the height of the tower.



Solution

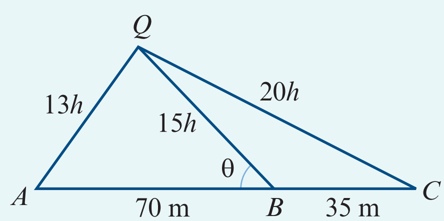
Let the height of the tower, PQ, be h m. Then

$$h = QA\tan\alpha = QB\tan\beta = QC\tan\gamma$$

which implies that

$$QA = 13h, \quad QB = 15h, \quad QC = 20h$$

Now consider the base triangle ACQ.



Using the cosine rule in $\triangle AQB$:

$$\cos\theta = \frac{(70)^2 + (15h)^2 - (13h)^2}{2(70)(15h)}$$

Using the cosine rule in $\triangle CQB$:

$$-\cos\theta = \cos(180° - \theta) = \frac{(35)^2 + (15h)^2 - (20h)^2}{2(35)(15h)}$$

Hence

$$\frac{(70)^2 + (15h)^2 - (13h)^2}{2(70)(15h)} = \frac{(20h)^2 - (15h)^2 - (35)^2}{2(35)(15h)}$$

$$4900 + 56h^2 = 2(175h^2 - 1225)$$

$$7350 = 294h^2$$

$$\therefore \quad h = 5$$

The height of the tower is 5 m.

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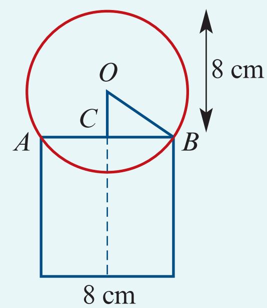
Example 20

A sphere rests on the top of a vertical cylinder which is open at the top. The inside diameter of the cylinder is 8 cm. The sphere projects 8 cm above the top of the cylinder. Find the radius length of the sphere.

Solution

This 3D problem can be represented by a 2D diagram without loss of information.

Let the radius length of the sphere be r cm. Then, in $\triangle OBC$, we have



$$OC = (8 - r) \text{ cm}, \quad BC = 4 \text{ cm}, \quad OB = r \text{ cm}$$

Using Pythagoras' theorem:

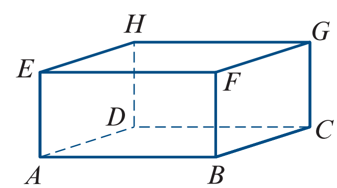
$$\begin{aligned}(8 - r)^2 + 4^2 &= r^2 \\ 64 - 16r + r^2 + 16 &= r^2 \\ -16r + 80 &= 0 \\ \therefore \quad r &= 5\end{aligned}$$

The radius length of the sphere is 5 cm.

Exercise 13G

Example 18

1 The diagram shows a rectangular prism. Assume that $AB = 4a$ units, $BC = 3a$ units, $GC = a$ units.



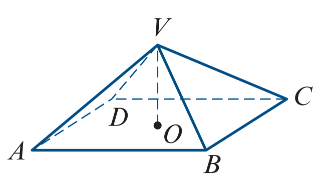
a Calculate the areas of the faces $ABFE$, $BCGF$ and $ABCD$.

b Calculate the magnitude of the angle which plane $GFAD$ makes with the base.

c Calculate the magnitude of the angle which plane $HGBA$ makes with the base.

d Calculate the magnitude of the angle which AG makes with the base.

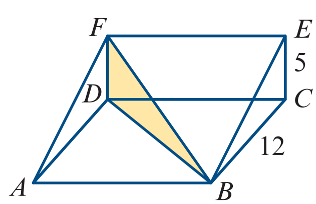
2 $VABCD$ is a right pyramid with square base $ABCD$, and with $AB = 2a$ and $OV = a$.



a Find the slope of the edge VA. That is, find the magnitude of $\angle VAO$.

b Find the slope of the face VBC.

3 A hill has gradient $\frac{5}{12}$. If BF makes an angle of 45° with the line of greatest slope, find:



a the gradient of BF

b the magnitude of $\angle FBD$.

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4 The cross-section of a right prism is an isosceles triangle ABC with $AB = BC = 16$ cm and $\angle ABC = 58°$. The equal edges AD, BE and CF are parallel and of length 12 cm. Calculate:

a the length of AC **b** the length of AE **c** the angle between AE and EC.

Example 19 **5** A vertical tower, AT, of height 50 m, stands at a point A on a horizontal plane. The points A, B and C lie on the same horizontal plane, where B is due west of A and C is due south of A. The angles of elevation of the top of the tower, T, from B and C are 25° and 30° respectively.

a Giving answers to the nearest metre, calculate the distances:

i AB **ii** AC **iii** BC

b Calculate the angle of elevation of T from the midpoint, M, of AB.

6 A right square pyramid, vertex O, stands on a square base $ABCD$. The height is 15 cm and the base side length is 10 cm. Find:

a the length of the slant edge

b the inclination of a slant edge to the base

c the inclination of a sloping face to the base

d the magnitude of the angle between two adjacent sloping faces.

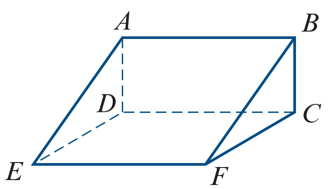
7 A post stands at one corner of a rectangular courtyard. The elevations of the top of the post from the nearest corners are 30° and 45°. Find the elevation from the diagonally opposite corner.

8 $VABC$ is a regular tetrahedron with base $\triangle ABC$. (All faces are equilateral triangles.) Find the magnitude of the angle between:

a a sloping edge and the base **b** adjacent sloping faces.

9 An observer at a point A at sea level notes an aircraft due east at an elevation of 35°. At the same time an observer at B, which is 2 km due south of A, reports the aircraft on a bearing of 50°. Calculate the altitude of the aircraft.

10 $ABFE$ represents a section of a ski run which has a uniform inclination of 30° to the horizontal, with $AE = 100$ m and $AB = 100$ m. A skier traverses the slope from A to F. Calculate:



a the distance that the skier has traversed

b the inclination of the skier's path to the horizontal.

Example 20 **11** A sphere of radius length 8 cm rests on the top of a hollow inverted cone of height 15 cm whose vertical angle is 60°. Find the height of the centre of the sphere above the vertex of the cone.

12 A cube has edge length a cm. What is the radius length, in terms of a, of:

a the sphere that just contains the cube

b the sphere that just fits inside the cube?

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Review

Chapter summary

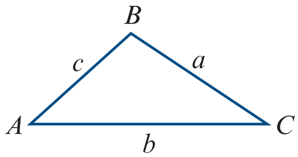
AS
Nrich

Triangles

Labelling triangles

- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.

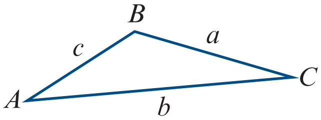
For example, the magnitude of angle BAC is denoted by A, and the length of side BC by a.



Sine rule

For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



The sine rule is used to find unknown quantities in a triangle in the following cases:

- one side and two angles are given
- two sides and a non-included angle are given.

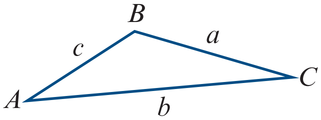
In the first case, the triangle is uniquely defined. But in the second case, there may be two triangles.

Cosine rule

For triangle ABC:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



The symmetrical results also hold:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

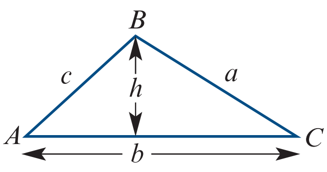
The cosine rule is used to find unknown quantities in a triangle in the following cases:

- two sides and the included angle are given
- three sides are given.

Area of a triangle

$$\text{Area} = \frac{1}{2}bh$$

$$\text{Area} = \frac{1}{2}bc\sin A$$



That is, the area of a triangle is half the product of the lengths of two sides and the sine of the angle included between them.

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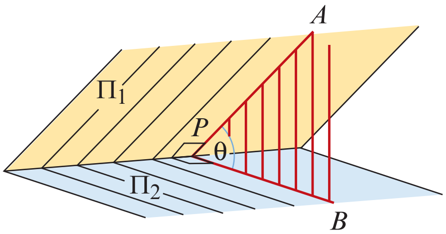
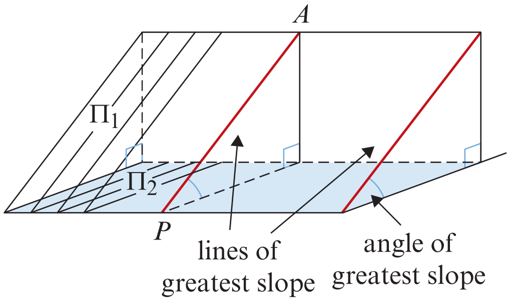
Review

Angle between planes

- Consider any point P on the common line of two planes Π_1 and Π_2. If lines PA and PB are drawn at right angles to the common line so that PA is in Π_1 and PB is in Π_2, then $\angle APB$ is the angle between Π_1 and Π_2.

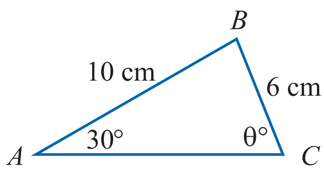


- If plane Π_2 is horizontal, then PA is called a **line of greatest slope** in plane Π_1.



Technology-free questions

1 For the diagram shown, find the value of $\sin(\theta°)$.



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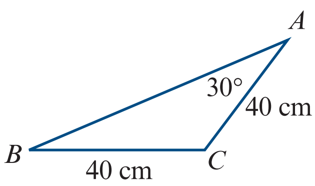
2 **a** Find $\angle ABC$ and $\angle ACB$.
b Find the length of side AB.
c Find the distance CM, where M is the midpoint of side AB.



3 Triangle ABC has $AB = BC = 10$ cm and $\angle ABC = 120°$. Find AC.

4 From a port P, a ship Q is 20 km away on a bearing of 120°, and a ship R is 12 km away on a bearing of 060°. Find the distance between the two ships.

5 Calculate the cosine of the largest angle in the triangle with sides of length 9 cm, 11 cm and 13 cm.

6 In a quadrilateral $ABCD$, $AB = 5$ cm, $BC = 5$ cm, $CD = 7$ cm, $B = 120°$ and $C = 90°$. Find:

CF

a the length of the diagonal AC
b the area of triangle ABC
c the area of triangle ADC
d the area of the quadrilateral.

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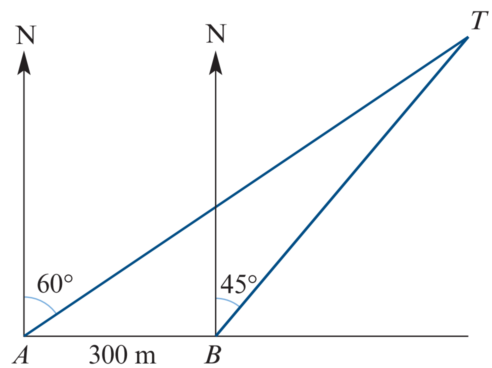
7 If $\sin x = \sin 37°$ and x is obtuse, find x.

8 A point T is 10 km due north of a point S. A point R, which is east of the straight line joining T and S, is 8 km from T and 7 km from S. Calculate the cosine of the bearing of R from S.

9 In $\triangle ABC$, $AB = 5$ cm, $\angle BAC = 60°$ and $AC = 6$ cm. Calculate the sine of $\angle ABC$.

10 An acute-angled triangle PQR has area 10.8 cm^2 and side lengths $PQ = 6$ cm and $QR = 4$ cm. Find $\sin(\angle PQR)$.

11 The diagram shows two survey points, A and B, which are on an east–west line on level ground. From point A, the bearing of a tower T is 060°, while from point B, the bearing of the tower is 045°.



a **i** Find the magnitude of $\angle TAB$.

ii Find the magnitude of $\angle ATB$.

b Given that $\sin 15° = \dfrac{\sqrt{6} - \sqrt{2}}{4}$, find the distances AT and BT.

12 A boat sails 11 km from a harbour on a bearing of 210°. It then sails 15 km on a bearing of 330°. How far is the boat from the harbour?

13 A helicopter leaves a heliport A and flies 2.4 km on a bearing of 150° to a checkpoint B. It then flies due east to its base C.

a If the bearing of C from A is 120°, find the distances AC and BC.

b The helicopter flies at a constant speed throughout and takes five minutes to fly from A to C. Find its speed.

14 From a cliff top 11 m above sea level, two boats are observed. One has an angle of depression of 45° and is due east, the other an angle of depression of 30° on a bearing of 120°. Calculate the distance between the boats.

Multiple-choice questions

1 In a triangle XYZ, $x = 21$ cm, $y = 18$ cm and $\angle YXZ = 62°$. The magnitude of $\angle XYZ$, correct to one decimal place, is

A 0.4° **B** 0.8° **C** 1.0° **D** 49.2° **E** 53.1°

2 In a triangle ABC, $a = 30$, $b = 21$ and $\cos C = \dfrac{51}{53}$. The value of c, to the nearest whole number, is

A 9 **B** 10 **C** 11 **D** 81 **E** 129

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3 In a triangle ABC, $a = 5.2$ cm, $b = 6.8$ cm and $c = 7.3$ cm. The magnitude of $\angle ACB$, correct to the nearest degree, is

A 43° **B** 63° **C** 74° **D** 82° **E** 98°

4 The area of the triangle ABC, where $b = 5$ cm, $c = 3$ cm, $\angle A = 30°$, is

A 2.75 cm^2 **B** 3.75 cm^2 **C** 6.5 cm^2 **D** 7.5 cm^2 **E** 8 cm^2

5 From a point on a cliff 500 m above sea level, the angle of depression to a boat is 20°. The distance from the foot of the cliff to the boat, to the nearest metre, is

A 182 m **B** 193 m **C** 210 m **D** 1374 m **E** 1834 m

6 A tower 80 m high is 1.3 km away from a point on the ground. The angle of elevation to the top of the tower from this point, correct to the nearest degree, is

A 1° **B** 4° **C** 53° **D** 86° **E** 89°

7 A man walks 5 km due east followed by 7 km due south. The bearing he must take to return to the start is

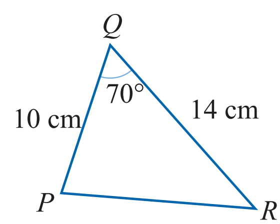
A 036° **B** 306° **C** 324° **D** 332° **E** 348°

8 A boat sails at a bearing of 215° from A to B. The bearing it must take from B to return to A is

A 035° **B** 055° **C** 090° **D** 215° **E** 250°

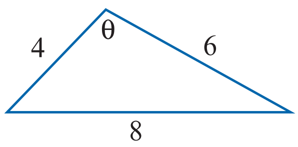
9 The area of triangle PQR is closest to

A 24 cm^2 **B** 54 cm^2 **C** 66 cm^2
D 70 cm^2 **E** 140 cm^2



10 For the diagram shown, the value of $\cos\theta$ is

A $-\frac{1}{4}$ **B** $-\frac{1}{2}$ **C** $\frac{1}{4}$
D $\frac{1}{2}$ **E** $\frac{3}{4}$



11 A hiker starts at a point P. She walks 5 km on a bearing of 030° to point Q, and then walks 10 km on a bearing of 330° to point R. How far west of point P is point R?

A 2.5 km **B** 5 km **C** 7.5 km **D** 10 km **E** 15 km

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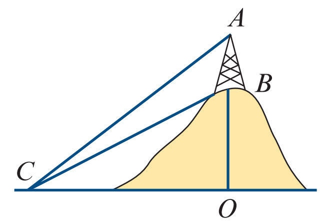
Extended-response questions

1 AB is a tower 60 m high on top of a hill. The magnitude of $\angle ACO$ is 49° and the magnitude of $\angle BCO$ is 37°.

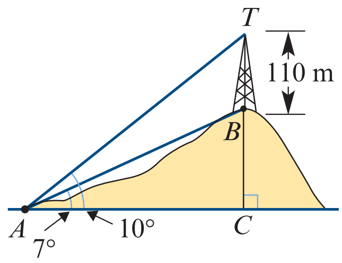


a Find the magnitudes of $\angle ACB$, $\angle CBO$ and $\angle CBA$.

b Find the length of BC.

c Find the height of the hill, i.e. the length of OB.

2 A tower 110 m high stands on the top of a hill. From a point A at the foot of the hill, the angle of elevation of the bottom of the tower is 7° and that of the top is 10°.

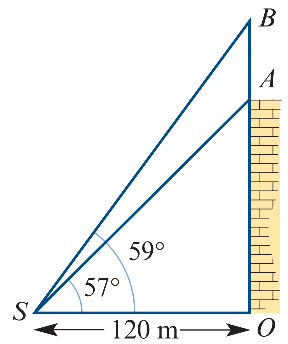


a Find the magnitudes of angles TAB, ABT and ATB.

b Use the sine rule to find the length of AB.

c Find CB, the height of the hill.

3 Point S is a distance of 120 m from the base of a building. On the building is an aerial, AB. The angle of elevation from S to A is 57°. The angle of elevation from S to B is 59°. Find:

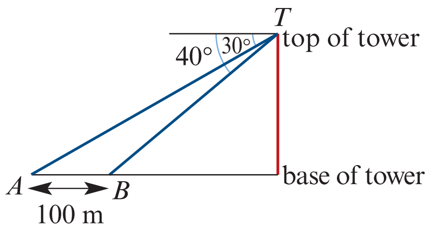


a the distance OA

b the distance OB

c the distance AB.

4 From the top of a communications tower, T, the angles of depression of two points A and B on a horizontal line through the base of the tower are 30° and 40°. The distance between the points is 100 m. Find:



a the distance AT

b the distance BT

c the height of the tower.

5 Point X is 10 km due south of point Y. Point Z, which lies to the east of the line XY, is 5 km from Y and 9 km from X.

a Calculate the bearing of Z from X.

b A fourth point, W, is on a bearing of 319° from X and it is 12 km from Y.

i Find $\angle XWY$.

ii Hence calculate the bearing of Y from W.

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Review

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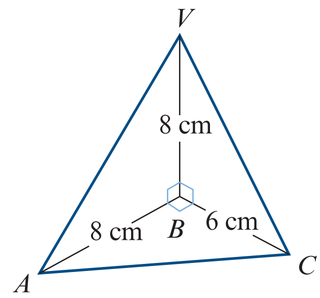
6 Angles VBA, VBC and ABC are right angles. Find:

a the distance VA

b the distance VC

c the distance AC

d the magnitude of angle VCA.



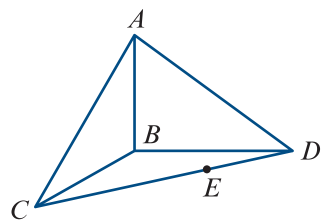
7 A surveyor measures the angle of elevation of the top of a mountain from a point at sea level as 20°. She then travels 1000 m along a road that slopes uniformly uphill towards the mountain. From this point, which is 100 m above sea level, she measures the angle of elevation as 23°. Find the height of the mountain above sea level, correct to the nearest metre.

8 In the diagram, the edge AB is vertical, $\triangle BCD$ is horizontal, $\angle CBD$ is a right angle and $AB = 20$ m, $BD = 40$ m, $BC = 30$ m. Calculate the inclination to the horizontal of:

a AD

b AE, where AE is the line of greatest slope

c AE, where E is the midpoint of CD.



9 The perimeter of a triangle ABC is L metres. Find the area of the triangle in terms of L and the triangle's angles α, β and γ.

Hint: Let $AB = x$. Using the sine rule, first find the other side lengths in terms of x.

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Chapter 14

14 Refresher on probability and discrete random variables

Objectives

- To revise the basic concepts of **probability**.
- To define **discrete random variables**.
- To define the **probability distribution** of a discrete random variable.
- To calculate and interpret **expected value** (**mean**) for a discrete random variable.
- To calculate and interpret **variance** and **standard deviation** for a discrete random variable.
- To illustrate the property that for many random variables **approximately 95% of the distribution is within two standard deviations of the mean**.

Uncertainty is involved in much of the reasoning we undertake every day of our lives. We are often required to make decisions based on the chance of a particular occurrence. Some events can be predicted from our present store of knowledge, such as the time of the next high tide. Others, such as whether a head or tail will show when a coin is tossed, are not predictable.

Ideas of uncertainty are pervasive in everyday life, and the use of chance and risk models makes an important impact on many human activities and concerns. Probability is the study of chance and uncertainty.

In this chapter, we revise important concepts from your study of probability and discrete random variables in Mathematical Methods Units 1 & 2. In particular, we will consider the concept of the probability distribution for a discrete random variable. Using this distribution, we can determine the theoretical values of two important parameters which describe the random variable: the mean and the standard deviation. We will see that, together, the mean and the standard deviation can tell us a lot about the distribution of a random variable.

ISBN 978-1-108-45164-2

14A Sample spaces and probability

In this section we will review the fundamental concepts of probability, the numerical value which we assign to give a measure of the likelihood of an outcome of an experiment. Probability takes a value between 0 and 1, where a probability of 0 means that the outcome is impossible, and a probability of 1 means that it is certain. Generally, the probability of an outcome will be somewhere in between, with a higher value meaning that the outcome is more likely.

► Sample spaces and events

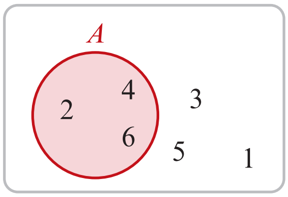
When a six-sided die is rolled, the possible outcomes are the numbers 1, 2, 3, 4, 5, 6. Rolling a six-sided die is an example of a **random experiment**, since while we can list all the possible outcomes, we do not know which one will be observed.

The possible outcomes are generally listed as the elements of a set, and the set of all possible outcomes is called the **sample space** and denoted by the Greek letter ε (epsilon). Thus, for this example:

$$\varepsilon = \{1, 2, 3, 4, 5, 6\}$$

An **event** is a subset of the sample space, usually denoted by a capital letter. If the event A is defined as ‘an even number when a six-sided die is rolled’, we write

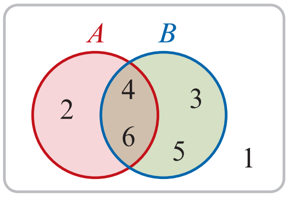
$$A = \{2, 4, 6\}$$



If A and B are two events, then the **union** of A and B, denoted by $A \cup B$, is equivalent to either event A or event B or both occurring.

Thus, if event A is ‘an even number when a six-sided die is rolled’ and event B is ‘a number greater than 2 when a six-sided die is rolled’, then $A = \{2, 4, 6\}$, $B = \{3, 4, 5, 6\}$ and

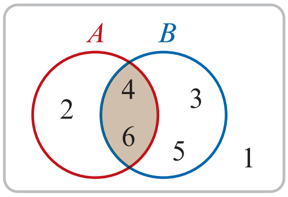
$$A \cup B = \{2, 3, 4, 5, 6\}$$



The **intersection** of A and B, denoted by $A \cap B$, is equivalent to both event A and event B occurring.

Thus, using the events A and B already described:

$$A \cap B = \{4, 6\}$$



In some experiments, it is helpful to list the elements of the sample space systematically by means of a tree diagram.

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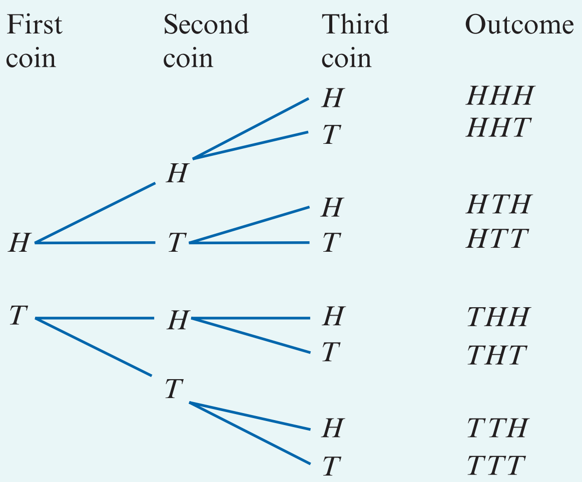
Example 1

Find the sample space when three coins are tossed and the results noted.

Solution

To list the elements of the sample space, construct a tree diagram:



Each path along the branches of the tree identifies an outcome, giving the sample space as

$$\varepsilon = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

▶ Determining probabilities for equally likely outcomes

Probability is a numerical measure of the chance of a particular event occurring. There are many approaches to determining probability, but often we assume that all of the possible outcomes are equally likely.

We require that the probabilities of all the outcomes in the sample space sum to 1, and that the probability of each outcome is a non-negative number. This means that the probability of each outcome must lie in the interval $[0, 1]$. Since six outcomes are possible when rolling a die, we can assign the probability of each outcome to be $\frac{1}{6}$. That is,

$$\Pr(1) = \Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = \Pr(6) = \frac{1}{6}$$

When the sample space is finite, the **probability of an event** is equal to the sum of the probabilities of the outcomes in that event.

For example, let A be the event that an even number is rolled on the die. Then $A = \{2, 4, 6\}$ and $\Pr(A) = \Pr(2) + \Pr(4) + \Pr(6) = \frac{1}{2}$. Since the outcomes are equally likely, we can calculate this more easily as

$$\Pr(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{3}{6} = \frac{1}{2}$$

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Equally likely outcomes

In general, if the sample space ε for an experiment contains n outcomes, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each of these outcomes.

Then the probability of any event A which contains m of these outcomes is the ratio of the number of elements in A to the number of elements in ε. That is,

$$\Pr(A) = \frac{n(A)}{n(\varepsilon)} = \frac{m}{n}$$

where the notation $n(S)$ is used to represent the number of elements in set S.

We will see that there are other methods of determining probabilities. But whichever method is used, the following rules of probability will hold:

- $\Pr(A) \geq 0$ for all events $A \subseteq \varepsilon$
- $\Pr(\varepsilon) = 1$
- The sum of the probabilities of all outcomes of an experiment is 1.
- $\Pr(\varnothing) = 0$, where $\varnothing$ represents the empty set
- $\Pr(A') = 1 - \Pr(A)$, where A' is the complement of A
- $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$, the **addition rule**

When two events A and B have no outcomes in common, i.e. when they cannot occur together, they are called **mutually exclusive** events. In this case, we have $\Pr(A \cap B) = 0$ and so the addition rule becomes:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B), \quad \text{the addition rule when } A \text{ and } B \text{ are mutually exclusive}$$

We illustrate some of these rules in the following example.

Example 2

If one card is chosen at random from a well-shuffled deck of 52 cards, what is the probability that the card is:

a an ace **b** not a heart **c** an ace or a heart **d** either a king or an ace?

Solution

a Let A be the event 'the card drawn is an ace'. A standard deck of cards contains four aces, so

$$\Pr(A) = \frac{4}{52} = \frac{1}{13}$$

b Let H be the event 'the card drawn is a heart'. There are 13 cards in each suit, so

$$\Pr(H) = \frac{13}{52} = \frac{1}{4}$$

and therefore

$$\Pr(H') = 1 - \Pr(H) = 1 - \frac{1}{4} = \frac{3}{4}$$

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c Using the addition rule:

$$\Pr(A \cup H) = \Pr(A) + \Pr(H) - \Pr(A \cap H)$$

Now $\Pr(A \cap H) = \frac{1}{52}$, since the event $A \cap H$ corresponds to drawing the ace of hearts. Therefore

$$\Pr(A \cup H) = \frac{1}{13} + \frac{1}{4} - \frac{1}{52} = \frac{4}{13}$$

d Let K be the event 'the card drawn is a king'. We observe that $K \cap A = \emptyset$. That is, the events K and A are mutually exclusive. Hence

$$\Pr(K \cup A) = \Pr(K) + \Pr(A) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$$

Example 3

500 people were questioned and classified according to age and whether or not they regularly use social media. The results are shown in the table.

Do you regularly use social media?

	Age < 25	Age ≥ 25	Total
Yes	200	100	300
No	40	160	200
Total	240	260	500

One person is selected from these 500. Find the probability that:

a the person regularly uses social media

b the person is less than 25 years of age

c the person is less than 25 years of age and does not regularly use social media.

Solution

a $\Pr(\text{Yes}) = \frac{300}{500} = \frac{3}{5}$

b $\Pr(\text{Age} < 25) = \frac{240}{500} = \frac{12}{25}$

c $\Pr(\text{No} \cap \text{Age} < 25) = \frac{40}{500} = \frac{2}{25}$

Explanation

There are 300 out of 500 people who say yes.

There are 240 out of 500 people who are less than 25 years of age.

There are 40 out of 500 people who are less than 25 years of age and say no.

► Other methods of determining probabilities

When we are dealing with a random experiment which does not have equally likely outcomes, other methods of determining probability are required.

ISBN 978-1-108-45164-2

Subjective probabilities

Sometimes, the probability is assigned a value on the basis of judgement. For example, a farmer may look at the weather conditions and determine that there is a 70% chance of rain that day, and take appropriate actions. Such probabilities are called **subjective probabilities**.

Probabilities from data

A better way to estimate an unknown probability is by experimentation: by performing the random experiment many times and recording the results. This information can then be used to estimate the chances of the event happening again in the future. The proportion of trials that resulted in this event is called the **relative frequency** of the event. (For most purposes we can consider proportion and relative frequency as interchangeable.) That is,

$$\text{Relative frequency of event } A = \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}}$$

This information can then be used to estimate the probability of the event.

When the number of trials is sufficiently large, the observed relative frequency of an event A becomes close to the probability $\Pr(A)$. That is,

$$\Pr(A) \approx \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}} \qquad \text{for a large number of trials}$$

If the experiment was repeated, it would generally be found that the results were slightly different. One might conclude that relative frequency is not a very good way of estimating probability. In many situations, however, experiments are the only way to get at an unknown probability. One of the most valuable lessons to be learnt is that such estimates are not exact, and will in fact vary from sample to sample.

Understanding the variation between estimates is extremely important in the study of statistics, and this is the topic of Chapter 18. At this stage it is valuable to realise that the variation does exist, and that the best estimates of the probabilities will result from using as many trials as possible.

Example 4

Suppose that a die is tossed 1000 times and the following outcomes observed:

Outcome	1	2	3	4	5	6
Frequency	135	159	280	199	133	97

a Use this information to estimate the probability of observing a 6 when this die is rolled.

b What outcome would you predict to be most likely the next time the die is rolled?

Solution

a $\Pr(6) \approx \dfrac{97}{1000} = 0.097$

b The most likely outcome is 3, since it has the highest relative frequency.

ISBN 978-1-108-45164-2

Probabilities from area

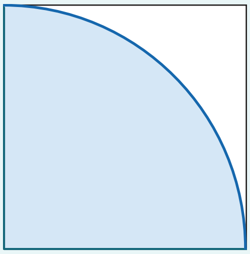
When we use the model of equally likely outcomes to determine probabilities, we count both the outcomes in the event and the outcomes in the sample space, and use the ratio to determine the probability of the event.

This idea can be extended to calculate probabilities when areas are involved, by assuming that the probabilities of all points in the region (which can be considered to be the sample space) are equally likely.

Example 5

A dartboard consists of a square of side length 2 metres containing a blue one-quarter of a circular disc centred at the bottom-left vertex of the square, as shown.

If a dart thrown at the square is equally likely to hit any part of the square, and it hits the square every time, find the probability of it hitting the blue region.



Solution

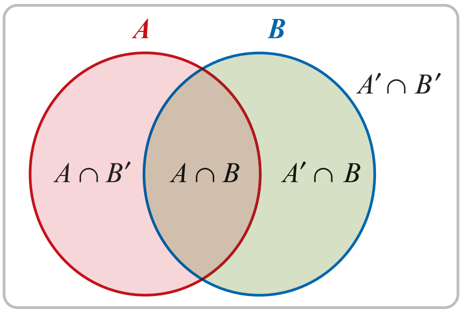
Area of blue region $= \frac{1}{4}\pi r^2 = \frac{1}{4}\pi \times 4 = \pi \text{ m}^2$

Area of dartboard $= 2 \times 2 = 4 \text{ m}^2$

$$\Pr(\text{hitting blue region}) = \frac{\text{area of blue region}}{\text{area of dartboard}} = \frac{\pi}{4}$$

Probability tables

A **probability table** is an alternative to a Venn diagram when illustrating a probability problem diagrammatically. Consider the Venn diagram which illustrates two intersecting sets A and B.



From the Venn diagram it can be seen that the sample space is divided by the sets into four disjoint regions: $A \cap B$, $A \cap B'$, $A' \cap B$ and $A' \cap B'$. These regions may be represented in a table as follows. Such a table is sometimes referred to as a **Karnaugh map**.

	B	B'
A	$A \cap B$	$A \cap B'$
A'	$A' \cap B$	$A' \cap B'$

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In a probability table, the entries give the probabilities of each of these events occurring.

	B	B'
A	$\Pr(A \cap B)$	$\Pr(A \cap B')$
A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$

Summing the rows and columns, we can complete the table as shown.

	B	B'	
A	$\Pr(A \cap B)$	$\Pr(A \cap B')$	$\Pr(A)$
A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$	$\Pr(A')$
	$\Pr(B)$	$\Pr(B')$	1

These tables can be useful when solving problems involving probability, as shown in the next example.

Example 6

Simone visits the dentist every 6 months for a checkup. The probability that she will need her teeth cleaned is 0.35, the probability that she will need a filling is 0.1 and the probability that she will need both is 0.05.

a What is the probability that she will not need her teeth cleaned on a visit, but will need a filling?

b What is the probability that she will not need either of these treatments?

Solution

The information in the question may be entered into a table as shown, where we use C to represent 'cleaning' and F to represent 'filling'.

	F	F'	
C	0.05		0.35
C'			
	0.1		1

All the empty cells in the table may now be filled in by subtraction:

	F	F'	
C	0.05	0.3	0.35
C'	0.05	0.6	0.65
	0.1	0.9	1

a The probability that she will not need her teeth cleaned but will need a filling is given by $\Pr(C' \cap F) = 0.05$.

b The probability that she will not need either of these treatments is $\Pr(C' \cap F') = 0.6$.

ISBN 978-1-108-45164-2

Section summary

- The **sample space**, ε, for a random experiment is the set of all possible outcomes.
- An **event** is a subset of the sample space. The probability of an event A occurring is denoted by $\Pr(A)$.
- Equally likely outcomes If the sample space ε for an experiment contains n outcomes, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each outcome. Then the probability of an event A is given by

$$\Pr(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{n(A)}{n(\varepsilon)}$$

- Estimates of probability When a probability is unknown, it can be estimated by the relative frequency obtained through repeated trials of the random experiment under consideration. In this case,

$$\Pr(A) \approx \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}} \qquad \text{for a large number of trials}$$

- Whichever method of determining probability is used, the rules of probability hold:
  - $\Pr(A) \geq 0$ for all events $A \subseteq \varepsilon$
  - $\Pr(\varnothing) = 0$ and $\Pr(\varepsilon) = 1$
  - The sum of the probabilities of all outcomes of an experiment is 1.
  - $\Pr(A') = 1 - \Pr(A)$, where A' is the complement of A
  - $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$, the **addition rule**
- If two events A and B are **mutually exclusive** (i.e. if A and B have no outcomes in common), then $\Pr(A \cap B) = 0$ and therefore $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Exercise 14A

SF

Example 1
1 An experiment consists of rolling a die and tossing a coin. Use a tree diagram to list the sample space for the experiment.

2 Two coins are tossed and a die is rolled. Use a tree diagram to show all the possible outcomes.

Example 2
3 If one card is chosen at random from a well-shuffled deck of 52 cards, what is the probability that the card is:

a a queen
b not a club
c a queen or a heart
d either a king or a queen?

ISBN 978-1-108-45164-2

SF

4 A blank six-sided die is marked with a 1 on two sides, a 2 on one side, and a 3 on the remaining three sides. Find the probability that when the die is rolled:

a a 3 shows **b** a 2 or a 3 shows.

5 Suppose that the probability that a student owns a smartphone is 0.7, the probability that they own a laptop is 0.6, and the probability that they own both is 0.5. What is the probability that a student owns either a smartphone or a laptop or both?

6 At a particular university, the probability that an Arts student studies a language is 0.3, literature is 0.6, and both is 0.25. What is the probability that an Arts student studies either a language or literature or both?

CF

7 A computer manufacturer notes that 5% of their computers are returned owing to faulty disk drives, 2% are returned owing to faulty keyboards, and 0.3% are returned because both disk drives and keyboards are faulty. Find the probability that the next computer manufactured will be returned with:

a a faulty disk drive or a faulty keyboard

b a faulty disk drive and a working keyboard.

8 A new drug has been released and produces some minor side effects: 8% of users suffer only loss of sleep, 12% of users suffer only nausea, and 75% of users have no side effects at all. What percentage of users suffer from both loss of sleep and nausea?

9 In a particular town, the probability that an adult owns a car is 0.7, while the probability that an adult owns a car and is employed is 0.6. If a randomly selected adult is found to own a car, what is the probability that he or she is also employed?

Example 3 **10** An insurance company analysed the records of 500 drivers to determine the relationship between age and accidents in the last year.

	Accidents in the last year				
Age	0	1	2	3	Over 3
Under 20	19	35	25	17	10
20–29	30	45	33	39	17
30–39	40	33	15	6	2
40–49	18	15	10	3	1
Over 49	21	25	17	13	11

What is the probability that a driver chosen from this group at random:

a is under 20 years old and has had three accidents in the last year

b is from 40 to 49 years old and has had no accidents in the last year

c is from 20 to 29 years old

d has had more than three accidents in the last year?

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CF

11 200 people were questioned and classified according to sex and whether or not they think private individuals should be allowed to carry guns. The results are shown in the table.

Do you think private individuals should be allowed to carry guns?

	Male	Female	Total
Yes	70	60	130
No	50	20	70
Total	120	80	200

One person is selected at random from these 200.

a What is the probability that the person thinks private individuals should be allowed to carry guns?

b What is the probability that the person is male and thinks private individuals should be allowed to carry guns?

Example 4 **12** Use the given data to estimate the probability of the specified event occurring:

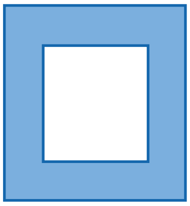
a Pr(head) if a coin is tossed 200 times and 114 heads observed

b Pr(ten) if a spinner is spun 380 times and lands on the 'ten' 40 times

c Pr(two heads) if two coins are tossed 200 times and two heads are observed on 54 occasions

d Pr(three sixes) if three dice are rolled 500 times and three sixes observed only twice

Example 5 **13** Suppose that a square dartboard consists of a white square of side length 30 cm inside a larger blue square of side length 50 cm, as shown. If a dart thrown at the board has equal chance of landing anywhere on the board, what is the probability it lands in the white area? (Ignore the possibility that it might land on the line or miss the board altogether.)

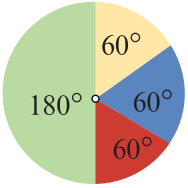


14 A spinner is as shown in the diagram. Find the probability that when spun the pointer will land on:

a the green section

b the yellow section

c any section except the yellow section.



Example 6 **15** In a particular country it has been established that the probability that a person drinks tea is 0.45, the probability that a person drinks coffee is 0.65, and the probability that a person drinks neither tea nor coffee is 0.22. Use the information to complete a probability table and hence determine the probability that a randomly selected person in that country:

a drinks tea but not coffee

b drinks tea and coffee.

ISBN 978-1-108-45164-2

CF

16 A chocolate is chosen at random from a box of chocolates. It is known that in this box:

- the probability that the chocolate is dark but not soft-centred is 0.15
- the probability that the chocolate is not dark but is soft-centred is 0.42
- the probability that the chocolate is not dark is 0.60.

Find the probability that the randomly chosen chocolate is:

a dark **b** soft-centred **c** not dark and not soft-centred.

17 Records indicate that, in Australia, 65% of secondary students participate in sport, and 71% of secondary students are Australian by birth. They also show that 53% of students are Australian by birth and participate in sport. Use this information to find the probability that a student selected at random:

a does not participate in sport
b is Australian by birth and does not participate in sport
c is not Australian by birth and participates in sport
d is not Australian by birth and does not participate in sport.

14B Conditional probability and independence

The probability of an event A occurring when it is known that some event B has occurred is called conditional probability and is written $\Pr(A \mid B)$. This is usually read as 'the probability of A given B', and can be thought of as a means of adjusting probability in the light of new information.

Sometimes, the probability of an event is not affected by knowing that another event has occurred. For example, if two coins are tossed, then the probability of the second coin showing a head is independent of whether the first coin shows a head or a tail. Thus,

$$\begin{aligned}&\Pr(\text{head on second coin} \mid \text{head on first coin})\\&= \Pr(\text{head on second coin} \mid \text{tail on first coin})\\&= \Pr(\text{head on second coin})\end{aligned}$$

For other situations, however, a previous result may alter the probability. For example, the probability of rain today given that it rained yesterday will generally be different from the probability that it will rain today given that it didn't rain yesterday.

The **conditional probability** of an event A, given that event B has already occurred, is given by

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \text{if } \Pr(B) \neq 0$$

This formula may be rearranged to give the **multiplication rule of probability**:

$$\Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B)$$

ISBN 978-1-108-45164-2

The probabilities associated with a multi-stage experiment can be calculated by constructing an appropriate tree diagram and multiplying along the relevant branches (from the multiplication rule).

Example 7

In a certain town, the probability that it rains on any Monday is 0.21. If it rains on Monday, then the probability that it rains on Tuesday is 0.83. If it does not rain on Monday, then the probability of rain on Tuesday is 0.3. For a given week, find the probability that it rains:

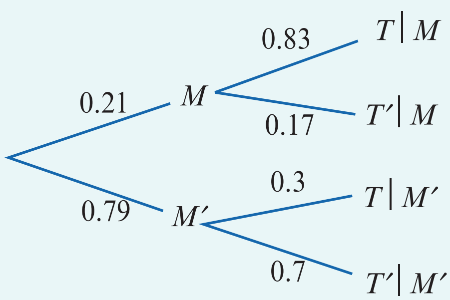
a on both Monday and Tuesday

b on Tuesday.

Solution

Let M represent the event 'rain on Monday' and T represent the event 'rain on Tuesday'.

The situation described in the question can be represented by a tree diagram. You can check that the probabilities are correct by seeing if they add to 1.



$\Pr(T \cap M) = 0.21 \times 0.83 = 0.1743$

$\Pr(T' \cap M) = 0.21 \times 0.17 = 0.0357$

$\Pr(T \cap M') = 0.79 \times 0.3 = 0.237$

$\Pr(T' \cap M') = 0.79 \times 0.7 = 0.553$

a The probability that it rains on both Monday and Tuesday is given by

$$\begin{aligned}\Pr(T \cap M) &= 0.21 \times 0.83 \\ &= 0.1743\end{aligned}$$

b The probability that it rains on Tuesday is given by

$$\begin{aligned}\Pr(T) &= \Pr(T \cap M) + \Pr(T \cap M') \\ &= 0.1743 + 0.237 \\ &= 0.4113\end{aligned}$$

The solution to part **b** of Example 7 is an application of a rule known as the law of total probability. This can be expressed in general terms as follows:

The **law of total probability** states that, in the case of two events A and B,

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$$

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Example 8

Adrienne, Regan and Michael are doing the dishes. Since Adrienne is the oldest, she washes the dishes 40% of the time. Regan and Michael each wash 30% of the time. When Adrienne washes the probability of at least one dish being broken is 0.01, when Regan washes the probability is 0.02, and when Michael washes the probability is 0.03. Their parents don't know who is washing the dishes one particular night.

a What is the probability that at least one dish will be broken?

b Given that at least one dish is broken, what is the probability that the person washing was Michael?

Solution

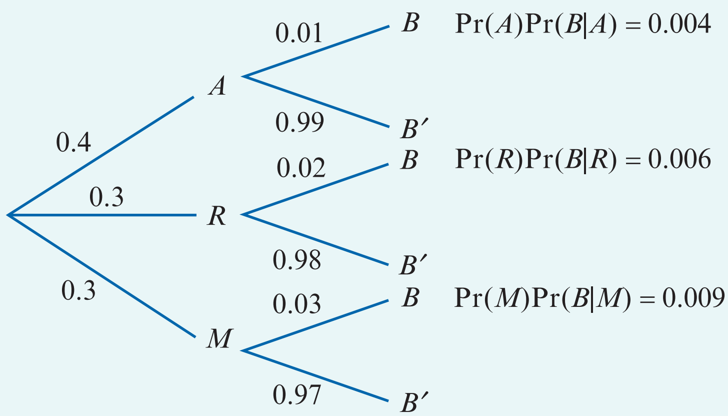
Let A be the event 'Adrienne washes the dishes', let R be the event 'Regan washes the dishes' and let M be the event 'Michael washes the dishes'. Then

$$\Pr(A) = 0.4, \qquad \Pr(R) = 0.3, \qquad \Pr(M) = 0.3$$

Let B be the event 'at least one dish is broken'. Then

$$\Pr(B \mid A) = 0.01, \qquad \Pr(B \mid R) = 0.02, \qquad \Pr(B \mid M) = 0.03$$

This information can be summarised in a tree diagram as shown:



a The probability of at least one dish being broken is

$$\begin{aligned}\Pr(B) &= \Pr(B \cap A) + \Pr(B \cap R) + \Pr(B \cap M)\\ &= \Pr(A)\Pr(B \mid A) + \Pr(R)\Pr(B \mid R) + \Pr(M)\Pr(B \mid M)\\ &= 0.004 + 0.006 + 0.009\\ &= 0.019\end{aligned}$$

b The required probability is

$$\begin{aligned}\Pr(M \mid B) &= \frac{\Pr(M \cap B)}{\Pr(B)}\\ &= \frac{0.009}{0.019} = \frac{9}{19}\end{aligned}$$

ISBN 978-1-108-45164-2

Example 9

As part of an evaluation of the school tuck shop, all students at a Senior Secondary College (Years 10–12) were asked to rate the tuck shop as poor, good or excellent. The results are shown in the table.

	Year			
Rating	10	11	12	Total
Poor	30	20	10	60
Good	80	65	35	180
Excellent	60	65	35	160
Total	170	150	80	400

What is the probability that a student chosen at random from this college:

a is in Year 12

b is in Year 12 and rates the tuck shop as excellent

c is in Year 12, given that they rate the tuck shop as excellent

d rates the tuck shop as excellent, given that they are in Year 12?

Solution

Let T be the event 'the student is in Year 12' and let E be the event 'the rating is excellent'.

a $\Pr(T) = \dfrac{80}{400} = \dfrac{1}{5}$

Explanation: From the table, we can see that there are 80 students in Year 12 and 400 students altogether.

b $\Pr(T \cap E) = \dfrac{35}{400} = \dfrac{7}{80}$

Explanation: From the table, there are 35 students who are in Year 12 and also rate the tuck shop as excellent.

c $\Pr(T \mid E) = \dfrac{35}{160} = \dfrac{7}{32}$

Explanation: From the table, a total of 160 students rate the tuck shop as excellent, and of these 35 are in Year 12.

d $\Pr(E \mid T) = \dfrac{35}{80} = \dfrac{7}{16}$

Explanation: From the table, there are 80 students in Year 12, and of these 35 rate the tuck shop as excellent.

Note: The answers to parts **c** and **d** could also have been found using the rule for conditional probability, but here it is easier to determine the probability directly from the table.

► Independent events

Two events A and B are **independent** if the probability of A occurring is the same, whether or not B has occurred.

Independent events

For events A and B with $\Pr(A) \neq 0$ and $\Pr(B) \neq 0$, the following three conditions are all equivalent conditions for the independence of A and B:

- $\Pr(A \mid B) = \Pr(A)$
- $\Pr(B \mid A) = \Pr(B)$
- $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

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Notes:

- Sometimes this definition of independence is referred to as **pairwise independence**.
- In the special case that $\Pr(A) = 0$ or $\Pr(B) = 0$, the condition $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$ holds since both sides are zero, and so we say that A and B are independent.

Example 10

The probability that Monica remembers to do her homework is 0.7, while the probability that Patrick remembers to do his homework is 0.4. If these events are independent, then what is the probability that:

a both will do their homework

b Monica will do her homework but Patrick forgets?

Solution

Let M be the event 'Monica does her homework' and let P be the event 'Patrick does his homework'. Since these events are independent:

a $$\begin{aligned}\Pr(M \cap P) &= \Pr(M) \times \Pr(P)\\ &= 0.7 \times 0.4\\ &= 0.28\end{aligned}$$

b $$\begin{aligned}\Pr(M \cap P') &= \Pr(M) \times \Pr(P')\\ &= 0.7 \times 0.6\\ &= 0.42\end{aligned}$$

Section summary

- Conditional probability
  - The probability of an event A occurring when it is known that some event B has already occurred is called conditional probability and is written $\Pr(A \mid B)$.
  - In general, the **conditional probability** of an event A, given that event B has already occurred, is given by

    $$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \quad \text{if } \Pr(B) \neq 0$$

    This formula may be rearranged to give the **multiplication rule of probability**:

    $$\Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B)$$

- Law of total probability

  The **law of total probability** states that, in the case of two events A and B,

  $$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$$

- Independence

  Two events A and B are **independent** if the occurrence of one event has no effect on the probability of the occurrence of the other, that is, if

  $$\Pr(A \mid B) = \Pr(A)$$

  Events A and B are independent if and only if

  $$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

ISBN 978-1-108-45164-2

Exercise 14B

Skillsheet
Example 7

1 In a certain town, the probability that it rains on any Saturday is 0.25. If it rains on Saturday, then the probability of rain on Sunday is 0.8. If it does not rain on Saturday, then the probability of rain on Sunday is 0.1. For a given week, find the probability that:

a it rains on both Saturday and Sunday
b it rains on neither day
c it rains on Sunday.

SF

2 Given that for two events A and B, $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(A \cap B) = 0.1$, find:

a $\Pr(B \mid A)$
b $\Pr(A \mid B)$

3 Given that for two events A and B, $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(B \mid A) = 0.1$, find:

a $\Pr(A \cap B)$
b $\Pr(A \mid B)$

4 In Alia's school, the probability that a student studies French is 0.5, and the probability that they study both French and Chinese is 0.3. Find the probability that a student studies Chinese, given that they study French.

Example 8

5 The chance that a harvest is poorer than average is 0.5, but if it is known that a certain disease D is present, this probability increases to 0.8. The disease D is present in 30% of harvests. Find the probability that, when a harvest is observed to be poorer than average, the disease D is present.

CF

Example 9

6 A group of 1000 eligible voters were asked their age and their preference in an upcoming election, with the following results.

	Age			
Preference	18–25	26–40	Over 40	Total
Candidate A	200	100	85	385
Candidate B	250	230	50	530
No preference	50	20	15	85
Total	500	350	150	1000

What is the probability that a person chosen from this group at random:

a is 18–25 years of age
b prefers candidate A
c is 18–25 years of age, given that they prefer candidate A
d prefers candidate A, given that they are 18–25 years of age?

ISBN 978-1-108-45164-2

7 The following data was derived from accident records on a highway noted for its above-average accident rate.

	Probable cause			
Type of accident	Speed	Alcohol	Other	Total
Fatal	42	61	12	115
Non-fatal	88	185	60	333
Total	130	246	72	448

Use the table to find:

a the probability that speed is the cause of the accident

b the probability that the accident is fatal

c the probability that the accident is fatal, given that speed is the cause

d the probability that the accident is fatal, given that alcohol is the cause.

Example 10

8 The probability of James winning a particular tennis match is independent of Sally winning another particular tennis match. If the probability of James winning is 0.8 and the probability of Sally winning is 0.3, find:

a the probability that they both win

b the probability that either or both of them win.

9 An experiment consists of drawing a number at random from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 3, 5, 7, 9\}$ and $C = \{4, 6, 8, 9\}$.

a Are A and B independent?

b Are A and C independent?

c Are B and C independent?

10 If A and B are independent events such that $\Pr(A) = 0.5$ and $\Pr(B) = 0.4$, find:

a $\Pr(A \mid B)$ **b** $\Pr(A \cap B)$ **c** $\Pr(A \cup B)$

11 Nathan knows that his probability of kicking more than four goals on a wet day is 0.3, while on a dry day it is 0.6. The probability that it will be wet on the day of the next game is 0.7. Calculate the probability that Nathan will kick more than four goals in the next game.

12 Find the probability that, in three tosses of a fair coin, there are three heads, given that there is at least one head.

13 The test used to determine if a person suffers from a particular disease is not perfect. The probability of a person with the disease returning a positive result is 0.95, while the probability of a person without the disease returning a positive result is 0.02. The probability that a randomly selected person has the disease is 0.03. What is the probability that a randomly selected person will return a positive result?

ISBN 978-1-108-45164-2

CF

14 Anya goes through three sets of traffic lights when she cycles to school each morning. The probability she stops at the first set is 0.6. If she stops at any one set, the probability that she has to stop at the next is 0.9. If she doesn't have to stop at any one set, the probability that she doesn't have to stop at the next is 0.7. Use a tree diagram to find the probability that:

a she stops at all three sets of lights
b she stops only at the second set of lights
c she stops at exactly one set of lights.

15 There are four red socks and two blue socks in a drawer. Two socks are removed at random. What is the probability of obtaining:

a two red socks
b two blue socks
c one of each colour?

16 A car salesperson was interested in the relationship between the size of the car a customer purchased and their marital status. From the sales records, the table on the right was constructed.

	Marital status		
Size of car	Married	Single	Total
Large	60	20	80
Medium	100	60	160
Small	90	70	160
Total	250	150	400

What is the probability that a person chosen at random from this group:

a drives a small car
b is single and drives a small car
c is single, given that they drive a small car
d drives a small car, given that they are single?

17 Jenny has two boxes of chocolates. Box A contains three white chocolates and four dark chocolates. Box B contains two white chocolates and five dark chocolates. Jenny first chooses a box at random and then selects a chocolate at random from it. Find the probability that:

a Jenny selects a white chocolate
b given that Jenny selects a white chocolate, it was chosen from box A.

18 At a particular petrol station, 30% of customers buy premium unleaded, 60% buy standard unleaded and 10% buy diesel. When a customer buys premium unleaded, there is a 25% chance they will fill the tank. Of the customers buying standard unleaded, 20% fill their tank. Of those buying diesel, 70% fill their tank.

a What is the probability that, when a car leaves the petrol station, it will not have a full tank?
b Given that a car leaving the petrol station has a full tank, what is the probability that the tank contains standard unleaded petrol?

19 A bag contains three red, four white and five black balls. If three balls are taken without replacement, what is the probability that they are all the same colour?

ISBN 978-1-108-45164-2

14C Discrete random variables

Suppose that three balls are drawn at random from a jar containing four white and six black balls, with replacement (i.e. each selected ball is replaced before the next draw). The sample space for this random experiment is as follows:

$$\varepsilon = \{WWW, WWB, WBW, BWW, WBB, BWB, BBW, BBB\}$$

Suppose the variable of interest is the number of white balls in the sample. This corresponds to a simpler sample space whose outcomes are numbers.

If X represents the number of white balls in the sample, then the possible values of X are 0, 1, 2 and 3. Since the actual value that X will take is the result of a random experiment, we say that X is a random variable.

A **random variable** is a function that assigns a number to each outcome in the sample space ε.

A random variable can be discrete or continuous:

- A **discrete random variable** is one that can take only a countable number of values. For example, the number of white balls in a sample of size three is a discrete random variable which may take one of the values 0, 1, 2, 3. Other examples include the number of children in a family, and a person's shoe size. (Note that discrete random variables do not have to take only whole-number values.)
- A **continuous random variable** is one that can take any value in an interval of the real number line, and is usually (but not always) generated by measuring. Height, weight, and the time taken to complete a puzzle are all examples of continuous random variables.

In this chapter we are interested in understanding more about discrete random variables.

Consider again the sample space for the random experiment described above. Each outcome in the sample space is associated with a value of X:

Experiment outcome	Value of X
WWW	$X = 3$
WWB	$X = 2$
WBW	$X = 2$
BWW	$X = 2$
WBB	$X = 1$
BWB	$X = 1$
BBW	$X = 1$
BBB	$X = 0$

Associated with each event is a probability. Since the individual draws of the ball from the jar are independent events, we can determine the probabilities by multiplying and adding appropriate terms.

 ISBN 978-1-108-45164-2

Example 11

A jar contains four white and six black balls. What is the probability that, if three balls are drawn at random from the jar, with replacement, a white ball will be drawn exactly once (i.e. the situations where $X = 1$ in the table)?

Solution

$X = 1$ corresponds to the outcomes WBB, BWB and BBW.

Since there are 10 balls in total, $\Pr(W) = \frac{4}{10} = 0.4$ and $\Pr(B) = \frac{6}{10} = 0.6$.

$$\begin{aligned} \text{Thus } \Pr(X = 1) &= \Pr(WBB) + \Pr(BWB) + \Pr(BBW) \\ &= (0.4 \times 0.6 \times 0.6) + (0.6 \times 0.4 \times 0.6) + (0.6 \times 0.6 \times 0.4) \\ &= 0.432 \end{aligned}$$

► Discrete probability distributions

The **probability distribution** for a discrete random variable consists of all the values that the random variable can take, together with the probability of each of these values. For example, if a fair die is rolled, then the probability distribution is:

x	1	2	3	4	5	6
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The probability distribution of a discrete random variable X is described by a function

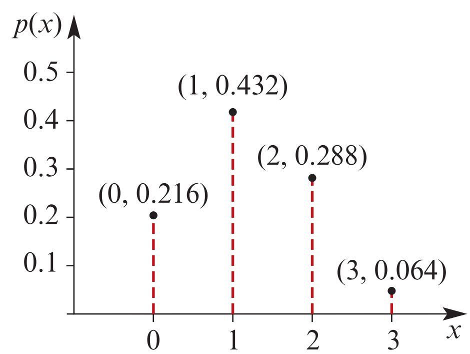
$$p(x) = \Pr(X = x)$$

This function is called a **discrete probability function** or a **probability mass function**.

Consider again the black and white balls from Example 11. The probability distribution for X, the number of white balls in the sample, is given by the following table:

x	0	1	2	3
$p(x)$	0.216	0.432	0.288	0.064

The probability distribution may also be given graphically, as shown on the right.



Note that the probabilities in the table sum to 1, which must occur if all values of the random variable have been listed.

We will use the following notation, which is discussed further in Appendix A:

- the sum of all the values of $p(x)$ is written as $\sum_{x} p(x)$
- the sum of the values of $p(x)$ for x between a and b inclusive is written as $\sum_{a \le x \le b} p(x)$

ISBN 978-1-108-45164-2

For any discrete probability function $p(x)$, the following two conditions must hold:

1 Each value of $p(x)$ belongs to the interval $[0, 1]$. That is,

$$0 \le p(x) \le 1 \quad \text{for all } x$$

2 The sum of all the values of $p(x)$ is 1. That is,

$$\sum_x p(x) = 1$$

To determine the probability that X takes a value in the interval from a to b (including the values a and b), add the values of $p(x)$ from $x = a$ to $x = b$:

$$\Pr(a \le X \le b) = \sum_{a \le x \le b} p(x)$$

Example 12

Consider the table shown.

x	0	1	2	3
$p(x)$	0.2	0.3	0.1	0.4

a Does this meet the conditions to be a discrete probability distribution?

b Use the table to find $\Pr(X \le 2)$.

Solution

a Yes, each value of $p(x)$ is between 0 and 1, and the values add to 1.

b $$\begin{aligned}\Pr(X \le 2) &= p(0) + p(1) + p(2)\\ &= 0.2 + 0.3 + 0.1\\ &= 0.6\end{aligned}$$

Example 13

Let X be the number of heads showing when a fair coin is tossed three times.

a Find the probability distribution of X and show that all the probabilities sum to 1.

b Find the probability that one or more heads show.

c Find the probability that more than one head shows.

Solution

a The sample space is $\varepsilon = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

Now $p(0) = \Pr(X = 0) = \Pr(\{TTT\}) = \frac{1}{8}$

$$p(1) = \Pr(X = 1) = \Pr(\{HTT, THT, TTH\}) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

$$p(2) = \Pr(X = 2) = \Pr(\{HHT, HTH, THH\}) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

$$p(3) = \Pr(X = 3) = \Pr(\{HHH\}) = \frac{1}{8}$$

ISBN 978-1-108-45164-2

Thus the probability distribution of X is:

x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b The probability that one or more heads shows is

$$\Pr(X \geq 1) = p(1) + p(2) + p(3) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$$

c The probability that more than one head shows is

$$\Pr(X > 1) = \Pr(X \geq 2) = p(2) + p(3) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Example 14

The random variable X represents the number of chocolate chips in a certain brand of biscuit, and is known to have the following probability distribution.

x	2	3	4	5	6	7
$p(x)$	0.01	0.25	0.40	0.30	0.02	0.02

Find:

a $\Pr(X \geq 4)$ **b** $\Pr(X \geq 4 \mid X > 2)$ **c** $\Pr(X < 5 \mid X > 2)$

Solution

a
$$\begin{aligned}\Pr(X \geq 4) &= \Pr(X = 4) + \Pr(X = 5) + \Pr(X = 6) + \Pr(X = 7)\\ &= 0.4 + 0.3 + 0.02 + 0.02\\ &= 0.74\end{aligned}$$

b
$$\begin{aligned}\Pr(X \geq 4 \mid X > 2) &= \frac{\Pr(X \geq 4)}{\Pr(X > 2)}\\ &= \frac{0.74}{0.99} \qquad \text{since } \Pr(X > 2) = 1 - 0.01 = 0.99\\ &= \frac{74}{99}\end{aligned}$$

c
$$\begin{aligned}\Pr(X < 5 \mid X > 2) &= \frac{\Pr(2 < X < 5)}{\Pr(X > 2)}\\ &= \frac{\Pr(X = 3) + \Pr(X = 4)}{\Pr(X > 2)}\\ &= \frac{0.65}{0.99}\\ &= \frac{65}{99}\end{aligned}$$

ISBN 978-1-108-45164-2

Section summary

- For any discrete probability function $p(x)$, the following two conditions must hold:
  1. Each value of $p(x)$ belongs to the interval $[0, 1]$. That is,
     $$0 \le p(x) \le 1 \quad \text{for all } x$$
  2. The sum of all the values of $p(x)$ is 1. That is,
     $$\sum_x p(x) = 1$$
- To determine the probability that X takes a value in the interval from a to b (including the values a and b), add the values of $p(x)$ from $x = a$ to $x = b$:
  $$\Pr(a \le X \le b) = \sum_{a \le x \le b} p(x)$$

Exercise 14C

Skillshett

SF

1 Which of the following random variables are discrete?

a the number of people in your family

b waist measurement

c shirt size

d the number of times a die is rolled before obtaining a six

2 Which of the following random variables are discrete?

a your age

b your height to the nearest centimetre

c the time you will wait to be served at the bank

d the number of people in the queue at the bank

Example 11

3 A fair coin is tossed three times and the number of heads noted.

a List the sample space.

b List the possible values of the random variable X, the number of heads, together with the corresponding outcomes.

c Find $\Pr(X \ge 2)$.

Example 12

4 Consider the following table:

x	0	1	2	3	4
$p(x)$	0.1	0.2	0.1	0.4	0.2

a Does this meet the conditions to be a discrete probability distribution?

b Use the table to find $\Pr(X \le 3)$.

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Example 13

5 A jar contains four red and five blue balls. A ball is withdrawn, its colour is observed, and it is then replaced. This is repeated three times. Let X be the number of red balls among the three balls withdrawn.

a Find the probability distribution of X and show that all the probabilities sum to 1.

b Find the probability that one or more red balls are obtained.

c Find the probability that more than one red ball is obtained.

CF

Example 14

6 Two dice are rolled and the numbers noted.

a List the sample space.

b A random variable Y is defined as the total of the numbers showing on the two dice. List the possible values of Y, together with the corresponding outcomes.

c Find:

i $\Pr(Y < 5)$ **ii** $\Pr(Y = 3 \mid Y < 5)$ **iii** $\Pr(Y \leq 3 \mid Y < 7)$

iv $\Pr(Y \geq 7 \mid Y > 4)$ **v** $\Pr(Y = 7 \mid Y > 4)$ **vi** $\Pr(Y = 7 \mid Y < 8)$

SF

7 A die is weighted as follows:

$$\Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = 0.2, \qquad \Pr(1) = \Pr(6) = 0.1$$

CF

The die is rolled twice, and the smaller of the numbers showing is noted. Let Y represent this value.

a List the sample space.

b List the possible values of Y.

c Find $\Pr(Y = 1)$.

8 Suppose that three balls are selected at random, with replacement, from a jar containing four white and six black balls. If X is the number of white balls in the sample, find:

a $\Pr(X = 2)$ **b** $\Pr(X = 3)$

c $\Pr(X \geq 2)$ **d** $\Pr(X = 3 \mid X \geq 2)$

9 A fair die is rolled twice and the numbers noted. Define the following events:

A = 'a four on the first roll'

B = 'a four on the second roll'

C = 'the sum of the two numbers is at least eight'

D = 'the sum of the two numbers is at least 10'

a List the sample space obtained.

b Find $\Pr(A)$, $\Pr(B)$, $\Pr(C)$ and $\Pr(D)$.

c Find $\Pr(A \mid B)$, $\Pr(A \mid C)$ and $\Pr(A \mid D)$.

d Which of the following pairs of events are independent?

i A and B **ii** A and C **iii** A and D

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CF

10 Consider the table shown on the right.

x	0	1	2	3
$p(x)$	0.1	0.4	0.2	0.3

a Does this meet the conditions to be a discrete probability distribution?

b Use the table to find $\Pr(X \geq 2)$.

SF

11 Which of the following is *not* a probability distribution?

a

x	1	3	5	7
$p(x)$	0.1	0.3	0.5	0.7

b

x	−1	0	1	2
$p(x)$	0.25	0.25	0.25	0.25

c

x	0.25	0.5	0.75	1.0
$p(x)$	−0.5	−0.25	0.25	0.5

d

x	10	20	30	40
$p(x)$	10%	20%	30%	40%

CF

12 Three balls are selected from a jar containing four black and six red balls. Find the probability distribution of the number of black balls in the sample:

a if the ball chosen is replaced after each selection

b if the ball chosen is not replaced after each selection.

13 A coin is known to be biased such that the probability of obtaining a head on any toss is 0.4. Find the probability distribution of X, the number of heads observed when the coin is tossed twice.

14 A spinner is numbered from 1 to 5, and each of the five numbers is equally likely to come up. Find:

a the probability distribution of X, the number showing on the spinner

b $\Pr(X \geq 3)$, the probability that the number showing on the spinner is three or more

c $\Pr(X \leq 3 \mid X \geq 3)$

15 Two dice are rolled and the numbers noted.

a List the sample space for this experiment.

b Find the probability distribution of X, the sum of the numbers showing on the two dice.

c Draw a graph of the probability distribution of X.

d Find $\Pr(X \geq 9)$, the probability that the sum of the two numbers showing is nine or more.

e Find $\Pr(X \leq 10 \mid X \geq 9)$.

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CF

16 Two dice are rolled and the numbers noted.

a List the sample space for this experiment.

b Find the probability distribution of Y, the remainder when the larger number showing is divided by the smaller number. (Note that, if the two numbers are the same, then $Y = 0$.)

c Draw a graph of the probability distribution of Y.

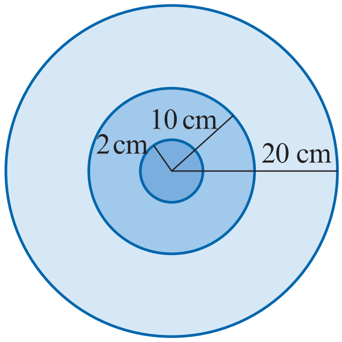
17 Suppose that two socks are drawn without replacement from a drawer containing four red and six black socks. Let X represent the number of red socks obtained.

a Find the probability distribution for X.

b From the probability distribution, determine the probability that a pair of socks is obtained.

18 A dartboard consists of three circular sections, with radii of 2 cm, 10 cm and 20 cm respectively, as shown in the diagram.

When a dart lands in the centre circle the score is 100 points, in the middle circular section the score is 20 points and in the outer circular section the score is 10 points. Assume that all darts thrown hit the board, each dart is equally likely to land at any point on the dartboard, and none lands on the lines.



a Find the probability distribution for X, the number of points scored on one throw.

b Find the probability distribution for Y, the total score when two darts are thrown.

19 Erin and Nick are going to play a tennis match. Suppose that they each have an equal chance of winning any set (0.5) and that they plan to play until one player has won three sets. Let X be the number of sets played until the match is complete.

a Find $\Pr(X = 3)$.

b List the outcomes that correspond to $X = 4$, and use this to find $\Pr(X = 4)$.

c Hence, or otherwise, find $\Pr(X = 5)$.

14D Expected value, variance and standard deviation

From your studies of statistics, you may already be familiar with the mean as a measure of centre and with the variance and the standard deviation as measures of spread. When these are calculated from a set of data, they are termed ‘sample statistics’. It is also possible to use the probability distribution to determine the theoretically ‘true’ values of the mean, variance and standard deviation. When they are calculated from the probability distribution, they are called ‘population parameters’. Determining the values of these parameters is the topic for this section.

ISBN 978-1-108-45164-2

▶ Expected value

When the mean of a random variable is determined from the probability distribution, it is generally called the **expected value** of the random variable. Expected value has a wide variety of applications. The concept of expected value first arose in gambling problems, where gamblers wished to know how much they could expect to win or lose in the long run, in order to decide whether or not a particular game was a good investment.

Example 15

A person may buy a lucky ticket for \$1. They have a 20% chance of winning \$2, a 5% chance of winning \$11, and otherwise they lose. Is this a good game to play?

Solution

Let P be the amount the person will profit from each game. As it costs \$1 to play, the person can lose \$1 ($P = -1$), win \$1 ($P = 1$) or win \$10 ($P = 10$). Thus the amount that the person may win, \$$P$, has a probability distribution given by:

p	-1	1	10
$\Pr(P = p)$	0.75	0.20	0.05

Suppose you played the game 1000 times. You would expect to lose \$1 about 750 times, to win \$1 about 200 times and to win \$10 about 50 times. Thus, you would win about

$$\frac{-1 \times 750 + 1 \times 200 + 10 \times 50}{1000} = -\$0.05 \text{ per game}$$

Thus your 'expectation' is to lose 5 cents per game, and we write this as

$$\mathrm{E}(P) = -0.05$$

Note: This value gives an indication of the worth of the game: in the long run, you would expect to lose about 5 cents per game. This is called the **expected value** of P (or the **mean** of P). It is not the amount we expect to profit on any one game. (You cannot lose 5 cents in one game!) It is the amount that we expect to win on average per game in the long run.

Example 15 demonstrates how the expected value of a random variable X is determined.

The **expected value** of a discrete random variable X is determined by summing the products of each value of X and the probability that X takes that value.

That is,

$$\begin{aligned}\mathrm{E}(X) &= \sum_x x \cdot \Pr(X = x) \\ &= \sum_x x \cdot p(x)\end{aligned}$$

The expected value $\mathrm{E}(X)$ may be considered as the long-run average value of X. It is generally denoted by the Greek letter μ (mu), and is also called the **mean** of X.

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Example 16

A coin is biased in favour of heads such that the probability of obtaining a head on any single toss is 0.6. The coin is tossed three times and the results noted. If X is the number of heads obtained on the three tosses, find $E(X)$, the expected value of X.

Solution

The following probability distribution can be found by listing the outcomes in the sample space and determining the value of X and the associated probability for each outcome.

x	0	1	2	3
$p(x)$	0.064	0.288	0.432	0.216

$$\begin{aligned}\mu = E(X) &= \sum_x x \cdot p(x)\\ &= (0 \times 0.064) + (1 \times 0.288) + (2 \times 0.432) + (3 \times 0.216)\\ &= 0.288 + 0.864 + 0.648\\ &= 1.8\end{aligned}$$

Note: This means that, if the experiment were repeated many times, then an average of 1.8 heads per three tosses would be observed.

Sometimes we wish to find the expected value of a function of X. This is determined by calculating the value of the function for each value of X, and then summing the products of these values and the associated probabilities.

The expected value of $g(X)$ is given by

$$E[g(X)] = \sum_x g(x) \cdot p(x)$$

Example 17

For the random variable X defined in Example 16, find:

a $E(3X + 1)$ **b** $E(X^2)$

Solution

a $$\begin{aligned}E(3X + 1) &= \sum_x (3x + 1) \cdot p(x)\\ &= (1 \times 0.064) + (4 \times 0.288) + (7 \times 0.432) + (10 \times 0.216)\\ &= 6.4\end{aligned}$$

b $$\begin{aligned}E(X^2) &= \sum_x x^2 \cdot p(x)\\ &= (0^2 \times 0.064) + (1^2 \times 0.288) + (2^2 \times 0.432) + (3^2 \times 0.216)\\ &= 3.96\end{aligned}$$

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Let us compare the values found in Example 17 with the value of E(X) found in Example 16. In part **a**, we found that $E(3X + 1) = 6.4$. Since $3E(X) + 1 = 3 \times 1.8 + 1 = 6.4$, we see that

$$E(3X + 1) = 3E(X) + 1$$

In part **b**, we found that $E(X^2) = 3.96$. Since $[E(X)]^2 = 1.8^2 = 3.24$, we see that

$$E(X^2) \neq [E(X)]^2$$

These two examples illustrate an important point concerning expected values.

In general, the expected value of a function of X is not equal to that function of the expected value of X. That is,

$$E[g(X)] \neq g[E(X)]$$

An exception is when the function is linear:

Expected value of $aX + b$

$$E(aX + b) = aE(X) + b \qquad \text{(for } a, b \text{ constant)}$$

This is illustrated in Example 18.

Example 18

For a \$5 monthly fee, a TV repair company guarantees customers a complete service. The company estimates the probability that a customer will require one service call in a month as 0.05, the probability of two calls as 0.01 and the probability of three or more calls as 0.00. Each call costs the repair company \$40. What is the TV repair company's expected monthly gain from such a contract?

Solution

We may summarise the given information in the following table.

Calls	0	1	2	≥ 3
Gain, g	5	−35	−75	
$\Pr(G = g)$	0.94	0.05	0.01	0.00

$$E(G) = \sum_{g=0}^{2} g \cdot \Pr(G = g)$$

$$= 5 \times 0.94 - 35 \times 0.05 - 75 \times 0.01$$

$$= 2.20$$

Thus, the company can expect to gain \$2.20 per month on each contract sold.

Alternative solution

An alternative method of solution uses the formula for the expected value of $aX + b$, as follows.

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Let X be the number of calls received. Then

$$G = 5 - 40X$$

and so $\quad \mathrm{E}(G) = 5 - 40 \times \mathrm{E}(X)$

Since $\quad \mathrm{E}(X) = 1 \times 0.05 + 2 \times 0.01 = 0.07$

we have $\quad \mathrm{E}(G) = 5 - 40 \times 0.07 = 2.20$ as previously determined.

Another useful property of expectation is that the expected value of the sum of two random variables is equal to the sum of their expected values. That is, if X and Y are two random variables, then

$$\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$$

Measures of variability: variance and standard deviation

As well as knowing the long-run average value of a random variable (the mean), it is also useful to have a measure of how close to this mean are the possible values of the random variable — that is, a measure of how spread out the probability distribution is. The most useful measures of variability for a discrete random variable are the variance and the standard deviation.

The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean or expected value μ. It is defined as

$$\mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$$

and may be considered as the long-run average value of the square of the distance from X to μ. The variance is usually denoted by σ^2, where σ is the lowercase Greek letter *sigma*.

From the definition,

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{E}[(X - \mu)^2] \\ &= \sum_x (x - \mu)^2 \cdot \Pr(X = x) \end{aligned}$$

Since the variance is determined by squaring the distance from X to μ, it is no longer in the units of measurement of the original random variable X. A measure of spread in the appropriate unit is found by taking the square root of the variance.

The **standard deviation** of X is defined as

$$\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

The standard deviation is usually denoted by σ.

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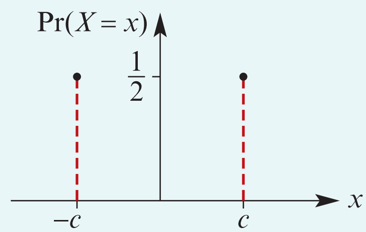
Example 19

Suppose that a discrete random variable X has the probability distribution shown in the following table, where $c > 0$.

x	$-c$	c
$\Pr(X = x)$	0.5	0.5

Find the standard deviation of X.

Solution



$$\begin{aligned} \mu = \mathrm{E}(X) &= \sum_x x \cdot \Pr(X = x) \\ &= (-c \times 0.5) + (c \times 0.5) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \sigma^2 = \mathrm{Var}(X) &= \mathrm{E}[(X - \mu)^2] \\ &= \mathrm{E}(X^2) \qquad \text{since } \mu = 0 \\ &= \sum_x x^2 \cdot \Pr(X = x) \\ &= (-c)^2 \times 0.5 + c^2 \times 0.5 \\ &= c^2 \end{aligned}$$

which is the average of (the distance from X to μ)^2.

Therefore $\sigma = \mathrm{sd}(X) = c$.

Using the definition is not always the easiest way to calculate the variance.

An alternative (computational) formula for variance is

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

Proof We already know that

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b \qquad (1)$$

and $$\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y) \qquad (2)$$

Hence
$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{E}[(X - \mu)^2] \\ &= \mathrm{E}(X^2 - 2\mu X + \mu^2) \\ &= \mathrm{E}(X^2) + \mathrm{E}(-2\mu X + \mu^2) && \text{using (2)} \\ &= \mathrm{E}(X^2) - 2\mu \mathrm{E}(X) + \mu^2 && \text{using (1)} \\ &= \mathrm{E}(X^2) - 2\mu^2 + \mu^2 && \text{since } \mu = \mathrm{E}(X) \\ &= \mathrm{E}(X^2) - \mu^2 \end{aligned}$$

ISBN 978-1-108-45164-2

Example 20

For the probability distribution shown, find $E(X^2)$ and $[E(X)]^2$ and hence find the variance of X.

x	0	1	2	3
$\Pr(X = x)$	0.08	0.18	0.4	0.34

Solution

We have

$$E(X) = 1 \times 0.18 + 2 \times 0.4 + 3 \times 0.34 = 2$$
$$[E(X)]^2 = \mu^2 = 4$$
$$E(X^2) = 1 \times 0.18 + 4 \times 0.4 + 9 \times 0.34 = 4.84$$

Hence

$$\text{Var}(X) = E(X^2) - \mu^2 = 4.84 - 4 = 0.84$$

Variance of $aX + b$

$$\text{Var}(aX + b) = a^2\text{Var}(X) \qquad \text{(for } a, b \text{ constant)}$$

Example 21

If X is a random variable such that $\text{Var}(X) = 9$, find:

a $\text{Var}(3X + 2)$

b $\text{Var}(-X)$

Solution

a $\text{Var}(3X + 2) = 3^2\text{Var}(X)$
$= 9 \times 9$
$= 81$

b $\text{Var}(-X) = \text{Var}(-1 \times X)$
$= (-1)^2\text{Var}(X)$
$= \text{Var}(X)$
$= 9$

Interpretation of standard deviation

We can make the standard deviation more meaningful by giving it an interpretation that relates to the probability distribution.

Example 22

The number of chocolate bars, X, sold by a manufacturer in any month has the following distribution:

x	100	150	200	250	300	400
$p(x)$	0.05	0.15	0.35	0.25	0.15	0.05

What is the probability that X takes a value in the interval $\mu - 2\sigma$ to $\mu + 2\sigma$?

ISBN 978-1-108-45164-2

Solution

First we must find the values of μ and σ.

$$\mu = \mathrm{E}(X) = \sum_x x \cdot p(x)$$
$$= 5 + 22.5 + 70 + 62.5 + 45 + 20$$
$$= 225$$

Before determining the standard deviation σ, we need to find the variance σ^2.

Now $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

and $\mathrm{E}(X^2) = \sum_x x^2 \cdot p(x)$

$$= 500 + 3375 + 14\,000 + 15\,625 + 13\,500 + 8000$$
$$= 55\,000$$

Thus $\mathrm{Var}(X) = 55\,000 - (225)^2$

$$= 4375$$

and so $\sigma = \mathrm{sd}(X)$

$$= \sqrt{4375}$$
$$= 66.14 \quad \text{(correct to two decimal places)}$$

Hence $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$

$= \Pr(92.72 \le X \le 357.28)$

$= \Pr(100 \le X \le 300)$ since X only takes the values in the table

$= 0.95$ from the probability distribution of X

In this example, 95% of the distribution lies within two standard deviations either side of the mean. While this is not always true, in many circumstances it is approximately true.

For many random variables X,

$$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$$

Example 23

A manufacturer knows that the mean number of faulty light bulbs in a batch of 10 000 is 12, with a standard deviation of 3. He wishes to claim to his clients that 95% of batches will contain between c_1 and c_2 faulty light bulbs (where c_1 and c_2 are symmetric about the mean). What are two possible values of c_1 and c_2?

Solution

Since $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$, we can say

$$c_1 = \mu - 2\sigma = 6 \quad \text{and} \quad c_2 = \mu + 2\sigma = 18$$

ISBN 978-1-108-45164-2

Section summary

- The **expected value** (or **mean**) of a discrete random variable X may be considered as the long-run average value of X. It is found by summing the products of each value of X and the probability that X takes that value. That is,

$$\mu = \mathrm{E}(X) = \sum_x x \cdot \Pr(X = x)$$
$$= \sum_x x \cdot p(x)$$

- The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean μ. It is defined as

$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$$

An alternative (computational) formula for variance is

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

- The **standard deviation** of a random variable X is defined as

$$\sigma = \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

- In general, for many random variables X,

$$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$$

Exercise 14D

Example 15

1 Tickets in a game of chance can be purchased for \$5. Each ticket has a 20% chance of winning \$5, a 1% chance of winning \$100, and otherwise loses. How much might you expect to win or lose if you play the game 50 times? CF

Example 16

2 For each of the following probability distributions, find the mean (expected value): SF

a

x	2	4	6	8
$p(x)$	0.1	0.2	0.3	0.4

b

x	−2	−1	0	1	2
$p(x)$	0.1	0.2	0.4	0.2	0.1

c

x	0	1	2	3	4
$p(x)$	0.09	0.22	0.26	0.13	0.30

3 A business consultant evaluates a proposed venture as follows. A company stands to make a profit of \$500 000 with probability 0.2, to make a profit of \$100 000 with probability 0.4, to break even with probability 0.2, and to lose \$200 000 with probability 0.2. Find the expected profit. CF

ISBN 978-1-108-45164-2

CF

4 A spinner is numbered from 1 to 20, and each of the 20 numbers has an equal chance of coming up. A player who bets \$1 on any number wins \$10 if that number comes up; otherwise the \$1 is lost. What is the player's expected profit on the game?

5 Suppose that the following frequency data for the number of female children in a three-child family have been determined from past records.

Number of females	0	1	2	3
Frequency	25	70	65	20

a Based on the data, construct a table giving the probability distribution of X, the number of female children in a three-child family.

b Based on this probability distribution, what is the mean number of female children in a three-child family?

6 A player throws a die with faces numbered from 1 to 6 inclusive. If the player obtains a 3, she throws the die a second time, and in this case her score is the sum of 3 and the second number; otherwise her score is the number first obtained. The player has no more than two throws. Let X be the random variable denoting the player's score. Determine the mean of X.

SF

Example 17 **7** The random variable X represents the number of chocolate chips in a certain brand of biscuit, and is known to have the following probability distribution.

x	2	3	4	5	6
$p(x)$	0.01	0.25	0.40	0.30	0.04

Calculate:

a $\mathrm{E}(X)$ **b** $\mathrm{E}(5X - 4)$

CF

Example 18 **8** Manuel is a car salesperson. In any week his probability of making sales is as follows:

Number of cars sold, x	2	3	4	5	6
$\Pr(X = x)$	0.45	0.25	0.20	0.08	0.02

If he is paid \$2000 commission on each car sold, what is his expected weekly income?

Example 20 **9** A discrete random variable X takes values 0, 1, 2, 3 with probabilities as shown in the table.

x	0	1	2	3
$\Pr(X = x)$	p	0.12	0.24	0.36

a Find p. **b** Find $\mathrm{E}(X)$. **c** Find $\mathrm{Var}(X)$.

10 A spinner is numbered from 1 to 8. The probability of any number coming up is proportional to that number. Thus, if X denotes the number obtained from one spin, then $\Pr(X = x) = kx$ for $x = 1, 2, 3, \dots, 8$, where k is a constant.

a Find the value of k. **b** Find $\mathrm{E}(X)$. **c** Find $\mathrm{Var}(X)$.

ISBN 978-1-108-45164-2

CF

11 A coin and a six-sided die are thrown simultaneously. The random variable X is defined as follows: If the coin shows a head, then X is the score on the die; if the coin shows a tail, then X is twice the score on the die.

a Find the expected value, μ, of X.

b Find $\Pr(X < \mu)$.

c Find $\mathrm{Var}(X)$.

SF

Example 22 **12** A random variable X has the probability distribution shown.

x	0	1	2	3	4	5
$\Pr(X = x)$	0.125	0.125	0.25	0.25	0.125	0.125

Find:

a $\mathrm{E}(X)$, the mean of X

b $\mathrm{Var}(X)$, the variance of X, and hence the standard deviation of X

c $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$

13 A random variable X has the probability distribution shown.

x	1	2	3	4	5
$\Pr(X = x)$	k	$2k$	$3k$	$4k$	$5k$

Find:

a the constant k

b $\mathrm{E}(X)$, the expectation of X

c $\mathrm{Var}(X)$, the variance of X

d $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$

14 Two dice are rolled. If X is the sum of the numbers showing on the two dice, find:

a $\mathrm{E}(X)$, the mean of X

b $\mathrm{Var}(X)$, the variance of X

c $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$

15 The number of heads, X, obtained when a fair coin is tossed six times has the following probability distribution.

x	0	1	2	3	4	5	6
$p(x)$	0.0156	0.0937	0.2344	0.3126	0.2344	0.0937	0.0156

Find:

a $\mathrm{E}(X)$, the mean of X

b $\mathrm{Var}(X)$, the variance of X

c $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$

CF

Example 23 **16** The random variable X, the number of heads observed when a fair coin is tossed 100 times, has a mean of 50 and a standard deviation of 5. If $\Pr(c_1 \le X \le c_2) \approx 0.95$, give possible values of c_1 and c_2.

ISBN 978-1-108-45164-2

Review

AS Nrich

Chapter summary

Probability

- Probability is a numerical measure of the chance of a particular event occurring and may be determined experimentally or by symmetry.
- Whatever method is used to determine the probability, the following rules will hold:
  - $0 \le \Pr(A) \le 1$ for all events $A \subseteq \varepsilon$
  - $\Pr(\varnothing) = 0$ and $\Pr(\varepsilon) = 1$
  - $\Pr(A') = 1 - \Pr(A)$, where A' is the complement of A
  - $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$, the **addition rule**.
- Two events A and B are **mutually exclusive** if $A \cap B = \varnothing$. In this case, we have $\Pr(A \cap B) = 0$ and therefore $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.
- The **conditional probability** of event A occurring, given that event B has already occurred, is

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \text{if } \Pr(B) \neq 0$$

giving $\quad \Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B) \quad$ (the **multiplication rule**)

- The **law of total probability** states that, in the case of two events A and B,

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$$

- Two events A and B are **independent** if $\Pr(A \mid B) = \Pr(A)$, so whether or not B has occurred has no effect on the probability of A occurring.
- Events A and B are independent if and only if $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$.

Discrete random variables

- A **random variable** associates a number with each outcome of a random experiment.
- A **discrete** random variable is one which can take only a countable number of values. Often these values are whole numbers, but not necessarily.
- The **probability distribution** of a discrete random variable X is a function

$$p(x) = \Pr(X = x)$$

that assigns a probability to each value of X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.

- For any discrete probability distribution, the following two conditions must hold:
  1. Each value of $p(x)$ belongs to the interval $[0, 1]$. That is,

  $$0 \le p(x) \le 1 \quad \text{for all } x$$

  2. The sum of all the values of $p(x)$ is 1. That is,

  $$\sum_x p(x) = 1$$

- To determine the probability that X takes a value in the interval from a to b (including the values a and b), add the values of $p(x)$ from $x = a$ to $x = b$:

$$\Pr(a \le X \le b) = \sum_{a \le x \le b} p(x)$$

ISBN 978-1-108-45164-2

Measures of centre and spread

- The **expected value** (or **mean**) of a discrete random variable X may be considered as the long-run average value of X. It is found by summing the products of each value of X and the probability that X takes that value. That is,

$$\mu = \mathrm{E}(X) = \sum_x x \cdot \Pr(X = x) = \sum_x x \cdot p(x)$$

- The expected value of a function of X is given by

$$\mathrm{E}[g(X)] = \sum_x g(x) \cdot p(x)$$

- The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean μ. It is defined as

$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$$

  An alternative (computational) formula for variance is

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

- The **standard deviation** of a random variable X is defined as

$$\sigma = \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

- Linear function of a discrete random variable:

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$
$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$$

- In general, for many random variables X,

$$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$$

Technology-free questions

CF

1 If $\Pr(A) = 0.5$, $\Pr(B) = 0.2$ and $\Pr(A \cup B) = 0.7$, are the events A and B mutually exclusive? Explain.

2 Show, using a diagram or otherwise, that $\Pr(A \cup B) = 1 - \Pr(A' \cap B')$. How would you describe this relationship in words?

3 A box contains five black and four white balls. Find the probability that two balls drawn at random are of different colours if:

a the first ball drawn is replaced before the second is drawn

b the balls are drawn without replacement.

4 A gambler has two coins, A and B; the probabilities of their turning up heads are 0.8 and 0.4 respectively. One coin is selected at random and tossed twice, and a head and a tail are observed. Find the probability that the coin selected was A.

ISBN 978-1-108-45164-2

Review

SF

5 The probability distribution of a discrete random variable X is given by the following table. Show that $p = 0.5$ or $p = 1$.

x	0	1	2	3
$\Pr(X = x)$	$0.4p^2$	0.1	0.1	$1 - 0.6p$

6 A random variable X has the following probability distribution.

x	−1	0	1	2	3	4
$\Pr(X = x)$	k	$2k$	$3k$	$2k$	k	k

Find:

a the constant k **b** $E(X)$, the mean of X **c** $\text{Var}(X)$, the variance of X

7 If X has a probability function given by

$$p(x) = \frac{1}{4}, \quad x = 2, 4, 16, 64$$

find:

a $E(X)$ **b** $\text{Var}(X)$ **c** $\text{sd}(X)$

CF

8 A manufacturer sells cylinders for \$$x$ each; the cost of the manufacture of each cylinder is \$2. If a cylinder is defective, it is returned and the purchase money refunded. A returned cylinder is regarded as a total loss to the manufacturer. The probability that a cylinder is returned is $\frac{1}{5}$.

a Let P be the profit per cylinder. Find the probability function of P.

b Find the mean of P in terms of x.

c How much should the manufacturer sell the cylinders for in order to make a profit in the long term?

9 A group of 1000 drivers were classified according to their age and the number of accidents they had been involved in during the previous year. The results are shown in the table.

	Age < 30	Age ≥ 30
At most one accident	130	170
More than one accident	470	230

a Calculate the probability that, if a driver is chosen at random from this group, the driver is aged less than 30 and has had more than one accident.

b Calculate the probability that a randomly chosen driver is aged less than 30, given that he or she has had more than one accident.

ISBN 978-1-108-45164-2

10 This year, 70% of the population have been immunised against a certain disease. Records indicate that an immunised person has a 5% chance of contracting the disease, whereas for a non-immunised person the chance is 60%. Calculate the overall percentage of the population who are expected to contract the disease.

11 Given $\Pr(A) = \frac{1}{2}$, $\Pr(B) = \frac{1}{4}$ and $\Pr(A \mid B) = \frac{1}{6}$, find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$ **c** $\Pr(A' \mid B)$ **d** $\Pr(A \mid B')$

Multiple-choice questions

1 Consider the following table:

x	-2	-1	0	1	2
$\Pr(X = x)$	$2k$	$3k$	0.1	$3k$	$2k$

For the table to represent a probability function, the value of k is

A 0.09 **B** 0.9 **C** 0.01 **D** 0.2 **E** 1

2 Suppose that the random variable X has the probability distribution given in the following table:

x	-3	-2	-1	0	1	2	3
$\Pr(X = x)$	0.07	0.15	0.22	0.22	0.17	0.12	0.05

$\Pr(-3 \le X < 0)$ is equal to

A 0.59 **B** 0.37 **C** 0.22 **D** 0.44 **E** 0.66

3

x	1	2	3	4	5
$\Pr(X = x)$	0.46	0.24	0.14	0.09	0.07

For this probability distribution, the expected value $E(X)$ is

A 2 **B** 1 **C** 1.59 **D** 2.07 **E** 5.87

4 A random variable X is such that $E(X) = 1.20$ and $E(X^2) = 1.69$. The standard deviation of X is equal to

A 1.3 **B** $\sqrt{3.13}$ **C** 0.25 **D** 0.7 **E** 0.5

5 Suppose that a random variable X is such that $E(X) = 100$ and $\text{Var}(X) = 100$. Suppose further that Y is a random variable such that $Y = 3X + 10$. Then

A $E(Y) = 300$ and $\text{Var}(Y) = 900$
B $E(Y) = 310$ and $\text{Var}(Y) = 300$
C $E(Y) = 310$ and $\text{Var}(Y) = 900$
D $E(Y) = 300$ and $\text{Var}(Y) = 30$
E $E(Y) = 310$ and $\text{Var}(Y) = 100\sqrt{3}$

ISBN 978-1-108-45164-2

Review

6 The random variable X has the probability distribution shown, where $0 < p < \frac{1}{3}$.
The mean of X is

x	-1	0	1
$\Pr(X = x)$	p	$2p$	$1 - 3p$

A 1 **B** 0 **C** $1 - 4p$ **D** $4p$ **E** $1 + 4p$

7 The random variable X has the probability distribution shown on the right.
If the mean of X is 0.2, then

x	-2	0	2
$\Pr(X = x)$	a	b	0.2

A $a = 0.2,\ b = 0.6$ **B** $a = 0.1,\ b = 0.7$ **C** $a = 0.4,\ b = 0.4$
D $a = 0.7,\ b = 0.1$ **E** $a = 0.5,\ b = 0.3$

Extended-response questions

CU

1 Given the following probability function:

x	1	2	3	4	5	6	7
$\Pr(X = x)$	c	$2c$	$2c$	$3c$	c^2	$2c^2$	$7c^2 + c$

a Find c.
b Evaluate $\Pr(X \geq 5)$.
c If $\Pr(X \leq k) > 0.5$, find the minimum value of k.

2 Five identical cards are placed face down on the table. Three of the cards are marked \$5 and the remaining two are marked \$10. A player picks two cards at random (without replacement) and is paid an amount equal to the sum of the values on the two cards. How much should the player pay to play if this is to be a fair game? (A fair game is considered to be one for which $\mathrm{E}(X) = 0$, where X is the profit from the game.)

3 A manufacturing company has three assembly lines: A, B and C. It has been found that 95% of the products produced on assembly line A will be free from faults, 98% from assembly line B will be free from faults and 99% from assembly line C will be free from faults. Assembly line A produces 50% of the day's output, assembly line B produces 30% of the day's output, and the rest is produced on assembly line C. If an item is chosen at random from the company's stock, find the probability that it:

a was produced on assembly line A
b is defective, given that it came from assembly line A
c is defective
d was produced on assembly line A, given that it was found to be defective.

ISBN 978-1-108-45164-2

4 A recent study found that P, the number of passengers per car entering a city on the freeway on a workday morning, is given by:

p	0	1	2	3	4	5
$\Pr(P = p)$	0.39	0.27	0.16	0.12	0.04	0.02

a **i** Compute $E(P)$, the mean number of passengers per car.
ii Compute $\text{Var}(P)$ and hence find the standard deviation of P.
iii Find $\Pr(\mu - 2\sigma \le P \le \mu + 2\sigma)$.

b The fees for cars at a toll booth on the freeway are as follows:

- Cars carrying no passengers pay \$1 toll.
- Cars carrying one passenger pay \$0.40 toll.
- Cars carrying two or more passengers pay no toll.

Let T be the toll paid by a randomly selected car on the freeway.

i Construct the probability distribution of T.
ii Find $E(T)$, the mean toll paid per car.
iii Find $\Pr(\mu - 2\sigma \le T \le \mu + 2\sigma)$.

5 The random variable Y, the number of cars sold in a week by a car salesperson, has the following probability distribution:

y	0	1	2	3	4	5	6	7	8
$\Pr(Y = y)$	0.135	0.271	0.271	0.180	0.090	0.036	0.012	0.003	0.002

a Compute $E(Y)$, the mean number of sales per week.
b Compute $\text{Var}(Y)$ and hence find the standard deviation of Y.
c The car salesperson is given a bonus as follows: If fewer than three cars are sold in the week, no bonus is given; if three or four cars are sold, a \$100 bonus is given; for more than four cars, the bonus is \$200. Let B be the bonus paid to the salesperson.

i Construct the probability distribution for B.
ii Find $E(B)$, the mean bonus paid.

6 A die is loaded such that the chance of throwing a 1 is $\frac{x}{4}$, the chance of a 2 is $\frac{1}{4}$ and the chance of a 6 is $\frac{1}{4}(1 - x)$. The chance of a 3, 4 or 5 is $\frac{1}{6}$. The die is thrown twice.

a Prove that the probability, p, of throwing a total of 7 is given by

$$p = \frac{9x - 9x^2 + 10}{72}$$

b Find the value of x which will make the probability p a maximum, and find this maximum probability.

ISBN 978-1-108-45164-2

Review

7 A game of chance consists of rolling a disc of diameter 2 cm on a horizontal square board. The board is divided into 25 small squares, each of side length 4 cm. A player wins a prize if, when the disc settles, it lies entirely within any one small square. There is a ridge around the outside edge of the board so that the disc always bounces back, cannot fall off and lies entirely within the boundary of the large square.
Prizes are awarded as follows:

Centre	(the middle square)	50c
Inner	(the eight squares surrounding the centre)	25c
Corner	(the four corner squares)	12c
Outer	(any other smaller square)	5c

When no skill is involved, the centre of the disc may be assumed to be randomly distributed over the accessible region.

a Calculate the probability in any one throw of winning:

i 50c **ii** 25c **iii** 12c **iv** 5c **v** no prize

b The proprietor wishes to make a profit in the long run, but is anxious to charge as little as possible to attract customers. He charges C cents, where C is an integer. Find the lowest value of C that will yield a profit.

8 **Discrete uniform distributions**

a Let X represent the number appearing on the uppermost face when a fair die is rolled. The probability distribution of X is shown in the table below.

i Find $\mathrm{E}(X)$.

ii Find $\mathrm{Var}(X)$.

x	1	2	3	4	5	6
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b In general, a random variable X with values $1, 2, 3, \dots, n$ has a **uniform distribution** if each value of X is equally likely, and therefore

$$\Pr(X = x) = \frac{1}{n}, \quad \text{for } x = 1, 2, 3, \dots, n$$

i Find $\mathrm{E}(X)$. Hint: Use the result $1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1)$.

ii Find $\mathrm{Var}(X)$. Hint: Use the result $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.

c A number is chosen randomly from the set $\{1, 2, 3, \dots, 10\}$. Let X represent the number chosen. Find:

i $\Pr(X = 4)$ **ii** $\Pr(X \le 4)$ **iii** $\mathrm{E}(X)$ **iv** $\mathrm{Var}(X)$

ISBN 978-1-108-45164-2

15 Bernoulli sequences and the binomial distribution

Objectives

- To define a **Bernoulli sequence** and a **Bernoulli random variable**.
- To define the **binomial probability distribution**.
- To investigate the shape of the graph of the binomial probability distribution for different values of the parameters.
- To calculate and interpret the **mean**, **variance** and **standard deviation** for the binomial probability distribution.
- To use the binomial probability distribution to solve problems.

The binomial distribution is important because it has very wide application. It is concerned with situations where there are two possible outcomes, and many 'real life' scenarios of interest fall into this category. For example:

- A political poll of voters is carried out. Each polled voter is asked whether or not they would vote for the present government.
- A poll of Year 12 students in Australia is carried out. Each student is asked whether or not they watch the ABC on a regular basis.
- The effectiveness of a medical procedure is tested by selecting a group of patients and recording whether or not it is successful for each patient in the group.
- Components for an electronic device are tested to see if they are defective or not.

The binomial distribution has application in each of these examples.

We will use the binomial distribution again in Chapter 18, where we further develop our understanding of sampling.

This chapter covers Unit 4 Topic 3: Discrete random variables 2.

ISBN 978-1-108-45164-2

15A Introduction to Bernoulli sequences and the binomial distribution

▶ Bernoulli sequences

An experiment often consists of repeated trials, each of which may be considered as having only two possible outcomes. For example, when a coin is tossed, the two possible outcomes are 'head' and 'tail'. When a die is rolled, the two possible outcomes are determined by the random variable of interest for the experiment. If the event of interest is a 'six', then the two outcomes are 'six' and 'not a six'.

A **Bernoulli sequence** is the name used to describe a sequence of repeated trials with the following properties:

- Each trial results in one of two outcomes, which are usually designated as either a success, S, or a failure, F.
- The probability of success on a single trial, p, is constant for all trials (and thus the probability of failure on a single trial is $1 - p$).
- The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).

Example 1

Suppose that a netball player has a probability of $\frac{1}{3}$ of scoring a goal each time she attempts to goal. She repeatedly has shots for goal. Is this a Bernoulli sequence?

Solution

In this example:

- Each trial results in one of two outcomes, goal or miss.
- The probability of scoring a goal ($\frac{1}{3}$) is constant for all attempts, as is the probability of a miss ($\frac{2}{3}$).
- The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).

Thus, the player's shots at goal can be considered a Bernoulli sequence.

▶ Bernoulli random variables

The outcome from a Bernoulli trial is represented by a **Bernoulli random variable**, which is a random variable that takes only the values 1 (indicating a success) and 0 (indicating a failure). Thus a Bernoulli random variable Y has a probability distribution of the form:

y	0	1
$\Pr(Y = y)$	$1 - p$	p

In Example 1, each shot at goal can be modelled by a Bernoulli random variable with $p = \frac{1}{3}$, where 1 represents a goal and 0 represents a miss.

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▶ The binomial probability distribution

We can think of a Bernoulli random variable as counting the number of successes in a single Bernoulli trial. What happens if we count the number of successes in a Bernoulli sequence?

The number of successes in a Bernoulli sequence of n trials is called a **binomial random variable** and is said to have a **binomial probability distribution**.

For example, consider rolling a fair six-sided die three times. Let the random variable X be the number of 3s observed. Let T represent a 3, and let N represent not a 3. Each roll meets the conditions of a Bernoulli trial. Thus X is a binomial random variable.

Now consider all the possible outcomes from the three rolls and their probabilities.

Outcome	Number of 3s	Probability	
TTT	$X = 3$	$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$	$\Pr(X = 3) = (\frac{1}{6})^3$
TTN	$X = 2$	$\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}$	
TNT	$X = 2$	$\frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}$	$\Pr(X = 2) = 3 \times (\frac{1}{6})^2 \times \frac{5}{6}$
NTT	$X = 2$	$\frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}$	
TNN	$X = 1$	$\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}$	
NTN	$X = 1$	$\frac{5}{6} \times \frac{1}{6} \times \frac{5}{6}$	$\Pr(X = 1) = 3 \times \frac{1}{6} \times (\frac{5}{6})^2$
NNT	$X = 1$	$\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$	
NNN	$X = 0$	$\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$	$\Pr(X = 0) = (\frac{5}{6})^3$

Thus the probability distribution of X is given by the following table.

x	0	1	2	3
$\Pr(X = x)$	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

Instead of listing all the outcomes to find the probability distribution, we can use our knowledge of selections from Mathematical Methods Units 1 & 2 (revised in Appendix A).

Consider the probability that $X = 1$, that is, when exactly one 3 is observed. We can see from the table that there are three ways this can occur. Since the 3 could occur on the first, second or third roll of the die, we can consider this as selecting one object from a group of three, which can be done in $\binom{3}{1}$ ways.

Consider the probability that $X = 2$, that is, when exactly two 3s are observed. Again from the table there are three ways this can occur. Since the two 3s could occur on any two of the three rolls of the die, we can consider this as selecting two objects from a group of three, which can be done in $\binom{3}{2}$ ways.

This leads us to a general formula for this probability distribution:

$$\Pr(X = x) = \binom{3}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{3-x} \qquad x = 0, 1, 2, 3$$

This is an example of the binomial distribution.

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If the random variable X is the number of successes in n independent trials, each with probability of success p, then X has a **binomial distribution** and the rule is

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \dots, n$$

where $\binom{n}{x} = \dfrac{n!}{x!\,(n-x)!}$

Example 2

Find the probability of obtaining exactly three heads when a fair coin is tossed seven times, correct to four decimal places.

Solution

Obtaining a head is considered a success here, and the probability of success on each of the seven independent trials is 0.5.

Let X be the number of heads obtained. In this case, the parameters are $n = 7$ and $p = 0.5$.

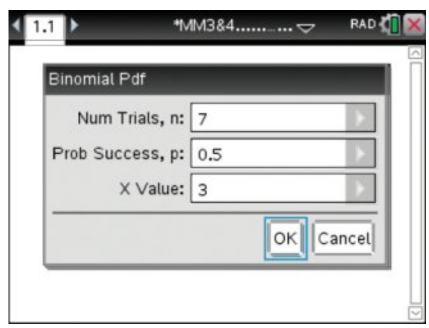
$$\Pr(X = 3) = \binom{7}{3}(0.5)^3(1-0.5)^{7-3}$$
$$= 35 \times (0.5)^7 = 0.2734$$

Using the TI-Nspire CX non-CAS

Use [menu] > **Probability** > **Distributions** > **Binomial Pdf** and complete as shown.

Use [tab] or ▼ to move between cells.

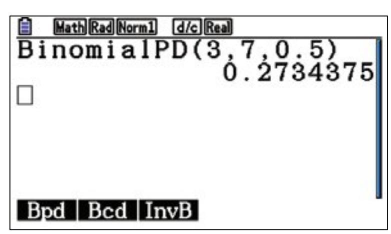
The result is shown below.



Note: You can also type in the command and the parameter values directly if preferred.

Using the Casio

- In **Run-Matrix** mode, go to the **Statistics** menu [OPTN] [F5].
- For the binomial probability distribution, select **Distributions** [F3], **Binomial** [F5], **Bpd** [F1].
- Complete by entering: 3, 7, 0.5)
- Press [EXE].



Note: The syntax for the binomial probability distribution is:
BinomialPD(*number of successes*, *number of trials*, *probability of success*)

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Example 3

The probability that a person currently in prison has ever been imprisoned before is 0.72. Find the probability that of five prisoners chosen at random at least three have been imprisoned before, correct to four decimal places.

Solution

If X is the number of prisoners who have been imprisoned before, then

$$\Pr(X = x) = \binom{5}{x}(0.72)^x(0.28)^{5-x} \qquad x = 0, 1, \ldots, 5$$

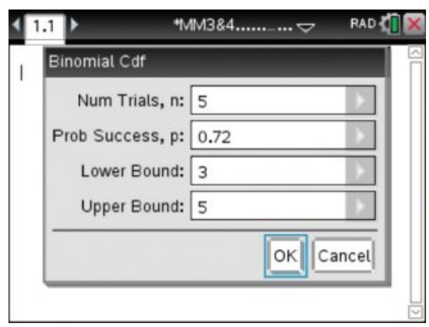
and so

$$\begin{aligned}\Pr(X \geq 3) &= \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)\\ &= \binom{5}{3}(0.72)^3(0.28)^2 + \binom{5}{4}(0.72)^4(0.28)^1 + \binom{5}{5}(0.72)^5(0.28)^0\\ &= 0.8624\end{aligned}$$

Using the TI-Nspire CX non-CAS

Use (menu) > **Probability** > **Distributions** > **Binomial Cdf** and complete as shown.

Use (tab) or ▼ to move between cells.

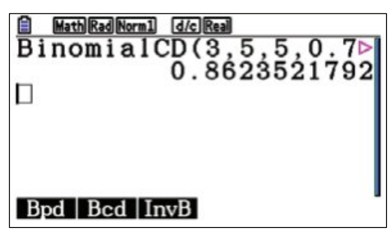


The result is shown below.

Note: You can also type in the command and the parameter values directly if preferred.

Using the Casio

- In **Run-Matrix** mode, go to the **Statistics** menu (OPTN) (F5).
- For the binomial cumulative distribution, select **Distributions** (F3), **Binomial** (F5), **Bcd** (F2).
- Complete by entering: 3, 5, 5, 0.72)
- Press (EXE).



Note: The syntax for the binomial cumulative distribution is:
BinomialCD(*lower bound*, *upper bound*, *number of trials*, *probability of success*)

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▶ The binomial distribution and conditional probability

We can use the binomial distribution to solve problems involving conditional probabilities.

Example 4

The probability of a netballer scoring a goal is 0.3. Find the probability that out of six attempts the netballer scores a goal:

a four times **b** four times, given that she scores at least one goal.

Solution

Let X be the number of goals scored.
Then X has a binomial distribution with $n = 6$ and $p = 0.3$.

a $$\Pr(X = 4) = \binom{6}{4}(0.3)^4(0.7)^2$$
$$= 15 \times 0.0081 \times 0.49$$
$$= 0.059535$$

b $$\Pr(X = 4 \mid X \geq 1) = \frac{\Pr(X = 4 \cap X \geq 1)}{\Pr(X \geq 1)}$$
$$= \frac{\Pr(X = 4)}{\Pr(X \geq 1)}$$
$$= \frac{0.059535}{1 - 0.7^6} \qquad \text{since } \Pr(X \geq 1) = 1 - \Pr(X = 0)$$
$$= 0.0675$$

Section summary

- A **Bernoulli sequence** is a sequence of trials with the following properties:
  - Each trial results in one of two outcomes, which are usually designated as either a success, S, or a failure, F.
  - The probability of success on a single trial, p, is constant for all trials (and thus the probability of failure on a single trial is $1 - p$).
  - The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).
- A **Bernoulli random variable** describes the outcome from a Bernoulli trial; it has a probability distribution of the form $\Pr(Y = 1) = p$ and $\Pr(Y = 0) = 1 - p$.
- The number of successes, X, in a Bernoulli sequence of n trials is called a **binomial random variable** and has a **binomial probability distribution**:

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \ldots, n$$

where $\binom{n}{x} = \dfrac{n!}{x!\,(n - x)!}$

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Exercise 15A

Skillsheet

Example 1

SF

1 Which of the following describes a Bernoulli sequence?

A tossing a fair coin many times

B drawing balls from an urn containing five red and three black balls, replacing the chosen ball each time

C selecting people at random from the population and noting their age

D selecting people at random from the population and noting their sex, male or female

Example 2

2 Find the probability of obtaining exactly four heads when a fair coin is tossed seven times, correct to four decimal places.

3 For a binomial distribution with $n = 4$ and $p = 0.2$, find the probability of:

a three successes **b** four successes.

4 For a binomial distribution with $n = 5$ and $p = 0.4$, find the probability of:

a no successes **b** three successes **c** five successes.

5 Suppose that a fair coin is tossed three times, and the number of heads observed.

a Write down a general rule for the probability distribution of the number of heads.

b Use the rule to calculate the probability of observing two heads.

CF

6 Suppose that X is the number of male children born into a family of six children. Assume that the distribution of X is binomial, with probability of success 0.48.

a Write down a general rule for the probability distribution of the number of male children.

b Use the rule to calculate the probability that a family with six children will have exactly two male children.

Example 3

7 A fair die is rolled six times and the number of 2s noted. Find the probability of:

a exactly three 2s **b** more than three 2s **c** at least three 2s.

8 Jo knows that each ticket has a probability of 0.1 of winning a prize in a lucky ticket competition. Suppose she buys 10 tickets.

a Write down a general rule for the probability distribution of the number of winning tickets.

b Use the rule to calculate the probability that Jo has:

i no wins **ii** at least one win.

9 Suppose that the probability that a person selected at random is left-handed is always 0.2. If 11 people are selected at random for the cricket team:

a Write down a general rule for the probability distribution of the number left-handed people on the team.

b Use the rule to calculate the probability of selecting:

i exactly two left-handers **ii** no left-handers **iii** at least one left-hander.

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CF

10 In a particular city, the probability of rain falling on any given day is $\frac{1}{5}$.

a Write down a general rule for the probability distribution of the number of days of rain in a week.

b Use the rule to calculate the probability that in a particular week rain will fall:

i every day **ii** not at all **iii** on two or three days.

11 The probability of a particular drug causing side effects in a person is 0.2. What is the probability that at least two people in a random sample of 10 people will experience side effects?

12 Records show that $x\%$ of people will pass their driver's licence on the first attempt. If six students attempt their driver's licence, write down in terms of x the probability that:

a all six students pass **b** only one fails **c** no more than two fail.

13 A supermarket has four checkouts. A customer in a hurry decides to leave without making a purchase if all the checkouts are busy. At that time of day the probability of each checkout being free is 0.25. Assuming that whether or not a checkout is busy is independent of any other checkout, calculate the probability that the customer will make a purchase.

14 A fair die is rolled 50 times. Find the probability of observing:

a exactly 10 sixes **b** no more than 10 sixes **c** at least 10 sixes.

15 Find the probability of getting at least nine successes in 100 trials for which the probability of success is $p = 0.1$.

16 A fair coin is tossed 50 times. If X is the number of heads observed, find:

a $\Pr(X = 25)$ **b** $\Pr(X \leq 25)$ **c** $\Pr(X \leq 10)$ **d** $\Pr(X \geq 40)$

17 A survey of the population in a particular city found that 40% of people regularly participate in sport. What is the probability that fewer than half of a random sample of six people regularly participate in sport?

18 An examination consists of six multiple-choice questions. Each question has four possible answers. At least three correct answers are required to pass the examination. Suppose that a student guesses the answer to each question.

a What is the probability the student guesses every question correctly?

b What is the probability the student will pass the examination?

Example 4 **19** The manager of a shop knows from experience that 60% of her customers will use a credit card to pay for their purchases. Find the probability that:

a the next three customers will use a credit card, and the three after that will not

b three of the next six customers will use a credit card

c at least three of the next six customers will use a credit card

d exactly three of the next six customers will use a credit card, given that at least three of the next six customers use a credit card.

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CF

20 A multiple-choice test has eight questions, each with five possible answers, only one of which is correct. Find the probability that a student who guesses the answer to every question will have:

a no correct answers

b six or more correct answers

c every question correct, given they have six or more correct answers.

21 The probability that a full forward in Australian Rules football will kick a goal from outside the 50-metre line is 0.15. If the full forward has 10 kicks at goal from outside the 50-metre line, find the probability that he will:

a kick a goal every time

b kick at least one goal

c kick more than one goal, given that he kicks at least one goal.

15B The graph, expectation and variance of a binomial distribution

We looked at the properties of discrete probability distributions in Chapter 14. We now consider these properties for the binomial distribution.

▶ The graph of a binomial probability distribution

As discussed in Chapter 14, a probability distribution may be represented as a rule, a table or a graph. We now investigate the shape of the graph of a binomial probability distribution for different values of the parameters n and p.

A method for plotting a binomial distribution with a graphics calculator can be found in the calculator appendices in the Interactive Textbook.

Example 5

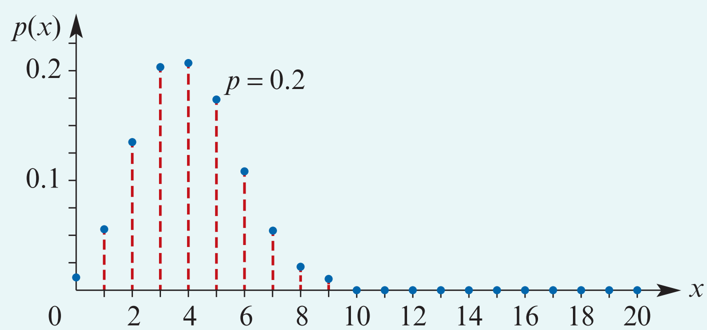
Construct and compare the graph of the binomial probability distribution for 20 trials ($n = 20$) with probability of success:

a $p = 0.2$ **b** $p = 0.5$ **c** $p = 0.8$

Solution

a For $p = 0.2$, the graph is positively skewed.

Mostly from 1 to 8 successes will be observed in 20 trials.



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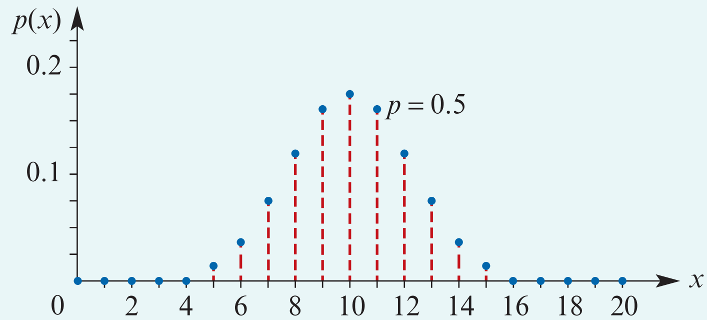
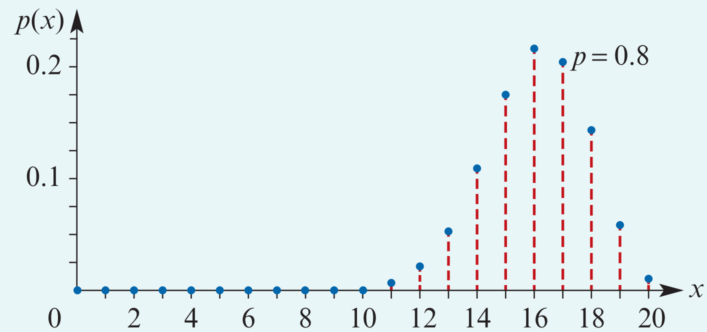
b For $p = 0.5$, the graph is symmetrical (as the probability of success is the same as the probability of failure).

Mostly from 6 to 14 successes will be observed in 20 trials.



c For $p = 0.8$, the graph is negatively skewed.

Mostly from 12 to 19 successes will be observed in 20 trials.



▶ Expectation and variance for a Bernoulli random variable

The table on the right shows the probability distribution for a Bernoulli random variable.

y	0	1
$\Pr(Y = y)$	$1 - p$	p

Thus $\quad \mathrm{E}(Y) = 0 \times (1 - p) + 1 \times p = p$

and $\quad \mathrm{E}(Y^2) = 0^2 \times (1 - p) + 1^2 \times p = p$

so $\quad \mathrm{Var}(Y) = p - p^2 = p(1 - p)$

Hence, if Y is a Bernoulli random variable with probability of success p, then

$$\mathrm{E}(Y) = p$$

$$\mathrm{Var}(Y) = p(1 - p)$$

▶ Expectation and variance for a binomial random variable

How many heads would you expect to obtain, on average, if a fair coin was tossed 10 times?

While the exact number of heads in the 10 tosses would vary, and could theoretically take values from 0 to 10, it seems reasonable that the long-run average number of heads would be 5. It turns out that this is correct. That is, for a binomial random variable X with $n = 10$ and $p = 0.5$,

$$\mathrm{E}(X) = \sum_x x \cdot \Pr(X = x) = 5$$

In general, the expected value of a binomial random variable is equal to the number of trials multiplied by the probability of success. The variance can also be calculated from the parameters n and p.

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If X is the number of successes in n trials, each with probability of success p, then the expected value and the variance of X are given by

$$\mathrm{E}(X) = np$$
$$\mathrm{Var}(X) = np(1-p)$$

Note: These formulas are consistent with those for a Bernoulli random variable, which is a special case of a binomial random variable where $n = 1$.

While it is not necessary in this course to be familiar with the derivations of these formulas, they are included for completeness in the final section of this chapter.

Example 6

An examination consists of 30 multiple-choice questions, each question having three possible answers. A student guesses the answer to every question. Let X be the number of correct answers.

a How many will she expect to get correct? That is, find $\mathrm{E}(X) = \mu$.

b Find $\mathrm{Var}(X)$.

Solution

The number of correct answers, X, is a binomial random variable with parameters $n = 30$ and $p = \frac{1}{3}$.

a The student has an expected result of $\mu = np = 10$ correct answers. (This is not enough to pass if the pass mark is 50%.)

b $$\mathrm{Var}(X) = np(1-p) = 30 \times \frac{1}{3} \times \frac{2}{3} = \frac{20}{3}$$

Section summary

If X is the number of successes in n trials, each with probability of success p, then the expected value and the variance of X are given by

- $\mathrm{E}(X) = np$
- $\mathrm{Var}(X) = np(1-p)$

Exercise 15B

Example 5 **1** Plot the graph of the probability distribution

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \dots, n$$

for $n = 8$ and $p = 0.25$.

SF

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SF

2 Plot the graph of the probability distribution

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

for $n = 12$ and $p = 0.35$.

3 **a** Plot the graph of the probability distribution

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

for $n = 10$ and $p = 0.2$.

b On the same axes, plot the graph of

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

for $n = 10$ and $p = 0.8$, using a different plotting symbol.

c Compare the two distributions.

d Comment on the effect of the value of p on the shape of the distribution.

Example 6

4 Find the mean and variance of the binomial random variables with parameters:

a $n = 25, \ p = 0.2$

b $n = 10, \ p = 0.6$

c $n = 500, \ p = \frac{1}{3}$

d $n = 40, \ p = 20\%$

5 A fair die is rolled six times.

a Find the expected value for the number of sixes obtained.

b Find the probability that more than the expected number of sixes is obtained.

CF

6 The survival rate for a certain disease is 75%. Of the next 50 people who contract the disease, how many would you expect would survive?

7 Records show that 60% of the students in a certain state attend government schools. If a group of 200 students are to be selected at random, find the mean and standard deviation of the number of students in the group who attend government schools.

8 A binomial random variable X has mean 12 and variance 9. Find the parameters n and p, and hence find $\Pr(X = 7)$.

9 A binomial random variable X has mean 30 and variance 21. Find the parameters n and p, and hence find $\Pr(X = 20)$.

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15C Finding the sample size

While we can never be absolutely certain about the outcome of a random experiment, sometimes we are interested in knowing what size sample would be required to observe a certain outcome. For example, how many times do you need to roll a die to be reasonably sure of observing a six, or how many lotto tickets must you buy to be reasonably sure that you will win a prize?

Example 7

The probability of winning a prize in a game of chance is 0.48.

a What is the least number of games that must be played to ensure that the probability of winning at least once is more than 0.95?

b What is the least number of games that must be played to ensure that the probability of winning at least twice is more than 0.95?

Solution

Since the probability of winning each game is the same each time the game is played, this is an example of a binomial distribution, with the probability of success $p = 0.48$.

a The required answer is the smallest value of n such that $\Pr(X \geq 1) > 0.95$.

$$\begin{aligned} & \Pr(X \geq 1) > 0.95 \\ \Leftrightarrow \quad & 1 - \Pr(X = 0) > 0.95 \\ \Leftrightarrow \quad & \Pr(X = 0) < 0.05 \\ \Leftrightarrow \quad & 0.52^n < 0.05 \quad \text{since } \Pr(X = 0) = 0.52^n \end{aligned}$$

This can be solved by taking logarithms of both sides:

$$\begin{aligned} \ln(0.52^n) &< \ln(0.05) \\ n \ln(0.52) &< \ln(0.05) \\ \therefore \quad n &> \frac{\ln(0.05)}{\ln(0.52)} \approx 4.58 \end{aligned}$$

Thus the game must be played at least five times to ensure that the probability of winning at least once is more than 0.95.

b The required answer is the smallest value of n such that $\Pr(X \geq 2) > 0.95$, or equivalently, such that $\Pr(X < 2) < 0.05$. We have

$$\begin{aligned} \Pr(X < 2) &= \Pr(X = 0) + \Pr(X = 1) \\ &= \binom{n}{0} 0.48^0 0.52^n + \binom{n}{1} 0.48^1 0.52^{n-1} \\ &= 0.52^n + 0.48n(0.52)^{n-1} \end{aligned}$$

So the answer is the smallest value of n such that

$$0.52^n + 0.48n(0.52)^{n-1} < 0.05$$

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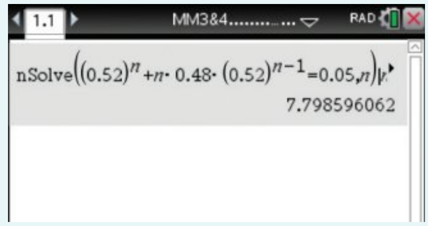
This equation cannot be solved algebraically; but a graphics calculator can be used to find the solution $n > 7.7985\ldots$.
Thus the game must be played at least eight times to ensure that the probability of winning at least twice is more than 0.95.



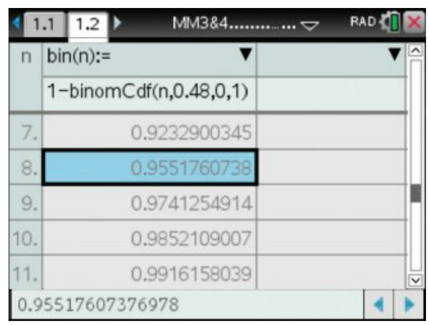
The following calculator inserts give a solution to part **b** of Example 7. Similar techniques can be used for part **a**.

TI-84

Using the TI-Nspire CX non-CAS

To find the smallest value of n such that $\Pr(X \geq 2) > 0.95$, where $p = 0.48$:

- Define the binomial CDF as shown.
  The last two parameters are the lower and upper bounds (inclusive) of the X value.
- Insert a **Lists & Spreadsheet** page. Press (ctrl) (T) to show the table of values.
- Scroll through the table to find where the probability is greater than 0.95. Hence $n = 8$.



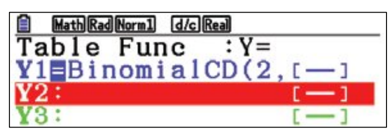
Using the Casio

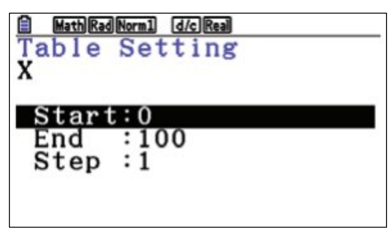
- Press (MENU) (7) to select **Table** mode.
- Select the binomial cumulative distribution:

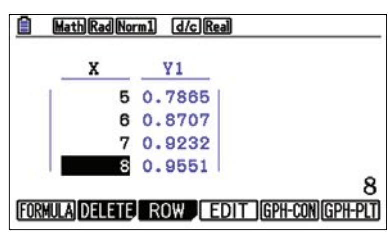
  (OPTN) (F6) (F3) (F1) (F5) (F2)
- Complete the rule as follows: 2, x, x, 0.48)
- Press (EXE).

Note that this rule gives the probability of winning between 2 and x times, when x games are played.

- Select **Set** (F5) and adjust the Table Settings as shown. Press (EXIT) to return to the function list.
- Select **Table** (F6) and scroll down until the probability exceeds 0.95.
- Hence, at least eight games must be played to ensure that the probability of winning at least twice is more than 0.95.







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Cambridge University Press

Exercise 15C

Skillsheet
Example 7

CF

1 The probability of a target shooter hitting the bullseye on any one shot is 0.2.

a If the shooter takes five shots at the target, find the probability of:

i missing the bullseye every time

ii hitting the bullseye at least once.

b What is the smallest number of shots the shooter should make to ensure a probability of more than 0.95 of hitting the bullseye at least once?

c What is the smallest number of shots the shooter should make to ensure a probability of more than 0.95 of hitting the bullseye at least twice?

2 The probability of winning a prize with a lucky ticket on a wheel of fortune is 0.1.

a If a person buys 10 lucky tickets, find the probability of:

i winning twice

ii winning at least once.

b What is the smallest number of tickets that should be bought to ensure a probability of more than 0.7 of winning at least once?

3 Rex is shooting at a target. His probability of hitting the target is 0.6. What is the minimum number of shots needed for the probability of Rex hitting the target exactly five times to be more than 25%?

4 Janet is selecting chocolates at random out of a box. She knows that 20% of the chocolates have hard centres. What is the minimum number of chocolates she needs to select to ensure that the probability of choosing exactly three hard centres is more than 10%?

5 The probability of winning a prize in a game of chance is 0.35. What is the fewest number of games that must be played to ensure that the probability of winning at least twice is more than 0.9?

6 Geoff has determined that his probability of hitting '4' off any ball when playing cricket is 0.07. What is the fewest number of balls he must face to ensure that the probability of hitting more than one '4' is more than 0.8?

7 Monique is practising goaling for netball. She knows from past experience that her chance of making any one shot is about 70%. Her coach has asked her to keep practising until she scores 50 goals. How many shots would she need to attempt to ensure that the probability of scoring at least 50 goals is more than 0.99?

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15D Proofs for the expectation and variance

In this section we give proofs of three important results on the binomial distribution.

> The probabilities of a binomial distribution sum to 1.

Proof The binomial theorem, discussed in Appendix A, states that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Now, using the binomial theorem, the sum of the probabilities for a binomial random variable X with parameters n and p is given by

$$\sum_{x=0}^{n} \Pr(X = x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$
$$= ((1-p)+p)^n = (1)^n = 1$$

> **Expected value**
>
> If X is a binomial random variable with parameters n and p, then $\mathrm{E}(X) = np$.

Proof By the definition of expected value:

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x (1-p)^{n-x} && \text{by the distribution formula} \\
&= \sum_{x=0}^{n} x \cdot \left(\frac{n!}{x!\,(n-x)!}\right) p^x (1-p)^{n-x} && \text{expanding } \tbinom{n}{x} \\
&= \sum_{x=1}^{n} x \cdot \left(\frac{n}{x!\,(n-x)!}\right) p^x (1-p)^{n-x} && \text{since the } x = 0 \text{ term is zero} \\
&= \sum_{x=1}^{n} x \cdot \left(\frac{n!}{x(x-1)!\,(n-x)!}\right) p^x (1-p)^{n-x} && \text{since } x! = x(x-1)! \\
&= \sum_{x=1}^{n} \left(\frac{n!}{(x-1)!\,(n-x)!}\right) p^x (1-p)^{n-x} && \text{cancelling the } x\text{s}
\end{aligned}$$

This expression is very similar to the probability function for a binomial random variable, and we know the probabilities sum to 1. Taking out factors of n and p from the expression and letting $z = x - 1$ gives

$$\mathrm{E}(X) = np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x}$$
$$= np \sum_{z=0}^{n-1} \binom{n-1}{z} p^{z} (1-p)^{n-1-z}$$

Note that this sum corresponds to the sum of all the values of the probability function for a binomial random variable Z, which is the number of successes in $n - 1$ trials each with probability of success p. Therefore the sum equals 1, and so

$$\mathrm{E}(X) = np$$

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Variance

If X is a binomial random variable with parameters n and p, then $\mathrm{Var}(X) = np(1-p)$.

Proof The variance of the binomial random variable X may be found using

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2, \qquad \text{where } \mu = np$$

Thus, to find the variance, we need to determine $\mathrm{E}(X^2)$:

$$\begin{aligned}\mathrm{E}(X^2) &= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} x^2 \left(\frac{n!}{x!\,(n-x)!}\right) p^x (1-p)^{n-x}\end{aligned}$$

But x^2 is not a factor of $x!$ and so we cannot proceed as in the previous proof for expected value.

The strategy used here is to determine $\mathrm{E}[X(X-1)]$:

$$\begin{aligned}\mathrm{E}[X(X-1)] &= \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} x(x-1) \left(\frac{n!}{x!\,(n-x)!}\right) p^x (1-p)^{n-x} \\ &= \sum_{x=2}^{n} x(x-1) \left(\frac{n!}{x!\,(n-x)!}\right) p^x (1-p)^{n-x}\end{aligned}$$

since the first and second terms of the sum equal zero (when $x = 0$ and $x = 1$).

Taking out a factor of $n(n-1)p^2$ and letting $z = x - 2$ gives

$$\begin{aligned}\mathrm{E}[X(X-1)] &= n(n-1)p^2 \sum_{x=2}^{n} \left(\frac{(n-2)!}{(x-2)!\,(n-x)!}\right) p^{x-2} (1-p)^{n-x} \\ &= n(n-1)p^2 \sum_{z=0}^{n-2} \binom{n-2}{z} p^z (1-p)^{n-2-z}\end{aligned}$$

Now the sum corresponds to the sum of all the values of the probability function for a binomial random variable Z, which is the number of successes in $n - 2$ trials each with probability of success p, and is thus equal to 1. Hence

$$\begin{aligned}\mathrm{E}[X(X-1)] &= n(n-1)p^2 \\ \therefore \quad \mathrm{E}(X^2) - \mathrm{E}(X) &= n(n-1)p^2 \\ \therefore \quad \mathrm{E}(X^2) &= n(n-1)p^2 + \mathrm{E}(X) \\ &= n(n-1)p^2 + np\end{aligned}$$

This is an expression for $\mathrm{E}(X^2)$ in terms of n and p, as required. Thus

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mu^2 \\ &= n(n-1)p^2 + np - (np)^2 \\ &= np(1-p)\end{aligned}$$

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Review

Chapter summary

AS
Nrich

- A **Bernoulli sequence** is a sequence of trials with the following properties:
  - Each trial results in one of two outcomes, which are usually designated as either a success, S, or a failure, F.
  - The probability of success on a single trial, p, is constant for all trials (and thus the probability of failure on a single trial is $1 - p$).
  - The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).
- A **Bernoulli random variable** describes the outcome from a Bernoulli trial; it has a probability distribution of the form $\Pr(Y = 1) = p$ and $\Pr(Y = 0) = 1 - p$.
- If X is the number of successes in n Bernoulli trials, each with probability of success p, then X is called a **binomial random variable** and is said to have a **binomial probability distribution** with parameters n and p. The probability of observing x successes in the n trials is given by

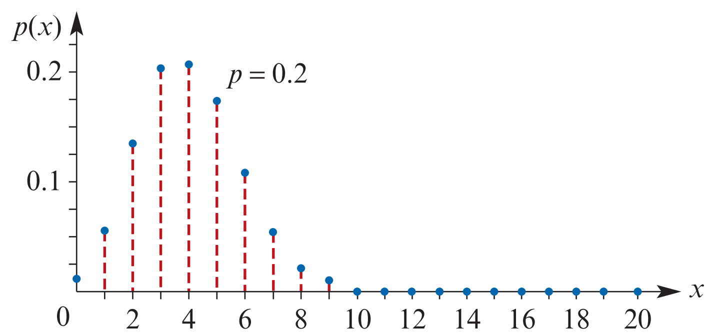
$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

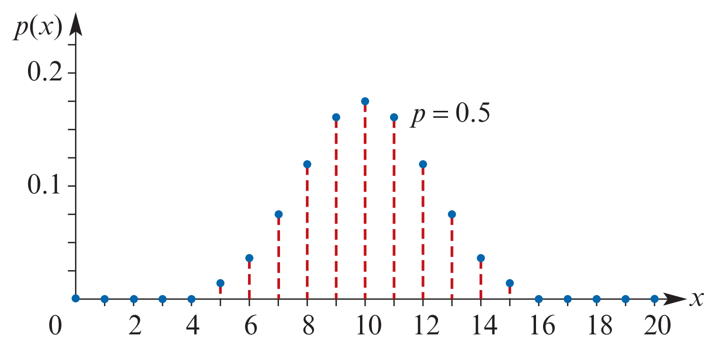
where $\binom{n}{x} = \dfrac{n!}{x!\,(n - x)!}$

- If X has a binomial probability distribution with parameters n and p, then

$$\mathrm{E}(X) = np$$
$$\mathrm{Var}(X) = np(1 - p)$$

- The shape of the graph of a binomial probability function depends on the values of n and p.





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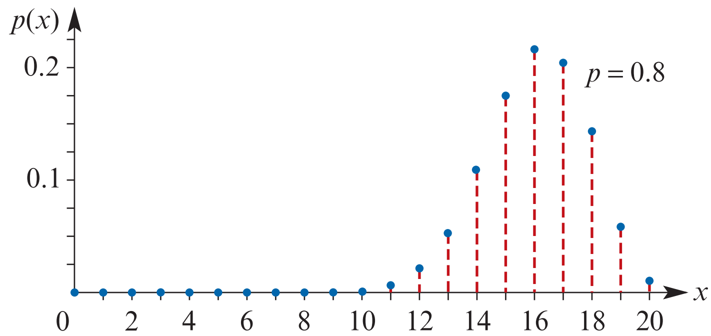

Review

Technology-free questions

SF

1 If X is a binomial random variable with parameters $n = 4$ and $p = \frac{1}{3}$, find:

a $\Pr(X = 0)$ **b** $\Pr(X = 1)$ **c** $\Pr(X \leq 1)$ **d** $\Pr(X \geq 1)$

CF

2 A salesperson knows that 60% of the people who enter a particular shop will make a purchase. What is the probability that of the next three people who enter the shop exactly two will make a purchase?

3 If 10% of patients fail to improve on a certain medication, find the probability that of five patients selected at random one or more will fail to show improvement.

4 A machine has a probability of 0.1 of manufacturing a defective part.

a What is the expected number of defective parts in a random sample of 20 parts manufactured by the machine?

b What is the standard deviation of the number of defective parts?

5 An experiment consists of four independent trials. Each trial results in either a success or a failure. The probability of success in a trial is p. Find the probability of each of the following in terms of p:

a no successes **b** one success **c** at least one success

d four successes **e** at least two successes.

6 A coin is tossed 10 times. The probability of three heads is $m \times (\frac{1}{2})^{10}$. State the value of m.

7 An experiment consists of five independent trials. Each trial results in either a success or a failure. The probability of success in a trial is p. Find, in terms of p, the probability of exactly one success given at least one success.

8 A die is rolled five times. What is the probability of obtaining an even number on the uppermost face on exactly three of the rolls?

9 In a particular city, the probability of rain on any day in June is $\frac{1}{5}$. What is the probability of it raining on three of five days?

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Review

Multiple-choice questions

1 A coin is biased such that the probability of a head is 0.6. The probability that exactly three heads will be observed when the coin is tossed five times is

A 0.6×3 **B** $(0.6)^3$ **C** $(0.6)^3(0.4)^2$ **D** $10 \times (0.6)^3(0.4)^2$ **E** $\binom{5}{3}(0.6)^5$

2 The probability that the 8:25 train arrives on time is 0.35. What is the probability that the train is on time at least once during a working week (Monday to Friday)?

A $1 - (0.65)^5$ **B** $(0.35)^5$ **C** $1 - (0.35)^5$
D $5 \times (0.35)^1(0.65)^4$ **E** $(0.65)^5$

3 A fair die is rolled four times. The probability that a number greater than 4 is observed on two occasions is

A $\frac{1}{4}$ **B** $\frac{16}{81}$ **C** $\frac{1}{9}$ **D** $\frac{1}{81}$ **E** $\frac{8}{27}$

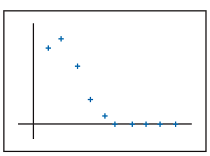
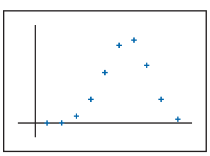
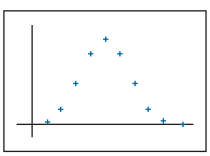
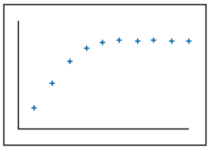
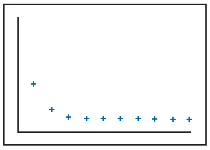
4 The probability that a person in a certain town has a tertiary education is 0.4. What is the probability that, if 80 people are chosen at random from this town, less than 30 will have a tertiary education?

A 0.7139 **B** 0.2861 **C** 0.0827 **D** 0.3687 **E** 0.3750

5 If X is a binomial random variable with parameters $n = 18$ and $p = \frac{1}{3}$, then the mean and variance of X are closest to

A $\mu = 6,\ \sigma^2 = 4$ **B** $\mu = 9,\ \sigma^2 = 4$ **C** $\mu = 6,\ \sigma^2 = 2$
D $\mu = 6,\ \sigma^2 = 16$ **E** $\mu = 18,\ \sigma^2 = 6$

6 Which one of the following best represents the shape of the probability distribution of a binomial random variable X with 10 independent trials and probability of success 0.7?

A

B

C

D

E


7 Suppose that X is a binomial random variable with mean $\mu = 10$ and standard deviation $\sigma = 2$. The probability of success, p, in any trial is

A 0.4 **B** 0.5 **C** 0.6 **D** 0.7 **E** 0.8

8 Suppose that X is the number of heads observed when a coin known to be biased towards heads is tossed 10 times. If $\text{Var}(X) = 1.875$, then the probability of a head on any one toss is

A 0.25 **B** 0.55 **C** 0.75 **D** 0.65 **E** 0.80

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Questions 9 and 10 refer to the following information.

The probability of Thomas beating William in a set of tennis is 0.24, and Thomas and William decide to play a set of tennis every day for n days.

9 What is the fewest number of days on which they should play to ensure that the probability of Thomas winning at least one set is more than 0.95?

A 7 **B** 8 **C** 9 **D** 10 **E** 11

10 What is the fewest number of days on which they should play to ensure that the probability of Thomas winning at least two sets is more than 0.95?

A 12 **B** 18 **C** 17 **D** 21 **E** 14

Extended-response questions

CU

1 In a test to detect learning disabilities, a child is asked 10 questions, each of which has possible answers labelled A, B and C. Children with a disability of type 1 almost always answer A or B on every question, while children with a disability of type 2 almost always answer C on every question. Children without either disability have an equal chance of answering A, B or C for each question.

a What is the probability that the answers given by a child without either disability will be all As and Bs, thereby indicating a type 1 disability?

b A child is further tested for type 2 disability if he or she answers C five or more times. What is the probability that a child without either disability will test positive for type 2 disability?

2 An inspector takes a random sample of 10 items from a very large batch. If none of the items is defective, he accepts the batch; otherwise, he rejects the batch. What is the probability that a batch is accepted if the fraction of defective items is 0, 0.01, 0.02, 0.05, 0.1, 0.2, 0.5, 1? Plot these probabilities against the corresponding fraction defective. Is the inspection method a good one or not?

3 It has been found in the past that 4% of the CDs produced in a certain factory are defective. A sample of 10 CDs is drawn randomly from each hour's production and the number of defective CDs is noted.

a What percentage of these hourly samples would contain at least two defective CDs?

b Find the mean and standard deviation of the number of defective CDs in a sample, and calculate $\mu \pm 2\sigma$.

c A particular sample is found to contain three defective CDs. Would this cause you to have doubts about the production process?

4 A pizza company claims that they deliver 90% of orders within 30 minutes. In a particular 2-hour period, the supervisor notes that there are 67 orders, and of these 12 orders are delivered late. If the company claim is correct, and 90% of orders are delivered on time, what is the probability that at least 12 orders are delivered late?

 ISBN 978-1-108-45164-2

CU

5 **a** A sample of six objects is to be drawn from a large population in which 20% of the objects are defective. Find the probability that the sample contains:

i three defectives **ii** fewer than three defectives.

b Another large population contains a proportion p of defective items.

i Write down an expression in terms of p for P, the probability that a sample of six items contains exactly two defectives.

ii By differentiating to find $\frac{dP}{dp}$, show that P is greatest when $p = \frac{1}{3}$.

6 Groups of six people are chosen at random and the number, x, of people in each group who normally wear glasses is recorded. The table gives the results from 200 groups.

Number wearing glasses, x	0	1	2	3	4	5	6
Number of occurrences	17	53	65	45	18	2	0

a Calculate, from the above data, the mean value of x.

b Assuming that the situation can be modelled by a binomial distribution having the same mean as the one calculated above, state the appropriate values for the binomial parameters n and p.

c Calculate the theoretical frequencies corresponding to those in the table.

7 A sampling inspection scheme is devised as follows. A sample of size 10 is drawn at random from a large batch of articles and all 10 articles are tested. If the sample contains fewer than two faulty articles, the batch is accepted; if the sample contains three or more faulty articles, the batch is rejected; but if the sample contains exactly two faulty articles, a second sample of size 10 is taken and tested. If this second sample contains no faulty articles, the batch is accepted; but if it contains any faulty articles, the batch is rejected. Previous experience has shown that 5% of the articles in a batch are faulty.

a Find the probability that the batch is accepted after the first sample is taken.

b Find the probability that the batch is rejected.

c Find the expected number of articles to be tested.

8 Assume that dates of birth in a large population are distributed such that the probability of a randomly chosen person's birthday being in any particular month is $\frac{1}{12}$.

a Find the probability that of six people chosen at random exactly two will have a birthday in January.

b Find the probability that of eight people at least one will have a birthday in January.

c N people are chosen at random. Find the least value of N such that the probability that at least one will have a birthday in January exceeds 0.9.

9 Suppose that, in flight, aeroplane engines fail with probability q, independently of each other, and that a plane will complete the flight successfully if at least half of its engines are still working. For what values of q is a two-engine plane to be preferred to a four-engine one?

ISBN 978-1-108-45164-2

Chapter 16

16 Continuous random variables

Objectives

- To introduce **continuous random variables**.
- To use relative frequencies to estimate probabilities associated with continuous random variables.
- To use **probability density functions** to specify the distributions for continuous random variables.
- To relate the probability for an interval to an area under the graph of a probability density function.
- To use calculus to find probabilities for intervals from a probability density function.
- To use technology to find probabilities for intervals from a probability density function.
- To calculate and interpret the **expectation** (**mean**), **median**, **variance** and **standard deviation** for a continuous random variable.
- To investigate the mean and variance for a linear function of a random variable.
- To use **cumulative distribution functions** to specify the distributions for continuous random variables.

In this chapter we extend our knowledge of probability to include continuous random variables, which can take any value in an interval of the real number line. Examples include the time taken to complete a puzzle and the height of an adult.

We also introduce the concept of the probability density function to describe the distribution of a continuous random variable. We shall see that probabilities associated with a continuous random variable are described by areas under the probability density function, and thus integration is an important skill required to determine these probabilities.

Chapters 16 and 17 cover Unit 4 Topic 4: Continuous random variables and the normal distribution.

ISBN 978-1-108-45164-2

16A Introduction to continuous random variables

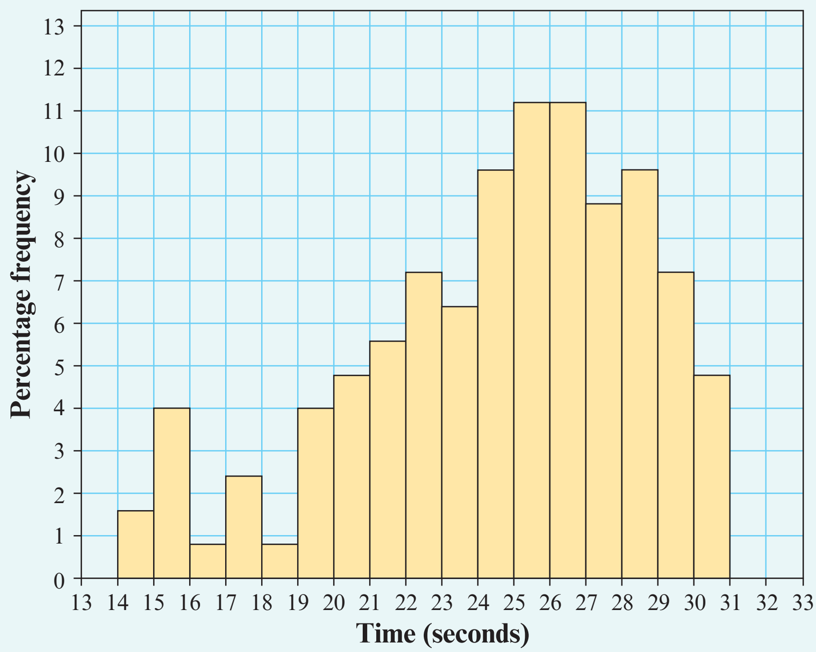
A **continuous random variable** is one that can take any value in an interval of the real number line. For example, if X is the random variable which takes its values as 'distance in metres that a parachutist lands from a marker', then X is a continuous random variable, and here the values which X may take are the non-negative real numbers.

► Using data to approximate a continuous random variable

Relative frequencies obtained from data can be used to approximate the probabilities associated with a continuous random variable.

Example 1

Let T represent the time (in seconds) that it takes a student to complete a particular puzzle. The following percentage frequency histogram was obtained by recording the times taken to complete the puzzle by 500 students, with each recorded time rounded down to a whole number of seconds.



Use the histogram to estimate:

a $\Pr(19 \le T < 22)$ **b** $\Pr(T \ge 28)$

Solution

a $\Pr(19 \le T < 22) \approx 4\% + 4.8\% + 5.6\%$
$= 14.4\%$
$= 0.144$

b $\Pr(T \ge 28) \approx 9.6\% + 7.2\% + 4.8\%$
$= 21.6\%$
$= 0.216$

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▶ An example of a continuous random variable

A continuous random variable has no limit as to the accuracy with which it can be measured. For example, let W be the random variable with values 'a person's weight in kilograms' and let W_i be the random variable with values 'a person's weight in kilograms measured to the ith decimal place'.

Then $W_0 = 83$ implies $82.5 \leq W < 83.5$

$W_1 = 83.3$ implies $83.25 \leq W < 83.35$

$W_2 = 83.28$ implies $83.275 \leq W < 83.285$

$W_3 = 83.281$ implies $83.2805 \leq W < 83.2815$

and so on. Thus, the random variable W cannot take an exact value, since it is always rounded to the limits imposed by the method of measurement used. Hence, the probability of W being exactly equal to a particular value is zero, and this is true for all continuous random variables.

That is,

$$\Pr(W = w) = 0 \quad \text{for all } w$$

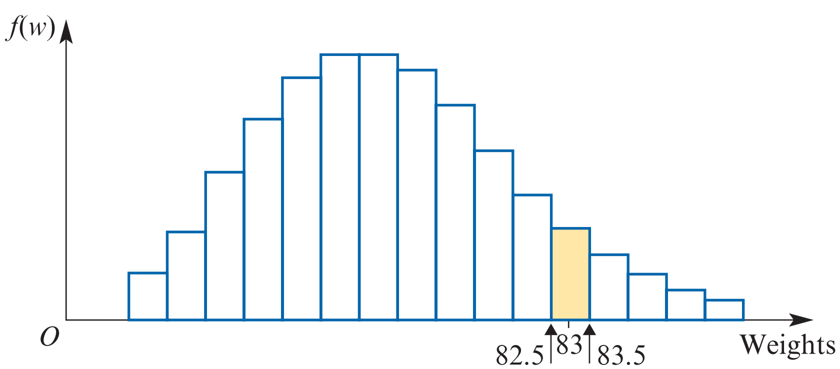
In practice, considering W_i taking a particular value is equivalent to W taking a value in an appropriate interval.

Thus, from above:

$$\Pr(W_0 = 83) = \Pr(82.5 \leq W < 83.5)$$

To determine the value of this probability, you could begin by measuring the weight of a large number of randomly chosen people, and determine the proportion of the people in the group who have weights in this interval.

Suppose after doing this a histogram of weights was obtained as shown.



From this histogram:

$$\Pr(W_0 = 83) = \Pr(82.5 \leq W < 83.5)$$

$$= \frac{\text{shaded area from 82.5 to 83.5}}{\text{total area}}$$

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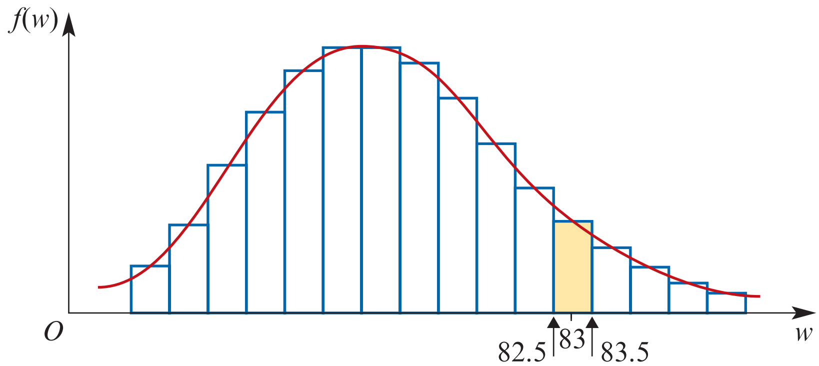
If the histogram is scaled so that the total area under the blocks is 1, then

$$\Pr(W_0 = 83) = \Pr(82.5 \le W < 83.5)$$
$$= \text{area under block from 82.5 to 83.5}$$

Now suppose that the sample size gets larger and that the class interval width gets smaller. If theoretically this process is continued so that the intervals are arbitrarily small, then the histogram can be modelled by a smooth curve, as shown in the following diagram.



The curve obtained here is of great importance for a continuous random variable.

The function f whose graph models the histogram as the number of intervals is increased is called the **probability density function**. The probability density function f is used to describe the probability distribution of a continuous random variable X.

Now, the probability of interest is no longer represented by the area under the histogram, but by the area under the curve. That is,

$$\Pr(W_0 = 83) = \Pr(82.5 \le W < 83.5)$$
$$= \text{area under the graph of the function with rule } f(w) \text{ from 82.5 to 83.5}$$
$$= \int_{82.5}^{83.5} f(w)\, dw$$

▶ Probability density functions

In general, a **probability density function** f is a function with domain some interval (e.g. domain $[c, d]$ or $\mathbb{R}$) such that:

1 $f(x) \ge 0$ for all x in the interval, and

2 the area under the graph of the function is equal to 1.

If the domain of f is $[c, d]$, then the second condition corresponds to $\int_c^d f(x)\, dx = 1$.

In many cases, however, the domain of f will be an 'unbounded' interval such as $[1, \infty)$ or $\mathbb{R}$. Therefore, some new notation is necessary.

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- If the probability density function f has domain $[1, \infty)$, then $\int_1^\infty f(x)\,dx = 1$. This integral is computed as $\lim_{k\to\infty} \int_1^k f(x)\,dx$.
- If the probability density function f has domain $\mathbb{R}$, then $\int_{-\infty}^\infty f(x)\,dx = 1$. This integral is computed as $\lim_{k\to\infty} \int_{-k}^k f(x)\,dx$.

Note: Definite integrals which have one or both limits infinite are called **improper integrals**. There are possible complications with such integrals which we avoid in this course; you will only need the methods of evaluation illustrated in Examples 2 and 4.

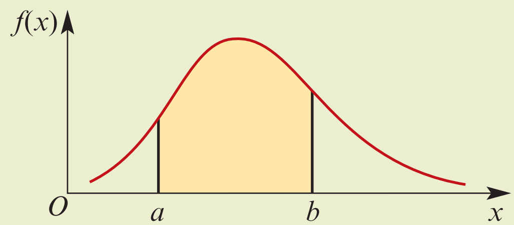
The probability density function of a random variable

Now consider a continuous random variable X with range $[c, d]$. (Alternatively, the range of X may be an unbounded interval such as $(-\infty, d]$, $[c, \infty)$ or $\mathbb{R}$.) Let f be a probability density function with domain $[c, d]$. Then:

We say that f is the **probability density function of X** if

$$\Pr(a < X < b) = \int_a^b f(x)\,dx$$

for all $a < b$ in the range of X.



Notes:

- The values of a probability density function f are not probabilities, and $f(x)$ may take values greater than 1.
- The probability of any specific value of X is 0. That is, $\Pr(X = a) = 0$. It follows that all of the following expressions have the same numerical value:
  - $\Pr(a < X < b)$
  - $\Pr(a \le X < b)$
  - $\Pr(a < X \le b)$
  - $\Pr(a \le X \le b)$
- If f has domain $[c, d]$ and $a \in [c, d]$, then $\Pr(X < a) = \Pr(X \le a) = \int_c^a f(x)\,dx$.

The natural extension of a probability density function

Any probability density function f with domain $[c, d]$ (or any other interval) may be extended to a function f^* with domain $\mathbb{R}$ by defining

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in [c, d] \\ 0 & \text{if } x \notin [c, d] \end{cases}$$

This leads to the following:

A probability density function f (or its natural extension) must satisfy the following two properties:

1 $f(x) \ge 0$ for all x **2** $\int_{-\infty}^\infty f(x)\,dx = 1$

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Example 2

Consider the function f with the rule:

$$f(x) = \begin{cases} 1.5(1 - x^2) & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \text{ or } x < 0 \end{cases}$$

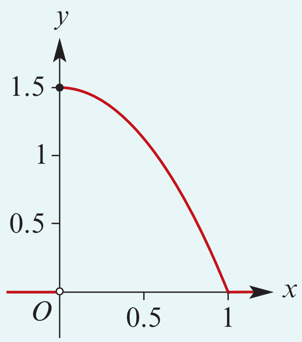
a Sketch the graph of f.

b Show that f is a probability density function.

c Find $\Pr(X > 0.5)$, where the random variable X has probability density function f.

Solution

a For $0 \le x \le 1$, the graph of $y = f(x)$ is part of a parabola with intercepts at $(0, 1.5)$ and $(1, 0)$.



b From the graph, we can see that $f(x) \ge 0$ for all x, and so the first condition holds.

The second condition to check is that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Now $\int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 1.5(1 - x^2)\, dx$ since $f(x) = 0$ elsewhere

$$= 1.5\left[x - \frac{x^3}{3}\right]_0^1$$

$$= 1.5\left(1 - \frac{1}{3}\right)$$

$$= 1$$

Thus the second condition holds, and hence f is a probability density function.

c
$$\Pr(X > 0.5) = \int_{0.5}^{1} 1.5(1 - x^2)\, dx$$

$$= 1.5\left[x - \frac{x^3}{3}\right]_{0.5}^{1}$$

$$= 1.5\left[\left(1 - \frac{1}{3}\right) - \left(0.5 - \frac{0.125}{3}\right)\right]$$

$$= 0.3125$$

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Example 3

Suppose that the random variable X has the probability density function with rule:

$$f(x) = \begin{cases} cx & \text{if } 0 \le x \le 2 \\ 0 & \text{if } x > 2 \text{ or } x < 0 \end{cases}$$

a Find the value of c that makes f a probability density function.

b Find $\Pr(X > 1.5)$.

c Find $\Pr(1 \le X \le 1.5)$.

Solution

a Since f is a probability density function, we know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Now $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^2 cx\,dx$ since $f(x) = 0$ elsewhere

$$= \left[\frac{cx^2}{2}\right]_0^2$$

$$= 2c$$

Therefore $2c = 1$ and so $c = 0.5$.

b

$$\Pr(X > 1.5) = \int_{1.5}^{2} 0.5x\,dx$$

$$= 0.5\left[\frac{x^2}{2}\right]_{1.5}^{2}$$

$$= 0.5\left(\frac{4}{2} - \frac{2.25}{2}\right)$$

$$= 0.4375$$

c

$$\Pr(1 \le X \le 1.5) = \int_{1}^{1.5} 0.5x\,dx$$

$$= 0.5\left[\frac{x^2}{2}\right]_{1}^{1.5}$$

$$= 0.5\left(\frac{2.25}{2} - \frac{1}{2}\right)$$

$$= 0.3125$$

▶ Probability density functions with unbounded domain

Some intervals for which definite integrals need to be evaluated are of the form $(-\infty, a]$ or $[a, \infty)$ or $(-\infty, \infty)$. For a function f with non-negative values, such integrals are defined as follows (provided the limits exist):

- To integrate over the interval $(-\infty, a]$, find $\lim_{k\to-\infty} \int_k^a f(x)\,dx$.
- To integrate over the interval $[a, \infty)$, find $\lim_{k\to\infty} \int_a^k f(x)\,dx$.
- To integrate over the interval $(-\infty, \infty)$, find $\lim_{k\to\infty} \int_{-k}^k f(x)\,dx$.

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Example 4

Consider the exponential probability density function f with the rule:

$$f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

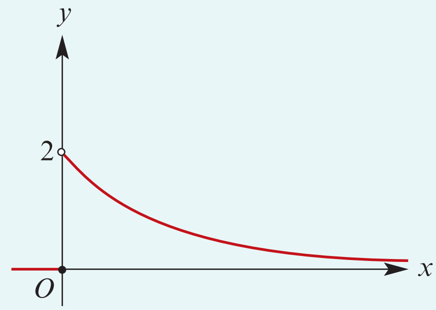
a Sketch the graph of f.

b Show that f is a probability density function.

c Find $\Pr(X > 1)$, where the random variable X has probability density function f.

Solution

a For $x > 0$, the graph of $y = f(x)$ is part of the graph of an exponential function with y-axis intercept 2. As $x \to \infty$, $y \to 0$.



b Since $f(x) \ge 0$ for all x, the first condition holds.

The second condition to check is that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Now $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} 2e^{-2x}\,dx$ since $f(x) = 0$ elsewhere

$$= \lim_{k\to\infty} \int_0^k 2e^{-2x}\,dx$$

$$= \lim_{k\to\infty} \left[\frac{2e^{-2x}}{-2}\right]_0^k$$

$$= \lim_{k\to\infty} \left[-e^{-2x}\right]_0^k$$

$$= \lim_{k\to\infty} \left((-e^{-2k}) - (-e^{-0})\right)$$

$$= 0 + e^0$$

$$= 1$$

Thus f satisfies the two conditions for a probability density function.

c $\Pr(X > 1) = \lim_{k\to\infty} \int_1^k 2e^{-2x}\,dx$

$$= \lim_{k\to\infty} \left[-e^{-2x}\right]_1^k$$

$$= \lim_{k\to\infty} \left((-e^{-2k}) - (-e^{-2})\right)$$

$$= 0 + e^{-2}$$

$$= \frac{1}{e^2}$$

$= 0.1353$ correct to four decimal places

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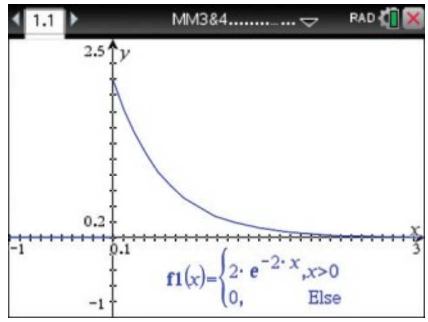
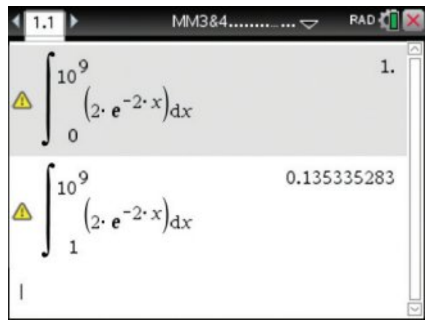
Using the TI-Nspire CX non-CAS

This is an application of integration.

a The graph is as shown. The piecewise function template {⁝⁝ has been used in this example; access the template using (templates).



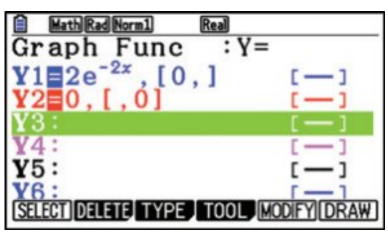
b, c The two required integrations are shown. Use 10^9 for ∞ as the upper terminal value.

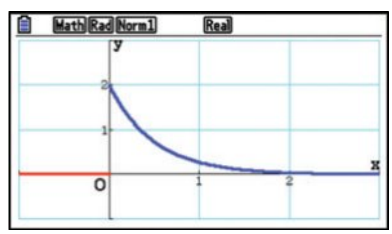


Using the Casio

a To sketch the graph of f:

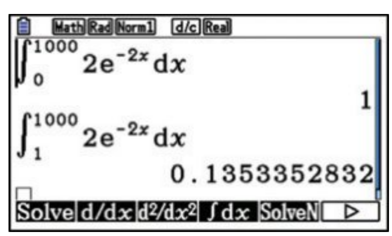
- In **Graph** mode, enter the rule for f as shown below.
- Select **Draw** (F6). Adjust the View Window (SHIFT) (F3) if required.





b To show that f is a probability density function:

- In **Run-Matrix** mode, go to the **Calculation** menu (OPTN) (F4) and select the integral template **∫dx** (F4).
- Complete the integral template as shown. Use $x = 1000$ for the upper terminal value.



c To find $\Pr(X > 1)$, calculate the integral $\int_1^{1000} 2e^{-2x}\,dx$.

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Conditional probability

Next is an example involving conditional probability with continuous random variables.

Example 5

The time (in seconds) that it takes a student to complete a puzzle is a random variable X with a density function given by

$$f(x) = \begin{cases} \dfrac{5}{x^2} & x \geq 5 \\ 0 & x < 5 \end{cases}$$

a Find the probability that a student takes less than 12 seconds to complete the puzzle.

b Find the probability that a student takes between 8 and 10 seconds to complete the puzzle, given that he takes less than 12 seconds.

Solution

a
$$\begin{aligned}\Pr(X < 12) &= \int_5^{12} f(x)\,dx \\ &= \int_5^{12} \frac{5}{x^2}\,dx \\ &= \left[-\frac{5}{x}\right]_5^{12} \\ &= -\frac{5}{12} + 1 \\ &= \frac{7}{12}\end{aligned}$$

b
$$\begin{aligned}&\Pr(8 < X < 10 \mid X < 12) \\ &= \frac{\Pr(8 < X < 10 \cap X < 12)}{\Pr(X < 12)} \\ &= \frac{\Pr(8 < X < 10)}{\Pr(X < 12)} \\ &= \frac{\int_8^{10} f(x)\,dx}{\int_5^{12} f(x)\,dx} \\ &= \frac{-\frac{1}{2} + \frac{5}{8}}{\frac{7}{12}} = \frac{3}{14}\end{aligned}$$

Section summary

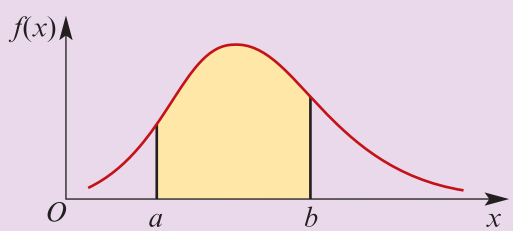
- A probability density function f (or its natural extension) must satisfy the following two properties:

  1 $f(x) \geq 0$ for all x **2** $\int_{-\infty}^{\infty} f(x)\,dx = 1$

- If X is a continuous random variable with density function f, then

  $$\Pr(a < X < b) = \int_a^b f(x)\,dx$$

  which is the area of the shaded region.



- Definite integrals may need to be evaluated over unbounded intervals:
  - To integrate over the interval $(-\infty, a]$, find $\lim\limits_{k\to-\infty} \int_k^a f(x)\,dx$.
  - To integrate over the interval $[a, \infty)$, find $\lim\limits_{k\to\infty} \int_a^k f(x)\,dx$.
  - To integrate over the interval $(-\infty, \infty)$, find $\lim\limits_{k\to\infty} \int_{-k}^k f(x)\,dx$.

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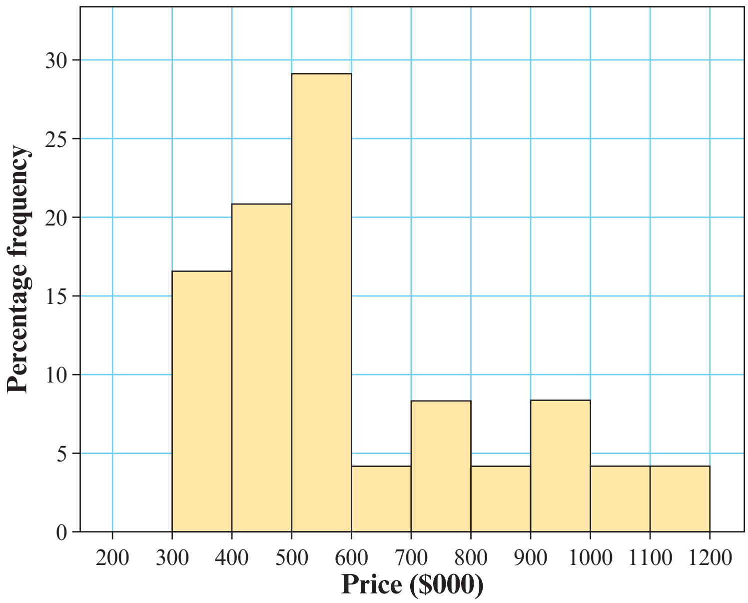
Exercise 16A

Skillsheet
Example 1

1 The following percentage frequency histogram summarises the prices, $\$X$, of apartments in a certain town.

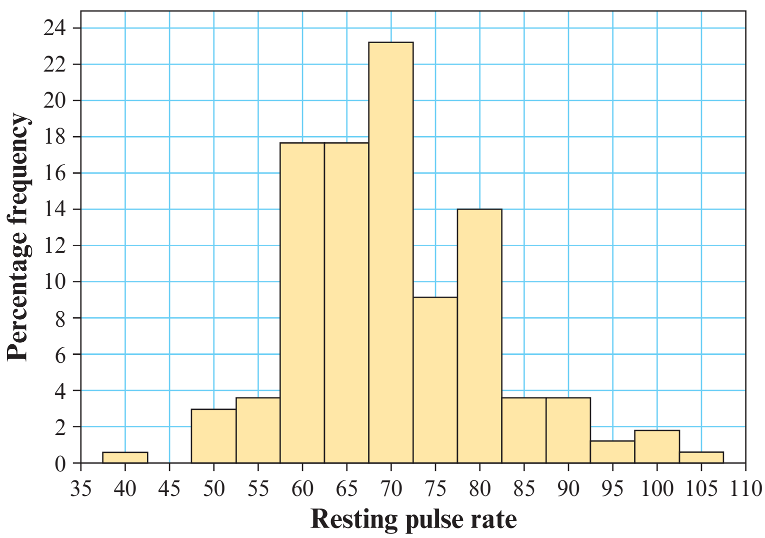


Use the histogram to estimate:

a $\Pr(X \geq 900\,000)$

b $\Pr(600\,000 \leq X < 1\,000\,000)$

2 Let X be the resting pulse rate (in beats per minute) of a randomly chosen person. The following percentage frequency data was obtained from a random sample of people, with each recorded pulse rate rounded to the nearest 5 beats per minute.



Use the histogram to estimate:

a $\Pr(62.5 \leq X < 67.5)$

b $\Pr(67.5 \leq X < 87.5)$

SF

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Example 2 **3** Show that the function f with the following rule is a probability density function:

$$f(x) = \begin{cases} \dfrac{24}{x^3} & 3 \le x \le 6 \\ 0 & x < 3 \text{ or } x > 6 \end{cases}$$

Example 3 **4** Let X be a continuous random variable with the following probability density function:

$$f(x) = \begin{cases} x^2 + kx + 1 & 0 \le x \le 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$$

Determine the constant k such that f is a valid probability density function.

SF

5 Consider the random variable X having the probability density function with the rule:

$$f(x) = \begin{cases} 12x^2(1 - x) & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Sketch the graph of $y = f(x)$.

b Find $\Pr(X < 0.5)$.

c Shade the region which represents this probability on your sketch graph.

CF

Example 4 **6** Consider the random variable Y with the probability density function:

$$f(y) = \begin{cases} ke^{-y} & y \ge 0 \\ 0 & y < 0 \end{cases}$$

a Find the constant k.

b Find $\Pr(Y \le 2)$.

Example 5 **7** The quarantine period for a certain disease is between 5 and 11 days after contact. The probability of showing the first symptoms at various times during the quarantine period is described by the probability density function:

$$f(t) = \frac{1}{36}(t - 5)(11 - t)$$

a Sketch the graph of the function.

b Find the probability that the symptoms appear within 7 days.

c Find the probability that the symptoms appear within 7 days, given that they appear after 5.5 days.

d Find the probability that the symptoms appear within 7 days, given that they appear within 10 days.

8 A probability model for the mass, X kg, of a 2-year-old child is given by

$$f(x) = k\sin\left(\frac{\pi(x - 7)}{10}\right), \quad 7 \le x \le 17$$

a Show that $k = \dfrac{\pi}{20}$.

b Hence find the percentage of 2-year-old children whose mass is:

i greater than 16 kg

ii between 12 kg and 13 kg.

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CF

9 A probability density function for the lifetime, T hours, of Electra light bulbs has rule

$$f(t) = ke^{\left(\frac{-t}{200}\right)}, \quad t > 0$$

a Find the value of the constant k.

b Find the probability that an Electra light bulb will last more than 1000 hours.

10 A random variable X has a probability density function given by

$$f(x) = \begin{cases} k(1+x) & -1 \le x \le 0 \\ k(1-x) & 0 < x \le 1 \\ 0 & x < -1 \text{ or } x > 1 \end{cases}$$

where $k > 0$.

a Sketch the graph of the probability density function. **b** Evaluate k.

c Find the probability that X lies between -0.5 and 0.5.

11 Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Sketch the graph of $y = f(x)$.

b Find $\Pr(0.25 < X < 0.75)$ and illustrate this on your graph.

12 A random variable X has a probability density function f with the rule:

$$f(x) = \begin{cases} \dfrac{1}{100}(10+x) & \text{if } -10 < x \le 0 \\ \dfrac{1}{100}(10-x) & \text{if } 0 < x \le 10 \\ 0 & \text{if } x \le -10 \text{ or } x > 10 \end{cases}$$

a Sketch the graph of f. **b** Find $\Pr(-1 \le X < 1)$.

13 The life, X hours, of a type of light bulb has a probability density function with the rule:

$$f(x) = \begin{cases} \dfrac{k}{x^2} & x > 1000 \\ 0 & x \le 1000 \end{cases}$$

a Evaluate k. **b** Find the probability that a bulb will last at least 2000 hours.

14 The weekly demand for petrol, X (in thousands of litres), at a particular service station is a random variable with probability density function:

$$f(x) = \begin{cases} 2\left(1 - \dfrac{1}{x^2}\right) & 1 \le x \le 2 \\ 0 & x < 1 \text{ or } x > 2 \end{cases}$$

a Determine the probability that more than 1.5 thousand litres are bought in one week.

b Determine the probability that the demand for petrol in one week is less than 1.8 thousand litres, given than it is more than 1.5 thousand litres.

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CF

15 The length of time, X minutes, between the arrival of customers at an ATM is a random variable with probability density function:

$$f(x) = \begin{cases} \frac{1}{5}e^{-\frac{x}{5}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a Find the probability that more than 8 minutes elapses between successive customers.

b Find the probability that more than 12 minutes elapses between successive customers, given that more than 8 minutes has passed.

16 A random variable X has density function given by

$$f(x) = \begin{cases} 0.2 & -1 < x \leq 0 \\ 0.2 + 1.2x & 0 < x \leq 1 \\ 0 & x \leq -1 \text{ or } x > 1 \end{cases}$$

a Find $\Pr(X \leq 0.5)$.

b Hence find $\Pr(X > 0.5 \mid X > 0.1)$.

17 The continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a Sketch the graph of f.

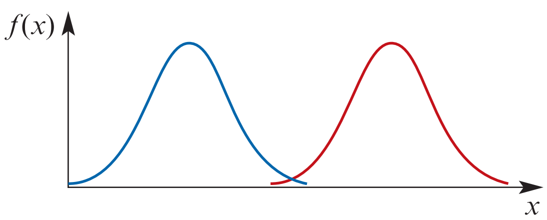
b Find:

i $\Pr(X < 0.5)$ **ii** $\Pr(X \geq 1)$ **iii** $\Pr(X \geq 1 \mid X > 0.5)$

16B Mean and median for a continuous random variable

The centre is an important summary feature of a probability distribution.

The following diagram shows two probability distributions which are identical except for their centres.



More than one measure of centre may be determined for a continuous random variable, and each gives useful information about the random variable under consideration. The most generally useful measure of centre is the mean.

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▶ Mean

We defined the mean for a discrete random variable in Section 14D. We can also define the mean for a continuous random variable.

For a continuous random variable X with probability density function f, the **mean** or **expected value** of X is given by

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

provided the integral exists. The mean is denoted by the Greek letter μ (mu).

If $f(x) = 0$ for all $x \notin [c, d]$, then $\mathrm{E}(X) = \int_{c}^{d} x f(x)\, dx$.

This definition is consistent with the definition of the expected value for a discrete random variable. As in the case of a discrete random variable, the expected value of a continuous random variable is the long-run average value of the variable. For example, consider the daily demand for petrol at a service station. The mean of this variable tells us the average daily demand for petrol over a very long period of time.

Example 6

Find the expected value of the random variable X which has probability density function with rule:

$$f(x) = \begin{cases} 0.5x & 0 \le x \le 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$$

Solution

By definition,

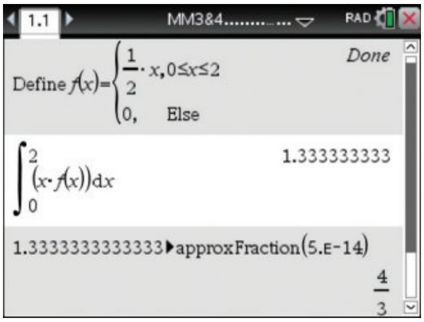
$$\begin{aligned} \mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx &= \int_0^2 x \times 0.5x\, dx \qquad \text{since } f(x) = 0 \text{ elsewhere} \\ &= 0.5 \int_0^2 x^2\, dx \\ &= 0.5 \left[\frac{x^3}{3} \right]_0^2 \\ &= \frac{4}{3} \end{aligned}$$

Using the TI-Nspire CX non-CAS

Define the function f as shown; access the piecewise function template using [template key].

Notes:

- Leave the domain for the last function piece blank; it will autofill as 'Else'.
- To convert the answer to a fraction, use (menu) > **Number** > **Approximate to Fraction**.



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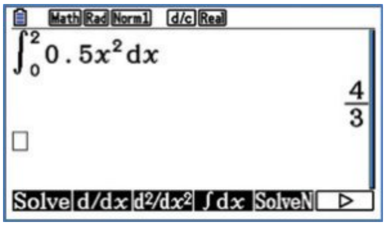
Using the Casio

- In **Run-Matrix** mode, go to the **Calculation** menu OPTN F4 and select **∫dx** F4.
- Complete the template as shown to integrate $xf(x)$ over the interval where $f(x)$ is non-zero.



The mean of a function of X is calculated as follows. (In this case, the function of X is denoted by $g(X)$ and is the composition of the random variable X followed by the function g.)

The expected value of $g(X)$ is given by

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

provided the integral exists.

Generally, as in the case of a discrete random variable, the expected value of a function of X is not equal to that function of the expected value of X. That is,

$$\mathrm{E}[g(X)] \neq g[\mathrm{E}(X)]$$

Example 7

Let X be a random variable with probability density function f given by

$$f(x) = \begin{cases} 0.5x & 0 \le x \le 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$$

Find:

a the expected value of X^2

b the expected value of e^X.

Solution

a
$$\begin{aligned} \mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\, dx \\ &= \int_0^2 x^2 \times 0.5x\, dx \\ &= 0.5 \int_0^2 x^3\, dx \\ &= 0.5 \left[\frac{x^4}{4} \right]_0^2 \\ &= 2 \end{aligned}$$

b
$$\begin{aligned} \mathrm{E}(e^X) &= \int_{-\infty}^{\infty} e^x f(x)\, dx \\ &= \int_0^2 e^x f(x)\, dx \\ &= \int_0^2 e^x \times 0.5x\, dx \\ &= 4.195 \end{aligned}$$

correct to three decimal places.

A case where the equality does hold is where g is a linear function:

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b \qquad \text{(for } a, b \text{ constant)}$$

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► Percentiles and the median

Another value of interest is the value of X which bounds a particular area under the probability density function. For example, a teacher may wish determine the mark, p, below which lie 75% of all students' marks. This is called the 75th percentile of the population, and is found by solving

$$\int_{-\infty}^{p} f(x)\,dx = 0.75$$

This can be stated more generally:

Percentiles

The value p of X which is the solution of an equation of the form

$$\int_{-\infty}^{p} f(x)\,dx = q$$

is called a **percentile** of the distribution.

For example, the 75th percentile is the value p found by taking $q = 75\% = 0.75$.

Example 8

The duration of telephone calls to the order department of a large company is a random variable, X minutes, with probability density function:

$$f(x) = \begin{cases} \dfrac{1}{3}e^{-\frac{x}{3}} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Find the value of a such that 90% of phone calls last less than a minutes.

Solution

To find the value of a, solve the equation:

$$\int_{0}^{a} \frac{1}{3}e^{-\frac{x}{3}}\,dx = 0.9$$

$$\left[-e^{-\frac{x}{3}}\right]_0^a = 0.9$$

$$1 - e^{-\frac{a}{3}} = 0.9$$

$$-\frac{a}{3} = \ln 0.1$$

$$\therefore \quad a = 3\ln 10$$

$$= 6.908 \quad \text{(correct to three decimal places)}$$

So 90% of the calls to this company last less than 6.908 minutes.

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A percentile of special interest is the **median**, or 50th percentile. The median is the middle value of the distribution. That is, the probability of X taking a value below the median is 0.5, and the probability of X taking a value above the median is 0.5. Thus, if m is the median value of the distribution, then

$$\Pr(X \leq m) = \Pr(X > m) = 0.5$$

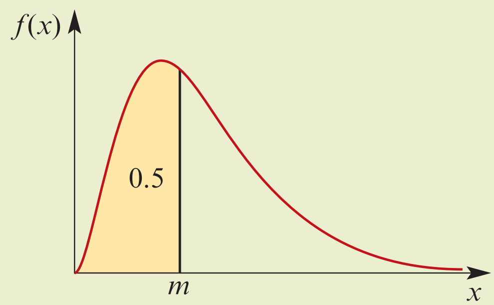
Graphically, the median is the value of the random variable which divides the area under the probability density function in half.

The median

The median is another measure of centre for a continuous probability distribution.

The median, m, of a continuous random variable X is the value of X such that

$$\int_{-\infty}^{m} f(x)\,dx = 0.5$$



Example 9

Suppose the probability density function of weekly sales of topsoil, X (in tonnes), is given by the rule:

$$f(x) = \begin{cases} 2(1-x) & 0 \leq x \leq 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

Find the median value of X, and interpret.

Solution

The median m is such that

$$\int_0^m 2(1-x)\,dx = 0.5$$

$$2\left[x - \frac{x^2}{2}\right]_0^m = 0.5$$

$$2m - m^2 = 0.5$$

$$m^2 - 2m + 0.5 = 0$$

$$\therefore \quad m = 0.293 \text{ or } m = 1.707$$

But since $0 \leq x \leq 1$, the median is $m = 0.293$ tonnes.

This means that, in the long run, 50% of weekly sales will be less than 0.293 tonnes, and 50% will be more.

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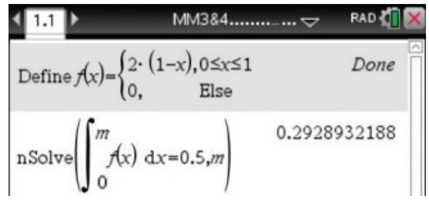
TI-84

Using the TI-Nspire CX non-CAS

This is an application of integration.

- Solve the definite integral equal to 0.5 as shown to find m (the median value).
- The median is $m = 0.293$.

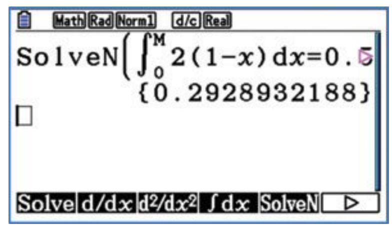


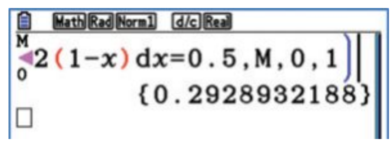
Using the Casio

- In **Run-Matrix** mode, go to the **Calculation** menu (OPTN) (F4) and select **SolveN** (F5).
- Then select **∫dx** (F4) and complete the integral template as $\int_0^M 2(1-x)\,dx$:

  (2) (( ) (1) (−) (X,θ,T) ( )) ► (0) ► (ALPHA) (7)
- Equate the integral to 0.5 and solve for M between 0 and 1:

  ► (SHIFT) (·) (0) (·) (5) (,)
  (ALPHA) (7) (,) (0) (,) (1) ( )) (EXE) (EXIT)





Section summary

For a continuous random variable X with probability density function f:

- the **mean** or **expected value** of X is given by $\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx$
- the expected value of $g(X)$ is given by $\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$
- the **median** of X is the value m such that $\int_{-\infty}^{m} f(x)\,dx = 0.5$

Exercise 16B

Skillsheet
Example 6

SF

1 Find the mean, E(X), of the continuous random variables with the following probability density functions:

a $f(x) = 2x, \quad 0 < x < 1$

b $f(x) = \dfrac{1}{2\sqrt{x}}, \quad 0 < x < 1$

c $f(x) = 6x(1-x), \quad 0 < x < 1$

d $f(x) = \dfrac{1}{x^2}, \quad x \geq 1$

2 For each of the following, use your calculator to check that f is a probability density function and to find the mean, E(X), of the corresponding continuous random variable:

a $f(x) = \sin x, \quad 0 < x < \dfrac{\pi}{2}$

b $f(x) = \ln x, \quad 1 < x < e$

c $f(x) = \dfrac{1}{\sin^2 x}, \quad \dfrac{\pi}{4} < x < \dfrac{\pi}{2}$

d $f(x) = -4x \ln x, \quad 0 < x < 1$

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SF

3 A continuous random variable X has the probability density function given by

$$f(x) = \begin{cases} 2x^3 - x + 1 & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Find μ, the mean value of X.

b Find the probability that X takes a value less than or equal to the mean.

4 Consider the probability density function given by

$$f(x) = \frac{1}{2\pi}(1 + \cos x), \quad -\pi \le x \le \pi$$

Find the expected value of X.

CF

5 A random variable Y has the probability density function:

$$f(y) = \begin{cases} Ay & 0 \le y \le B \\ 0 & y < 0 \text{ or } y > B \end{cases}$$

Find A and B if the mean of Y is 2.

Example 7

6 A random variable X has the probability density function given by

$$f(x) = \begin{cases} 12x^2(1 - x) & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Find $\mathrm{E}\left(\frac{1}{X}\right)$. **b** Find $\mathrm{E}(e^X)$.

Example 8

7 The time, X seconds, between arrivals of particles at a radiation counter has been found to have a probability density function f with the rule:

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \ge 0 \end{cases}$$

a Find $\Pr(X \le 1)$. **b** Find $\Pr(1 \le X \le 2)$. **c** Find the median, m, of X.

8 The random variable X has a probability density function given by

$$f(x) = \begin{cases} k & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Find the value of k. **b** Find the median, m, of X.

9 A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} 5(1 - x)^4 & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

Find the median, m, of X correct to four decimal places.

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CF

10 Suppose that the time (in minutes) between telephone calls received at a pizza restaurant has the probability density function:

$$f(x) = \begin{cases} \frac{1}{4}e^{-\frac{x}{4}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Find the median time between calls.

Example 9 **11** A continuous random variable X has the probability density function given by

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & x < 0 \text{ or } x \geq 2 \end{cases}$$

a Find μ, the expected value of X.
b Find m, the median value of X.

12 Let the probability density function of X be given by

$$f(x) = \begin{cases} 30x^4(1 - x) & 0 < x < 1 \\ 0 & x \leq 0 \text{ or } x \geq 1 \end{cases}$$

a Find the expected value, μ, of X.
b Find the median value, m, of X and hence show the mean is less than the median.

13 A probability model for the mass, X kg, of a 2-year-old child is given by

$$f(x) = \frac{\pi}{20}\sin\left(\frac{\pi(x - 7)}{10}\right), \quad 7 \leq x \leq 17$$

Find the median value, m, of X.

14 A random variable X has density function given by

$$f(x) = \begin{cases} 0.2 & -1 \leq x \leq 0 \\ 0.2 + 1.2x & 0 < x \leq 1 \\ 0 & x < -1 \text{ or } x > 1 \end{cases}$$

a Find μ, the expected value of X.
b Find m, the median value of X.

15 The exponential probability distribution describes the distribution of the time between random events, such as phone calls. The general form of the exponential distribution with parameter λ is

$$f(x) = \begin{cases} \frac{1}{\lambda}e^{-\frac{x}{\lambda}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a Differentiate $(kx + 1)e^{-kx}$ and hence find an anti-derivative of kxe^{-kx}.
b Show that the mean of an exponential random variable is λ.
c On the same axes, sketch the graphs of the distribution for $\lambda = \frac{1}{2}$, $\lambda = 1$ and $\lambda = 2$.
d Describe the effect of varying the value of λ on the graph of the distribution.

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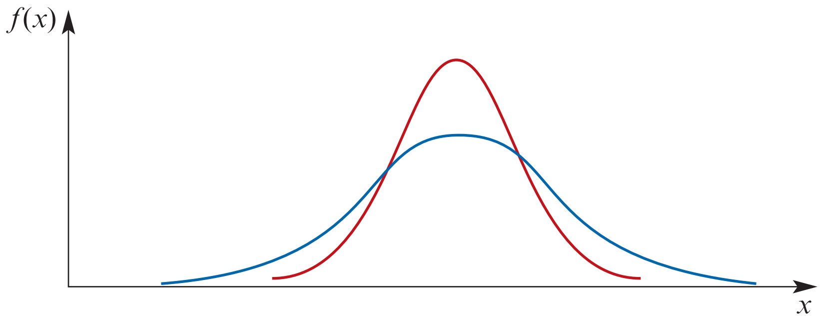
16C Measures of spread

Another important summary feature of a distribution is variation or spread. The following diagram shows two distributions that are identical except for their spreads.



As in the case of centre, there is more than one measure of spread. The most commonly used is the variance, together with its companion measure, the standard deviation. Others that you may be familiar with are the range and the interquartile range.

► Variance and standard deviation

The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean or expected value μ. It is defined as:

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}[(X-\mu)^2]\\ &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx\end{aligned}$$

As for discrete random variables, the variance is usually denoted by σ^2, where σ is the lowercase Greek letter *sigma*.

Variance may be considered as the long-run average value of the square of the distance from X to μ. This means that the variance is not in the same units of measurement as the original random variable X. A measure of spread in the appropriate unit is found by taking the square root of the variance.

The **standard deviation** of X is defined as:

$$\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

The standard deviation is usually denoted by σ.

As in the case of discrete random variables, an alternative (computational) formula for variance is generally used.

To calculate variance, use

$$\mathrm{Var(X)} = \mathrm{E}(X^2) - \mu^2$$

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Proof The computational form of the expression for variance is derived as follows:

$$\begin{aligned}\text{Var}(X) &= \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx \\ &= \int_{-\infty}^{\infty}(x^2-2\mu x+\mu^2)f(x)\,dx \\ &= \int_{-\infty}^{\infty}x^2 f(x)\,dx - \int_{-\infty}^{\infty}2\mu x f(x)\,dx + \int_{-\infty}^{\infty}\mu^2 f(x)\,dx \\ &= \text{E}(X^2) - 2\mu\int_{-\infty}^{\infty}x f(x)\,dx + \mu^2\int_{-\infty}^{\infty}f(x)\,dx\end{aligned}$$

Since $\int_{-\infty}^{\infty} x f(x)\,dx = \mu$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$, we obtain

$$\begin{aligned}\text{Var}(X) &= \text{E}(X^2) - 2\mu^2 + \mu^2 \\ &= \text{E}(X^2) - \mu^2\end{aligned}$$

Example 10

Find the variance and standard deviation of the random variable X which has the probability density function f with rule:

$$f(x) = \begin{cases} 0.5x & 0 \le x \le 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$$

Solution

Use the computational formula $\text{Var}(X) = \text{E}(X^2) - \mu^2$.

First evaluate $\text{E}(X^2)$:

$$\begin{aligned}\text{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx \\ &= \int_0^2 x^2 \times 0.5x\,dx \\ &= 0.5\int_0^2 x^3\,dx \\ &= 0.5\left[\frac{x^4}{4}\right]_0^2 \\ &= 0.5 \times 4 \\ &= 2\end{aligned}$$

Since $\text{E}(X) = \dfrac{4}{3}$ from Example 6, we now have

$$\text{Var}(X) = 2 - \left(\frac{4}{3}\right)^2 = \frac{2}{9}$$

and $\quad \text{sd}(X) = \sqrt{\dfrac{2}{9}} = \dfrac{\sqrt{2}}{3} = 0.471$ (correct to three decimal places)

It helps to make the standard deviation more meaningful to give it an interpretation which relates to the probability distribution. As already stated for discrete random variables, it is also the case for many continuous random variables that about 95% of the distribution lies within two standard deviations either side of the mean.

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In general, for many continuous random variables X,

$$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$$

Example 11

The life of a certain brand of battery, X hours, is a continuous random variable with mean 50 and variance 16. Find an (approximate) interval for the time period for which 95% of the batteries would be expected to last.

Solution

$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$

Since $\mu = 50$ and $\sigma = \sqrt{16} = 4$, we expect 95% of the batteries to last between 42 hours and 58 hours.

Interquartile range

The **interquartile range** is the range of the middle 50% of the distribution; it is the difference between the 75th percentile (also known as Q3) and the 25th percentile (also known as Q1).

Example 12

Determine the interquartile range of the random variable X which has the probability density function:

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

Solution

To find the 25th percentile a, solve:

$$\int_0^a 2x\,dx = 0.25$$
$$[x^2]_0^a = 0.25$$
$$a^2 = 0.25$$
$$\therefore \quad a = \sqrt{0.25} = 0.5$$

To find the 75th percentile b, solve:

$$\int_0^b 2x\,dx = 0.75$$
$$[x^2]_0^b = 0.75$$
$$b^2 = 0.75$$
$$\therefore \quad b = \sqrt{0.75} \approx 0.866$$

Thus the interquartile range is $0.866 - 0.5 = 0.366$, correct to three decimal places.

Note that the negative solutions to these equations were not appropriate, as $0 \le x \le 1$.

Section summary

- To calculate the **variance** of a continuous random variable X, use

  $$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$$

- The **standard deviation** of X is defined by $\sigma = \sqrt{\mathrm{Var}(X)}$.

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- The **interquartile range** of X is

  $$\text{IQR} = b - a$$

  where a and b are such that

  $$\int_{-\infty}^{a} f(x)\,dx = 0.25 \quad \text{and} \quad \int_{-\infty}^{b} f(x)\,dx = 0.75$$

  and where f is the probability density function of X.

Exercise 16C

SF

Example 10 **1** A random variable X has probability density function:

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & x \le 0 \text{ or } x \ge 1 \end{cases}$$

Find the variance of X, and hence find the standard deviation of X.

CF

Example 11 **2** The life of a certain brand of light bulb, X hours, is a continuous random variable with mean 400 and variance 64. Find an (approximate) interval for the time period for which 95% of the light bulbs would be expected to last.

Example 12 **3** A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Find a such that $\Pr(X \le a) = 0.25$.

b Find b such that $\Pr(X \le b) = 0.75$.

c Find the interquartile range of X.

4 A random variable X has the probability density function given by

$$f(x) = \begin{cases} 0.5e^{x} & x \le 0 \\ 0.5e^{-x} & x > 0 \end{cases}$$

a Sketch the graph of $y = f(x)$.

b Find the interquartile range of X, giving your answer correct to three decimal places.

5 A continuous random variable X has probability density function given by

$$f(x) = \begin{cases} \dfrac{k}{x} & 1 \le x \le 9 \\ 0 & x < 1 \text{ or } x > 9 \end{cases}$$

a Find the value of k.

b Find the mean and variance of X, giving your answer correct to three decimal places.

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CF

6 A continuous random variable X has density function f given by

$$f(x) = \begin{cases} 0 & x < 0 \\ 2 - 2x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

a Find the interquartile range of X.

b Find the mean and variance of X.

7 A random variable X has probability density function f with the rule:

$$f(x) = \begin{cases} 0 & x < 0 \\ 2xe^{-x^2} & x \ge 0 \end{cases}$$

Find the interquartile range of X.

8 A random variable X has a probability density function given by

$$f(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{2} & 0 \le x < 2 \\ 0 & x \ge 2 \end{cases}$$

a Find the interquartile range of X.

b Find the mean and variance of X.

9 The queuing time, X minutes, of a traveller at the ticket office of a large railway station has probability density function f defined by

$$f(x) = \begin{cases} kx(100 - x^2) & 0 \le x \le 10 \\ 0 & x > 10 \text{ or } x < 0 \end{cases}$$

a Find the value of k.

b Find the mean of the distribution.

c Find the standard deviation of the distribution, correct to two decimal places.

10 A probability density function is given by

$$f(x) = \begin{cases} k(a^2 - x^2) & -a \le x \le a \\ 0 & x > a \text{ or } x < -a \end{cases}$$

a Find k in terms of a.

b Find the value of a which gives a standard deviation of 2.

11 A continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} k(3 - x) & 0 \le x \le 3 \\ k(x - 3) & 3 < x \le 6 \\ 0 & x > 6 \text{ or } x < 0 \end{cases}$$

where k is a constant.

a Sketch the graph of f.

b Hence, or otherwise, find the value of k.

c Verify that the mean of X is 3.

d Find $\text{Var}(X)$.

ISBN 978-1-108-45164-2

16D Properties of mean and variance*

It has already been stated that the expected value of a function of X is not necessarily equal to that function of the expected value of X. That is, in general,

$$\mathrm{E}[g(X)] \neq g[\mathrm{E}(X)]$$

An exception is the case where the function g is linear: the mean of a linear function of X is equal to the linear function of the mean of X.

▶ The mean and variance of $aX + b$

For any continuous random variable X,

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$

Proof The validity of this statement can be readily demonstrated:

$$\begin{aligned}
\mathrm{E}(aX + b) &= \int_{-\infty}^{\infty} (ax + b) f(x)\,dx \\
&= \int_{-\infty}^{\infty} ax f(x)\,dx + \int_{-\infty}^{\infty} b f(x)\,dx \\
&= a \int_{-\infty}^{\infty} x f(x)\,dx + b \int_{-\infty}^{\infty} f(x)\,dx \\
&= a\mathrm{E}(X) + b \qquad \left(\text{since } \int_{-\infty}^{\infty} f(x)\,dx = 1\right)
\end{aligned}$$

We can also obtain a formula for the variance of a linear function of X.

For any continuous random variable X,

$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$$

Proof Consider the variance of a linear function of X:

$$\mathrm{Var}(aX + b) = \mathrm{E}[(aX + b)^2] - [\mathrm{E}(aX + b)]^2$$

Now $[\mathrm{E}(aX + b)]^2 = [a\mathrm{E}(X) + b]^2 = (a\mu + b)^2 = a^2\mu^2 + 2ab\mu + b^2$

and
$$\begin{aligned}
\mathrm{E}[(aX + b)^2] &= \mathrm{E}(a^2X^2 + 2abX + b^2) \\
&= a^2\mathrm{E}(X^2) + 2ab\mu + b^2
\end{aligned}$$

Thus
$$\begin{aligned}
\mathrm{Var}(aX + b) &= a^2\mathrm{E}(X^2) + 2ab\mu + b^2 - a^2\mu^2 - 2ab\mu - b^2 \\
&= a^2\mathrm{E}(X^2) - a^2\mu^2 \\
&= a^2\mathrm{Var}(X)
\end{aligned}$$

Although initially the absence of b in the variance may seem surprising, on reflection it makes sense that adding a constant merely translates the probability density function, and has no effect on its spread.

* This section is not part of the syllabus for Mathematical Methods Units 3 & 4, but is included for completeness as several important concepts in the following two chapters build on this section.

 ISBN 978-1-108-45164-2

The probability density function of $aX + b$

We can also describe the probability density function of a linear function of X. This idea will underpin much of our discussion of the normal distribution, which is an important continuous probability distribution that is studied in depth in the next chapter.

The random variable $X + b$

If the probability density function of X has rule $f(x)$, then the probability density function of $X + b$ is obtained by the translation $(x, y) \to (x + b, y)$ and so has rule $f(x - b)$.

The random variable aX

Similarly, multiplying by a is similar to a dilation of factor a from the y-axis. However, there has to be an adjustment to determine the rule for the probability density function of aX, as the transformation must be area-preserving. The rule is $\frac{1}{a} f\left(\frac{x}{a}\right)$.

The random variable $aX + b$

Thus, if the probability density function of X has rule $f(x)$, then the probability density function of $aX + b$ has rule $\frac{1}{a} f\left(\frac{x-b}{a}\right)$. The transformation is described by

$$(x, y) \to \left(ax + b, \frac{y}{a}\right)$$

In the case that a and b are positive, this is a dilation of factor a from the y-axis and factor $\frac{1}{a}$ from the x-axis, followed by a translation of b units in the positive direction of the x-axis.

Example 13

Suppose that X is a continuous random variable with mean $\mu = 10$ and variance $\sigma^2 = 2$.

a Find $E(2X + 1)$.

b Find $Var(1 - 3X)$.

c If X has probability density function f, describe the rule of a probability density function g for $2X + 1$.

Solution

a $E(2X + 1) = 2E(X) + 1$
$= 2 \times 10 + 1$
$= 21$

b $Var(1 - 3X) = (-3)^2 Var(X)$
$= 9 \times 2$
$= 18$

c The rule is $g(x) = \frac{1}{a} f\left(\frac{x-b}{a}\right)$ where $a = 2$ and $b = 1$. Therefore $g(x) = \frac{1}{2} f\left(\frac{x-1}{2}\right)$.

Section summary

Linear function of a continuous random variable:

$$E(aX + b) = aE(X) + b$$
$$Var(aX + b) = a^2 Var(X)$$

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Exercise 16D

This exercise is not part of the syllabus, but is included to provide a greater understanding of concepts introduced in the following two chapters.

CF

1 The amount of flour used each day in a bakery is a continuous random variable X with a mean of 4 tonnes. The cost of the flour is $C = 300X + 100$. Find $E(C)$.

2 The mean weight of the oranges grown at a certain orchard is 100 g, with a standard deviation of 5 g. The oranges are sold in bags of 10. If the weight of the empty bag is 10 g, find the mean and standard deviation of the weight of a bag of oranges.

3 For certain glass ornaments, the proportion of impurities per ornament, X, is a random variable with a density function given by

$$f(x) = \begin{cases} \dfrac{3x^2}{2} + x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x < 0 \text{ or } x > 1 \end{cases}$$

The value of each ornament (in dollars) is $V = 100 - 1.5X$.

a Find $E(X)$ and $Var(X)$.

b Hence find the mean and standard deviation of V.

Example 13

4 Let X be a random variable with probability density function:

$$f(x) = \begin{cases} \dfrac{3x^2}{2} & \text{if } -1 \le x \le 1 \\ 0 & \text{if } x < -1 \text{ or } x > 1 \end{cases}$$

Find:

a $E(3X)$ and $Var(3X)$

b $E(3 - X)$ and $Var(3 - X)$

c $E(3X + 1)$ and $Var(3X + 1)$

d the rule of a probability density function for $3X$

e the rule of a probability density function for $3X + 1$.

5 Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} x^2\left(2x + \dfrac{3}{2}\right) & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a **i** Find $E(X)$.

ii Hence find $E(V)$, where $V = 2X + 3$.

b **i** Find $E\left(\dfrac{1}{X}\right)$ and $Var\left(\dfrac{1}{X}\right)$.

ii Hence find $Var(Y)$, where $Y = \dfrac{2}{X} + 3$.

ISBN 978-1-108-45164-2

Cambridge University Press

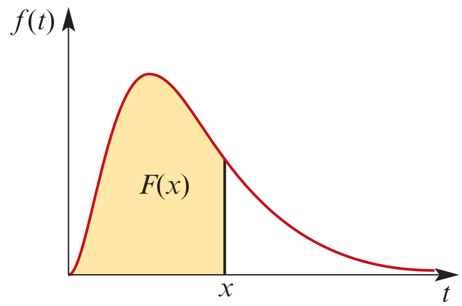
16E Cumulative distribution functions

Another function of importance in describing a continuous random variable is the **cumulative distribution function** or **CDF**. For a continuous random variable X, with probability density function f defined on the interval $[c, d]$, the cumulative distribution function F is given by

$$\begin{aligned} F(x) &= \Pr(X \le x) \\ &= \int_c^x f(t)\, dt \end{aligned}$$

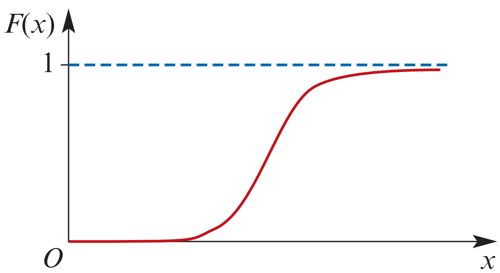
where t is the variable of integration. The cumulative distribution function at a particular value x gives the probability that the random variable X takes a value less than or equal to x.

The diagram on the right shows the relationship between the probability density function f and the cumulative distribution function F.



The function F describes the area under the graph of the probability density function between the lower bound of the domain of f and x. (In the diagram, the lower bound is 0.)

For every continuous random variable X, the cumulative distribution function F is continuous.



Using the general version of the fundamental theorem of calculus, it can be shown that the derivative of the cumulative distribution function is the density function. More precisely, we have $F'(x) = f(x)$, for each value of x at which f is continuous.

There are three important properties of a cumulative distribution function.

For a continuous random variable X with range $[c, d]$:

1. The probability that X takes a value less than or equal to c is 0. That is, $F(c) = 0$.
2. The probability that X takes a value less than or equal to d is 1. That is, $F(d) = 1$.
3. If x_1 and x_2 are values of X with $x_1 \le x_2$, then $\Pr(X \le x_1) \le \Pr(X \le x_2)$. That is,

$$x_1 \le x_2 \quad \text{implies} \quad F(x_1) \le F(x_2)$$

The function F is a **non-decreasing** function.

For a probability density function f defined on $\mathbb{R}$, the cumulative distribution is given by

$$\begin{aligned} F(x) &= \Pr(X \le x) \\ &= \int_{-\infty}^x f(t)\, dt \end{aligned}$$

In this case, we have $F(x) \to 0$ as $x \to -\infty$, and $F(x) \to 1$ as $x \to \infty$.

The importance of the cumulative distribution function is that probabilities for various intervals can be computed directly from $F(x)$.

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Example 14

The time, X seconds, that it takes a student to complete a puzzle is a random variable with density function given by

$$f(x) = \begin{cases} \dfrac{5}{x^2} & x \ge 5 \\ 0 & x < 5 \end{cases}$$

a Find $F(x)$, the cumulative distribution function of X.

b Use the cumulative distribution function to find:

i $\Pr(X \le 7)$ **ii** $\Pr(X \ge 6)$ **iii** $\Pr(10 \le X \le 20)$

Solution

a
$$\begin{aligned} F(x) &= \int_5^x f(t)\,dt \\ &= \int_5^x \frac{5}{t^2}\,dt \\ &= \left[\frac{-5}{t}\right]_5^x \\ &= \frac{-5}{x} + 1 \end{aligned}$$

Thus $F(x) = 1 - \dfrac{5}{x}$ for $x \ge 5$.

b **i**
$$\begin{aligned} \Pr(X \le 7) &= F(7) \\ &= 1 - \frac{5}{7} = \frac{2}{7} \end{aligned}$$

ii
$$\begin{aligned} \Pr(X \ge 6) &= 1 - \Pr(X < 6) \\ &= 1 - F(6) \\ &= 1 - \left(1 - \frac{5}{6}\right) = \frac{5}{6} \end{aligned}$$

iii
$$\begin{aligned} &\Pr(10 \le X \le 20) \\ &= \Pr(X \le 20) - \Pr(X < 10) \\ &= F(20) - F(10) \\ &= \left(1 - \frac{5}{20}\right) - \left(1 - \frac{5}{10}\right) = \frac{1}{4} \end{aligned}$$

Example 15

The time to failure (in hundreds of hours) for a certain electronic component is a random variable X with cumulative distribution function F given by

$$F(x) = \begin{cases} 1 - e^{-x^2} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

Find the rule for a probability density function f for X.

Solution

To find $f(x)$, we differentiate $F(x)$:

$$\begin{aligned} f(x) = F'(x) &= -e^{-x^2} \cdot (-2x) \\ &= 2xe^{-x^2} \qquad \text{(for } x \ge 0\text{)} \end{aligned}$$

Hence

$$f(x) = \begin{cases} 2xe^{-x^2} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

ISBN 978-1-108-45164-2

Section summary

- The **cumulative distribution function** of a continuous random variable X is defined by

  $$F(x) = \Pr(X \le x)$$

- For a probability density function f defined on $\mathbb{R}$, we have

  $$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

- For each value of x at which f is continuous, we have

  $$F'(x) = f(x)$$

Exercise 16E

SF

Example 14

1 The probability density function for a random variable X is given by

$$f(x) = \begin{cases} \dfrac{1}{5} & \text{if } 0 < x < 5 \\ 0 & \text{if } x \le 0 \text{ or } x \ge 5 \end{cases}$$

a Find $F(x)$, the cumulative distribution function of X.

b Hence find $\Pr(X \le 3)$.

CF

2 A continuous random variable X has probability density function

$$f(x) = \begin{cases} \dfrac{1}{4} & \text{if } 0 \le x < 1 \\ \dfrac{x^3}{5} & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a Find $F(x)$, the cumulative distribution function of X.

b Solve the equation $F(x) = 0.5$ for x to find the median value of X.

3 A random variable X has the cumulative distribution function with rule:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x^2} & x \ge 0 \end{cases}$$

a Sketch the graph of $y = F(x)$.

b Find $\Pr(X \ge 2)$.

c Find $\Pr(X \ge 2 \mid X < 3)$.

4 The continuous random variable X has cumulative distribution function F given by

$$F(x) = \begin{cases} 0 & x < 0 \\ kx^2 & 0 \le x \le 6 \\ 1 & x > 6 \end{cases}$$

a Determine the value of the constant k.

b Calculate $\Pr\left(\dfrac{1}{2} \le X \le 1\right)$.

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CF

5 The cumulative distribution function of a random variable X is given by

$$F(x) = \begin{cases} 1 - \dfrac{10}{x} & x > 10 \\ 0 & x \le 10 \end{cases}$$

Use the cumulative distribution function to determine:

a $\Pr(X < 30)$

b the median of X

c the interquartile range of X.

Example 15

6 The cumulative distribution function of a continuous random variable X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ 4x^3 - 3x^4 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

Find a probability density function for X.

7 A continuous random variable X has cumulative distribution function given by

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - (1 - x)^5 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

Find a probability density function for X.

8 Let X be a random variable with the cumulative distribution function

$$F(x) = \begin{cases} 0.5e^{x} & x \le 0 \\ 1 - 0.5e^{-x} & x > 0 \end{cases}$$

Find a probability density function for X.

9 The maximum daily temperature, X°C, at a ski resort during winter has the cumulative distribution function

$$F(x) = \frac{1}{1 + e^{-0.5x}} \quad \text{for } x \in \mathbb{R}$$

a With the help of your calculator, sketch the graph of F.

b Given that the probability density function f of X is continuous everywhere, determine the rule for $f(x)$ and sketch the graph of f.

c Find the interquartile range of X. Give your answer correct to three decimal places.

ISBN 978-1-108-45164-2

Review

Chapter summary

AS

Nrich

- A **continuous random variable** is one that can take any value in an interval of the real number line.
- A continuous random variable can be described by a **probability density function** f. There are many different probability density functions with different shapes and properties. However, they all have the following two fundamental properties:
  1. For any value of x, the value of $f(x)$ is non-negative. That is,
  $$f(x) \geq 0 \text{ for all } x$$
  2. The total area enclosed by the graph of f and the x-axis is equal to 1. That is,
  $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

  The probability of X taking a value in the interval (a, b) is found by determining the area under the probability density curve between a and b. That is,
  $$\Pr(a < X < b) = \int_a^b f(x)\,dx$$
- The **mean** or **expected value** of a continuous random variable X with probability density function f is given by
  $$\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$
  provided the integral exists.
- If $g(X)$ is a function of X, then the expected value of $g(X)$ is given by
  $$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$$
  provided the integral exists. In general, $\mathrm{E}[g(X)] \neq g[\mathrm{E}(X)]$.
- The **median** of a continuous random variable X is the value m such that
  $$\int_{-\infty}^{m} f(x)\,dx = 0.5$$
- The **variance** of a continuous random variable X with probability density function f is defined by
  $$\begin{aligned}\sigma^2 = \mathrm{Var}(X) &= \mathrm{E}[(X-\mu)^2] \\ &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx\end{aligned}$$
  provided the integral exists. To calculate the variance, use
  $$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$$
- The **standard deviation** of X is defined by
  $$\sigma = \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$
- Linear function of a continuous random variable:
  $$\begin{aligned}\mathrm{E}(aX + b) &= a\mathrm{E}(X) + b \\ \mathrm{Var}(aX + b) &= a^2\mathrm{Var}(X)\end{aligned}$$
- In general, for many continuous random variables X,
  $$\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.95$$

ISBN 978-1-108-45164-2

- The **interquartile range** of X is IQR $= b - a$, where a and b are such that

$$\int_{-\infty}^{a} f(x)\,dx = 0.25 \quad \text{and} \quad \int_{-\infty}^{b} f(x)\,dx = 0.75$$

and where f is the probability density function of X.

- The **cumulative distribution function** of a continuous random variable X is defined by

$$F(x) = \Pr(X \le x)$$

- For a probability density function f defined on $\mathbb{R}$, we have $F(x) = \int_{-\infty}^{x} f(t)\,dt$.
- For each value of x at which f is continuous, we have $F'(x) = f(x)$.

Technology-free questions

SF

1 The probability density function of X is given by

$$f(x) = \begin{cases} kx & \text{if } 1 \le x \le \sqrt{2} \\ 0 & \text{otherwise} \end{cases}$$

a Find k. **b** Find $\Pr(1 < X < 1.1)$. **c** Find $\Pr(1 < X < 1.2)$.

CF

2 If the probability density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & x > 1 \text{ or } x < 0 \end{cases}$$

and $\mathrm{E}(X) = \dfrac{2}{3}$, find a and b.

3 The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{\sin x}{2} & 0 \le x \le \pi \\ 0 & x > \pi \text{ or } x < 0 \end{cases}$$

Find the median of X.

4 The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{1}{4} & 1 \le x < 5 \\ 0 & x < 1 \text{ or } x \ge 5 \end{cases}$$

a Find $\Pr(1 < X < 3)$. **b** Find $\Pr(X > 2 \mid 1 < X < 3)$. **c** Find $\Pr(X > 4 \mid X > 2)$.

5 Consider the random variable X having the probability density function given by

$$f(x) = \begin{cases} 12x^2(1 - x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Sketch the graph of $y = f(x)$.
b Find $\Pr(X < 0.5)$ and illustrate this probability on your sketch graph.
c Find $F(x)$, the cumulative distribution function of X.

ISBN 978-1-108-45164-2

Review

CF

6 The probability density function of a random variable X is

$$f(x) = \begin{cases} kx^2(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Determine k.

b Find the probability that X is less than $\frac{2}{3}$.

c Find the probability that X is less than $\frac{1}{3}$, given that X is less than $\frac{2}{3}$.

7 Let X be a continuous random variable with probability density function:

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Find $\Pr(X < 0.2)$. **b** Find $\Pr(X < 0.2 \mid X < 0.3)$.

8 A continuous random variable X has probability density function:

$$f(x) = \begin{cases} \dfrac{\pi}{4}\cos\left(\dfrac{\pi x}{4}\right) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the median value, m, of X.

9 The probability density function f of a random variable X is given by

$$f(x) = \begin{cases} \dfrac{x+2}{16} & \text{if } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a Find $E(X)$. **b** Find a such that $\Pr(X \le a) = \dfrac{5}{32}$.

10 The probability density function f of a random variable X is given by

$$f(x) = \begin{cases} c(1-x^2) & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Find c.

b Find $E(X)$.

c Find $F(x)$, the cumulative distribution function of X.

11 Define the function f by

$$f(x) = \begin{cases} n(1-x)^{n-1} & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

where the constant n is a natural number.

a Show that f is a probability density function.

b Find the corresponding cumulative distribution function F.

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12 The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{1}{x} & 1 \le x \le e \\ 0 & x > e \text{ or } x < 1 \end{cases}$$

a Find $F(x)$, the cumulative distribution function of X.

b Hence find the median value of X and the interquartile range of X.

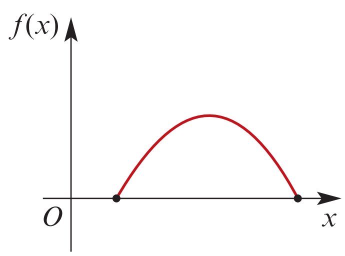
13 The amount of fluid, X mL, in a can of soft drink is a continuous random variable with mean 330 and standard deviation 5. Find an (approximate) interval for the amount of soft drink contained in 95% of the cans.

14 The weight, X g, of cereal in a packet is a continuous random variable with mean 250 and variance 4. Find an (approximate) interval for the weight of cereal contained in 95% of the packets.

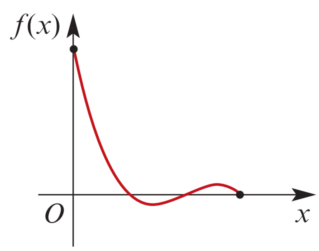
Multiple-choice questions

1 Which of the following graphs could *not* represent a probability density function f?

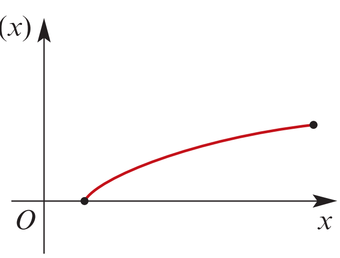
A



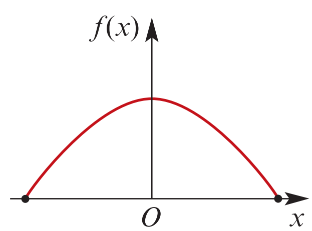
B



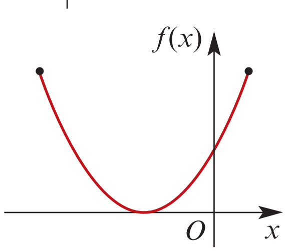
C



D



E



2 If the function $f(x) = 4x$ represents a probability density function, then which of the following could be the domain of f?

A $0 \le x \le 0.25$ **B** $0 \le x \le 0.5$ **C** $0 \le x \le 1$

D $0 \le x \le \dfrac{1}{\sqrt{2}}$ **E** $\dfrac{1}{\sqrt{2}} \le x \le \dfrac{2}{\sqrt{2}}$

ISBN 978-1-108-45164-2

Review

3 If a random variable X has probability density function given by

$$f(x) = \begin{cases} \frac{1}{2}\sin x & 0 < x < k \\ 0 & x \geq k \text{ or } x \leq 0 \end{cases}$$

then k is equal to

A 1 **B** $\frac{\pi}{2}$ **C** 2 **D** π **E** 2π

The following information relates to Questions 4, 5 and 6.

A random variable X has probability density function:

$$f(x) = \begin{cases} \frac{3}{4}(x^2 - 1) & 1 < x < 2 \\ 0 & x \leq 1 \text{ or } x \geq 2 \end{cases}$$

4 $\Pr(X \leq 1.3)$ is closest to

A 0.0743 **B** 0.4258 **C** 0.3 **D** 0.25 **E** 0.9258

5 The mean, $E(X)$, of X is equal to

A 1 **B** $\frac{3}{2}$ **C** $\frac{9}{4}$ **D** $\frac{27}{32}$ **E** $\frac{27}{16}$

6 The variance of X is

A $\frac{27}{16}$ **B** $\frac{67}{1280}$ **C** $\frac{81}{16}$ **D** $\frac{81}{256}$ **E** $\frac{729}{256}$

7 If a random variable X has a probability density function given by

$$f(x) = \begin{cases} \frac{x^3}{4} & 0 \leq x \leq 2 \\ 0 & x > 2 \text{ or } x < 0 \end{cases}$$

then the median of X is closest to

A 1.5 **B** 1.4142 **C** 1.6818 **D** 1.2600 **E** 1

8 If a random variable X has a probability density function given by

$$f(x) = \begin{cases} \frac{3}{2}(x-1)(x-2)^2 & 1 \leq x \leq 3 \\ 0 & x < 1 \text{ or } x > 3 \end{cases}$$

then the mean of X is

A 1 **B** 1.333 **C** 2 **D** 2.6 **E** 3

9 If the consultation time (in minutes) at a surgery is represented by a random variable X which has probability density function

$$f(x) = \begin{cases} \frac{x}{40\,000}(400 - x^2) & 0 \leq x \leq 20 \\ 0 & x < 0 \text{ or } x > 20 \end{cases}$$

then the expected consultation time (in minutes) for three patients is

A $10\frac{2}{3}$ **B** 30 **C** 32 **D** 42 **E** $43\frac{2}{3}$

ISBN 978-1-108-45164-2

10 The top 10% of students in an examination will be awarded an 'A'. If the distribution of scores on the examination is a random variable X with probability density function

$$f(x) = \begin{cases} \dfrac{\pi}{100}\sin\left(\dfrac{\pi x}{50}\right) & 0 \le x \le 50 \\ 0 & x < 0 \text{ or } x > 50 \end{cases}$$

then the minimum score required to be awarded an 'A' is closest to

A 40 **B** 41 **C** 42 **D** 43 **E** 44

11 The cumulative distribution function gives the probability

A that a random variable takes a particular value

B that a random variable takes a value less than or equal to a particular value

C that a random variable takes a value more than a particular value

D of two or more events occurring at once

E that a random variable takes a particular value given that another event has occurred

12 Suppose that X is a continuous random variable with cumulative distribution function given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \dfrac{1}{8}x & \text{if } 0 \le x < 1 \\ \dfrac{1}{8}(3x-2) & \text{if } 1 \le x < 2 \\ \dfrac{1}{2}(x-1) & \text{if } 2 \le x < 3 \\ 1 & \text{if } x \ge 3 \end{cases}$$

Then $\Pr(1 < X < 2.5)$ is equal to

A $\dfrac{1}{8}$ **B** $\dfrac{5}{16}$ **C** $\dfrac{5}{8}$ **D** $\dfrac{3}{4}$ **E** $\dfrac{7}{8}$

Extended-response questions

1 The distribution of X, the life of a certain electronic component in hours, is described by the following probability density function: CF

$$f(x) = \begin{cases} \dfrac{a}{100}\left(1 - \dfrac{x}{100}\right) & 100 < x < 1000 \\ 0 & x \le 100 \text{ and } x \ge 1000 \end{cases}$$

a What is the value of a?

b Find the expected value of the life of the components.

c Find the median value of the life of the components.

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CF

2 The cumulative distribution function of a continuous random variable X is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x - x^2 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

a Find $\Pr(X > 0.5)$.

b Find a such that $\Pr(X < a) = 0.8$.

c Find $E(X)$ and $E(\sqrt{X})$.

3 The probability density function of X is given by

$$f(x) = \begin{cases} \frac{\pi}{20}\cos\left(\frac{\pi(x-6)}{10}\right) & 1 \leq x \leq 11 \\ 0 & x < 1 \text{ or } x > 11 \end{cases}$$

a Find $F(x)$, the cumulative distribution function of X.

b Find the median and the interquartile range of X.

c Find the mean and the variance of X.

CU

4 A hardware shop sells a certain size nail either in a small packet at \$1 per packet, or loose at \$4 per kilogram. On any shopping day, the number, X, of packets sold is a binomial random variable with number of trials $n = 8$ and probability of success $p = 0.6$, and the weight, Y kg, of nails sold loose is a continuous random variable with probability density function f given by

$$f(y) = \begin{cases} \frac{2(y-1)}{25} & 1 \leq y \leq 6 \\ 0 & y < 1 \text{ or } y > 6 \end{cases}$$

a Find $F(y)$, the cumulative distribution function of Y.

b Hence find the probability that the weight of nails sold loose on any shopping day will be between 4 kg and 5 kg.

c Calculate the expected money received on any shopping day from the sale of this size nail in the shop.

5 The continuous random variable X has the probability density function f, where

$$f(x) = \begin{cases} \frac{x-2}{2} & 2 \leq x \leq 4 \\ 0 & x < 2 \text{ or } x > 4 \end{cases}$$

By first expanding $(X - c)^2$, or otherwise, find two values of c such that

$$E[(X-c)^2] = \frac{2}{3}$$

ISBN 978-1-108-45164-2

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CF Review

6 The continuous random variable X has probability density function f, where

$$f(x) = \begin{cases} \dfrac{k}{12(x+1)^3} & 0 \le x \le 4 \\ 0 & x < 0 \text{ or } x > 4 \end{cases}$$

a Find k.

b Evaluate $\mathrm{E}(X+1)$. Hence, find the mean of X.

c Use your calculator to verify your answer to part **b**.

d Find the value of $c > 0$ for which $\Pr(X \le c) = c$.

7 The yield of a variety of corn has probability density function:

$$f(x) = \begin{cases} kx & 0 \le x < 2 \\ k(4-x) & 2 \le x \le 4 \\ 0 & x < 0 \text{ or } x > 4 \end{cases}$$

a Find k.

b Find the expected value, μ, and the variance of the yield of corn.

c Find the probability $\Pr(\mu - 1 < X < \mu + 1)$.

d Find the value of a such that $\Pr(X > a) = 0.6$, giving your answer correct to one decimal place.

8 **Continuous uniform distributions**

a A continuous random variable X has probability density function given by

$$f(x) = \begin{cases} \dfrac{1}{7} & \text{if } 1 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}$$

i Sketch the graph of f.

ii Find $F(x)$, the cumulative distribution function of X.

iii Find $\mathrm{E}(X)$, the expected value of X.

iv Find $\mathrm{Var}(X)$, the variance of X.

b In general, a continuous random variable X is said to have a **uniform distribution** if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

where a and b are real constants with $a < b$. Find:

i the cumulative distribution function of X **ii** $\mathrm{E}(X)$ **iii** $\mathrm{Var}(X)$

c Suppose that the amount of time that a person must wait for a bus is uniformly distributed between 0 and 15 minutes, inclusive. Let X be the waiting time in minutes. Find:

i the cumulative distribution function of X **ii** $\mathrm{E}(X)$ **iii** $\mathrm{Var}(X)$

ISBN 978-1-108-45164-2

Chapter 17

17 The normal distribution

Objectives

- To introduce the **standard normal distribution**.
- To introduce the family of normal distributions as transformations of the standard normal distribution.
- To investigate the effect that changing the values of the parameters defining the normal distribution has on the graph of the probability density function.
- To recognise the **mean**, **median**, **variance** and **standard deviation** of a normal distribution.
- To use technology to determine **probabilities for intervals** in the solution of problems where the normal distribution is appropriate.

The most useful continuous distribution, and one that occurs frequently, is the normal distribution. The probability density functions of normal random variables are symmetric, single-peaked, bell-shaped curves.

Data sets occurring in nature will often have such a bell-shaped distribution, as measurements on many random variables are closely approximated by a normal probability distribution.

Variables such as height, weight, IQ and the volume of milk in a milk carton are all examples of normally distributed random variables.

As well as helping us to understand better the behaviour of many real-world variables, the normal distribution also underpins the development of statistical estimation, which is the topic of Chapter 18.

Chapters 16 and 17 cover Unit 4 Topic 4: Continuous random variables and the normal distribution.

ISBN 978-1-108-45164-2

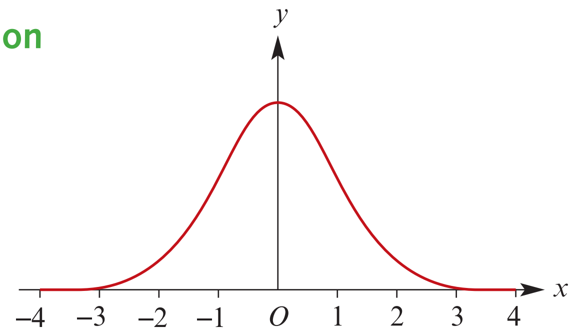
17A The normal distribution

▶ The standard normal distribution

The simplest form of the normal distribution is a random variable with probability density function f given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

The domain of f is $\mathbb{R}$.



Because it is the simplest form of the normal distribution, it is given a special name: the **standard normal distribution**. The graph of the standard normal distribution is as shown.

The graph of the standard normal probability density function f is symmetric about $x = 0$, since $f(-x) = f(x)$. That is, the function f is even.

The line $y = 0$ is an asymptote: as $x \to \pm\infty$, $y \to 0$. Almost all of the area under the probability density function lies between $x = -3$ and $x = 3$.

The mean and standard deviation of the standard normal distribution

It can be seen from the graph that the mean and median of this distribution are the same, and are equal to 0. While the probability density function for the standard normal distribution cannot be integrated exactly, the value of the mean can be verified by observing the symmetry of the two integrals formed below. One is just the negative of the other.

$$\begin{aligned} \mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}x^2}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_0^{\infty} x e^{-\frac{1}{2}x^2}\, dx + \int_{-\infty}^{0} x e^{-\frac{1}{2}x^2}\, dx \right) \end{aligned}$$

Thus the mean, $\mathrm{E}(X)$, of the standard normal distribution is 0.

What can be said about the standard deviation of this distribution? It can be shown that

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}x^2}\, dx = 1$$

Therefore

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 1 - 0 = 1 \quad \text{and} \quad \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)} = 1$$

Standard normal distribution

A random variable with the standard normal distribution has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

The standard normal distribution has mean $\mu = 0$ and standard deviation $\sigma = 1$.

Henceforth, we will denote the random variable of the standard normal distribution by Z.

Cambridge Senior Maths for Queensland Mathematical Methods 3&4 ISBN 978-1-108-45164-2

► The general normal distribution

The normal distribution does not apply just to the special circumstances where the mean is 0 and the standard deviation is 1.

Transformations of the standard normal distribution

The graph of the probability density function for a normal distribution with mean μ and standard deviation σ may be obtained from the graph of the probability density function for the standard normal distribution by the transformation with rule:

$$(x, y) \to \left(\sigma x + \mu, \frac{y}{\sigma}\right)$$

This is a dilation of factor σ from the y-axis and a dilation of factor $\frac{1}{\sigma}$ from the x-axis, followed by a translation of μ units in the positive direction of the x-axis, for $\mu > 0$. (In Section 16D, this was discussed for probability density functions in general.)

Conversely, the transformation which maps the graph of a normal distribution with mean μ and standard deviation σ to the graph of the standard normal distribution is given by

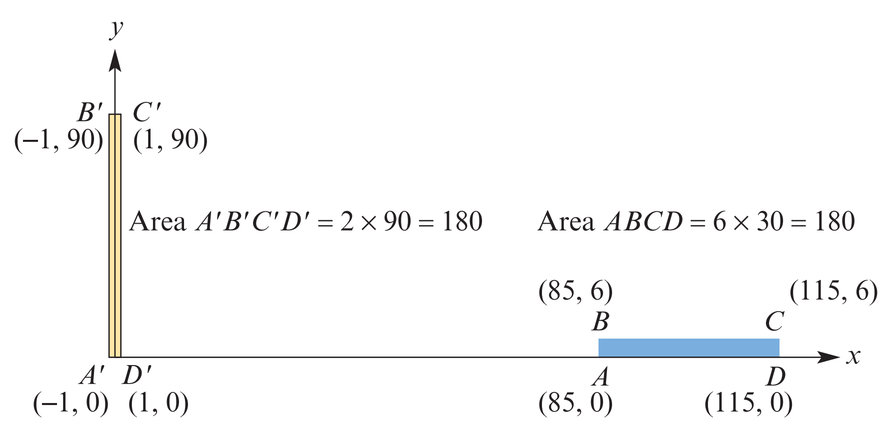
$$(x, y) \to \left(\frac{x - \mu}{\sigma}, \sigma y\right)$$

This is a translation of μ units in the negative direction of the x-axis, followed by a dilation of factor $\frac{1}{\sigma}$ from the y-axis and a dilation of factor σ from the x-axis.

For example, if $\mu = 100$ and $\sigma = 15$, then this transformation is

$$(x, y) \to \left(\frac{x - 100}{15}, 15y\right)$$

This transformation is area-preserving. In the following diagram, the rectangle $ABCD$ is mapped to $A'B'C'D'$. Both rectangles have an area of 180 square units.



This property enables the probabilities of any normal distribution to be determined from the probabilities of the standard normal distribution.

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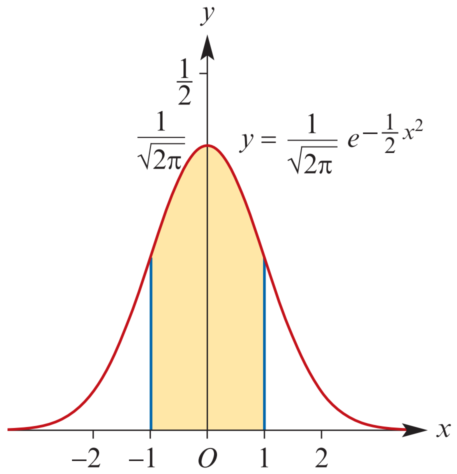

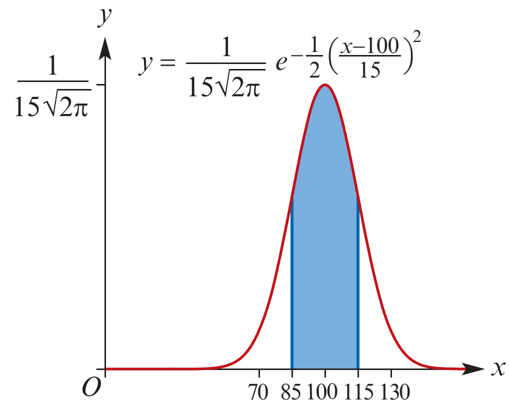

The shaded regions are of equal area.

This leads to the general rule for the family of normal probability distributions.

The rule for the general normal distribution

If X is a **normally distributed random variable** with mean μ and standard deviation σ, then the probability density function of X is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

and

$$\Pr(X \le a) = \Pr\left(Z \le \frac{a-\mu}{\sigma}\right)$$

where Z is the random variable of the standard normal distribution.

The general form of the normal density function involves two parameters, μ and σ, which are the mean (μ) and the standard deviation (σ) of that particular distribution.

When a random variable has a distribution described by a normal density function, the random variable is said to have a **normal distribution**.

As with all probability density functions, the normal density function has the fundamental properties that:

- probability corresponds to an area under the curve
- the total area under the curve is 1.

However, it has some additional special properties.

The graph of a normal density function is symmetric and bell-shaped:

- its **centre** is determined by the **mean** of the distribution
- its **width** is determined by the **standard deviation** of the distribution.

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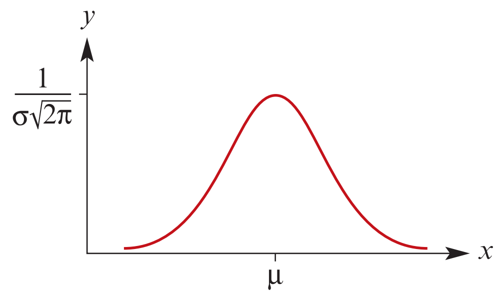
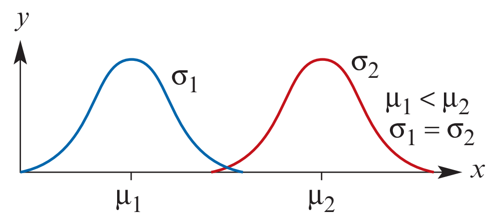
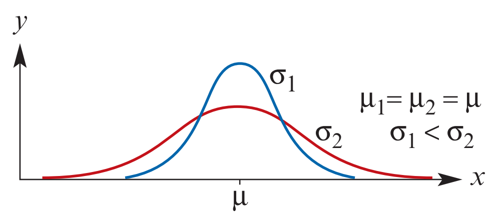
The graph of $y = f(x)$ is shown on the right.

The graph is symmetric about the line $x = \mu$, and has a maximum value of $\dfrac{1}{\sigma\sqrt{2\pi}}$, which occurs when $x = \mu$.



Thus the *location* of the curve is determined by the value of μ, and the *steepness* of the curve by the value of σ.





Irrespective of the values of the mean and standard deviation of a particular normal density function, the area under the curve within a given number of standard deviations from the mean is always the same.

Example 1

On the same set of axes, sketch the graphs of the probability density functions of the standard normal distribution and the normal distribution with:

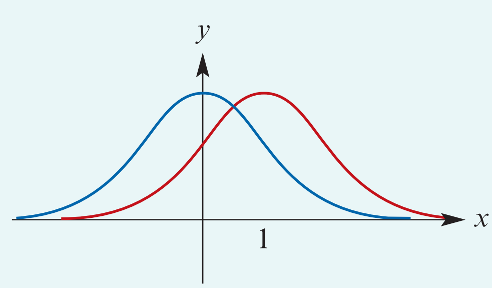
a mean 1 and standard deviation 1

b mean 1 and standard deviation 2.

(A calculator can be used to help.)

Solution

a The graph has been translated 1 unit in the positive direction of the x-axis.



The rules of the two density functions are $y = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ and $y = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2}$.

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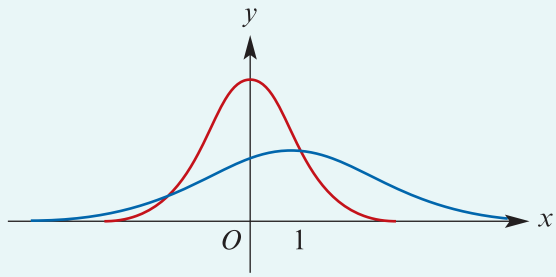
b The graph has been dilated from the y-axis by factor 2 and from the x-axis by factor $\frac{1}{2}$, and then translated 1 unit in the positive direction of the x-axis.



The rules of the two density functions are $y = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ and $y = \dfrac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-1}{2}\right)^2}$.

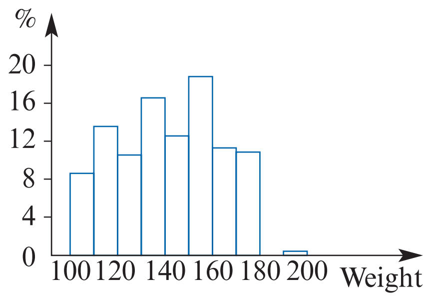
Exercise 17A

SF

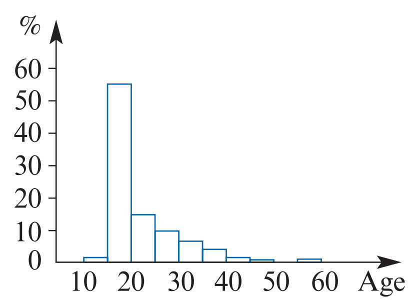
Example 1 **1** Both the random variables X_1 and X_2 are normally distributed, with means μ_1 and μ_2 and standard deviations σ_1 and σ_2, respectively. If $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$, sketch both distributions on the same diagram.

2 Which of the following data distributions are approximately normally distributed?

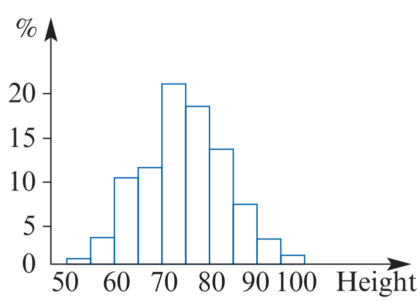
A



B



C



CF

3 Consider the normal probability density function:

$$f(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2}, \quad x \in \mathbb{R}$$

a Use your calculator to find $\int_{-\infty}^{\infty} f(x)\, dx$.

b **i** Express $\mathrm{E}(X)$ as an integral.

ii Use your calculator to evaluate the integral found in **i**.

c **i** Write down an expression for $\mathrm{E}(X^2)$. **ii** What is the value of $\mathrm{E}(X^2)$?

iii What is the value of σ?

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CF

4 Consider the normal probability density function:

$$f(x) = \frac{1}{5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2}, \quad x \in \mathbb{R}$$

a Use your calculator to find $\int_{-\infty}^{\infty} f(x)\, dx$.

b **i** Express $E(X)$ as an integral.

ii Use your calculator to evaluate the integral found in **i**.

c **i** Write down an expression for $E(X^2)$.

ii What is the value of $E(X^2)$?

iii What is the value of σ?

SF

5 The probability density function of a normal random variable X is given by

$$f(x) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-3}{10}\right)^2}$$

a Write down the mean and the standard deviation of X.

b Sketch the graph of $y = f(x)$.

6 The probability density function of a normal random variable X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+3)^2}$$

a Write down the mean and the standard deviation of X.

b Sketch the graph of $y = f(x)$.

7 The probability density function of a normal random variable X is given by

$$f(x) = \frac{1}{\sqrt{18\pi}} e^{-\frac{1}{2}\left(\frac{x}{3}\right)^2}$$

a Write down the mean and the standard deviation of X.

b Sketch the graph of $y = f(x)$.

8 Describe the sequence of transformations which takes the graph of the probability density function of the standard normal distribution to the graph of the probability density function of the normal distribution with:

a $\mu = 3$ and $\sigma = 2$ **b** $\mu = 3$ and $\sigma = \frac{1}{2}$ **c** $\mu = -3$ and $\sigma = 2$

9 Describe the sequence of transformations which takes the graph of the probability density function of the normal distribution with the given mean and standard deviation to the graph of the probability density function of the standard normal distribution:

a $\mu = 3$ and $\sigma = 2$ **b** $\mu = 3$ and $\sigma = \frac{1}{2}$ **c** $\mu = -3$ and $\sigma = 2$

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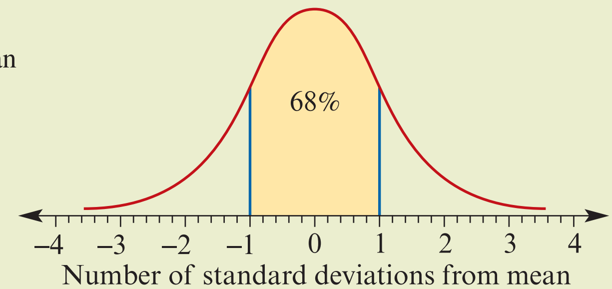
17B Standardisation

For a set of data values that are normally distributed, approximately 68% of the values will lie within one standard deviation of the mean, approximately 95% of the values will lie within two standard deviations of the mean, and almost all (99.7%) within three standard deviations. This gives rise to what is known as the **68–95–99.7% rule**.

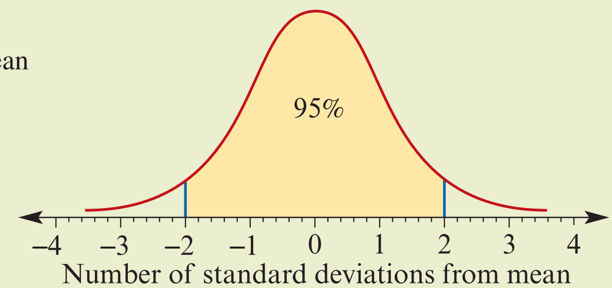
The 68–95–99.7% rule

For a normally distributed random variable, approximately:

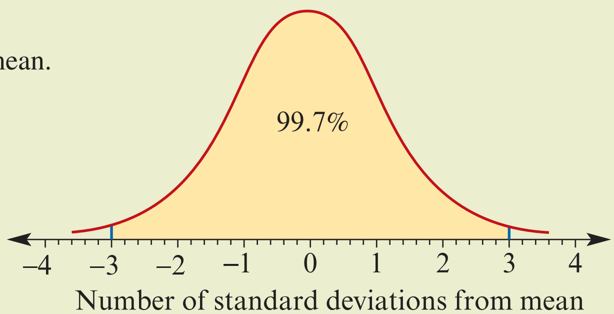
- 68% of the values lie within one standard deviation of the mean



- 95% of the values lie within two standard deviations of the mean



- 99.7% of the values lie within three standard deviations of the mean.



If we know that a random variable is approximately normally distributed, and we know its mean and standard deviation, then we can use the 68–95–99.7% rule to quickly make some important statements about the way in which the data values are distributed.

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Example 2

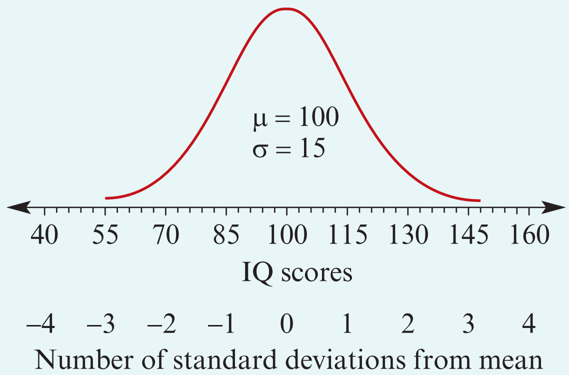
Experience has shown that the scores obtained on a commonly used IQ test can be assumed to be normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$.

Approximately what percentage of the distribution lies within one, two or three standard deviations of the mean?

Solution

Since the scores are normally distributed with $\mu = 100$ and $\sigma = 15$, the 68–95–99.7% rule means that approximately:

- 68% of the scores will lie between 85 and 115
- 95% of the scores will lie between 70 and 130
- 99.7% of the scores will lie between 55 and 145.



Note: In this example, we are using a continuous distribution to model a discrete situation.

Statements can also be made about the percentage of scores that lie in the tails of the distribution, by using the symmetry of the distribution and noting that the total area under the curve is 100%.

Example 3

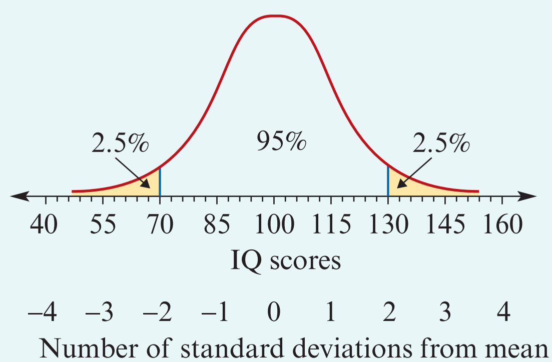
From Example 2, we know that 95% of the scores in the IQ distribution lie between 70 and 130 (that is, within two standard deviations of the mean). What percentage of the scores are *more* than two standard deviations above or below the mean (in this instance, less than 70 or greater than 130)?

Solution

If we focus our attention on the tails of the distribution, we see that 5% of the IQ scores lie outside this region.

Using the symmetry of the distribution, we can say that 2.5% of the scores are below 70, and 2.5% are above 130.

That is, if you obtained a score greater than 130 on this test, you would be in the top 2.5% of the group.



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▶ Standardised values

Clearly, the standard deviation is a natural measuring stick for normally distributed data. For example, a person who obtained a score of 112 on an IQ test with a mean of $\mu = 100$ and a standard deviation of $\sigma = 15$ is less than one standard deviation from the mean. Their score is typical of the group as a whole, as it lies well within the middle 68% of scores. In contrast, a person who scored 133 has done exceptionally well; their score is more than two standard deviations from the mean and this puts them in the top 2.5%.

Because of the additional insight provided, it is usual to convert normally distributed data to a new set of units which shows the number of standard deviations each data value lies from the mean of the distribution. These new values are called **standardised values** or ***z*-values**. To standardise a data value x, we first subtract the mean μ of the normal random variable from the value and then divide the result by the standard deviation σ. That is,

$$\text{standardised value} = \frac{\text{data value} - \text{mean of the normal curve}}{\text{standard deviation of the normal curve}}$$

or symbolically,

$$z = \frac{x - \mu}{\sigma}$$

Standardised values can be positive or negative:

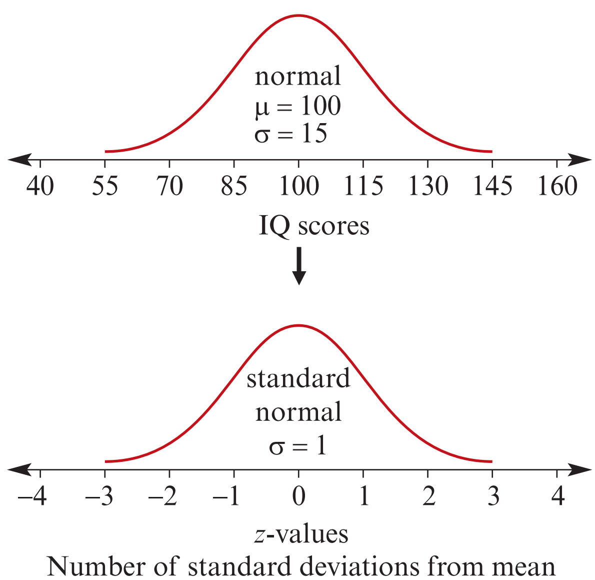
- A **positive *z*-value** indicates that the data value it represents lies **above** the mean.
- A **negative *z*-value** indicates that the data value lies **below** the mean.

For example, an IQ score of 90 lies *below* the mean and has a standardised value of

$$z = \frac{x - \mu}{\sigma} = \frac{90 - 100}{15} = \frac{-10}{15} \approx -0.67$$

There are as many different normal curves as there are values of μ and σ. But if the measurement scale is changed to 'standard deviations from the mean' or z-values, all normal curves reduce to the same normal curve with mean $\mu = 0$ and standard deviation $\sigma = 1$.

The figures on the right show how standardising IQ scores transforms a normal distribution with mean $\mu = 100$ and standard deviation $\sigma = 15$ into the standard normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$.



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Exercise 17B

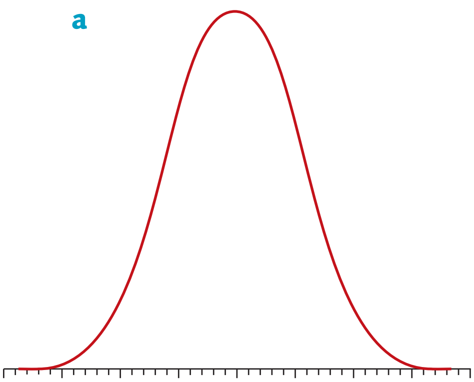
Example 2 **1** The scores obtained on an IQ test can be assumed to be normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$. What percentage of scores lie:

a above 115
b below 85
c above 130
d below 70?

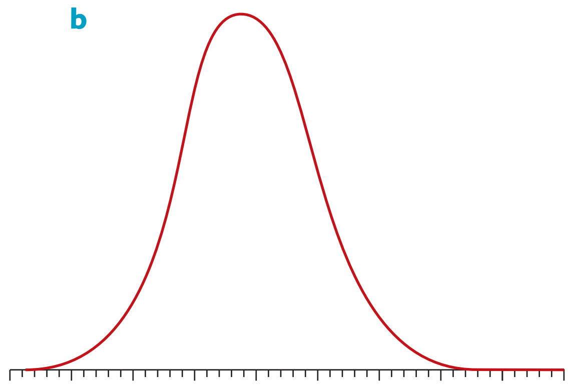
SF

2 State the values of the mean and standard deviation of the normal distributions shown:

a



b



Example 3 **3** The heights of young women are normally distributed with mean $\mu = 160$ cm and standard deviation $\sigma = 8$ cm. What percentage of the women would you expect to have heights:

a between 152 cm and 168 cm
b greater than 168 cm
c less than 136 cm?

4 Fill in the blanks in the following paragraph.

The age at marriage of males in the US in the 1980s was approximately normally distributed with a mean of $\mu = 27.3$ years and a standard deviation of $\sigma = 3.1$ years. From this data, we can conclude that in the 1980s about 95% of males married between the ages of ☐ and ☐.

5 Fill in the blanks in the following statement of the 68–95–99.7% rule.

For any normal distribution, about:

- 68% of the values lie within ☐ standard deviation of the mean
- ☐% of the values lie within two standard deviations of the mean
- ☐% of the values lie within ☐ standard deviations of the mean.

6 If you are told that in Australian adults, nostril width is approximately normally distributed with a mean of $\mu = 2.3$ cm and a standard deviation of $\sigma = 0.3$ cm, find the percentage of people with nostril widths less than 1.7 cm.

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CF

7 The distribution of IQ scores for the inmates of a certain prison is approximately normal with mean $\mu = 85$ and standard deviation $\sigma = 15$.

a What percentage of the prison population have an IQ of 100 or higher?

b If someone with an IQ of 70 or less can be classified as having special needs, what percentage of the prison population could be classified as having special needs?

8 The distribution of the heights of navy officers was found to be normal with a mean of $\mu = 175$ cm and a standard deviation of $\sigma = 5$ cm. Determine:

a the percentage of navy officers with heights between 170 cm and 180 cm

b the percentage of navy officers with heights greater than 180 cm

c the approximate percentage of navy officers with heights greater than 185 cm.

9 The distribution of blood pressures (systolic) among women of similar ages is normal with a mean of 120 (mm of mercury) and a standard deviation of 10 (mm of mercury). Determine the percentage of women with a systolic blood pressure:

a between 100 and 140

b greater than 130

c greater than 120

d between 90 and 150.

10 The heights of women are normally distributed with mean $\mu = 160$ cm and standard deviation $\sigma = 8$ cm. What is the standardised value for the height of a woman who is:

a 160 cm tall

b 150 cm tall

c 172 cm tall?

11 The length of pregnancy for a human is approximately normally distributed with a mean of $\mu = 270$ days and a standard deviation of $\sigma = 10$ days. How many standard deviations away from the mean is a pregnancy of length:

a 256 days

b 281 days

c 305 days?

12 Michael scores 85 on the mathematics section of a scholastic aptitude test, the results of which are known to be normally distributed with a mean of 78 and a standard deviation of 5. Cheryl sits for a different mathematics ability test and scores 27. The scores from this test are normally distributed with a mean of 18 and a standard deviation of 6. Assuming that both tests measure the same kind of ability, who has the better score?

13 The following table gives a student's results in Biology and History. For each subject, the table gives the student's mark (x) and also the mean (μ) and standard deviation (σ) for the class.

	Mark (x)	Mean (μ)	Standard deviation (σ)	Standardised mark (z)
Biology	77	68.5	4.9	
History	79	75.3	4.1	

Complete the table by calculating the student's standardised mark for each subject, and use this to determine in which subject the student did best *relative* to her peers.

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14 Three students took different tests in French, English and Mathematics:

Student	Subject	Mark (x)	Mean (μ)	Standard deviation (σ)	Standardised mark (z)
	French	19	15	4	
Mary	English	42	35	8	
	Mathematics	20	20	5	
	French	21	23	4	
Steve	English	39	42	3	
	Mathematics	23	18	4	
	French	15	15	5	
Sue	English	42	35	10	
	Mathematics	19	20	5	

a Determine the standardised mark for each student on each test.

b Who is the best student in:

i French **ii** English **iii** Mathematics?

c Who is the best student overall? Give reasons for your answer.

17C Determining normal probabilities

A graphics calculator can be used to determine areas under normal curves, allowing us to find probabilities for ranges of values other than one, two or three standard deviations from the mean. The following example is for the standard normal distribution, but the same procedures can be used for any normal distribution by entering the appropriate values for μ and σ.

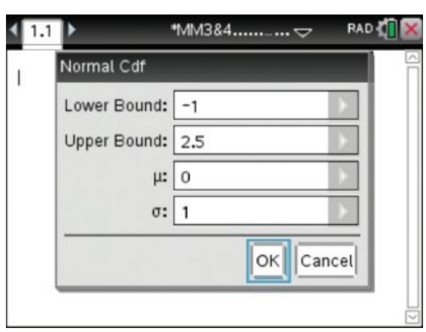
Example 4

Suppose that Z is a standard normal random variable (that is, it has mean $\mu = 0$ and standard deviation $\sigma = 1$). Find:

a $\Pr(-1 < Z < 2.5)$ **b** $\Pr(Z > 1)$

Using the TI-Nspire CX non-CAS

a Use [menu] > **Probability** > **Distributions** > **Normal Cdf** and complete as shown. (Use [tab] or ▼ to move between cells.)



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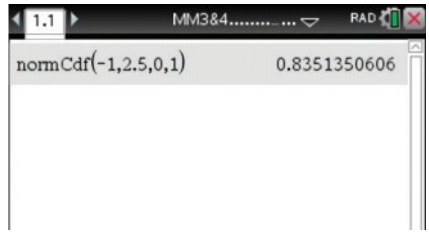
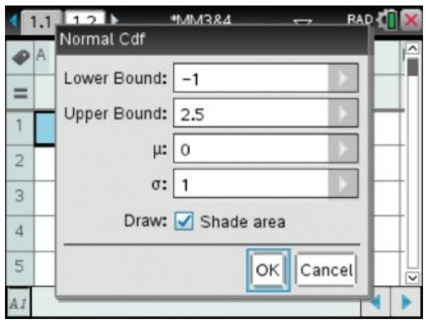
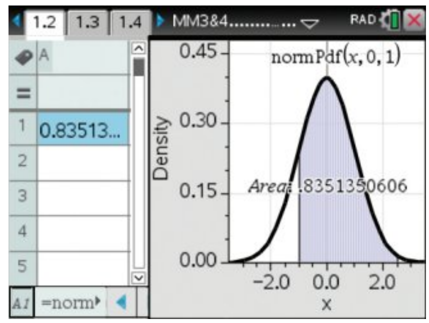
The answer is:

$\Pr(-1 < Z < 2.5) = 0.8351$



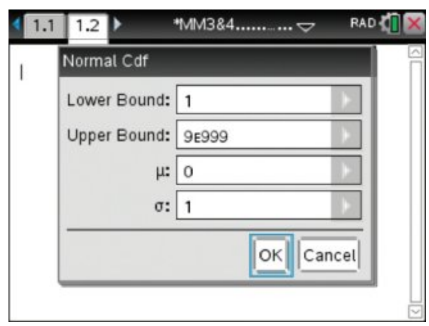
Note: Alternatively, you can solve this problem in a **Lists & Spreadsheet** page and plot the graph. Use (menu) > **Statistics** > **Distributions** > **Normal Cdf** and complete as shown below.





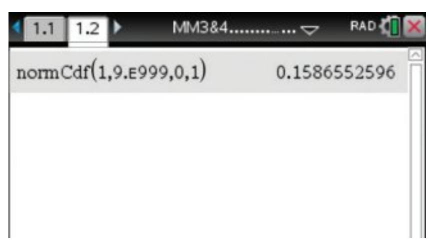
b Use (menu) > **Probability** > **Distributions** > **Normal Cdf** and complete as shown.

Note: By default, the calculator uses $-9E999$ for $-\infty$ and $9E999$ for ∞. The symbol E can be accessed from the Symbols palette ((ctrl) (📖)). For practical purposes, it suffices to use -10^9 for $-\infty$ and 10^9 for ∞.



The answer is:

$\Pr(Z > 1) = 0.1587$



Note: You can enter the commands and parameters directly if preferred. The commands are not case sensitive.

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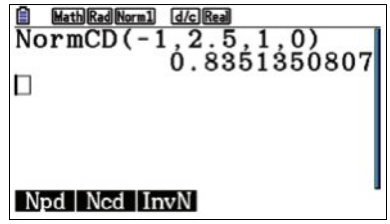
Using the Casio

Method 1: Using Run-Matrix mode

To find $\Pr(-1 < Z < 2.5)$:

- In **Run-Matrix** mode, go to the **Statistics** menu OPTN F5.
- For the normal cumulative distribution, select **Distributions** F3, **Normal** F1, **Ncd** F2.
- Complete by entering: $-1, 2.5, 1, 0)$
- Press EXE.

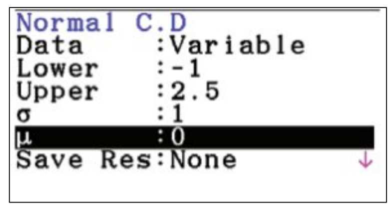


Note: The syntax for the normal cumulative distribution is:
NormCD(*lower bound*, *upper bound*, *standard deviation*, *mean*)

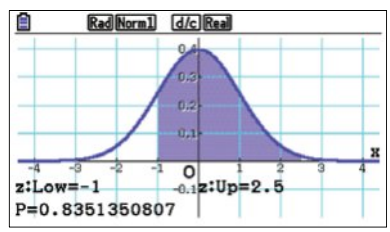
Method 2: Using Statistics mode

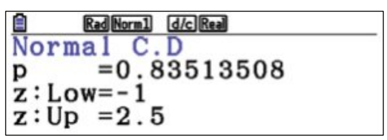
a To find $\Pr(-1 < Z < 2.5)$:

- Press MENU 2 to select **Statistics** mode.
- For the normal cumulative distribution, select **Distributions** F5, **Normal** F1, **Ncd** F2.
- Enter the information as shown.



- Scroll down and select **Calc** or **Draw**.
- Hence $\Pr(-1 < Z < 2.5) = 0.8351$.

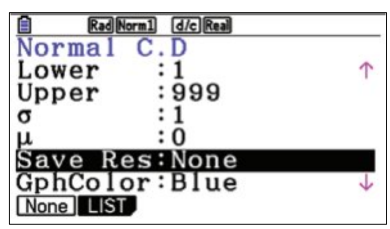




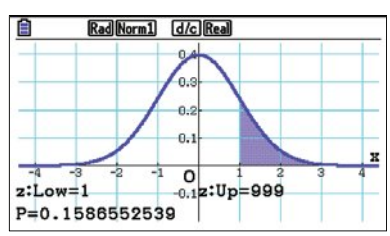
b To find $\Pr(Z > 1)$:

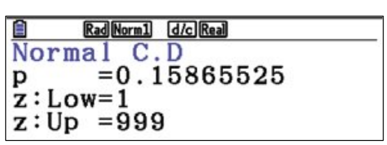
- In **Statistics** mode, select the normal cumulative distribution again.
- Enter the information as shown.

Note: Use a large value for the upper bound.



- Scroll down and select **Calc** or **Draw**.
- Hence $\Pr(Z > 1) = 0.1587$.





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The calculator can also be used to determine **percentiles** of any normal distribution.

Example 5

Suppose X is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 6$.

Find k such that $\Pr(X \leq k) = 0.95$.

Using the TI-Nspire CX non-CAS

Use (menu) > **Probability** > **Distributions** > **Inverse Normal** and complete as shown.

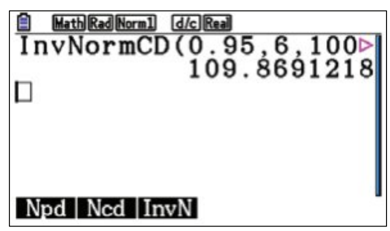
The value of k is 109.869.

Note: You can enter the command and parameters directly if preferred. The command is not case sensitive.

Using the Casio

Method 1: Using Run-Matrix mode

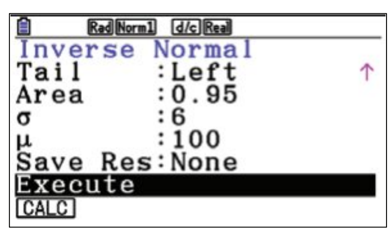
- In **Run-Matrix** mode, go to **Statistics** (OPTN) (F5).
- For the inverse normal distribution, select **Distributions** (F3), **Normal** (F1), **InvN** (F3).
- Complete by entering: 0.95, 6, 100)
- Press (EXE).



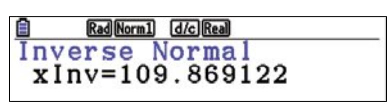
Note: The syntax for the inverse normal distribution is:
InvNormCD(*area*, *standard deviation*, *mean*)

Method 2: Using Statistics mode

- In **Statistics** mode, select the inverse normal distribution by going to **Distributions** (F5), **Normal** (F1), **InvN** (F3).
- Enter the information as shown.



- Scroll down and select **Calc**.
- Hence $k = 109.869$.



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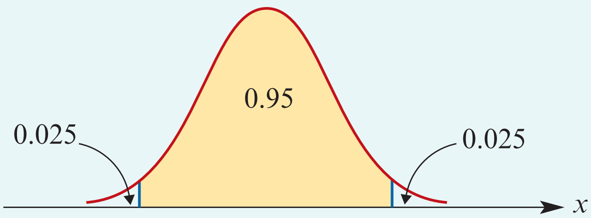
Example 6

Suppose X is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 6$.

Find values of c_1 and c_2 (symmetric about the mean) such that $\Pr(c_1 < X < c_2) = 0.95$.

Solution

Examining the normal curve, we see that there are (infinitely) many intervals which enclose an area of 0.95. By convention, we choose the interval which leaves equal areas in each tail.



To find c_1 using the inverse-normal facility of your calculator, enter 0.025 as the area. To find c_2, enter 0.975.

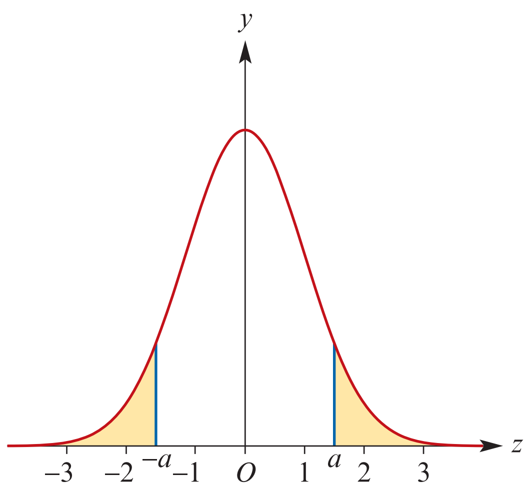
This will give the answer $c_1 = 88.240$ and $c_2 = 111.760$.

► Symmetry properties

Probabilities associated with a normal distribution can often be determined by using its symmetry properties.

Here we work with the standard normal distribution, as it is easiest to use the symmetry properties in this situation:

- $\Pr(Z > a) = 1 - \Pr(Z \leq a)$
- $\Pr(Z < -a) = \Pr(Z > a)$
- $\Pr(-a < Z < a) = 1 - 2\Pr(Z \geq a)$
  $= 1 - 2\Pr(Z \leq -a)$



Exercise 17C

Example 4

1 Suppose Z is a standard normal random variable (that is, it has mean $\mu = 0$ and standard deviation $\sigma = 1$). Find the following probabilities, drawing an appropriate diagram in each case: SF

a $\Pr(Z < 2)$ **b** $\Pr(Z < 2.5)$ **c** $\Pr(Z \leq 2.5)$ **d** $\Pr(Z < 2.53)$

e $\Pr(Z \geq 2)$ **f** $\Pr(Z > 1.5)$ **g** $\Pr(Z \geq 0.34)$ **h** $\Pr(Z > 1.01)$

2 Suppose Z is a standard normal random variable. Find the following probabilities, drawing an appropriate diagram in each case:

a $\Pr(Z > -2)$ **b** $\Pr(Z > -0.5)$ **c** $\Pr(Z > -2.5)$ **d** $\Pr(Z \geq -1.283)$

e $\Pr(Z < -2)$ **f** $\Pr(Z < -2.33)$ **g** $\Pr(Z \leq -1.8)$ **h** $\Pr(Z \leq -0.95)$

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SF

3 Suppose Z is a standard normal random variable. Find the following probabilities, drawing an appropriate diagram in each case:

a $\Pr(-1 < Z < 1)$ **b** $\Pr(-2 < Z < 2)$ **c** $\Pr(-3 < Z < 3)$

How do these results compare with the 68–95–99.7% rule discussed in Section 17B?

4 Suppose Z is a standard normal random variable. Find the following probabilities, drawing an appropriate diagram in each case:

a $\Pr(2 < Z < 3)$ **b** $\Pr(-1.5 < Z < 2.5)$

c $\Pr(-2 < Z < -1.5)$ **d** $\Pr(-1.4 < Z < -0.8)$

Example 5

5 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $\Pr(Z \leq c) = 0.9$.

6 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $\Pr(Z \leq c) = 0.75$.

7 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $\Pr(Z \leq c) = 0.975$.

8 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $\Pr(Z \geq c) = 0.95$.

9 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $\Pr(Z \geq c) = 0.8$.

10 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then the value c such that $\Pr(Z \leq c) = 0.10$.

11 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $\Pr(Z \leq c) = 0.025$.

12 Let X be a normal random variable with mean $\mu = 100$ and standard deviation $\sigma = 6$. Find:

a $\Pr(X < 110)$ **b** $\Pr(X < 105)$ **c** $\Pr(X > 110)$ **d** $\Pr(105 < X < 110)$

13 Let X be a normal random variable with mean $\mu = 40$ and standard deviation $\sigma = 5$. Find:

a $\Pr(X < 48)$ **b** $\Pr(X < 36)$ **c** $\Pr(X > 32)$ **d** $\Pr(32 < X < 36)$

14 Let X be a normal random variable with mean $\mu = 6$ and standard deviation $\sigma = 2$.

a Find c such that $\Pr(X < c) = 0.95$.

b Find k such that $\Pr(X < k) = 0.90$.

15 Let X be a normal random variable with mean $\mu = 10$ and standard deviation $\sigma = 3$.

a Find c such that $\Pr(X < c) = 0.50$.

b Find k such that $\Pr(X < k) = 0.975$.

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16 The 68–95–99.7% rule tells us approximately the percentage of a normal distribution which lies within one, two or three standard deviations of the mean. If Z is the standard normal random variable find, correct to two decimal places:

a a such that $\Pr(-a < Z < a) = 0.68$
b b such that $\Pr(-b < Z < b) = 0.95$
c c such that $\Pr(-c < Z < c) = 0.997$

17 Given that X is a normally distributed random variable with a mean of 22 and a standard deviation of 7, find:

a $\Pr(X < 26)$
b $\Pr(25 < X < 27)$
c $\Pr(X < 26 \mid 25 < X < 27)$
d c such that $\Pr(X < c) = 0.95$
e k such that $\Pr(X > k) = 0.9$
f c_1 and c_2 such that $\Pr(c_1 < X < c_2) = 0.95$

18 Let X be a normal random variable with mean $\mu = 10$ and standard deviation $\sigma = 0.5$. Find:

a $\Pr(X < 11)$
b $\Pr(X < 11 \mid X < 13)$
c c such that $\Pr(X < c) = 0.95$
d k such that $\Pr(X < k) = 0.2$
e c_1 and c_2 such that $\Pr(c_1 < X < c_2) = 0.95$

17D Solving problems using the normal distribution

The normal distribution can be used to solve many practical problems.

Example 7

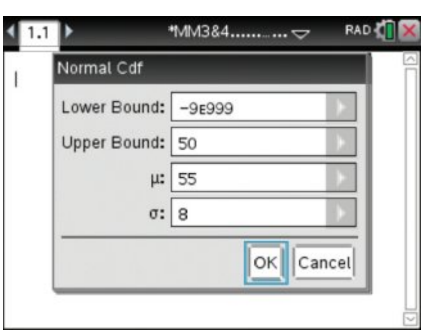
The time taken to complete a logical reasoning task is normally distributed with a mean of 55 seconds and a standard deviation of 8 seconds.

a Find the probability, correct to four decimal places, that a randomly chosen person will take less than 50 seconds to complete the task.

b Find the probability, correct to four decimal places, that a randomly chosen person will take less than 50 seconds to complete the task, if it is known that this person took less than 60 seconds to complete the task.

Using the TI-Nspire CX non-CAS

a Use (menu) > **Probability** > **Distributions** > **Normal Cdf** and complete as shown.



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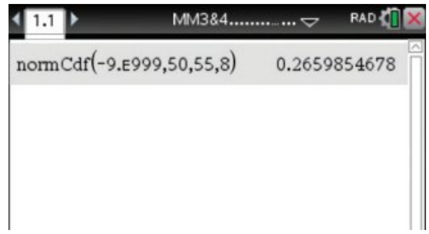
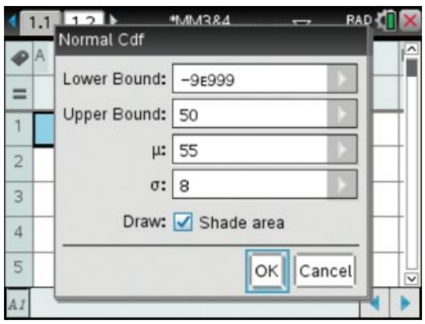
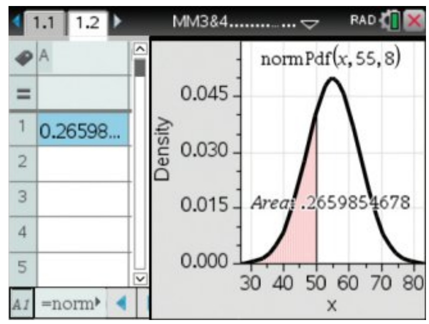
The answer is:

$\Pr(X < 50) = 0.2660$



Note: Alternatively, you can solve this problem in a **Lists & Spreadsheet** page and plot the graph. Use (menu) > **Statistics** > **Distributions** > **Normal Cdf** and complete as shown below.





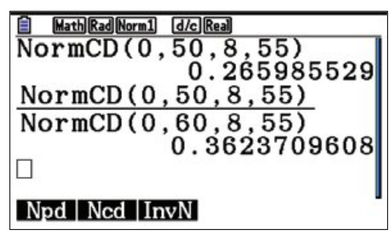
b $\Pr(X < 50 \mid X < 60) = \dfrac{\Pr(X < 50 \cap X < 60)}{\Pr(X < 60)}$

$= \dfrac{\Pr(X < 50)}{\Pr(X < 60)} = \dfrac{0.2660}{0.7340} = 0.3624$

Using the Casio

a In **Run-Matrix** mode:

- Go to the **Statistics** menu (OPTN) (F5) and select **Distributions** (F3), **Normal** (F1), **Ncd** (F2).
- Complete as shown.
- Hence $\Pr(X < 50) = 0.2660$.



b Use the fraction template as shown to find:

$\Pr(X < 50 \mid X < 60) = \dfrac{\Pr(X < 50 \cap X < 60)}{\Pr(X < 60)}$

$= \dfrac{\Pr(X < 50)}{\Pr(X < 60)}$

$= 0.3624$

Note: Recall that the syntax for the normal cumulative distribution is:

NormCD(*lower bound*, *upper bound*, *standard deviation*, *mean*)

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When the mean and standard deviation of a normal distribution are unknown, it is sometimes necessary to transform to the standard normal distribution. This is demonstrated in the following example.

Example 8

Limits of acceptability imposed on the lengths of a certain batch of metal rods are 1.925 cm and 2.075 cm. It is observed that, on average, 5% are rejected as undersized and 5% are rejected as oversized.

Assuming that the lengths are normally distributed, find the mean and standard deviation of the distribution.

Solution

It is given that

$$\Pr(X > 2.075) = 0.05$$
$$\Pr(X < 1.925) = 0.05$$

Symmetry tells us that the mean is equal to

$$\mu = \frac{2.075 + 1.925}{2} = 2$$

Transforming to the standard normal gives

$$\Pr\left(Z > \frac{2.075 - \mu}{\sigma}\right) = 0.05 \quad \text{and} \quad \Pr\left(Z < \frac{1.925 - \mu}{\sigma}\right) = 0.05$$

The first equality can be rewritten as

$$\Pr\left(Z < \frac{2.075 - \mu}{\sigma}\right) = 0.95$$

Use the inverse-normal facility of your calculator to obtain

$$\frac{2.075 - \mu}{\sigma} = 1.6448\ldots \qquad \text{and} \qquad \frac{1.925 - \mu}{\sigma} = -1.6448\ldots$$

These equations confirm that $\mu = 2$.

Substitute $\mu = 2$ into the first equation and solve for σ:

$$\frac{2.075 - 2}{\sigma} = 1.6448\ldots$$

$$\therefore \qquad \sigma = 0.045596\ldots$$

Thus $\sigma = 0.0456$, correct to four decimal places.

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Exercise 17D

Skillsheet

Example 7

1 Suppose that IQ scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$.

a What is the probability that a person chosen at random has an IQ:

i greater than 110

ii less than 75

iii greater than 130, given that they have an IQ greater than 110?

b To be allowed to join an elite club, a potential member must have an IQ in the top 5% of the population. What IQ score would be necessary to join this club?

2 The heights of women are normally distributed with a mean of $\mu = 160$ cm and a standard deviation of $\sigma = 8$ cm.

a What is the probability that a woman chosen at random would be:

i taller than 155 cm

ii shorter than 170 cm

iii taller than 170 cm, given that her height is between 168 cm and 174 cm?

b What height would put a woman among the tallest 10% of the population?

c What height would put a woman among the shortest 20% of the population?

3 The results of a mathematics exam are normally distributed with mean $\mu = 50$ and standard deviation $\sigma = 7$.

a What is the probability that a student chosen at random has an exam mark:

i greater than 60

ii less than 75

iii greater than 60, given that they passed? (Assume a pass mark of 50.)

b The top 15% of the class are to be awarded a distinction. What mark would be required to gain a distinction in this exam?

4 The lengths of a species of fish are normally distributed with a mean length of 40 cm and a standard deviation of 4 cm. Find the percentage of these fish having lengths:

a greater than 45 cm

b between 35.5 cm and 45.5 cm.

Example 8

5 The weights of cats are normally distributed. It is known that 10% of cats weigh more than 1.8 kg, and 15% of cats weigh less than 1.35 kg. Find the mean and the standard deviation of this distribution.

6 The marks of a large number of students in a statistics examination are normally distributed with a mean of 48 marks and a standard deviation of 15 marks.

a If the pass mark is 53, find the percentage of students who passed the examination.

b If 8% of students gained an A on the examination by scoring a mark of at least c, find the value of c.

CF

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CF

7 The height of a certain population of adult males is normally distributed with mean 176 cm and standard deviation 7 cm.

a Find the probability that the height of a randomly selected male will exceed 190 cm.

b If two males are selected at random, find the probability that both of their heights will exceed 190 cm.

c Suppose 10 males are selected at random. Find the probability that at least two will have heights that exceed 190 cm.

8 **a** Machine A is packaging bags of mints with a mean weight of 300 grams. The bags are considered underweight if they weigh less than 295 grams. It is observed that, on average, 5% of bags are rejected as underweight. Assuming that the weights of the bags are normally distributed, find the standard deviation of the distribution.

b In the same factory, machine B is packaging bags of liquorice. The bags from this machine are considered underweight if they weigh less than 340 grams. It is observed that, on average, 2% of bags from machine B are rejected as underweight. Assuming that the weights are normally distributed with a standard deviation of 5 grams, find the mean of the distribution.

9 The volume of soft drink in a 1-litre bottle is normally distributed. The soft drink company needs to calibrate its filling machine. They don't want to put too much soft drink into each bottle, as it adds to their expense. However, they know they will be fined if more than 2% of bottles are more than 2 millilitres under volume. The standard deviation of the volume dispensed by the filling machine is 2.5 millilitres. What should they choose as the target volume (i.e. the mean of the distribution)? Give your answer to the nearest millilitre.

10 The weights of pumpkins sold to a greengrocer are normally distributed with a mean of 1.2 kg and a standard deviation of 0.4 kg. The pumpkins are sold in three sizes:

Small: under 0.8 kg   Medium: from 0.8 kg to 1.8 kg   Large: over 1.8 kg

a Find the proportions of pumpkins in each of the three sizes.

b The prices of the pumpkins are $2.80 for a small, $3.50 for a medium, and $5.00 for a large. Find the expected cost for 100 pumpkins chosen at random from the greengrocer's supply.

11 Potatoes are delivered to a chip factory in semitrailer loads. A sample of 1 kg of the potatoes is chosen from each load and tested for starch content. From past experience it is known that the starch content is normally distributed with a standard deviation of 2.1.

a For a semitrailer load of potatoes with a mean starch content of 22.0:

i What is the probability that the test reading is 19.5 or less?

ii What reading will be exceeded with a probability of 0.98?

b If the starch content is greater than 22.0, the potatoes cannot be used for chips, and so the semitrailer load is rejected. What is the probability that a load with a mean starch content of 18.0 will be rejected?

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12 The amount of a certain chemical in a type A cell is normally distributed with a mean of 10 and a standard deviation of 1. The amount in a type B cell is normally distributed with a mean of 14 and a standard deviation of 2. To determine whether a cell is type A or type B, the amount of chemical in the cell is measured. The cell is classified as type A if the amount is less than a specified value c, and as type B otherwise.

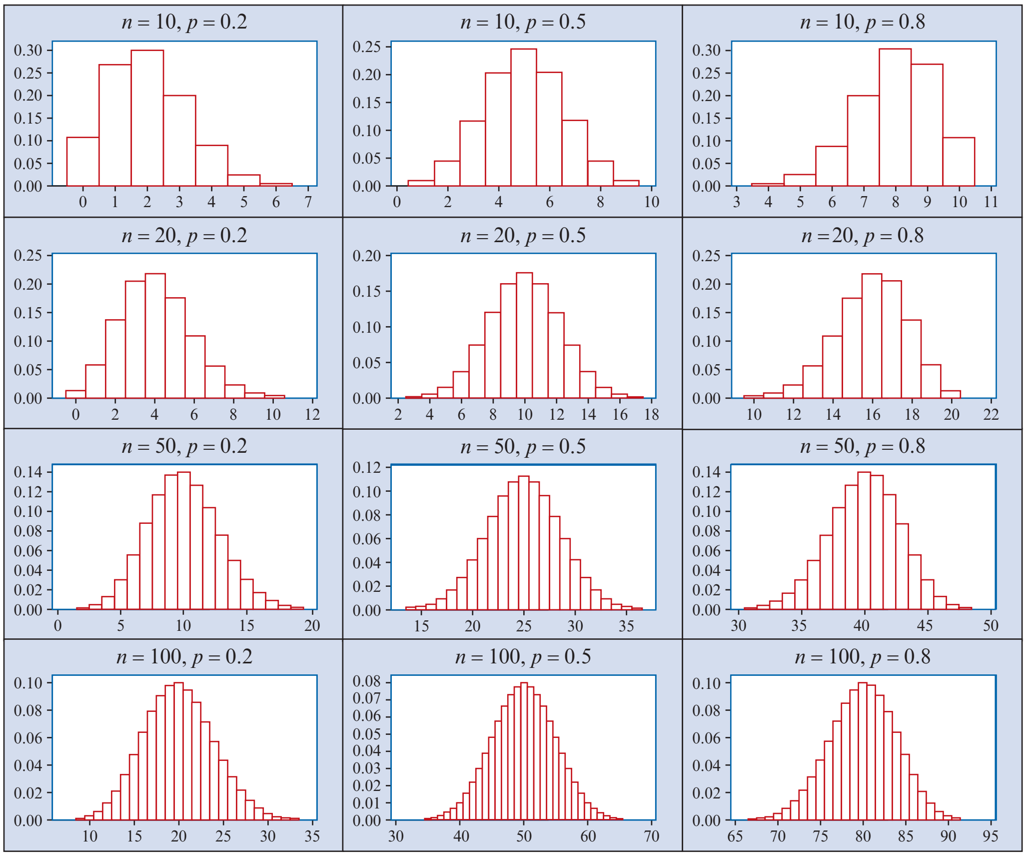
a If $c = 12$, calculate the probability that a type A cell will be misclassified, and the probability that a type B cell will be misclassified.

b Find the value of c for which the two probabilities of misclassification are equal.

17E The normal approximation to the binomial distribution

In many cases, it is possible to use the normal distribution to approximate the binomial distribution. This approximation is fundamental to our understanding of confidence intervals, which are introduced in Chapter 18.

We saw in Chapter 15 that the shape of the binomial distribution depends on n and p. The following plots show the binomial distribution for $n = 10, 20, 50, 100$ and $p = 0.2, 0.5, 0.8$.



We can see that, if n is small and p is close to 0 or 1, these distributions are skewed. Otherwise, they look remarkably symmetric. In fact, if n is large enough and p is not too close to 0 or 1, the binomial distribution is approximately normal. Moreover, the mean and standard deviation of this normal distribution agree with those of the binomial distribution.

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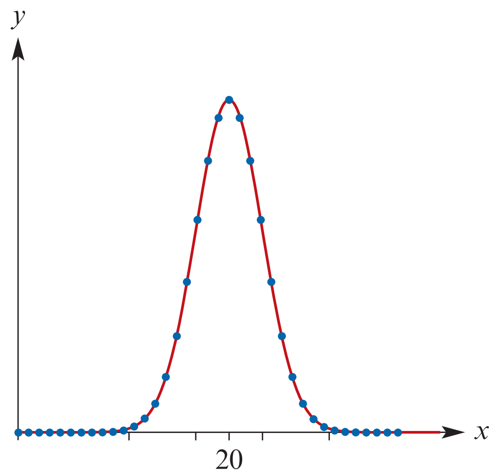
In the figure opposite, the binomial distribution with $n = 40$ and $p = 0.5$ is plotted (the blue points). This distribution has mean $\mu = 20$ and standard deviation $\sigma = \sqrt{10}$.

On the same axes, the probability density function of the normal distribution with mean $\mu = 20$ and standard deviation $\sigma = \sqrt{10}$ is drawn (the red curve).

We will see that this approximation has important uses in statistics.



When is it appropriate to use the normal approximation?

If n is large enough, the skew of the binomial distribution is not too great. In this case, the normal distribution can be used as a reasonable approximation to the binomial distribution. The approximation is generally better for larger n and when p is not too close to 0 or 1.

If n is sufficiently large, the binomial random variable X will be approximately normally distributed, with a mean of $\mu = np$ and a standard deviation of $\sigma = \sqrt{np(1-p)}$.

One rule of thumb is that:

Both np and $n(1-p)$ must be greater than 5 for a satisfactory approximation.

In the example shown in the figure above, we have $np = 20$ and $n(1-p) = 20$. There are ways of improving this approximation but we will not go into that here.

Example 9

A sample of 1000 people from a certain city were asked to indicate whether or not they were in favour of the construction of a new freeway. It is known that 30% of people in this city are in favour of the new freeway. Find the approximate probability that between 270 and 330 people in the sample were in favour of the new freeway.

Solution

Let X be the number of people in the sample who are in favour of the freeway. Then we can assume that X is a binomial random variable with $n = 1000$ and $p = 0.3$.

Therefore

$$\mu = np = 1000 \times 0.3 = 300$$

and

$$\sigma = \sqrt{np(1-p)} = \sqrt{1000 \times 0.3 \times 0.7} = \sqrt{210}$$

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Thus

$$\Pr(270 < X < 330) \approx \Pr\left(\frac{270 - 300}{\sqrt{210}} < Z < \frac{330 - 300}{\sqrt{210}}\right)$$
$$\approx \Pr(-2.070 < Z < 2.070)$$
$$\approx 0.9616$$

Note: When we calculate this probability directly using the binomial distribution, we find that $\Pr(270 \le X \le 330) = 0.9648$ and $\Pr(270 < X < 330) = 0.9583$.

Exercise 17E

In each of the following questions, use the normal approximation to the binomial distribution.

SF

Example 9

1 A die is rolled 100 times. What is the probability that more than 10 sixes will be observed?

2 If 50% of the voting population in a particular state favour candidate A, what is the approximate probability that more than 156 in a sample of 300 will favour that candidate?

3 A sample of 100 people is drawn from a city in which it is known that 10% of the population is over 65 years of age. Find the approximate probability that the sample contains:

a at least 15 people who are over 65 years of age

b no more than 8 people over 65 years of age.

4 A manufacturing process produces on average 40 defective items per 1000. What is the approximate probability that a random sample of size 400 contains:

a at least 10 and no more than 20 defective items

b 25 or more defective items?

5 A survey of the entire population in a particular city found that 40% of people regularly participate in sport. What is the approximate probability that fewer than 38% of a random sample of 200 people regularly participate in sport?

CF

6 An examination consists of 25 multiple-choice questions. Each question has four possible answers. At least 10 correct answers are required to pass the examination. Suppose that a student guesses the answer to each question.

a What is the approximate probability that the student will pass the examination?

b What is the approximate probability that the student guesses from 12 to 14 answers correctly?

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Chapter summary

AS

Nrich

- A special continuous random variable X, called a **normal random variable**, has a probability density function given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where μ and σ are the mean and standard deviation of X.
- In the special case that $\mu = 0$ and $\sigma = 1$, this probability density function defines the **standard normal distribution**. A random variable with this distribution is usually denoted by Z.
- The graph of a normal density function is a symmetric, bell-shaped curve; its centre is determined by the mean, μ, and its width by the standard deviation, σ.
- The **68–95–99.7% rule** states that, for any normal distribution:
  - approximately 68% of the values lie within one standard deviation of the mean
  - approximately 95% of the values lie within two standard deviations of the mean
  - approximately 99.7% of the values lie within three standard deviations of the mean.
- If X is a normally distributed random variable with mean μ and standard deviation σ, then to **standardise** a value x of X we subtract the mean and divide by the standard deviation:

$$z = \frac{x-\mu}{\sigma}$$

The standardised value z indicates the number of standard deviations that the value x lies above or below the mean.
- A calculator can be used to evaluate the cumulative distribution function of a normal random variable – that is, to find the area under the normal curve up to a specified value.
- The inverse-normal facility of a calculator can be used to find the value of a normal random variable corresponding to a specified area under the normal curve.

Technology-free questions

1 Given that $\Pr(Z \leq a) = p$ for the standard normal random variable Z, find in terms of p: SF

a $\Pr(Z > a)$ **b** $\Pr(Z < -a)$ **c** $\Pr(-a \leq Z \leq a)$

2 Let X be a normal random variable with mean 4 and standard deviation 1. Let Z be the standard normal random variable.

a If $\Pr(X < 3) = \Pr(Z < a)$, then $a =$ ☐.

b If $\Pr(X > 5) = \Pr(Z > b)$, then $b =$ ☐.

c $\Pr(X > 4) =$ ☐

3 A normal random variable X has mean 8 and standard deviation 3. Give the rule for a transformation that maps the graph of the density function of X to the graph of the density function for the standard normal distribution. CF

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4 Let X be a normal random variable with mean μ and standard deviation σ. If $\mu < a < b$ with $\Pr(X < b) = p$ and $\Pr(X < a) = q$, find: CF

a $\Pr(X < a \mid X < b)$ **b** $\Pr(X < 2\mu - a)$ **c** $\Pr(X > b \mid X > a)$

5 Let X be a normal random variable with mean 4 and standard deviation 2. Write each of the following probabilities in terms of Z: SF

a $\Pr(X < 5)$ **b** $\Pr(X < 3)$ **c** $\Pr(X > 5)$

d $\Pr(3 < X < 5)$ **e** $\Pr(3 < X < 6)$

In Questions 6 to 8, you will use the following:

$$\Pr(Z < 1) = 0.84$$
$$\Pr(Z < 2) = 0.98$$
$$\Pr(Z < 0.5) = 0.69$$

6 A machine produces metal rods with mean diameter 2.5 mm and standard deviation 0.05 mm. Let X be the random variable of the normal distribution. Find:

a $\Pr(X < 2.55)$ **b** $\Pr(X < 2.5)$

c $\Pr(X < 2.45)$ **d** $\Pr(2.45 < X < 2.55)$

7 Nuts are packed in tins such that the mean weight of the tins is 500 g and the standard deviation is 5 g. The weights are normally distributed with random variable W. Find:

a $\Pr(W > 505)$ **b** $\Pr(500 < W < 505)$

c $\Pr(W > 505 \mid W > 500)$ **d** $\Pr(W > 510)$

8 A random variable X has a normal distribution with mean 6 and standard deviation 1. Find:

a $\Pr(X < 6.5)$ **b** $\Pr(6 < X < 6.5)$

c $\Pr(6.5 < X < 7)$ **d** $\Pr(5 < X < 7)$

9 Suppose that three tests were given in your mathematics course. The class means and standard deviations, together with your scores, are listed in the table. CF

	μ	σ	Your score
Test A	50	11	62
Test B	47	17	64
Test C	63	8	73

On which test did you do best and on which did you do worst?

10 Let X be a normally distributed random variable with mean 10 and variance 4, and let Z be a random variable with the standard normal distribution.

a Find $\Pr(X > 10)$.

b Find b such that $\Pr(X > 13) = \Pr(Z < b)$.

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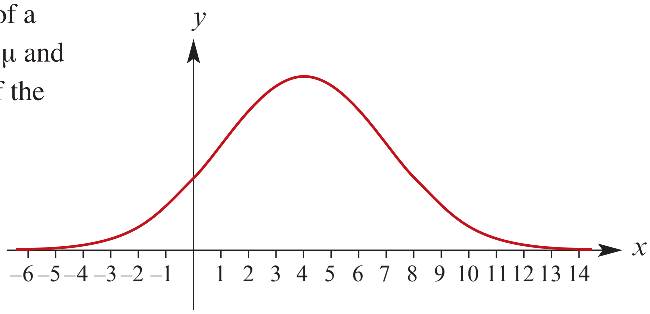
Review

Multiple-choice questions

1 The diagram shows the graph of a normal distribution with mean μ and standard deviation σ. Which of the following statements is true?



A $\mu = 4$ and $\sigma = 3$
B $\mu = 3$ and $\sigma = 4$
C $\mu = 4$ and $\sigma = 2$
D $\mu = 3$ and $\sigma = 2$
E $\mu = 4$ and $\sigma = 4$

2 If Z is a standard normal random variable, then $\Pr(Z > 1.45) =$

A 0.1394 **B** 0.8606 **C** 0.0735 **D** 0.9625 **E** 0.0925

3 If Z is a standard normal random variable and $\Pr(Z < c) = 0.25$, then the value of c is closest to

A 0.6745 **B** −0.6745 **C** 0.3867 **D** 0.5987 **E** −0.5987

4 The random variable X has a normal distribution with mean 12 and variance 9. If Z is a standard normal random variable, then the probability that X is more than 15 is equal to

A $\Pr(Z < 1)$ **B** $\Pr(Z > 1)$ **C** $\Pr\left(Z > \frac{1}{3}\right)$
D $1 - \Pr\left(Z > \frac{1}{3}\right)$ **E** $1 - \Pr(Z > 1)$

5 Suppose that, in a board-game tournament, the length of time taken for each game is normally distributed with a mean of 102 minutes and a standard deviation of 3 minutes. The percentage of games that last more than 110 minutes is approximately

A 96.2% **B** 81.3% **C** 2.7% **D** 18.7% **E** 0.38%

6 If the number of goals that the Suns score in a match is a normally distributed random variable with a mean of 16 and a standard deviation of 2, then in what percentage of their matches (approximately) do they score from 10 to 22 goals?

A 5% **B** 16% **C** 68% **D** 95% **E** 99.7%

7 If X is a normally distributed random variable with mean $\mu = 6$ and standard deviation $\sigma = 3$, then the transformation which maps the graph of the density function f of X to the graph of the standard normal distribution is

A $(x, y) \to \left(\frac{x-3}{6}, 6y\right)$ **B** $(x, y) \to \left(\frac{x-6}{3}, \frac{y}{3}\right)$ **C** $(x, y) \to \left(\frac{x-6}{3}, 3y\right)$
D $(x, y) \to (3(x+6), 3y)$ **E** $(x, y) \to \left(3(x+6), \frac{y}{3}\right)$

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8 The amount of water that Steve uses to water the garden is normally distributed with a mean of 100 litres and a standard deviation of 14 litres. On 20% of occasions it takes him more than k litres to water the garden. What is the value of k?

A 88.2 **B** 110.7 **C** 120.0 **D** 111.8 **E** 114.0

9 The marks achieved by Angie in Mathematics, Indonesian and Politics, together with the mean and standard deviation for each subject, are given in the following table:

Subject	Mark	Mean (μ)	Standard deviation (σ)
Mathematics	72	72	5
Indonesian	57	59	2
Politics	68	64	4

Which of the following statements is correct?

A Angie's best subject was Politics, followed by Mathematics and then Indonesian.
B Angie's best subject was Mathematics, followed by Politics and then Indonesian.
C Angie's best subject was Politics, followed by Indonesian and then Mathematics.
D Angie's best subject was Mathematics, followed by Indonesian and then Politics.
E Angie's best subject was Indonesian, followed by Mathematics and then Politics.

10 Suppose that X is normally distributed with mean 11.3 and standard deviation 2.9. Values of c_1 and c_2 such that $\Pr(c_1 < X < c_2) = 0.90$ are closest to

A 5.5, 17.1 **B** 6.08, 16.52 **C** 15.02, 7.58 **D** 6.53, 16.07 **E** 5.62, 16.98

11 The volume of liquid in a 1-litre bottle of soft drink is a normally distributed random variable with a mean of μ litres and a standard deviation of 0.005 litres. To ensure that 99.9% of the bottles contain at least 1 litre of soft drink, the value of μ should be closest to

A 0.995 litres **B** 1.0 litres **C** 1.005 litres **D** 1.015 litres **E** 1.026 litres

12 The gestation period for human pregnancies in a certain country is normally distributed with a mean of 272 days and a standard deviation of σ days. If from a population of 1000 births there were 91 pregnancies of length less than 260 days, then σ is closest to

A 3 **B** 5 **C** 9 **D** 12 **E** 16

Extended-response questions

1 A test devised to measure mathematical aptitude gives scores that are normally distributed with a mean of 50 and a standard deviation of 10. If we wish to categorise the results so that the highest 10% of scores are designated as high aptitude, the next 20% as moderate aptitude, the middle 40% as average, the next 20% as little aptitude and the lowest 10% as no aptitude, then what ranges of scores will be covered by each of these five categories? CU

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Review

2 If X is normally distributed with $\mu = 10$ and $\sigma = 2$, find the value of k such that CF

$$\Pr(\mu - k \leq X \leq \mu + k) = 0.95$$

3 Records kept by a manufacturer of car tyres suggest that the distribution of the mileage from their tyres is normal, with mean 60 000 km and standard deviation 5000 km. CU

a What proportion of the company's tyres last:

i less that 55 000 km

ii more than 50 000 km but less than 74 000 km

iii more than 72 000 km, given that they have already lasted more than 60 000 km?

b The company's advertising manager wishes to claim that '90% of our tyres last longer than c km'. What should c be?

c What is the probability that a customer buys five tyres at the same time and finds that they all last longer than 72 000 km?

4 The owner of a new van complained to the dealer that he was using, on average, 18 litres of petrol to drive 100 km. The dealer pointed out that the 15 litres per 100 km referred to in an advertisement was 'just a guide and actual consumption will vary'. Suppose that the distribution of fuel consumption for this make of van is normal, with a mean of 15 litres per 100 km and a standard deviation of 0.75 litres per 100 km.

a How probable is it that such a van uses at least 18 litres per 100 km?

b What does your answer to **a** suggest about the manufacturer's claim?

c Find c_1 and c_2 such that the van's fuel consumption is more than c_1 but less than c_2 with a probability of 0.95.

5 Suppose that L, the useful life (in hours) of a fluorescent tube designed for indoor gardening, is normally distributed with a mean of 600 and a standard deviation of 4. The fluorescent tubes are sold in boxes of 10. Find the probability that at least three of the tubes in a randomly selected box last longer than 605 hours.

6 The amount of anaesthetic required to cause surgical anaesthesia in patients is normally distributed, with a mean of 50 mg and a standard deviation of 10 mg. The lethal dose is also normally distributed, with a mean of 110 mg and a standard deviation of 20 mg. If a dosage that brings 90% of patients to surgical anaesthesia were used, what percentage of patients would be killed by this dose?

7 In a given manufacturing process, components are rejected if they have a particular dimension greater than 60.4 mm or less than 59.7 mm. It is found that 3% are rejected as being too large and 5% are rejected as being too small. Assume that the dimension is normally distributed.

a Find the mean and standard deviation of the distribution of the dimension, correct to one decimal place.

b Use the result of **a** to find the percentage of rejects if the limits for acceptance are changed to 60.3 mm and 59.6 mm.

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8 The hardness of a metal may be determined by impressing a hardened point into the surface of the metal and then measuring the depth of penetration of the point. Suppose that the hardness of a particular alloy is normally distributed with mean 70 and standard deviation 3.

a If a specimen is acceptable only if its hardness is between 65 and 75, what is the probability that a randomly chosen specimen has an acceptable hardness?

b If the acceptable range of hardness was $(70 - c, 70 + c)$, for what value of c would 95% of all specimens have acceptable hardness?

c If the acceptable range is the same as in **a**, and the hardness of each of 10 randomly selected specimens is independently determined, what is the expected number of acceptable specimens among the 10?

d What is the probability that at most eight out of 10 randomly selected specimens have a hardness less than 73.84?

e The profit on an acceptable specimen is \$20, while unacceptable specimens result in a loss of \$5. If $\$P$ is the profit on a randomly selected specimen, find the mean and variance of P.

9 The weekly error (in seconds) of a brand of watch is known to be normally distributed. Only those watches with an error of less than 5 seconds are acceptable.

a Find the mean and standard deviation of the distribution of error if 3% of watches are rejected for losing time and 3% are rejected for gaining time.

b Determine the probability that fewer than two watches are rejected in a batch of 10 such watches.

10 A brand of detergent is sold in bottles of two sizes: standard and large. For each size, the content (in litres) of a randomly chosen bottle is normally distributed with mean and standard deviation as given in the table:

	Mean	Standard deviation
Standard bottle	0.760	0.008
Large bottle	1.010	0.009

a Find the probability that a randomly chosen standard bottle contains less than 0.75 litres.

b Find the probability that a box of 10 randomly chosen standard bottles contains at least three bottles whose contents are each less than 0.75 litres.

c Using the results

$$\mathrm{E}(aX - bY) = a\mathrm{E}(X) - b\mathrm{E}(Y)$$
$$\mathrm{Var}(aX - bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$$

find the probability that there is more detergent in four randomly chosen standard bottles than in three randomly chosen large bottles. (Assume that $aX - bY$ is normally distributed.)

ISBN 978-1-108-45164-2

Chapter 18

18 Sampling and estimation

Objectives

- To understand **random samples** and how they may be obtained.
- To define the **population proportion** and the **sample proportion**.
- To introduce the concept of the sample proportion as a random variable.
- To investigate the **sampling distribution** of the sample proportion both exactly (for small samples) and through simulation.
- To use a **normal distribution** to approximate the sampling distribution of the sample proportion.
- To use the sample proportion as a **point estimate** of the population proportion.
- To find **confidence intervals** for the population proportion.
- To introduce the concept of **margin of error**, and illustrate how this varies both with level of confidence and with sample size.

There is more to a complete statistical investigation than data analysis. First, we should consider the method used to collect the data. The purpose of selecting a sample and analysing the information collected from the sample is to make some sort of conclusion, or inference, about the population from which the sample was drawn.

For example, consider the question:

- What proportion of Year 12 students in Australia intend to take a gap year?

While we can answer this question for a sample of students, we really want to know about the whole population of Year 12 students. How can we generalise information gained from a sample to the population, and how confident can we be in that generalisation?

This chapter covers Unit 4 Topic 5: Interval estimates for proportions.

ISBN 978-1-108-45164-2

18A Populations and samples

The set of all eligible members of a group which we intend to study is called a **population**. For example, if we are interested in the IQ scores of the Year 12 students at ABC Secondary College, then this group of students could be considered a population; we could collect and analyse all the IQ scores for these students. However, if we are interested in the IQ scores of all Year 12 students across Australia, then this becomes the population.

Often, dealing with an entire population is not practical:

- The population may be too large – for example, all Year 12 students in Australia.
- The population may be hard to access – for example, all blue whales in the Pacific Ocean.
- The data collection process may be destructive – for example, testing every battery to see how long it lasts would mean that there were no batteries left to sell.

Nevertheless, we often wish to make statements about a property of a population when data about the entire population is unavailable.

The solution is to select a subset of the population – called a **sample** – in the hope that what we find out about the sample is also true about the population it comes from. Dealing with a sample is generally quicker and cheaper than dealing with the whole population, and a well-chosen sample will give much useful information about this population. How to select the sample then becomes a very important issue.

▶ Random samples

Suppose we are interested in investigating the effect of sustained computer use on the eyesight of a group of university students. To do this we go into a lecture theatre containing the students and select all the students sitting in the front two rows as our sample. This sample may be quite inappropriate, as students who already have problems with their eyesight are more likely to be sitting at the front, and so the sample may not be typical of the population. To make valid conclusions about the population from the sample, we would like the sample to have a similar nature to the population.

While there are many sophisticated methods of selecting samples, the general principle of sample selection is that the method of choosing the sample should not favour or disfavour any subgroup of the population. Since it is not always obvious if the method of selection will favour a subgroup or not, we try to choose the sample so that every member of the population has an equal chance of being in the sample. In this way, all subgroups have a chance of being represented. The way we do this is to choose the sample at random.

A sample of size n is called a **simple random sample** if it is selected from the population in such a way that every subset of size n has an equal chance of being chosen as the sample. In particular, every member of the population must have an equal chance of being included in the sample.

To choose a sample from the group of university students, we could put the name of every student in a hat and then draw out, one at a time, the names of the students who will be in the sample.

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Choosing the sample in an appropriate manner is critical in order to obtain useable results.

Example 1

A researcher wishes to evaluate how well the local library is catering to the needs of a town's residents. To do this she hands out a questionnaire to each person entering the library over the course of a week. Will this method result in a random sample?

Solution

Since the members of the sample are already using the library, they are possibly satisfied with the service available. Additional valuable information might well be obtained by finding out the opinion of those who do not use the library.

A better sample would be obtained by selecting at random from the town's entire population, so the sample contains both people who use the library and people who do not.

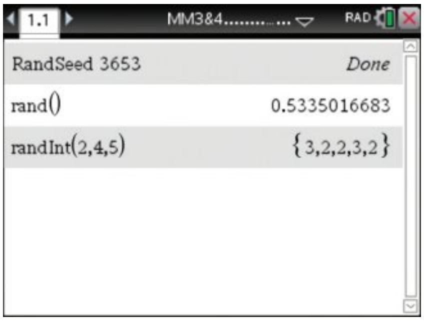
Thus, we have a very important consideration when sampling if we wish to generalise from the results of the sample.

In order to make valid conclusions about a population from a sample, we would like the sample chosen to be representative of the population as a whole. This means that all the different subgroups present in the population appear in the sample in similar proportions as they do in the population.

One very useful method for drawing random samples is to generate random numbers using a calculator or a computer.

Using the TI-Nspire CX non-CAS

- In a **Calculator** page, go to (menu) > **Probability** > **Random** > **Seed** and enter the last 4 digits of your phone number. This ensures that your random-number starting point differs from the calculator default.
- For a random number between 0 and 1, use (menu) > **Probability** > **Random** > **Number**.
- For a random integer, use (menu) > **Probability** > **Random** > **Integer**. To obtain five random integers between 2 and 4 inclusive, use the command randInt(2, 4, 5) as shown.



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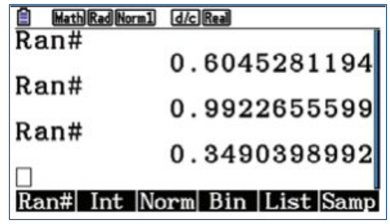
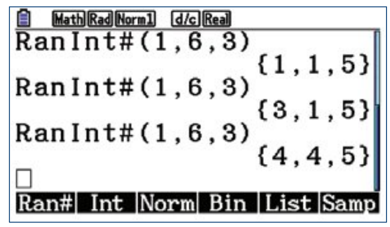
Using the Casio

In **Run-Matrix** mode, go to the **Probability** menu OPTN F6 F3. For random number generation, select **Rand** F4.

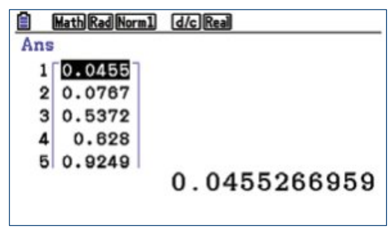
- To generate a random number between 0 and 1:
  - Select **Ran#** F1 and press EXE.
  - Press EXE again to generate another random number.



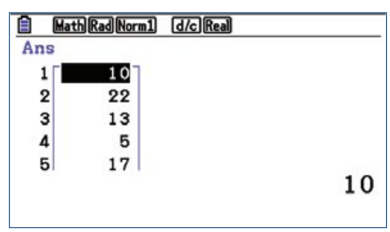
- To generate three random integers between 1 and 6 inclusive:
  - Select **Int** F2.
  - Type: 1, 6, 3)
  - Press EXE.
  - Press EXE again to generate another three random integers.



- To generate a list of 10 random numbers between 0 and 1:
  - Select **List** F5.
  - Type: 10)
  - Press EXE.
  - To view the list as a table, select the list and press EXE.
  - To exit the table, press EXIT.



- To generate a list of 20 random integers between 1 and 30 inclusive:
  - Select **Int** F2.
  - Type: 1, 30, 20)
  - Press EXE.
  - To view the list as a table, select the list and press EXE.
  - To exit the table, press EXIT.



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Example 2

Use a random number generator to select a group of six students from the following class:

- Denise
- Matt
- Teresa
- Sue
- Sharyn
- Mark
- Peter
- Nick
- Miller
- William
- Anne
- Darren
- Tom
- David
- Sally
- Janelle
- Steven
- Jane
- Georgia
- Jaimie

Solution

First assign a number to each member of the class:

- Denise (1)
- Matt (2)
- Teresa (3)
- Sue (4)
- Sharyn (5)
- Mark (6)
- Peter (7)
- Nick (8)
- Miller (9)
- William (10)
- Anne (11)
- Darren (12)
- Tom (13)
- David (14)
- Sally (15)
- Janelle (16)
- Steven (17)
- Jane (18)
- Georgia (19)
- Jaimie (20)

Generating six random integers from 1 to 20 gives on this occasion: 4, 19, 9, 2, 13, 14. The sample chosen is thus:

Sue, Georgia, Miller, Matt, Tom, David

Note: In this example, we want a list of six random integers without repeats. We do not add a randomly generated integer to our list if it is already in the list.

The sample proportion as a random variable

Suppose that our population of interest is the class of students from Example 2, and suppose further that we are particularly interested in the proportion of female students in the class. This is called the **population proportion** and is generally denoted by p. The population proportion p is constant for a particular population.

$$\text{Population proportion } p = \frac{\text{number in population with attribute}}{\text{population size}}$$

Since there are 10 males and 10 females, the proportion of female students in the class is

$$p = \frac{10}{20} = \frac{1}{2}$$

Now consider the proportion of female students in the sample chosen:

Sue, Georgia, Miller, Matt, Tom, David

The proportion of females in the sample may be calculated by dividing the number of females in the sample by the sample size. In this case, the proportion of female students in the sample is $\frac{2}{6} = \frac{1}{3}$. This value is called the **sample proportion** and is denoted by $\hat{p}$. (We say ‘p hat’.)

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> Sample proportion $\hat{p} = \dfrac{\text{number in sample with attribute}}{\text{sample size}}$

Note that different symbols are used for the sample proportion and the population proportion, so that we don't confuse them.

In this particular case, $\hat{p} = \frac{1}{3}$, which is not the same as the population proportion $p = \frac{1}{2}$. This does not mean there is a problem. In fact, each time a sample is selected the number of females in the sample will vary. Sometimes the sample proportion $\hat{p}$ will be $\frac{1}{2}$, and sometimes it will not.

- The population proportion p is a **population parameter**; its value is constant.
- The sample proportion $\hat{p}$ is a **sample statistic**; its value is not constant, but varies from sample to sample.

Example 3

Use a random number generator to select another group of six students from the same class, and determine the proportion of females in the sample.

Denise (1)	Sharyn (5)	Miller (9)	Tom (13)	Steven (17)
Matt (2)	Mark (6)	William (10)	David (14)	Jane (18)
Teresa (3)	Peter (7)	Anne (11)	Sally (15)	Georgia (19)
Sue (4)	Nick (8)	Darren (12)	Janelle (16)	Jaimie (20)

Solution

Generating another six random integers from 1 to 20 gives 19, 3, 11, 9, 15, 1.

The sample chosen is thus:

Georgia, Teresa, Anne, Miller, Sally, Denise

For this sample, we have

$$\hat{p} = \frac{5}{6}$$

We can see from Examples 2 and 3 that the random sampling process has resulted in two quite different samples. As a result, the values of the sample proportion, $\hat{p}$, for the two samples are also different. In fact, the sample proportion could take any of the values

$$0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1$$

Since $\hat{p}$ varies according to the contents of the random sample, we can consider these possible values of the sample proportion $\hat{p}$ as being the values of a random variable, which we will denote by $\hat{P}$. We investigate this idea further in the next section.

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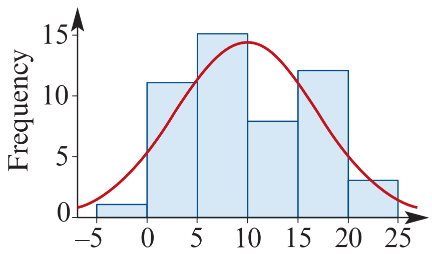
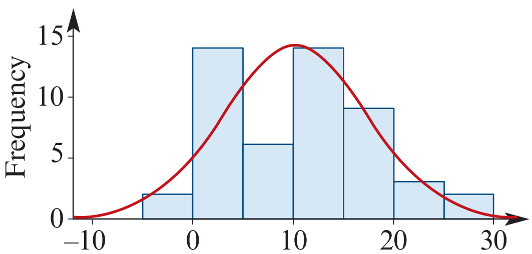
Investigating the variability of random samples

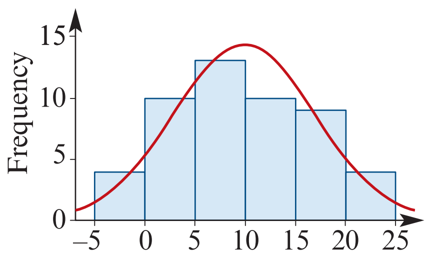
We can use simulation to investigate random samples from a variety of different distributions. For example, suppose that we draw a random sample of size 50 from a normal distribution with a mean of 10 and a standard deviation of 7.

A histogram of the sample data is shown on the right, together with the graph of the normal distribution from which the random sample has been drawn.



Repeating the process of drawing a random sample of size 50 from the same distribution twice more gave rise to the following two histograms:





Comparing these three histograms illustrates again a very important idea: the contents of a random sample will vary from sample to sample. Understanding the nature of this variation, and what a random sample can tell us about the population from which it has been drawn, are important ideas which we explore in more detail in the remainder of this chapter.

Section summary

- A **population** is the set of all eligible members of a group which we intend to study.
- A **sample** is a subset of the population which we select in order to make inferences about the population. Generalising from the sample to the population will not be useful unless the sample is representative of the population.
- A sample of size n is called a **simple random sample** if it is selected from the population in such a way that every subset of size n has an equal chance of being chosen as the sample. In particular, every member of the population must have an equal chance of being included in the sample.
- The **population proportion** p is the proportion of individuals in the entire population possessing a particular attribute, and is constant.
- The **sample proportion** $\hat{p}$ is the proportion of individuals in a particular sample possessing the attribute, and varies from sample to sample.
- The sample proportions $\hat{p}$ are the values of a random variable $\hat{P}$.

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Exercise 18A

CF

Example 1 **1** In order to determine the sort of film in which to invest his money, a producer waits outside a theatre and asks people as they leave whether they prefer comedy, drama, horror or science fiction. Do you think this is an appropriate way of selecting a random sample of movie goers? Explain your answer.

2 A market researcher wishes to find out how people spend their leisure time. She positions herself in a shopping mall and asks shoppers as they pass to fill out a short questionnaire.

a Do you think this sample will be representative of the general population? Explain.

b How would you suggest that the sample could be chosen?

3 To investigate people's attitudes to control of gun ownership, a television station conducts a phone-in poll, where people are asked to telephone one number if they are in favour of tighter gun control, and another if they are against. Is this an appropriate method of choosing a random sample? Give reasons for your answer.

4 A researcher wishes to select five guinea pigs at random from a large cage containing 20 guinea pigs. In order to select her sample, she reaches into the cage and (gently) pulls out five guinea pigs.

a Do you think this sample will be representative of the general population? Explain.

b How would you suggest the sample could be chosen?

5 In order to estimate how much money young people spend on takeaway food, a questionnaire is sent to several schools randomly chosen from a list of all schools in the state, to be given to a random selection of students in the school. Is this an appropriate method of choosing a random sample? Give reasons for your answer.

Example 2 **6** Use a random number generator to select a random sample of size 3 from the following list of people:

- Karen
- Kate
- Trinh
- Paul
- Lina
- Alexander
- Juliet
- Craig
- Conrad
- Fairlie
- Kylie
- Edward
- Kelly
- Rani
- Maree
- Janet
- Fleur
- Connie
- Aden
- Wolfgang
- Zoe
- Cara
- Noel
- Judy
- Andrew

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CF

7 In a survey to obtain adults' views on unemployment, people were stopped by interviewers as they came out of:

a a travel agency

b a supermarket

c an employment-services centre.

What is wrong with each of the methods of sampling listed here? Describe a better method of choosing the sample.

8 A marine biologist wishes to estimate the total number of crabs on a rock platform which is 10 metres square. It would be impossible to count them all individually, so she places a 1-metre-square frame at five random locations on the rock platform, and counts the number of crabs in the frame. To estimate the total number, she will multiply the average number in the frame by the total area of the rock platform.

a Explain how a random number generator could be used to select the five locations for the frame.

b Will this give a good estimate of the crab population?

9 In order to survey the attitude of parents to the current uniform requirements, the principal of a school selected 100 students at random from the school roll, and then interviewed their parents. Do you think this group of parents would form a simple random sample?

10 A television station carried out a poll to find out if the public felt that mining should be allowed in a particular area. People were asked to ring one number to register a 'yes' vote and another to register a 'no' vote. The results showed that 77% of people were in favour of mining proceeding. Comment on the results.

11 A market-research company decided to collect information concerning the way people use their leisure time by phoning a randomly chosen group of 1000 people at home between 7 p.m. and 10 p.m. on weeknights. The final report was based on the responses of only the 550 people of those sampled who could be found at home. Comment on the validity of this report.

12 In a certain school, 35% of the students travel on the school bus. A group of 100 students were selected in a random sample, and 42 of them travel on the school bus. In this example:

a What is the population?

b What is the value of the population proportion p?

c What is the value of the sample proportion $\hat{p}$?

ISBN 978-1-108-45164-2

CF

13 Of a random sample of 100 homes, 22 were found to have central heating.

a What proportion of these homes have central heating?

b Is this the value of the population proportion p or the sample proportion $\hat{p}$?

Example 3 **14** Use a random number generator to select another group of six students from the class listed below, and determine the proportion of females in the sample:

- Denise (1)
- Matt (2)
- Teresa (3)
- Sue (4)
- Sharyn (5)
- Mark (6)
- Peter (7)
- Nick (8)
- Miller (9)
- William (10)
- Anne (11)
- Darren (12)
- Tom (13)
- David (14)
- Sally (15)
- Janelle (16)
- Steven (17)
- Jane (18)
- Georgia (19)
- Jaimie (20)

15 Consider a continuous random variable X that is uniformly distributed in the interval $[5, 10]$. The probability density function of X is given by

$$f(x) = \begin{cases} \frac{1}{5} & \text{if } 5 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

a Sketch the graph of the probability density function f.

b Simulate the selection of a random sample of size 25 from this random variable using appropriate technology. (This can be done by generating random numbers between 5 and 10. Depending on the calculator or spreadsheet used, an appropriate command may be of the form $5 \times \text{rand}() + 5$.)

c Construct a histogram or dotplot of the data, and compare with the graph of f.

d Simulate the selection of two more random samples of size 25 from this random variable, and construct a histogram or dotplot for each sample.

e Comment on the variability of the three random samples.

16 Consider a continuous random variable X that is normally distributed with a mean of 110 and a standard deviation of 15.

a Simulate the selection of a random sample of size 50 from this random variable using appropriate technology. (Calculators and spreadsheets enable you to simulate randomly selecting from a normal random variable.)

b Construct a histogram or dotplot of the data, and compare with the graph of the probability density function of X.

c Simulate the selection of two more random samples of size 50 from this random variable, and construct a histogram or dotplot for each sample.

d Comment on the variability of the three random samples.

ISBN 978-1-108-45164-2

18B The exact distribution of the sample proportion

We have seen that the sample proportion varies from sample to sample. We can use our knowledge of probability to further develop our understanding of the sample proportion.

▶ Sampling from a small population

Suppose we have a bag containing six blue balls and four red balls, and from the bag we take a sample of size 4. We are interested in the proportion of blue balls in the sample. We know that the population proportion is equal to $\frac{6}{10} = \frac{3}{5}$. That is,

$$p = 0.6$$

The probabilities associated with the possible values of the sample proportion $\hat{p}$ can be calculated either by direct consideration of the sample outcomes or by using our knowledge of selections. Recall that

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

is the number of different ways to select x objects from n objects.

Example 4

A bag contains six blue balls and four red balls. If we take a random sample of size 4, what is the probability that there is one blue ball in the sample ($\hat{p} = \frac{1}{4}$)?

Solution

Method 1

Consider selecting the sample by taking one ball from the bag at a time (without replacement). The favourable outcomes are RRRB, RRBR, RBRR and BRRR, with

$$\begin{aligned}
&\Pr(\{\text{RRRB}, \text{RRBR}, \text{RBRR}, \text{BRRR}\})\\
&= \left(\frac{4}{10}\times\frac{3}{9}\times\frac{2}{8}\times\frac{6}{7}\right)+\left(\frac{4}{10}\times\frac{3}{9}\times\frac{6}{8}\times\frac{2}{7}\right)+\left(\frac{4}{10}\times\frac{6}{9}\times\frac{3}{8}\times\frac{2}{7}\right)+\left(\frac{6}{10}\times\frac{4}{9}\times\frac{3}{8}\times\frac{2}{7}\right)\\
&= \frac{4}{35}
\end{aligned}$$

Method 2

In total, there are $\binom{10}{4} = 210$ ways to select 4 balls from 10 balls.

There are $\binom{4}{3} = 4$ ways of choosing 3 red balls from 4 red balls, and there are $\binom{6}{1} = 6$ ways of choosing one blue ball from 6 blue balls.

Thus the probability of obtaining 3 red balls and one blue ball is equal to

$$\frac{\binom{4}{3}\times\binom{6}{1}}{\binom{10}{4}} = \frac{24}{210} = \frac{4}{35}$$

ISBN 978-1-108-45164-2

Cambridge University Press

The following table gives the probability of obtaining each possible sample proportion $\hat{p}$ when selecting a random sample of size 4 from the bag.

Number of blue balls in the sample	0	1	2	3	4
Proportion of blue balls in the sample, $\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
Probability	$\frac{1}{210}$	$\frac{24}{210}$	$\frac{90}{210}$	$\frac{80}{210}$	$\frac{15}{210}$

We can see from the table that we can consider the sample proportion as a random variable, $\hat{P}$, and we can write:

- $\Pr(\hat{P} = 0) = \frac{1}{210}$
- $\Pr\left(\hat{P} = \frac{1}{4}\right) = \frac{24}{210}$
- $\Pr\left(\hat{P} = \frac{1}{2}\right) = \frac{90}{210}$
- $\Pr\left(\hat{P} = \frac{3}{4}\right) = \frac{80}{210}$
- $\Pr(\hat{P} = 1) = \frac{15}{210}$

The possible values of $\hat{p}$ and their associated probabilities together form a probability distribution for the random variable $\hat{P}$, which can be summarised as follows:

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{210}$	$\frac{24}{210}$	$\frac{90}{210}$	$\frac{80}{210}$	$\frac{15}{210}$

The distribution of a statistic which is calculated from a sample (such as the sample proportion) has a special name – it is called a **sampling distribution**.

Example 5

A bag contains six blue balls and four red balls. Use the sampling distribution in the previous table to determine the probability that the proportion of blue balls in a sample of size 4 is more than $\frac{1}{4}$.

Solution

$$\Pr\left(\hat{P} > \frac{1}{4}\right) = \Pr\left(\hat{P} = \frac{1}{2}\right) + \Pr\left(\hat{P} = \frac{3}{4}\right) + \Pr(\hat{P} = 1)$$
$$= \frac{90}{210} + \frac{80}{210} + \frac{15}{210}$$
$$= \frac{185}{210}$$
$$= \frac{37}{42}$$

ISBN 978-1-108-45164-2

Sampling from a large population

Generally, when we select a sample, it is from a population which is too large or too difficult to enumerate or even count – populations such as all the people in Australia, or all the cows in Texas, or all the people who will ever have asthma. When the population is so large, we assume that the probability of observing the attribute we are interested in remains constant with each selection, irrespective of prior selections for the sample.

Suppose we know that 70% of all 17-year-olds in Australia attend school. That is,

$$p = 0.7$$

We will assume that this probability remains constant for all selections for the sample.

Now consider selecting a random sample of size 4 from the population of all 17-year-olds in Australia. This time we can use our knowledge of binomial distributions to calculate the associated probability for each possible value of the sample proportion $\hat{p}$, using the probability function

$$\Pr(X = x) = \binom{4}{x} 0.7^x 0.3^{4-x} \qquad x = 0, 1, 2, 3, 4$$

The following table gives the probability of obtaining each possible sample proportion $\hat{p}$ when selecting a random sample of four 17-year-olds.

Number at school in the sample	0	1	2	3	4
Proportion at school in the sample, $\hat{p}$	0	0.25	0.5	0.75	1
Probability	0.0081	0.0756	0.2646	0.4116	0.2401

Once again, we can summarise the sampling distribution of the sample proportion as follows:

$\hat{p}$	0	0.25	0.5	0.75	1
$\Pr(\hat{P} = \hat{p})$	0.0081	0.0756	0.2646	0.4116	0.2401

The population that the sample of size $n = 4$ is being taken from is such that each item selected has a probability $p = 0.7$ of success. Thus we can define the random variable

$$\hat{P} = \frac{X}{4}$$

where X is a binomial random variable with parameters $n = 4$ and $p = 0.7$. To emphasise this we can write:

x	0	1	2	3	4
$\hat{p} = \frac{x}{4}$	0	0.25	0.5	0.75	1
$\Pr(\hat{P} = \hat{p}) = \Pr(X = x)$	0.0081	0.0756	0.2646	0.4116	0.2401

Note: The probabilities for the sample proportions, $\hat{p}$, correspond to the probabilities for the numbers of successes, x.

ISBN 978-1-108-45164-2

Example 6

Use the sampling distribution in the previous table to determine the probability that, in a random sample of four Australian 17-year-olds, the proportion attending school is less than 50%.

Solution

$$\begin{aligned}\Pr(\hat{P} < 0.5) &= \Pr(\hat{P} = 0) + \Pr(\hat{P} = 0.25)\\ &= 0.0081 + 0.0756\\ &= 0.0837\end{aligned}$$

▶ The mean and standard deviation of the sample proportion

Since the sample proportion $\hat{P}$ is a random variable with a probability distribution, we can determine values for the mean and standard deviation, as illustrated in the following example.

Example 7

Use the probability distribution to determine the mean and standard deviation of the sample proportion $\hat{P}$ from Example 6.

$\hat{p}$	0	0.25	0.5	0.75	1
$\Pr(\hat{P} = \hat{p})$	0.0081	0.0756	0.2646	0.4116	0.2401

Solution

By definition, the mean of $\hat{P}$ is given by

$$\begin{aligned}\mathrm{E}(\hat{P}) &= \sum \hat{p} \cdot \Pr(\hat{P} = \hat{p})\\ &= 0 \times 0.0081 + 0.25 \times 0.0756 + 0.5 \times 0.2646 + 0.75 \times 0.4116 + 1 \times 0.2401\\ &= 0.7\end{aligned}$$

Similarly, by definition,

$$\mathrm{sd}(\hat{P}) = \sqrt{\mathrm{E}(\hat{P}^2) - [\mathrm{E}(\hat{P})]^2}$$

We have

$$\begin{aligned}\mathrm{E}(\hat{P}^2) &= 0^2 \times 0.0081 + 0.25^2 \times 0.0756 + 0.5^2 \times 0.2646 + 0.75^2 \times 0.4116 + 1^2 \times 0.2401\\ &= 0.5425\end{aligned}$$

Thus

$$\mathrm{sd}(\hat{P}) = \sqrt{0.5425 - 0.7^2} = 0.2291$$

We can see from Example 7 that the mean of the sampling distribution in this case is actually the same as the value of the population proportion (0.7). Is this always true? Can we determine the mean and standard deviation of the sample proportion without needing to find the probability distribution?

ISBN 978-1-108-45164-2

If we are selecting a random sample of size n from a large population, then we can assume that the sample proportion is of the form

$$\hat{P} = \frac{X}{n}$$

where X is a binomial random variable with parameters n and p. From Chapter 15, the mean and variance of X are given by

$$\mathrm{E}(X) = np \qquad \text{and} \qquad \mathrm{Var}(X) = np(1-p)$$

Thus we can determine

$$\begin{aligned}\mathrm{E}(\hat{P}) &= \mathrm{E}\left(\frac{X}{n}\right)\\ &= \frac{1}{n}\mathrm{E}(X) && \text{since } \mathrm{E}(aX+b) = a\mathrm{E}(X)+b\\ &= \frac{1}{n}\times np\\ &= p\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(\hat{P}) &= \mathrm{Var}\left(\frac{X}{n}\right)\\ &= \left(\frac{1}{n}\right)^2\mathrm{Var}(X) && \text{since } \mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)\\ &= \frac{1}{n^2}\times np(1-p)\\ &= \frac{p(1-p)}{n}\end{aligned}$$

If we are selecting a random sample of size n from a large population, then the mean and standard deviation of the sample proportion $\hat{P}$ are given by

$$\mathrm{E}(\hat{P}) = p \quad \text{and} \quad \mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$$

(The standard deviation of a sample statistic is called the **standard error**.)

Example 8

Use these rules to determine the mean and standard deviation of the sample proportion $\hat{P}$ from Example 6. Are they the same as those found in Example 7?

Solution

$$\mathrm{E}(\hat{P}) = p = 0.7$$

$$\mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.7(1-0.7)}{4}} = 0.2291$$

These are the same as those obtained in Example 7.

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Example 9

Suppose that 70% of 17-year-olds in Australia attend school. If a random sample of size 20 is chosen from this population, find:

a the probability that the sample proportion is equal to the population proportion (0.7)

b the probability that the sample proportion lies within one standard deviation of the population proportion

c the probability that the sample proportion lies within two standard deviations of the population proportion.

Solution

a If the sample proportion is $\hat{p} = 0.7$ and the sample size is 20, then the number of school students in the sample is $0.7 \times 20 = 14$. Thus

$$\begin{aligned}\Pr(\hat{P} = 0.7) &= \Pr(X = 14)\\ &= \binom{20}{14} 0.7^{14} 0.3^{6} = 0.1916\end{aligned}$$

b We have

$$\begin{aligned}\mathrm{sd}(\hat{P}) &= \sqrt{\frac{p(1-p)}{n}}\\ &= \sqrt{\frac{0.7(1-0.7)}{20}} = 0.1025\end{aligned}$$

Since $0.7 - 0.1025 = 0.5975$ and $0.7 + 0.1025 = 0.8025$, we find

$$\begin{aligned}\Pr(0.5975 \le \hat{P} \le 0.8025) &= \Pr(11.95 \le X \le 16.05)\\ &= \Pr(12 \le X \le 16) \qquad \text{since } X \text{ takes integer values}\\ &= 0.7795\end{aligned}$$

c Since $0.7 - 2 \times 0.1025 = 0.495$ and $0.7 + 2 \times 0.1025 = 0.905$, we find

$$\begin{aligned}\Pr(0.495 \le \hat{P} \le 0.905) &= \Pr(9.9 \le X \le 18.1)\\ &= \Pr(10 \le X \le 18)\\ &= 0.9752\end{aligned}$$

Section summary

- The distribution of a statistic which is calculated from a sample is called a **sampling distribution**.
- The **sample proportion** $\hat{P} = \dfrac{X}{n}$ is a random variable, where X is the number of favourable outcomes in a sample of size n.
- The distribution of $\hat{P}$ is known as the **sampling distribution** of the sample proportion.
- When the population is *small*, the sampling distribution of the sample proportion $\hat{P}$ can be determined using our knowledge of selections.

ISBN 978-1-108-45164-2

- When the population is *large*, the sampling distribution of the sample proportion $\hat{P}$ can be determined by assuming that X is a binomial random variable with parameters n and p. In this case, the mean and standard deviation of $\hat{P}$ are given by

$$\mathrm{E}(\hat{P}) = p \quad \text{and} \quad \mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$$

Exercise 18B

Skillsheet

Example 4, 5

1 Consider a bag containing five blue and five red balls.

a What is p, the proportion of blue balls in the bag?

b If samples of size 3 are taken from the bag, without replacement, then a sample could contain 0, 1, 2 or 3 blue balls. What are the possible values of the sample proportion $\hat{p}$ of blue balls associated with each of these samples?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of blue balls when samples of size 3 are taken from the bag, without replacement.

d Use the sampling distribution from **c** to determine the probability that the proportion of blue balls in the sample is more than 0.5. That is, find $\Pr(\hat{P} > 0.5)$.

SF

2 A company employs a sales team of 20 people, consisting of 12 men and 8 women.

a What is p, the proportion of men in the sales team?

b Five salespeople are to be selected at random to attend an important conference. What are the possible values of the sample proportion $\hat{p}$ of men in the sample?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of men when samples of size 5 are selected from the sales team.

d Use the sampling distribution from **c** to determine the probability that the proportion of men in the sample is more than 0.7.

e Find $\Pr(0 < \hat{P} < 0.7)$ and hence find $\Pr(\hat{P} < 0.7 \mid \hat{P} > 0)$.

CF

3 A pond contains eight gold and eight black fish.

a What is p, the proportion of gold fish in the pond?

b Three fish are to be selected at random. What are the possible values of the sample proportion $\hat{p}$ of gold fish in the sample?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of gold fish when samples of size 3 are selected from the pond.

d Use the sampling distribution from **c** to determine the probability that the proportion of gold fish in the sample is more than 0.25.

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CF

4 A random sample of three items is selected from a batch of 10 items which contains four defectives.

a What is p, the proportion of defectives in the batch?

b What are the possible values of the sample proportion $\hat{p}$ of defectives in the sample?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of defectives in the sample.

d Use the sampling distribution from **c** to determine the probability that the proportion of defectives in the sample is more than 0.5.

e Find $\Pr(0 < \hat{P} < 0.5)$ and hence find $\Pr(\hat{P} < 0.5 \mid \hat{P} > 0)$.

Example 6

5 Suppose that a fair coin is tossed four times, and the number of heads observed.

a What is p, the probability that a head is observed when a fair coin is tossed?

b What are the possible values of the sample proportion $\hat{p}$ of heads in the sample?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of heads in the sample.

d Use the sampling distribution from **c** to determine the probability that the proportion of heads in the sample is more than 0.7.

6 Suppose that the probability of a male child is 0.5, and that a family has five children.

a What are the possible values of the sample proportion $\hat{p}$ of male children in the family?

b Construct a probability distribution table which summarises the sampling distribution of the sample proportion of male children in the family.

c Use the sampling distribution from **b** to determine the probability that the proportion of male children in the family is less than 0.4.

d Find $\Pr(\hat{P} > 0 \mid \hat{P} < 0.8)$.

7 Suppose that, in a certain country, the probability that a person is left-handed is 0.2. If four people are selected at random from that country:

a What are the possible values of the sample proportion $\hat{p}$ of left-handed people in the sample?

b Construct a probability distribution table which summarises the sampling distribution of the sample proportion of left-handed people in the sample.

c Find $\Pr(\hat{P} > 0.5 \mid \hat{P} > 0)$.

Example 7

8 Use the sampling distribution from Question 5 to determine the mean and standard deviation of the sample proportion $\hat{P}$ of heads observed when a fair coin is tossed four times.

9 Use the sampling distribution from Question 6 to determine the mean and standard deviation of the sample proportion $\hat{P}$ of male children in a family of five children.

10 Use the sampling distribution from Question 7 to determine the mean and standard deviation of the sample proportion $\hat{P}$ of left-handed people when a sample of four people are selected.

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Example 8 **11** Suppose that the probability of rain on any day is 0.3. Find the mean and standard deviation of the sample proportion of rainy days which might be observed in the month of June.

12 In a certain country, it is known that 40% of people speak more than one language. If a sample of 100 people is selected, find the mean and standard deviation of the sample proportion of people who speak more than one language.

13 An examination consists of 100 multiple-choice questions, each with five possible answers. Find the mean and standard deviation of the sample proportion of correct answers that will be achieved if a student guesses every answer.

Example 9 **14** Suppose that 65% of people in Australia support a BBL cricket team. If a random sample of size 20 is chosen from this population, find:

a the probability that the sample proportion is equal to the population proportion

b the probability that the sample proportion lies within one standard deviation of the population proportion

c the probability that the sample proportion lies within two standard deviations of the population proportion.

CF

18C Approximating the distribution of the sample proportion

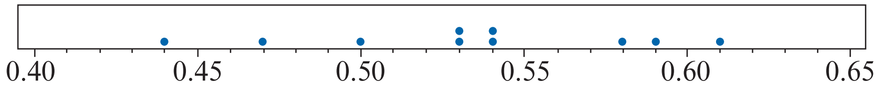
In the previous section, we used our knowledge of probability to determine the exact distribution of the sample proportion. Working out the exact probabilities associated with a sample proportion is really only practical when the sample size is quite small (say less than 10). In practice, we are rarely working with such small samples. But we can overcome this problem by approximating the distribution of the sample proportion.

Suppose, for example, we know that 55% of people in Australia have blue eyes ($p = 0.55$) and that we are interested in the values of the sample proportion $\hat{p}$ which might be observed when samples of size 100 are drawn at random from the population.

If we select one sample of 100 people and find that 50 people have blue eyes, then the value of the sample proportion is $\hat{p} = \dfrac{50}{100} = 0.5$.

If a second sample of 100 people is selected and this time 58 people have blue eyes, then the value of the sample proportion for this second sample is $\hat{p} = \dfrac{58}{100} = 0.58$.

Continuing in this way, after selecting 10 samples, the values of $\hat{p}$ that are observed might look like those in the following dotplot:



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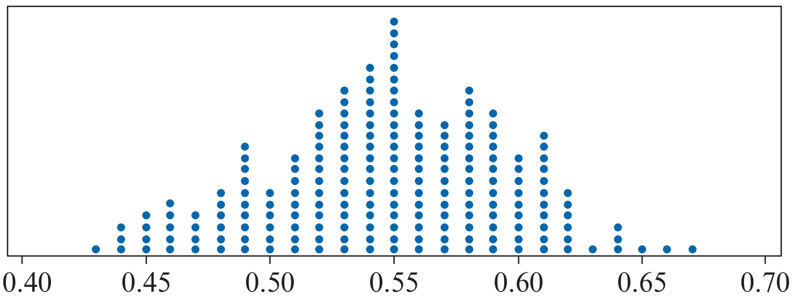
It is clear that the proportion of people with blue eyes in the sample, $\hat{p}$, is varying from sample to sample: from as low as 0.44 to as high as 0.61 for these particular 10 samples.

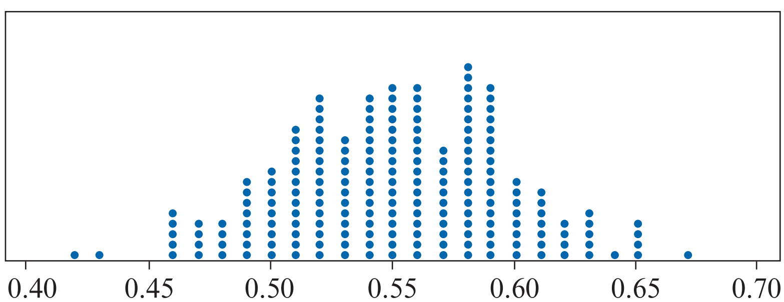
What does the distribution of the sample proportions look like if we continue with this sampling process?

The following dotplot summarises the values of $\hat{p}$ observed when 200 samples (each of size 100) were selected from a population in which the probability of having blue eyes is 0.55. We can see from the dotplot that the distribution is reasonably symmetric, centred at 0.55, and has values ranging from 0.43 to 0.67.



What does the distribution look like when another 200 samples (each of size 100) are selected at random from the same population?

The following dotplot shows the distribution obtained when this experiment was repeated. Again, the distribution is reasonably symmetric, centred at 0.55, and has values ranging from 0.42 to 0.67.



It seems reasonable to infer from these examples that, while there will be variation in the details of the distribution each time we take a collection of samples, the distribution of the values of $\hat{p}$ observed tends to conform to a predictable shape, centre and spread.

Actually, we already know from Chapter 17 that, when the sample size is large enough, the distribution of a binomial random variable is well approximated by the normal distribution. We have also seen that the rule of thumb for the normal approximation to the binomial distribution to apply is that both np and $n(1 - p)$ should be greater than 5.

The dotplots confirm the reasonableness of the normality assumption with regard to the sample proportion $\hat{P}$, which can be considered to be a linear function of a binomial random variable.

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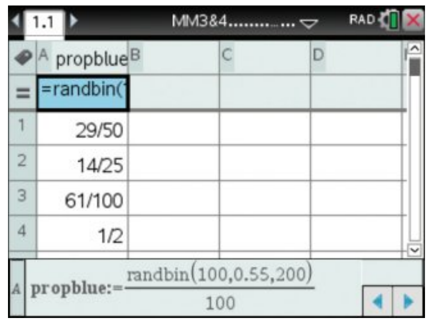
Repeated sampling can be investigated using a calculator.

Example 10

Assume that 55% of people in Australia have blue eyes. Use your calculator to illustrate a possible distribution of sample proportions $\hat{p}$ that may be obtained when 200 different samples (each of size 100) are selected from the population.

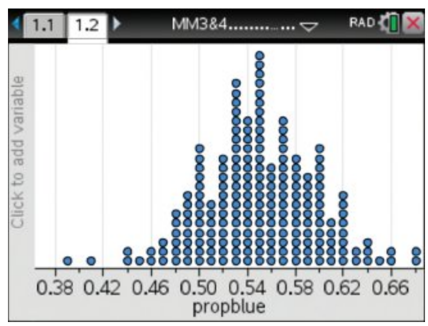
Using the TI-Nspire CX non-CAS

- To generate the sample proportions:
  - Start from a **Lists & Spreadsheet** page.
  - Name the list ‘propblue’ in Column A.
  - In the formula cell of Column A, enter the formula using (menu) > **Data** > **Random** > **Binomial** and complete as: = randbin(100, 0.55, 200)/100



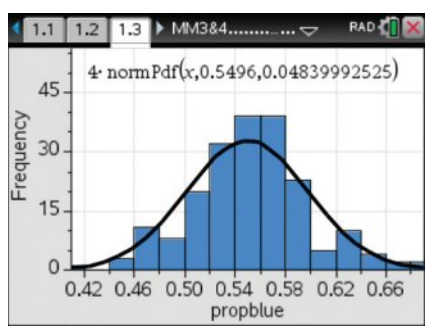
Note: The syntax is: randbin(*sample size*, *population proportion*, *number of samples*)
To calculate as a proportion, divide by the sample size.

- To display the distribution of sample proportions:
  - Insert a **Data & Statistics** page ((ctrl)(I) or (ctrl)(doc ▾)).
  - Click on ‘Click to add variable’ on the x-axis and select ‘propblue’. A dotplot is displayed.



Note: You can recalculate the random sample proportions by using (ctrl)(R) while in the **Lists & Spreadsheet** page.

- To fit a normal curve to the distribution:
  - (menu) > **Plot Type** > **Histogram**
  - (menu) > **Analyze** > **Show Normal PDF**



Note: The calculated Normal PDF, based on the data set, is superimposed on the plot, showing the mean and standard deviation of the sample proportion.

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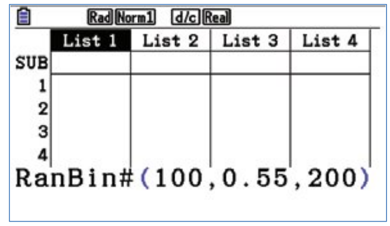
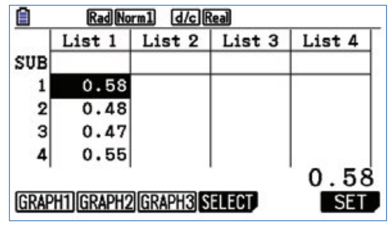
Using the Casio

- Press (MENU) (2) to select **Statistics** mode.
- To generate the sample proportions:
  - Move the cursor to the top of List1.
  - Select **RanBin#**: (OPTN) (F5) (F4) (F4)
  - Complete by entering: 100, 0.55, 200)/100

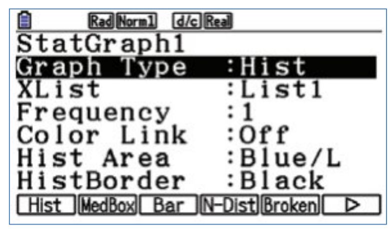




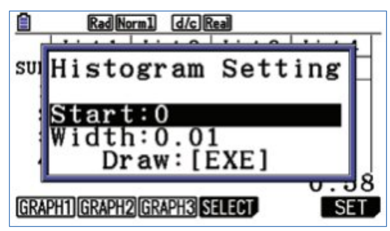
Note: The syntax is: RanBin#(*sample size*, *population proportion*, *number of samples*)
To calculate as a proportion, divide by the sample size.

- To construct a histogram to display the distribution of sample proportions:
  - Go to Graph Settings and select **Histogram** for the Graph Type:

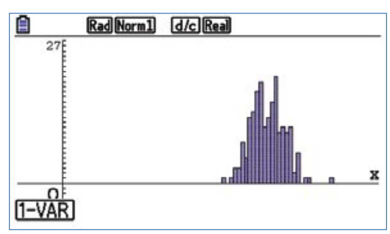
    (EXIT) (EXIT) (EXIT) (F1) (F6) ▼ (F6) (F1) (EXE)



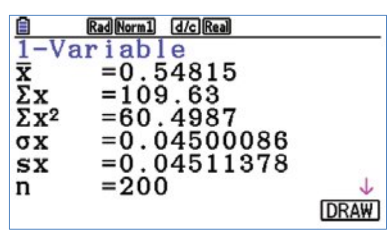
  - Enter a width of 0.01.



  - Press (EXE) to draw the graph.



  - To obtain statistics from the distribution, select **1-Var**.



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When the sample size n is large, the sample proportion $\hat{P}$ has an approximately normal distribution, with mean $\mu = p$ and standard deviation $\sigma = \sqrt{\dfrac{p(1-p)}{n}}$.

Thus, when samples of size $n = 100$ are selected from a population in which the proportion of people with blue eyes is $p = 0.55$, the distribution of the sample proportion $\hat{P}$ is approximately normal, with mean and standard deviation given by

$$\mu = \mathrm{E}(\hat{P}) = 0.55 \quad \text{and} \quad \sigma = \mathrm{sd}(\hat{P}) = \sqrt{\frac{0.55 \times 0.45}{100}} = 0.0497$$

Example 11

Assume that 60% of people have a driver's licence. Using the normal approximation, find the approximate probability that, in a randomly selected sample of size 200, more than 65% of people have a driver's licence.

Solution

Here $n = 200$ and $p = 0.6$. Since n is large, the distribution of $\hat{P}$ is approximately normal, with mean $\mu = p = 0.6$ and standard deviation

$$\sigma = \sqrt{\frac{0.6(1-0.6)}{200}} = 0.0346$$

Thus the probability that more than 65% of people in the sample have a driver's licence is

$$\Pr(\hat{P} > 0.65) = 0.0745 \quad \text{(correct to four decimal places)}$$

Section summary

When the sample size n is large, the sample proportion $\hat{P}$ has an approximately normal distribution, with mean $\mu = p$ and standard deviation $\sigma = \sqrt{\dfrac{p(1-p)}{n}}$.

Exercise 18C

In each of the following questions, use the normal approximation to the binomial distribution.

SF

Example 11

1. Find the approximate probability that, in the next 50 tosses of a fair coin, the proportion of heads observed will be less than or equal to 0.46.

2. In a large city, 12% of the workforce are unemployed. If 300 people from the workforce are selected at random, find the approximate probability that more than 10% of the people surveyed are unemployed.

3. It is known that on average 50% of the children born at a particular hospital are female. Find the approximate probability that more than 60% of the next 25 children born at that hospital will be female.

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SF

4 A car manufacturer expects 10% of cars produced to require minor adjustments before they are certified as ready for sale. What is the approximate probability that more than 15% of the next 200 cars inspected will require minor adjustments?

5 Past records show that on average 30% of the workers at a particular company have had one or more accidents in the workplace. What is the approximate probability that less than 20% of a random sample of 50 workers have had one or more accidents?

CF

6 Sacha is shooting at a target which she has a probability of 0.6 of hitting. What is the approximate probability that:

- **a** the proportion of times she hits the target in her next 100 attempts is less than 0.8
- **b** the proportion of times she hits the target in her next 100 attempts is between 0.6 and 0.8
- **c** the proportion of times she hits the target in her next 100 attempts is between 0.7 and 0.8, given that it is more than 0.6?

7 Find the approximate probability that, in the next 100 tosses of a fair coin, the proportion of heads will be between 0.4 and 0.6.

8 A machine has a probability of 0.1 of producing a defective item.

- **a** What is the approximate probability that, in the next batch of 1000 items produced, the proportion of defective items will be between 0.08 and 0.12?
- **b** What is the approximate probability that, in the next batch of 1000 items produced, the proportion of defective items will be between 0.08 and 0.12, given that we know that it is greater than 0.10?

9 The proportion of voters in the population who favour Candidate A is 52%. Of a random sample of 400 voters, 230 indicated that they would vote for Candidate A at the next election.

- **a** What is the value of the sample proportion, $\hat{p}$?
- **b** Find the approximate probability that, in a random sample of 400 voters, the proportion who favour Candidate A is greater than or equal to the value of $\hat{p}$ observed in this particular sample.

10 A manufacturer claims that 90% of their batteries will last more than 100 hours. Of a random sample of 250 batteries, 212 lasted more than 100 hours.

- **a** What is the value of the sample proportion, $\hat{p}$?
- **b** Find the approximate probability that, in a random sample of 250 batteries, the proportion lasting more than 100 hours is less than or equal to the value of $\hat{p}$ observed in this particular sample.
- **c** Does your answer to **b** cause you to doubt the manufacturer's claim?

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18D Confidence intervals for the population proportion

In practice, the reason we analyse samples is to further our understanding of the population from which they are drawn. That is, we know what is in the sample, and from that knowledge we would like to infer something about the population.

► Point estimates

Suppose, for example, we wish to know the proportion of primary school children in Australia who regularly use social media. The value of the population proportion p is unknown. As already mentioned, collecting information about the whole population is generally not feasible, and so a random sample must suffice. What information can be obtained from a single sample? Certainly, the sample proportion $\hat{p}$ gives some indication of the value of the population proportion p, and can be used when we have no other information.

The value of the sample proportion $\hat{p}$ can be used to estimate the population proportion p. Since this is a single-valued estimate, it is called a **point estimate** of p.

Thus, if we select a random sample of 20 Australian primary school children and find that the proportion who use social media is 0.7, then the value $\hat{p} = 0.7$ serves as an estimate of the unknown population proportion p.

► Interval estimates

The value of the sample proportion $\hat{p}$ obtained from a single sample is going to change from sample to sample, and while sometimes the value will be close to the population proportion p, at other times it will not. To use a single value to estimate p can be rather risky. What is required is an interval that we are reasonably sure contains the parameter value p.

An **interval estimate** for the population proportion p is called a **confidence interval** for p.

We have already seen that, when the sample size n is large, the sample proportion $\hat{P}$ has an approximately normal distribution with $\mu = p$ and $\sigma = \sqrt{\dfrac{p(1-p)}{n}}$.

By standardising, we can say that the distribution of the random variable

$$\frac{\hat{P} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

is approximated by that of a standard normal random variable Z.

We know that $\Pr(-1.96 < Z < 1.96) = 0.95$, correct to two decimal places, and therefore

$$\Pr\left(-1.96 < \frac{\hat{P} - p}{\sqrt{\frac{p(1-p)}{n}}} < 1.96\right) \approx 0.95$$

Multiplying through gives

$$\Pr\left(-1.96\sqrt{\frac{p(1-p)}{n}} < \hat{P} - p < 1.96\sqrt{\frac{p(1-p)}{n}}\right) \approx 0.95$$

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Further simplifying, we obtain

$$\Pr\left(\hat{P} - 1.96\sqrt{\frac{p(1-p)}{n}} < p < \hat{P} + 1.96\sqrt{\frac{p(1-p)}{n}}\right) \approx 0.95$$

Remember that what we want to do is to use the value of the sample proportion $\hat{p}$ obtained from a single sample to calculate an interval that we are fairly certain (say 95% certain) contains the true population proportion p (which we do not know).

In order to do this, we need to make one further approximation, and substitute $\hat{p}$ for p in our estimate of the standard deviation σ of $\hat{P}$.

An approximate **95% confidence interval** for p is given by

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

where:

- p is the population proportion (unknown)
- $\hat{p}$ is a value of the sample proportion
- n is the size of the sample from which $\hat{p}$ was calculated.

Note: In order to use this rule to calculate a confidence interval, the criteria for the normal approximation to the binomial distribution must apply. Therefore, from Chapter 17, we require both np and $n(1-p)$ to be greater than 5.

Example 12

Find an approximate 95% confidence interval for the proportion p of primary school children in Australia who regularly use social media, if we select a random sample of 20 children and find the sample proportion $\hat{p}$ to be 0.7.

Solution

Since $\hat{p} = 0.7$ and $n = 20$, we have

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.7 \times 0.3}{20}} = 0.1025$$

and so a 95% confidence interval for p is

$$(0.7 - 1.96 \times 0.1025,\ 0.7 + 1.96 \times 0.1025) = (0.499, 0.901)$$

Thus, based on a sample of size 20 and a sample estimate of 0.7, an approximate 95% confidence interval for the population proportion p is $(0.499, 0.901)$.

▶ Interpretation of confidence intervals

The confidence interval found in Example 12 should not be interpreted as meaning that $\Pr(0.499 < p < 0.901) = 0.95$. In fact, such a statement is meaningless, as p is a constant and either does or does not lie in the stated interval.

The particular confidence interval found is just one of any number of confidence intervals which could be found for the population proportion p, each one depending on the particular value of the sample proportion $\hat{p}$.

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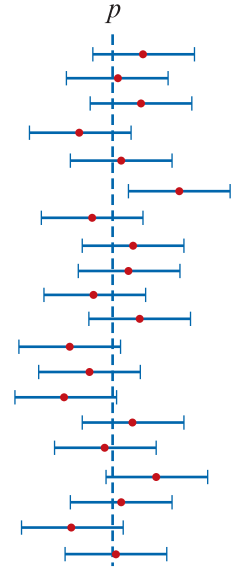
The correct interpretation of the confidence interval is that we expect approximately 95% of such intervals to contain the population proportion p. Whether or not the particular confidence interval obtained contains the population proportion p is generally not known.



If we were to repeat the process of taking a sample and calculating a confidence interval many times, the result would be something like that indicated in the diagram.

The diagram shows the confidence intervals obtained when 20 different samples were drawn from the same population. The round dot indicates the value of the sample estimate in each case. The intervals vary, because the samples themselves vary. The value of the population proportion p is indicated by the vertical line, and it is of course constant.

It is quite easy to see from the diagram that none of the values of the sample estimate is exactly the same as the population proportion, but that all the intervals except one (19 out of 20, or 95%) have captured the value of the population proportion, as would be expected in the case of a 95% confidence interval.

▶ Using a calculator to determine confidence intervals

Example 13

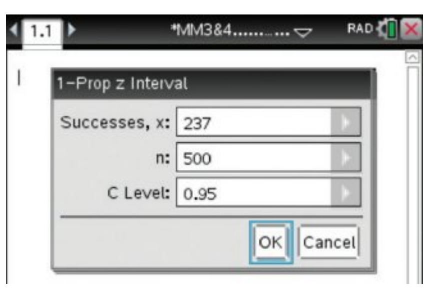
A survey found that 237 out of 500 undergraduate university students questioned intended to take a postgraduate course in the future. Find a 95% confidence interval for the proportion of undergraduates intending to take a postgraduate course.

TI-84

Using the TI-Nspire CX non-CAS

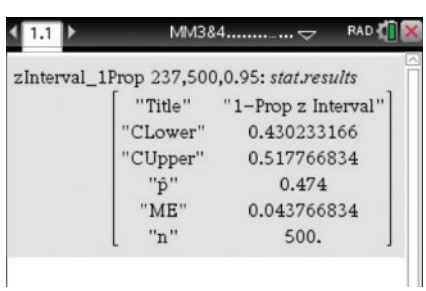
In a **Calculator** page:

- Use (menu) > **Statistics** > **Confidence Intervals** > **1–Prop z Interval**.
- Enter the values $x = 237$ and $n = 500$ as shown.



- The 'CLower' and 'CUpper' values give the 95% confidence interval (0.43, 0.52).

Note: 'ME' stands for margin of error, which is covered in the next subsection.



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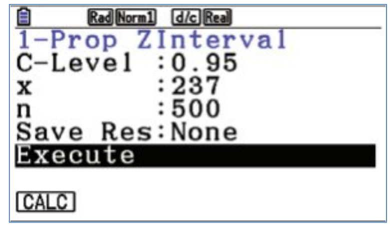
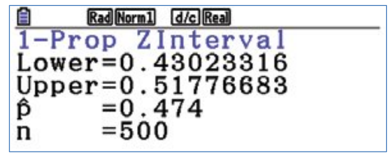
Using the Casio

- Press (MENU) (2) to select **Statistics** mode.
- For the Confidence Interval menu, select **Intr** (F4).
- Select **Z** (F1) and then **1-Prop** (F3).
- Enter the values $x = 237$ and $n = 500$ as shown.



- The 'Lower' and 'Upper' values give the 95% confidence interval $(0.43, 0.52)$.



▶ Precision and margin of error

In Example 12, we found an approximate 95% confidence interval $(0.499, 0.901)$ for the proportion p of primary school children in Australia who use social media, based on a sample of size 20. Therefore we predict that the population proportion p is somewhere in the range of approximately 50% to 90%! But this interval is so wide as to be not very helpful.

Example 14

Find an approximate 95% confidence interval for the proportion p of primary school children in Australia who regularly use social media, if we select a random sample of 200 children and find the sample proportion $\hat{p}$ to be 0.7.

Solution

Since $\hat{p} = 0.7$ and $n = 200$, we have

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.7 \times 0.3}{200}}$$
$$= 0.0324$$

and so a 95% confidence interval for p is

$$(0.7 - 1.96 \times 0.0324,\ 0.7 + 1.96 \times 0.0324) = (0.636, 0.764)$$

Thus, based on a sample of size 200 and a sample estimate of 0.7, an approximate 95% confidence interval for the population proportion p is $(0.636, 0.764)$.

Note: This interval is much narrower than the one determined in Example 12, which was based on a sample of size 20.

Often we discuss the confidence interval in terms of its width or, more formally, in terms of the distance between the sample estimate and the endpoints of the confidence interval.

That is, we find it useful to make statements such as 'we predict the proportion of people who will vote Labor in the next election as 52% ± 2%'. Here the sample estimate is 52%, and the distance between the sample estimate and the endpoints is 2%.

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The distance between the sample estimate and the endpoints of the confidence interval is called the **margin of error** (E). For a 95% confidence interval,

$$E = 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

We can see from this rule that the margin of error is a function of the sample size n, and that one way to make the interval narrower (that is, to increase the precision of the estimate) is to increase the sample size.

Example 15

Determine the sample size required to achieve a margin of error of 2% in an approximate 95% confidence interval for the proportion p of primary school children in Australia who use social media, if the sample proportion $\hat{p}$ is found to be 0.7.

Solution

Substituting $E = 0.02$ and $\hat{p} = 0.7$ in the expression for the margin of error gives

$$0.02 = 1.96\sqrt{\frac{0.7 \times 0.3}{n}}$$

Solving for n:

$$\left(\frac{0.02}{1.96}\right)^2 = \frac{0.7 \times 0.3}{n}$$

$$\therefore \quad n = 0.7 \times 0.3 \times \left(\frac{1.96}{0.02}\right)^2 = 2016.84$$

Thus, to achieve a margin of error of 2%, we need a sample of size 2017.

Of course, it is highly unlikely that we will know the value of the sample proportion $\hat{p}$ before we have selected the sample. Thus it is usual to substitute an estimated value into the equation in order to determine the sample size before we select the sample. This estimate can be based on our prior knowledge of the population or on a pilot study. If we denote this estimated value for the sample proportion by p^*, we can write

$$E = 1.96\sqrt{\frac{p^*(1-p^*)}{n}}$$

Rearranging to make n the subject of the equation, we find

$$E^2 = 1.96^2\left(\frac{p^*(1-p^*)}{n}\right)$$

$$\therefore \quad n = \left(\frac{1.96}{E}\right)^2 p^*(1-p^*)$$

A 95% confidence interval for a population proportion p will have margin of error approximately equal to a specified value of E when the sample size is

$$n = \left(\frac{1.96}{E}\right)^2 p^*(1-p^*)$$

where p^* is an estimated value for the population proportion p.

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Changing the level of confidence

So far we have only considered 95% confidence intervals, but in fact we can choose any level of confidence for a confidence interval. What is the effect of changing the level of confidence?

Consider again a 95% confidence interval:

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

From our knowledge of the normal distribution, we have $\Pr(-2.58 < Z < 2.58) = 0.99$, and so we can say that a 99% confidence interval will be given by

$$\left(\hat{p} - 2.58\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 2.58\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

In general:

- An approximate ***C*% confidence interval** for p is given by

$$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

  where the value of z is chosen so that $\Pr(-z < Z < z) = C\%$.
- The associated margin of error is

$$E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Note: The values of z (to two decimal places) for commonly used confidence intervals are:

- 90% $z = 1.64$
- 95% $z = 1.96$
- 99% $z = 2.58$

Example 16

Consider the proportion p of primary school children in Australia who use social media.

a Calculate and compare 90%, 95% and 99% confidence intervals for p, if we select a random sample of 200 children and find the sample proportion $\hat{p}$ to be 0.7.

b Determine the sample size required to achieve a margin of error of 2% in an approximate 99% confidence interval for p.

Solution

a From Example 14, we know that the 95% confidence interval is (0.636, 0.764).

The 90% confidence interval is

$$\left(0.7 - 1.64\sqrt{\frac{0.7 \times 0.3}{200}},\ 0.7 + 1.64\sqrt{\frac{0.7 \times 0.3}{200}}\right) = (0.647, 0.753)$$

The 99% confidence interval is

$$\left(0.7 - 2.58\sqrt{\frac{0.7 \times 0.3}{200}},\ 0.7 + 2.58\sqrt{\frac{0.7 \times 0.3}{200}}\right) = (0.616, 0.784)$$

It is helpful to use a diagram to compare these confidence intervals.

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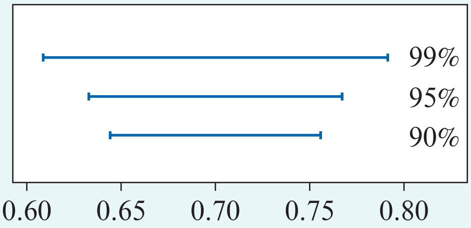
From the diagram, it can be clearly seen that the effect of being more confident that the confidence interval captures the true value of the population proportion means that a wider interval is required.



b Based on the sample from part **a**, we estimate the new sample proportion as $\hat{p} = 0.7$.

Substitute $E = 0.02$, $z = 2.58$ and $\hat{p} = 0.7$ in the expression for the margin of error and solve for n:

$$0.02 = 2.58\sqrt{\frac{0.7 \times 0.3}{n}}$$

$$\left(\frac{0.02}{2.58}\right)^2 = \frac{0.7 \times 0.3}{n}$$

$$\therefore \quad n = 0.7 \times 0.3 \times \left(\frac{2.58}{0.02}\right)^2 = 3494.61$$

Thus, to achieve a margin of error of 2%, we need a sample of size 3495.

Note: Comparing this answer with Example 15, we see that to achieve the same margin of error in a 99% confidence interval as in a 95% confidence interval, we require a much larger sample size.

In general, the width of the confidence interval (and hence the margin of error) will increase as the level of confidence increases. To be more confident that the interval will capture the true value of the population proportion, a wider confidence interval will be required.

Example 17

Suppose that we toss a fair coin 20 times, and determine the proportion of heads observed in this sample. Suppose further that this is repeated 10 times.

a Use your calculator to generate 10 values of the sample proportion $\hat{p}$ of heads in 20 coin tosses.

b Use your calculator to find an approximate 90% confidence interval for the population proportion p from each of these values of the sample proportion $\hat{p}$.

c How many of these intervals contain the value of the population proportion p?

d How many of these intervals would you expect to contain the value of the population proportion p?

Solution

a One set of simulations gave the following values for $\hat{p}$:

- 0.5
- 0.6
- 0.4
- 0.55
- 0.5
- 0.6
- 0.45
- 0.7
- 0.55
- 0.55

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b Confidence intervals based on these values of $\hat{p}$ are

- (0.32, 0.68)
- (0.42, 0.78)
- (0.22, 0.58)
- (0.37, 0.73)
- (0.32, 0.68)
- (0.42, 0.78)
- (0.27, 0.63)
- (0.53, 0.87)
- (0.37, 0.73)
- (0.37, 0.73)

c Here we find that 9 of the 10 intervals contain $p = 0.5$.

d On average, we would expect $0.9 \times 10 = 9$ intervals to contain the value of p.

Section summary

- The value of the sample proportion $\hat{p}$ can be used to estimate the population proportion p. Since this is a single-valued estimate, it is called a **point estimate** of p.
- An **interval estimate** for the population proportion p is called a **confidence interval** for p.
- An approximate ***C*% confidence interval** for p is given by

$$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

where:
  - p is the population proportion (unknown)
  - $\hat{p}$ is a value of the sample proportion
  - n is the size of the sample from which $\hat{p}$ was calculated
  - z is such that $\Pr(-z < Z < z) = C\%$.
- Values of z (to two decimal places) for commonly used confidence intervals:
  - 90% $z = 1.64$
  - 95% $z = 1.96$
  - 99% $z = 2.58$
- The distance between the sample estimate and the endpoints of the confidence interval is called the **margin of error**:

$$E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- A $C\%$ confidence interval for a population proportion p will have margin of error approximately equal to a specified value of E when the sample size is

$$n = \left(\frac{z}{E}\right)^2 p^*(1-p^*)$$

where p^* is an estimated value for the population proportion p.

Exercise 18D

Skillsheet

Example 12

1 A quality-control engineer in a factory needs to estimate the proportion of bags of potato chips packed by a certain machine that are underweight. The engineer takes a random sample of 100 bags and finds that eight of them are underweight. SF

a Find a point estimate for p, the proportion of bags packed by the machine that are underweight.

b Calculate a 95% confidence interval for p.

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SF

2 A newspaper wants to estimate the proportion of its subscribers who believe that the government should be allowed to tap telephones without a court order. It selects a random sample of 250 subscribers, and finds that 48 of them believe that the government should have this power.

a Find a point estimate for p, the proportion of subscribers who believe that the government should be allowed to tap telephones without a court order.

b Calculate a 95% confidence interval for p.

3 The lengths of stay in hospital among patients is of interest to health planners. A random sample of 100 patients was investigated, and 20 were found to have stayed longer than 7 days.

a Find a point estimate for p, the proportion of patients who stay in hospital longer than 7 days.

b Calculate a 95% confidence interval for p.

CF

Example 13

4 Given that 132 out of 400 randomly selected adult males are cigarette smokers, find a 95% confidence interval for the proportion of adult males in the population who smoke.

5 Of a random sample of 400 voters in a particular electorate, 210 indicated that they would vote for the Labor party at the next election.

a Use this information to find a 95% confidence interval for the proportion of Labor voters in the electorate.

b A random sample of 4000 voters from the same electorate was taken, and this time 2100 indicated that they would vote for Labor at the next election. Find a 95% confidence interval for the proportion of Labor voters in the electorate.

c Compare your answers to parts **a** and **b**.

Example 15

6 Determine the size of sample required to achieve a margin of error of 2% in an approximate 95% confidence interval when the sample proportion $\hat{p}$ is 0.8.

7 Samar is conducting a survey to estimate the proportion of people in Australia who would support reducing the driving age to 16. He knows from previous studies that this proportion is about 30%.

a Determine the size of sample required for the survey to achieve a margin of error of 3% in an approximate 95% confidence interval for this proportion.

b Determine the size of sample required for the survey to achieve a margin of error of 2% in an approximate 95% confidence interval for this proportion.

c Compare your answers to parts **a** and **b**.

Example 16a

8 When a coin thought to be biased was tossed 100 times, it came up heads 60 times. Calculate and compare 90%, 95% and 99% confidence intervals for the probability of observing a head when that coin is tossed.

9 As part of a survey, a total of 537 people from a random sample of 1000 people answered no to the question ‘Do you think the government is doing enough to address global warming?’ Calculate and compare 90%, 95% and 99% confidence intervals for the proportion of people in Australia who would answer no to that question.

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CF

10 A manufacturer claims that 90% of their batteries will last more than 50 hours.

a Of a random sample of 250 batteries, 212 lasted more than 50 hours. Use this information to find a 99% confidence interval for the proportion of batteries lasting more than 50 hours.

b An inspector requested further information. A random sample of 2500 batteries was selected and this time 2120 lasted more than 50 hours. Use this information to find a 99% confidence interval for the proportion of batteries lasting more than 50 hours.

c Compare your answers to parts **a** and **b**.

Example 16b **11** Determine the size of sample required to achieve a margin of error of 5% in an approximate 90% confidence interval when the sample proportion $\hat{p}$ is 0.2.

12 Bob is thinking of expanding his pizza delivery business to include a range of desserts. He would like to know the proportion of his clients who would order dessert from him, and so he intends to ask a number of his clients what they think.

a Bob thinks that the proportion of his clients who would order dessert is around 0.3. Determine the size of sample required for Bob to achieve a margin of error of 2% in an approximate 95% confidence interval for this proportion.

b Bob's business partner Phil thinks that the proportion of clients who would order dessert is around 0.5. Determine the size of sample required to achieve a margin of error of 2% in an approximate 95% confidence interval for this proportion.

c What is the effect on the margin of error if:

i Bob is correct, but they use the sample size from Phil's estimate

ii Phil is correct, but they use the sample size from Bob's estimate?

d What sample size would you recommend that Bob and Phil use?

Example 17 **13** Jelly beans are packed in boxes of 50, and the overall proportion of black jelly beans is set by the manufacturer to be 0.2. Suppose that 10 boxes of jelly beans are selected at random, and the proportion of black jelly beans in each box determined.

a Use your calculator to generate 10 values of the sample proportion $\hat{p}$ of black jelly beans in a box.

b Use your calculator to find an approximate 80% confidence interval for the population proportion p from each of these values of the sample proportion $\hat{p}$.

c How many of these intervals contain the value of the population proportion p?

d How many of these intervals would you expect to contain the value of the population proportion p?

e Suppose that we generate 50 approximate 80% confidence intervals for p. How many of these intervals would you expect to contain the value of the population proportion p?

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Review

Chapter summary

AS

Sampling

- A **population** is the set of all eligible members of a group which we intend to study.
- A **sample** is a subset of the population which we select in order to make inferences about the population. Generalising from the sample to the population will not be useful unless the sample is representative of the population.
- A sample of size n is called a **simple random sample** if it is selected from the population in such a way that every subset of size n has an equal chance of being chosen as the sample. In particular, every member of the population must have an equal chance of being included in the sample.
- The **population proportion** p is the proportion of individuals in the entire population possessing a particular attribute, and is constant.
- The **sample proportion** $\hat{p}$ is the proportion of individuals in a particular sample possessing the attribute, and varies from sample to sample.
- The sample proportion $\hat{P} = \dfrac{X}{n}$ is a random variable, where X is the number of favourable outcomes in a sample of size n. The distribution of the random variable $\hat{P}$ is known as the **sampling distribution** of the sample proportion.
- When the population is *small*, the sampling distribution of the sample proportion $\hat{P}$ can be determined using our knowledge of selections.
- When the population is *large*, the sampling distribution of the sample proportion $\hat{P}$ can be determined by assuming that X is a binomial random variable with parameters n and p. In this case, the mean and standard deviation of $\hat{P}$ are given by

$$\mathrm{E}(\hat{P}) = p \quad \text{and} \quad \mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$$

- When the sample size n is large, the sample proportion $\hat{P}$ has an approximately normal distribution, with mean $\mu = p$ and standard deviation $\sigma = \sqrt{\dfrac{p(1-p)}{n}}$.

Estimation

- If the value of the sample proportion $\hat{p}$ is used as an estimate of the population proportion p, then it is called a **point estimate** of p.
- An approximate ***C*% confidence interval** for p is given by

$$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

  where:
  - p is the population proportion (unknown)
  - $\hat{p}$ is a value of the sample proportion
  - n is the size of the sample from which $\hat{p}$ was calculated
  - z is such that $\Pr(-z < Z < z) = C\%$.
- Values of z (to two decimal places) for commonly used confidence intervals:
  - 90% $z = 1.64$
  - 95% $z = 1.96$
  - 99% $z = 2.58$

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- The distance between the sample estimate and the endpoints of the confidence interval is called the **margin of error**:

$$E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- A $C\%$ confidence interval for a population proportion p will have margin of error approximately equal to a specified value of E when the sample size is

$$n = \left(\frac{z}{E}\right)^2 p^*(1-p^*)$$

where p^* is an estimated value for the population proportion p.

Technology-free questions

SF

1 A company has 2000 employees, 700 of whom are female. A random sample of 100 employees was selected, and 40 of them were female. In this example:

a What is the population?

b What is the value of the population proportion p?

c What is the value of the sample proportion $\hat{p}$?

2 To study the effectiveness of yoga for reducing stress levels, a researcher measured the stress levels of 50 people who had just enrolled in a 10-week introductory yoga course, and then measured their stress levels at the end the course.

a Do you think that this sample will be representative of the general population? Explain your answer.

b How would you suggest that the sample could be chosen?

3 A coin is tossed 100 times, and k heads observed.

a Give a point estimate for p, the probability of observing a head when the coin is tossed.

b Write down an expression for a 95% confidence interval for p.

CF

4 A sample of n people were asked whether they thought that income tax in Australia was too high, and 90% said yes.

a What is the value of the sample proportion $\hat{p}$?

b Write down an expression for E, the margin of error for this estimate at the 95% confidence level, in terms of n.

c If the number of people in the sample were doubled, what would be the effect on the margin of error E?

5 Suppose that 40 independent random samples are taken from a large population, and a 95% confidence interval for the population proportion p is computed from each sample.

a How many of the 95% confidence intervals would you expect to contain the population proportion p?

b Write down an expression for the probability that all 40 confidence intervals contain the population proportion p.

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CF

6 Suppose that 50 independent random samples were taken from a large population, and that a 90% confidence interval for the population proportion p was computed from each of these samples.

a How many of the 90% confidence intervals would you expect to contain the population proportion p?

b Write down an expression for the probability that at least 49 of the 50 confidence intervals contain the population proportion p.

7 A newspaper determined that an approximate 95% confidence interval for the proportion of people in Australia who regularly read the news online was $(0.50, 0.70)$.

a What was the value of $\hat{p}$ which was used to determine this confidence interval?

b What is the margin of error?

c How could the newspaper increase the precision of their study?

Multiple-choice questions

1 In order to estimate the ratio of males to females at a school, a teacher determines the number of males and the number of females in a particular class. The ratio that he then calculates is called a

A sample **B** sample statistic **C** population parameter
D population **E** sample parameter

2 In a complete census of the population of a particular community, it is found that 59% of families have two or more children. Here '59%' represents the value of a

A sample **B** sample statistic **C** population parameter
D population **E** sample parameter

3 From a random sample, a 90% confidence interval for the population proportion p is found to be $(0.7, 0.8)$. This means that

A the population proportion $p = 0.75$
B the probability that the population proportion p lies in the interval $(0.7, 0.8)$ is 0.9
C the probability that the population proportion p lies in the interval $(0.7, 0.8)$ is 0.1
D 90% of random samples lead to confidence intervals which contain the population proportion p
E none of these

4 A survey showed that 15 out of a random sample of 50 football supporters attend at least one match per season. If this information is used to find a 95% confidence interval for the proportion of all football supporters who attend at least one match per season, then the margin of error will be

A 0.3 **B** 0.004 **C** 0.065 **D** 0.254 **E** 0.127

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5 Of a random sample of 50 golfers, four were found to play golf left-handed. A 95% confidence interval for the proportion of golfers in the population who play left-handed is given by

A (0.053, 0.107) **B** (0.026, 0.134) **C** (0.005, 0.155)
D (0.006, 0.154) **E** (0.075, 0.085)

6 Fourteen of a random sample of 88 people said they prefer to watch the news on a particular channel. A 90% confidence interval for the proportion of people in the population who prefer to watch the news on that channel is given by

A (0.083, 0.236) **B** (0.083, 0.197) **C** (0.085, 0.233)
D (0.095, 0.223) **E** (0.120, 0.198)

7 If the sample proportion remains unchanged, then an increase in the level of confidence will lead to a confidence interval which is

A narrower **B** wider **C** unchanged **D** asymmetric
E cannot be determined from the information given

8 Which of the following statements is true?

I The centre of a confidence interval is a population parameter.
II The bigger the margin of error, the smaller the confidence interval.
III The confidence interval is a type of point estimate.
IV A population proportion is an example of a point estimate.

A I only **B** II only **C** III only **D** IV only **E** none of these

9 If a researcher increases her sample size by a factor of 4, then the width of a 95% confidence interval would

A increase by a factor of 2 **B** increase by a factor of 4 **C** decrease by a factor of 2
D decrease by a factor of 4 **E** none of these

10 The Education Department in a certain state wishes to determine the percentage of teachers who are considering leaving the profession in the next two years. They believe it to be about 25%. How large a sample should be taken to find the answer to within ±3% at the 95% confidence level?

A 6 **B** 33 **C** 534 **D** 752 **E** 897

11 Which of the following statements is true?

A We use sample statistics to estimate population parameters.
B We use sample parameters to estimate population statistics.
C We use population parameters to estimate sample statistics.
D We use population statistics to estimate sample parameters.
E none of the above

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12 A sampling distribution can best be described as a distribution which

- **A** gives the possible range of values of the sample statistic
- **B** describes how a statistic's value will change from sample to sample
- **C** describes how samples do not give reliable estimates
- **D** gives the distribution of the values observed in a particular sample
- **E** none of the above

13 A survey is conducted to determine the percentage of students in Year 12 who intend to go straight to university after they finish secondary school. In a random sample of 100 students, 78% indicated this intention. A 95% confidence interval for the percentage of students in Year 12 who intend to go straight to university is

- **A** 68.2% to 87.8%
- **B** 68.5% to 87.5%
- **C** 69.9% to 86.1%
- **D** 71.2% to 84.8%
- **E** 73.9% to 82.1%

14 In Question 13, what is one way to decrease the width of the confidence interval?

- **A** increase the sample size
- **B** use a smaller confidence level
- **C** use a higher confidence level
- **D** both A and B are correct
- **E** both A and C are correct

Extended-response questions

1 A survey is being planned to estimate the proportion of people in Australia who think that university fees should be abolished. The organisers of the survey want the error in the approximate 95% confidence interval for this proportion to be no more than ±2%. They have no prior information about the value of the proportion.

CU

- **a** Plot the sample size, $n = \left(\frac{1.96}{E}\right)^2 p^*(1 - p^*)$, against p^* for $0 \le p^* \le 1$.
- **b** For what value of p^* is the sample size the maximum?
- **c** What value of n would you recommend be used for the survey?
- **d** Show that the maximum sample size required for the error in an approximate 95% confidence interval to be no more than E is approximately $n = \frac{1}{E^2}$.

2 It is known that 60% of the voters in a particular electorate support the Liberal party. A sample of 100 voters is taken. Let $\hat{P}$ be the random variable for the sampling distribution of the sample proportion. Use the normal approximation to find:

- **a** $\Pr(\hat{P} > 0.65)$
- **b** $\Pr(0.5 < \hat{P} < 0.65)$

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3 **a** Summer is investigating the probability that a drawing pin will land point-up when tossed. She tosses the drawing pin 100 times, and finds that it lands point-up 57 times. Determine an approximate 95% confidence interval for the probability that the drawing pin lands point-up when tossed.

b Four of Summer's friends decide to repeat her investigation, each tossing the drawing pin 100 times. They each calculate an approximate 95% confidence interval based on their own data, making five confidence intervals in all.

i What is the probability that all five confidence intervals contain the true value of p, the probability that the drawing pin will land point-up when tossed?

ii What is the probability that none of the confidence intervals contain p?

iii What is the probability that at least one of the confidence intervals does not contain p?

iv How many of these five confidence intervals would you expect to contain p?

c Summer's four friends obtained the following results, each based on tossing the drawing pin 100 times and counting the number of times that it lands point-up:

- Emma 67
- Chloe 72
- Maddie 55
- Regan 60

Summer suggests that the best estimate of p would be obtained by pooling their results. Based on all the data collected, determine an approximate 95% confidence interval for p.

4 A landscape gardener wishes to estimate how many carp live in his very large ornamental lake. He is advised that the best way to do this is through capture–recapture sampling.

a Suppose that there are N carp in the lake and he captures 500 of them, tags them and then releases them back into the lake. Write down an expression for the proportion of tagged carp in the lake.

b The next day, a sample of 400 carp is captured from the lake, and he finds that there are 60 tagged carp in this sample. What is the proportion of tagged carp in the second sample?

c If the second sample is representative of the population, we expect the proportion of tagged carp in the second sample to be the same as the proportion of tagged carp in the lake. That is,

$$\frac{60}{400} \approx \frac{500}{N}$$

Use this equation to find an estimate for the number of carp in the lake.

d Show that an expression for a 95% confidence interval for the proportion of tagged carp in the lake can be written as

$$0.15 - 1.96\sqrt{\frac{0.1275}{400}} < \frac{500}{N} < 0.15 + 1.96\sqrt{\frac{0.1275}{400}}$$

e Use this inequality to find an approximate 95% confidence interval for the number of carp in the lake.

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Revision

19 Revision of Unit 4

19A Technology-free questions

The second derivative and applications

SF

1 Find the second derivative of each of the following:

a $x^3 + 2x^2 + 5x - 2$ **b** $e^{-\frac{x}{4}}$ **c** xe^{-5x}

d $\sin\left(\frac{x}{5}\right)$ **e** $\cos\left(\frac{\pi x}{2}\right)$ **f** $e^{2x} - e^{-2x}$

2 A particle moves in a straight line so that after t seconds its position, x m, relative to a fixed point O on the line is given by $x = 2t^3 - 24t$. Find the velocity and acceleration of the particle at time t seconds.

CF

3 Let $f(x) = xe^{2x^2-4x}$.

a Find $f'(x)$.

b Show that $f'(x) \geq 0$ for all x.

c Hence show that the only stationary point on the graph of f is a stationary point of inflection.

SF

4 For $f(x) = x^3 - 3x^2 + 1$, find the values of x for which the graph of f is concave up.

5 Let $f(x) = 4x - \sin(2x)$ for $0 \leq x \leq 2\pi$. Find the coordinates of the points of inflection on the graph of $y = f(x)$.

6 A polynomial function f has derivative f' with rule $f'(x) = (x-1)^2(x+1)(x-3)$.

a Solve the equation $f'(x) = 0$ for x.

b Given that $f''(x) = 4(x-1)(x^2 - 2x - 1)$, determine the nature of the stationary points corresponding to the x-values found in part **a**.

c Find the x-values at which there is a point of inflection.

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7 Sketch the graph of $f(x) = x^4 - 10x^2 + 9$, locating the stationary points and the points of inflection.

Trigonometry using the sine and cosine rules

8 A triangle has sides 6 cm, 7 cm and 10 cm. Find the cosine of the smallest angle of this triangle.

9 Triangle ABC has area 20 cm^2. The length of AB is 10 cm and the length of AC is 12 cm. Find $\sin(\angle BAC)$.

10 The three side lengths of a triangle are $\sqrt{2}$, 2 and $1 + \sqrt{3}$. Find the cosine of the largest angle of this triangle.

11 Consider triangle ABC with $AB = 6$ cm, $AC = 4$ cm and $\angle BAC = 60°$.
(For each of the following, give your answer in simplest surd form as $m\sqrt{n}$, where m and n are natural numbers and n has no factors that are perfect squares.)

a Find the length BC.
b Find the area of triangle ABC.

12 Consider triangle ABC with $AB = 6$ cm, $AC = 4$ cm and $\angle BAC = 120°$.
(For each of the following, give your answer in simplest surd form.)

a Find the length BC.
b Find the area of triangle ABC.

Discrete random variables and the binomial distribution

13 The random variable X has the following probability distribution.

x	0	1	2	3	4
$\Pr(X = x)$	0.3	0.2	0.1	0.3	0.1

Find:

a $\Pr(X > 3 \mid X > 1)$
b $\Pr(X > 1 \mid X \leq 3)$
c the mean of X
d the variance of X

14 The random variable X has probability distribution:

x	1	2	3	4	5
$\Pr(X = x)$	a	0.3	0.1	0.2	b

Given that $E(X) = 2.34$, find the values of a and b.

15 A biased coin is tossed three times. On each toss, the probability of a head is p.

a Find, in terms of p, the probability that all three tosses show tails.
b If the probability of three tails is equal to 8 times the probability of three heads, find the value of p.

16 A binomial distribution with parameters n and p has mean 6 and variance 3.6. Find the values of n and p.

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Revision

CF

17 A fair die is rolled three times.

a What is the probability that a 6 is obtained on all three rolls?

b What is the probability that a 6 is obtained on at least two of the three rolls?

18 Let p be the probability that a randomly chosen person believes in ghosts. Eight people are asked whether they believe in ghosts. Write an expression for the probability that:

a all eight people believe in ghosts

b none of the eight people believes in ghosts

c at least one of the eight people believes in ghosts.

19 Suppose that X is a binomial random variable with mean 3 and variance 2.
Express $\Pr(X = 1)$ in the form $\dfrac{a^b}{c^d}$, where a and c are prime numbers.

20 Consider a binomial random variable X with parameters $n = 20$ and $p = \dfrac{1}{5}$.

a Write expressions for $\Pr(X = k)$ and $\Pr(X = k + 1)$.

b Show that $\dfrac{\Pr(X = k + 1)}{\Pr(X = k)} = \dfrac{20 - k}{4(k + 1)}$.

c Use this result to find the value of k for which $\Pr(X = k)$ is a maximum.

Continuous random variables and the normal distribution

21 The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} kx(6 - x) & \text{if } 0 < x < 6 \\ 0 & \text{otherwise} \end{cases}$$

Find:

a the value of k

b $\Pr(X < 4)$

c the median of X

d the mean of X

e $\Pr(X < 2 \mid X < 3)$

f $\Pr(X > 2 \mid X < 4)$

22 The random variable X has probability density function:

$$f(x) = \begin{cases} (x - a)(2a - x) & \text{if } a \le x \le 2a \\ 0 & \text{otherwise} \end{cases}$$

a Show that $a^3 = 6$.

b Find $\mathrm{E}(X)$.

23 The probability density function of a random variable X is given by

$$f(x) = \begin{cases} 2(1 - x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Find $F(x)$, the cumulative distribution function of X.

b Find the value a of X such that $\Pr(X \le a) = \dfrac{3}{4}$.

24 The random variable X is normally distributed with mean 40 and standard deviation 2. If $\Pr(36 < X < 44) = q$, find $\Pr(X > 44)$ in terms of q.

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25 The random variable X is normally distributed with mean 50 and standard deviation 5. If $\Pr(X < 60) = q$, find $\Pr(40 < X < 60)$ in terms of q.

26 Let X be a normal random variable with mean $\mu = 92$ and standard deviation $\sigma = 10$. Let Z be the standard normal random variable. Given that $\Pr(Z < 1) = 0.84$, find:

a $\Pr(X > 82)$ **b** $\Pr(82 < X < 92)$

27 Let X be a normally distributed random variable with a mean of 10 and a variance of 9. Let Z be the standard normal random variable.

a Find $\Pr(X > 10)$.

b Find the value of a such that $\Pr(X > 4) = \Pr(Z < a)$.

Sampling and estimation

28 Suppose that an approximate 95% confidence interval for the population proportion is given by the interval (a, b).

a Write down an expression for the sample proportion in terms of a and b.

b Write down an expression for the margin of error for this confidence interval in terms of a and b.

29 In a large population of marsupials, the proportion of animals with a particular disease is 25%. Let $\hat{P}$ be the random variable that represents the sample proportion of animals with the disease for samples of size n drawn from this population. Find the smallest value of n such that the standard deviation of $\hat{P}$ is less than or equal to $\frac{1}{50}$.

30 A committee of four is going to be randomly chosen from a group of five men and four women. Let $\hat{P}$ be the random variable for the proportion of males in the committee.

a List the possible values of $\hat{p}$, the proportion of males in the committee.

b Find:

i $\Pr(\hat{P} = 0)$ **ii** $\Pr(\hat{P} > 0)$ **iii** $\Pr\left(\hat{P} = \frac{21}{2}\right)$

31 An investigation was carried out to determine the most popular model of car in a particular town. The data was collected by observing the model of each car entering the car park of a supermarket in this town during a 2 hour period. Comment on the validity of this data-collection method.

32 Suppose that 20 independent random samples were taken from a large population, and that a 90% confidence interval for the population proportion p was computed from each of these samples.

a How many of these 90% confidence intervals would you expect to contain the population proportion p?

b Write down an expression for the probability that all 20 confidence intervals contain the population proportion p.

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33 Over 30 000 people are attending a concert, with four in nine of these people under 18 years of age. A simple random sample of 2000 concertgoers is selected. Find the standard deviation of $\hat{P}$, the proportion of people in the sample under 18 years of age. CF

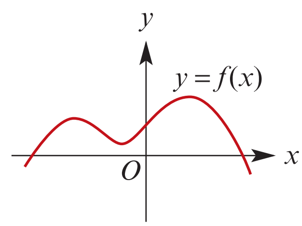
19B Multiple-choice questions

1 The second derivative of $3x^2 + 2x - \dfrac{4}{x^2}$ with respect to x is

A $6x + 2 - \dfrac{8}{x^3}$ **B** $6x + 2 + \dfrac{8}{x^3}$ **C** $6 - \dfrac{24}{x^4}$ **D** $6 + \dfrac{24}{x^4}$ **E** $3 - \dfrac{4}{x^4}$

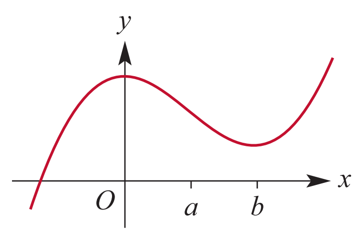
2 For the graph of $y = f(x)$ shown, the number of points of inflection is

A 3 points **B** 2 points **C** 5 points
D 0 points **E** none of these



3 The curve shown is concave down on the interval

A (a, ∞) **B** $(-\infty, a)$ **C** $(0, b)$
D $(0, \infty)$ **E** $(-\infty, b)$

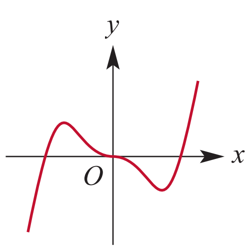


4 A particular polynomial function f has the following three properties:

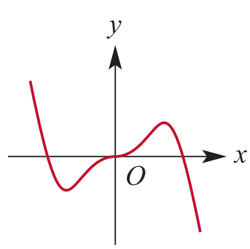
- $f'(x) = 0$ for $x = -2$ and $x = 2$ only
- $f''(x) > 0$ for all $x < 0$
- $f''(x) < 0$ for all $x > 0$.

Which of the following could be the graph of f?

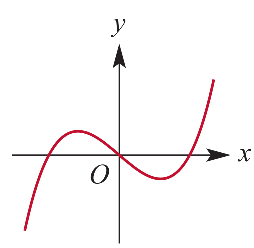
A



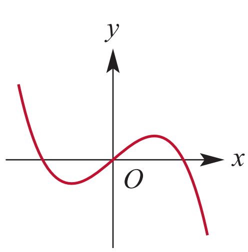
B



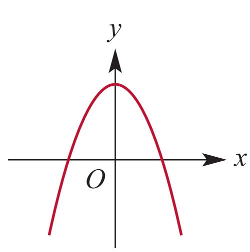
C



D



E



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5 Which of the following graphs has no point of inflection?

A $y = \sin x$ **B** $y = \tan x$ **C** $y = x^3 - 3x$ **D** $y = 2x^2 - 1$ **E** $y = x^4 - x$

6 If $y = x^2e^x$, then the minimum value of y is

A -2 **B** 0 **C** $4e^{-2}$ **D** $-4e^{-2}$ **E** e

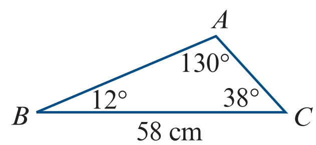
7 Given that $x + y = 1$, the maximum value of $P = x^2 + xy - y^2$ occurs for x equal to

A 2 **B** -1 **C** $\frac{3}{2}$ **D** 1 **E** $\frac{2}{3}$

8 A function f with domain $(0, 4)$ satisfies $f'(x) = x \sin x - \cos x$. The graph of f has a point of inflection when x is approximately

A 0.9 **B** 1.2 **C** 2.3 **D** 3.4 **E** 3.7

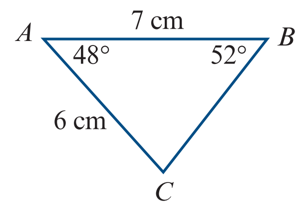
9 Which one of the following gives the correct value for c?



A $\dfrac{58 \cos 38°}{\cos 130°}$ **B** $\dfrac{58 \sin 38°}{\sin 130°}$ **C** $58 \sin 38°$

D $\dfrac{58 \cos 130°}{\cos 38°}$ **E** $\dfrac{58 \sin 130°}{\sin 38°}$

10 Which one of the following expressions will give the area of triangle ABC?

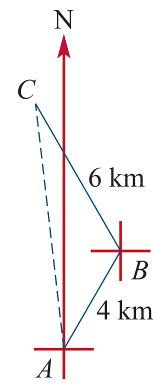


A $\frac{1}{2} \times 6 \times 7 \sin 48°$ **B** $\frac{1}{2} \times 6 \times 7 \cos 48°$

C $\frac{1}{2} \times 6 \times 7 \sin 52°$ **D** $\frac{1}{2} \times 6 \times 7 \cos 52°$

E $\frac{1}{2} \times 6 \times 7 \tan 48°$

11 A hiker walks 4 km from point A on a bearing of 030° to point B, and then walks 6 km on a bearing of 330° to point C. The distance AC, in kilometres, is



A $\dfrac{4}{\sin 30°}$

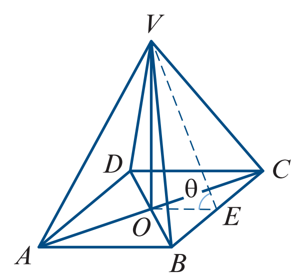
B $\sqrt{6^2 + 4^2 - 48 \cos 120°}$

C $\sqrt{6^2 + 4^2 + 48 \cos 120°}$

D $6 \sin 60°$

E $\sqrt{52}$

12 $VABCD$ is a right square pyramid with base length 80 mm and perpendicular height 100 mm. The angle θ between a sloping face and the base $ABCD$, to the nearest degree, is



A 22° **B** 29° **C** 51°

D 61° **E** 68°

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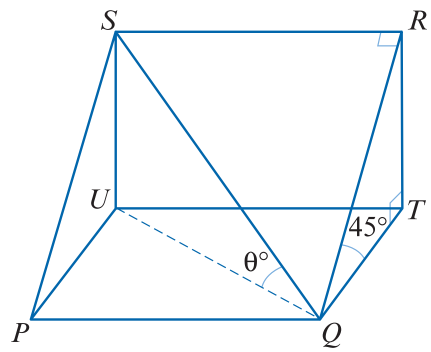
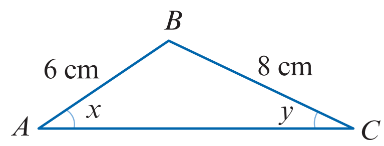
13 In this figure, the rectangle $PQRS$ is inclined at an angle of 45° to the horizontal plane $PQTU$. The magnitude of $\angle PQS$ is 60°. Let θ° be the angle of inclination of QS to the horizontal plane. The value of $\sin(\theta°)$ is



A $\frac{1}{4}$ **B** $\frac{\sqrt{2}}{4}$ **C** $\frac{\sqrt{2}}{2}$ **D** $\frac{\sqrt{3}}{2}$ **E** $\frac{\sqrt{6}}{4}$

14 Consider triangle ABC shown on the right. If $\sin x = \frac{3}{7}$, then $\sin y =$



A $\frac{1}{7}$ **B** $\frac{9}{28}$ **C** $\frac{1}{2}$ **D** $\frac{4}{7}$ **E** $\frac{3}{4}$

15 A box contains 12 red balls and 4 green balls. A ball is selected at random from the box and not replaced, and then a second ball is drawn. The probability that the two balls are both green is equal to

A $\frac{1}{4}$ **B** $\frac{1}{16}$ **C** $\frac{3}{64}$ **D** $\frac{1}{8}$ **E** $\frac{1}{20}$

16 A test consists of six true/false questions. The probability that a student who guesses will obtain six correct answers is

A 0.9844 **B** 0.0278 **C** 0.5 **D** 0.0156 **E** 0.17

17 A random variable X has the following probability distribution.

x	1	2	3	4
$\Pr(X = x)$	$4c^2$	$5c^2$	$4c^2$	$3c^2$

The value of c is

A 0.5 **B** 0.0263 **C** 0.1622 **D** 0.25 **E** 0.0625

Questions 18–19 refer to the following probability distribution.

x	4	6	7	9
$\Pr(X = x)$	0.3	0.2	0.1	0.4

18 For this probability distribution, the mean, $\mathrm{E}(X)$, is equal to

A 6.7 **B** 0.275 **C** 6.5 **D** 2.75 **E** 2.59

19 For this probability distribution, the variance, $\mathrm{Var}(X)$, is equal to

A 19.45 **B** 4.41 **C** 6.7 **D** 2.1 **E** 0.61

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20 If a random variable X is such that $E(X) = 11$ and $E(X^2) = 202$, then the standard deviation of X is equal to

A 191 **B** 13.82 **C** 9 **D** 3.72 **E** 81

21 If three fair coins are tossed, what is the probability that there are at least two heads?

A $\frac{1}{3}$ **B** $\frac{6}{7}$ **C** $\frac{1}{4}$ **D** $\frac{1}{2}$ **E** $\frac{1}{8}$

22 Let X be a binomial random variable with parameters $n = 400$ and $p = 0.1$. Then $E(X)$, the mean of X, is equal to

A 36 **B** 6 **C** 40 **D** 6.32 **E** 360

23 Which of the following does *not* define a binomial random variable?

A A die is rolled 10 times, and the number of sixes observed.

B A die is rolled until a six is obtained, and the number of rolls counted.

C A die is rolled five times, and the number of even numbers showing observed.

D A sample of 20 people is chosen from a large population, and the number of females counted.

E A student guesses the answer to every question on a multiple-choice test, and the number of correct answers is noted.

24 Let X be a binomial random variable with parameters $n = 900$ and $p = 0.2$. The standard deviation of X is equal to

A 18 **B** 144 **C** 180 **D** 13.42 **E** 12

25 Let X be a binomial random variable with a variance of 9.4248. If $n = 42$, then the probability of success p is equal to

A 0.45 **B** 0.22 **C** 0.34 **D** 0.68 **E** 0.34 or 0.66

26 If p is the probability of success in one trial, then $\binom{7}{5} p^5(1-p)^2$ is the probability in seven trials of

A exactly two failures **B** exactly two successes

C at least two failures **D** exactly five failures

E more failures than successes

27 The proportion of female students at a particular university is 0.2. A sample of 10 students is chosen at random from the entire student population. What is the probability that the sample contains exactly four female students?

A 0.0881 **B** 0.5000 **C** 0.0328 **D** 0.0016 **E** 0.9672

28 Mai decides to call five friends to invite each of them to a party. The probability of a friend not being at home when Mai calls is 0.4. What is the probability that Mai finds at least one of her friends at home?

A 0.0778 **B** 0.9222 **C** 0.0102 **D** 0.9898 **E** 0.0768

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29 If a random variable X has probability density function given by

$$f(x) = \begin{cases} kx^3 + \frac{3}{4}x & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

then k is equal to

A $-\frac{3}{16}$ **B** $\frac{6}{25}$ **C** $-\frac{9}{16}$ **D** $-\frac{1}{8}$ **E** $-\frac{3}{8}$

30 If a random variable X has probability density function given by

$$f(x) = \begin{cases} \frac{1}{9}(4x - x^2) & \text{if } 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

then $\Pr(X \leq 2)$ is closest to

A 0.6667 **B** 0.4074 **C** 0.5926 **D** 0.4444 **E** 0.5556

31 A random variable X has probability density function:

$$f(x) = \begin{cases} \frac{8}{3}(1 - x) & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

The median of X is closest to

A 0.222 **B** 0.667 **C** 0.250 **D** 1.791 **E** 0.209

32 A random variable X has probability density function:

$$f(x) = \begin{cases} 2(1 - x^{-2}) & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

The mean of X is closest to

A 1 **B** 1.614 **C** 2 **D** 1.5 **E** 0.609

33 The probability of obtaining a z-value which falls between $z = -1.0$ and $z = 0$ for a standard normal distribution is approximately

A 0.05 **B** 0.20 **C** 0.34 **D** 0.68 **E** 0.16

34 If X is a normally distributed random variable with mean $\mu = 2$ and standard deviation $\sigma = 2$, then the probability that X is less than -2 is

A 0.1587 **B** 0.8413 **C** 0.9772 **D** 0.1228 **E** 0.0228

35 If X is a normally distributed random variable with mean $\mu = 2$ and variance $\sigma^2 = 4$, then $\Pr(1 < X < 2.5)$ is

A 0.5987 **B** 0.2902 **C** 0.6915 **D** 0.4013 **E** 0.3085

36 An automatic dispensing machine fills cups with cordial. If the amount of cordial in the cup is a normally distributed random variable with a mean of 50 mL and a standard deviation of 2 mL, then 90% of the cups contain more than

A 44.87 mL **B** 53.29 mL **C** 46.71 mL **D** 52.56 mL **E** 47.44 mL

 ISBN 978-1-108-45164-2

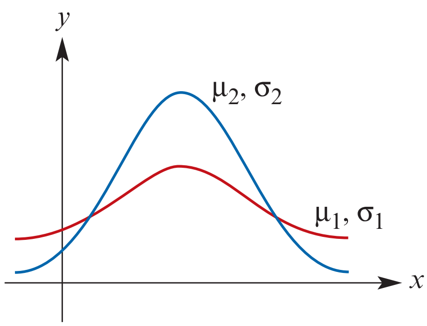
37 Assume that X is a normally distributed random variable with mean $\mu = 1$ and variance $\sigma^2 = 2.25$. If $\Pr(\mu - k < X < \mu + k) = 0.7$, then $k =$

A 1.555 **B** 1.037 **C** 0.787 **D** 0.524 **E** 2.332

38 The weight of a packet of biscuits is known to be normally distributed with a mean of 1 kg. If a packet is more than 0.05 kg underweight, it is unacceptable. If it is found that 3% of packets are unacceptable, then the standard deviation of the weight is

A 1.881 **B** 0.027 **C** 10.488 **D** 0.030 **E** 37.616

39 The diagram shows the probability density functions of two normally distributed random variables, one with mean μ_1 and standard deviation σ_1, and the other with mean μ_2 and standard deviation σ_2.
Which of the following statements is true?



A $\mu_1 = \mu_2,\ \sigma_1 < \sigma_2$ **B** $\mu_1 = \mu_2,\ \sigma_1 > \sigma_2$
C $\mu_1 > \mu_2,\ \sigma_1 = \sigma_2$ **D** $\mu_1 < \mu_2,\ \sigma_1 = \sigma_2$
E $\mu_1 = \mu_2,\ \sigma_1 = \sigma_2$

40 In a random sample of 200 people, 38% said they would rather watch tennis on television than attend the match. An approximate 95% confidence interval for the proportion of people in the population who prefer to watch tennis on television is

A $(0.136, 0.244)$ **B** $(0.313, 0.447)$ **C** $(0.285, 0.475)$
D $(0.255, 0.505)$ **E** $(0.292, 0.468)$

41 For a fixed sample, an increase in the level of confidence will lead to a confidence interval which is

A narrower **B** wider **C** unchanged **D** asymmetric
E cannot be determined from the information given

42 Which of the following statements are true?

I The lower the level of confidence, the smaller the confidence interval.
II The larger the sample size, the smaller the confidence interval.
III The smaller the sample size, the smaller the confidence interval.
IV The higher the level of confidence, the smaller the confidence interval.

A I and II **B** I and III **C** II only **D** II and IV **E** none of these

43 If a researcher decreases her sample size by a factor of 2, then the width of a 95% confidence interval would

A increase by a factor of 2 **B** increase by a factor of $\sqrt{2}$
C decrease by a factor of $\sqrt{2}$ **D** decrease by a factor of 4
E none of these

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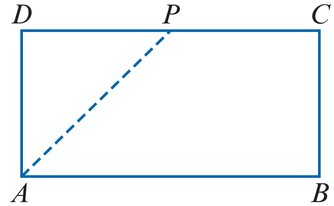
Revision

19C Extended-response questions

CU

1 A Queensland resort has a large swimming pool as illustrated, with $AB = 75$ m and $AD = 30$ m.



A boy can swim at 1 m/s and run at $1\frac{2}{3}$ m/s. He starts at A, swims to a point P on DC, and runs from P to C. He takes 2 seconds to pull himself out of the pool.

Let $DP = x$ m and the total time taken be T s.

a Show that

$$T = \sqrt{x^2 + 900} + \frac{3}{5}(75 - x) + 2$$

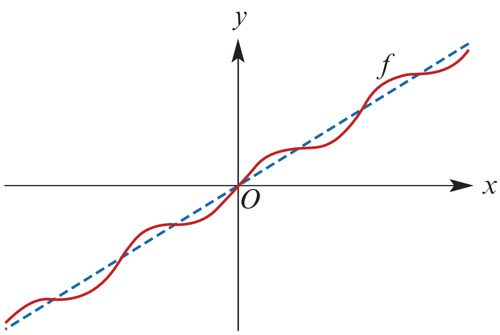
b Find $\frac{dT}{dx}$.

c **i** Find the value of x for which the time taken is a minimum.

ii Find the minimum time.

d Find the time taken if the boy runs from A to D and then from D to C.

2 Consider the function f given by $f(x) = x + \sin x$ for $-4\pi \le x \le 4\pi$.



a Find $f'(x)$ and $f''(x)$.

b Show that $f'(x) \ge 0$ for all x.

c Solve the equation $f''(x) = 0$ for $-4\pi \le x \le 4\pi$.

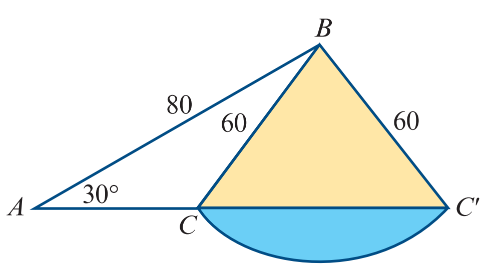
d Find the coordinates of the stationary points of inflection on the graph of f.

Now consider $g(x) = \frac{x}{2} + \sin x$ for $-2\pi \le x \le 2\pi$.

e Solve the equation $g'(x) = 0$ for $-2\pi \le x \le 2\pi$.

f Find the coordinates of the stationary points on the graph of g.

3 For $\triangle ABC$ in the diagram, $A = 30°$, $a = 60$ and $c = 80$.



a Find the magnitudes of angles:

i BCA and ABC

ii $BC'A$ and ABC'

b Find the lengths of line segments:

i AC **ii** AC' **iii** CC'

c Show that the magnitude of $\angle CBC'$ is 96.38° (correct to two decimal places). Then using this value:

i find the area of triangle BCC'

ii find the area of the shaded sector

iii find the area of the shaded segment.

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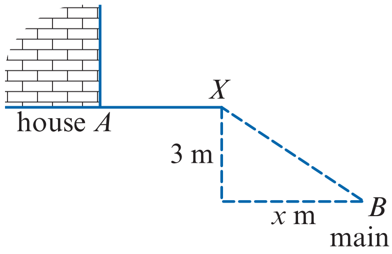
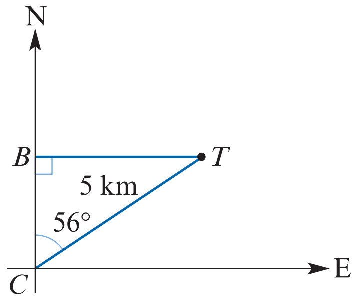
4 To connect a house to a gas supply, a pipe must be installed connecting the point A on the house to the point B on the main, where B is 3 m below ground level and at a horizontal distance of 4 m from the building. If it costs \$25 per metre to lay pipe underground and \$10 per metre on the surface, find the length of pipe which should be on the surface to minimise costs.



5 Adam notices a distinctive tree while orienteering on a flat horizontal plane. From where he is standing, the tree is 200 m away on a bearing of 050°. Two other people, Brian and Colin, who are both standing due east of Adam, each claim that the tree is 150 m away from them. Given that their claims are true and that Brian and Colin are not standing in the same place, how far apart are they? Give your answer to the nearest metre.

6 Cora is standing at point C. She can see a beacon due north at point B. She can also see a tower at point T, which is 5 km away on a bearing of 056°. The tower at T is due east of the beacon at B.

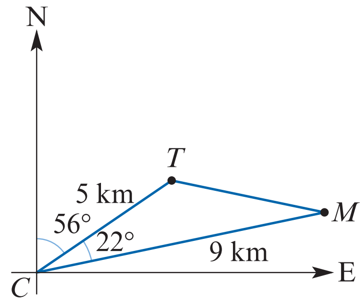
a Find the distance between the beacon and the tower, correct to three decimal places.



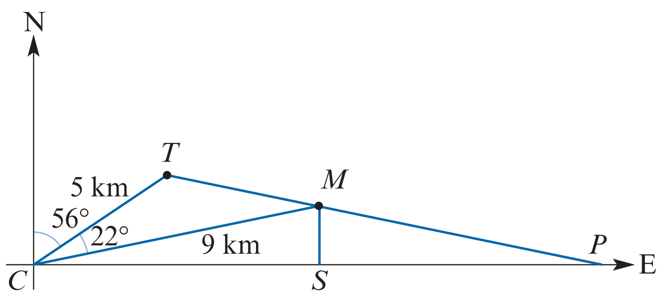
b By looking a further 22° east of the tower at T, Cora can see a radio mast at point M, which is 9 km away.

i Find the bearing of the mast at M from C.

ii Find the distance between the tower and the mast, correct to three decimal places.



c Cora now walks due east.



i How far must she walk until she is at point S due south of the mast? Give your answer correct to three decimal places.

ii What is the bearing of the mast at M from the tower at T?

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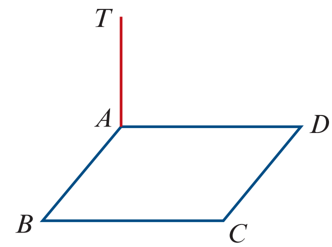
Revision

7 A vertical tower of height 10 m stands in one corner of a rectangular courtyard. From the top of the tower, T, the angles of depression to the nearest corners B and D are 32° and 19° respectively. Find:



CU

a AB, correct to two decimal places

b AD, correct to two decimal places

c the angle of depression from T to the corner C diagonally opposite the tower, correct to the nearest degree.

8 In a sequence of trials, the probability of a certain event occurring in the first trial is $\frac{1}{2}$. In subsequent trials, the probability is $\frac{1}{2}$ if the event did not occur in the previous trial, and 0 if it did.

CF

a Show that the probability of this event occurring:

i exactly twice in three trials is $\frac{1}{4}$ **ii** exactly twice in four trials is $\frac{1}{2}$.

b What is the probability of this event occurring exactly twice in five trials?

9 Katia and Mikki play a game in which a fair six-sided die is thrown five times:

CU

- Katia will receive \$1 from Mikki if there is an odd number of sixes
- Mikki will receive \$$x$ from Katia if there is an even number of sixes.

Find the value of x so that the game is fair. (Note that the number 0 is even.)

10 The random variable X has probability function given by

CF

$$\Pr(X = x) = \begin{cases} \dfrac{c}{x} & \text{if } x = 1, 2, \dots, 6 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant. Find the value of:

a c **b** $E(X)$ **c** $Var(X)$

11 The lifetime of a certain brand of light bulb is normally distributed with a mean of $\mu = 400$ hours and a standard deviation of $\sigma = 50$ hours.

CU

a Find the probability that a randomly chosen light bulb will last more than 375 hours.

b The light bulbs are sold in boxes of 10. Find the probability that at least nine of the bulbs in a randomly selected box will last more than 375 hours.

12 The weight of cereal in boxes, packed by a particular machine, is normally distributed with a mean of μ g and a standard deviation of $\sigma = 5$ g.

a A box is considered underweight if it weighs less than 500 g.

i Find the proportion of boxes that will be underweight if $\mu = 505$ g.

ii Find the value of μ required to ensure that only 1% of boxes are underweight.

b As a check on the setting of the machine, a random sample of five boxes is chosen and the setting changed if more than one of them is underweight. Find the probability that the setting of the machine is changed if $\mu = 505$ g.

ISBN 978-1-108-45164-2

13 A factory has two machines that produce widgets. The time taken, X seconds, to produce a widget using machine A is normally distributed, with a mean of 10 seconds and a standard deviation of 2 seconds. The time taken, Y seconds, to produce a widget using machine B has probability density function given by

$$f(y) = \begin{cases} k(y-8) & \text{if } 8 < y < 12 \\ 0 & \text{otherwise} \end{cases}$$

a **i** Find the value of k.

ii Show that machine A has a greater probability of producing a widget in less than 11 seconds than machine B.

iii Find which machine, on average, is quicker in producing widgets.

b Suppose that 60% of the widgets manufactured at the factory are produced by machine A, and 40% by machine B. If a widget selected at random is known to have been produced in less than 10 seconds, what is the probability that it was produced by machine A?

14 The queuing time, X minutes, at the box office of a movie theatre has probability density function:

$$f(x) = \begin{cases} kx(100-x^2) & \text{if } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

a Find:

i the value of k

ii the mean of X

iii the probability that a moviegoer will have to queue for more than 3 minutes

iv the probability that a moviegoer will have to queue for more than 3 minutes, given that she queues for less than 7 minutes.

b If 10 moviegoers go independently to the theatre, find the probability that at least five of them will be required to queue for more than 3 minutes.

15 Electronic sensors of a certain type fail when they become too hot. The temperature at which a randomly chosen sensor fails is T°C, where T is modelled as a normal random variable with mean μ and standard deviation σ.

a In a laboratory test, 98% of a random sample of sensors continued working at a temperature of 80°C, but only 4% continued working at 104°C.

i Show the given information on a sketch of the distribution of T.

ii Determine estimates for the values of μ and σ.

b More extensive tests confirmed that T is normally distributed, but with $\mu = 94.5$ and $\sigma = 5.7$. Use these values in the rest of the question.

i What proportion of sensors will operate in boiling water (i.e. at 100°C)?

ii The manufacturers wish to quote a safe operating temperature at which 99% of the sensors will work. What temperature should they quote?

ISBN 978-1-108-45164-2

16 A flight into an airport is declared to be 'on time' if it touches down within 3 minutes either side of the advertised arrival time; otherwise, it is declared early or late. On any one occasion, the probability that a flight is on time is 0.5 and the probability that it is late is 0.3. The time of arrival of a particular flight on any one day is independent of the time of arrival on any other day.

a Calculate the probability that:

i on any given day, the flight arrives early

ii on any given day, the flight does not arrive late

iii the flight arrives on time on three consecutive days

iv in any given week, the flight arrives late on Monday, but is on time for all the remaining four weekdays.

b In a given week of five days, find the probability that:

i the flight is late exactly once

ii the flight is early exactly twice.

c The airline is reported to the authority if the flight is late on more than two occasions in a five-day week. Find the probability that this happens.

17 An electronic game comes with five batteries. The game only needs four batteries to work. But because the batteries are sometimes faulty, the manufacturer includes five of them with the game. Suppose that X is the number of good batteries included with the game. The probability distribution of X is given in the following table.

x	0	1	2	3	4	5
$\Pr(X = x)$	0.01	0.02	0.03	0.04	0.45	0.45

a Use the information in the table to:

i find μ, the expected value of X

ii find σ, the standard deviation of X, correct to one decimal place

iii find, exactly, the proportion of the distribution that lies within two standard deviations of the mean

iv find the probability that a randomly selected game works, i.e. find $\Pr(X \geq 4)$.

b The electronic games are packed in boxes of 20. Whether or not an electronic game in a box will work is independent of any other game in the box working. Let Y be the number of working games in a box.

i Name the distribution of Y.

ii Find the expected number of working games in a box.

iii Find the standard deviation of the number of working games in a box.

iv Find the probability that a randomly chosen box will contain at least 19 working games.

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18 In a study of the prevalence of red hair in a certain country, researchers collected data from a random sample of 1800 adults.

a Of the 1000 females in the sample, they found that 10% had red hair. Calculate an approximate 95% confidence interval for the proportion with red hair in the female population.

b Of the 800 males in the sample, they found that 10% had red hair. Calculate an approximate 95% confidence interval for the proportion with red hair in the male population.

c Why is the width of the confidence interval for males different from the width of the confidence interval for females?

d How should the sample of 1800 adults be chosen to ensure that the widths of the two confidence intervals are the same when the sample proportions are the same?

e Assume that there are 1000 females and 800 males in the sample, and that the proportion of females in the sample with red hair is 10%. What sample proportion of red-headed males would result in the 95% confidence interval for the proportion with red hair in the female population and the 95% confidence interval for the proportion with red hair in the male population being of the same width?

19D Degree-of-difficulty classified questions

▶ Simple familiar questions

1 The bearing of A from B is 290° and the bearing of C from B is 040°. Given that $AB = BC$, determine:

a the magnitude of $\angle BCA$ **b** the bearing of A from C

2 In triangle ABC, $AB = 6$ cm, $AC = 16$ cm and $\angle BAC = 40°$. Find each of the following correct to three decimal places:

a the length of BC **b** the magnitude of $\angle ACB$

c the magnitude of $\angle ABC$ **d** the area of triangle ABC

3 Find the second derivative of each of the following with respect to x:

a $x\sin\left(\frac{x}{2}\right)$ **b** $x^2\ln(2x)$ **c** $e^{-2x}\cos(2x)$

4 Find the coordinates of the points of inflection on each of the following graphs:

a $y = x^3(x - 4)$ **b** $y = x^3\ln x$ **c** $y = 4x^2 + \sin(4x),\ 0 \le x \le \pi$

5 A Bernoulli sequence consists of eight independent trials, each with probability of success 0.3. Find the probability that there are exactly two successes in the eight trials.

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6 In a particular country, 55% of the babies born are boys.

a Write down a general rule for the probability distribution of the number of boys in a family of four children.

b Use this rule to calculate the probability that a family of four children consists of one boy and three girls.

7 A multiple-choice test has eight questions, each with five alternatives.

a What is the probability that a student who guesses:

i gets all of the questions correct

ii gets none of the questions correct

iii passes the test (that is, gets at least half of the questions correct)?

b How many questions would you expect a student who guesses to get correct?

8 Consider the function f given by

$$f(x) = \begin{cases} \dfrac{24}{x^3} & \text{if } 3 \le x \le 6 \\ 0 & \text{if } x < 3 \text{ or } x > 6 \end{cases}$$

Show that f is a probability density function.

9 A continuous random variable X has the probability density function

$$f(x) = \begin{cases} 12x^2(1 - x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathrm{E}(X)$.

10 The time, X minutes, between calls at a pizza restaurant is a random variable with probability density function

$$f(x) = \begin{cases} \dfrac{1}{5}e^{-\frac{x}{5}} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

Find the median time between calls.

11 The cumulative distribution function F of a continuous random variable X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{4} & 0 \le x < 1 \\ \dfrac{1}{20}(x^4 + 4) & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

Find a probability density function for X.

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12 The probability density function f of a normal random variable X is given by

$$f(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-10}{3}\right)^2} \quad \text{for } x \in \mathbb{R}$$

a Write down the mean and standard deviation of X.

b Sketch the graph of f.

c On the graph, shade the region that represents $\Pr(X > 3)$.

13 The lengths of similar components produced at a factory are approximated by a normal distribution model with a mean of 10 cm and a standard deviation of 0.1 cm. A component is chosen at random.

a What is the probability that its length is more than 10.2 cm?

b What is the probability that its length is between 9.95 cm and 10.05 cm?

14 The annual salaries of employees in a large company are approximately normally distributed, with a mean of \$80 000 and a standard deviation of \$15 000. Find the proportion of employees in this company who earn:

a less than \$50 000

b more than \$90 000

c between \$60 000 and \$100 000.

15 This table summarises a student's results in four different subjects.

For each subject, the table gives the student's mark and also gives the mean (μ) and standard deviation (σ) for the class.

Subject	Mark	μ	σ
Mathematics	68	60.5	8.5
English	70	65.2	7.2
Chemistry	55	50.9	3.3
Psychology	81	71.9	9.9

a Determine the student's standardised mark for each subject.

b List the four subjects in order from best result to worst result, based on this student's performance relative to the class.

16 A market researcher monitored a random sample of 500 people who entered a shop on a certain day and noted that 375 people made a purchase.

a Find a 95% confidence interval for the proportion of people entering the shop on the following day who will make a purchase.

b Find a 99% confidence interval for the proportion of people entering the shop on the following day who will make a purchase.

17 A newspaper wishes to carry out a survey to investigate how people will vote in an upcoming referendum. An estimated value of the proportion of 'yes' voters is 0.3. Determine the sample size that should be used to achieve a margin of error of 1% in:

a an approximate 90% confidence interval for the proportion of 'yes' voters

b an approximate 95% confidence interval for the proportion of 'yes' voters.

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Revision

Complex familiar questions

1 For each of the following, find the largest open intervals for which:

i f is increasing
ii f is decreasing
iii f is concave up
iv f is concave down.

a $f(x) = x^3 - 12x + 24$
b $f(x) = x^3 - 12x^2 + 24$
c $f(x) = \cos x, \quad 0 < x < 2\pi$
d $f(x) = \sin^2 x, \quad 0 < x < \pi$
e $f(x) = 3x^4 - 4x^3 + 6$
f $f(x) = 2x^2 \ln(2x), \quad x > 0$

2 Let $f(x) = (x - a)^3(x - b)$, where a and b are real constants.

a Find $f'(x)$ and $f''(x)$.
b Find the coordinates of the points of inflection on the graph of f.
c Find the equation of the tangent to the graph of f at the point where $x = \dfrac{a+b}{2}$, and find the x-axis intercept of this tangent.

3 For the graph of $y = x^4 - 6x^2 + 8x + 24$, find the coordinates of:

a the local minimum
b the stationary point of inflection
c the other point of inflection.

4 Let $f(x) = \ln(x^2 + 4)$.

a Find $f'(x)$ and $f''(x)$.
b Find the coordinates of the stationary point on the graph of f.
c Find the coordinates of the points of inflection on the graph of f.
d Sketch the graph of f.

5 In $\triangle ABC$, $\angle BAC = 60°$, $AB = 8$ and $BC = 7$. Given that $AC < 4$, find AC.

6 A yacht starts from a marina and sails in a straight line on a bearing of 245° for 5 km to a nearby island.

a How far south has the yacht sailed?
b How far west has the yacht sailed?
c What is the bearing of the marina from the island?
d The yacht now sails from the island on a bearing of 310° and stops at a headland. The marina is on a bearing of 115° from the headland. Find the distance to the headland from the island.

7 A rectangular board has size 10 cm by 6 cm. One side of the board is propped up, so that the 10 cm sides are horizontal and the 6 cm sides are inclined at 30° to the horizontal. Find the angle of inclination to the horizontal of a diagonal of the board.

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8 A parallelogram has side lengths 4 cm and 5 cm, and an acute angle of $47.8°$. Find the lengths of the two diagonals of the parallelogram.

9 In a particular city, the probability of rain on any day in June is 0.2, independent of any other day.

a Write down a general rule for the probability distribution of the number of days on which it rains during one week in June.

b Use this rule to calculate the probability that during one week in June:

i it rains on three days

ii it rains on two or three days

iii it rains on three days, given that it rains on at least two days.

10 Marcel knows that his probability of hitting the bullseye when throwing a dart is 0.4.

a What is the probability that he hits the bullseye exactly once in five throws?

b What is the minimum number of darts he should throw to ensure that the probability that he hits the bullseye at least once is more than 0.99?

11 The time (in minutes) that it takes a worker to assemble a component is a random variable X with probability density function f given by

$$f(x) = \begin{cases} \dfrac{6}{x^2} & x \geq 6 \\ 0 & x < 6 \end{cases}$$

a Find the probability that it takes a worker less than 10 minutes to assemble a component.

b Find the probability that it takes less than 20 minutes to assemble a component, given that it takes more than 10 minutes.

12 Suppose that the distribution of scores on an examination can be modelled by a random variable X with probability density function:

$$f(x) = \begin{cases} \dfrac{\pi}{200} \sin\left(\dfrac{\pi x}{100}\right) & \text{if } 0 \leq x \leq 100 \\ 0 & \text{otherwise} \end{cases}$$

The top 20% of students who sit the examination will receive an 'A'.

a Find (correct to one decimal place) the minimum score required to receive an 'A'.

b Find the expected number of students who receive an 'A' in a class of 25 students.

13 Suppose that, in a 95% confidence interval for the population proportion p, the margin of error is equal to 0.216. What is the margin of error for the 99% confidence interval for p that is determined using the same information?

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▶ Complex unfamiliar questions

1 $ABCDPQRS$ is a rectangular prism on a horizontal base $ABCD$. The vertical edges AP, BQ, CR and DS each have length 6 cm. The edge AB has length 8 cm, and the edge BC has length 9 cm. Find the angle between each of the following pairs of lines:

a AR and BS **b** AR and CR **c** AS and DS

2 An aeroplane is flying in a horizontal circle of radius 1 km. A person is standing on the ground, within the projection of this circle onto the ground. He measures the angle of elevation to the aeroplane as it flies around him, and observes that the greatest angle is 25° and the least angle is 15°. Find the height of the aeroplane above the ground.

3 A function h has a rule of the form $h(x) = (ax^2 + b)e^{cx}$. Find the values of the constants a, b and c, given that the function has the following three properties:

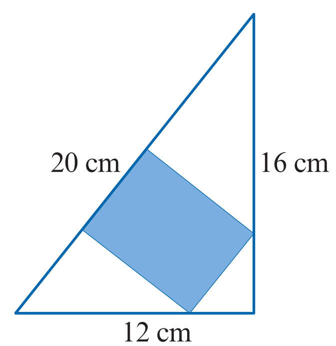
- $h(0) = -4$
- $h'(0) = 8$
- the graph of h has a local minimum at $x = -1$.

4 Tower A is 4 km due west of tower B. There are flagpoles at points C and D.

- The bearing of C from A is 025°, and the bearing of C from B is 350°.
- The bearing of D from A is 035°, and the bearing of D from B is 345°.

Find the distance between the two flagpoles.

5 A right-angled triangle has sides 12 cm, 16 cm and 20 cm as shown. A rectangle is inscribed in the triangle with one side along the hypotenuse and a vertex on each of the other two sides of the triangle. What are the dimensions of the largest such rectangle?



6 A chocolatier randomly chooses chocolates to fill boxes of 12. The chocolates are either soft-centred or hard-centred. Let p be the proportion of hard-centred chocolates. Let the random variable X represent the number of hard-centred chocolates in a box of 12.

a If $p = 0.25$, find:

i $\Pr(X < 5)$ **ii** $\Pr(X \geq 7)$

b If $\Pr(X = 0) = 0.05$, find the value of p correct to three decimal places.

c If $\text{Var}(X) = 1.92$, find the possible values of p.

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7 When a telemarketer makes a phone call, the probability that someone will answer the phone is 0.1.

a Find the probability that none of her next 10 calls are answered.

b Find the probability that more than two of her next 20 calls are answered.

c The telemarketer is required to speak to at least four people per day on average. Find the least number of calls per day that she should make to meet this requirement.

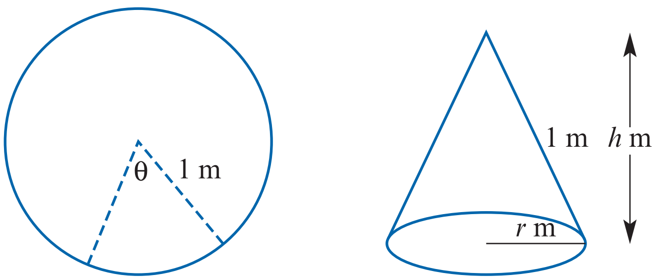
d Find the least number of calls that she needs to make for the probability that at least one person answers the phone to be greater than 0.95.

8 A random variable X has probability density function

$$f(x) = \begin{cases} Ax^2 & \text{if } 0 \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

Given that $\mathrm{E}(X) = 0.75$, find the values of A and B.

9 A cone is made by cutting out a sector with central angle θ from a circular piece of cardboard of radius 1 m and joining the two cut edges to form a cone of slant height 1 m as shown in the following diagrams.



The volume of a cone is given by the formula $V = \frac{1}{3}\pi r^2 h$.

a **i** Find r in terms of θ.

ii Find h in terms of θ.

iii Show that $V = \frac{1}{3}\pi\left(\frac{2\pi - \theta}{2\pi}\right)^2\sqrt{1 - \left(\frac{2\pi - \theta}{2\pi}\right)^2}$.

b Find the value of V when $\theta = \frac{\pi}{4}$.

c Find the value(s) of θ for which the volume of the cone is 0.3 m^3.

d **i** Use a calculator to determine the value of θ that maximises the volume of the cone.

ii Find the maximum volume.

e Determine the maximum volume using calculus.

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20 Revision of Units 3 & 4

20A Technology-free questions

SF

1 If $\sin x = 0.3$ and $\cos\theta = 0.6$, find the value of each of the following:

a $\sin(-x)$ **b** $\sin(\pi - x)$ **c** $\cos(\pi + \theta)$

d $\cos(2\pi - \theta)$ **e** $\sin\left(\frac{\pi}{2} - \theta\right)$ **f** $\cos\left(\frac{\pi}{2} + x\right)$

2 For each of the following derivative functions, find an expression for y in terms of x using the given information:

a $\frac{dy}{dx} = \sin x + 1$ given that $y = 2$ when $x = 0$

b $\frac{dy}{dx} = 2e^{\frac{x}{3}} - e$ given that $y = 2e$ when $x = 3$

c $\frac{dy}{dx} = \frac{1}{2x - 3}$ given that $y = 4$ when $x = 2$

3 Let $f(x) = x^2 + 6$ and $g(x) = 3x + 1$. Find $f(g(x))$.

4 **a** Let $f(x) = (5x^3 - 3x)^7$. Find $f'(x)$. **b** Let $f(x) = 2xe^{4x}$. Evaluate $f'(0)$.

5 **a** For $f(x) = \frac{\sin x}{2x + 1}$, find $f'\left(\frac{\pi}{2}\right)$. **b** For $f(x) = 3x\sin(2x)$, find $f'\left(\frac{\pi}{3}\right)$.

6 **a** Find the second derivative of $x^2\ln(2x)$ with respect to x.

b Let $f(x) = e^{\sin(2x)}$. Find $f''(x)$.

7 Let $f(x) = 4\sin\left(2\left(x + \frac{\pi}{6}\right)\right)$ for $-\pi \le x \le \pi$.

a Write down the amplitude and period of the function f.

b Sketch the graph of the function f. Label the axis intercepts and the endpoints with their coordinates.

ISBN 978-1-108-45164-2

8 Sketch the graph of the function $f(x) = 1 - \dfrac{4}{x-2}$ for $x \in [-1, \infty) \setminus \{2\}$.
Label all axis intercepts, and label each asymptote with its equation.

SF

9 Consider the graph of the function f with rule $f(x) = x^3 - \dfrac{1}{2}x^4$.

CF

a Find the x-axis intercepts.
b Find the coordinates of any stationary points.
c Find the coordinates of any points of inflection.
d Use the second derivative to establish the nature of each stationary point.
e Sketch the graph of f, showing all features.

10 A random variable X has a probability density function f with rule:

$$f(x) = \begin{cases} 1 - kx^2 & \text{if } 0 \le x \le 2 \\ 0 & \text{if } x < 0 \text{ or } x > 2 \end{cases}$$

a Find the value of k. **b** Find $\Pr(0 \le X \le 1)$. **c** Find the mean of X.

11 A function f has rule $f(x) = \dfrac{2x-1}{x+2}$.

a If the rule of f is written as $f(x) = a + \dfrac{b}{x+2}$, find the values of a and b.
b Find the area between the graph of $y = f(x)$ and the x-axis for $0 \le x \le 2$.
c Let $g(x) = \dfrac{1}{x+2}$. Solve the equation $f(x) = g(x)$ for x.
d Find the area between the graphs of $y = f(x)$ and $y = g(x)$ for $0 \le x \le 1$.

12 For each of the following, solve for x in terms of y:

SF

a $y = 5e^{x-1} - 3$ **b** $y = \ln(\sqrt{3} - x)$

13 Solve the equation $\cos\left(\dfrac{5x}{2}\right) = \dfrac{1}{2}$ for $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

14 Find the average value of $y = e^x$ over the interval $[0, 4]$.

15 The graph of $y = ax^3 + bx + c$ has intercepts $(0, 6)$ and $(-2, 0)$ and has a turning point where $x = -1$.

CF

a Find the value of c.
b Write down two simultaneous equations in a and b from the given information.
c Hence find the values of a and b.

16 Triangle ABC has $AB = AC = 10$ cm and $\angle BAC = 150°$. Find:

SF

a the length BC **b** the area of triangle ABC

17 A hiker starts at point A. He walks for 10 km on a bearing of 330° and then walks for 10 km on a bearing of 060° to arrive at point B. How far is B from A, and what is the bearing of B from A?

CF

ISBN 978-1-108-45164-2

Revision

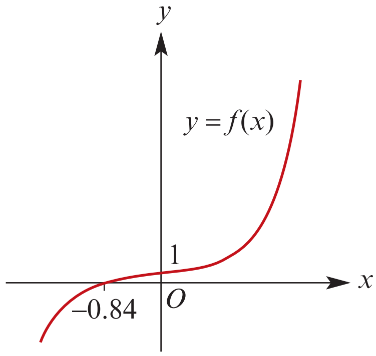
18 The graph of the piecewise-defined function

$$f(x) = \begin{cases} -2x^4 + 1 & \text{if } x \le 0 \\ 2x^4 + 1 & \text{otherwise} \end{cases}$$

is shown.

a Draw the graph of the derivative function f'.

b Write down a rule for the derivative function.



CF

19 Find the coordinates of the points of inflection on the graph of $f(x) = x^2(4 - x^2)$.

SF

20 Find an anti-derivative of $\frac{1}{1-3x}$ with respect to x, for $x < \frac{1}{3}$.

21 Let X be a normally distributed random variable with a mean of 84 and a standard deviation of 6. Let Z be the standard normal random variable.

CF

a Find the probability that X takes a value greater than 84.

b Use the result that $\Pr(Z < 1) = 0.84$ to find the probability that $78 < X < 90$.

c Find the probability that $X < 78$ given that $X < 84$.

22 The probability density function of a random variable X is given by

$$f(x) = \begin{cases} \frac{x}{24} & \text{if } 1 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$$

a Find $\Pr(X < 3)$.

b If $b \in [1, 7]$ and $\Pr(X \ge b) = \frac{3}{8}$, find b.

23 A tangent to the graph of $y = x^{\frac{1}{3}}$ has equation $y = \frac{1}{3}x + a$. Find the value(s) of a.

SF

24 Triangle ABC has $AB = 10$ cm, $BC = 8$ cm and $\angle BAC = 30°$.

CU

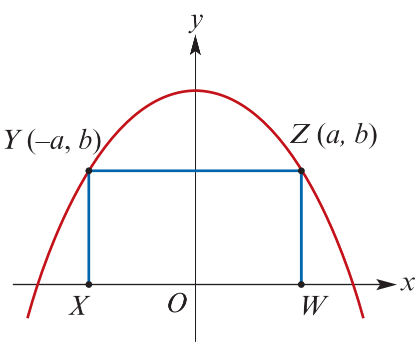
a Find the sine of $\angle ACB$.

b Let $x = AC$. Use the cosine rule to write a quadratic equation in x. Explain why there are two solutions to this equation.

25 A rectangle $XYZW$ has two vertices on the x-axis and the other two vertices on the graph of $y = 16 - 4x^2$, as shown in the diagram.

a Find the area, A, of rectangle $XYZW$ in terms of a.

b Find the maximum value of A and the value of a for which this occurs.



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SF

26 Let $f(x) = -3x^2 + 2bx + 9$ with $\int_{-1}^{3} f(x)\,dx = 32$. Find the value of b.

27 A player in a game of chance can win \$0, \$1, \$2 or \$3. The amount won, \$$X$, is a random variable with probability distribution given by:

x	0	1	2	3
$\Pr(X = x)$	0.6	0.2	0.15	0.05

a Find the mean of X.

b What is the probability that a player wins the same amount from two games?

CF

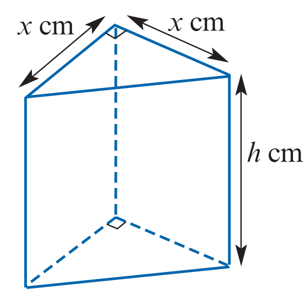
28 A biased coin is tossed four times. On each toss, the probability of a head is p.

a Find, in terms of p, the probability that from the four tosses there are:

i exactly two heads **ii** exactly three heads.

b Given that $0 < p < 1$ and that the probability of two heads is equal to the probability of three heads, find the value of p.

29 A brick is made in the shape of a right triangular prism. The triangular end is a right-angled isosceles triangle, with the equal sides of length x cm. The height of the brick is h cm. The volume of the brick is 2000 cm^3.



a Find an expression for h in terms of x.

b Show that the total surface area, $A\text{ cm}^2$, of the brick is given by

$$A = \frac{4000\sqrt{2} + 8000}{x} + x^2$$

c Find the value of x^3 if the brick has minimum surface area.

30 In order to measure the effect of alcohol on reaction time, an investigator selects a random sample of subjects from a group of diners in a restaurant.

a Do you think this sample will be representative of the general population? Explain your answer.

b How would you suggest that the sample could be chosen?

SF

31 A coin is tossed 100 times, and 53 heads observed.

a Give a point estimate for p, the probability of a head when the coin is tossed.

b Write down an expression for a 95% confidence interval for p.

CF

32 A sample of n people were asked whether they thought that Australians had access to adequate hospital care, and 37% said no.

a What is the value of the sample proportion, $\hat{p}$?

b Write down an expression for E, the margin of error for this estimate at the 95% confidence level, in terms of n.

c If the number of people in the sample were halved, what would be the effect on E?

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Revision

CF

33 Suppose that 20% of camels from a large population have two humps instead of one. A random sample of three camels is selected for inspection.

a What are the possible values of $\hat{p}$, the proportion of camels in the sample with two humps?

b Construct a probability distribution table for the sample proportion $\hat{P}$.

c Use this table to determine the probability that the proportion of camels in the sample with two humps is at least 0.3.

20B Multiple-choice questions

1 Define the function f by

$$f(x) = \begin{cases} 5x + 1 & \text{if } x \geq -\dfrac{4}{5} \\ -5x - 7 & \text{if } x < -\dfrac{4}{5} \end{cases}$$

Which of the following statements is *not true* about this function?

A The graph of f is continuous everywhere.
B The graph of f' is continuous everywhere.
C $f(x) \geq -3$ for all values of x.
D $f'(x) = 5$ for all $x > 0$.
E $f'(x) = -5$ for all $x < -2$.

2 Let $k = \int_2^6 \frac{2}{x}\,dx$. Then e^k is equal to

A $2\ln 3$ **B** 1 **C** $\frac{1}{9}$ **D** 9 **E** $\frac{1}{3}$

3 The average value of the function with rule $f(x) = \ln(x + 2)$ over the interval $[-1, 3]$ is

A $\frac{-1}{5}$ **B** $\ln 6$ **C** $\frac{\ln 5}{4}$ **D** $\frac{5\ln 5 - 4}{4}$ **E** $\frac{5\ln 5 - 3\ln 3 - 4}{4}$

4 The average value of the function $y = \sin(2x)$ over the interval $\left[0, \frac{\pi}{2}\right]$ is

A $\frac{2}{\pi}$ **B** $\frac{\pi}{2}$ **C** 0.5 **D** 0 **E** π

5 If $f(x) = e^{3x}$, for all real x, and $[f(x)]^3 = f(y)$, then y is equal to

A e^{9x} **B** $9x$ **C** $3x$ **D** $9x^3$ **E** $(3x)^3$

6 The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \sin(2x) & \text{if } 0 \leq x \leq \dfrac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

The value of a such that $\Pr(X > a) = 0.25$ is closest to

A 0.25 **B** 0.75 **C** 1.04 **D** 1.05 **E** 1.09

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7 The function f is a probability density function, with rule

$$f(x) = \begin{cases} 1 + 2e^{\frac{x}{k}} & \text{if } 0 \le x \le 2k \\ 0 & \text{otherwise} \end{cases}$$

Hence k is equal to

A $\frac{1}{2}e^{-2}$ **B** $1 + e^2$ **C** e^{-2} **D** $1 - e^{-2}$ **E** 1

8 The random variable X has a normal distribution with a mean of 8 and a standard deviation of 0.25. If Z has the standard normal distribution, then the probability that X is less than 7.5 is equal to

A $\Pr(Z > 2)$ **B** $\Pr(Z < -1.5)$ **C** $\Pr(Z < 1)$ **D** $\Pr(Z \ge 1.5)$ **E** $\Pr(Z < -4)$

9 The graph of $y = 2kx - 2$ intersects the graph of $y = x^2 + 12x$ at two points for

A $k = 12$ **B** $k > 6 + \sqrt{2}$ or $k < 6 - \sqrt{2}$
C $4 < k < 7$ **D** $5 < k < 7$ **E** $6 - \sqrt{2} < k < 6 + \sqrt{2}$

10 The values of x that satisfy the equation $e^{4x} - 7e^{2x} + 12 = 0$ are

A $3,\ 4$ **B** $-4,\ -3$ **C** $-2,\ -\sqrt{3},\ \sqrt{3},\ 2$
D $\ln\sqrt{3},\ \ln 2$ **E** $-\ln\sqrt{3},\ \ln\sqrt{3},\ \ln 2$

11 Assume that $f'(x) = g'(x)$ with $f(1) = 2$ and $g(x) = -xf(x)$. Then $f(x) =$

A $g(x) + 4x + 4$ **B** $g'(x) + 4$ **C** $g(x) + 4x$
D $\dfrac{4 - 4x}{x + 1}$ **E** $g(x) + 4$

12 The number of points of inflection on the graph of $y = x^4 - 4x^2$ is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

13 The graph of $y = 7x^{\frac{1}{2}}$ is reflected in the x-axis and then translated 3 units to the right and 4 units down. The equation of the new graph is

A $y = 7(x - 3)^{\frac{1}{2}} + 4$ **B** $y = -7(x - 3)^{\frac{1}{2}} - 4$ **C** $y = -7(x + 3)^{\frac{1}{2}} - 1$
D $y = -7(x - 4)^{\frac{1}{2}} + 3$ **E** $y = 7(x - 4)^{\frac{1}{2}} + 3$

14 If a random variable X has probability density function given by

$$f(x) = \begin{cases} \frac{1}{8}x & \text{if } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

then $\mathrm{E}(X)$ is equal to

A $\frac{1}{2}$ **B** 1 **C** $\frac{8}{3}$ **D** $\frac{16}{3}$ **E** 2

15 A yacht sails for 7 km on a bearing of 107°. Correct to one decimal place, how far south of its starting point does the yacht finish?

A 2.0 km **B** 2.1 km **C** 2.3 km **D** 6.7 km **E** 23.9 km

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16 The random variable X has the following probability distribution.

x	0	1	2
$\Pr(X = x)$	a	b	0.6

If the mean of X is 1.6, then

A $a = 0.3$ and $b = 0.7$ **B** $a = 0.2$ and $b = 0.2$ **C** $a = 0.4$ and $b = 0.4$
D $a = 0.1$ and $b = 0.3$ **E** $a = 0$ and $b = 0.4$

17 A hockey player attempts to score a goal in a practice session. The probability of each shot scoring is $\frac{1}{8}$, independently of the outcome of any other shot. The probability that the player hits four goals out of six shots is

A $\left(\frac{1}{8}\right)^4\left(\frac{7}{8}\right)^2$ **B** $15\left(\frac{7}{8}\right)^4\left(\frac{1}{8}\right)^2$ **C** $20\left(\frac{7}{8}\right)^4\left(\frac{1}{8}\right)^2$ **D** $\left(\frac{7}{8}\right)^4\left(\frac{1}{8}\right)^2$ **E** $15\left(\frac{1}{8}\right)^4\left(\frac{7}{8}\right)^2$

18 A continuous random variable X has cumulative distribution function F given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2(3 - 2x) & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Which of the following is a possible rule for a probability density function f of X?

A $f(x) = \begin{cases} \frac{1}{2}x^3(2 - x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

B $f(x) = \begin{cases} -4x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

C $f(x) = \begin{cases} x(3 - 2x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

D $f(x) = \begin{cases} 6x(1 - x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

E $f(x) = \begin{cases} \frac{1}{3}x^4(3 - x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

19 Triangle ABC is isosceles with $AB = BC = 12$ cm and $\angle ABC = 50°$. The area of this triangle is closest to

A 62 cm^2 **B** 55 cm^2 **C** 65 cm^2 **D** 61 cm^2 **E** 46 cm^2

20 A mug of hot coffee is left on a bench. The rate of change of the temperature of the coffee, T°C, with respect to time, t minutes, is given by

$$\frac{dT}{dt} = -9e^{-\frac{1}{8}t} \quad \text{for } t \ge 0$$

Correct to the nearest degree, the change in temperature of the coffee over the first 15 minutes is

A −52°C **B** −57°C **C** −60°C **D** −61°C **E** −64°C

ISBN 978-1-108-45164-2

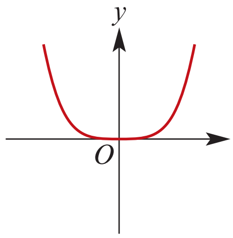
Revision

21 A triangle has sides of length 2 m, 5 m and 6 m. The magnitude of the smallest angle of this triangle is approximately

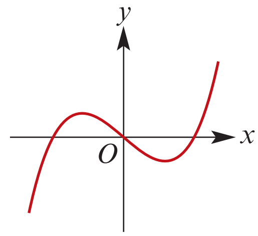
A 18° **B** 23° **C** 24° **D** 49° **E** 72°

22 A polynomial function f is such that $f''(x) = 0$ for $x = 0$ only. Which of the following *could not* be the graph of f?

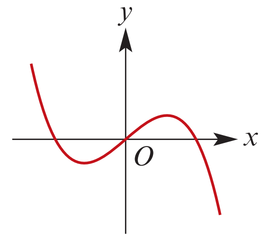
A



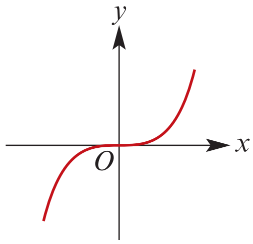
B



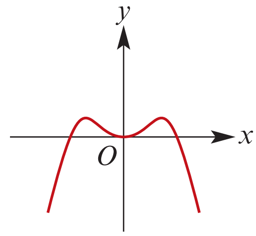
C



D



E



Questions 23 and 24 are based on the following information:

An exit poll of 1000 randomly selected voters found that 520 favoured candidate A.

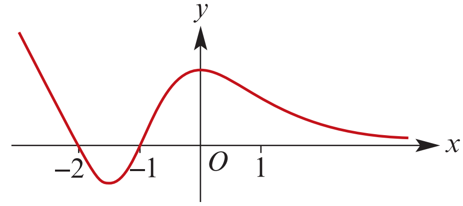
23 An approximate 95% confidence interval for the proportion of voters in favour of candidate A is

A $(0.484, 0.546)$ **B** $(0.422, 0.618)$ **C** $(0.494, 0.546)$
D $(0.489, 0.551)$ **E** $(0.479, 0.561)$

24 On the basis of this confidence interval, what would be your prediction for the result of the election?

A predict a win for candidate A
B predict a loss for candidate A
C too close to make any prediction
D cannot tell as we do not know the number of candidates
E none of the above

25 The graph of a function is shown on the right. On which of the following intervals is the graph concave up?



A $(1, \infty)$ **B** $(-\infty, 0)$
C $(-1, \infty)$ **D** $(-1, 1)$
E $(0, \infty)$

ISBN 978-1-108-45164-2

20C Extended-response questions

CU

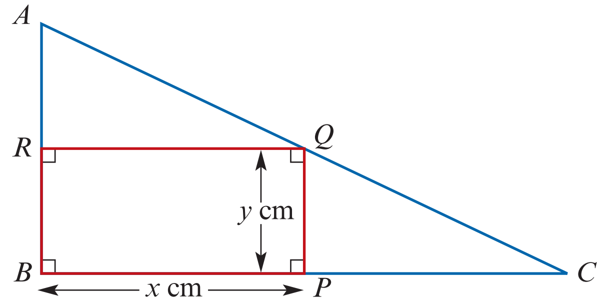
1 **a** **i** Find the coordinates of the stationary point for the curve with equation

$$y = \frac{16x^3 + 4x^2 + 1}{2x^2}$$

ii Determine the nature of this stationary point.

b The right-angled triangle ABC shown in the diagram has side lengths $AB = 5$ cm and $AC = 13$ cm.

The rectangle $BPQR$ is such that its vertices P, Q and R lie on the line segments BC, CA and AB respectively.



i Given that $BP = x$ cm and $PQ = y$ cm, show that $y = \dfrac{60 - 5x}{12}$.

ii Find the area of the rectangle, A cm^2, in terms of x.

iii Find the maximum value of this area as x varies.

2 A theoretical model of the relationship between two variables, x and y, predicts the values given in the table.

x	0	1	3
y	6	0	0

a An equation of the form $y = k(x - p)(x - q)$ is suggested, where p, q and k are constants and $p < q$.
Use the information in the table to find p, q and k.

b A series of experiments is carried out to test this model. The values of y when $x = 0, 1, 3$ are found to be as predicted. But when $x = 2$, the value of y is found to be 2. After further discussion, a new model is proposed with an equation of the form

$$y = m(x - p)^2(x - q)$$

where p and q have the values already calculated and m is a constant.

i Find the value of m.

ii Obtain the equation of this new model in the form $y = ax^3 + bx^2 + cx + d$.

iii Sketch the graph of y against x. State the coordinates of the stationary points and the nature of each of these points.

3 The bearing of buoy A from a point X at the foot of a cliff is 133°, and the bearing of buoy B from X is 219°. The cliff is 70 m high. Point Y is at the top of the cliff vertically above X. The angle of depression of A from Y is 9.5°, and the angle of depression of B from Y is 7.4°.

a Find $\angle AXB$.

b Find AX and BX.

c Find AB, the distance between the two buoys.

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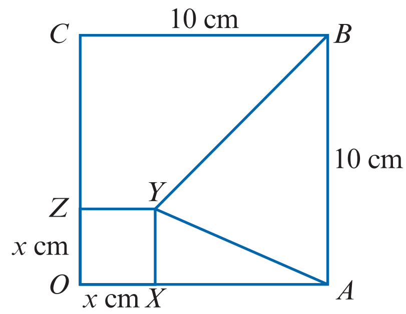
4 A square piece of card $OABC$, of side length 10 cm, is cut into four pieces by removing a square $OXYZ$ of side length x cm as shown, and then cutting out the triangle ABY.



a **i** Find A cm^2, the sum of the areas of $OXYZ$ and ABY, in terms of x.

ii Find the domain of the function which determines this area.

iii Sketch the graph of the function, with domain determined in **ii**.

iv State the minimum value of this area.

b **i** Find the rule for the function of x which represents the area of triangle AXY.

ii Sketch the graph of this function for a suitable domain.

c Find the ratio of the areas of the four pieces when the area of triangle AXY is a maximum.

5 A curve C has equation $y = ax - x^2$, where a is a positive constant.

a Sketch C, showing clearly the coordinates of the axis intercepts.

b Calculate the area of the finite region bounded by C and the x-axis, giving your answer in terms of a.

c The lines $x = \frac{1}{3}a$ and $x = \frac{2}{3}a$ intersect C at the points A and B respectively.

i Find, in terms of a, the y-coordinates of A and B.

ii Calculate the area of the finite region bounded by C and the straight line AB, giving your answer in terms of a.

6 The number of people unemployed in a particular population can be modelled by the function

$$f(t) = 1000(t^2 - 10t + 44)e^{\frac{-t}{10}}$$

where t is the number of months after January 2017 and $0 \le t \le 35$.

a Use this function to find an expression for:

i the rate of increase of the number unemployed

ii the rate of increase of this rate of increase.

b Find the values of t for which:

i the number unemployed was increasing

ii the rate of increase of the number unemployed was going down

iii the number unemployed was increasing and the rate of increase of the number unemployed was going down.

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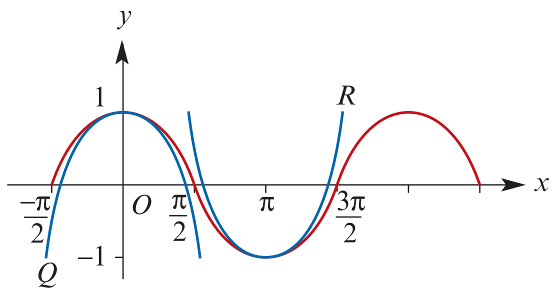
7 The diagram shows part of the graph of $y = \cos x$ and the graphs of two quadratic functions, denoted by Q and R, which approximate to the cosine function around $x = 0$ and $x = \pi$ respectively.

The equation of Q is $y = 1 - \frac{1}{2}x^2$.



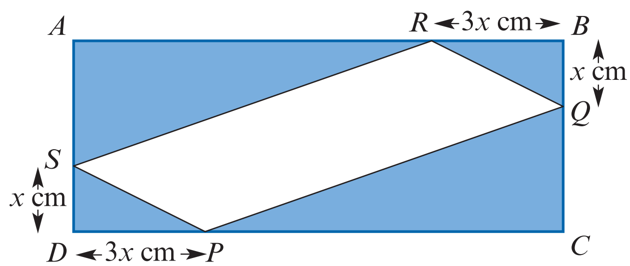
a **i** Find an estimate of $\cos 0.1$ by using the approximation $y = 1 - \frac{1}{2}x^2$.

ii Find an approximation for the solution to the equation $\cos x = 0.98$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, by solving the quadratic equation $1 - \frac{1}{2}x^2 = 0.98$.

b **i** The graph Q can be transformed into R by a reflection in the x-axis, followed by a translation. Use this fact to find an equation for the graph R.

ii Estimate the value of $\cos 3$ using this approximation.

8 In the figure, $ABCD$ is a rectangle with $AB = 30$ cm and $AD = 10$ cm. The shaded portions are cut away, leaving the parallelogram $PQRS$, where $BQ = SD = x$ cm and $RB = DP = 3x$ cm.



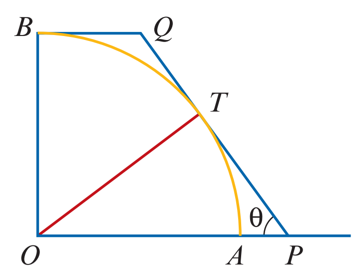
a Find the area, S cm^2, of the parallelogram in terms of x.

b Find the allowable values of x.

c Find the value of x for which S is a maximum.

d Sketch the graph of S against x for a suitable domain.

9 In the figure, OAB is a quadrant of a circle of radius 1 unit. The line segment OA is extended to a point P. From P, a tangent to the quadrant is drawn, touching it at T and meeting another tangent, BQ, at Q. Let $\angle OPQ = \theta$.



a **i** Find the length OP as a function of θ.

ii Find the length BQ as a function of θ.

b Show that the area, S, of trapezium $OPQB$ is given by $\dfrac{2 - \cos\theta}{2\sin\theta}$.

c Show that $\dfrac{dS}{d\theta} = \dfrac{2 - 4\cos\theta}{4\sin^2\theta}$.

d Find the minimum value of S and the distance AP when S is a minimum.

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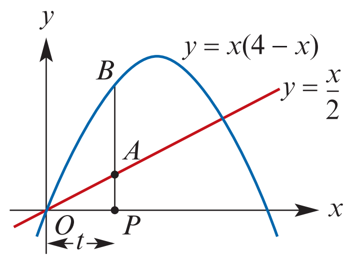
10 The point P has coordinates $(t, 0)$, where $0 < t < \frac{7}{2}$.
The line PAB is parallel to the y-axis.



a Let Z be the length of AB. Find Z in terms of t.

b Sketch the graph of Z against t.

c State the maximum value of Z and the value of t for which it occurs.

11 A study is being conducted of the numbers of male and female children in families in a certain population.

a A simple model is that each child in any family is equally likely to be male or female, and that the sex of each child is independent of the sex of any previous children in the family. Using this model, calculate the probability that in a randomly chosen family of four children:

i there will be two males and two females

ii there will be exactly one female, given that there is at least one female.

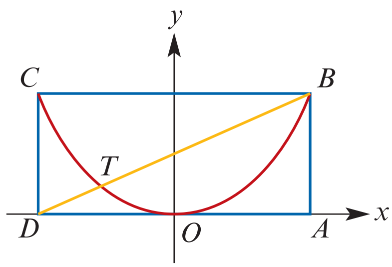
b An alternative model is that the first child in any family is equally likely to be male or female, but that, for any subsequent children, the probability that they will be of the same sex as the previous child is $\frac{3}{5}$. Using this model, calculate the probability that in a randomly chosen family of four children:

i all four will be of the same sex

ii no two consecutive children will be of the same sex

iii there will be two males and two females.

12 In the figure, $ABCD$ is a rectangle with $OA = OD = a$ and $AB = b$.
The equation of the parabola BOC is $y = kx^2$.



a Express k in terms of a and b.

b If BD cuts the parabola at T, find:

i the equation of the straight line BD

ii the coordinates of T.

c Show that the area bounded by the parabola and the line BC is $\frac{4}{3}ab$ square units.

d Let S_1 be the area of the region bounded by the line segment BT and the curve BOT. Let S_2 be the area of the region bounded by the curve CT and the line segments BC and BT. Find the ratio $S_1 : S_2$.

ISBN 978-1-108-45164-2

Revision

CU

13 A certain type of brass washer is manufactured as follows. A length of brass rod is cut cross-sectionally into pieces of mean thickness 0.25 cm, with a standard deviation of 0.002 cm. These brass slices are then put through a machine that punches out a circular hole of mean diameter 0.5 cm through the middle of the slice, with a standard deviation of 0.05 cm. The thickness of the washers and the diameters of the holes are known to be normally distributed, and do not depend on each other.

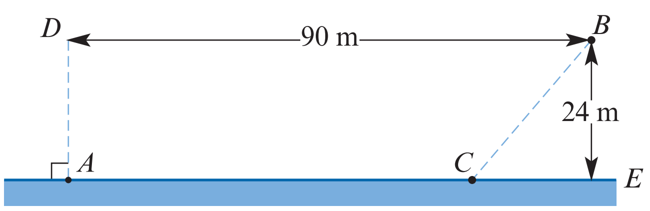
a Find the probability that a randomly selected washer will:

- **i** have a thickness of less than 0.253 cm
- **ii** have a thickness of less than 0.247 cm
- **iii** have a hole punched with a diameter greater than 0.56 cm
- **iv** have a hole punched with a diameter less than 0.44 cm.

b The brass washers are acceptable only if they are between 0.247 cm and 0.253 cm in thickness with a hole of diameter between 0.44 cm and 0.56 cm. Find:

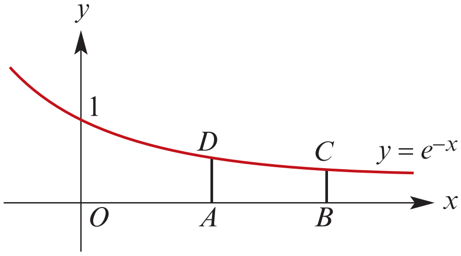
- **i** the percentage of washers that are rejected
- **ii** the expected number of washers of acceptable thickness in a batch of 1000 washers
- **iii** the expected number of washers of acceptable thickness that will be rejected in a batch of 1000 washers.

14 A ditch is to be dug to connect the points A and B in the figure. The earth on the same side of AE as B is hard, and the earth on the other side is soft.



The cost of digging hard earth is \$200 per metre and soft earth is \$100 per metre. Find the position of point C, where the turn is made, that will minimise the cost.

15 The diagram shows the graph of $y = e^{-x}$. The points A and B have coordinates $(n, 0)$ and $(n + 1, 0)$ respectively, and the points C and D on the curve are such that AD and BC are parallel to the y-axis.



a **i** Find the equation of the tangent to $y = e^{-x}$ at the point D.

ii Find the intercept of the tangent with the x-axis.

b **i** Find the area of the region $ABCD$.

ii The line BD divides the region into two parts. Find the ratio of the areas of these two parts.

ISBN 978-1-108-45164-2

16 A manufacturer sells cylinders whose diameters are normally distributed with mean 3 cm and standard deviation 0.002 cm. The selling price is \$$s$ per cylinder and the cost of manufacture is \$1 per cylinder. A cylinder is returned and the purchase money is refunded if the diameter of the cylinder is found to differ from 3 cm by more than d cm. A returned cylinder is regarded as a total loss to the manufacturer. The probability that a cylinder is returned is 0.25.

- **a** Find d.
- **b** The profit, \$$Q$, per cylinder is a random variable. Give the possible values of Q in terms of s, and the probabilities of these values.
- **c** Express the mean and standard deviation of Q in terms of s.

17 The length of a certain species of worm has a normal distribution with mean 20 cm and standard deviation 1.5 cm.

- **a** Find the probability that a randomly selected worm has a length greater than 22 cm.
- **b** If the lengths of the worms are measured to the nearest centimetre, find the probability that a randomly selected worm has its length measured as 20 cm.
- **c** If five worms are randomly selected, find the probability that exactly two will have their lengths measured as 20 cm (to the nearest centimetre).

18 The amount of coal, P tonnes, produced by x miners in one shift is given by the rule:

$$P = \frac{x^2}{90}(56 - x) \quad \text{where } 1 \le x \le 40$$

- **a** Find $\frac{dP}{dx}$.
- **b** **i** Sketch the graph of P against x for $1 \le x \le 40$.
  - **ii** State the maximum value of P.
- **c** Write down an expression in terms of x for the average production per miner in the shift. Denote the average production per miner by A (in tonnes).
  - **i** Sketch the graph of A against x for $1 \le x \le 40$.
  - **ii** State the maximum value of A and the value of x for which it occurs.

19 A straight road passes by a hill. The angle of elevation to the top of the hill is measured from three points A, B and C along the road. Point B is between points A and C such that $AB = BC = 1200$ m. The angle of elevation is 12.5° from point A and point B, and 9.5° from point C.

Let h m be the height of the hill. Let Y be the top point of the hill and let X be the point vertically below Y at the same level as the road.

- **a** Find AX, BX and CX in terms of h.
- **b** Let M be the midpoint of AB. Find MX^2 in terms of h.
- **c** Use Pythagoras' theorem in triangle CMX to find h.
- **d** Find the perpendicular distance from the point X to the road.
- **e** Find the distance of each of the points A, B and C from point X.

ISBN 978-1-108-45164-2

Revision

CU

20 Consider the family of quadratic functions with rules of the form

$$f(x) = (k+2)x^2 + (6k-4)x + 2$$

where k is an arbitrary constant.

a Sketch the graph of f when:

i $k = 0$ **ii** $k = -2$ **iii** $k = -4$

b Find the coordinates of the turning point of the graph of $y = f(x)$ in terms of k. If the coordinates of the turning point are (a, b), find the values of k such that:

i $a > 0$ **ii** $a = 0$ **iii** $b > 0$ **iv** $b < 0$

c For what values of k is the turning point a local maximum?

d By using the discriminant, state the values of k for which:

i $f(x)$ is a perfect square **ii** there are no solutions to the equation $f(x) = 0$.

21 **a** Find the solution to the equation $e^{2-2x} = 2e^{-x}$.

b Let $y = e^{2-2x} - 2e^{-x}$.

i Find $\dfrac{dy}{dx}$. **ii** Solve the equation $\dfrac{dy}{dx} = 0$.

iii State the coordinates of the turning points of $y = e^{2-2x} - 2e^{-x}$.

iv Sketch the graph of $y = e^{2-2x} - 2e^{-x}$ for $x \geq 0$.

c State the values of k for which the equation $e^{2-2x} - 2e^{-x} = k$ has two distinct positive solutions.

22 **a** Sketch, on a single clear diagram, the graphs of:

i $y = x^2$ **ii** $y = (x+a)^2$ **iii** $y = b(x+a)^2$ **iv** $y = b(x+a)^2 + c$

where a, b and c are positive constants with $b > 1$.

b Show that $\dfrac{2x^2+4x+5}{x^2+2x+1} = \dfrac{3}{(x+1)^2} + 2$, for all values except $x = -1$.

c Hence state precisely a sequence of transformations by which the graph of $y = \dfrac{2x^2+4x+5}{x^2+2x+1}$ may be obtained from the graph of $y = \dfrac{1}{x^2}$.

d Evaluate $\displaystyle\int_0^1 \frac{2x^2+4x+5}{x^2+2x+1}\,dx$.

e Sketch the graphs of $y = \dfrac{1}{x^2}$ and $y = \dfrac{3}{(x+1)^2} + 2$ on the one set of axes, and indicate the region for which the area has been determined in **d**.

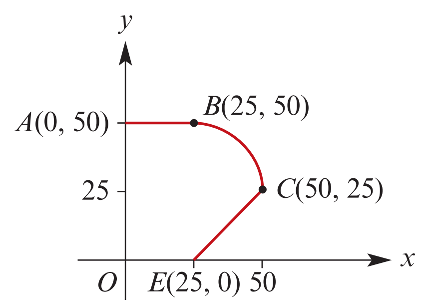
23 The length of an engine part must be between 4.81 cm and 5.20 cm. In mass production, it is found that 0.8% are too short and 3% are too long. Assume that the lengths are normally distributed.

a Find the mean and standard deviation of this distribution.

b Each part costs \$4 to produce; those that turn out to be too long are shortened at an extra cost of \$2, and those that turn out to be too short are rejected. Find the expected total cost of producing 100 parts that meet the specifications.

ISBN 978-1-108-45164-2

24 A real-estate agent has a block of land to sell. An x–y coordinate grid is placed with the origin at O, as shown in the diagram. The block of land is $OABCE$, where OA, AB, CE and EO are straight line segments and the curve through points B and C is part of a parabola with equation of the form $y = ax^2 + 4x + c$.



a Find the equation of line segments:

i AB **ii** EC

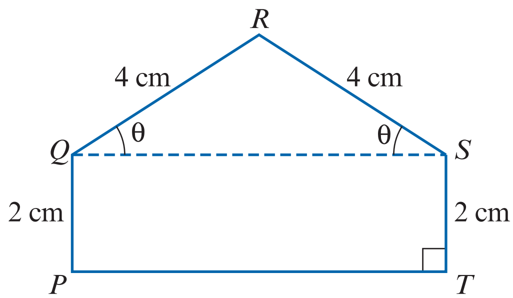
b Find the values of a and c and hence find the equation of the parabola through points B and C.

c Find the area of:

i the rectangle $OEBA$ **ii** the region EBC (with boundaries as defined above)

iii the block of land.

25 In the diagram, $PQRST$ is a thin metal plate, where $PQST$ is a rectangle with $PQ = 2$ cm and QRS is an isosceles triangle with $QR = RS = 4$ cm.



a Show that the area of the metal plate, A cm^2, is given by

$$A = 16(\cos\theta + \cos\theta\sin\theta)$$

for $0 < \theta < \frac{\pi}{2}$.

b Show that $\frac{dA}{d\theta} = 16(1 - \sin\theta - 2\sin^2\theta)$.

c Solve the equation $\frac{dA}{d\theta} = 0$ for $0 < \theta < \frac{\pi}{2}$ by first solving $16(1 - a - 2a^2) = 0$ for a.

d Sketch the graph of A against θ for $0 < \theta < \frac{\pi}{2}$, and state the maximum value of A.

26 The temperature, T°C, of water in a kettle at time t minutes is given by the formula

$$T = \theta + Ae^{-kt}$$

where θ°C is the temperature of the room in which the kettle sits.

a Assume that the room is of constant temperature 21°C. At 2:23 p.m., the water in the kettle boils at 100°C. After 10 minutes, the temperature of the water in the kettle is 84°C. Use this information to find the values of k and A, giving your answer correct to two decimal places.

b At what time will the temperature of the water in the kettle be 70°C?

c Sketch the graph of T against t for $t \ge 0$.

d Find the average rate of change of temperature for the time interval [0, 10].

e Find the instantaneous rate of change of temperature when:

i $t = 6$ **ii** $T = 60$

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CU

27 Large batches of similar components are delivered to a company. A sample of five articles is taken at random from each batch and tested. If at least four of the five articles are found to be good, the batch is accepted. Otherwise, the batch is rejected.

a If the fraction of defectives in a batch is $\frac{1}{2}$, find the probability of the batch being accepted.

b If the fraction of defectives in a batch is p, show that the probability of the batch being accepted is given by a function of the form

$$A(p) = (1 - p)^4(1 + bp), \quad 0 \leq p \leq 1$$

and find the value of b.

c Sketch the graph of A against p for $0 \leq p \leq 1$. (Using a calculator would be appropriate.)

d Find correct to two decimal places:

i the value of p for which $A(p) = 0.95$

ii the value of p for which $A(p) = 0.05$.

e **i** Find $A'(p)$, for $0 \leq p \leq 1$.

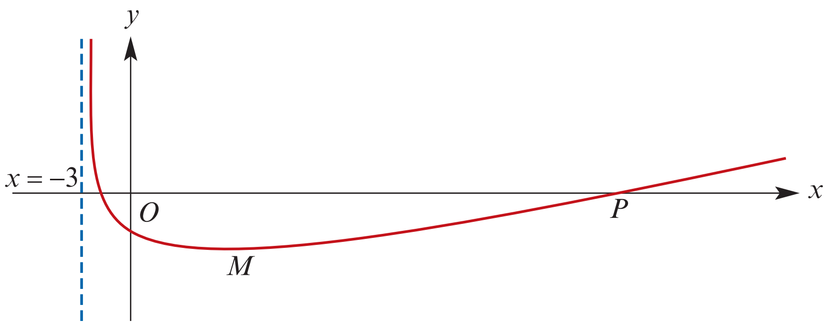
ii Sketch the graph of $A'(p)$ against p.

iii For what value of p is $A'(p)$ a minimum?

iv Describe what the result of **iii** means.

28 The diagram shows a sketch graph of

$$y = \frac{x}{10} - \ln(x + 3), \quad x > -3$$



a Find the x-coordinate of the local minimum at M.

b Show that the gradient of the curve is always less than $\frac{1}{10}$.

c Find the equation of the straight line through M with a gradient of $\frac{1}{10}$.

d **i** Hence show that the value of the x-axis intercept at P is greater than $10 \ln 10$.

ii Find, correct to three decimal places, the value of the intercept at P.

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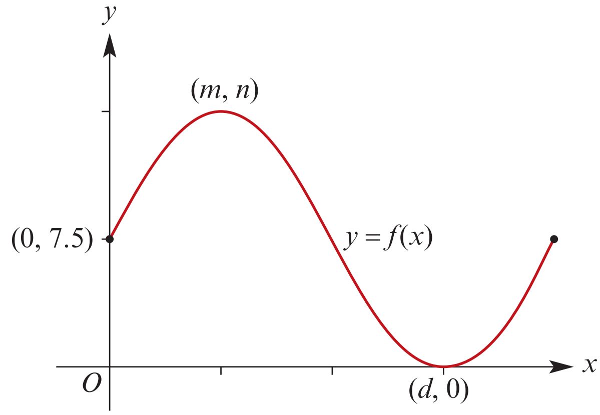
29 A section of a creek bank can be modelled by the function:

$$f(x) = a + b\sin\left(\frac{2\pi x}{50}\right), \quad x \in [0, 50]$$

where units are in metres.



a **i** Find the values of a, b, d, m and n.

ii The other bank of the creek can be modelled by the function $y = f(x) + 4$. Sketch the graph of this new function.

b Find the coordinates of the points on the first bank with y-coordinate 10.

c A particular river has a less severe bend than this creek. It is found that a section of the bank of the river can be modelled by the function:

$$g(x) = 2f\left(\frac{x}{5}\right), \quad x \in [0, 250]$$

Sketch the graph of this function; label the turning points with their coordinates.

d Over the years, the river bank moves. The shape of the bends are maintained, but there is a translation of 10 metres in the positive direction of the x-axis.

i Give the rule that describes this section of the river bank after the translation (relative to the original axes).

ii Sketch the graph of this new function.

30 A machine produces ball-bearings with a mean diameter of 3 mm. It is found that 6.3% of the production is being rejected as below the lower tolerance limit of 2.9 mm, and a further 6.3% is being rejected as above the upper tolerance limit of 3.1 mm. Assume that the diameters are normally distributed.

a Calculate the standard deviation of the distribution.

b A sample of eight ball-bearings is taken. Find the probability that:

i at least one is rejected

ii two are rejected.

c The setting of the machine now 'wanders' such that the standard deviation remains the same, but the mean changes to 3.05 mm.

i Calculate the total percentage of the production that will now fall outside the given tolerance limits.

ii Find the value of c such that the probability that the diameter lies in the interval $(3.05 - c, 3.05 + c)$ is 0.9.

 ISBN 978-1-108-45164-2

Revision

31 There is a probability of 0.8 that a boarding student will miss breakfast if he oversleeps. There is a probability of 0.3 that the student will miss breakfast even if he does not oversleep. The student has a probability of 0.4 of oversleeping. CU

a On a random day, what is the probability of:

i the student oversleeping and missing breakfast

ii the student not oversleeping and still missing breakfast

iii the student not missing breakfast?

b Given that the student misses breakfast, find the probability that he overslept.

c It is found that 10 students in the boarding house have identical probabilities for sleeping in and missing breakfast to the student mentioned above. Find the probability that:

i exactly two of the 10 students miss breakfast

ii at least one of the 10 students misses breakfast

iii at least eight of the students don't miss breakfast.

32 The continuous random variable X has probability density function f given by CF

$$f(x) = \begin{cases} k(5-2x) & \text{if } 2 < x \le \frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$$

a Find the value of k.

b **i** Find $\mathrm{E}(X)$.

ii Find the median of X.

iii Find σ, the standard deviation of X, correct to two decimal places.

c Find $\Pr(X < \mu - \sigma)$, where $\mu = \mathrm{E}(X)$.

33 The lifetime, X days, of a particular type of computer component has a probability density function given by

$$f(x) = \begin{cases} k(a-x) & \text{if } 0 < x \le a \\ 0 & \text{if } x \le 0 \text{ or } x > a \end{cases}$$

where k and a are positive constants.

a Find k in terms of a.

b Find the mean, μ, and the variance, σ^2, of X in terms of a.

c Find $\Pr(X > \mu + 2\sigma)$.

d Find the value of a if the median lifetime is 1000 days.

ISBN 978-1-108-45164-2

34 A particle is moving along a path with equation $y = \sqrt{x^2 + 24}$.

a Find $\dfrac{dy}{dx}$.

b Find the coordinates of the local minimum of the curve.

c Does this rule define an even function?

d As $x \to \infty$, $y \to x$ and as $x \to -\infty$, $y \to -x$. Sketch the graph of $y = \sqrt{x^2 + 24}$, showing the asymptotes.

e Find the equation of the normal to the curve at the point with coordinates $(1, 5)$, and sketch the graph of this normal with the graph of **d**.

f Show that

$$\frac{d}{dx}\left(12\ln\left(\sqrt{x^2 + 24} + x\right) + \frac{x\sqrt{x^2 + 24}}{2}\right) = \sqrt{x^2 + 24}$$

for $x > 0$.

g Use this result to find the area of the region bounded by the curve, the x-axis and the lines $x = 2$ and $x = 5$.

35 The boxplot is a display used to describe the distribution of a data set. Located on the boxplot are the minimum, the lower quartile, the median, the upper quartile and the maximum. Boxplots also show outliers. These are values which are more than 1.5 interquartile ranges below the lower quartile or above the upper quartile.

a Suppose that a random variable Z is normally distributed with a mean of 0 and a standard deviation of 1.

- **i** Find the value of the median, i.e. find m such that $\Pr(Z \leq m) = 0.5$.
- **ii** Find the value of the lower quartile, i.e. find q_1 such that $\Pr(Z \leq q_1) = 0.25$.
- **iii** Find the value of the upper quartile, i.e. find q_3 such that $\Pr(Z \leq q_3) = 0.75$.
- **iv** Hence find the interquartile range (IQR) for this distribution.
- **v** Find $\Pr(q_1 - 1.5 \times \text{IQR} < Z < q_3 + 1.5 \times \text{IQR})$.
- **vi** What percentage of data values would you expect to be designated as outliers for this distribution?

b Suppose that a random variable X is normally distributed with a mean of μ and a standard deviation of σ.

- **i** Find the value of the median, i.e. find m such the $\Pr(X \leq m) = 0.5$.
- **ii** Find the value of the lower quartile, i.e. find q_1 such that $\Pr(X \leq q_1) = 0.25$.
- **iii** Find the value of the upper quartile, i.e. find q_3 such that $\Pr(X \leq q_3) = 0.75$.
- **iv** Hence find the interquartile range (IQR) for this distribution.
- **v** Find $\Pr(q_1 - 1.5 \times \text{IQR} < X < q_3 + 1.5 \times \text{IQR})$.
- **vi** What percentage of data values would you expect to be designated as outliers for this distribution?

ISBN 978-1-108-45164-2

Revision

CF

36 A coin is tossed 1000 times, and 527 heads observed.

a Give a point estimate for p, the probability of observing a head when the coin is tossed.

b Determine an approximate 95% confidence interval for p.

c What level of confidence would be given by a confidence interval for p which is half the width of the approximate 95% confidence interval?

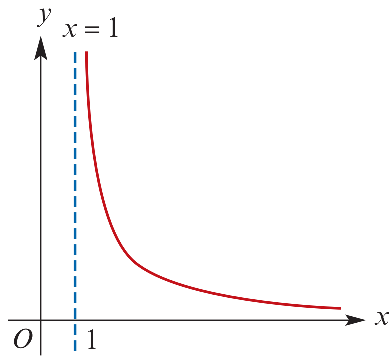
d What level of confidence would be given by a confidence interval for p which is twice the width of the approximate 95% confidence interval?

CU

37 The diagram shows the graph of the function

$$g(x) = \frac{1}{x-1}, \quad x > 1$$

The line segment AB is drawn from the point $A(2, 1)$ to the point $B(b, g(b))$, where $b > 2$.



a **i** What is the gradient of AB?

ii At what value of x between 1 and b does the tangent to the graph of g have the same gradient as AB?

b **i** Calculate $\int_2^{e+1} g(x)\, dx$.

ii Let c be a real number with $1 < c < 2$. Find the exact value of c such that $\int_c^{e+1} g(x)\, dx = 8$.

c **i** What is the area of the trapezium bounded by the line segment AB, the x-axis and the lines $x = 2$ and $x = b$?

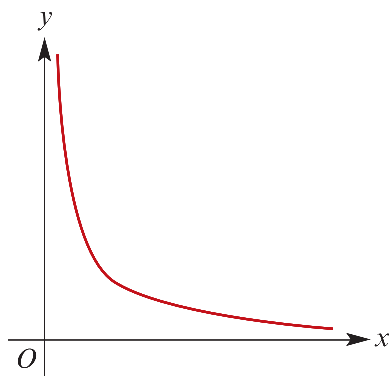
ii For what exact value of b does this area equal 8?

d Given that $\int_2^{mn+1} g(x)\, dx + \int_2^{\frac{m}{n}+1} g(x)\, dx = 2$, where $n > 0$, find the value of m.

38 The diagram shows the graph of the function

$$f(x) = \frac{1}{x^2}, \quad x > 0$$

The line segment AB is drawn from the point $A(1, 1)$ to the point $B(b, f(b))$, where $b > 1$.



a **i** What is the gradient of AB?

ii At what value of x between 1 and b does the tangent to the graph of f have the same gradient as AB?

b **i** What is the area, $S(b)$, of the trapezium bounded by the line segment AB, the x-axis and the lines $x = 1$ and $x = b$?

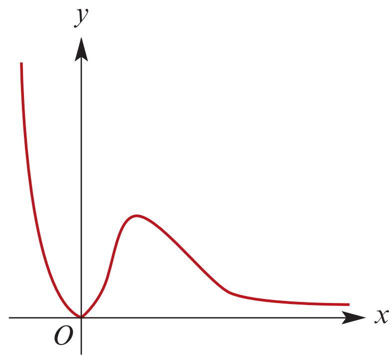
ii For what exact value of b does this area equal $\frac{10}{9}$?

iii Show that $\int_1^b f(x)\, dx < 1$ for $b > 1$.

c Show that the function $D(b) = S(b) - \int_1^b f(x)\, dx$ is strictly increasing for $b > 1$.

ISBN 978-1-108-45164-2

39 Define the function $f(x) = x^m e^{-nx+n}$, where the constants m and n are positive integers. The graph of $y = f(x)$ is as shown.



a Find the coordinates of the stationary point not at the origin in terms of n, and state its nature.

b Find the coordinates of the point on the graph at which the tangent of f passes through the origin.

c Consider the continuous probability density function with rule

$$f(x) = \begin{cases} kx^2e^{-2x+2} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where k is a positive real number.

i Find the value of k.

ii Find $\Pr(X < 1)$, where X is the associated random variable.

40 Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} ke^{-qx} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where q is a positive real number.

a **i** Find the value of k in terms of q.

ii Find $\mathrm{E}(X)$ in terms of q.

iii Find $\mathrm{Var}(X)$ in terms of q.

iv Show that the median of the distribution is $m = \frac{1}{q}\ln 2$.

b Find $\Pr\left(X > \frac{1}{q}\ln 3 \;\middle|\; X > \frac{1}{q}\ln 2\right)$.

c The distance, X metres, between flaws in a certain type of yarn is a continuous random variable with probability density function $f(x) = 0.01e^{-0.01x}$ for $x \geq 0$.

i Sketch the graph of $y = f(x)$.

ii Find the probability, correct to two decimal places, that the distance between consecutive flaws is more than 100 m.

iii Find the median value of this distribution, correct to two decimal places.

ISBN 978-1-108-45164-2

Revision

20D Degree-of-difficulty classified questions

▶ Simple familiar questions

1 Find the derivative of each of the following, given that the function g is differentiable for all real numbers:

a $x^3 g(x)$ **b** $(2x + 3)\, g(x)$ **c** $(g(x))^3$ **d** $\dfrac{g(x)}{x^2}$

2 If $f(4) = 6$ and $f'(4) = 2$, find $g'(4)$ where:

a $g(x) = \sqrt{x}\, f(x)$ **b** $g(x) = \dfrac{f(x)}{x}$

3 Given that $f'(x) = \sin\left(2x - \dfrac{\pi}{3}\right)$ and $g(x) = 3x$, find $F'(x)$ if $F(x) = f(g(x))$.

4 If $f(x) = 4x^2 - 5x + 6$, find:

a $f'(x)$ **b** $f'(0)$ **c** the value of x for which $f'(x) = 1$.

5 Find the derivative of $\ln(3f(x))$ with respect to x.

6 The tangent to the graph of $y = \sqrt{a - x}$ at $x = 1$ has a gradient of -6. Find the value of a.

7 Let $f(x) = e^x \sin(x)$.

a Find:

i $f'(x)$ **ii** $f'(\pi)$ **iii** $f'(0)$

b Find the equation of the tangent to the graph of $y = f(x)$ at:

i $x = 0$ **ii** $x = \pi$

c Find the equation of the tangent to the graph of $y = f(x - \pi)$ at:

i $x = 0$ **ii** $x = \pi$

8 Solve each of the following equations for x:

a $e^{2x} = 1$ **b** $2^{x-1} \times 4^x = 16$

c $\log_2(3x + 1) = -1$ **d** $\ln(x) - 2\ln(2x) = 1$

9 Sketch graphs for the following functions, labelling the axis intercepts:

a $y = \dfrac{1}{2}e^{-x} - 1$ **b** $y = 2\log_3(x + 1) + 1$

10 Differentiate each of the following with respect to x:

a $e^{2x}\cos(3x)$ **b** $(x - 2)^2 \ln(x - 2)$ **c** $\dfrac{\ln(x^2)}{x}$ **d** $\cos(2x)\sin(3x)$

e $\dfrac{1}{\sin(2x)}$ **f** $\sin^3(2 - 5x)$ **g** $x^3\cos^3(3x)$ **h** $x^3\ln(x^3)$

 ISBN 978-1-108-45164-2

11 Find the gradient of each of the following curves at the stated value of x:

a $y = 4 - e^{2x}, \quad x = 1$

b $y = e^{x^3 - 1}, \quad x = 0$

c $y = 4e^{2x} + x^3, \quad x = 1$

d $y = e^{-x} + 5, \quad x = 0$

12 A triangular area of land has boundary side lengths of 10 m, 12 m and 17 m.

a Find the largest internal angle in degrees, correct to two decimal places.

b Find the area of the land, correct to the nearest square metre.

13 Find the area bounded by the graph of the function and the x-axis for the given domain:

a $y = -3\sin(2x)$ for $0 \le x \le \dfrac{\pi}{2}$

b $y = e^{-x} - \dfrac{1}{e}$ for $0 \le x \le 2$

14 The probability that Billy will be late to school on Monday is 0.2. If he is late on Monday, then the probability that he will be late on Tuesday is 0.3; otherwise this probability is 0.1.

a Find the probability that:

i Billy is late on both Monday and Tuesday

ii Billy is late on Tuesday

b If Billy was late on Tuesday, find the probability that he was late on Monday.

15 Laura works as a salesperson for an electric-car company. The following table gives the probability distribution for the number, X, of cars that Laura sells in a day.

x	0	1	2	3
$\Pr(X = x)$	0.3	0.4	0.2	k

a State the value of k.

b Find the probability that on a given day:

i Laura sells at least one car

ii Laura sells exactly one car, given that she sells at least one car.

c Find the mean, μ, and the standard deviation, σ, for the number of cars that Laura sells in one day.

d Find $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$.

16 The weights of goldfish are normally distributed with a mean of 40 g and a standard deviation of 5 g.

a Find the probability, correct to three decimal places, that the weight of a randomly selected goldfish is:

i between 30 g and 50 g

ii more than 45 g, given that it is less than 50 g.

b What weight, correct to one decimal place, would put a goldfish in:

i the bottom 20% of the population by weight

ii the top 40% of the population by weight?

ISBN 978-1-108-45164-2

Complex familiar questions

1 Find the implied domain for the function $f(x) = \sqrt{x-3} + \sqrt{5-x}$.

2 Strontium-90 decays at a rate proportional to the amount present. Therefore, if $A(t)$ is the amount of strontium-90 present after t years, then

$$A(t) = A_0 e^{-kt}$$

where A_0 is the initial amount and k is a positive constant. It takes 28.8 years for an amount of strontium-90 to reduce by half.

a Find the value of k.

b How many years does it take for an amount of strontium-90 to reduce by 75%?

3 A particle starts from rest at a fixed point O and moves in a straight line towards a point A. The particle's velocity, v m/s, is given by

$$v = 8 - 8e^{2t}$$

where t is the time in seconds after leaving O.

a Find the particle's acceleration at time t seconds.

b Given that the particle reaches point A at time $t = \ln 2$, find the distance OA.

4 Among 200 households that had the newspaper delivered on Sunday, there were 40 households that did not open their paper.

a If 10 of the 200 households are surveyed, find the probability that at least three of the surveyed households did not open their paper. (Answer correct to three decimal places.)

b Determine a 95% confidence interval for the proportion of households that do not open their Sunday paper. (Round to two decimal places.)

5 Solve for x:

a $2 \times 2^{2x} = 17 \times 2^x - 8$

b $\ln(2x-3) - \ln(x) = 2$

6 A triangle ABC has $AB = 2$ m, $AC = 3$ m, $\angle ACB = 30°$ and $\angle ABC$ is obtuse.

a Find the two internal angles $\angle ABC$ and $\angle BAC$, correct to one decimal place.

b Find the area of $\triangle ABC$, correct to one decimal place.

c Find the perimeter of $\triangle ABC$, correct to one decimal place.

7 A particle moves in a straight line such that its acceleration, a m/s^2, at time t seconds is given by $a = 3t + 2$. The particle starts at rest 10 metres to the right of a fixed point O on the line. Find the position of the particle relative to O after 5 seconds.

8 An exponential function has a rule of the form $y = A \times 3^{bx}$. Given that the graph of the function passes through the points $(0, 2)$ and $(1, 6)$, find the values of A and b.

ISBN 978-1-108-45164-2

► Complex unfamiliar questions

1 A water-cooling device has a system of water circulation for the first 30 minutes of its operation. The circulation follows the following sequence:

- For the first 3 minutes water is flowing in.
- For the second 3 minutes water is flowing out.
- For the third 3 minutes water is flowing in.

This pattern is continued for the first 30 minutes. The rate of flow of water is given by the function

$$R(t) = 10e^{\frac{-t}{10}} \sin\left(\frac{\pi t}{3}\right)$$

where $R(t)$ litres per minute is the rate of flow at time t minutes. Initially there are 4 litres of water in the device.

a **i** Find $R(0)$. **ii** Find $R(3)$.

b Find $R'(t)$.

c **i** Solve the equation $R'(t) = 0$ for $t \in [0, 12]$.

ii Find the coordinates of the stationary points of $y = R(t)$ for $t \in [0, 12]$.

d Solve the equation $R(t) = 0$ for $t \in [0, 12]$.

e Sketch the graph of $y = R(t)$ for $t \in [0, 12]$.

f **i** How many litres of water flowed into the device for $t \in [0, 3]$?

ii How many litres of water flowed out of the device for $t \in [3, 6]$?

iii How many litres of water are in the device when $t = 6$? (Remember there are initially 4 litres of water.)

g How many litres of water are there in the device when $t = 30$?

2 The cross-section of the curved surface of a skate ramp is modelled by the function

$$f(x) = \frac{2}{(x-1)^2} + 1, \quad 2 \le x \le 5$$

where the x-axis represents ground level and units are in metres.

a Describe a sequence of transformations that takes the graph of $y = \frac{1}{x^2}$ to the graph of $y = f(x)$.

b Find the area of the cross-section above the ground.

c Find the gradient of the ramp at its steepest point.

To strengthen the skate ramp, a steel strut will be added within the cross-section. The strut will reach from the ground to the point on the ramp surface where $x = 3$, and will meet the ramp surface at an angle of 90° to the surface.

d Find the equation of a straight line representing the strut.

e Hence, find the length of the strut.

ISBN 978-1-108-45164-2

Revision

3 The sales revenue, in dollars, that a manufacturer receives for selling x units of a particular product can be approximated by the function

$$R(x) = 90\,000 \ln\left(1 + \frac{x}{100}\right), \quad 10 \le x \le 30\,000$$

Each unit costs the manufacturer \$10 to produce, and the initial cost of adjusting the machinery for a production run is \$2000. So the total cost, in dollars, of producing x units is given by

$$C(x) = 2000 + 10x, \quad 10 \le x \le 30\,000$$

a Write down the profit, $P(x)$ in dollars, from the production and sale of x units.

b Find $P'(x)$.

c Find the value of x that gives the maximum profit and find this maximum profit.

d Sketch the graph of $y = P(x)$ for $10 \le x \le 30\,000$.

4 The wingspans of a large population of eagles are normally distributed with mean 1.2 m and standard deviation 20 cm.

a Approximately what percentage of these eagles have a wingspan greater than 1 m?

b Find the probability, correct to three decimal places, that a randomly selected eagle from this population has a wingspan:

i greater than 90 cm

ii greater than 90 cm, given that it is less than 1.1 m.

A random sample of three eagles was selected from this population. For each of the following, give your answer correct to three decimal places.

c Find the probability that at least two eagles in the sample have a wingspan greater than 1 m.

d Find the probability that the proportion of eagles in the sample with a wingspan greater than 1 m is less than 0.4.

In a separate experiment, a random sample of 100 eagles was selected from a different population of eagles, and 80 of them had a wingspan greater than 1 m.

e Find an approximate 95% confidence interval for p, the proportion of eagles in this population with a wingspan greater than 1 m. (Round to two decimal places.)

f Find the sample size that is required to achieve a margin of error of 2% in a 95% confidence interval for p.

ISBN 978-1-108-45164-2

Counting methods and the binomial theorem

A1 Counting methods

► The addition rule

In general, to choose between alternatives simply add up the number of choices available for each alternative.

Example 1

At the library Alan is having trouble deciding which book to borrow. He has a choice of three mystery novels, three biographies or two science fiction books. How many choices of book does he have?

Solution

As he is choosing between alternatives (mystery novels *or* biographies *or* science fiction), he has a total of $3 + 3 + 2 = 8$ choices.

► The multiplication rule

When sequential choices are involved, the total number of possibilities is found by multiplying the number of options at each successive stage.

Example 2

Sandi has six choices of T-shirts or tank tops, and seven choices of shorts or skirts. How many choices does she have for a complete outfit?

Solution

As Sandi will wear either a T-shirt or a tank top *and* shorts or a skirt, we cannot consider these to be alternative choices. We could draw a tree diagram to list the possibilities, but this would be arduous. Using the multiplication rule, however, we can quickly determine the number of choices to be $6 \times 7 = 42$.

ISBN 978-1-108-45164-2

Cambridge University Press

Permutations or arrangements

The number of arrangements of n objects in groups of size r is denoted ${}^nP_r$ and given by

$$\begin{aligned}{}^nP_r &= \frac{n!}{(n-r)!} \\ &= n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)\end{aligned}$$

Example 3

How many different four-digit numbers can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 if each digit may be used only once?

Solution

The number of arrangements of 9 digits in groups of size 4 is

$${}^9P_4 = \frac{9!}{5!} = 9 \times 8 \times 7 \times 6 = 3024$$

Combinations or selections

The number of combinations of n objects in groups of size r is

$$\begin{aligned}{}^nC_r &= \frac{{}^nP_r}{r!} \\ &= \frac{n!}{r!\,(n-r)!}\end{aligned}$$

A commonly used alternative notation for ${}^nC_r$ is $\binom{n}{r}$.

Example 4

Four flavours of ice-cream – vanilla, chocolate, strawberry and caramel – are available at the school tuck shop. How many different double-scoop selections are possible if two different flavours must be used?

Solution

The number of combinations of 4 flavours in groups of size 2 is

$${}^4C_2 = \frac{4!}{2!\,2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 6$$

Example 5

A team of three boys and three girls is to be chosen from a group of eight boys and five girls. How many different teams are possible?

Solution

We can choose three boys from eight in ${}^8C_3$ ways *and* three girls from five in ${}^5C_3$ ways. Thus the total number of possible teams is

$${}^8C_3 \times {}^5C_3 = 56 \times 10 = 560$$

ISBN 978-1-108-45164-2

Exercise A1

SF

1 A student needs to select a two-unit study for her course: one unit in each semester. In Semester 1 she has a choice of two mathematics units, three language units and four science units. In Semester 2 she has a choice of two history units, three geography units and two art units. How many choices does she have for her two-unit study?

2 In order to travel from Adelaide to Brisbane, Dominic is given the following choices. He can fly directly from Adelaide to Brisbane on one of three airlines, or he can fly from Adelaide to Sydney on one of four airlines and then travel from Sydney to Brisbane with one of five bus lines, or he can go on one of three bus lines directly from Adelaide to Brisbane. In how many ways could he travel from Adelaide to Brisbane?

3 If there are eight swimmers in the final of the 1500 m freestyle event, in how many ways can the first three places be filled?

4 In how many ways can the letters of the word TROUBLE be arranged:

a if they are all used

b in groups of three?

5 In how many ways can the letters of the word PANIC be arranged:

a if they are all used

b in groups of four?

6 A student has the choice of three mathematics subjects and four science subjects. In how many ways can she choose to study one mathematics and two science subjects?

7 A survey is to be conducted, and eight people are to be chosen from a group of 30.

a How many different groups of eight people could be chosen?

b If the group contains 10 men and 20 women, how many groups of eight people containing exactly two men are possible?

8 From a standard 52-card deck, how many seven-card hands have exactly five spades and two hearts?

9 In how many ways can a committee of five be selected from eight women and four men:

a without restriction

b if there must be exactly three women on the committee?

10 Six females and five males are interviewed for five positions. If all are found to be acceptable for any position, in how many ways could the following combinations be selected?

a three females and two males

b four females and one male

c five females

d five people regardless of sex

e at least four females

ISBN 978-1-108-45164-2

Cambridge University Press

A2 Summation notation

Suppose that m and n are integers with $m < n$. Then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

This notation, which is called **summation notation** or **sigma notation**, is very convenient for concisely representing sums. These sums will arise throughout the course. The notation uses the symbol Σ, which is the uppercase Greek letter *sigma*.

The notation

$$\sum_{i=m}^{n} a_i$$

is read 'the sum of the numbers a_i from i equals m to i equals n'.

The expression $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$ is called the **expanded form** of $\sum_{i=m}^{n} a_i$.

Example 6

Write $\sum_{i=1}^{5} 2^i$ in expanded form and evaluate.

Solution

$$\sum_{i=1}^{5} 2^i = 2^1 + 2^2 + 2^3 + 2^4 + 2^5$$
$$= 2 + 4 + 8 + 16 + 32$$
$$= 62$$

Example 7

Write $1^2 + 2^2 + 3^2 + \cdots + 30^2$ using summation notation.

Solution

$$1^2 + 2^2 + 3^2 + \cdots + 30^2 = \sum_{k=1}^{30} k^2$$

Example 8

Write $x_1 + x_2 + x_3 + \cdots + x_{10}$ using summation notation.

Solution

$$x_1 + x_2 + x_3 + \cdots + x_{10} = \sum_{i=1}^{10} x_i$$

ISBN 978-1-108-45164-2

Exercise A2

1 Write each of the following in expanded form and evaluate:

a $\sum_{i=1}^{4} i^3$ **b** $\sum_{k=1}^{5} k^3$ **c** $\sum_{i=1}^{5} (-1)^i\, i$ **d** $\frac{1}{5}\sum_{i=1}^{5} i$

e $\sum_{i=1}^{6} i$ **f** $\sum_{k=1}^{4} (k-1)^2$ **g** $\frac{1}{3}\sum_{i=1}^{4} (i-2)^2$ **h** $\sum_{i=1}^{6} i^2$

SF

2 Write each of the following using summation notation:

a $1 + 2 + 3 + \cdots + n$ **b** $x_1 + x_2 + x_3 + \cdots + x_{11}$

c $\dfrac{x_1 + x_2 + x_3 + \cdots + x_{10}}{10}$ **d** $1^4 + 2^4 + 3^4 + \cdots + n^4 + (n+1)^4$

e $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

3 Write each of the following in expanded form:

a $\sum_{i=1}^{n} x^i$ **b** $\sum_{i=0}^{5} x^i \cdot 2^{5-i}$ **c** $\sum_{i=0}^{6} (2x)^i \cdot 3^{6-i}$ **d** $\sum_{i=0}^{4} (x - x_i)^i$

4 Write each of the following using summation notation:

a $x^5 + 3x^4 + 9x^3 + 27x^2 + 81x + 243$ **b** $x^5 - 3x^4 + 9x^3 - 27x^2 + 81x - 243$

c $4x^2 + 2x + 1$ **d** $8x^3 + 12x^2 + 18x + 27$

A3 The binomial theorem

Consider the expansions of binomial powers shown below:

$$(x+b)^0 = 1$$
$$(x+b)^1 = 1x + 1b$$
$$(x+b)^2 = 1x^2 + 2xb + 1b^2$$
$$(x+b)^3 = 1x^3 + 3x^2b + 3xb^2 + 1b^3$$
$$(x+b)^4 = 1x^4 + 4x^3b + 6x^2b^2 + 4xb^3 + 1b^4$$
$$(x+b)^5 = 1x^5 + 5x^4b + 10x^3b^2 + 10x^2b^3 + 5xb^4 + 1b^5$$

The coefficients can be arranged in a triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

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This array is known as **Pascal's triangle**, and can also be constructed from combinations:

Row 0: $\binom{0}{0}$

Row 1: $\binom{1}{0}$ $\binom{1}{1}$

Row 2: $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

Row 3: $\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

Row 4: $\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

Row 5: $\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$

Remember that $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$.

The expansion of $(x+b)^6$ can be written by using this observation:

$$(x+b)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5b + \binom{6}{2}x^4b^2 + \binom{6}{3}x^3b^3 + \binom{6}{4}x^2b^4 + \binom{6}{5}xb^5 + \binom{6}{6}b^6$$

In summation notation:

$$(x+b)^6 = \sum_{k=0}^{6}\binom{6}{k}x^{6-k}b^k$$

In general:

$$(x+b)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}b^k \qquad \text{and} \qquad (ax+b)^n = \sum_{k=0}^{n}\binom{n}{k}(ax)^{n-k}b^k$$

The first term of the expansion of $(ax+b)^n$ is $\binom{n}{0}(ax)^n$ and the second term is $\binom{n}{1}(ax)^{n-1}b$.

In general, the $(r+1)$st term is $\binom{n}{r}(ax)^{n-r}b^r$.

By convention, the expansion of $(ax+b)^n$ is written with decreasing powers of x.

Example 9

Expand $(2x+3)^5$.

Solution

$$(2x+3)^5 = \sum_{k=0}^{5}\binom{5}{k}(2x)^{5-k}\,3^k$$

$$= (2x)^5 + \binom{5}{1}(2x)^4\cdot 3 + \binom{5}{2}(2x)^3\cdot 3^2 + \binom{5}{3}(2x)^2\cdot 3^3 + \binom{5}{4}(2x)\cdot 3^4 + 3^5$$

$$= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$$

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Example 10

Find the eighth term in the expansion of $(2x - 4)^{10}$.

Solution

The $(r + 1)$st term is $\binom{10}{r}(2x)^{10-r}(-4)^r$.

Therefore the 8th term is

$$\binom{10}{7}(2x)^3(-4)^7 = -15\,728\,640x^3$$

Example 11

Find the coefficient of x^{20} in the expansion of $(x + 2)^{30}$.

Solution

The $(r + 1)$st term is $\binom{30}{r}x^{30-r}\,2^r$.

When $30 - r = 20$, $r = 10$. Therefore the term with x^{20} is $\binom{30}{10}2^{10}x^{20}$.

Hence the coefficient of x^{20} is $\binom{30}{10}2^{10}$.

Exercise A3

SF

1 Expand each of the following using the binomial theorem:

a $(x + 6)^6$ **b** $(2x + 1)^5$ **c** $(2x - 1)^5$ **d** $(2x + 3)^6$
e $(2x - 6)^6$ **f** $(2x - 3)^4$ **g** $(x - 2)^6$ **h** $(x + 1)^{10}$

2 Find the eighth term of each expansion (where descending powers of x are assumed):

a $(2x - 1)^{10}$ **b** $(2x + 1)^{10}$ **c** $(1 - 2x)^{10}$
d $(3x + 1)^{12}$ **e** $(x + 3)^{12}$ **f** $(2x - b)^{12}$

3 Find the third term in the expansion of $\left(2 - \frac{1}{3}x\right)^9$, assuming descending powers of x.

4 Find the sixth term in the expansion of $(3x - 1)^{11}$, assuming descending powers of x.

5 Expand $(1 - x)^{11}$.

6 Find the coefficient of x^3 in the expansion of each of the following:

a $(x + 2)^5$ **b** $(2x - 1)^6$ **c** $(1 - 2x)^5$
d $(4x - 3)^7$ **e** $(3x + 4)^4$ **f** $(3x - 2)^5$

7 Find the coefficient of x^{10} in the expansion of $(2x - 3)^{14}$.

8 Find the coefficient of x^5 in the expansion of $(4 - 2x)^6$.

ISBN 978-1-108-45164-2

Glossary

A

Absolute maximum and minimum [p. 455]
For a continuous function f defined on an interval $[a, b]$:

- the *absolute maximum* is the value M of the function f such that $f(x) \le M$ for all $x \in [a, b]$
- the *absolute minimum* is the value N of the function f such that $f(x) \ge N$ for all $x \in [a, b]$.

Acceleration [p. 319] the rate of change of a particle's velocity with respect to time

Acceleration, average [p. 319] The average acceleration of a particle for the time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.

Acceleration, instantaneous [pp. 319, 442]

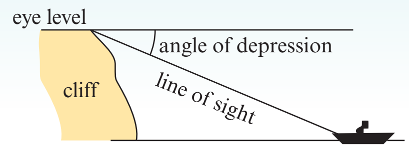
$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Addition rule for choices [p. 743] To determine the total number of choices from disjoint alternatives, simply add up the number of choices available for each alternative.

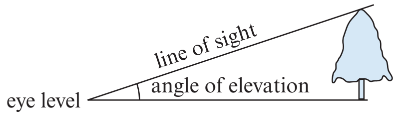
Addition rule for probability [p. 518] The probability of A or B or both occurring is given by
$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Amplitude of trigonometric functions [p. 140]
The distance between the mean position and the maximum position is called the amplitude.
The graph of $y = \sin x$ has an amplitude of 1.

Angle of depression [p. 497] the angle between the horizontal and a direction below the horizontal



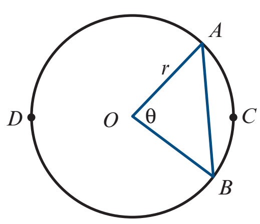
Angle of elevation [p. 497] the angle between the horizontal and a direction above the horizontal



Anti-derivative [p. 347] To find the general anti-derivative of $f(x)$: If $F'(x) = f(x)$, then
$\int f(x)\,dx = F(x) + c$
where c is an arbitrary real number.

Arc [MM1&2] Two points on a circle divide the circle into arcs; the shorter is the *minor arc*, and the longer is the *major arc*.

Arc length, ℓ [MM1&2]
The length of arc ACB is given by $\ell = r\theta$, where $\theta^c = \angle AOB$.



Note: The glossary contains some terms from Mathematical Methods Units 1 & 2 [MM1&2].

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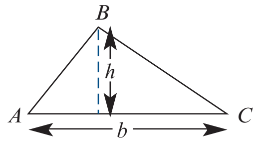
Area of a triangle [p. 494] given by half the product of the lengths of two sides and the sine of the angle included between them.

$$\text{Area} = \frac{1}{2}bh$$

$$\text{Area} = \frac{1}{2}bc\sin A$$



Arithmetic sequence [MM1&2] a sequence in which each successive term is found by adding a fixed amount to the previous term; e.g. $2, 5, 8, 11, \dots$. An arithmetic sequence has a recurrence relation of the form $t_n = t_{n-1} + d$, where d is the common difference. The nth term can be found using $t_n = a + (n-1)d$, where $a = t_1$.

Arithmetic series [MM1&2] the sum of the terms in an arithmetic sequence. The sum of the first n terms is given by the formula

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big)$$

where $a = t_1$ and d is the common difference.

Arrangements [p. 744] counted when order is important. The number of ways of selecting and arranging r objects from a total of n objects is

$$\frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \dots \times (n-r+1)$$

Asymptote [MM1&2] A straight line is an asymptote of the graph of a function $y = f(x)$ if the graph of $y = f(x)$ gets arbitrarily close to the straight line. An asymptote can be horizontal, vertical or oblique.

Average value [p. 402] The average value of a continuous function f for an interval $[a, b]$ is defined as $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

B

Bearing [p. 498] the compass bearing; the direction measured from north clockwise

Bernoulli random variable [p. 560] a random variable that takes only the values 1 (indicating a success) and 0 (indicating a failure)

Bernoulli sequence [p. 560] a sequence of repeated trials with the following properties:

- Each trial results in one of two outcomes, usually designated as a success or a failure.
- The probability of success on a single trial, p, is constant for all trials.
- The trials are independent. (The outcome of a trial is not affected by outcomes of other trials.)

Binomial distribution [p. 562] The probability of observing x successes in n independent trials, each with probability of success p, is given by

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n$$

where $\dbinom{n}{x} = \dfrac{n!}{x!\,(n-x)!}$

Binomial expansion [p. 748]

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} a^k$$

$$= x^n + \binom{n}{1} x^{n-1} a + \binom{n}{2} x^{n-2} a^2 + \dots + a^n$$

The $(r+1)$st term is $\dbinom{n}{r} x^{n-r} a^r$.

Binomial experiment [p. 562] a Bernoulli sequence of n independent trials, each with probability of success p

C

Chain rule [p. 282] The chain rule can be used to differentiate a complicated function $y = f(x)$ by transforming it into two simpler functions, which are 'chained' together:

$$x \xrightarrow{h} u \xrightarrow{g} y$$

Using Leibniz notation, the chain rule is stated as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Change of base [p. 211] $\log_a b = \dfrac{\log_c b}{\log_c a}$

Circle, general equation [p. 7] The general equation for a circle is $(x-h)^2 + (y-k)^2 = r^2$, where the centre is (h, k) and the radius is r.

Coefficient [p. 90] the number that multiplies a power of x in a polynomial. E.g. for $2x^5 - 7x^2 + 4$, the coefficient of x^2 is -7.

Combinations [p. 744] *see* selections

Common difference, *d* [MM1&2] the difference between two consecutive terms of an arithmetic sequence, i.e. $d = t_n - t_{n-1}$

Common ratio, *r* [MM1&2] the quotient of two consecutive terms of a geometric sequence, i.e.

$$r = \frac{t_n}{t_{n-1}}$$

Compass bearing [p. 498] the direction measured from north clockwise

Complement, *A′* [p. 518] the set of outcomes that are in the sample space, ε, but not in A. The probability of the event A' is $\Pr(A') = 1 - \Pr(A)$.

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Complementary relationships [p. 138]

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

Composition of functions [p. 23] For two functions f and g, the function with rule $h(x) = f(g(x))$ is the composition of f with g. We write $h = f \circ g$. For example, if $f(x) = x^4$ and $g(x) = x + 1$, then $(f \circ g)(x) = f(g(x)) = (x + 1)^4$.

Compound interest [MM1&2] is calculated at regular intervals on the total of the amount originally invested and the interest accumulated over the previous years. If $\$P$ is invested at $R\%$ p.a. compounded annually, then the value of the investment after n years, $\$A_n$, is given by

$A_n = Pr^n$, where $r = 1 + \frac{R}{100}$

Concavity [p. 447]

- If $f''(x) > 0$ for all $x \in (a, b)$, then the gradient of the curve is increasing over the interval; the curve is said to be *concave up*.
- If $f''(x) < 0$ for all $x \in (a, b)$, then the gradient of the curve is decreasing over the interval; the curve is said to be *concave down*.

Conditional probability [p. 526] the probability of an event A occurring when it is known that some event B has occurred, given by

$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

Confidence interval [p. 678] an interval estimate for the population proportion p based on the value of the sample proportion $\hat{p}$

Congruence tests [p. 480] Two triangles are congruent if one of the following conditions holds:

- **SSS** the three sides of one triangle are equal to the three sides of the other triangle
- **SAS** two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle
- **AAS** two angles and one side of one triangle are equal to two angles and the matching side of the other triangle
- **RHS** the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle.

Congruent figures [p. 480] have exactly the same shape and size

Constant function [MM1&2] a function with a rule of the form $f(x) = a$; e.g. $f(x) = 7$

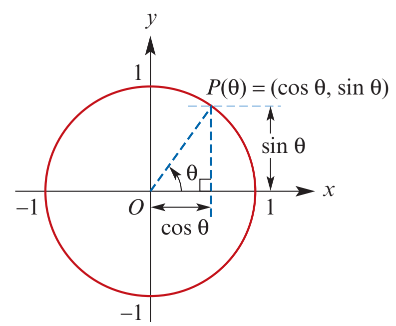
Continuous function [p. 274] A function f is continuous at the point $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

Continuous random variable [p. 582] a random variable X that can take any value in an interval of the real number line

Convergent series [MM1&2] An infinite series $t_1 + t_2 + t_3 + \cdots$ is convergent if the sum of the first n terms, S_n, approaches a limiting value as $n \to \infty$. An infinite geometric series is convergent if $-1 < r < 1$, where r is the common ratio.

Coordinates [MM1&2] an ordered pair of numbers that identifies a point in the Cartesian plane; the first number identifies the position with respect to the x-axis, and the second number identifies the position with respect to the y-axis

Cosine function [p. 131] cosine θ is defined as the x-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.

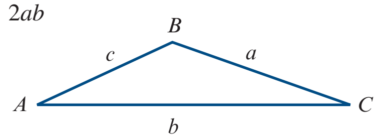


Cosine rule [p. 490] For triangle ABC:

$c^2 = a^2 + b^2 - 2ab\cos C$

or equivalently

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$



The cosine rule is used to find unknown quantities in a triangle given two sides and the included angle, or given three sides.

Cubic function [p. 103] a polynomial of degree 3. A cubic function f has a rule of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

Cumulative distribution function [p. 610] gives the probability that the random variable X takes a value less than or equal to x; that is,

$F(x) = \Pr(X \le x) = \int_{-\infty}^{x} f(t)\,dt$

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D

Definite integral [pp. 378, 384] $\int_a^b f(x)\,dx$ denotes the signed area enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$.

Degree of a polynomial [p. 90] given by the highest power of x with a non-zero coefficient; e.g. the polynomial $2x^5 - 7x^2 + 4$ has degree 5.

Dependent variable [MM1&2] If one variable, y, can be expressed as a function of another variable, x, then the value of y depends on the value of x. We say that y is the *dependent variable* and that x is the *independent variable*.

Derivative function [p. 256] also called the gradient function. The derivative f' of a function f is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives, basic [pp. 259, 271, 285–296]

$f(x)$	$f'(x)$	
c	0	where c is a constant
x^a	ax^{a-1}	where $a \in \mathbb{R} \setminus \{0\}$
e^{kx}	ke^{kx}	
$\ln(kx)$	$\frac{1}{x}$	
$\sin(kx)$	$k\cos(kx)$	
$\cos(kx)$	$-k\sin(kx)$	

Difference of two cubes [p. 99]

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Difference of two squares [MM1&2]

$x^2 - y^2 = (x - y)(x + y)$

Differentiable [p. 275] A function f is said to be differentiable at the point $x = a$ if $\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ exists.

Differentiation rules [p. 259]

- Sum: $f(x) = g(x) + h(x)$, $f'(x) = g'(x) + h'(x)$
- Multiple: $f(x) = k\,g(x)$, $f'(x) = k\,g'(x)$

see also chain rule, product rule, quotient rule

Dilation from the *x*-axis [p. 50] A dilation of factor b from the x-axis is described by the rule $(x, y) \to (x, by)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = bf(x)$.

Dilation from the *y*-axis [p. 50] A dilation of factor a from the y-axis is described by the rule $(x, y) \to (ax, y)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f\left(\frac{x}{a}\right)$.

Discontinuity [p. 274] A function is said to be discontinuous at a point if it is not continuous at that point.

Discrete random variable [p. 534] a random variable X which can take only a countable number of values, usually whole numbers

Discriminant, Δ, of a quadratic [p. 82] the expression $b^2 - 4ac$, which is part of the quadratic formula. For the quadratic equation $ax^2 + bx + c = 0$:

- If $b^2 - 4ac > 0$, there are two solutions.
- If $b^2 - 4ac = 0$, there is one solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

Disjoint [p. 2] Two sets A and B are disjoint if they have no elements in common, i.e. $A \cap B = \emptyset$.

Displacement [p. 316] The displacement of a particle moving in a straight line is defined as the change in position of the particle.

Distance between two points [p. 43] The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Domain [p. 6] the set of all the first coordinates of the ordered pairs in a relation

E

Element [p. 2] a member of a set.

- If x is an element of a set A, we write $x \in A$.
- If x is *not* an element of a set A, we write $x \notin A$.

Empty set, ∅ [p. 2] the set that has no elements

Equating coefficients [p. 92] Two polynomials P and Q are equal only if their corresponding coefficients are equal. For example, two cubic polynomials $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$ are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

Euler's number, *e* [p. 196] the natural base for exponential and logarithmic functions:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2.718\,281\ldots$$

Even function [p. 18] A function f is even if $f(-x) = f(x)$ for all x in the domain of f; the graph is symmetric about the y-axis.

Event [p. 516] a subset of the sample space (that is, a set of outcomes)

Expected value of a random variable, E(*X*) [pp. 542, 595] also called the mean, μ.

For a discrete random variable X:

$$\mathrm{E}(X) = \sum_x x \cdot \Pr(X = x) = \sum_x x \cdot p(x)$$

For a continuous random variable X:

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

Exponential function [p. 190] a function $f(x) = ka^x$, where k is a non-zero constant and the base a is a positive real number other than 1

Glossary D → E

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F

Factor [MM1&2] a number or expression that divides another number or expression without remainder

Factor theorem [p. 98] If $\beta x + \alpha$ is a factor of a polynomial $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$. Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.

Factorise [MM1&2] express as a product of factors

Formula [MM1&2] an equation containing symbols that states a relationship between two or more quantities; e.g. $A = \ell w$ (area = length × width). The value of A, the subject of the formula, can be found by substituting given values of ℓ and w.

Function [p. 8] a relation such that for each x-value there is only one corresponding y-value. This means that, if (a, b) and (a, c) are ordered pairs of a function, then $b = c$.

Function, vertical-line test [p. 8] used to identify whether a relation is a function or not. If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a *function*.

Fundamental theorem of calculus [p. 380] If f is a continuous function on an interval $[a, b]$, then

$\int_a^b f(x)\,dx = F(b) - F(a)$

where F is any anti-derivative of f and $\int_a^b f(x)\,dx$ is the definite integral from a to b.

G

Geometric sequence [MM1&2] a sequence in which each successive term is found by multiplying the previous term by a fixed amount; e.g. 2, 6, 18, 54, ... A geometric sequence has a recurrence relation of the form $t_n = rt_{n-1}$, where r is the common ratio. The nth term can be found using $t_n = ar^{n-1}$, where $a = t_1$.

Geometric series [MM1&2] the sum of the terms in a geometric sequence. The sum of the first n terms is given by the formula

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$$

where $a = t_1$ and r is the common ratio.

Gradient function *see* derivative function

Gradient of a line [p. 43] The gradient is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are the coordinates of two points on the line. The gradient of a vertical line (parallel to the y-axis) is undefined.

I

Implied domain *see* maximal domain

Indefinite integral *see* anti-derivative

Independence [p. 529] Two events A and B are independent if and only if

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

Independent variable [MM1&2] If one variable, y, can be expressed as a function of another variable, x, then the value of y depends on the value of x. We say that y is the *dependent variable* and that x is the *independent variable*.

Index laws [p. 200]

- To multiply two powers with the same base, add the indices: $a^x \times a^y = a^{x+y}$
- To divide two powers with the same base, subtract the indices: $a^x \div a^y = a^{x-y}$
- To raise a power to another power, multiply the indices: $(a^x)^y = a^{x \times y}$
- Rational indices: $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$
- For base $a \in \mathbb{R}^+ \setminus \{1\}$, if $a^x = a^y$, then $x = y$.

Inequality [MM1&2] a mathematical statement that contains an inequality symbol rather than an equals sign; e.g. $2x + 1 < 4$

Infinite geometric series [MM1&2] If $-1 < r < 1$, then the sum to infinity is given by

$$S_\infty = \frac{a}{1 - r}$$

where $a = t_1$ and r is the common ratio.

Integers [p. 3] $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Integrals, basic [pp. 348–358]

$f(x)$	$\int f(x)\,dx$	
x^r	$\frac{x^{r+1}}{r+1} + c$	where $r \in \mathbb{Q} \setminus \{-1\}$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln(ax+b) + c$	for $ax + b > 0$
e^{kx}	$\frac{1}{k}e^{kx} + c$	
$\sin(kx)$	$-\frac{1}{k}\cos(kx) + c$	
$\cos(kx)$	$\frac{1}{k}\sin(kx) + c$	

Integration, properties [p. 348]

- $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\int kf(x)\,dx = k\int f(x)\,dx$

Integration (definite), properties [p. 388]

- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

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Intersection of sets [pp. 2, 516] The intersection of two sets A and B, written $A \cap B$, is the set of all elements common to A and B.

Interval [p. 4] a subset of the real numbers of the form $[a, b]$, $[a, b)$, (a, ∞), etc.

Irrational number [p. 3] a real number that is not rational; e.g. π and $\sqrt{2}$

Iterative rule [MM1&2] *see* recurrence relation

K

Karnaugh map [p. 521] a probability table

L

Law of total probability [p. 527] In the case of two events, A and B:

$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$

Left-endpoint method [p. 373] gives an estimate for the area under the graph of $y = f(x)$ between $x = a$ and $x = b$:

$$L_n = \frac{b-a}{n}\Big[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\Big]$$

Limit [p. 255] The notation $\lim_{x \to a} f(x) = p$ says that the limit of $f(x)$, as x approaches a, is p. We can also say: 'As x approaches a, $f(x)$ approaches p.'

Linear equation [MM1&2] a polynomial equation of degree 1; e.g. $2x + 1 = 0$

Linear function [MM1&2] a function with a rule of the form $f(x) = mx + c$; e.g. $f(x) = 3x + 1$

Linear function of a random variable [p. 607]

- $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$
- $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$

Literal equation [p. 116] an equation for the variable x in which the coefficients of x, including the constants, are pronumerals; e.g. $ax + b = c$

Logarithm [p. 202] If $a \in \mathbb{R}^+ \setminus \{1\}$ and $x \in \mathbb{R}$, then the statements $a^x = y$ and $\log_a y = x$ are equivalent.

Logarithm, natural [p. 203] The natural logarithm function is given by

$\ln x = \log_e x$

where the base e is Euler's number.

Logarithm laws [p. 204]

- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a\left(\frac{1}{x}\right) = -\log_a x$
- $\log_a(x^p) = p\log_a x$

Logarithmic scale [p. 227] a measurement scale that uses the logarithm of a quantity; e.g. Richter scale for earthquakes

M

Margin of error, E [p. 682] the distance between the sample estimate and the endpoints of the confidence interval

Maximal domain [p. 16] When the rule for a relation is given and no domain is specified, then the domain taken is the largest for which the rule has meaning.

Maximum and minimum value [p. 455] *see* absolute maximum and minimum

Mean of a random variable, μ [pp. 542, 595] *see* expected value of a random variable, $\mathrm{E}(X)$

Median of a random variable, m [p. 598] the middle value of the distribution. For a continuous random variable, the median is the value m such that $\int_{-\infty}^{m} f(x)\,dx = 0.5$.

Midpoint of a line segment [p. 43] If $P(x, y)$ is the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$, then

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2}$$

Model [p. 236] a mathematical representation of a real-world situation. For example, an equation that describes the relationship between two physical quantities is a *mathematical model*.

Multiplication rule for choices [p. 743] When sequential choices are involved, the total number of possibilities is found by multiplying the number of options at each successive stage.

Multiplication rule for probability [p. 526] the probability of events A and B both occurring is

$\Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B)$

Multi-stage experiment [p. 527] an experiment that could be considered to take place in more than one stage; e.g. tossing two coins

Mutually exclusive [p. 518] Two events are said to be mutually exclusive if they have no outcomes in common.

N

$n!$ [p. 744] read as 'n factorial', the product of all the natural numbers from n down to 1:

$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$

Natural logarithm [p. 203] *see* logarithm

Natural numbers [p. 3] $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

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^nC_r [p. 744] the number of combinations of n objects in groups of size r:

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

An alternative notation for ^nC_r is $\binom{n}{r}$.

Normal distribution [p. 625] a symmetric, bell-shaped distribution that often occurs for a measure in a population (e.g. height, weight, IQ); its centre is determined by the mean, μ, and its width by the standard deviation, σ.

Normal line, equation [p. 305] Let (x_1, y_1) be a point on the curve $y = f(x)$. If f is differentiable at $x = x_1$, the equation of the normal at (x_1, y_1) is

$$y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)$$

O

Odd function [p. 18] A function f is odd if $f(-x) = -f(x)$ for all x in the domain of f; the graph has rotational symmetry about the origin.

Optimisation problem [p. 458] a problem where a quantity is to be maximised or minimised under given constraints; e.g. to maximise the area of land enclosed by a fixed length of fencing

Ordered pair [p. 6] a pair of elements, denoted (x, y), where x is the first coordinate and y is the second coordinate

P

Pascal's triangle [p. 748] a triangular pattern of numbers formed by the binomial coefficients ^nC_r

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Percentile [p. 597] For a continuous random variable X, the value p such that $\Pr(X \le p) = q\%$ is called the qth percentile of X, and is found by solving $\int_{-\infty}^{p} f(x)\,dx = \frac{q}{100}$.

Period of a function [p. 140] A function f with domain $\mathbb{R}$ is periodic if there is a positive constant a such that $f(x + a) = f(x)$ for all x. The smallest such a is called the period of f.

- Sine and cosine have period 2π.
- Tangent has period π.
- A function of the form $y = a\cos(nx + \varepsilon) + b$ or $y = a\sin(nx + \varepsilon) + b$ has period $\frac{2\pi}{n}$.

Permutations [p. 744] *see* arrangements

Piecewise-defined function [p. 17] a function which has different rules for different subsets of its domain

Point estimate [p. 678] If the value of the sample proportion $\hat{p}$ is used as an estimate of the population proportion p, then it is called a point estimate of p.

Point of inflection [p. 447] a point where a curve changes from concave up to concave down or from concave down to concave up. That is, a point of inflection occurs where the sign of the second derivative changes.

Polynomial function [p. 90] A polynomial has a rule of the type

$$y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad n \in \mathbb{N} \cup \{0\}$$

where $a_0, a_1, \ldots, a_n$ are numbers called coefficients.

Population [p. 655] the set of all eligible members of a group which we intend to study

Population parameter [p. 659] a statistical measure that is based on the whole population; the value is constant for a given population

Population proportion, p [p. 658] the proportion of individuals in the entire population possessing a particular attribute

Position [p. 316] For a particle moving in a straight line, the position of the particle relative to a point O on the line is determined by its distance from O and whether it is to the right or left of O. The direction to the right of O is positive.

Power function [p. 26] a function of the form $f(x) = x^r$, where r is a non-zero real number

Probability [p. 516] a numerical value assigned to the likelihood of an event occurring. If the event A is impossible, then $\Pr(A) = 0$; if the event A is certain, then $\Pr(A) = 1$; otherwise $0 < \Pr(A) < 1$.

Probability density function [p. 585] usually denoted $f(x)$; describes the probability distribution of a continuous random variable X such that $\Pr(a < X < b) = \int_a^b f(x)\,dx$

Probability function (discrete) [p. 535] denoted by $p(x)$ or $\Pr(X = x)$, a function that assigns a probability to each value of a discrete random variable X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.

Probability table [p. 521] a table used for illustrating a probability problem diagrammatically

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Product of functions [p. 21] $(fg)(x) = f(x)\,g(x)$ and $\text{dom}(fg) = \text{dom}\,f \cap \text{dom}\,g$

Product rule [p. 298] If $h(x) = f(x)g(x)$ then $h'(x) = f(x)g'(x) + f'(x)g(x)$
In Leibniz notation:
If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Pythagorean identity [p. 138]
$\cos^2\theta + \sin^2\theta = 1$

Q

$\mathbb{Q}$ [p. 3] the set of all rational numbers

Quadratic, turning point form [p. 77] The turning point form of a quadratic function is $y = a(x - h)^2 + k$, where (h, k) is the turning point.

Quadratic formula [p. 81] The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic function [p. 76] A quadratic has a rule of the form $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

Quartic function [p. 107] a polynomial of degree 4. A quartic function f has a rule of the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a \neq 0$.

Quotient rule [p. 302] If $h(x) = \dfrac{f(x)}{g(x)}$ then

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

In Leibniz notation:

If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

R

$\mathbb{R}^+$ [p. 3] $\{\, x : x > 0 \,\}$, positive real numbers

$\mathbb{R}^-$ [p. 3] $\{\, x : x < 0 \,\}$, negative real numbers

$\mathbb{R} \setminus \{0\}$ [p. 3] the set of real numbers excluding 0

$\mathbb{R}^2$ [p. 46] $\{\, (x, y) : x, y \in \mathbb{R} \,\}$; i.e. $\mathbb{R}^2$ is the set of all ordered pairs of real numbers

Radian [p. 129] One radian (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit:

$$1^c = \frac{180^\circ}{\pi} \quad \text{and} \quad 1^\circ = \frac{\pi^c}{180}$$

Random experiment [p. 516] an experiment, such as the rolling of a die, in which the outcome of a single trial is uncertain but observable

Random sample [p. 655] A sample of size n is called a *simple random sample* if it is selected from the population in such a way that every subset of size n has an equal chance of being chosen as the sample. In particular, every member of the population must have an equal chance of being included in the sample.

Random variable [p. 534] a variable that takes its value from the outcome of a random experiment; e.g. the number of heads observed when a coin is tossed three times

Range [p. 6] the set of all the second coordinates of the ordered pairs in a relation

Rational number [p. 3] a number that can be written as $\dfrac{p}{q}$, for some integers p and q with $q \neq 0$

Rectangular hyperbola [p. 27] The basic rectangular hyperbola has equation $y = \dfrac{1}{x}$.

Recurrence relation [MM1&2] a rule which enables each subsequent term of a sequence to be found from previous terms; e.g. $t_1 = 1$, $t_n = t_{n-1} + 2$

Reflection in the x-axis [p. 52] A reflection in the x-axis is described by the rule $(x, y) \to (x, -y)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = -f(x)$.

Reflection in the y-axis [p. 52] A reflection in the y-axis is described by the rule $(x, y) \to (-x, y)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f(-x)$.

Regression [p. 236] the process of fitting a mathematical model to data

Relation [p. 6] a set of ordered pairs; e.g. $\{\, (x, y) : y = x^2 \,\}$

Relative growth rate [pp. 222, 313] The relative growth rate of a function f is $\dfrac{f'(x)}{f(x)}$.
For an exponential function $f(x) = Ae^{kx}$, the constant k is the relative growth rate.

Remainder theorem [p. 97] When a polynomial $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\dfrac{\alpha}{\beta}\right)$.

Right-endpoint method [p. 373] gives an estimate for the area under the graph of $y = f(x)$ between $x = a$ and $x = b$:

$$R_n = \frac{b - a}{n}\Big[f(x_1) + f(x_2) + \cdots + f(x_n)\Big]$$

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S

Sample [p. 655] a subset of the population which we select in order to make inferences about the whole population

Sample proportion, $\hat{p}$ [p. 659] the proportion of individuals in a particular sample possessing a particular attribute. The sample proportions $\hat{p}$ are the values of a random variable $\hat{P}$.

Sample space, ε [p. 516] the set of all possible outcomes for a random experiment

Sample statistic [p. 659] a statistical measure that is based on a sample from the population; the value varies from sample to sample

Sampling distribution [p. 665] the distribution of a statistic which is calculated from a sample

Scientific notation [MM1&2] A number is in standard form when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 6.626×10^{-34}.

Secant [p. 254] a straight line that passes through two points $(a, f(a))$ and $(b, f(b))$ on the graph of a function $y = f(x)$

Second derivative [p. 441]

- The second derivative of a function f with rule $f(x)$ is denoted by f'' and has rule $f''(x)$.
- The second derivative of y with respect to x is denoted by $\dfrac{d^2y}{dx^2}$.

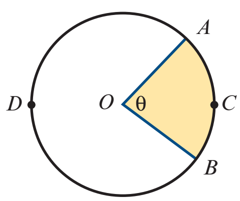
Second derivative test [p. 449]

- If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum.
- If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum.
- If $f''(a) = 0$, then further investigation is necessary.

Sector [MM1&2] Two radii and an arc define a region called a sector. In this diagram, the shaded region is a *minor sector* and the unshaded region is a *major sector*.

Area of sector $= \dfrac{1}{2}r^2\theta$

where $\theta^c = \angle AOB$



Selections [p. 744] counted when order is not important. The number of ways of selecting r objects from a total of n objects is

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

An alternative notation for ${}^nC_r$ is $\dbinom{n}{r}$.

Sequence [MM1&2] a list of numbers, with the order being important; e.g. 1, 1, 2, 3, 5, 8, 13, . . . The numbers of a sequence are called its *terms*, and the nth term is often denoted by t_n.
see also arithmetic sequence, geometric sequence

Series [MM1&2] the sum of the terms in a sequence
see also arithmetic series, geometric series

Set difference [p. 3] The set $A \setminus B$ contains all the elements of A that are not in B. For example, $\mathbb{R} \setminus \{0\}$ is the set of all real numbers excluding 0.

Set notation [p. 2]

$\in$	means 'is an element of'
$\notin$	means 'is not an element of'
$\subseteq$	means 'is a subset of'
$\cap$	means 'intersection'
$\cup$	means 'union'
$\varnothing$	is the empty set, containing no elements

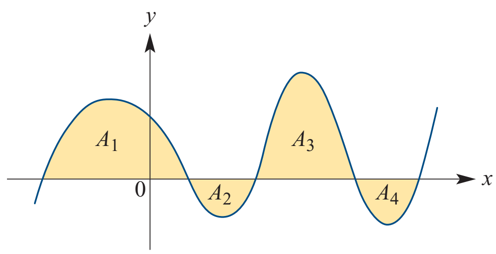
Sets of numbers [p. 3]

$\mathbb{N}$	is the set of natural numbers
$\mathbb{Z}$	is the set of integers
$\mathbb{Q}$	is the set of rational numbers
$\mathbb{R}$	is the set of real numbers

Signed area [p. 383]

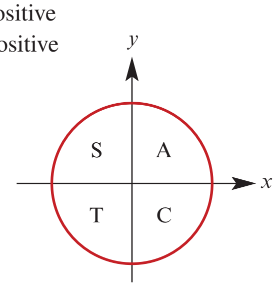
- Regions *above* the x-axis are defined to have *positive* signed area.
- Regions *below* the x-axis are defined to have *negative* signed area.

For example, the signed area of the shaded region in the following graph is $A_1 - A_2 + A_3 - A_4$.



Signs of trigonometric functions [p. 134]

1st quadrant	all are positive
2nd quadrant	sin is positive
3rd quadrant	tan is positive
4th quadrant	cos is positive



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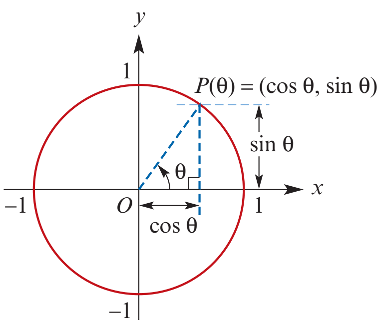
Simple interest [MM1&2] is always calculated on the amount originally invested (the *principal*). If $\$P$ is invested at $R\%$ p.a., then the value of the investment after n years, $\$A_n$, is given by

$$A_n = P + nP\frac{R}{100}$$

Simulation [p. 660] using technology (calculators or computers) to repeat a random process many times; e.g. random sampling

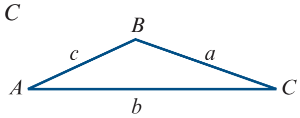
Simultaneous equations [p. 117] equations of two or more lines or curves in the Cartesian plane, the solutions of which are the points of intersection of the lines or curves

Sine function [p. 131] sine θ is defined as the y-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



Sine rule [p. 486] For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



The sine rule is used to find unknown quantities in a triangle given one side and two angles, or given two sides and a non-included angle.

Speed [p. 317] the magnitude of velocity

Speed, average [p. 317]

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

Standard deviation of a random variable, σ [pp. 545, 602] a measure of the spread or variability, given by $\text{sd}(X) = \sqrt{\text{Var}(X)}$

Standard form [MM1&2] *see* scientific notation

Standard normal distribution [p. 623] a special case of the normal distribution where $\mu = 0$ and $\sigma = 1$

Stationary point [p. 322] A point $(a, f(a))$ on a curve $y = f(x)$ is a stationary point if $f'(a) = 0$.

Straight line, equation given two points [p. 43]
$y - y_1 = m(x - x_1)$, where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Straight line, gradient–intercept form [p. 43]
$y = mx + c$, where m is the gradient and c is the y-axis intercept

Straight lines, parallel [MM1&2]
Two non-vertical straight lines are parallel to each other if and only if they have the same gradient.

Straight lines, perpendicular [p. 43]
Two straight lines are perpendicular to each other if and only if the product of their gradients is -1 (or if one is horizontal and the other vertical).

Strictly decreasing [pp. 26, 265] A function f is strictly decreasing on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

Strictly increasing [pp. 26, 265] A function f is strictly increasing on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.

Subset [p. 2] A set B is called a subset of set A if every element of B is also an element of A. We write $B \subseteq A$.

Sum of functions [p. 21] $(f + g)(x) = f(x) + g(x)$ and $\text{dom}(f + g) = \text{dom}\, f \cap \text{dom}\, g$

Sum of two cubes [p. 99]
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Sum to infinity [MM1&2] The sum to infinity of an infinite geometric series exists provided $-1 < r < 1$ and is given by

$$S_\infty = \frac{a}{1 - r}$$

where $a = t_1$ and r is the common ratio.

T

Tangent function [p. 131] The tangent function is given by

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Tangent line, equation [p. 305] Let (x_1, y_1) be a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.

Total change [p. 405] Given the rule for $f'(x)$, the total change in the value of $f(x)$ between $x = a$ and $x = b$ can be found using

$$f(b) - f(a) = \int_a^b f'(x)\, dx$$

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Translation [p. 46] A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule $(x, y) \to (x + h, y + k)$, where $h, k > 0$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y - k = f(x - h)$.

Trapezoidal rule [p. 375] gives an estimate for the area under the graph of $y = f(x)$ between $x = a$ and $x = b$:

$$T_n = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Tree diagram [p. 527] a diagram representing the outcomes of a multi-stage experiment

Trigonometric functions [p. 131] the sine, cosine and tangent functions

Trigonometric functions, exact values [p. 133]

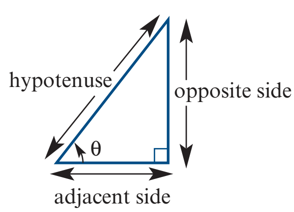
θ^c	$\theta°$	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0°	0	1	0
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90°	1	0	undefined

Trigonometric ratios [p. 481]

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



U

Uniform distribution [pp. 558, 621]

- A discrete random variable X with n values $x_1, x_2, x_3, \ldots, x_n$ has a uniform distribution if each value of X is equally likely, and therefore

  $$\Pr(X = x) = \frac{1}{n}, \text{ for } x = x_1, x_2, x_3, \ldots, x_n$$

- A continuous random variable X has a uniform distribution if its probability density function is given by

  $$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

  where a and b are real constants with $a < b$.

Union of sets [pp. 2, 516] The union of two sets A and B, written $A \cup B$, is the set of all elements which are in A or B or both.

V

Variance of a random variable, σ^2 [pp. 545, 602] a measure of the spread or variability, defined by $\text{Var}(X) = \text{E}[(X - \mu)^2]$. An alternative (computational) formula is $\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$

Velocity [p. 317] the rate of change of a particle's position with respect to time

Velocity, average [p. 317]

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

Velocity, instantaneous [p. 317] $v = \dfrac{dx}{dt}$

Vertical-line test [p. 8] *see* function

Z

$\mathbb{Z}$ [p. 3] the set of all integers

Zero polynomial [p. 90] The number 0 is called the zero polynomial.

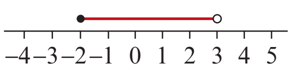
ISBN 978-1-108-45164-2

Answers

Chapter 1

Exercise 1A

1 **a** $\{8, 11\}$ **b** $\{8, 11\}$
c $\{1, 3, 8, 11, 18, 22, 23, 24, 25, 30\}$
d $\{3, 8, 11, 18, 22, 23, 24, 25, 30, 32\}$
e $\{3, 8, 11, 18, 22, 23, 24, 25, 30, 32\}$
f $\{1, 8, 11, 25, 30\}$

2 **a**

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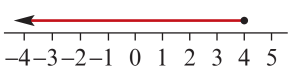
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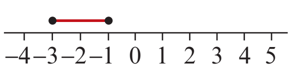
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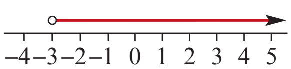
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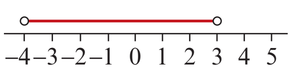
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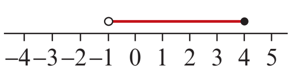
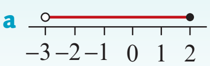


3 **a** $\{7, 9\}$ **b** $\{7, 9\}$
c $\{2, 3, 5, 7, 9, 11, 15, 19, 23\}$ **d** $\{2, 3, 5, 11\}$
e $\{2\}$ **f** $\{2, 7, 9\}$ **g** $\{2, 3, 5, 7\}$
h $\{7\}$ **i** $\{7, 9, 15, 19, 23\}$ **j** $(3, \infty)$

4 **a** $\{a, e\}$ **b** $\{a, b, c, d, e, i, o, u\}$
c $\{b, c, d\}$ **d** $\{i, o, u\}$

5 **a** $[-3, 1)$ **b** $(-4, 5]$ **c** $(-\sqrt{2}, 0)$
d $\left(-\frac{1}{\sqrt{2}}, \sqrt{3}\right)$ **e** $(-\infty, -3)$ **f** $(0, \infty)$
g $(-\infty, 0)$ **h** $[-2, \infty)$

6 **a** $(-2, 3)$ **b** $[-4, 1)$ **c** $[-1, 5]$ **d** $(-3, 2]$

7 **a**

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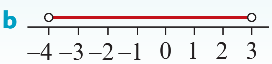
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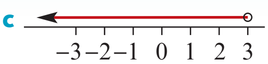
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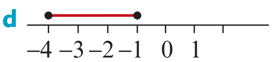
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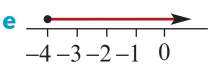
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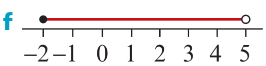
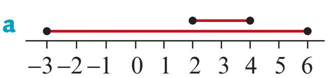
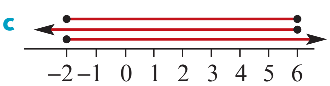


8 **a**

b

c

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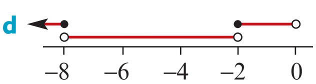

Exercise 1B

1 **a** Domain = $\mathbb{R}$ Range = $[-2, \infty)$
b Domain = $(-\infty, 2]$ Range = $\mathbb{R}$
c Domain = $(-2, 3)$ Range = $[0, 9)$
d Domain = $(-3, 1)$ Range = $(-6, 2)$
e Domain = $[-4, 0]$ Range = $[0, 4]$
f Domain = $\mathbb{R}$ Range = $(-\infty, 2)$

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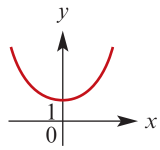
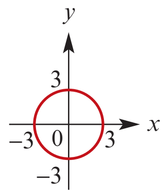
2 a



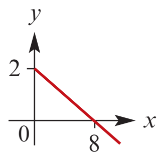
Domain = $\mathbb{R}$
Range = $[1, \infty)$

b



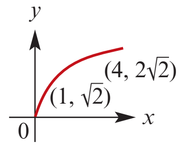
Domain = $[-3, 3]$
Range = $[-3, 3]$

c



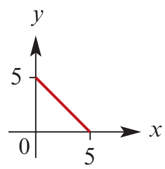
Domain = $[0, \infty)$
Range = $(-\infty, 2]$

d



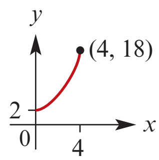
Domain = $[0, \infty)$
Range = $[0, \infty)$

e



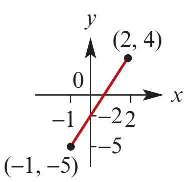
Domain = $[0, 5]$
Range = $[0, 5]$

f



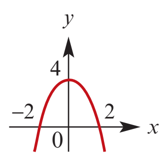
Domain = $[0, 4]$
Range = $[2, 18]$

g



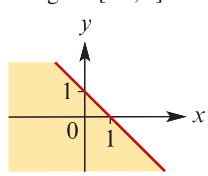
Domain = $[-1, 2]$
Range = $[-5, 4]$

h



Domain = $\mathbb{R}$
Range = $(-\infty, 4]$

i



Domain = $\mathbb{R}$
Range = $\mathbb{R}$

3 a Not a function; Domain = $\{-1, 1, 2, 3\}$;
Range = $\{1, 2, 3, 4\}$
b A function; Domain = $\{-2, -1, 0, 1, 2\}$;
Range = $\{-4, -1, 0, 3, 5\}$
c Not a function; Domain = $\{-2, -1, 2, 4\}$;
Range = $\{-2, 1, 2, 4, 6\}$
d A function; Domain = $\{-1, 0, 1, 2, 3\}$;
Range = $\{4\}$

4 Functions: a, c

5 a A function; Domain = $\mathbb{R}$; Range = $\{4\}$
b Not a function; Domain = $\{2\}$; Range = $\mathbb{Z}$
c A function; Domain = $\mathbb{R}$; Range = $\mathbb{R}$
d Not a function; Domain = $\mathbb{R}$; Range = $\mathbb{R}$
e Not a function; Domain = $[-4, 4]$;
Range = $[-4, 4]$

6 $y = \sqrt{x+2},\ x \ge -2$; Range = $[0, \infty)$
$y = -\sqrt{x+2},\ x \ge -2$; Range = $(-\infty, 0]$

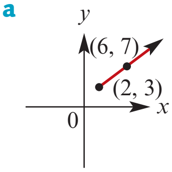
7 a $f(-1) = -2$, $f(2) = 16$, $f(-3) = 6$,
$f(2a) = 8a^2 + 8a$
b $g(-1) = -10$, $g(2) = 14$, $g(3) = 54$,
$g(a-1) = 2a^3 - 6a^2 + 8a - 10$

8 a $g(-2) = 10$, $g(4) = 46$
b i $12x^2 - 2$ **ii** $3x^2 - 12x + 10$
iii $3x^2 + 12x + 10$ **iv** $3x^4 - 2$

9 a 3 **b** 7 **c** $x = -\frac{3}{2}$ **d** $x > 3$

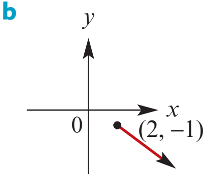
10 a $x = -3$ **b** $x > -3$ **c** $x = \frac{2}{3}$

11 a



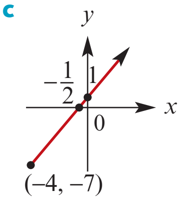
Range = $[3, \infty)$

b



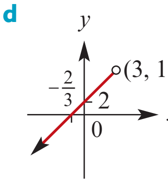
Range = $(-\infty, -1]$

c



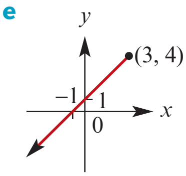
Range = $[-7, \infty)$

d



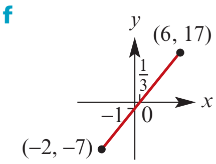
Range = $(-\infty, 11)$

e



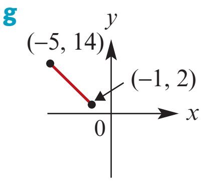
Range = $(-\infty, 4]$

f



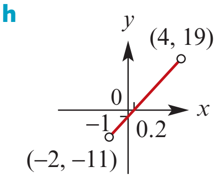
Range = $[-7, 17]$

g



Range = $[2, 14]$

h



Range = $(-11, 19)$

12 a $f(2) = -3$, $f(-3) = 37$, $f(-2) = 21$
b $g(-2) = 7$, $g(1) = 1$, $g(-3) = 9$
c i $f(a) = 2a^2 - 6a + 1$
ii $f(a+2) = 2a^2 + 2a - 3$
iii $g(-a) = 3 + 2a$ **iv** $g(2a) = 3 - 4a$
v $f(5-a) = 21 - 14a + 2a^2$
vi $f(2a) = 8a^2 - 12a + 1$
vii $g(a) + f(a) = 2a^2 - 8a + 4$
viii $g(a) - f(a) = 2 + 4a - 2a^2$

13 a $x = \frac{2}{3}$ or $x = -1$ **b** $x = \pm\sqrt{\frac{2}{3}}$
c $x = 0$ or $x = -\frac{1}{3}$ **d** $x < -1$ or $x > \frac{2}{3}$
e $x < -\sqrt{\frac{2}{3}}$ or $x > \sqrt{\frac{2}{3}}$ **f** $-\frac{1}{3} \le x \le 0$

14 a $f(-2) = 2$ **b** $f(2) = 6$
c $f(-a) = a^2 - a$ **d** $f(a) + f(-a) = 2a^2$
e $f(a) - f(-a) = 2a$ **f** $f(a^2) = a^4 + a^2$

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15 **a** $x = 2$ **b** $x > 2$ **c** $x = \frac{a+2}{3}$
d $x = -\frac{8}{3}$ **e** $x = 1$ **f** $x = \frac{13}{18}$

16 **a** $\frac{4}{3}$ **b** 6 **c** $-\frac{7}{3}$ **d** 9 **e** $\frac{1}{3}$ **f** $-\frac{2}{9}$

17 **a** $\frac{6}{5}$ **b** $\frac{1}{5}$ **c** $\pm\frac{1}{3}$ **d** 1 **e** $-1, 2$

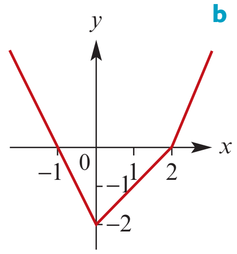
Exercise 1C

1 **a** Domain $= \mathbb{R}$ Range $= \mathbb{R}$
b Domain $= [0, \infty)$ Range $= [0, \infty)$
c Domain $= \mathbb{R}$ Range $= [-2, \infty)$
d Domain $= [-4, 4]$ Range $= [0, 4]$
e Domain $= \mathbb{R} \setminus \{0\}$ Range $= \mathbb{R} \setminus \{0\}$
f Domain $= \mathbb{R}$ Range $= (-\infty, 4]$
g Domain $= [3, \infty)$ Range $= [0, \infty)$

2 **a** Domain $= \mathbb{R}$ Range $= \mathbb{R}$
b Domain $= \mathbb{R}$ Range $= [-2, \infty)$
c Domain $= [-3, 3]$ Range $= [0, 3]$
d Domain $= \mathbb{R} \setminus \{1\}$ Range $= \mathbb{R} \setminus \{0\}$

3 **a** $\mathbb{R} \setminus \{3\}$ **b** $(-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$
c $\mathbb{R}$ **d** $[4, 11]$ **e** $\mathbb{R} \setminus \{-1\}$
f $(-\infty, -1] \cup [2, \infty)$ **g** $\mathbb{R} \setminus \{-1, 2\}$
h $(-\infty, -2) \cup [1, \infty)$ **i** $\left[0, \frac{1}{3}\right]$
j $[-5, 5]$ **k** $[3, 12]$

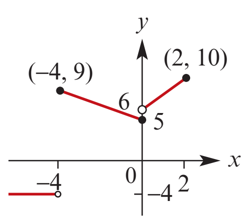
4 **a**



b Range $= [-2, \infty)$

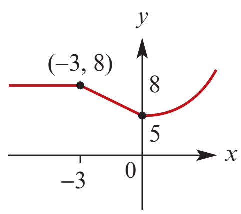
5 **a** Domain $= (-3, 0] \cup [1, 3)$; Range $= [-2, 3)$
b Domain $= [-5, 4]$; Range $= [-4, 0) \cup [2, 5]$

6 **a**



b Domain $= (-\infty, 2]$; Range $= [5, 10] \cup \{-4\}$

7 **a**



b Range $= [5, \infty)$

8 **a** $f(-4) = -8$ **b** $f(0) = 0$
c $f(4) = \frac{1}{4}$

d $f(a+3) = \begin{cases} \frac{1}{a+3}, & a > 0 \\ 2(a+3), & a \le 0 \end{cases}$

e $f(2a) = \begin{cases} \frac{1}{2a}, & a > \frac{3}{2} \\ 4a, & a \le \frac{3}{2} \end{cases}$

f $f(a-3) = \begin{cases} \frac{1}{a-3}, & a > 6 \\ 2(a-3), & a \le 6 \end{cases}$

9 $y = \begin{cases} -x-4, & x < -2 \\ \frac{1}{2}x - 1, & -2 \le x \le 3 \\ -\frac{1}{2}x + 2, & x > 3 \end{cases}$

10 **a** Even **b** Odd **c** Neither
d Even **e** Odd **f** Neither

11 **a** Even **b** Even **c** Odd
d Odd **e** Neither **f** Even
g Neither **h** Neither **i** Even

Exercise 1D

1 **a** $(f+g)(x) = 4x + 2$
$(fg)(x) = 3x^2 + 6x$
dom $= \mathbb{R}$

b $(f+g)(x) = 1$
$(fg)(x) = x^2 - x^4$
dom $= (0, 2]$

c $(f+g)(x) = \frac{x+1}{\sqrt{x}}$
$(fg)(x) = 1$
dom $= [1, \infty)$

d $(f+g)(x) = x^2 + \sqrt{4-x}$
$(fg)(x) = x^2\sqrt{4-x}$
dom $= [0, 4]$

2 **a** **i** $\frac{f}{g}(x) = \frac{3x}{x+2}$, dom $= \mathbb{R} \setminus \{-2\}$
ii $\frac{g}{f}(x) = \frac{x+2}{3x}$, dom $= \mathbb{R} \setminus \{0\}$

b **i** $\frac{f}{g}(x) = \frac{1-x^2}{x^2}$, dom $= (0, 2]$
ii $\frac{g}{f}(x) = \frac{x^2}{1-x^2}$, dom $= (0, 1) \cup (1, 2]$

c **i** $\frac{f}{g}(x) = x$, dom $= [1, \infty)$
ii $\frac{g}{f}(x) = \frac{1}{x}$, dom $= [1, \infty)$

d **i** $\frac{f}{g}(x) = \frac{x^2}{\sqrt{4-x}}$, dom $= [0, 4)$
ii $\frac{g}{f}(x) = \frac{\sqrt{4-x}}{x^2}$, dom $= (0, 4]$

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3 a i Even **ii** Odd **iii** Even **iv** Odd

b $(f+h)(x) = x^2 + 1 + \frac{1}{x^2}$, even;

$(gk)(x) = 1$, even; $(fh)(x) = 1 + \frac{1}{x^2}$, even;

$(f+g)(x) = x^2 + x + 1$, neither;

$(g+k)(x) = x + \frac{1}{x}$, odd;

$(fg)(x) = x^3 + x$, odd

4 a $f(g(x)) = 4x - 1$, $g(f(x)) = 4x - 2$

b $f(g(x)) = 8x + 5$, $g(f(x)) = 8x + 3$

c $f(g(x)) = 4x - 7$, $g(f(x)) = 4x - 5$

d $f(g(x)) = 2x^2 - 1$, $g(f(x)) = (2x-1)^2$

e $f(g(x)) = 2(x-5)^2 + 1$, $g(f(x)) = 2x^2 - 4$

f $f(g(x)) = 2x^2 + 1$, $g(f(x)) = (2x+1)^2$

5 a $f(h(x)) = 6x + 3$ **b** $h(f(x)) = 6x - 1$

c 15 **d** 11 **e** 21 **f** −7 **g** 3

6 a $9x^2 + 12x + 3$ **b** $3x^2 + 6x + 1$

c 120 **d** 46 **e** 3 **f** 1

7 a $h(x) = g(f(x))$, $f(x) = x^2 - 1$, $g(x) = x^4$

b $h(x) = g(f(x))$, $f(x) = x^4 + 3$, $g(x) = \sqrt{x}$

c $h(x) = g(f(x))$, $f(x) = x^2 - 2x$, $g(x) = x^n$

d $h(x) = g(f(x))$, $f(x) = 2x + 3$, $g(x) = \frac{1}{x}$

e $h(x) = g(f(x))$, $f(x) = x^2 - 2x$, $g(x) = x^3 - 2x$

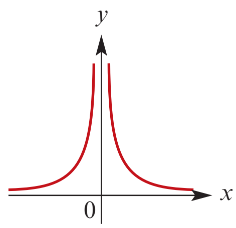
f $h(x) = f(f(x))$, $f(x) = 2x^2 + 1$

Exercise 1E

1 a Maximal domain = $\mathbb{R} \setminus \{0\}$; Range = $\mathbb{R}^+$

b i $\frac{1}{16}$ **ii** $\frac{1}{16}$ **iii** 16 **iv** 16

c

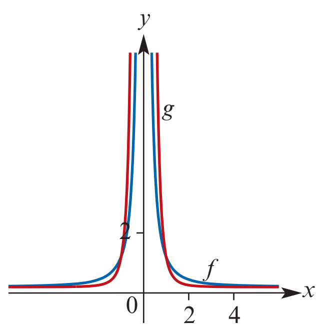


2 a Odd **b** Even **c** Odd

d Odd **e** Even **f** Odd

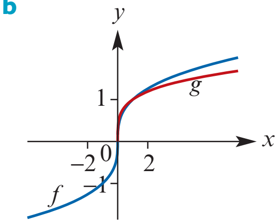
3 a $x = 1$ or $x = -1$

b



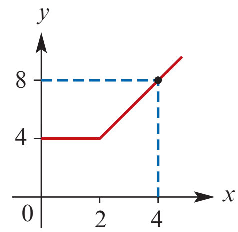
4 a $x = 1$ or $x = 0$

b



Exercise 1F

1 $f(x) = \begin{cases} 4 & \text{if } 0 \le x \le 2 \\ 2x & \text{if } x > 2 \end{cases}$

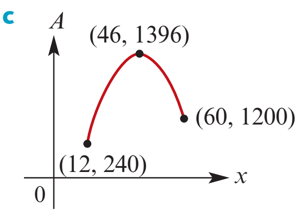


2 $V(x) = 4x(10 - x)(18 - x)$, domain = [0, 10]

3 a $A(x) = -x^2 + 92x - 720$

b $12 \le x \le 60$

c



d Maximum area 1396 m^2 occurs when $x = 46$ and $y = 34$

4 a i $S = 2x^2 + 6xh$ **ii** $S = 2x^2 + \frac{3V}{x}$

b Maximal domain = $(0, \infty)$

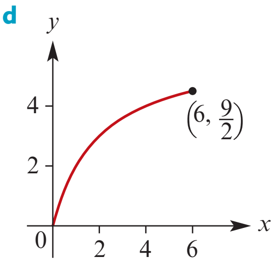
c Maximum value of S = 1508 m^2

5 Area = $x\sqrt{4a^2 - x^2}$, domain = $[0, 2a]$

6 a $A = \frac{6a}{a+2}$

b Domain = [0, 6]; Range = $\left[0, \frac{9}{2}\right]$

c $\frac{9}{2}$ **d**



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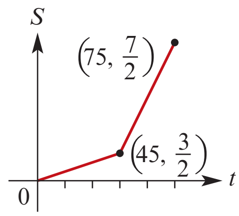
7 **a** $a = \frac{1}{30}$, $b = \frac{1}{15}$, $c = 45$, $d = -\frac{3}{2}$, $e = 75$

b



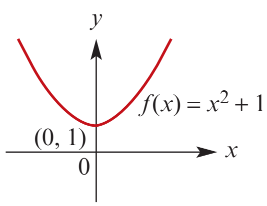
c Range $= \left[0, \frac{7}{2}\right]$

Chapter 1 review

Technology-free questions

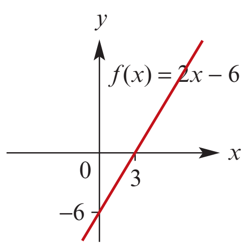
1 **a** Domain $= \mathbb{R}$

Range $= [1, \infty)$



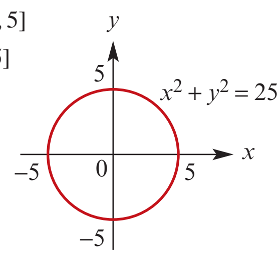
b Domain $= \mathbb{R}$

Range $= \mathbb{R}$



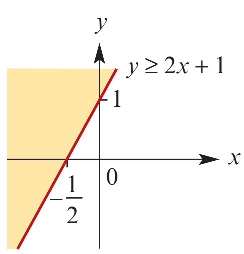
c Domain $= [-5, 5]$

Range $= [-5, 5]$



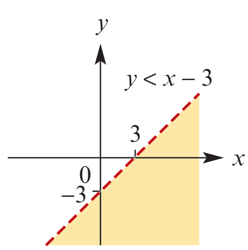
d Domain $= \mathbb{R}$

Range $= \mathbb{R}$

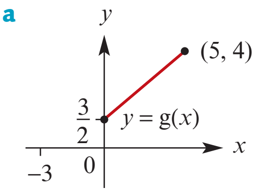


e Domain $= \mathbb{R}$

Range $= \mathbb{R}$



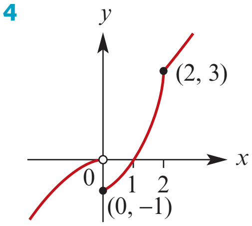
2 **a**



b $\text{ran } g = \left[\frac{3}{2}, 4\right]$ **c** $x = 5$

3 **a** $x = \frac{1}{5}$ **b** $x \leq \frac{1}{5}$ **c** $x = -\frac{1}{10}$

4



5 **a** $\mathbb{R} \setminus \{3\}$ **b** $\mathbb{R} \setminus [-\sqrt{5}, \sqrt{5}]$ **c** $\mathbb{R} \setminus \{1, -2\}$
d $[-5, 5]$ **e** $[5, 15]$ **f** $\mathbb{R} \setminus \{2\}$

6 $(f + g)(x) = x^2 + 5x + 1$,
$(fg)(x) = (x - 3)(x + 2)^2$

7 $(f + g)(x) = x^2 + 1, \ x \in [1, 5]$
$(fg)(x) = 2x(x - 1)^2, \ x \in [1, 5]$

8 **a** $(f + g)(x) = -x^2 + 2x + 3$
b $(fg)(x) = -x^2(2x + 3)$
c $x = -1$ or $x = 3$

9 **a** $f(g(x)) = -2x^3 + 3$
b $g(f(x)) = -(2x + 3)^3$
c $g(g(x)) = x^9$ **d** $f(f(x)) = 4x + 9$

10 **a** $h(x) = g(f(x)), f(x) = x^3 - 1, g(x) = x^{\frac{1}{3}}$
b $h(x) = g(f(x)), f(x) = x^2 + 1, g(x) = \frac{1}{x}$
c $h(x) = g(f(x)), f(x) = x^2 - 1, g(x) = \frac{1}{x}$

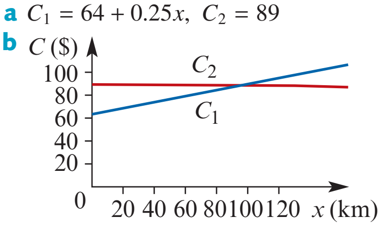
Multiple-choice questions

1 E	**2** B	**3** E	**4** E	**5** C
6 E	**7** C	**8** B	**9** C	**10** B
11 B	**12** C	**13** C	**14** D	

Extended-response questions

1 **a** $C_1 = 64 + 0.25x$, $C_2 = 89$

b



c $x > 100$ km

2 **a** $S = 6x^2$ **b** $S = 6V^{\frac{2}{3}}$

3 **a** $A = \frac{\sqrt{3}s^2}{4}$ **b** $A = \frac{\sqrt{3}h^2}{3}$

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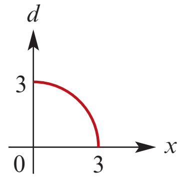
4 a $d(x) = \sqrt{9 - x^2}$ **b** dom $= [0, 3]$, ran $= [0, 3]$

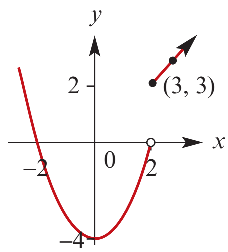


5 $S(x) = \dfrac{160x}{x + 80}$

6 a $V_1(h) = \pi h\left(36 - \dfrac{h^2}{4}\right),\ h \in (0, 12)$

b $V_2(r) = 2\pi r^2\sqrt{36 - r^2},\ r \in (0, 6)$

7 a



b $f(-1) = -3,\ f(3) = 3$

c $f(h(x)) = \begin{cases} 4x^2 - 4 & \text{if } x < 1 \\ 2x & \text{if } x \geq 1 \end{cases}$

$h(f(x)) = \begin{cases} 2x^2 - 8 & \text{if } x < 2 \\ 2x & \text{if } x \geq 2 \end{cases}$

8 $A(t) = \begin{cases} \dfrac{3t^2}{2}, & 0 < t \leq 1 \\ 3t - \dfrac{3}{2}, & t > 1 \end{cases}$

Domain $= (0, \infty)$; Range $= (0, \infty)$

9 a i $YB = r$ **ii** $ZB = r$ **iii** $AZ = x - r$ **iv** $CY = 3 - r$

b $r = \dfrac{x + 3 - \sqrt{x^2 + 9}}{2}$

c i $r = 1$ **ii** $x = 1.25$

10 b $f(x) = \dfrac{q}{x}$ **c** $x = 3 \pm \sqrt{17}$

11 a i $f(2) = 3,\ f(f(2)) = 2,\ f(f(f(2))) = 3$

ii $f(f(x)) = x$

b $f(f(x)) = \dfrac{-x - 3}{x - 1},\ f(f(f(x))) = x$

Chapter 2

Exercise 2A

1 a $\sqrt{205}$ **b** $\left(1, -\dfrac{1}{2}\right)$ **c** $-\dfrac{13}{6}$

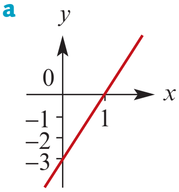
d $13x + 6y = 10$ **e** $13x + 6y = 43$

f $13y - 6x = -\dfrac{25}{2}$

2 a $(3, 7\frac{1}{2})$ **b** $\left(-\dfrac{5}{2}, -2\right)$ **c** $\left(\dfrac{3}{2}, \dfrac{1}{2}\right)$

3 a $(4, 7)$ **b** $(5, -2)$ **c** $(2, 19)$

4 a



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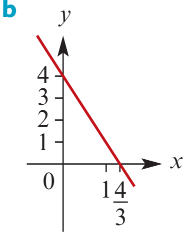

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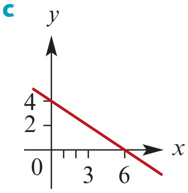
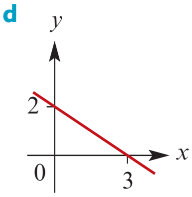
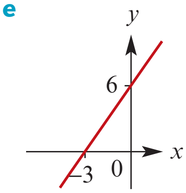


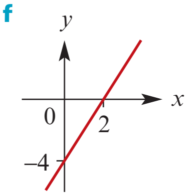
d



e



f

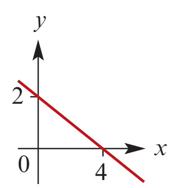


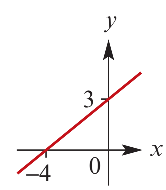
5 a $y = 2x - 6$ **b** $y = -3x - 5$

c $3y - 4x = 5$ **d** $y = 2x + 1$

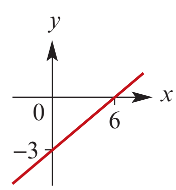
6 a $\dfrac{y}{2} - \dfrac{x}{3} = 1$ **b** $\dfrac{x}{4} + \dfrac{y}{6} = 1$ **c** $\dfrac{x}{6} - \dfrac{y}{2} = 1$

7 a $\dfrac{x}{4} + \dfrac{y}{2} = 1$ **b** $\dfrac{y}{3} - \dfrac{x}{4} = 1$





c $\dfrac{x}{6} - \dfrac{y}{3} = 1$



8 $C = \dfrac{11}{200}n + 2$, \$57

9 a $C = 5n + 175$ **b** Yes **c** \$175

10 a $\sqrt{5} \approx 2.236$ **b** $\sqrt{20} \approx 4.472$ **c** 5

11 a i $y = 2x + 4$ **ii** $2y + x = 13$

b i $y = -2x + 7$ **ii** $2y - x = 4$

12 $y = 2x - 3$

13 $y = -5$ or $y = 3$

14 a $32.01°$ **b** $153.43°$ **c** $56.31°$ **d** $120.96°$

15 $45°$

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16 $a = -12$ or $a = 8$

17 a $y = 3x - 6$ **b** $(2, 0)$

18 $k = 5$ and $h = 4$, or $k = -2$ and $h = -3$

19 a $a + 2$ **b** $\frac{4}{5}$

20 a $m = \frac{1}{2}$ **b** $(5, 7)$

c $AB = \sqrt{13}$, $AC = 2\sqrt{13}$

21 a $(2, 3)$

b $y + 5x = 13$

c i $2y = 3x - 13$ **ii** $(3, -2)$ **iii** $(1, 8)$

Exercise 2B

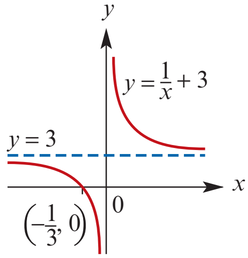
1 a $(-1, 3)$ **b** $(-5, 10)$ **c** $(-5, 7)$ **d** $(-3, 6)$

2 a $y = \frac{1}{x-2} - 3$ **b** $y = \frac{1}{x+2} + 3$

c $y = \frac{1}{x - \frac{1}{2}} + 4 = \frac{2}{2x-1} + 4$

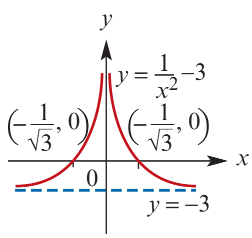
3 a Domain $= \mathbb{R} \setminus \{0\}$

Range $= \mathbb{R} \setminus \{3\}$



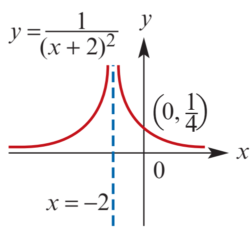
b Domain $= \mathbb{R} \setminus \{0\}$

Range $= (-3, \infty)$



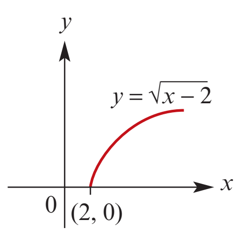
c Domain $= \mathbb{R} \setminus \{-2\}$

Range $= \mathbb{R}^+$



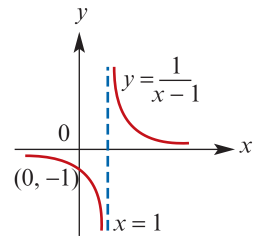
d Domain $= [2, \infty)$

Range $= [0, \infty)$



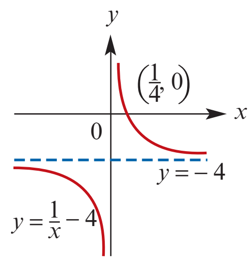
e Domain $= \mathbb{R} \setminus \{1\}$

Range $= \mathbb{R} \setminus \{0\}$



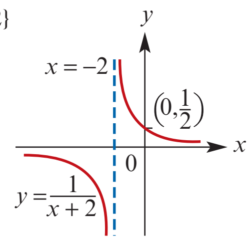
f Domain $= \mathbb{R} \setminus \{0\}$

Range $= \mathbb{R} \setminus \{-4\}$



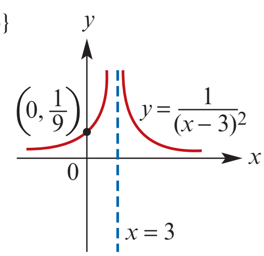
g Domain $= \mathbb{R} \setminus \{-2\}$

Range $= \mathbb{R} \setminus \{0\}$



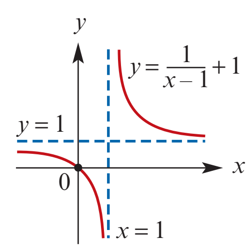
h Domain $= \mathbb{R} \setminus \{3\}$

Range $= \mathbb{R}^+$

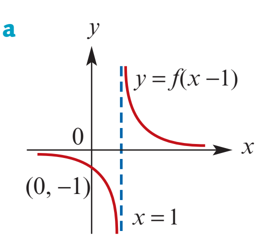


i Domain $= \mathbb{R} \setminus \{1\}$

Range $= \mathbb{R} \setminus \{1\}$



4 a



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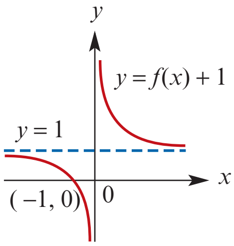
b



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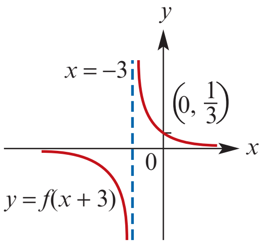

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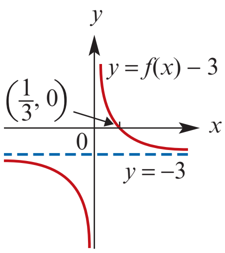

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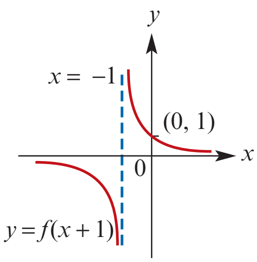

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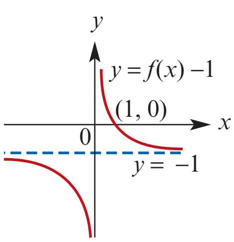


5 **a** Translation $(x, y) \to (x - 5, y)$
b Translation $(x, y) \to (x, y + 2)$
c Translation $(x, y) \to (x, y + 4)$
d Translation $(x, y) \to (x, y + 3)$
e Translation $(x, y) \to (x - 3, y)$

6 **a** **i** $y = (x - 7)^{\frac{1}{4}} + 1$ **ii** $y = (x + 2)^{\frac{1}{4}} - 6$
iii $y = (x - 2)^{\frac{1}{4}} - 3$ **iv** $y = (x + 1)^{\frac{1}{4}} + 4$
b **i** $y = \sqrt[3]{x - 7} + 1$ **ii** $y = \sqrt[3]{x + 2} - 6$
iii $y = \sqrt[3]{x - 2} - 3$ **iv** $y = \sqrt[3]{x + 1} + 4$
c **i** $y = \dfrac{1}{(x - 7)^3} + 1$ **ii** $y = \dfrac{1}{(x + 2)^3} - 6$
iii $y = \dfrac{1}{(x - 2)^3} - 3$ **iv** $y = \dfrac{1}{(x + 1)^3} + 4$
d **i** $y = \dfrac{1}{(x - 7)^4} + 1$ **ii** $y = \dfrac{1}{(x + 2)^4} - 6$
iii $y = \dfrac{1}{(x - 2)^4} - 3$ **iv** $y = \dfrac{1}{(x + 1)^4} + 4$

7 **a** $y = (x + 1)^2 + 5$ **b** $y = 2x^2$
c $y = \dfrac{1}{(x - 6)^2} + 1$

8 **a** $(x, y) \to (x + 2, y + 3)$
b $(x, y) \to (x - 2, y - 3)$
c $(x, y) \to (x - 4, y + 2)$

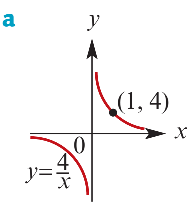
Exercise 2C

1 **a** $y = \dfrac{3}{x}$ **b** $y = \dfrac{3}{x}$

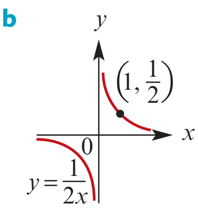
2 **a** $y = 2\sqrt{x}$ **b** $y = \sqrt{\dfrac{x}{2}}$

3 **a** $y = 2x^3$ **b** $y = \dfrac{x^3}{8}$

4 **a**



b



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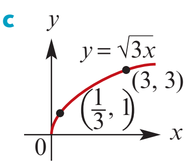
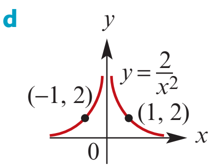
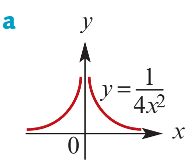


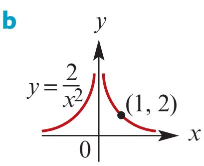
d



5 **a**



b



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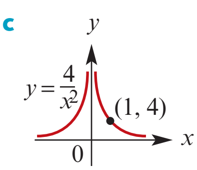

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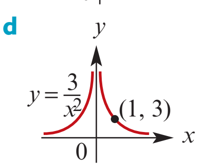

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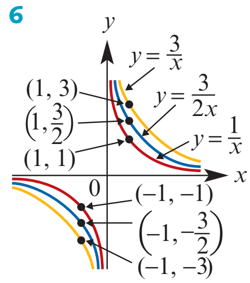

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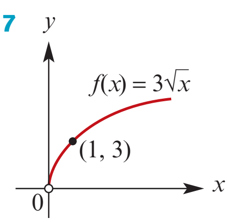

Range $= \mathbb{R}^+$

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8 **a** $\frac{1}{5}$ **b** $\sqrt{5}$

9 **a** Dilation of factor 5 from the x-axis

b Dilation of factor $\frac{1}{5}$ from the y-axis

c Dilation of factor $\frac{1}{3}$ from the y-axis

d Dilation of factor 2 from the y-axis

10 **a** **i** $y = 4x^2$ **ii** $y = \frac{2}{3}x^2$

iii $y = 4x^2$ **iv** $y = \frac{1}{25}x^2$

b **i** $y = \frac{4}{x^2}$ **ii** $y = \frac{2}{3x^2}$

iii $y = \frac{1}{4x^2}$ **iv** $y = \frac{25}{x^2}$

c **i** $y = 4\sqrt[3]{x}$ **ii** $y = \frac{2}{3} \times \sqrt[3]{x}$

iii $y = \sqrt[3]{2x}$ **iv** $y = \sqrt[3]{\frac{x}{5}}$

d **i** $y = \frac{4}{x^4}$ **ii** $y = \frac{2}{3x^4}$

iii $y = \frac{1}{16x^4}$ **iv** $y = \frac{625}{x^4}$

e **i** $y = 4x^{\frac{1}{5}}$ **ii** $y = \frac{2}{3}x^{\frac{1}{5}}$

iii $y = (2x)^{\frac{1}{5}}$ **iv** $y = \left(\frac{x}{5}\right)^{\frac{1}{5}}$

11 **a** $y = -(x-1)^2$ **b** $y = (x+1)^2$

12 Reflection in the y-axis

13 **a** **i** $y = -x^3$ **ii** $y = -x^3$

b **i** $y = -\sqrt[3]{x}$ **ii** $y = -\sqrt[3]{x}$

c **i** $y = \frac{-1}{x^3}$ **ii** $y = \frac{-1}{x^3}$

d **i** $y = \frac{-1}{x^4}$ **ii** $y = \frac{1}{x^4}$

e **i** $y = -x^{\frac{1}{4}}$ **ii** $y = (-x)^{\frac{1}{4}}$

Exercise 2D

1 **a** **i** $y = 2(x-2)^2 - 3$ **ii** $y = \left(\frac{x+2}{3}\right)^2 - 4$

iii $y = 2x^2$

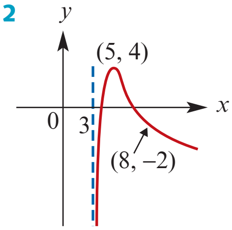
b **i** $y = 2\sqrt[3]{x-2} - 3$ **ii** $y = \sqrt[3]{\frac{x+2}{3}} - 4$

iii $y = -2\sqrt[3]{x}$

c **i** $y = \frac{2}{(x-2)^2} - 3$ **ii** $y = \frac{9}{(x+2)^2} - 4$

iii $y = \frac{2}{x^2}$

2



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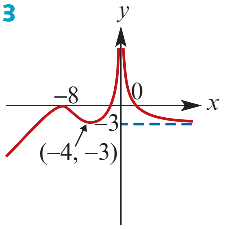


4 **a** **i** $y = -2(x-3)^2 - 4$

ii $y = -2(x-3)^2 + 4$

iii $y = -2(x-3)^2 - 4$

iv $y = -2(x-3)^2 - 8$

v $y = -2(x-3)^2 + 8$

vi $y = -2(x-3)^2 + 8$

b **i** $y = -2\sqrt[3]{x-3} - 4$

ii $y = -2\sqrt[3]{x-3} + 4$

iii $y = -2\sqrt[3]{x-3} - 4$

iv $y = -2\sqrt[3]{x-3} - 8$

v $y = -2\sqrt[3]{x-3} + 8$

vi $y = -2\sqrt[3]{x-3} + 8$

c **i** $y = \frac{-2}{x-3} - 4$ **ii** $y = \frac{-2}{x-3} + 4$

iii $y = \frac{-2}{x-3} - 4$ **iv** $y = \frac{-2}{x-3} - 8$

v $y = \frac{-2}{x-3} + 8$ **vi** $y = \frac{-2}{x-3} + 8$

d **i** $y = \frac{-2}{(x-3)^3} - 4$ **ii** $y = \frac{-2}{(x-3)^3} + 4$

iii $y = \frac{-2}{(x-3)^3} - 4$ **iv** $y = \frac{-2}{(x-3)^3} - 8$

v $y = \frac{-2}{(x-3)^3} + 8$ **vi** $y = \frac{-2}{(x-3)^3} + 8$

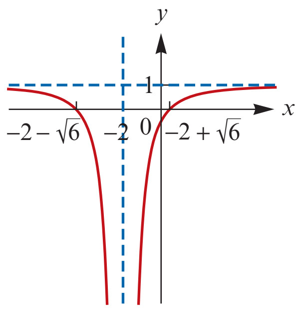
e **i** $y = \frac{-2}{(x-3)^2} - 4$ **ii** $y = \frac{-2}{(x-3)^2} + 4$

iii $y = \frac{-2}{(x-3)^2} - 4$ **iv** $y = \frac{-2}{(x-3)^2} - 8$

v $y = \frac{-2}{(x-3)^2} + 8$ **vi** $y = \frac{-2}{(x-3)^2} + 8$

5 $y = -\sqrt{\frac{x+12}{3}}$

6 **a**



b $y = 1 - \frac{6}{(x+2)^2}$

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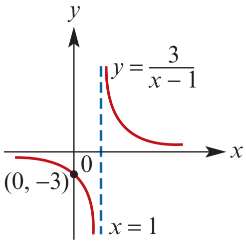
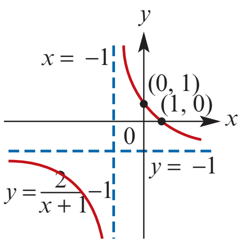
Exercise 2E

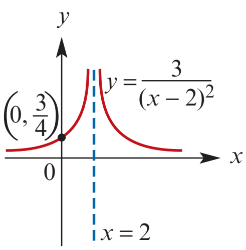
1 a Range = $\mathbb{R} \setminus \{0\}$



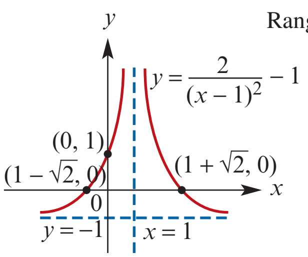
b Range = $\mathbb{R} \setminus \{-1\}$



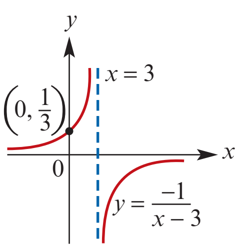
c Range = $\mathbb{R}^+$



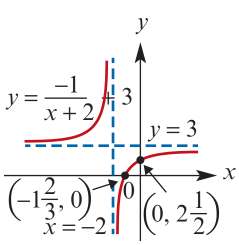
d Range = $(-1, \infty)$



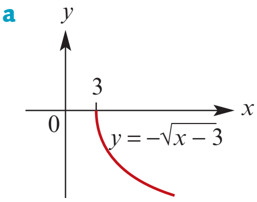
e Range = $\mathbb{R} \setminus \{0\}$



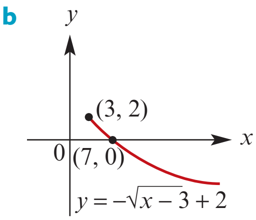
f Range = $\mathbb{R} \setminus \{3\}$



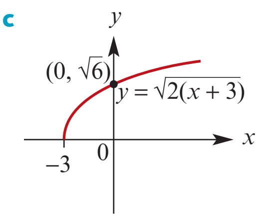
2 a Range = $(-\infty, 0]$



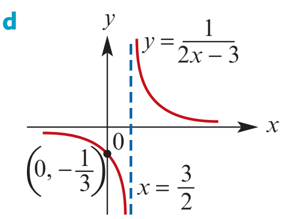
b Range = $(-\infty, 2]$



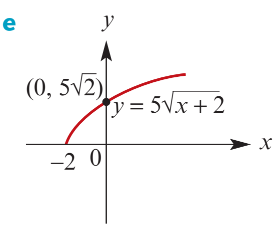
c Range = $[0, \infty)$



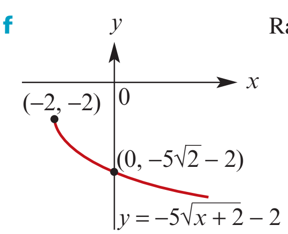
d Range = $\mathbb{R} \setminus \{0\}$



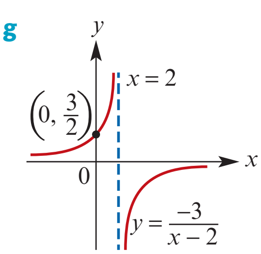
e Range = $[0, \infty)$



f Range = $(-\infty, -2]$



g Range = $\mathbb{R} \setminus \{0\}$



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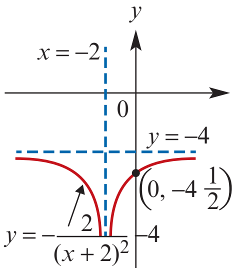
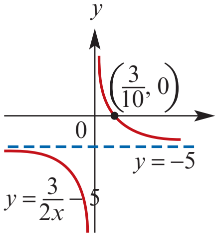
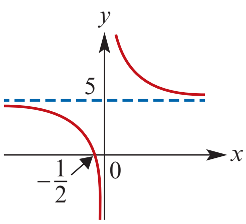
h Range = $(-\infty, -4)$



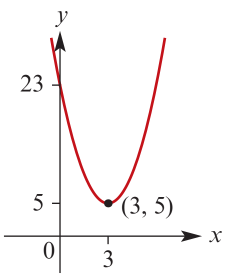
i Range = $\mathbb{R} \setminus \{-5\}$



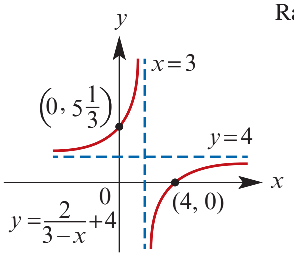
j Range = $\mathbb{R} \setminus \{5\}$



k Range = $[5, \infty)$



l Range = $\mathbb{R} \setminus \{4\}$



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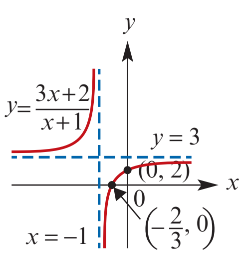

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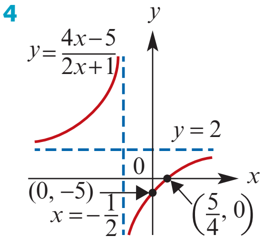
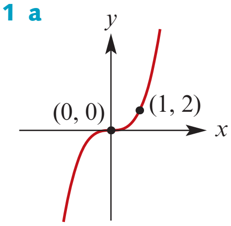
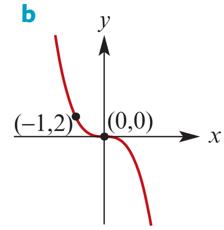


Exercise 2F

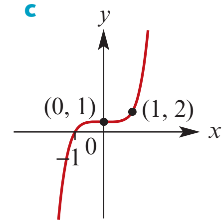
1 a



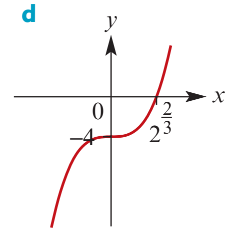
b



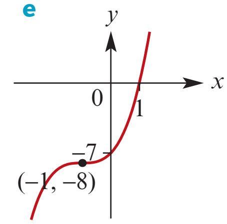
c



d



e



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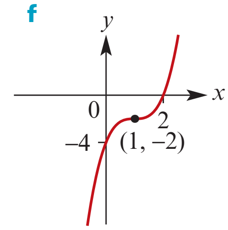

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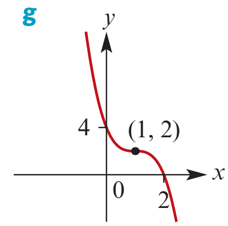

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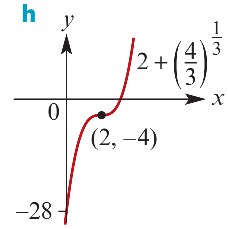

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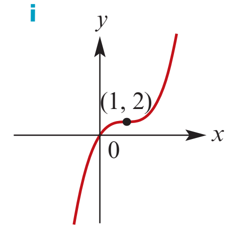

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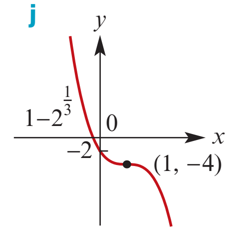

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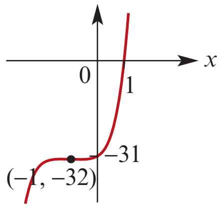
k



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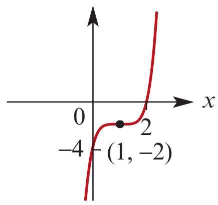


2 $a = -3,\ h = 0,\ k = 4$

3 **a** $y = 3x^3$ **b** $y = (x+1)^3 + 1$

c $y = -(x-2)^3 - 3$ **d** $y = 2(x+1)^3 - 2$

e $y = \dfrac{x^3}{27}$

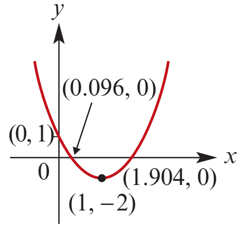
4 **a** $y = \dfrac{(3-x)^3}{27} + 1$

b Dilation of factor 3 from the x-axis, reflection in the x-axis, then translation 1 unit to the left and 4 units up

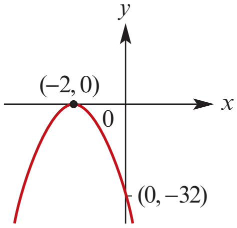
5 **a** $y = \dfrac{(x+2)^4}{16} - 1$

b Dilation of factor 3 from the x-axis, reflection in the x-axis, then translation 1 unit to the left and 5 units up

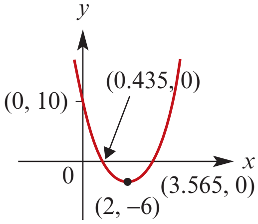
6 **a**



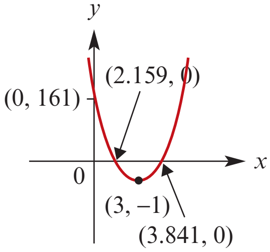
b



c



d



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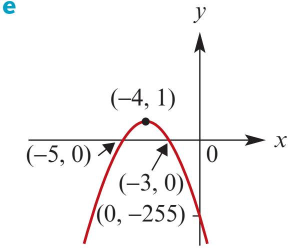

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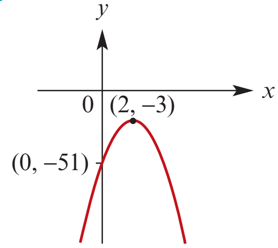


7 $a = -\dfrac{9}{16},\ h = -2,\ k = 3$

8 $a = 16,\ h = 1,\ k = 7$

Exercise 2G

1 $a = \dfrac{9}{2}, b = -\dfrac{1}{2}$ **2** $A = 1, b = -1, B = 2$

3 $a = \dfrac{5}{2}, b = -\dfrac{3}{2}$ **4** $A = 2, B = 3$

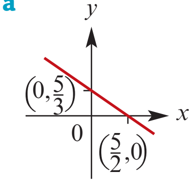
5 $A = 2, B = -1$ **6** $A = 8, b = 2, B = -3$

7 $a = -2, b = 1$ **8** $a = -6, b = -2$

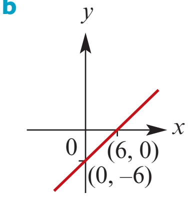
Chapter 2 review

Technology-free questions

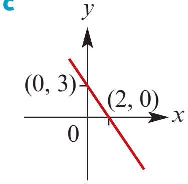
1 **a**



b



c



2 $\sqrt{13}$

3 $(1, 7)$

4 **a** $y = -2x + 5$ **b** $y = 2x + 2$

c $y = \dfrac{1}{2}x + \dfrac{1}{2}$ **d** $y = -2x + 3$

5 **a** $(22, 4)$ **b** $(5, -12)$

6 $y = 24$ or $y = 0$

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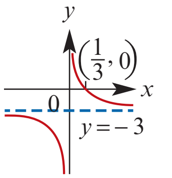
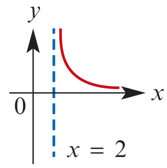
7 a



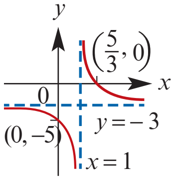
Range = $\mathbb{R} \setminus \{-3\}$

b



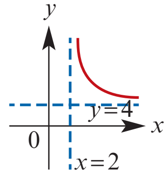
Range = $(0, \infty)$

c



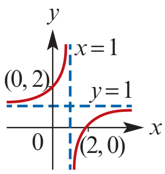
Range = $\mathbb{R} \setminus \{-3\}$

d



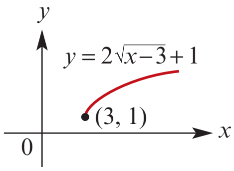
Range = $(4, \infty)$

e

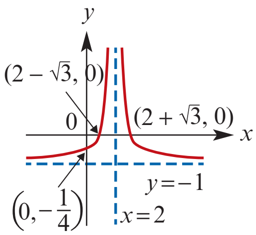


Range = $\mathbb{R} \setminus \{1\}$

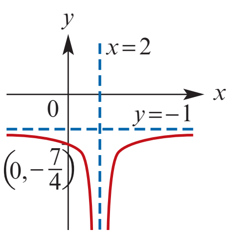
8 a



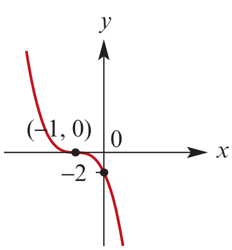
b



c

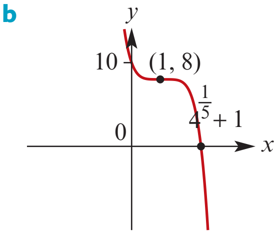


9 a



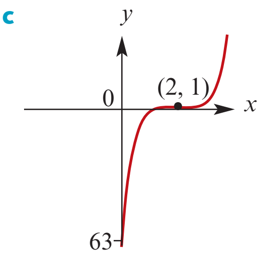
Point of zero gradient $(-1, 0)$;
Axis intercepts $(-1, 0)$, $(0, -2)$

b



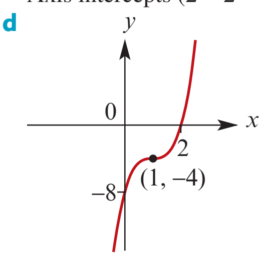
Point of zero gradient $(1, 8)$;
Axis intercepts $(4^{\frac{1}{5}} + 1, 0)$, $(0, 10)$

c



Point of zero gradient $(2, 1)$;
Axis intercepts $(2 - 2^{-\frac{1}{5}}, 0)$, $(0, -63)$

d



Point of zero gradient $(1, -4)$;
Axis intercepts $(2, 0)$, $(0, -8)$

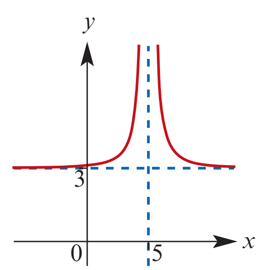
10 $a = 2$, $b = 4$

11 $a = -6$, $b = 9$

12 a $y = 6 - \dfrac{(x-4)^2}{4}$

b Reflection in the x-axis, dilation of factor 4 from the x-axis, then translation 1 unit to the right and 6 units up

13 Dilation of factor 3 from the x-axis, then translation 5 units to the right and 3 units up



Asymptotes $x = 5$, $y = 3$; Intercept $\left(0, \dfrac{78}{25}\right)$

14 Dilation of factor $\frac{1}{2}$ from the x-axis, then translation $\frac{3}{2}$ units up

15 Dilation of factor $\frac{1}{2}$ from the x-axis, then translation 3 units to the left and 2 units down

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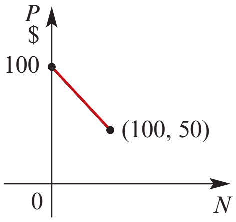
Multiple-choice questions

1 E **2** E **3** D **4** C **5** A **6** A
7 C **8** B **9** C **10** D **11** C **12** B
13 B **14** E **15** D **16** B **17** A

Extended-response questions

1 a

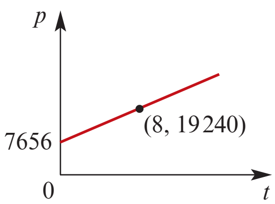


b $P = -\frac{1}{2}N + 100$

c i \$56 **ii** $N = 80$

2 a $p = 1448t + 7656$

b



c 22 136 **d** 1448 people per year

3 a $y = \frac{5}{3}x - 4$ **b** $\left(\frac{66}{7}, \frac{82}{7}\right)$

c $\frac{5}{3}$ **d** 15 **e** $\frac{629}{14}$ square units

4 a $y = \frac{4}{7}x + \frac{31}{14}$ **b** $\frac{59}{14}$ **c** $\sqrt{65}$

d $\frac{65}{28}$ square units

5 a $(1, -\frac{1}{2})$ **b** $\sqrt{269}$ **c** $\sqrt{269}$

d $y = -\frac{13}{10}x + \frac{4}{5}$ **e** $y = \frac{10}{13}x - \frac{33}{26}$

f $\left(\frac{7}{2}, -\frac{15}{4}\right)$ **g** $(26, -33)$

6 a i $\frac{3}{125}$ **ii** $(x, y) \to (x, -y)$

iii $(x, y) \to (x + 25, y + 15)$

iv $(x, y) \to \left(x + 25, \frac{-3}{125}y + 15\right)$

b i $y = \frac{-3}{125}(x - 25)^2 + 15$

ii $(x, y) \to (x + 50, y)$

iii $y = \frac{-3}{125}(x - 75)^2 + 15$

c i $(x, y) \to \left(x + \frac{m}{2}, -\frac{4n}{m^2}y + n\right)$

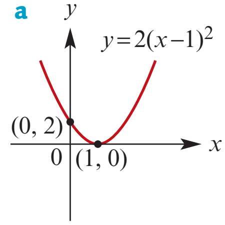
ii $y = -\frac{4n}{m^2}\left(x - \frac{m}{2}\right)^2 + n$

iii $y = -\frac{4n}{m^2}\left(x - \frac{3m}{2}\right)^2 + n$

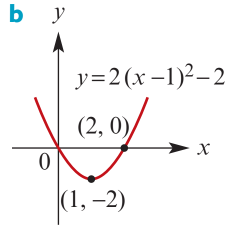
Chapter 3

Exercise 3A

1 a



b



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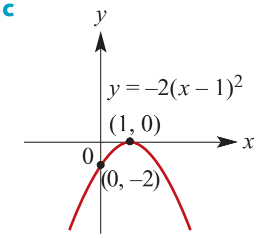
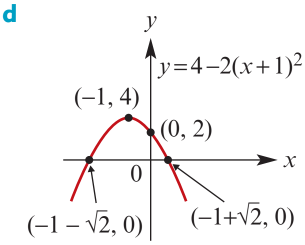
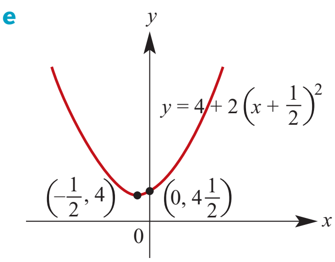


d



e



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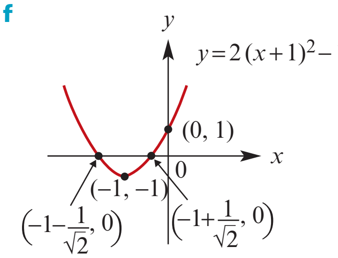
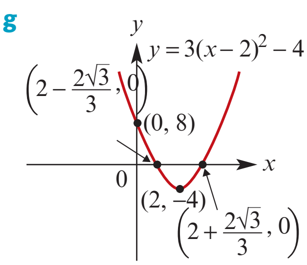


g



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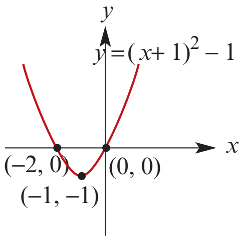
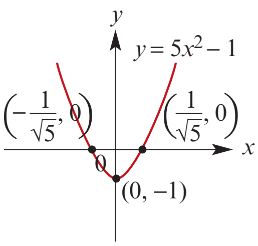
h



i



2 a $f(x) = (x + \frac{3}{2})^2 - 4\frac{1}{4}$

Minimum $= -4\frac{1}{4}$; Range $= [-4\frac{1}{4}, \infty)$

b $f(x) = (x - 3)^2 - 1$

Minimum $= -1$; Range $= [-1, \infty)$

c $f(x) = 2(x + 2)^2 - 14$

Minimum $= -14$; Range $= [-14, \infty)$

d $f(x) = 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8}$

Minimum $= -\frac{25}{8}$; Range $= \left[-\frac{25}{8}, \infty\right)$

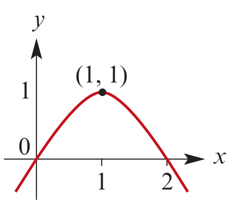
e $f(x) = -3\left(x + \frac{1}{3}\right)^2 + \frac{22}{3}$

Maximum $= \frac{22}{3}$; Range $= \left(-\infty, \frac{22}{3}\right]$

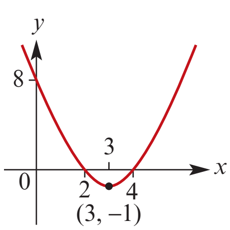
f $f(x) = -2\left(x - \frac{9}{4}\right)^2 + \frac{169}{8}$

Maximum $= \frac{169}{8}$; Range $= \left(-\infty, \frac{169}{8}\right]$

3 a



b



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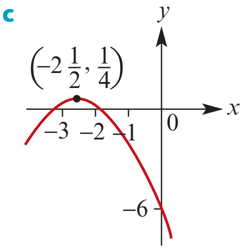
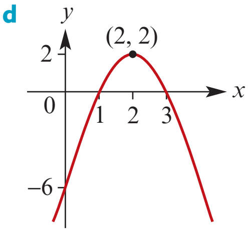
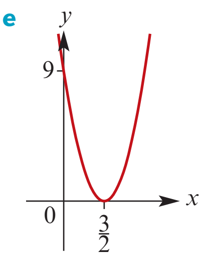


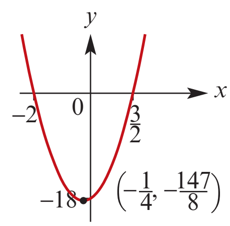
d



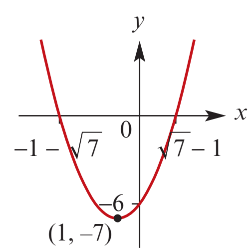
e



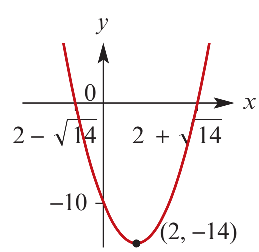
f



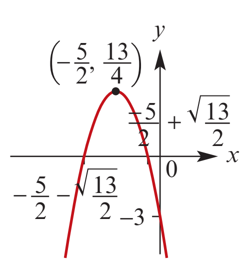
4 a $y = (x + 1)^2 - 7$



b $y = (x - 2)^2 - 14$



c $y = \frac{13}{4} - \left(x + \frac{5}{2}\right)^2$



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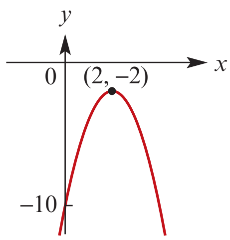
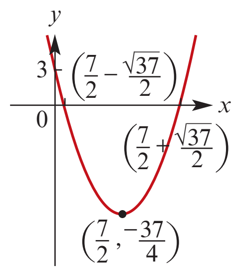
d $y = -2(x-2)^2 - 2$



e $y = \left(x - \frac{7}{2}\right)^2 - \frac{37}{4}$



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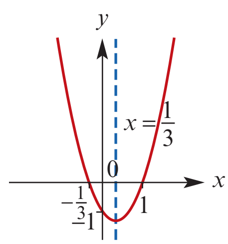

6 **a** C **b** B **c** D **d** A

7 **a** Crosses the x-axis
b Does not cross the x-axis
c Just touches the x-axis
d Crosses the x-axis
e Does not cross the x-axis
f Does not cross the x-axis

8 **a** $m > 3$ or $m < 0$ **b** $m = 3$

9 $m = 2$ or $m = -\frac{2}{9}$

10 $a < -6$

11 Show that $\Delta > 0$ for all a

12 Show that $\Delta \geq 0$ for all k

13 **a** $k < -5$ or $k > 0$ **b** $k = -5$

14 **a** $k > -6$ **b** $k = -6$

15 Show that $\Delta \geq 0$ for all a, b

Exercise 3B

1 $y = -2(x+3)(x+2)$ **2** $y = (x+3)(2x+3)$

3 $y = \frac{3}{2}(x+2)^2 + 4$ **4** $y = -2(x+2)^2 - 3$

5 $y = -5x^2 + 6x + 18$ **6** $y = -2x^2 - 8x + 10$

7 **a** $y = 4 - \frac{4}{25}x^2$ **b** $y = -x^2$
c $y = x^2 + 2x$ **d** $y = 2x - x^2$
e $y = x^2 - 5x + 4$ **f** $y = x^2 - 4x - 5$
g $y = x^2 - 2x - 1$ **h** $y = x^2 - 4x + 6$

8 $y = -\frac{1}{8}x^2 + x + 1,\ y = \frac{1}{8}x^2 + x - 5$

9 $A = 1,\ b = 2,\ B = 4$

Exercise 3C

1 **a** 3 **b** −5 **c** 7 **d** −21 **e** $\frac{17}{8}$ **f** $-\frac{9}{8}$

2 **a** 6 **b** 6 **c** 18 **d** 12
e $a^3 + 3a^2 - 4a + 6$ **f** $8a^3 + 12a^2 - 8a + 6$

3 **a** $a = -5$ **b** $a = \frac{40}{9}$ **c** $c = 8$
d $a = -23,\ b = -4$ **e** $a = -17,\ b = 42$

4 **a** $2x^3 - x^2 + 2x + 2$ **b** $2x^3 + 5x$
c $2x^3 - x^2 + 4x - 2$ **d** $6x^3 - 3x^2 + 9x$
e $-2x^4 + 5x^3 - 5x^2 + 6x$ **f** $4x - x^3$
g $2x^3 + 4x + 2$
h $2x^5 + 3x^4 + x^3 + 6x^2$

5 **a** $x^3 - 5x^2 + 10x - 8$ **b** $x^3 - 7x^2 + 13x - 15$
c $2x^3 - x^2 - 7x - 4$
d $x^2 + (b+2)x^2 + (2b+c)x + 2c$
e $2x^3 - 9x^2 - 2x + 3$

6 **a** $x^3 + (b+1)x^2 + (c+b)x + c$
b $b = -2$ and $c = -4$
c $(x+1)(x+\sqrt{5}-1)(x-\sqrt{5}-1)$

7 **a** $a = 2$ and $b = 5$
b $a = -2,\ b = -2$ and $c = -3$

8 $A = 1,\ B = 3$

9 **a** $A = 1,\ B = -2,\ C = 6$
b $A = 4,\ B = -\frac{3}{2},\ C = 5$
c $A = 1,\ B = -3,\ C = 5$

Exercise 3D

1 **a** $x^2 - 5x + 6$ **b** $2x^2 + 7x - 4$

2 **a** $x^2 - 4x - 3 + \frac{34}{x+3}$
b $2x^2 + 6x + 14 + \frac{54}{x-3}$

3 **a** $x^2 - \frac{5}{2}x - \frac{15}{4} + \frac{145}{4(2x+3)}$
b $2x^2 + 6x + 7 + \frac{33}{2x-3}$

4 **a** $2x^2 - x + 12 + \frac{33}{x-3}$
b $5x^4 + 8x^3 - 8x^2 + 6x - 6$

5 **a** $x^2 - 9x + 27 - \frac{26(x-2)}{x^2-2}$ **b** $x^2 + x + 2$

6 **a** −16 **b** $a = 4$

7 **a** 28 **b** 0 **c** $(x+2)(3x+1)(2x-3)$

8 **b** $k = \frac{11}{2}$

9 **a** $a = 3,\ b = 8$ **b** $2x - 1,\ x - 1$

10 $a = \frac{-92}{9},\ b = 9$

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11 81

12 **b** $6x - 4$

13 $x - 3,\ 2x - 1$

14 **b** $x^2 - 3,\ x^2 + x + 2$

15 **a** $(2a + 3b)(4a^2 - 6ab + 9b^2)$
b $(4 - a)(a^2 + 4a + 16)$
c $(5x + 4y)(25x^2 - 20xy + 16y^2)$
d $2a(a^2 + 3b^2)$

16 **a** $-4, 2, 3$ **b** $0, 2$ **c** $\frac{1}{2}, 2$ **d** $-2, 2$
e $0, -2, 2$ **f** $0, -3, 3$
g $1, -2, -\frac{1}{4}, \frac{1}{3}$ **h** $1, -2$ **i** $1, -2, \frac{1}{3}, \frac{3}{2}$

17 **a** $(-1, 0),\ (0, 0),\ (2, 0)$
b $(-2, 0),\ (0, 6),\ (1, 0),\ (3, 0)$
c $(-1, 0),\ (0, 6),\ (2, 0),\ (3, 0)$
d $\left(-\frac{1}{2}, 0\right),\ (0, 2),\ (1, 0),\ (2, 0)$
e $(-2, 0),\ (-1, 0),\ (0, -2),\ (1, 0)$
f $(-1, 0),\ \left(-\frac{2}{3}, 0\right),\ (0, -6),\ (3, 0)$
g $(-4, 0),\ (0, -16),\ \left(-\frac{2}{5}, 0\right),\ (2, 0)$
h $\left(-\frac{1}{2}, 0\right), (0, 1), \left(\frac{1}{3}, 0\right),\ (1, 0)$
i $(-2, 0), \left(-\frac{3}{2}, 0\right),\ (0, -30),\ (5, 0)$

18 $p = 1, q = -6$

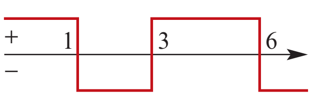
19 -33

20 **a** $(x - 9)(x - 13)(x + 11)$
b $(x + 11)(x - 9)(x - 11)$
c $(x + 11)(2x - 9)(x - 11)$
d $(x + 11)(2x - 13)(2x - 9)$

21 **a** $(x - 1)(x + 1)(x - 7)(x + 6)$
b $(x - 3)(x + 4)(x^2 + 3x + 9)$

22 **a** $(x - 9)(x - 5)(2x^2 + 3x + 9)$
b $(x + 5)(x + 9)(x^2 - x + 9)$
c $(x - 3)(x + 5)(x^2 + x + 9)$
d $(x - 4)(x - 3)(x + 5)(x + 6)$

Exercise 3E

1 **a**


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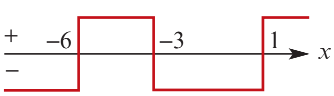

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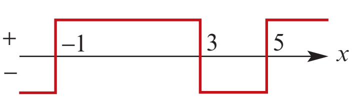

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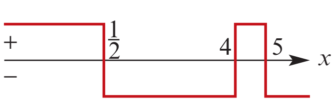

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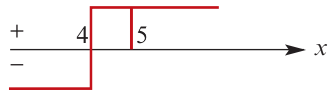

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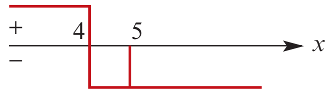
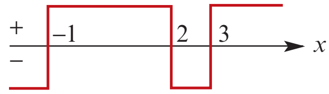
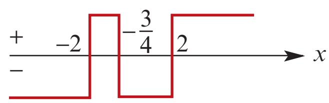


2 **a**


b


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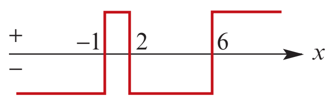
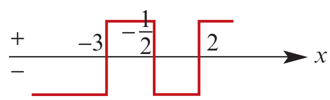
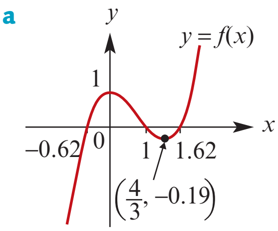
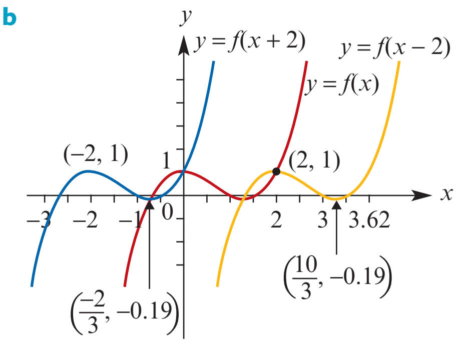
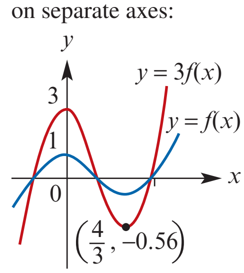


d


3 **a**


b


For clarity the graph of $y = 3f(x)$ is shown on separate axes:



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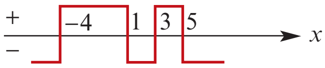
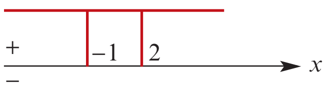
Exercise 3F

1 a



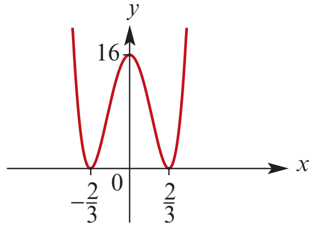
b



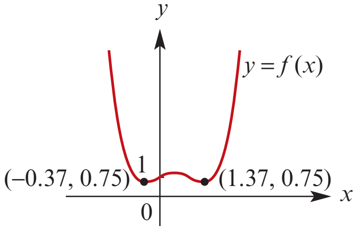
2 (0, 16)

$\left(\frac{2}{3}, 0\right)$

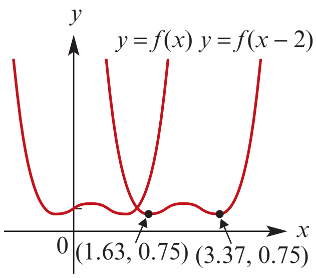
$\left(-\frac{2}{3}, 0\right)$



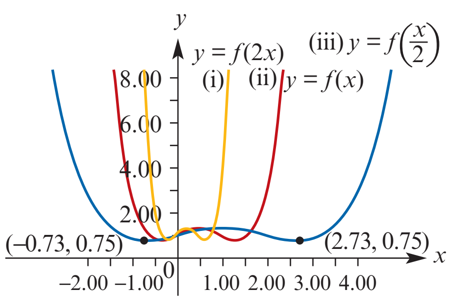
3 a



b

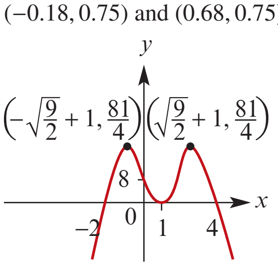


For clarity the graphs of the dilations are shown on separate axes:

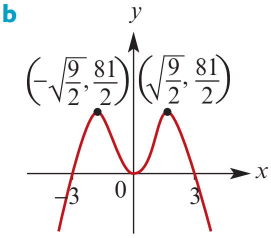


Turning points for $y = f(2x)$ are at $(-0.18, 0.75)$ and $(0.68, 0.75)$

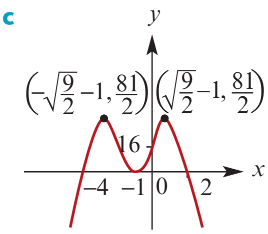
4 a



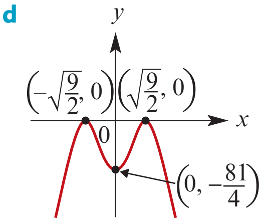
b



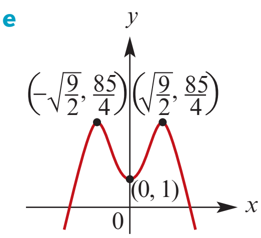
c



d



e



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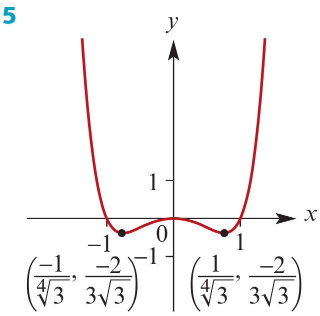

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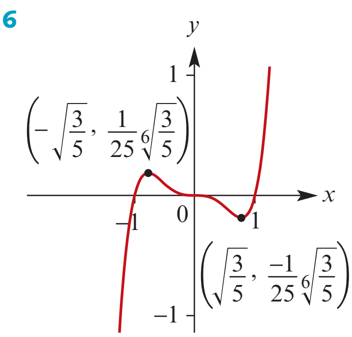

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Exercise 3G

1 **a** $a = -2$ **b** $a = 3$ **c** $a = \frac{10}{3},\ b = -\frac{70}{3}$

2 $a = -3,\ b = 2,\ c = -4,\ d = 5$

3 $y = \frac{11}{60}(x+5)(x+2)(x-6)$

4 $y = \frac{5}{9}(x+1)(x-3)^2$

5 **a** $y = x^3 + x + 1$ **b** $y = x^3 - x + 1$
c $y = 2x^3 - x^2 + x - 2$

6 **a** $y = (2x+1)(x-1)(x-2)$
b $y = x^2(x+1)$ **c** $y = x^3 + 2x^2 - x - 2$
d $y = (x+2)(x-3)^2$

7 **a** $y = -2x^3 - 25x^2 + 48x + 135$
b $y = 2x^3 - 30x^2 + 40x + 13$

8 **a** $y = -2x^4 + 22x^3 - 10x^2 - 37x + 40$
b $y = x^4 - x^3 + x^2 + 2x + 8$
c $y = \frac{31}{36}x^4 + \frac{5}{4}x^3 - \frac{157}{36}x^2 - \frac{5}{4}x + \frac{11}{2}$

Exercise 3H

1 **a** $x = \frac{6}{q-p}$ **b** $x = \frac{m+n}{m-n}$ **c** $x = \frac{a^2}{a-1}$
d $x = \frac{-1 \pm \sqrt{1-4k^2}}{2k}$, for $k \in [-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}$
e $x = 0,\ x = 3a$ or $x = 4a$
f $x = 0$ or $x = a^{\frac{1}{3}}$
g $x = \frac{k \pm \sqrt{k^2-4k}}{2}$, for $k \geq 4$ or $k \leq 0$
h $x = 0$ or $x = \pm\sqrt{a}$ if $a > 0$ **i** $x = \pm a$
j $x = a$ or $x = b$
k $x = a$ or $x = a^{\frac{1}{3}}$ or $x = \pm\sqrt{a}$ if $a \geq 0$

2 **a** $x = \sqrt[3]{\frac{2c-b}{a}}$ **b** $x = \pm\sqrt{\frac{c+b}{a}}$
c $x = \pm\sqrt{\frac{a-c}{b}}$ **d** $x = a^3$
e $x = (a-c)^n$ **f** $x = 2b + \sqrt[3]{\frac{c}{a}}$
g $x = \left(\frac{b}{a}\right)^3$ **h** $x = (c+d)^{\frac{1}{3}}$

3 **a** $(0,0),\ (1,1)$ **b** $(\frac{1}{2}, \frac{1}{2}),\ (0,0)$
c $\left(\frac{3+\sqrt{13}}{2}, \sqrt{13}+4\right),\ \left(\frac{3-\sqrt{13}}{2}, 4-\sqrt{13}\right)$

4 **a** $(13,3),\ (3,13)$ **b** $(10,5),\ (5,10)$
c $(11,8),\ (-8,-11)$ **d** $(9,4),\ (4,9)$
e $(9,5),\ (-5,-9)$

5 **a** $(17,11),\ (11,17)$ **b** $(37,14),\ (14,37)$
c $(14,9),\ (-9,-14)$

6 $(2,4),\ (0,0)$

7 $\left(\frac{\sqrt{5}+5}{2}, \frac{\sqrt{5}+5}{2}\right),\ \left(\frac{5-\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}\right)$

8 $\left(\frac{-130-80\sqrt{2}}{41}, \frac{60-64\sqrt{2}}{41}\right),$
$\left(\frac{80\sqrt{2}-130}{41}, \frac{64\sqrt{2}+60}{41}\right)$

9 $\left(\frac{1+\sqrt{21}}{2}, \frac{-1-\sqrt{21}}{2}\right),\ \left(\frac{1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}\right)$

10 $\left(\frac{4}{9}, 2\right)$

11 $(3,2),\ (0,-1)$

12 **a** $(3,2),\ \left(\frac{-8}{5}, \frac{-15}{4}\right)$ **b** $\left(\frac{27}{2}, \frac{10}{3}\right),\ (5,9)$
c $(6,4),\ \left(\frac{-12}{5}, -10\right)$

13 $c^2 - ac + b = 0$

14 $y = -7x + 14,\ y = 5x + 2$

15 $m < -7$ or $m > 1$

16 $c = -8$ or $c = 4$

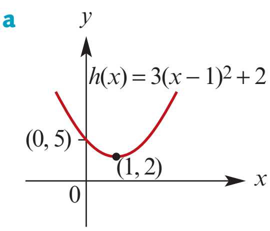
17 **a** $x = \frac{5 \pm \sqrt{4m+25}}{2m}$
b $m = \frac{-25}{4},\ \left(-\frac{2}{5}, \frac{5}{2}\right)$
c $m < \frac{-25}{4}$

18 $y = 3x + 3,\ y = -x + 3$

Chapter 3 review

Technology-free questions

1 **a**



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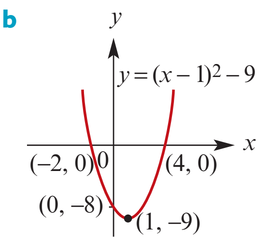
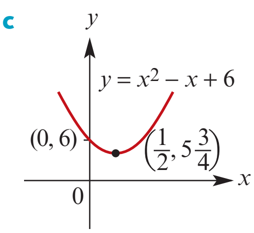


c



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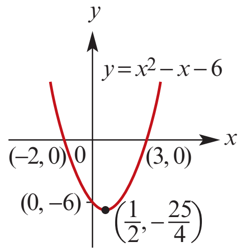
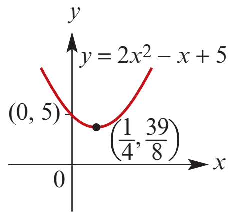
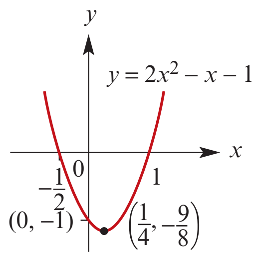
d



e



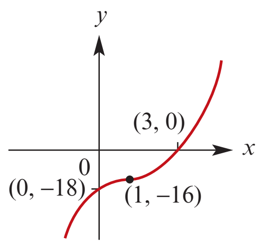
f



2 $y = \frac{4}{3}x^2 - \frac{1}{3};\ a = \frac{4}{3},\ b = -\frac{1}{3}$

3 $\frac{1}{3}(1 \pm \sqrt{31})$

4 a



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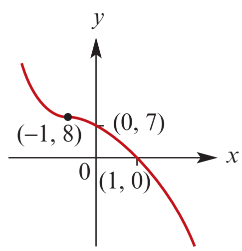
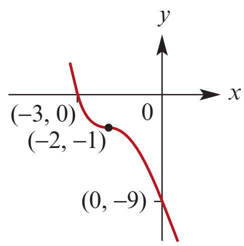
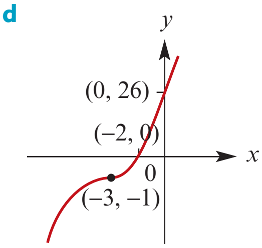


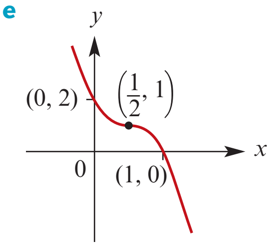
c



d



e

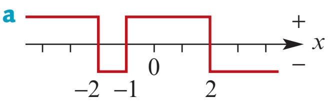


5 a $(x+2)^2 - 4$ **b** $3(x+1)^2 - 3$

c $(x-2)^2 + 2$ **d** $2\left(x - \frac{3}{2}\right)^2 - \frac{17}{2}$

e $2\left(x - \frac{7}{4}\right)^2 - \frac{81}{8}$ **f** $-\left(x - \frac{3}{2}\right)^2 - \frac{7}{4}$

6 a



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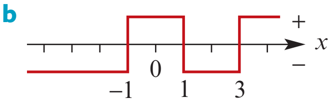
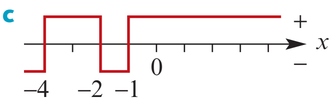
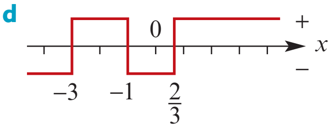


c



d



7 a 8 **b** 0 **c** 0

8 $y = (x-7)(x+3)(x+2)$

9 a $(x-2)(x+1)(x+3)$

b $(x-1)(x+1)(x-3)$

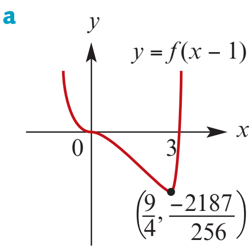
c $(x-1)(x+1)(x-3)(x+2)$

d $\frac{1}{4}(x-1)(2x+3+\sqrt{13})(2x+3-\sqrt{13})$

10 $x^2 + 4 = 1 \times (x^2 - 2x + 2) + 2x + 2$

11 $a = -6$

12 a



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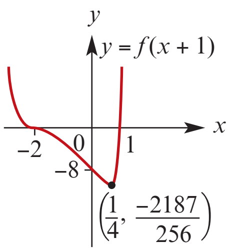
b



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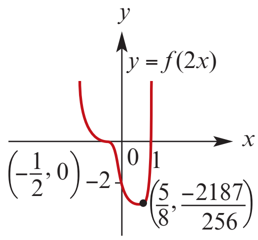
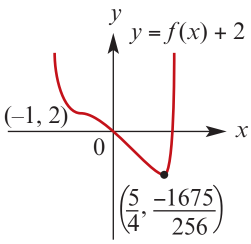


d



13 $k = \pm 8$

14 $(4, -5), (3, 9)$

15 $a = 3, \ b = \frac{5}{6}, \ c = -\frac{13}{12}$

16 $64x^3 + 144x^2 + 108x + 27$

17 $a = 1, \ b = -1, \ c = 4$

18 $-2 < p < 6$

19 $y = -x^3 + 7x^2 - 11x + 6$

Multiple-choice questions

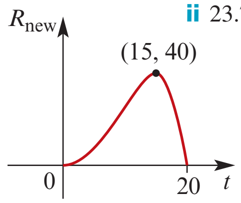
1 E **2** D **3** E **4** C **5** E **6** A
7 C **8** E **9** C **10** C **11** C **12** B

Extended-response questions

1 a $k = \frac{4}{3375}$

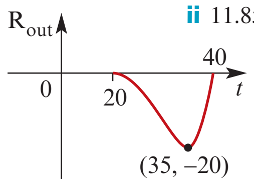
b 11.852 mL/min

c i



ii 23.704 mL/min

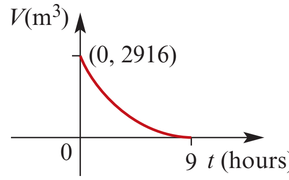
d i



ii 11.852 mL/min out

2 a i 2916 m^3 **ii** 0 m^3

b

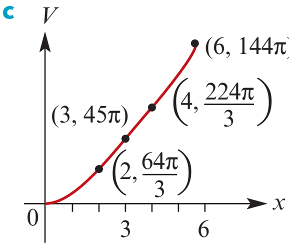


c 3.96 hours

3 a i $\frac{64\pi}{3}$ cm^3 **ii** 45π cm^3 **iii** $\frac{224\pi}{3}$ cm^3

b 144π cm^3

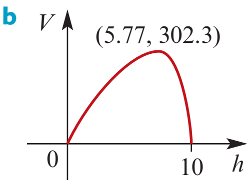
c



d $x = 5$; depth is 5 cm

4 a $r = \sqrt{25 - \frac{h^2}{4}}$

b



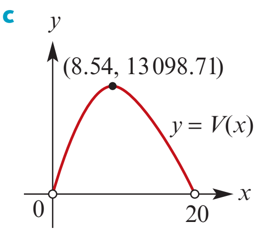
c $V = 96\pi$ cm^3

d $h = 2, r = 2\sqrt{6}$, i.e. height = 2 cm and radius = $2\sqrt{6}$ cm, or $h = 8.85$ and $r = 2.33$

5 a $V = (84 - 2x)(40 - 2x)x$

b $(0, 20)$

c



d i $x = 2, V = 5760$

ii $x = 6, V = 12\,096$

iii $x = 8, V = 13\,056$

iv $x = 10, V = 12\,800$

e $x = 13.50$ or $x = 4.18$

f 13 098.71 cm^3

6 a i $A = 2x(16 - x^2)$ **ii** $(0, 4)$

b i 42 **ii** $x = 0.82$ or $x = 3.53$

c i $V = 2x^2(16 - x^2)$ **ii** $x = 2.06$ or $x = 3.43$

7 a $A = \frac{\pi}{2}x^2 + yx$

b i $y = 100 - \pi x$ **ii** $A = 100x - \frac{\pi}{2}x^2$

iii $\left(0, \frac{100}{\pi}\right)$

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c $x = 12.43$

d **i** $V = \frac{x^2}{50}\left(100 - \frac{\pi}{2}x\right),\ x \in \left(0, \frac{100}{\pi}\right)$

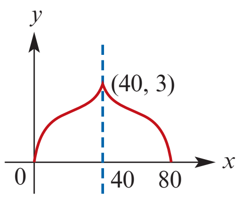
ii 248.5 m^3 **iii** $x = 18.84$

8 **a** $y = \frac{1}{12000}x^3 - \frac{1}{200}x^2 + \frac{17}{120}x$

b $x = 20$

d $y = -\frac{1}{6000}x^3 + \frac{29}{3000}x^2 - \frac{1}{20}x$

e **i**



ii Second section of graph is formed reflecting the graph of $y = f(x)$, $x \in [0, 40]$, in the line $x = 40$

Chapter 4

Exercise 4A

1 **a** $\frac{5\pi}{18}$ **b** $\frac{34\pi}{45}$ **c** $\frac{25\pi}{18}$
d $\frac{17\pi}{9}$ **e** $\frac{7\pi}{3}$ **f** $\frac{49\pi}{18}$

2 **a** $60°$ **b** $150°$ **c** $240°$
d $140°$ **e** $630°$ **f** $252°$

3 **a** $45.84°$ **b** $93.97°$ **c** $143.24°$
d $226.89°$ **e** $239.50°$ **f** $340.91°$

4 **a** 0.65 **b** 1.29 **c** 2.01
d 2.13 **e** 5.93 **f** 2.31

Exercise 4B

1 **a** 0 **b** 0 **c** −1 **d** −1 **e** 0
f 0 **g** 0 **h** 0 **i** −1 **j** 0
k −1 **l** 0 **m** 0 **n** 1

2 **a** 0.99 **b** 0.52 **c** −0.87 **d** 0.92
e −0.67 **f** −0.23 **g** −0.99 **h** 0.44
i −34.23 **j** −2.57 **k** 0.95 **l** 0.75

3 **a** $\frac{1}{\sqrt{2}}$ **b** $-\frac{1}{2}$ **c** $-\frac{\sqrt{3}}{2}$ **d** $\frac{1}{2}$
e $-\frac{1}{2}$ **f** $-\frac{1}{\sqrt{2}}$ **g** $-\frac{1}{\sqrt{2}}$ **h** $\frac{1}{2}$
i $\frac{1}{2}$ **j** $\frac{\sqrt{3}}{2}$ **k** $\frac{1}{2}$ **l** $\frac{\sqrt{3}}{2}$
m $\frac{1}{\sqrt{2}}$ **n** $-\frac{1}{2}$ **o** $\frac{1}{\sqrt{2}}$ **p** $-\frac{1}{2}$
q $-\sqrt{3}$ **r** $-\sqrt{3}$ **s** $\frac{1}{\sqrt{3}}$ **t** $\sqrt{3}$
u −1 **v** $-\frac{1}{\sqrt{3}}$

4 **a** 0.52 **b** −0.68 **c** 0.52 **d** 0.4
e −0.52 **f** 0.68 **g** −0.4 **h** −0.68
i −0.52 **j** 0.68 **k** −0.4

5 **a** 0.4 **b** −0.7 **c** 0.4 **d** 1.2
e −0.4 **f** 0.7 **g** −1.2 **h** −0.7
i −0.4 **j** 0.7 **k** −1.2

6 **a** $\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$ **b** $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1$
c $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1$ **d** $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}$
e $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1$ **f** $-\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$

Exercise 4C

1 **a** 0.6 **b** 0.6 **c** −0.7 **d** 0.3
e −0.3 **f** $\frac{10}{7}$ **g** −0.3 **h** 0.6
i −0.6 **j** −0.3 **k** $\frac{10}{7}$ **l** 0.3

2 **a** $-\frac{4}{5}, -\frac{4}{3}$ **b** $-\frac{12}{13}, -\frac{5}{12}$
c $-\frac{2\sqrt{6}}{5}, -2\sqrt{6}$ **d** $-\frac{5}{13}, \frac{12}{5}$
e $-\frac{3}{5}, -\frac{3}{4}$ **f** $-\frac{12}{13}, \frac{5}{12}$ **g** $-\frac{3}{5}, -\frac{3}{4}$

Exercise 4D

1 **a** $2\pi, 3$ **b** $\frac{2\pi}{3}, 5$ **c** $\pi, \frac{1}{2}$ **d** $6\pi, 2$
e $\frac{\pi}{2}, 3$ **f** $2\pi, \frac{1}{2}$ **g** $4\pi, 3$ **h** $3\pi, 2$

2 **a** Dilation of factor 4 from the x-axis, dilation of factor $\frac{1}{3}$ from the y-axis;
Amplitude = 4; Period = $\frac{2\pi}{3}$

b Dilation of factor 5 from the x-axis, dilation of factor 3 from the y-axis;
Amplitude = 5; Period = 6π

c Dilation of factor 6 from the x-axis, dilation of factor 2 from the y-axis;
Amplitude = 6; Period = 4π

d Dilation of factor 4 from the x-axis, dilation of factor $\frac{1}{5}$ from the y-axis;
Amplitude = 4; Period = $\frac{2\pi}{5}$

3 **a** Dilation of factor 2 from the x-axis, dilation of factor $\frac{1}{3}$ from the y-axis;
Amplitude = 2; Period = $\frac{2\pi}{3}$

b Dilation of factor 3 from the x-axis, dilation of factor 4 from the y-axis;
Amplitude = 3; Period = 8π

c Dilation of factor 6 from the x-axis, dilation of factor 5 from the y-axis;
Amplitude = 6; Period = 10π

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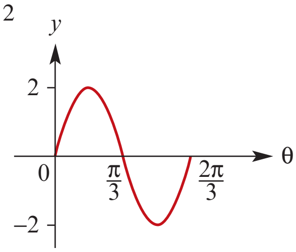
d Dilation of factor 3 from the x-axis, dilation of factor $\frac{1}{7}$ from the y-axis;

Amplitude = 3; Period = $\frac{2\pi}{7}$

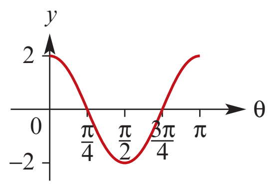
4 a Amplitude = 2

Period = $\frac{2\pi}{3}$



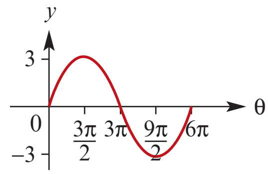
b Amplitude = 2

Period = π



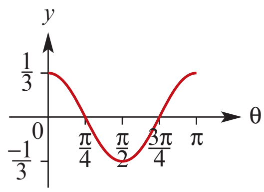
c Amplitude = 3

Period = 6π



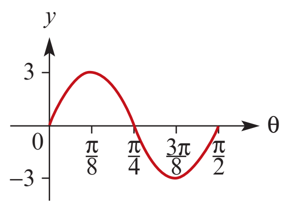
d Amplitude = $\frac{1}{3}$

Period = π



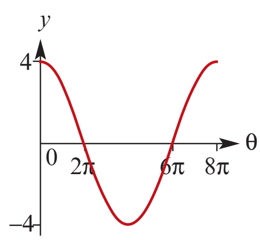
e Amplitude = 3

Period = $\frac{\pi}{2}$



f Amplitude = 4

Period = 8π



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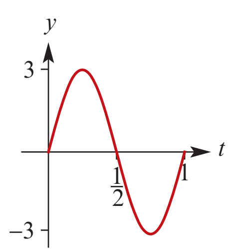

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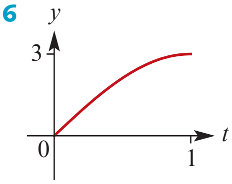

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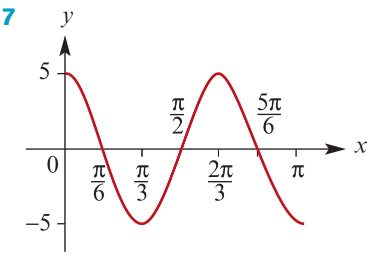

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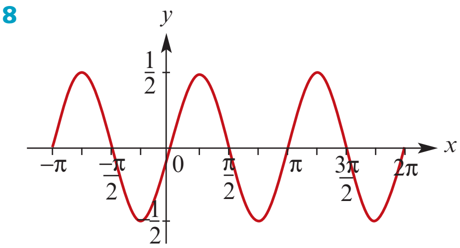

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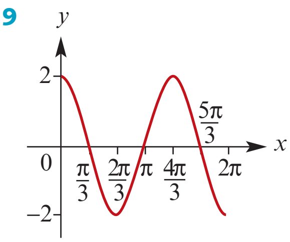
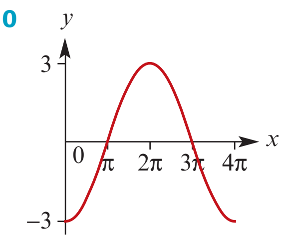


10



11 $y = 2\sin\left(\frac{x}{3}\right)$

12 $y = \frac{1}{2}\cos\left(\frac{x}{3}\right)$

13 $y = \frac{1}{2}\sin\left(\frac{x}{2}\right)$

Exercise 4E

1 a $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$ **b** $\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$

c $\frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}$ **d** $\pi, 3\pi$

2 a $-\frac{5\pi}{6}, -\frac{\pi}{6}$ **b** $-\frac{\pi}{6}, \frac{\pi}{6}$

c $-\frac{5\pi}{6}, \frac{5\pi}{6}$ **d** $\frac{\pi}{2}$

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3 **a** $\frac{\pi}{4}, \frac{3\pi}{4}$ **b** $\frac{3\pi}{4}, \frac{5\pi}{4}$ **c** $\frac{5\pi}{6}, \frac{7\pi}{6}$
d $\frac{7\pi}{6}, \frac{11\pi}{6}$ **e** $\frac{\pi}{4}, \frac{7\pi}{4}$ **f** $\frac{2\pi}{3}, \frac{4\pi}{3}$

4 **a** 0.643501, 2.49809 **b** 0.643501, 5.63968
c 3.60836, 5.81642 **d** 1.77215, 4.51103

5 **a** 17.46°, 162.54° **b** 66.42°, 293.58°
c 233.13°, 306.87° **d** 120°, 240°

6 **a** 60°, 300° **b** 60°, 120° **c** 225°, 315°
d 120°, 240° **e** 60°, 120° **f** 150°, 210°

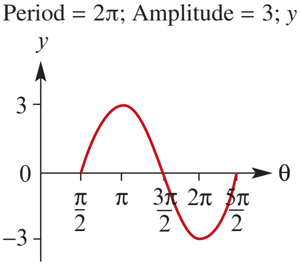
7 **a** $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$
b $\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$
c $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$
d $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}$
e $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$
f $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

8 **a** $\frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}$
b $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$
c $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}$
d $\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$
e $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$
f $\frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}$
g $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}$
h $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
i $\frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$

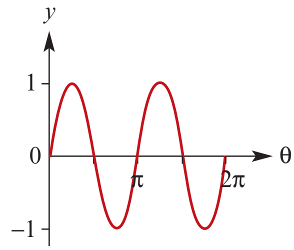
9 **a** 2.03444, 2.67795, 5.17604, 5.81954
b 1.89255, 2.81984, 5.03414, 5.96143
c 0.57964, 2.56195, 3.72123, 5.70355
d 0.309098, 1.7853, 2.40349, 3.87969, 4.49789, 5.97409

Exercise 4F

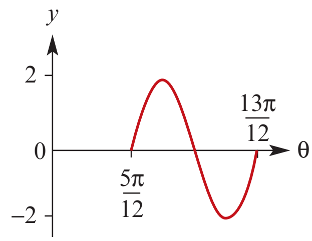
1 **a** Period = 2π; Amplitude = 3; $y = \pm 3$



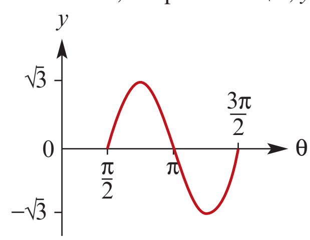
b Period = π; Amplitude = 1; $y = \pm 1$



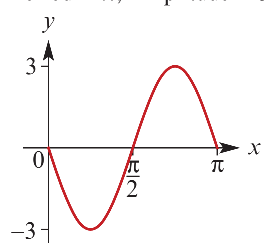
c Period = $\frac{2\pi}{3}$; Amplitude = 2; $y = \pm 2$



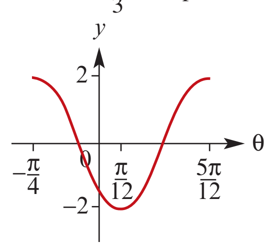
d Period = π; Amplitude = $\sqrt{3}$; $y = \pm\sqrt{3}$



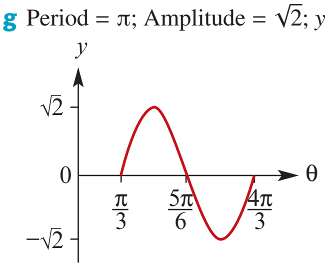
e Period = π; Amplitude = 3; $y = \pm 3$



f Period = $\frac{2\pi}{3}$; Amplitude = 2; $y = \pm 2$



g Period = π; Amplitude = $\sqrt{2}$; $y = \pm\sqrt{2}$



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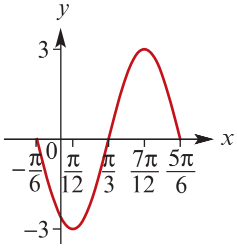
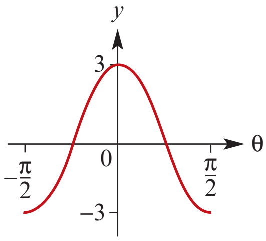
h Period = π; Amplitude = 3; $y = \pm 3$

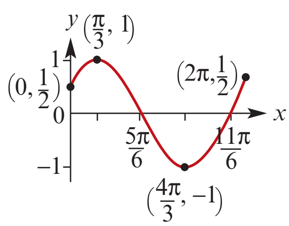


i Period = π; Amplitude = 3; $y = \pm 3$



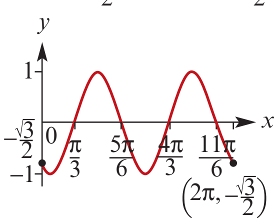
2 a $f(0) = \frac{1}{2}$, $f(2\pi) = \frac{1}{2}$

b



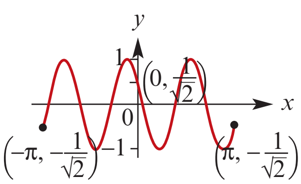
3 a $f(0) = -\frac{\sqrt{3}}{2}$, $f(2\pi) = -\frac{\sqrt{3}}{2}$

b



4 a $f(-\pi) = -\frac{1}{\sqrt{2}}$, $f(\pi) = -\frac{1}{\sqrt{2}}$

b



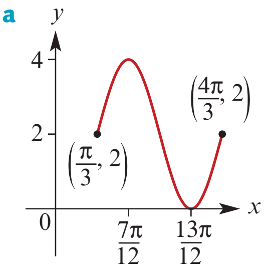
5 a $y = 3\sin\left(\frac{x}{2}\right)$ **b** $y = 3\sin(2x)$

c $y = 2\sin\left(\frac{x}{3}\right)$ **d** $y = \sin 2\left(x - \frac{\pi}{3}\right)$

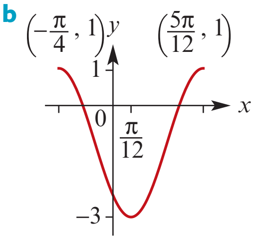
e $y = \sin\frac{1}{2}\left(x + \frac{\pi}{3}\right)$

Exercise 4G

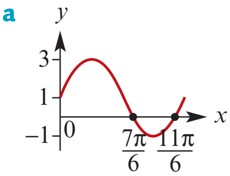
1 a



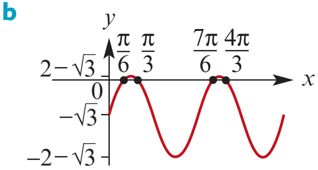
b



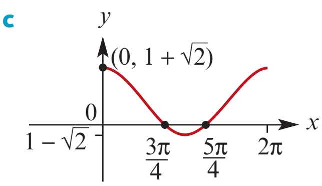
2 a



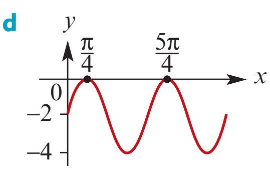
b



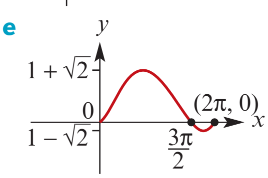
c



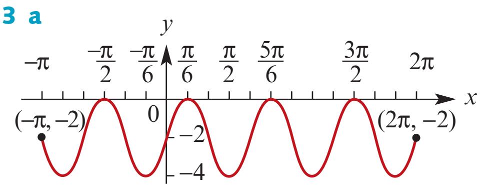
d



e



3 a



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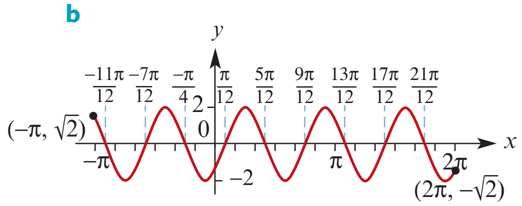
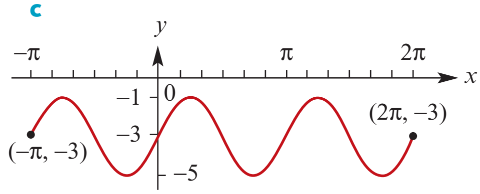
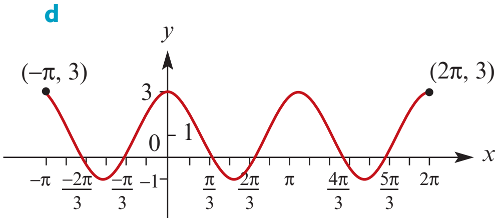
b



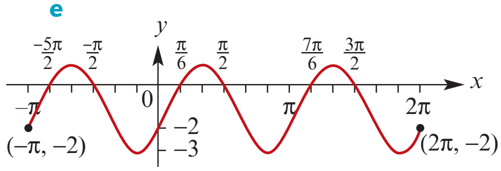
c



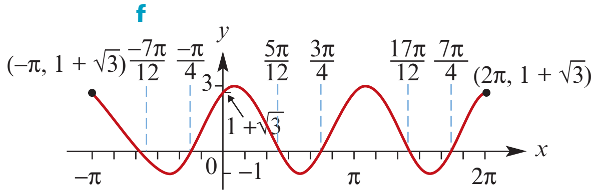
d



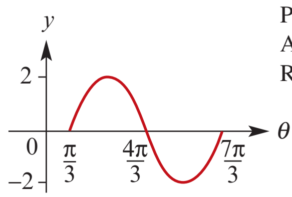
e



f

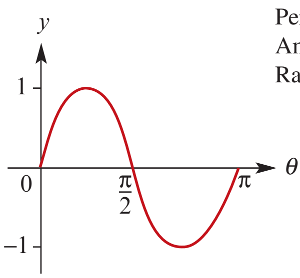


4 a



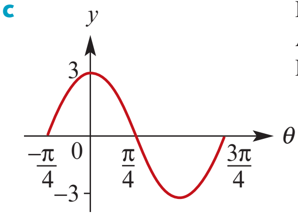
Period = 2π
Amplitude = 2
Range = [−2, 2]

b



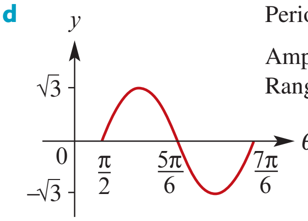
Period = π
Amplitude = 1
Range = [−1, 1]

c



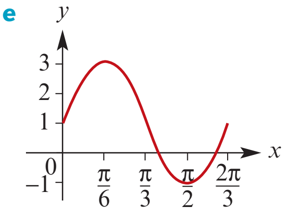
Period = π
Amplitude = 3
Range = [−3, 3]

d



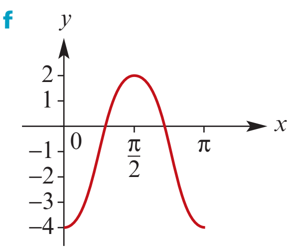
Period = $\frac{2\pi}{3}$
Amplitude = $\sqrt{3}$
Range = $[-\sqrt{3}, \sqrt{3}]$

e



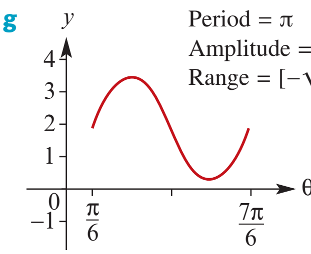
Period = $\frac{2\pi}{3}$
Amplitude = 2
Range = [−1, 3]

f



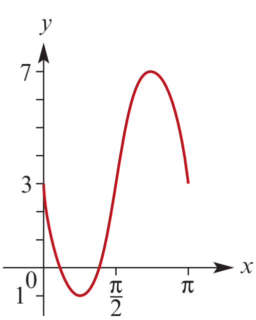
Period = π
Amplitude = 3
Range = [−4, 2]

g



Period = π
Amplitude = $\sqrt{2}$
Range = $[-\sqrt{2} + 2, 2 + \sqrt{2}]$

h



Period = π
Amplitude = 4
Range = [−1, 7]

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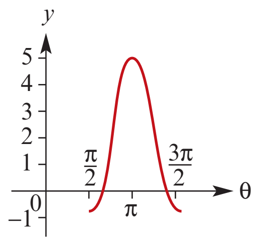
i



Period = π
Amplitude = 3
Range = [−1, 5]

5 a $y = \frac{1}{2}\cos\left(\frac{1}{3}\left(x - \frac{\pi}{4}\right)\right)$

b $y = 2\cos\left(x - \frac{\pi}{4}\right)$

c $y = -\frac{1}{3}\cos\left(x - \frac{\pi}{3}\right)$

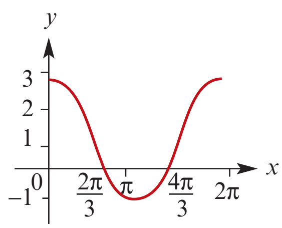
6 a
- Dilation of factor 3 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Reflection in the x-axis

b
- Dilation of factor 3 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Reflection in the x-axis
- Translation $\frac{\pi}{3}$ units to the right

c
- Dilation of factor 3 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Translation $\frac{\pi}{3}$ units to the right and 2 units up

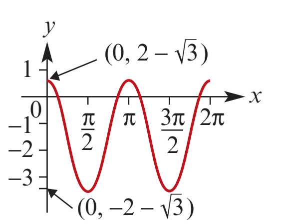
d
- Dilation of factor 2 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Reflection in the x-axis
- Translation $\frac{\pi}{3}$ units to the right and 5 units up

7 a



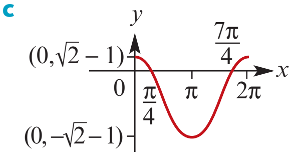
Intercepts: $\left(\frac{2\pi}{3}, 0\right), \left(\frac{4\pi}{3}, 0\right)$

b



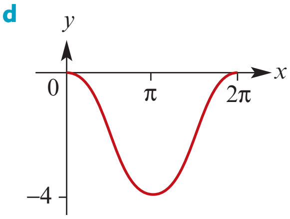
Intercepts: $\left(\frac{\pi}{12}, 0\right), \left(\frac{11\pi}{12}, 0\right), \left(\frac{13\pi}{12}, 0\right), \left(\frac{23\pi}{12}, 0\right)$

c



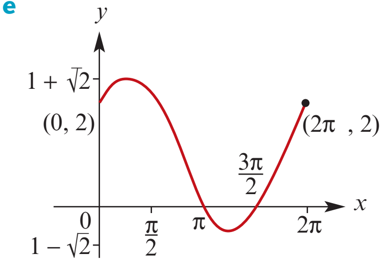
Intercepts: $\left(\frac{\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

d



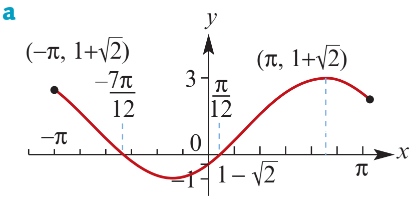
Intercepts: $(0, 0), (2\pi, 0)$

e

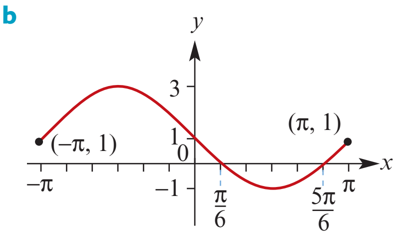


Intercepts: $(\pi, 0), \left(\frac{3\pi}{2}, 0\right)$

8 a



b



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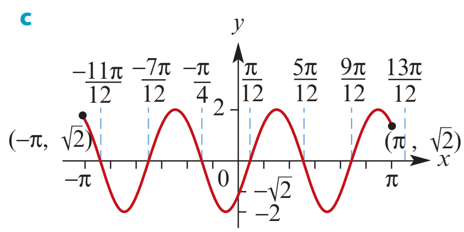

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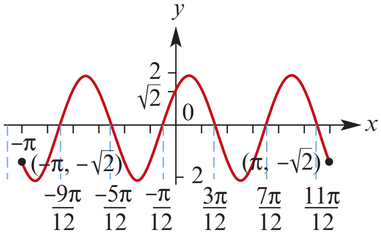
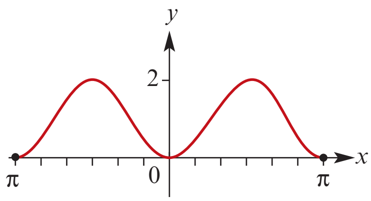
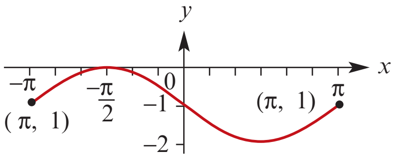
d



e



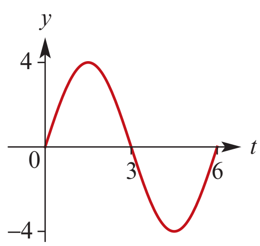
f



Exercise 4H

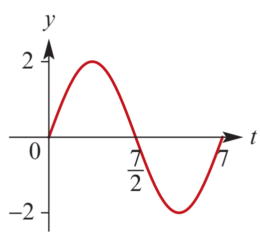
1 a $n = \frac{\pi}{3}$

$A = 4$



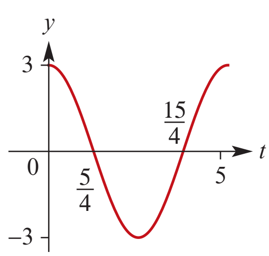
b $n = \frac{2\pi}{7}$

$A = 2$



c $n = \frac{2\pi}{5}$

$A = 3$



2 a $A = 3, \ n = \frac{\pi}{4}$ **b** $A = -4, \ n = \frac{\pi}{6}$

3 $A = 0.5, \ \varepsilon = \frac{-\pi}{3}$

4 $A = 3, \ n = 3, \ b = 5$

5 $A = 4, \ n = \frac{\pi}{4}, \ \varepsilon = \frac{-\pi}{2}$

(*Note:* ε can take infinitely many values)

6 $A = 2, \ n = \frac{\pi}{3}, \ \varepsilon = \frac{-\pi}{6}$

(*Note:* ε can take infinitely many values)

7 $A = 4, \ n = \frac{\pi}{4}, \ d = 2, \ \varepsilon = \frac{-\pi}{2}$

(*Note:* ε can take infinitely many values)

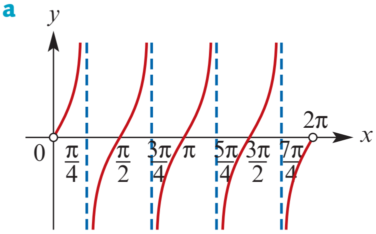
8 $A = 2, \ n = \frac{\pi}{3}, \ d = 2, \ \varepsilon = \frac{-\pi}{6}$

(*Note:* ε can take infinitely many values)

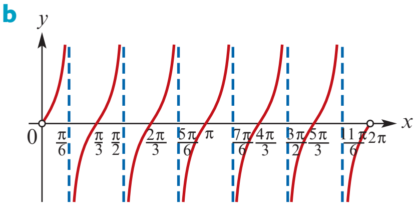
Exercise 4I

1 a $\frac{\pi}{3}$ **b** 2π **c** $\frac{2\pi}{3}$ **d** 1 **e** 2

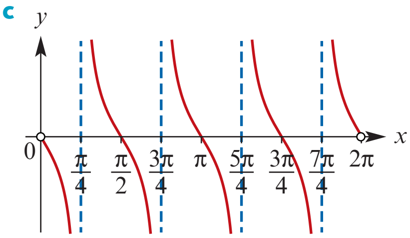
2 a



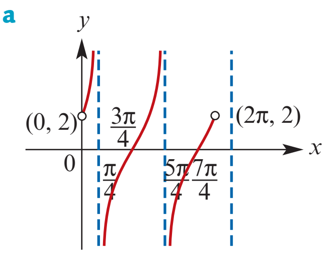
b



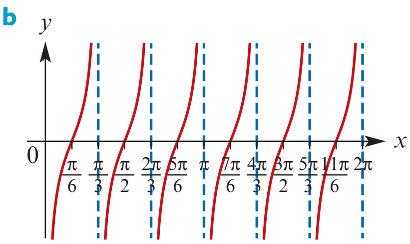
c



3 a



b



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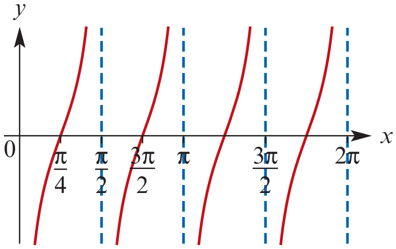
c 

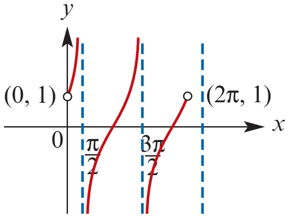
4 a $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$ **b** $-\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$

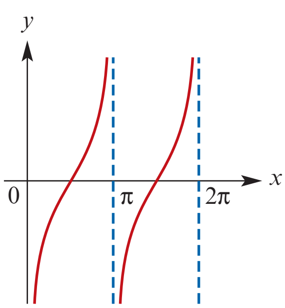
c $-\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$ **d** $-\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$

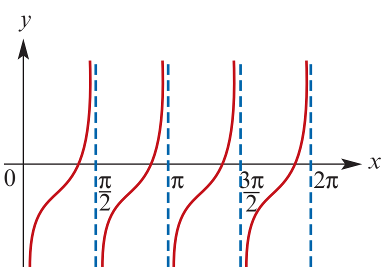
e $-\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}$

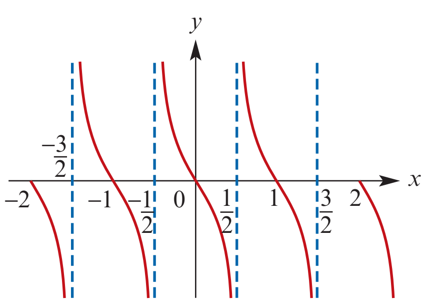
5 $\frac{11\pi}{24}, \frac{23\pi}{24}, \frac{35\pi}{24}, \frac{47\pi}{24}$

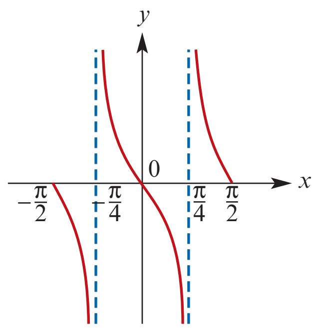
6 $\frac{7\pi}{12}, \frac{19\pi}{12}$

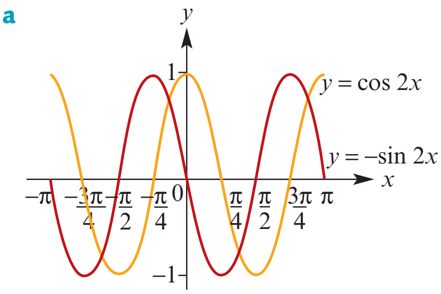
7 a 

b 

c 

8 

9 

10 a 

b $\left(\frac{-5\pi}{8}, \frac{-1}{\sqrt{2}}\right), \left(\frac{-\pi}{8}, \frac{1}{\sqrt{2}}\right), \left(\frac{3\pi}{8}, \frac{-1}{\sqrt{2}}\right), \left(\frac{7\pi}{8}, \frac{1}{\sqrt{2}}\right)$

11 a $\frac{\pi}{6}, \frac{7\pi}{6}$

b $\frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \frac{17\pi}{16}, \frac{21\pi}{16}, \frac{25\pi}{16}, \frac{29\pi}{16}$

c $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$

d $\frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$

e $\frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}, \frac{23\pi}{12}$

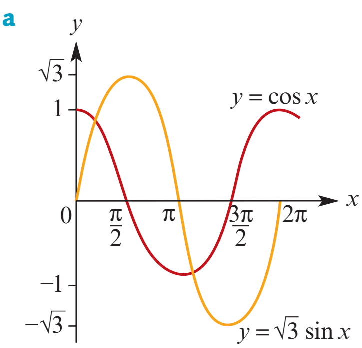
f 0.4636, 3.6052

g 1.1071, 4.2487

h $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

i $\frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}$

j $\frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$

12 a 

b $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right), \left(\frac{7\pi}{6}, -\frac{\sqrt{3}}{2}\right)$

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13 **a** $\frac{7\pi}{24}, \frac{19\pi}{24}, \frac{31\pi}{24}, \frac{43\pi}{24}$
b $\frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$
c $\frac{11\pi}{36}, \frac{23\pi}{36}, \frac{35\pi}{36}, \frac{47\pi}{36}, \frac{59\pi}{36}, \frac{71\pi}{36}$

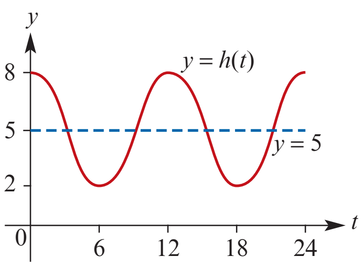
14 $A = 5,\ n = 3$

15 $A = 6,\ n = \frac{\pi}{2}$

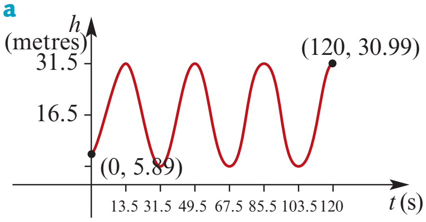
Exercise 4J

1 **a** **i** Amplitude $= 1\frac{1}{2}$
ii Period $= 12$
iii $d(t) = 3.5 - 1.5\cos\left(\frac{\pi}{6}t\right)$
iv 1.5 m
b $t \in [0, 3) \cup (9, 15) \cup (21, 24]$

2 **a** $A = 3, n = \frac{\pi}{6}, b = 5, \varepsilon = \frac{\pi}{2}$
b 2:21 a.m., 9:39 a.m., 2:21 p.m., 9:39 p.m.
c



3 **a**



b 5.89 m **c** 27.51 s **d** 6 times
e 20 times **f** 4.21 m **g** 13.9 m

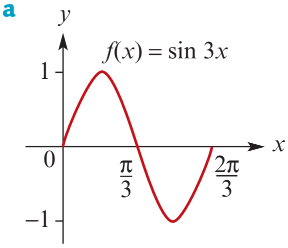
Chapter 4 review

Technology-free questions

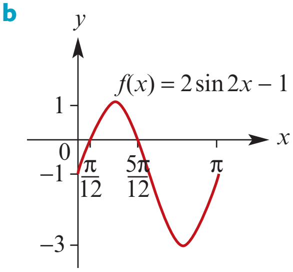
1 **a** −0.4 **b** −0.6 **c** 2 **d** −0.7 **e** $\frac{1}{5}$ **f** $\frac{3}{5}$

2 **a** $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{-2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$
c $\frac{-\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}$ **d** $\frac{-3\pi}{4}, \frac{-\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
e $\frac{-\pi}{6}, \frac{-5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **f** $\frac{-\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$
g $\frac{-3\pi}{8}, \frac{-5\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$
h $\frac{-7\pi}{18}, \frac{-11\pi}{18}, \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$

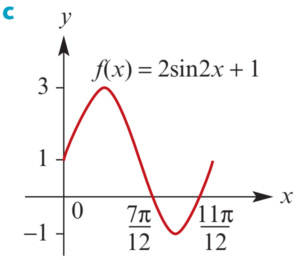
3 **a**



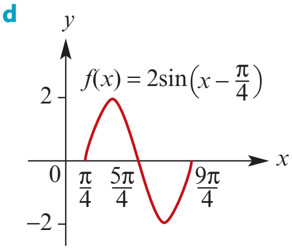
b



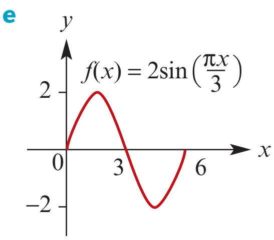
c



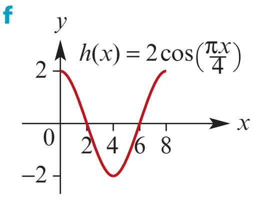
d



e



f



4 **a** 30, 150 **b** 45, 135, 225, 315
c 240, 300 **d** 90, 120, 270, 300
e 120, 240

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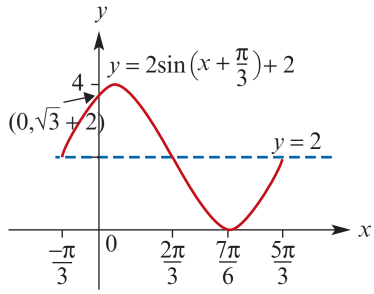
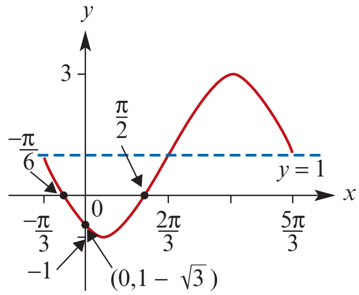
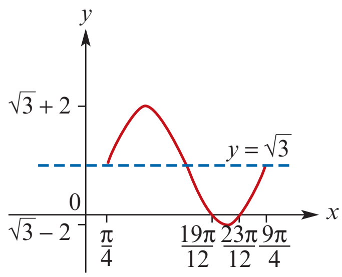
5 a



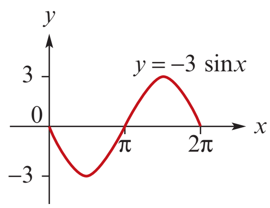
b



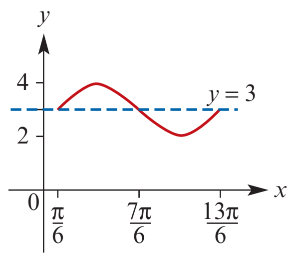
c



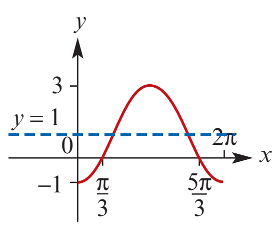
d



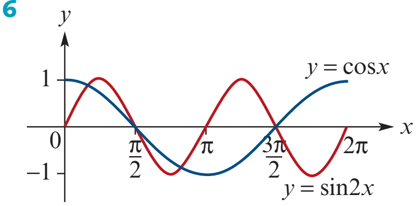
e



f

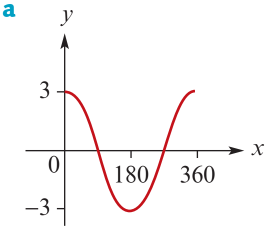


6

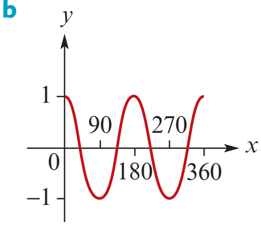


a 4 solutions **b** 4 solutions **c** 2 solutions

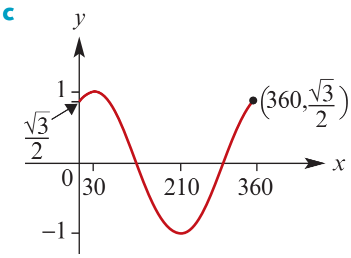
7 a



b



c



8 a $\frac{-2\pi}{3}, \frac{\pi}{3}$ **b** $\frac{-\pi}{4}, \frac{3\pi}{4}$

c $\frac{-5\pi}{8}, \frac{-\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$ **d** $\frac{-2\pi}{3}, \frac{-\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$

9 $-\frac{2\pi}{3}, \frac{\pi}{3}$

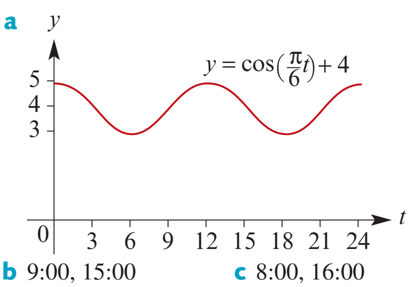
10 a $\frac{1}{\sqrt{3}}$ **b** $-\frac{5\pi}{6}$

Multiple-choice questions

1 C **2** A **3** D **4** C **5** B
6 E **7** B **8** C **9** A **10** C

Extended-response questions

1 a



b 9:00, 15:00 **c** 8:00, 16:00

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2 a Maximum = 210 cm; Minimum = 150 cm; Mean = 180 cm

b $A = 30,\ n = \dfrac{\pi}{6},\ \varepsilon = \dfrac{-\pi}{2},\ b = 180$

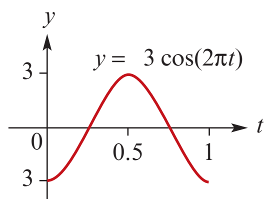
c **i** 165 cm

ii $180 - 15\sqrt{3} \approx 154$ cm

d $\approx$ 4:24, $\approx$ 7:36

3 a $a = -3,\ n = 2\pi$

b

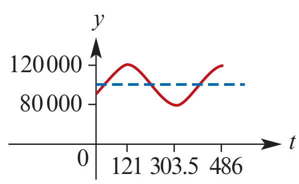


c **i** $t = \dfrac{1}{3}$ second **ii** $t = \dfrac{1}{6}$ second

d $t = 0.196$ seconds

4 a $a = 20\,000,\ b = 100\,000,\ n = \dfrac{2\pi}{365},\ \varepsilon \approx 5.77$

b



c **i** $t = 242.7,\ t = 364.3$

ii $t = 60.2,\ t = 181.8$

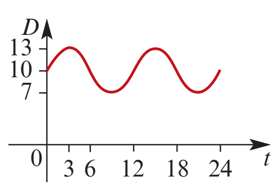
d $\approx 117\,219\ \text{m}^3$/day

5 a **i** 1.83×10^{-3} hours

ii 11.79 hours

b 25 April ($t = 3.856$), 14 August ($t = 7.477$)

6 a



b $t \in [0, 7] \cup [11, 19] \cup [23, 24]$

c 12.9 m

7 a i $25\sqrt{3}$ **ii** 30

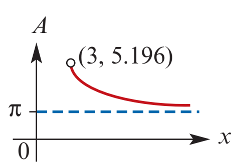
b 2.27, 0.53

d $b = 8$

e $\theta = 0.927$ or 1.837

f $a = 4\sqrt{3}$

8 c



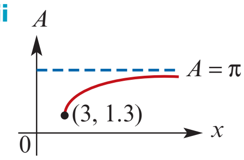
d **i** 2.055 **ii** 0.858

iii 0.0738 **iv** 0.0041

e $nr\tan\left(\dfrac{\pi}{n}\right)$

f **i** $n\sin\left(\dfrac{\pi}{n}\right)\cos\left(\dfrac{\pi}{n}\right)$

ii



Chapter 5

Technology-free questions

1 a Domain = $\mathbb{R} \setminus \{0\}$; Range = $\mathbb{R} \setminus \{2\}$

b Domain = $\left[\dfrac{2}{3}, \infty\right)$; Range = $(-\infty, 3]$

c Domain = $\mathbb{R} \setminus \{2\}$; Range = $(3, \infty)$

d Domain = $\mathbb{R} \setminus \{2\}$; Range = $\mathbb{R} \setminus \{4\}$

e Domain = $[2, \infty)$; Range = $[-5, \infty)$

2 $x = -\dfrac{7\pi}{9}$ or $x = \dfrac{\pi}{9}$ or $x = \dfrac{5\pi}{9}$

3 a Range = $[2, 8]$; Period = 6

b $x = \dfrac{\pi}{12}$ or $x = \dfrac{3\pi}{4}$

4 $a = -2$ and $b = 1$

5 a $m = \pm 2\sqrt{2}$ **b** $m > 2\sqrt{2}$ or $m < -2\sqrt{2}$

c $-2\sqrt{2} < m < 2\sqrt{2}$

6 a i $a = -3$ **ii** $a = 5$ or $a = 1$ **iii** $a = -3$

b $5y - 3x + 4 = 0,\quad \tan^{-1}\left(\dfrac{3}{5}\right)$

7 a i 4 **ii** $\sqrt{5}$ **iii** $2 - 2a$ **iv** $\sqrt{2a - 5}$

b i $x = -8$ **ii** $x = \dfrac{103}{2}$ **iii** $x < 1$

8 a **i** $f(g(x)) = 4x^2 + 8x - 3$

ii $g(f(x)) = 16x^2 - 16x + 3$

b Dilation of factor $\frac{1}{4}$ from the y-axis, then translation $\frac{3}{4}$ units to the right

c Translation 1 unit to the left and 1 unit down

9 $x = \dfrac{3\pi}{2}$ or $x = \dfrac{11\pi}{6}$

10 a $-2 \le x \le \dfrac{1}{2}$ or $x \ge 3$ **b** $x < 0$

Multiple-choice questions

1 D	**2** B	**3** E	**4** A	**5** C
6 C	**7** A	**8** B	**9** B	**10** A
11 B	**12** C	**13** C	**14** D	**15** D
16 E	**17** E	**18** A	**19** D	**20** A
21 E	**22** D	**23** E	**24** A	**25** A
26 E	**27** C	**28** D	**29** C	**30** B
31 B	**32** C	**33** D	**34** D	**35** D
36 A	**37** A	**38** D	**39** E	**40** B
41 D	**42** E			

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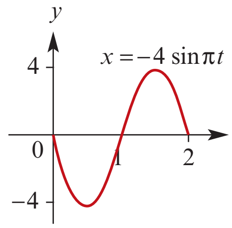
Extended-response questions

1 a $a = -0.09$, $b = 9$ **b** $DE = 2.79$ m
c Length $= 2\sqrt{30} \approx 10.95$ m

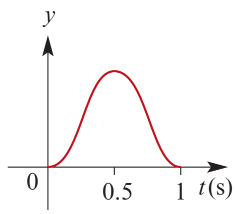
2 a $a = -3$ **b** $x = -1,\ x = -\frac{1}{2},\ x = 2$
c ii $b = \frac{7}{2},\ c = \frac{3}{2}$

3 a



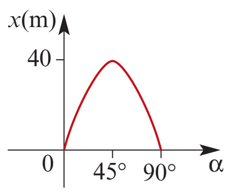
b i $x = 0$ **ii** $x = -4$ **iii** $x = 0$
c $t = \frac{7}{6}$
d Period $= \frac{2\pi}{\pi} = 2$ seconds

4 a i 0 **ii** 2.5 **iii** 0
b 1 second **c**



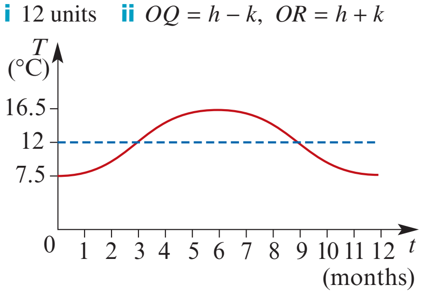
d $t = 0.35$ seconds

5 a 62.5 metres **b**



c 24.3° or 65.7°

6 a i 12 units **ii** $OQ = h - k,\ OR = h + k$
b



c $h = 12,\ k = 4.5$

7 a $D = 0.05t^2 - 0.25t + 1.8$ **b** \$3 000 000

8 $a = \frac{-7}{48},\ b = \frac{23}{24},\ c = 7.5$
Rainfall at noon was $\frac{35}{6}$ mm per hour
Rainfall greatest at $\frac{23}{7}$ hours after 4 a.m.
(approx 7:17 a.m.)

Degree-of-difficulty classified questions

Simple familiar questions

1 a $f(x) = 2(x-3)^2 - 11$ **b** -11
c $[-11, \infty)$

2 a 0 **b** 1 **c** $\frac{\sqrt{3}}{2}$ **d** $\frac{\sqrt{2}}{2}$ **e** $\frac{1}{2}$

3 $x = \frac{-1 \pm \sqrt{5}}{2}$

4 $[0, 9]$

5 a $\frac{2\pi}{5}$ **b** 8
c i
- Dilation of factor 8 from the x-axis
- Dilation of factor $\frac{1}{5}$ from the y-axis

ii
- Dilation of factor 8 from the x-axis
- Translation $\frac{\pi}{2}$ units to the right
- Reflection in the y-axis
- Dilation of factor $\frac{1}{5}$ from the y-axis

6 a $\frac{2\sqrt{2}}{3}$ **b** $\frac{\sqrt{2}}{4}$

7 20°, 100°, 140°, 220°, 260°, 340°

8 −10, −2, 2, 10

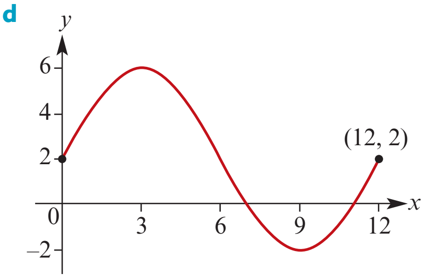
9 $\frac{1}{27}(3x - y)(9x^2 + 3xy + y^2)$

10 $(x-1)(x+2)(x^2 + x + 3)$

11 $\left(-\infty, \frac{1}{5}\right]$

12 $h(x) = f(5x - 7) + 3$

13 a Period = 12; Amplitude = 4
b Maximum = 6; Minimum = −2
c $t = 0$, 6 or 12
d



Complex familiar questions

1 $(-\infty, -4] \cup [2, \infty)$

2 a $a = 20,\ b = 29$ **b** -14

3 $n = 3$

4 a $M\left(3, \frac{\lambda - 2}{2}\right)$ **b** $\lambda = 4$ or $\lambda = 6$

5 $a = -3$ or $a = 2$ or $a = -1$

6 a $a = -6,\ b = 13$
b $P(x) = (x-1)^2(x-2)^2$

7 $a = -6,\ b = 4,\ c = 3$

8 $a = -18,\ b = 30$

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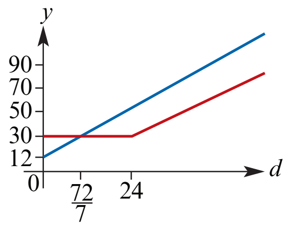
9 a $a = 2,\ b = -\frac{1}{2}$ **b** $2\sqrt{5}$

10 a $\left(\frac{3+\alpha}{2}, \frac{10+\beta}{2}\right)$ **b** $\alpha = 5,\ \beta = 8$

11 $p = -2,\ x = -3$

12 $a = 1$ or $a = 2$

13 c



d i \$41.75 **ii** \$30

e Thrifty Taxi

f Greater than $\frac{72}{7}$ km

Complex unfamiliar questions

1 a $a = 5$ **b** $b = -6$

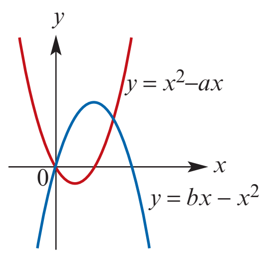
2 a $(a - b)^2$

4 a $h = (a - 1)x - x^2$ **b** $\frac{a-1}{2}$ **c** $\frac{(a-1)^2}{4}$

d i 2 **ii** 3 **iii** $1 + 2\sqrt{5}$ **iv** 7 **v** $1 + 2\sqrt{10}$

5 a $(0, 0), \left(\frac{a+b}{2}, \frac{b^2 - a^2}{4}\right)$

b



c $(a + b)x - 2x^2$ **d** $\frac{(a+b)^2}{8}$

6 b Min area $= \frac{147}{2}$ when $x = \frac{7}{2},\ y = \frac{7}{2}$

c Min area $= \frac{3a^2}{2}$ when $x = \frac{a}{2},\ y = \frac{a}{2}$

7 a $a = \frac{v^2}{10}$

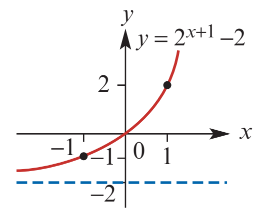
b i $a = \frac{125}{2}$ **ii** $\left(\frac{125}{4}, \frac{125}{8}\right)$ **iii** $\frac{25}{2}$, 50

c i $\left(\frac{45\sqrt{3}}{2}, \frac{135}{4}\right)$ **ii** $a = 45\sqrt{3}$

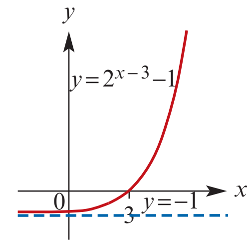
Chapter 6

Exercise 6A

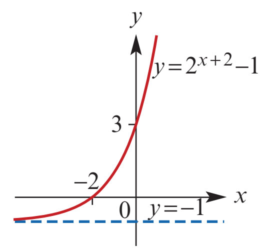
1 a Range $= (-2, \infty)$



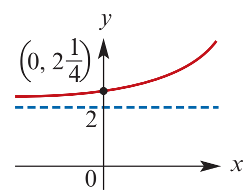
b Range $= (-1, \infty)$



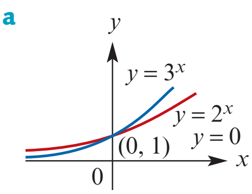
c Range $= (-1, \infty)$



d Range $= (2, \infty)$



2 a



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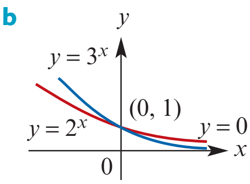

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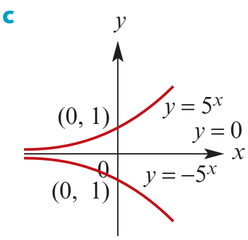

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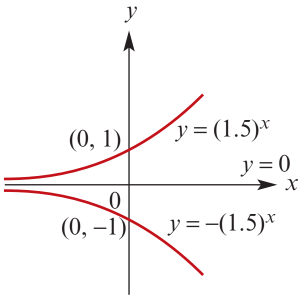
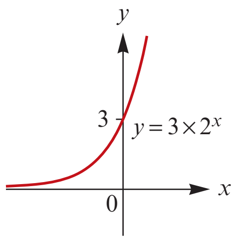
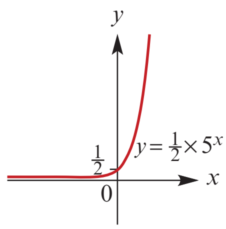
d



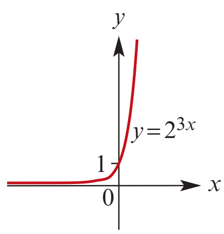
3 **a** Range = $(0, \infty)$



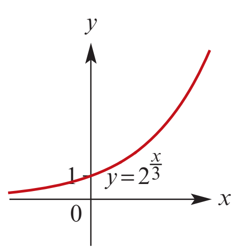
b Range = $(0, \infty)$



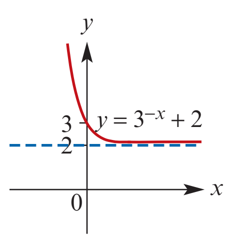
c Range = $(0, \infty)$



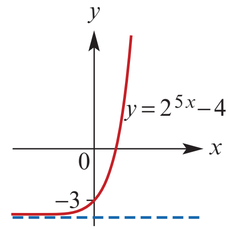
d Range = $(0, \infty)$



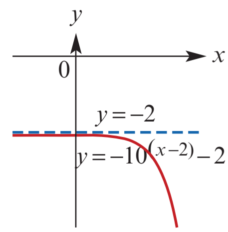
4 **a** Range = $(2, \infty)$



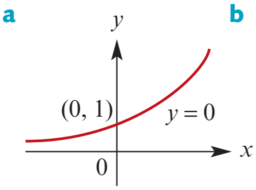
b Range = $(-4, \infty)$



c Range = $(-\infty, -2)$

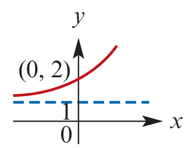


5 **a**



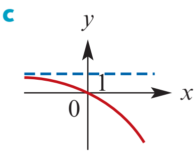
Range = $\mathbb{R}^+$

b



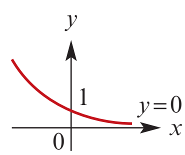
Range = $(1, \infty)$

c



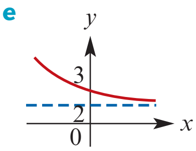
Range = $(-\infty, 1)$

d



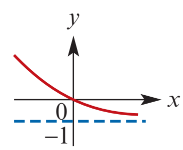
Range = $\mathbb{R}^+$

e



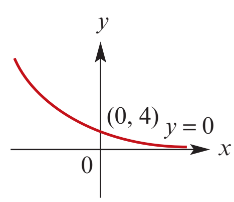
Range = $(2, \infty)$

f

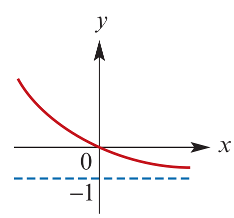


Range = $(-1, \infty)$

6 **a** Range = $\mathbb{R}^+$



b Range = $(-1, \infty)$



Answers 6A

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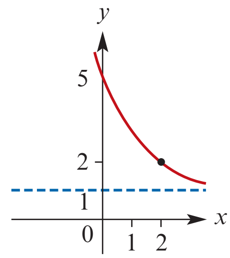
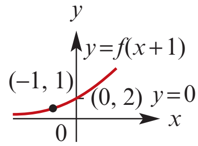
c Range = $(1, \infty)$



7 a



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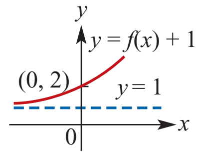

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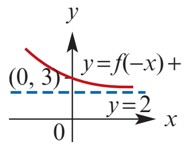

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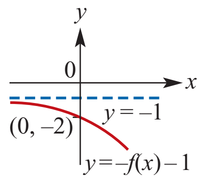

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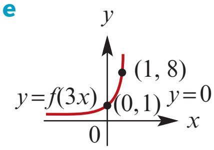

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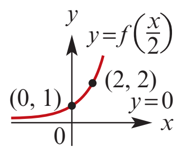

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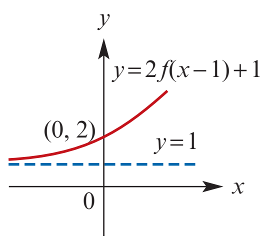

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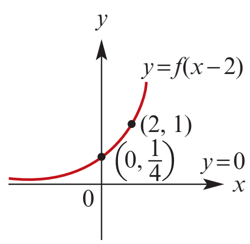
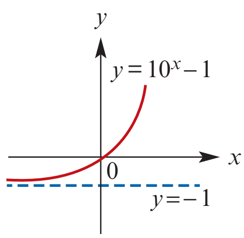
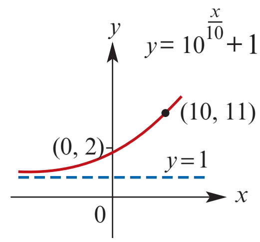


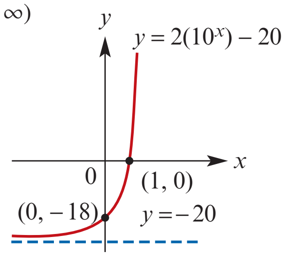
8 a Range = $(-1, \infty)$



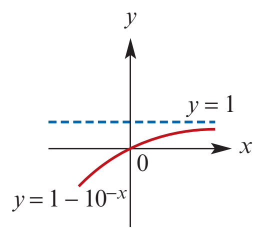
b Range = $(1, \infty)$



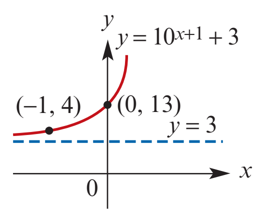
c Range = $(-20, \infty)$



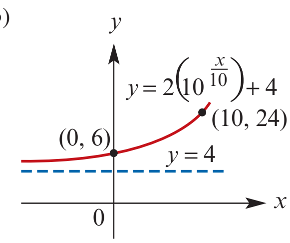
d Range = $(-\infty, 1)$



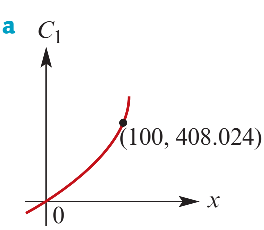
e Range = $(3, \infty)$



f Range = $(4, \infty)$



9 a



b **i** \$408.02 **ii** \$1274.70

c 239 days

d **ii** 302 days

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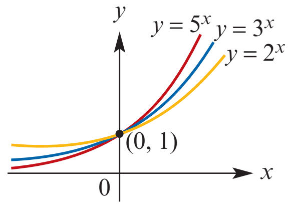
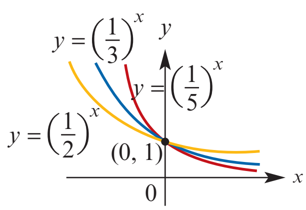
10 36 days

11 a i



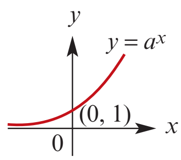
ii $x < 0$ **iii** $x > 0$ **iv** $x = 0$

b i

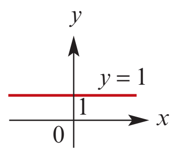


ii $x > 0$ **iii** $x < 0$ **iv** $x = 0$

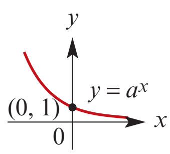
c i $a > 1$



ii $a = 1$

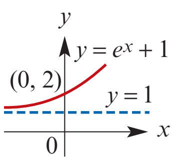


iii $0 < a < 1$



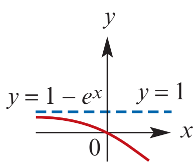
Exercise 6B

1 a



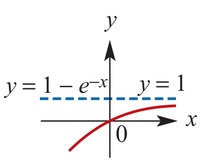
Range = $(1, \infty)$

b



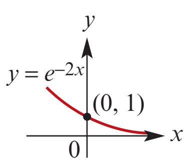
Range = $(-\infty, 1)$

c



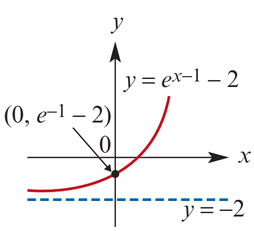
Range = $(-\infty, 1)$

d



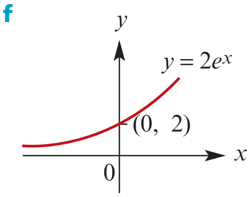
Range = $(0, \infty)$

e



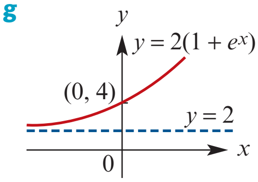
Range = $(-2, \infty)$

f



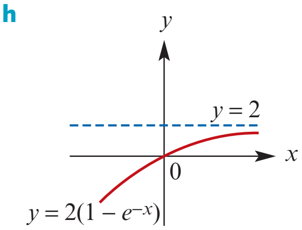
Range = $(0, \infty)$

g



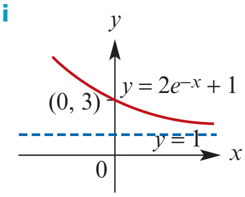
Range = $(2, \infty)$

h



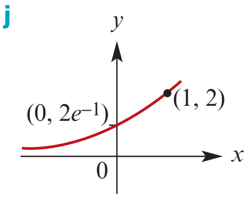
Range = $(-\infty, 2)$

i



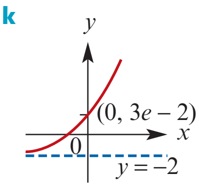
Range = $(1, \infty)$

j



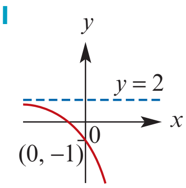
Range = $(0, \infty)$

k



Range = $(-2, \infty)$

l



Range = $(-\infty, 2)$

2 a Translation 2 units to the left and 3 units down

b Dilation of factor 3 from the x-axis, then translation 1 unit to the left and 4 units down

c Dilation of factor 5 from the x-axis and factor $\frac{1}{2}$ from the y-axis, then translation $\frac{1}{2}$ unit to the left

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d Reflection in the x-axis, then translation 1 unit to the right and 2 units up

e Dilation of factor 2 from the x-axis, reflection in the x-axis, then translation 2 units to the left and 3 units up

f Dilation of factor 4 from the x-axis and factor $\frac{1}{2}$ from the y-axis, then translation 1 unit down

3 a $y = -2e^{x-3} - 4$ **b** $y = 4 - 2e^{x-3}$
c $y = -2e^{x-3} - 4$ **d** $y = -2e^{x-3} - 8$
e $y = 8 - 2e^{x-3}$ **f** $y = -2e^{x-3} + 8$

4 a Translation 2 units to the right and 3 units up

b Translation 1 unit to the right and 4 units up, then dilation of factor $\frac{1}{3}$ from the x-axis

c Translation $\frac{1}{2}$ unit to the right, then dilation of factor $\frac{1}{5}$ from the x-axis and factor 2 from the y-axis

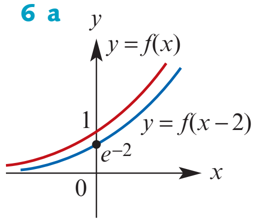
d Translation 1 unit to the left and 2 units down, then reflection in the x-axis

e Translation 2 units to the right and 3 units down, then dilation of factor $\frac{1}{2}$ from the x-axis and reflection in the x-axis

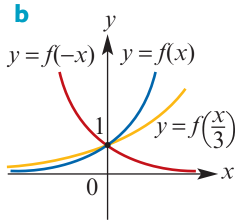
f Translation 1 unit up, then dilation of factor $\frac{1}{4}$ from the x-axis and factor 2 from the y-axis

5 a $x = 1.146$ or $x = -1.841$
b $x = -0.443$
c $x = -0.703$
d $x = 1.857$ or $x = 4.536$

6 a



b



Exercise 6C

1 a $6x^6y^9$ **b** $3x^6$ **c** $\dfrac{6y^2}{x^2}$ **d** 8
e 16 **f** $\dfrac{5x^{28}}{y^6}$ **g** $24x^5y^{10}$ **h** $2xy^2$
i x^2y^2

2 a 4 **b** $\dfrac{1}{2}$ **c** 8 **d** $\dfrac{1}{4}$ **e** $\dfrac{3}{5}$ **f** 3
g $\dfrac{5}{2}$ **h** 6 **i** 4

3 a 1 **b** 1 **c** $-\dfrac{3}{2}$ **d** 3 **e** -2 **f** 4
g $-\dfrac{10}{3}$ **h** $-\dfrac{3}{2}$ **i** 6 **j** $\dfrac{3}{5}$ **k** $\pm\dfrac{1}{2}$

4 a 1 **b** 2 **c** 1 **d** 1, 2 **e** 0, 1
f 2, 4 **g** 0, 1 **h** $-1, 2$ **i** $-1, 0$

Exercise 6D

1 a 3 **b** -4 **c** -3 **d** 6
e 6 **f** -7

2 a $\log_{10} 6$ **b** $\log_{10} 4$ **c** $\ln(10^6) = 6 \ln 10$
d $\ln 7$ **e** $\ln \dfrac{1}{60} = -\ln 60$
f $\ln(u^3v^6) = 3\ln(uv^2)$ **g** $\ln(x^7) = 7\ln x$
h $\ln 1 = 0$

3 a $x = 100$ **b** $x = 16$ **c** $x = 6$ **d** $x = 64$
e $x = e^3 - 5 \approx 15.086$ **f** $x = \frac{1}{2}$ **g** $x = -1$
h $x = 10^{-3} = \dfrac{1}{1000}$ **i** $x = 36$

4 a $x = 15$ **b** $x = 5$ **c** $x = 4$
d $x = 1$ ($x = -\frac{1}{2}$ is not an allowable solution)
e $x = \frac{3}{2}$

5 a $\log_{10} 27$ **b** $\log_2 4 = 2$
c $\dfrac{1}{2}\log_{10}\left(\dfrac{a}{b}\right) = \log_{10}\sqrt{\dfrac{a}{b}}$ **d** $\log_{10}\left(\dfrac{10a}{b^{\frac{1}{3}}}\right)$
e $\log_{10}\left(\dfrac{1}{8}\right) = -3\log_{10} 2$

6 a 1 **b** 1 **c** $2\frac{1}{2}$ **d** 3 **e** 0

7 a $-x$ **b** $2\log_2 x$ **c** 0

8 a $x = 4$ **b** $x = \dfrac{3e}{5 + 2e} \approx 0.7814$

9 a $x = \dfrac{-1 + \sqrt{1 + 12e}}{6}$, i.e. $x \approx 0.7997$
b $x = \ln 2 \approx 0.6931$

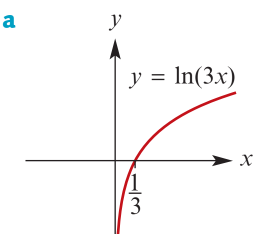
10 a $x = 3$ **b** $x = \dfrac{1}{2}$

11 $\dfrac{1}{4}, 2$

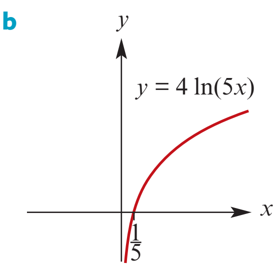
12 $N = \left(\dfrac{2}{3}\right)^3 = \dfrac{8}{27}$

Exercise 6E

1 a



b



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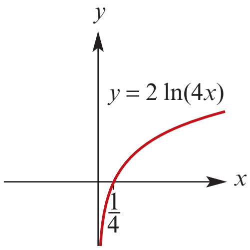
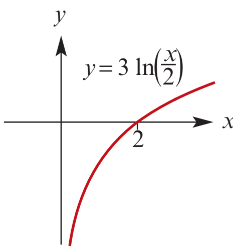
c

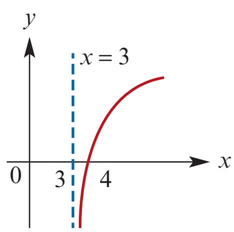


d



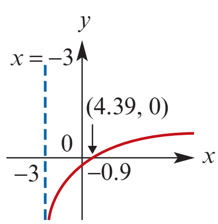
2 a Domain = $(3, \infty)$

Range = $\mathbb{R}$



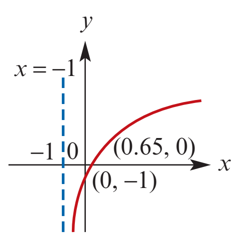
b Domain = $(-3, \infty)$

Range = $\mathbb{R}$



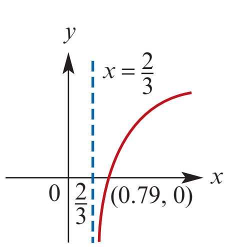
c Domain = $(-1, \infty)$

Range = $\mathbb{R}$



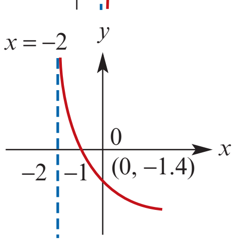
d Domain = $\left(\frac{2}{3}, \infty\right)$

Range = $\mathbb{R}$



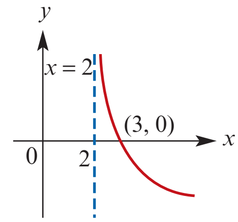
e Domain = $(-2, \infty)$

Range = $\mathbb{R}$



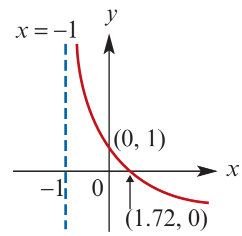
f Domain = $(2, \infty)$

Range = $\mathbb{R}$



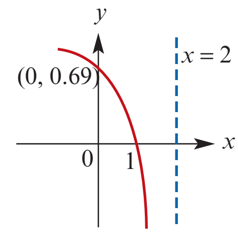
g Domain = $(-1, \infty)$

Range = $\mathbb{R}$



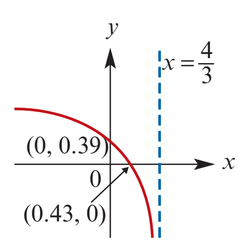
h Domain = $(-\infty, 2)$

Range = $\mathbb{R}$

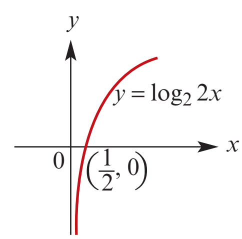


i Domain = $\left(-\infty, \frac{4}{3}\right)$

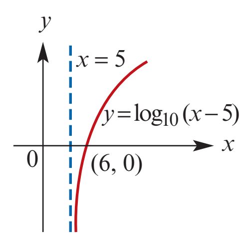
Range = $\mathbb{R}$



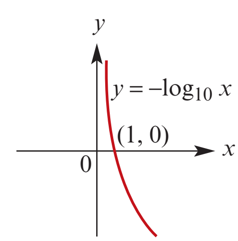
3 a Domain = $\mathbb{R}^+$



b Domain = $(5, \infty)$



c Domain = $\mathbb{R}^+$



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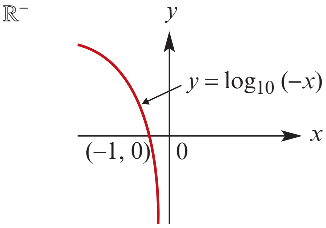
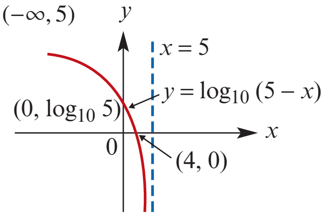
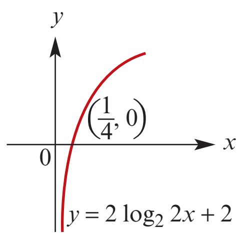
d Domain = $\mathbb{R}^-$



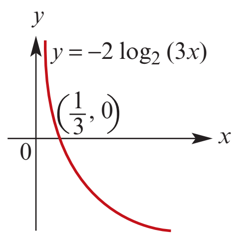
e Domain = $(-\infty, 5)$



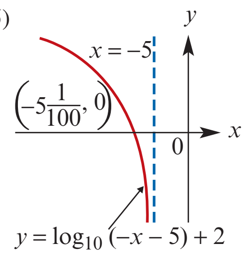
f Domain = $\mathbb{R}^+$



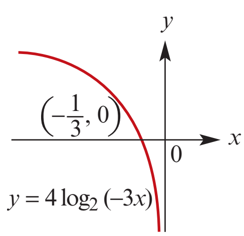
g Domain = $\mathbb{R}^+$



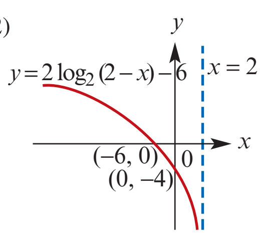
h Domain = $(-\infty, -5)$



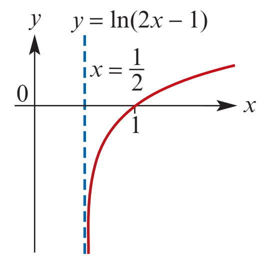
i Domain = $\mathbb{R}^-$



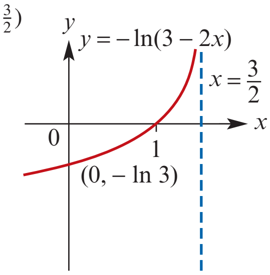
j Domain = $(-\infty, 2)$



k Domain = $(\frac{1}{2}, \infty)$

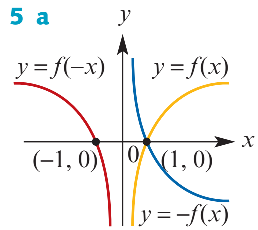


l Domain = $(-\infty, \frac{3}{2})$

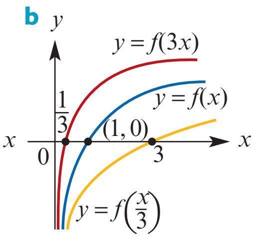


4 **a** $x = 1.557$ **b** $x = 1.189$

5 **a**



b



6 A dilation of factor ln 3 from the y-axis

7 A dilation of factor $\dfrac{1}{\ln 2}$ from the y-axis

Exercise 6F

1 $a = \dfrac{6}{e^4 - 1}$, $b = \dfrac{5e^4 - 11}{e^4 - 1}$

2 $a = \dfrac{2}{\ln 6}$, $b = -4$ **3** $a = 2$, $b = 4$

4 $a = \dfrac{14}{e - 1}$, $b = \dfrac{14}{1 - e}$ $(a \approx 8.148,\ b \approx -8.148)$

5 $a = 250$, $b = \dfrac{1}{3}\ln 5$ **6** $a = 200$, $b = 500$

7 $a = 2$, $b = 4$ **8** $a = 3$, $b = 5$

9 $a = 2$, $b = \dfrac{1}{3}\ln 5$ **10** $a = 2$, $b = 3$

11 $b = 1$, $a = \dfrac{2}{\ln 2}$, $c = 8$ $(a \approx 2.885)$

12 $a = \dfrac{2}{\ln 2}$, $b = 4$

Exercise 6G

1 **a** 2.58 **b** −0.32 **c** 2.18 **d** 1.16
e −2.32 **f** −0.68 **g** −2.15 **h** −1.38
i 2.89 **j** −1.70 **k** −4.42 **l** 5.76
m −6.21 **n** 2.38 **o** 2.80

2 **a** $x < 2.81$ **b** $x > 1.63$
c $x < -0.68$ **d** $x \geq 0.57$

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3 **a** $\log_2 5$ **b** $\frac{1}{2}(\log_3 8 + 1)$
c $\frac{1}{3}(\log_7 20 - 1)$ **d** $\log_3 7$ **e** $\log_3 6$
f $\log_5 6$ **g** $x = \log_3 8$ or $x = 0$ **h** $x = 1$

4 **a** $x > \log_7 52$ **b** $x < \frac{1}{2}\log_3 120$
c $x \geq \frac{1}{6}\log_2\left(\frac{5}{4}\right)$ **d** $x \leq \log_3 7290$
e $x < \log_3 106$ **f** $x < \log_5\left(\frac{3}{5}\right)$

5 **a** $x = \frac{19}{2}$ **b** $x = \frac{16}{5}$

6 **a** $x = \frac{2}{3}$ **b** $x = \frac{25}{11}$

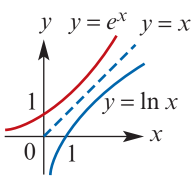
7 **a** $x = \frac{9}{5}$ **b** $x = \frac{2}{9}$ **c** $x = \frac{7}{2}$

8 **a** $k = \frac{1}{\log_2 7}$ **b** $x = \frac{\log_2 7 - 4}{\log_2 7}$
c $x = \frac{\ln 7 - 1}{\ln 14}$

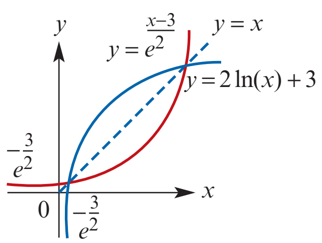
9 $x = e^{\frac{y+4}{3}}$

10 **a** $x = \frac{1}{2}e^y$ **b** $x = \frac{1}{2}e^{\frac{y-1}{3}}$
c $x = \ln(y - 2)$ **d** $x = \ln(y) - 2$
e $x = \frac{1}{2}(e^y - 1)$ **f** $x = \frac{1}{3}\left(e^{\frac{y}{4}} - 2\right)$
g $x = 10^y - 1$ **h** $x = \ln\left(\frac{y}{2}\right) + 1$

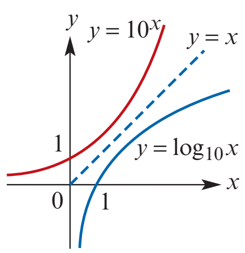
11 **a** **i**



ii



iii



b The graphs of $y = f(x)$ and $y = g(x)$ are reflections of each other in the line $y = x$

12 $t = \frac{-1}{k}\ln\left(\frac{P-b}{A}\right)$

13 **a** $x = e^{\frac{y-5}{2}}$ **b** $x = -\frac{1}{6}\ln\left(\frac{P}{A}\right)$
c $n = \frac{\ln\left(\frac{y}{a}\right)}{\ln x}$ **d** $x = \log_{10}\left(\frac{y}{5}\right)$
e $x = \frac{1}{2}e^{\frac{5-y}{3}}$ **f** $n = \frac{\ln\left(\frac{y}{6}\right)}{2\ln x}$
g $x = \frac{1}{2}(e^y + 1)$ **h** $x = \ln\left(\frac{5}{5-y}\right)$

14 **a** $a \approx 0.544$ **b** $k \approx 549.3$

16 **a** 9^u **b** $u + \frac{1}{2}$ **c** $\frac{2}{u}$

17 $x = 625$ or $x = \frac{1}{625}$

18 $\frac{2}{p}$

19 **b** $x = 4$

Exercise 6H

1 **a** $N = 1000 \times 2^{\frac{t}{15}}$ **b** 50 minutes

2 $d_0 = 52\left(\frac{13}{20}\right)^{\frac{1}{2}}$, $m = \frac{1}{2}\log_{10}\left(\frac{20}{13}\right)$

3 **a** **i** $N_0 = 20\,000$ **ii** -0.223
b 6.2 years

4 **a** $M_0 = 10$, $k = 4.95 \times 10^{-3}$
b 7.07 grams **c** 325 days

5 **a** $k = \frac{1}{1690}\ln 2$ **b** 3924 years

6 55 726 years

7 7575 years

8 **a** 16 471 **b** 35 years on from 2002

9 18.4 years

10 **a** 607 millibars **b** 6.389 km

11 21.82 hours

12 6.4°C

13 $k = 0.349$, $N_0 = 50.25$

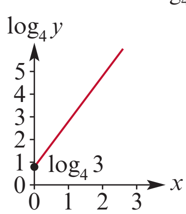
14 **a** $k = \ln\left(\frac{5}{4}\right)$ **b** 7.21 hours

15 **a** $a = 1000$, $b = 15^{\frac{1}{5}}$ **b** 3 hours
c 13 hours **d** 664 690

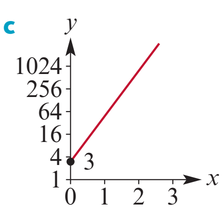
Exercise 6I

1 **a** $m = 2$ and $c = \log_4 3$

b



c



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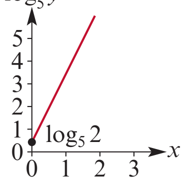
2 **a** $m = 3$ and $c = \log_5 2$

b



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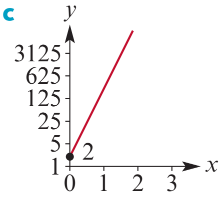


3 Street 10^{-9} watt/cm^2; Quiet car 10^{-11} watt/cm^2

4 **a** Increases by $10\log_{10} 2 \approx 3$ dB
b Increases by $10\log_{10} 10 = 10$ dB
c $P_{\text{new}} = (P_{\text{old}})^3 \times 10^{32}$
d $P = 10^{-16}$ **e** $P = 10^{-6}$

5 $P_1 = 10^{\frac{\lambda}{10}} \times P_2$ **6** $5 + \log_{10} 5 \approx 5.7$

7 $7.3 - \log_{10} 4 \approx 6.7$ **8** $[10^{-4}, 10^{-2}]$

9 **a** 10^{-7} moles per litre **b** 2.3
c 1.48 litres **d** 2.66

10 **a** $R = \frac{2}{3}(\log_{10} E - 4.4)$
b **i** 7.943×10^{11} J **ii** 2.512×10^{13} J
iii 7.943×10^{14} J **iv** 2.512×10^{16} J
v 1.995×10^{17} J
c 5.73

11 $100 + 10\log_{10} 2 \approx 103$ dB

12 **a** **i** \$4024.23 **ii** \$6500.00
iii \$11 903.09 **iv** \$17 647.05
b $1.5\log_{10} x + \frac{2000}{x}$
c **i** \$8.05 **ii** \$6.50 **iii** \$5.95 **iv** \$5.88

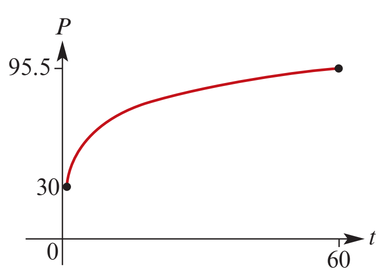
13 **a** $T_0 = 80$, $c = \frac{4}{\ln(\frac{8}{5})} \approx 8.511$
b 5.90 minutes **c** 8.36 minutes

14 **a** $y = e^2 \cdot e^{\frac{-2}{3}x}$ **b** $y \approx 7.39(0.51)^x$

15 **a** 7.3 **b** 63.55

16 **a** **i** 15.75 days **ii** 18.62 days
iii 30.40 days **iv** 34.80 days
b $n = \frac{2000}{1 + 1999e^{-\frac{t}{4}}}$

17 **a**



b 78% **c** 22.76 minutes
d $t = e^{\frac{P-30}{16}}$

18 $t = \frac{1}{k}\ln\left(\frac{T_0 - T_s}{T - T_s}\right)$

Exercise 6J

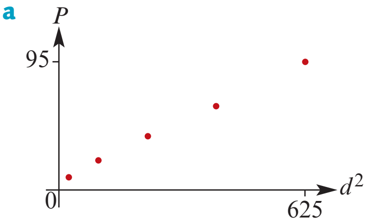
1 $a = 4.77$, $b = 0.976$

2 $y = 0.145x^2 + 2.065x + 6.011$

3 **a** $y = 1.04 \times 1.25^x$ **b** $y = 1.1 \times 1.05^x$

4 **a** $y = 3 + \ln x$ **b** $y = 5.23 + 8.5\ln x$
c $y = 6.28 + 9.49\ln x$

5 **a**

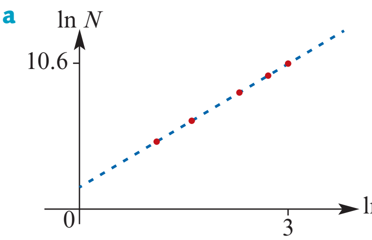


b Answers will vary. Linear regression gives ($P = 0.14d^2 + 7.4$)

6 **a** $y = 3.14\sin(3.05x + 0.04) - 0.14$
b $y = 4.97\sin(3.01x + 3.14) + 4.97$

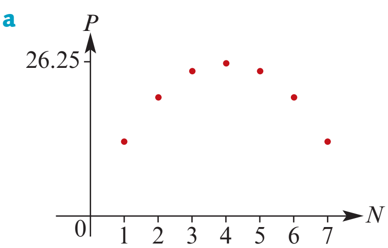
7 $a = 751.733$, $k = 0.35$

8 **a**



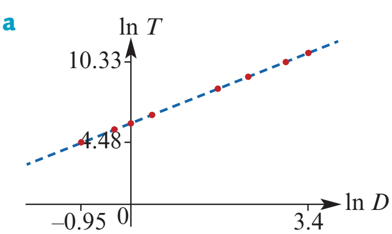
b $a = 2.98$, $b = 1.65$ **c** $A = 2.98$, $B = 5.21$

9 **a**



b $a = -1.5$, $b = 12$, $c = 2.25$

10 **a**



b $\ln T = 1.5\ln D + 5.90$
c $T = 365 \times D^{1.5}$ **d** $T^2 \propto D^3$

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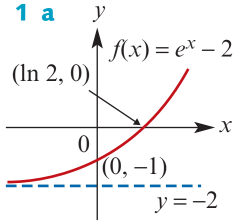
Chapter 6 review

Technology-free questions

1 a



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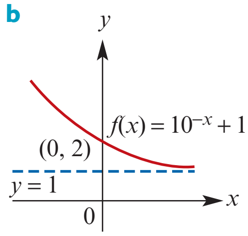

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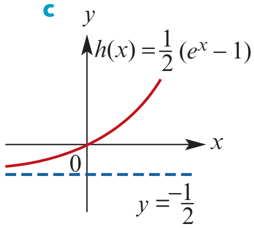

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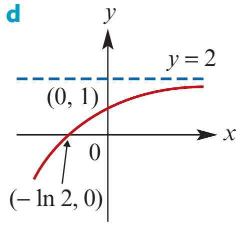

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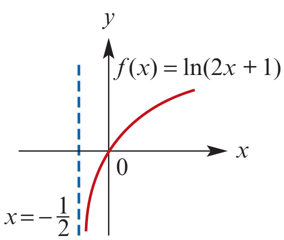

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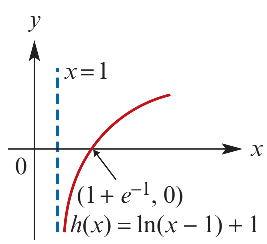

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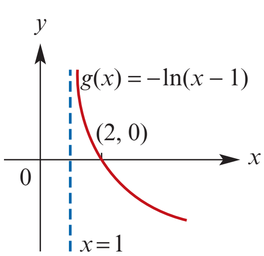

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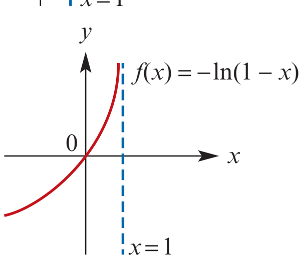


2 a $y = e^2 x$ **b** $y = 10x$ **c** $y = 16x^3$

d $y = \dfrac{x^5}{10}$ **e** $y = \dfrac{e^3}{x}$ **f** $y = e^{2x-3}$

3 a $x = \dfrac{\ln 11}{\ln 3}$ **b** $x = \dfrac{\ln 0.8}{\ln 2}$ **c** $x = \dfrac{\ln 3}{\ln(\frac{2}{3})}$

4 a $x = 1$ **b** $x = \dfrac{2}{3}$ **c** $x = \dfrac{1}{20}$

d $x = \log_{10} 3$ or $x = \log_{10} 4$

5 $a = 2,\ b = 2^{-\frac{2}{3}} - 1$ **6** $10^{\frac{6}{5}} - 1$

7 $\dfrac{1}{3}\ln\left(\dfrac{287}{4}\right)$ **8** $2a$

9 $x = 3^y + 4$

10 $a = \ln 5,\ b = 5,\ k = 2$

11 $x = \dfrac{1}{3}\ln(y + 4)$ **13** 3

14 a $k = \dfrac{1}{7}$ **b** $q = \dfrac{3}{2}$

15 $y = e^a \cdot x^b$

16 a $f(-x) = f(x)$ **b** $2(e^u + e^{-u})$

c 0 **d** $e^{2u} + e^{-2u}$

e $g(-x) = -g(x)$ **f** $2e^x,\ 2e^{-x},\ e^{2x} - e^{-2x}$

Multiple-choice questions

1 C **2** D **3** B **4** A **5** C
6 B **7** A **8** C **9** D **10** A
11 C **12** C **13** B **14** D

Extended-response questions

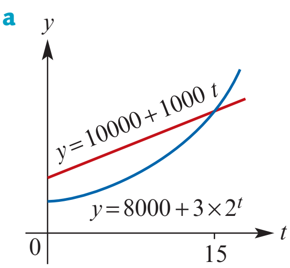
1 a 73.5366°C **b** 59.5946

2 a 75 **b** 2.37 **c** 0.646

3 a $k = 22\,497,\ \lambda = 0.22$ **b** \$11 627

4 $k = -0.5,\ A_0 = 100$

5 a



b i (12.210, 22 209.62)

ii $t = 12.21$ **iii** 22 210

c ii (12.21, 12.21)

d $c = 0.52$

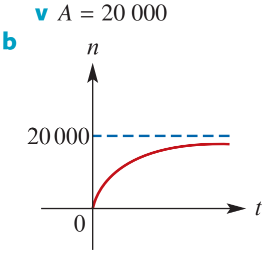
6 a iii $a = \dfrac{1}{2}$ or $a = 1$

iv If $a = 1$, then $e^{-2B} = 1$, and so $B = 0$;

If $a = \dfrac{1}{2}$, then $B = \dfrac{1}{2}\ln 2$

v $A = 20\,000$

b



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c $\dfrac{\ln 0.1}{\frac{1}{2}\ln(\frac{1}{2})} = \dfrac{2\ln 10}{\ln 2} \approx 6.644$

After 6.65 hours, the population is 18 000

7 a $k = 0.235$ **b** 22.7°C **c** 7.17 minutes

8 a **i** $5\ln\left(\dfrac{4}{3}\right) \approx 1.44$ minutes

ii $5\ln 2 \approx 3.47$ minutes

iii $5\ln 4 \approx 6.93$ minutes

b $8\left(1 - e^{-\frac{7}{5}}\right) \approx 6.03$ grams

c $x = 8\left(1 - e^{-\frac{t}{5}}\right)$

9 a $A = 10^{\frac{10R+75}{23}} - 4800$

b 2 010 537.68 km

c 7.69

d **i** 291.7 km **ii** 800.0 km **iii** 1320 km

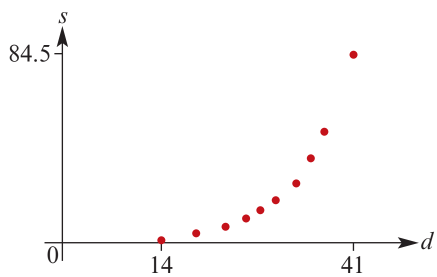
10 a $\alpha = -0.25,\ \beta = 3$

d	0	1	3	7	15	31
A	3	2.75	2.5	2.25	2	1.75

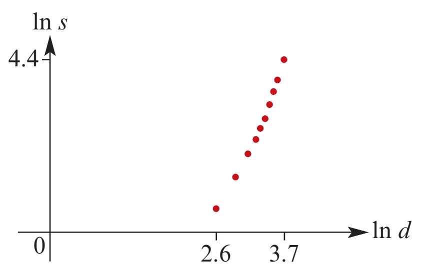
b 4095 cm

c 146 cm

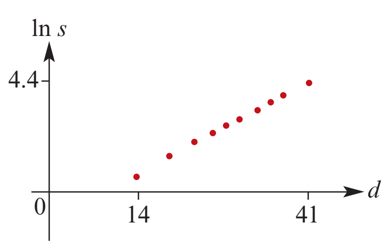
11 a



b



c



d $\ln s = 0.139d - 1.243,\ \ s = 0.289e^{0.139d}$

Chapter 7

Exercise 7A

1 -1

2 -1

3 a $h + 9$ **b** 9

4 a $x + 1$ **b** $2x^3 + 1$ **c** 40 **d** 0
e 5 **f** 1 **g** $2x + 1$ **h** $3x$
i $3x^3 + x$ **j** $6x$

5 a $2 + 3h + h^2$ **b** 2

6 $2x + h,\ 2x$

7 $h + 6,\ 6$

8 a $10x$ **b** 3 **c** 0 **d** $6x + 4$
e $15x^2$ **f** $10x - 6$

Exercise 7B

1 a $5x^4$ **b** $28x^6$ **c** 6 **d** $10x - 4$
e $12x^2 + 12x + 2$ **f** $20x^3 + 9x^2$
g $-4x + 4$ **h** $18x^2 - 4x + 4$

2 a -4 **b** -8 **c** -2 **d** -4

3 a -4 **b** -36

4 a $3t^2$ **b** $3t^2 - 2t$ **c** $x^3 + 9x^2$

5 a -2 **b** 0 **c** $15x^2 - 6x + 2$
d $\dfrac{6x^2 - 8}{5}$ **e** $4x - 5$ **f** $12x - 12$
g $50x^4$ **h** $27x^2 + 3$

6 a $4x - 15x^2$ **b** $-4z - 6$ **c** $18z^2 - 8z$
d $-2 - 15x^2$ **e** $-4z - 6$ **f** $-3z^2 - 8z$

7 a $(-\frac{1}{2}, 3\frac{1}{2})$ **b** $(2, 32), (-2, -32)$
c $(2, 6)$ **d** $(0, 0), (2, -4)$

8 a $(1, 7)$ **b** $\left(\dfrac{5}{4}, \dfrac{59}{8}\right)$

9 a $x = 1$ **b** $x = 0$ **c** $x = \dfrac{1 + \sqrt{3}}{2}$
d $x = \dfrac{3 + \sqrt{3}}{6}$ **e** $x = \dfrac{1 - \sqrt{3}}{2}$

10 a 78.69° **b** 0° **c** 45° **d** 135°
e 63.43° **f** 116.57°

11 a $8x - 4$ **b** $2x + 2$ **c** $6x^2 - 12x + 18$
d $x^2 - 2x + 1$

12 a $(3, 16)$, gradient $= 8$
b $(0, -1)$, gradient $= -1$
c $(-1, 6)$, gradient $= -8$
d $(4, 594)$, gradient $= 393$
e $(1, -28)$, gradient $= -92$
f $(2\frac{1}{2}, 0)$, gradient $= 0$

13 a $x = 1$ **b** $x = 1$ **c** $x > 1$ **d** $x < 1$
e $x = 2\frac{2}{3}$ **f** $x = 4$ or $x = -2$

14 a $x < -1$ or $x > 1$ **b** $-1 < x < 1$
c $x = -1$ or $x = 1$

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15 a $-1 < x < 0.5$ or $x > 2$
b $x < -1$ or $0.5 < x < 2$
c $x = -1$ or $x = 0.5$ or $x = 2$

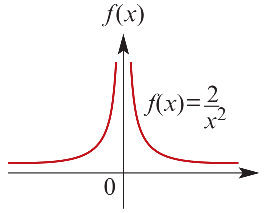
16 a $-\frac{1}{4} < x < 2$ or $x > 2$ **b** $x < -\frac{1}{4}$
c $x = -\frac{1}{4}$ or $x = 2$

17 a $(2, -12)$ **b** $(3, -11)$ **c** $\left(\frac{5}{4}, -\frac{183}{16}\right)$

21 a $(-\infty, -1]$ **b** $[2, \infty)$ **c** $(-\infty, 0]$
d $\left[\frac{3}{2}, \infty\right)$

Exercise 7C

1 a



b $\frac{-2(2+h)}{(1+h)^2}$ **c** -4

2 a $\frac{-1}{(x-3)^2}$ **b** $\frac{-1}{(x+2)^2}$

3 $-4x^{-5}$

4 a $-6x^{-3} - 5x^{-2}$ **b** $12x - \frac{15}{x^4}$
c $\frac{15}{x^4} - \frac{8}{x^3}$ **d** $-18x^{-4} - 6x^{-3}$ **e** $-\frac{2}{x^2}$

5 a $\frac{4}{z^2}$ **b** $\frac{-18-2z}{z^4}$ **c** $3z^{-4}$
d $\frac{-2z^3 + z^2 - 4}{z^2}$ **e** $\frac{6-12z}{z^4}$
f $-6x - \frac{6}{x^2}$

6 a $11\frac{3}{4}$ **b** $\frac{1}{32}$ **c** -1 **d** 5

7 $f'(x) = 10x^{-6} > 0$ for all $x \in \mathbb{R} \setminus \{0\}$

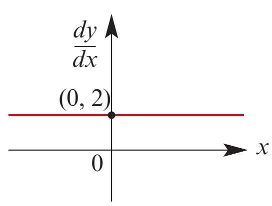
8 $\pm\frac{1}{2}$ **9** $a = -1,\ b = 4$

10 $\frac{1}{2}$ **11** $a = -9,\ b = 1$

12 $k = 0$ or $k = \frac{3}{2}$

Exercise 7D

1 a



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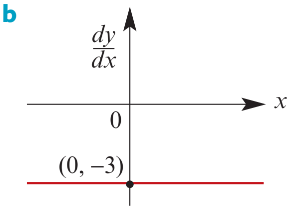

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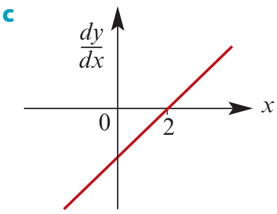

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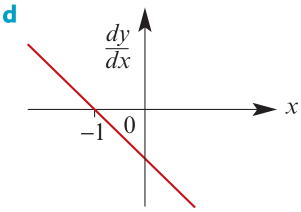

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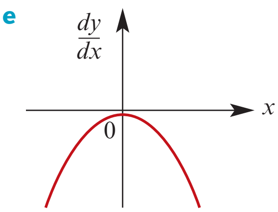

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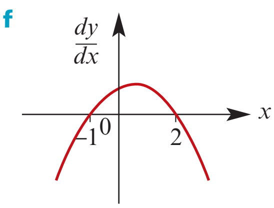
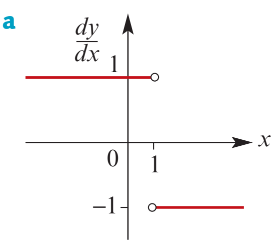


2 a



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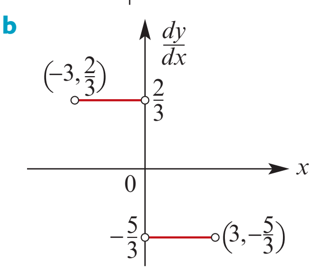

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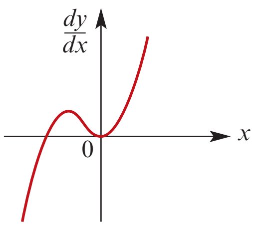

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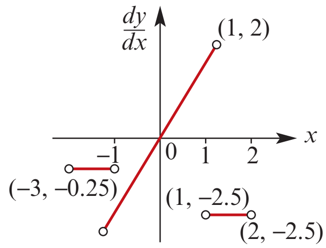

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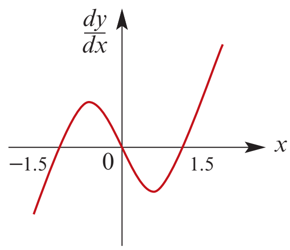

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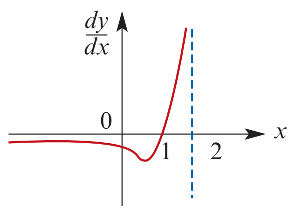

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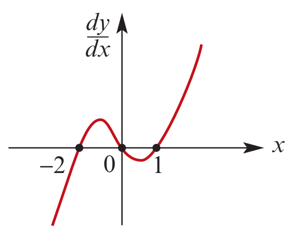

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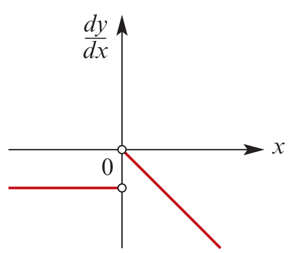

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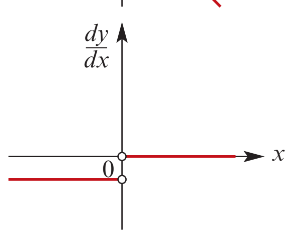
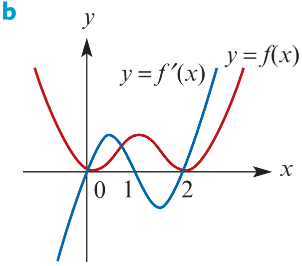


3 a D **b** F **c** B **d** C **e** A **f** E

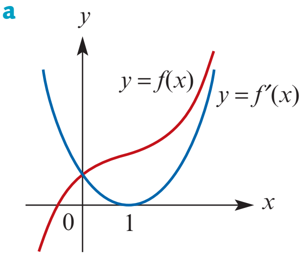
4 a b



c i 0 **ii** 0 **iii** 0 **iv** 96

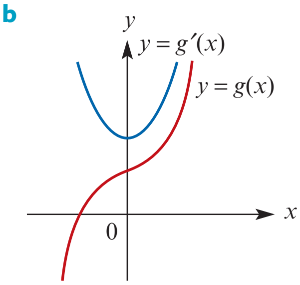
d i 1 **ii** 0.423

5 a



Gradient is 0 at $(1, \frac{4}{3})$;
Gradient is positive for $\mathbb{R} \setminus \{1\}$

b



Gradient is always positive;
Minimum gradient where $x = 0$

Chapter 7 review

Technology-free questions

1 a $h^2 + 6h + 12$ **b** 12

2 a 8 **b** −8

3 a −3 **b** $8x - 2$ **c** $x + \frac{3}{2}$

d $-\frac{10}{x^3} + \frac{3}{x^2} + 2x$ **e** $2x - \frac{4}{x^2}$

f $\frac{12}{x^3} - \frac{20}{x^5}$

4 a $4x - 3$ **b** −3 **c** $x = 1$

5 a $x = 0$ or $x = -2$ **b** $x > 0$ or $x < -2$

c $-2 < x < 0$

7 24π

8 $\left(\frac{5}{18}, \frac{5}{36}\right)$

9 $\frac{2}{3}$

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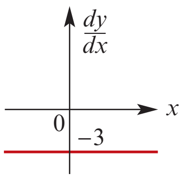
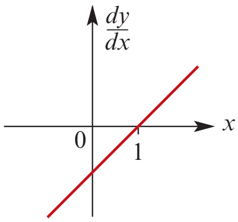
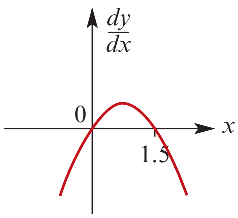
11 a



b



c



12 $0 < x < 2$

13 $x = -\frac{1}{2}$

14 $b = \frac{1}{4},\ c = 1$

Multiple-choice questions

1 B **2** D **3** A **4** C **5** D
6 A **7** D **8** B **9** D

Chapter 8

Exercise 8A

1 a $8x(x^2+1)^3$ **b** $20x(2x^2-3)^4$
c $24(6x+1)^3$ **d** $an(ax+b)^{n-1}$
e $2anx(ax^2+b)^{n-1}$ **f** $\frac{6x}{(1-x^2)^4}$
g $-3\left(x^2-\frac{1}{x^2}\right)^{-4}\left(2x+\frac{2}{x^3}\right)$
h $(1-x)^{-2}$

2 a $6(x+1)^5$ **b** $4x^3(3x+1)(x+1)^7$
c $4\left(6x^3+\frac{2}{x}\right)^3\left(18x^2-\frac{2}{x^2}\right)$
d $-4(x+1)^{-5}$

3 -10

4 $-\frac{1}{2}$ and $\frac{1}{2}$

5 $2x\sqrt{3x^2+1}$

6 a $n[f(x)]^{n-1}f'(x)$ **b** $\frac{-f'(x)}{[f(x)]^2}$

Exercise 8B

1 $x^{-\frac{1}{2}}$

2 a $\frac{x^{-\frac{4}{5}}}{5}$ **b** $\frac{5}{2}x^{\frac{3}{2}}$
c $\frac{5}{2}x^{\frac{3}{2}}-\frac{3}{2}x^{\frac{1}{2}}$ **d** $\frac{3}{2}x^{-\frac{1}{2}}-\frac{20}{3}x^{\frac{2}{3}}$
e $-\frac{6}{7}x^{-\frac{13}{7}}$ **f** $-\frac{1}{4}x^{-\frac{5}{4}}+2x^{-\frac{1}{2}}$

3 a $\frac{1}{27}$ **b** $\frac{1}{12}$ **c** $\frac{2}{9}$ **d** $\frac{5}{2}$

4 a $\frac{1}{\sqrt{2x+1}}$ **b** $\frac{-3}{2\sqrt{4-3x}}$
c $\frac{x}{\sqrt{x^2+2}}$ **d** $\frac{-1}{\sqrt[3]{(4-3x)^2}}$
e $\frac{3}{2}\sqrt{x}-\frac{1}{\sqrt{x^3}}$ **f** $3\sqrt{x}\left(\frac{5x+6}{2}\right)$

7 a $\frac{x}{\sqrt{x^2+2}}$ **b** $\frac{2x-5}{3\sqrt[3]{(x^2-5x)^2}}$
c $\frac{2x+2}{5\sqrt[5]{(x^2+2x)^4}}$

Exercise 8C

1 a $5e^{5x}$ **b** $-21e^{-3x}$
c $-12e^{-4x}+e^x-2x$ **d** e^x-e^{-x}
e $e^{-2x}(e^x-1)$ **f** $2e^{2x}-2e^{-2x}$

2 a $-6x^2e^{-2x^3}$ **b** $2xe^{x^2}+3$
c $(2x-4)e^{x^2-4x}+3$ **d** $(2x-2)e^{x^2-2x+3}-1$
e $-\frac{1}{x^2}e^{\frac{1}{x}}$ **f** $\frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}$

3 a $\frac{9}{2}$ **b** $\frac{1}{2}e^{\frac{1}{2}}+4$

4 a 5 **b** $5e^4+2$

5 a $2f'(x)e^{2f(x)}$ **b** $2e^{2x}f'(e^{2x})$

6 a $8e^{2x}(e^{2x}-1)^3$ **b** $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$ **c** $\frac{e^x}{2\sqrt{e^x-1}}$
d $\frac{2}{3}e^{x^{\frac{2}{3}}}x^{-\frac{1}{3}}$ **e** $(2x-3)e^{(x-1)(x-2)}$
f e^{e^x+x}

Exercise 8D

1 a $\frac{2}{x}$ **b** $\frac{2}{x}$ **c** $2x+\frac{3}{x}$
d $\frac{3x-1}{x^2}$ **e** $\frac{3+x}{x}$ **f** $\frac{1}{x+1}$
g $\frac{1}{x+2}$ **h** $\frac{3}{3x-1}$ **i** $\frac{6}{6x-1}$

2 a $\frac{3}{x}$ **b** $\frac{3(\ln x)^2}{x}$ **c** $\frac{2x+1}{x^2+x-1}$
d $\frac{3x^2+2x}{x^3+x^2}$ **e** $\frac{4}{2x+3}$ **f** $\frac{4}{2x-3}$

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3 a $\frac{2x}{x^2+1}$ **b** 1

4 a $(e, 1),\ m = \frac{1}{e}$

b $(e, \ln(e^2+1)),\ m = \frac{2e}{e^2+1}$

c $(-e, 1),\ m = -\frac{1}{e}$ **d** $(1, 1),\ m = 2$

e $(1, 0),\ m = 0$ **f** $(\frac{3}{2}, \ln 2),\ m = 1$

5 $\frac{1}{2}$ **6** $\frac{1+2x}{1+x+x^2}$ **7** $\frac{3}{5}$ **8** 2

Exercise 8E

1 a $5\cos(5x)$ **b** $-5\sin(5x)$

c $2\sin x\cos x$ **d** $-2x\sin(x^2+1)$

e $2\sin\left(x-\frac{\pi}{4}\right)\cos\left(x-\frac{\pi}{4}\right)$

f $-2\cos\left(x-\frac{\pi}{3}\right)\sin\left(x-\frac{\pi}{3}\right)$

g $6\sin^2\left(2x+\frac{\pi}{6}\right)\cos\left(2x+\frac{\pi}{6}\right)$

h $-6\sin\left(2x-\frac{\pi}{4}\right)\cos^2\left(2x-\frac{\pi}{4}\right)$

2 a $\frac{1}{\sqrt{2}},\ \sqrt{2}$ **b** 1, 0 **c** 2, 0

d 0, 0 **e** 1, 0 **f** 0, −4

3 a $-5\sin(x) - 6\cos(3x)$

b $-\sin x + \cos x$

4 a $-\frac{\pi}{90}\sin x°$ **b** $\frac{\pi}{60}\cos x°$ **c** $\frac{\pi}{60}\cos(3x)°$

5 a $\tan x$ **b** $-\frac{\cos x}{\sin x}$

c $2\cos(x)\,e^{2\sin x}$ **d** $-2\sin(2x)\,e^{\cos(2x)}$

Exercise 8F

1 a $20x^4 + 36x^2 + 4x$ **b** $9x^{\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}}$

c $3(2x-1)^2(8x-1)$ **d** $8(2x^2+1)(6x^2+1)$

e $5(3x+1)^{\frac{1}{2}}(3x+4)$ **f** $\frac{5x^2-8x+1}{\sqrt{2x-4}}$

g $x^2(3x^2+4x+3)(3x^2+2x+1)^{-2}$

h $2x^3(5x^2-2)(2x^2-1)^{-\frac{1}{2}}$

i $2x\sqrt[3]{x^2+2x} + \frac{2x^2(x+1)}{3\sqrt[3]{(x^2+2x)^2}}$

j $\frac{4(5x^2-4)^2(5x^2+2)}{x^3}$

k $\frac{3(x^6-16)}{x^4}$ **l** $\frac{2x^3(9x^2-8)}{5\sqrt[5]{(x^3-x)^4}}$

2 a $e^x(x^2+2x+1)$

b $e^{2x}(2x^3+3x^2+6x+5)$

c $2e^{4x+1}(x+1)(2x+3)$ **d** $\frac{-8x-7}{2e^{4x}\sqrt{x+1}}$

3 a $1+\ln x$ **b** $2x+4x\ln x$ **c** $e^x\ln x + \frac{e^x}{x}$

d $1+\ln(-x)$

4 a $\frac{2x^3(2-x)}{e^{2x}}$ **b** $2e^{2x+3}$

c $\frac{3}{2}(2e^{2x}+1)(e^{2x}+x)^{\frac{1}{2}}$ **d** $\frac{e^x(x-1)}{x^2}$

e $xe^{\frac{1}{2}x^2}$ **f** $-x^2e^{-x}$

5 a $e^x(f'(x)+f(x))$ **b** $\frac{e^x(f(x)-f'(x))}{[f(x)]^2}$

c $f'(x)e^{f(x)}$ **d** $2e^x f'(x)f(x) + [f(x)]^2e^x$

6 a $3x^2\cos(x) - x^3\sin(x)$

b $2x\cos x - (1+x^2)\sin x$

c $-e^{-x}\sin x + e^{-x}\cos x$

d $6\cos x - 6x\sin x$

e $3\cos(3x)\cos(4x) - 4\sin(4x)\sin(3x)$

f $12\sin x + 12x\cos x$

g $2xe^{\sin x} + x^2\cos x\,e^{\sin x}$

h $2x\cos^2 x - 2x^2\cos x\sin x$

7 a $-e^{\pi}$ **b** 0

8 2

Exercise 8G

1 a $\frac{4}{(x+4)^2}$ **b** $\frac{4x}{(x^2+1)^2}$ **c** $\frac{x^{-\frac{1}{2}} - x^{\frac{1}{2}}}{2(1+x)^2}$

d $\frac{(x+2)^2(x-3)(x-1)}{(x^2+1)^2}$ **e** $\frac{2+2x-x^2}{(x^2+2)^2}$

f $\frac{-4x}{(x^2-1)^2}$ **g** $\frac{x^2+4x+1}{(x^2+x+1)^2}$

h $\frac{-2(4x^3+3x^2+1)}{(2x^3+2x)^2}$

2 a 81, 378 **b** 0, 0 **c** 0, 0

d $\frac{1}{2}$, 0 **e** $\frac{3}{2}, -\frac{1}{2}$

3 a $\frac{2x^2+x+1}{\sqrt{x^2+1}}$ **b** $\frac{x(7x^3+3x+4)}{2\sqrt{x^3+1}}$

c $\frac{5}{(x+3)^2}$

4 a $\frac{3e^x-2e^{4x}}{(3+e^{3x})^2}$ **b** $-\left(\frac{(x+1)\sin(x)+\cos(x)}{(x+1)^2}\right)$

c $\frac{x-x\ln(x)+1}{x(x+1)^2}$

5 a $\frac{1-\ln x}{x^2}$ **b** $\frac{1+x^2-2x^2\ln x}{x(1+x^2)^2}$

6 a $\frac{9e^{3x}}{(3+e^{3x})^2}$ **b** $\frac{-2e^x}{(e^x-1)^2}$ **c** $\frac{-8e^{2x}}{(e^{2x}-2)^2}$

7 a −2 **b** -6π **c** $-e^{\pi}$ **d** $-\frac{1}{\pi}$

Exercise 8H

1 $y = 4x-5$

2 $y = -\frac{1}{3}x - 1$

3 $y = x-2$ and $y = -x+3$

4 $y = 18x+1,\ y = -\frac{1}{18}x+1$

5 $\left(\frac{3}{2}, -\frac{11}{4}\right),\ c = -\frac{29}{4}$

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6 a i $y = 2x - 3$ ii $y = -\frac{1}{2}x - \frac{1}{2}$
b i $y = -3x - 1$ ii $y = \frac{1}{3}x - 1$
c i $y = -x - 2$ ii $y = x$
d i $y = 8x + 2$ ii $y = -\frac{1}{8}x - \frac{49}{8}$
e i $y = \frac{3}{2}x + 1$ ii $y = -\frac{2}{3}x + 1$
f i $y = \frac{1}{2}x + \frac{1}{2}$ ii $y = -2x + 3$
g i $y = \frac{2}{3}x + \frac{4}{3}$ ii $y = -\frac{3}{2}x + \frac{7}{2}$
h i $y = 4x - 16$ ii $y = -\frac{1}{4}x - \frac{15}{2}$
i i $y = -2$ ii $x = 2$
j i $y = 4x - 4$ ii $y = -\frac{1}{4}x + \frac{1}{4}$

7 $y = 56x - 160$

8 a $y = -1$ **b** $y = \frac{3}{2}x + \frac{1}{2}$
c $y = -2x - 1$ **d** $y = -4x + 5$

9 a $y = 2x$ **b** $y = -1$
c $y = -x + \pi + 1$ **d** $y = x$

10 a $y = 2$ **b** $y = x$
c $y = 4e^2x - 3e^2$ **d** $y = \frac{e}{2}(x + 1)$
e $y = 3ex - 2e$ **f** $y = 4e^{-2}$

11 a $y = x - 1$ **b** $y = 2x - 1$ **c** $y = kx - 1$

12 a $x = 0$ **b** $x = 0$ **c** $x = 4$
d $x = -5$ **e** $x = -\frac{1}{2}$ **f** $x = -5$

13 $a = 1$

14 $a = e$

15 $a = 0$

16 $a = 0$ or $a = \frac{3}{2}$

17 $OA = \frac{\pi - 2}{2}$

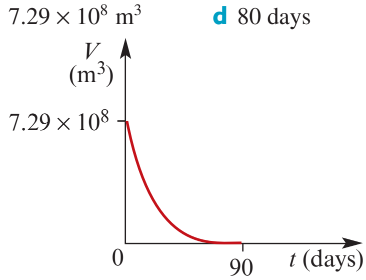
Exercise 8I

1 a 21 **b** $3h + 18$ **c** 18

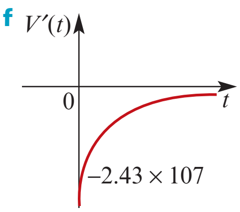
2 a $\frac{dV}{dt}$ **b** $\frac{dS}{dr}$ **c** $\frac{dV}{dx}$

3 Wanes by 0.006 units per day

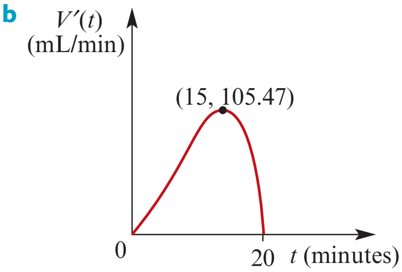
4 a $-3 \times 10^3(90 - t)^2$ **b** 90 days
c 7.29×10^8 m^3 **d** 80 days
e



f



5 a $V'(t) = \frac{t^3}{160}(20 - t)$
b



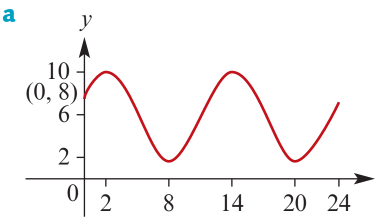
c $t = 15$

6 a $t \approx 100, t \approx 250, t \approx 500$
b $\approx 430\,000$ m^3/day **c** $\approx 270\,000$ m^3/day
d $100 < t < 250,\ 500 < t < 600$

7 a $\lambda = 0.1373,\ P_0 = 30$
b 9.625 hours
c i 4.120 units/hour ii 1.373 units/hour

8 a $-0.3(T - 15)$
b i -22.5°C/minute ii -13.5°C/minute
iii -4.5°C/minute

9 a

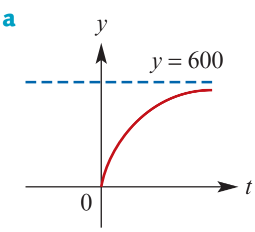


b 2 a.m. **c** $-\frac{2\sqrt{3}}{3}$ metres per hour
d $d'(t) = -\frac{2\pi}{3}\sin\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$
e 5 a.m., 11 a.m., 5 p.m., 11 p.m.
Maximum magnitude = $\frac{2\pi}{3}$ metres per hour

10 $\frac{dy}{dx} = 3 - 2\sin x$, gradient always positive

11 a 4.197 **b** -0.4

12 a



b 3.33

13 a $-2y$ **b** ky

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14 a 0.18 kg
b 3.47 hours
c i 6.93 hours **ii** 10.4 hours
d $0.2m$

15 a $A = 20,\ k = 0.1$ **b** 2 people per day
c After 39.12 days

Exercise 8J

1 a -6 cm/s **b** $t = 3,\ x = -1$
c -2 cm/s **d** $\frac{5}{2}$ cm/s

2 a -3 cm, 24 cm/s **b** $v = 3t^2 - 22t + 24$
c $t = \frac{4}{3},\ t = 6$ **d** $x = \frac{319}{27},\ x = -39$
e $\frac{476}{27}$ cm **f** $\frac{1988}{27}$ cm

3 a i 50 m **ii** 10 m
b $t = 9,\ t = 21$
c $v = \frac{10\pi}{3}\cos\left(\frac{\pi t}{6}\right),\ a = -\frac{5\pi^2}{9}\sin\left(\frac{\pi t}{6}\right)$
d $\frac{10\pi}{3}$ m/s; $t = 0, 6, 12, 18, 24;\ x = 30$
e $\frac{5\pi^2}{9}$ m/s^2; $t = 3, 9, 15, 21;\ x = 50, 10$
f 160 m

4 a $v = 6t^2 - 18t + 12$ **b** $a = 12t - 18$
c $t = 1, x = 5;\ t = 2, x = 4$
d $t = \frac{3}{2},\ x = \frac{9}{2}$

5 a 8 cm **b** 2 cm/s **c** $t = 1,\ x = 9$
d -2 cm/s^2

6 a $v = \frac{\sqrt{2}t}{\sqrt{t^2 + 1}}$ **b** 1 cm/s
c $a = \frac{\sqrt{2}}{\sqrt{(t^2 + 1)^3}}$ **d** $\frac{1}{2}$ cm/s^2

7 When $t = 0$: $v = 0.4$ m/s, $a = 0.4$ m/s^2;
When $t = 1$: $v = 0.4e$ m/s, $a = 0.4e$ m/s^2;
When $t = 2$: $v = 0.4e^2$ m/s, $a = 0.4e^2$ m/s^2

8 a $t = 0,\ t = \frac{\pi}{3},\ t = \frac{2\pi}{3}$
b $a = e^{-2t}(3\cos(3t) - 2\sin(3t))$
c $t = 0.328$

9 a $a = -\frac{1}{t + d}$ **b** $d = 10$ **c** $t = e^c - 10$

10 a $v = 5e^{-0.5t}$ **b** $a = -2.5e^{-0.5t}$
c $t = 2\ln 2,\ v = 2.5$ metres per minute

Exercise 8K

1 a $(2, -16), (-2, 16)$ **b** $(1, -2)$
c $(0, 0), (1, 1)$ **d** $(4, 48)$
e $(0, 0), \left(\frac{2}{\sqrt{3}}, \frac{16}{3}\right), \left(\frac{-2}{\sqrt{3}}, \frac{16}{3}\right)$ **f** $\left(\frac{1}{3}, \frac{14}{3}\right)$
g $(3, 2)$ **h** $(0, -10), (2, 6)$

2 a $(0, 1)$ **b** $\left(\frac{1}{3e}, -\frac{1}{3e}\right)$
c $(0, 1), (-\pi, 1), \left(-\frac{\pi}{2}, -1\right), \left(\frac{\pi}{2}, -1\right), (\pi, 1)$
d $(-1, -e^{-1})$ **e** $(0, 0), (2, 4e^{-2})$
f $(e^{-1}, -2e^{-1})$

3 $(\frac{1}{4}, -2\ln 2 - 1)$

4 $(2, \ln 2 + 1)$

5 $(0, 0), \left(\frac{2}{a}, \frac{4}{e^2a^2}\right)$

6 a $a = 6$ **b** $b = 3$

7 $b = -2,\ c = 1,\ d = 3$

8 $a = 2,\ b = -4,\ c = -1$

9 $a = \frac{2}{3},\ b = -2\frac{1}{2},\ c = -3,\ d = 7\frac{1}{2}$

10 a $a = 2$ and $b = 9$ **b** $(-1, -5)$

11 $x = \frac{1}{2}$ or $x = \frac{1 - 4n}{2n + 2}$

12 $x = \pm 1$ or $x = 0$

13 $(1, \frac{1}{2}),\ (-1, -\frac{1}{2})$

Exercise 8L

1 a $x = 0$

	0	
+	0	+
/	—	/
	inflexion	

b $x = 2, x = -5$

	−5		2	
+	0	−	0	+
/	—	\	—	/
	max.		min.	

c $x = -1, x = \frac{1}{2}$

	−1		$\frac{1}{2}$	
+	0	−	0	+
/	—	\	—	/
	max.		min.	

d $x = -3, x = 4$

	−3		4	
−	0	+	0	−
\	—	/	—	\
	min.		max.	

e $x = -3, x = 4$

	−3		4	
+	0	−	0	+
/	—	\	—	/
	max.		min.	

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f $x = 0,\ x = \frac{27}{5}$

	0		$\frac{27}{5}$	
$+$	0	$-$	0	$+$
/	—	\	—	/
	max.		min.	

g $x = 1,\ x = 3$

	1		3	
$+$	0	$-$	0	$+$
/	—	\	—	/
	max.		min.	

h $x = 1,\ x = 3$

	1		3	
$-$	0	$+$	0	$-$
\	—	/	—	\
	min.		max.	

2 a $x = -2$ (max), $x = 2$ (min)

b $x = 0$ (min), $x = 2$ (max)

c $x = \frac{1}{3}$ (max), $x = 3$ (min)

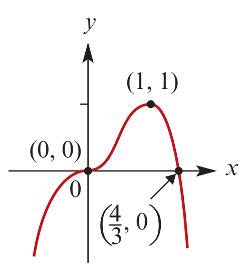
d $x = 0$ (inflection)

e $x = -2$ (inflection), $x = 0$ (min)

f $x = -\frac{1}{\sqrt{3}}$ (max), $x = \frac{1}{\sqrt{3}}$ (min)

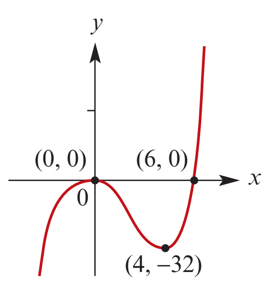
3 a i $(0, 0),\ (\frac{4}{3}, 0)$

ii $(0, 0)$ inflection, $(1, 1)$ max



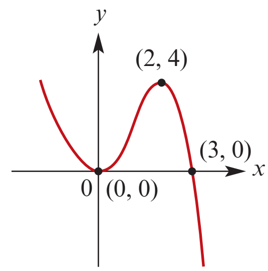
b i $(0, 0),\ (6, 0)$

ii $(0, 0)$ max, $(4, -32)$ min



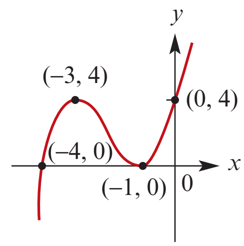
c i $(0, 0),\ (3, 0)$

ii $(0, 0)$ min, $(2, 4)$ max



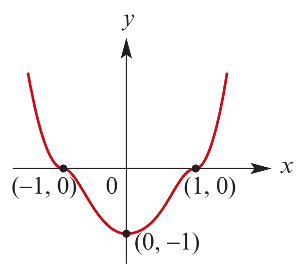
d i $(-4, 0),\ (-1, 0),\ (0, 4)$

ii $(-3, 4)$ max, $(-1, 0)$ min



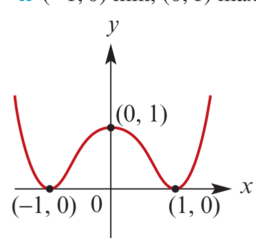
e i $(-1, 0),\ (0, -1),\ (1, 0)$

ii $(-1, 0)$ infl, $(0, -1)$ min, $(1, 0)$ infl



f i $(-1, 0),\ (0, 1),\ (1, 0)$

ii $(-1, 0)$ min, $(0, 1)$ max, $(1, 0)$ min

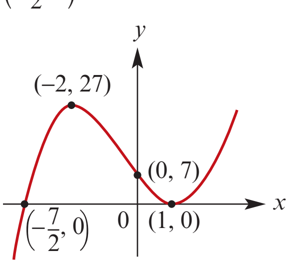


4 a $(-2, 27)$ max, $(1, 0)$ min

b $(1, 0)$ is a turning point

c $\left(-\frac{7}{2}, 0\right),\ (0, 7)$

d



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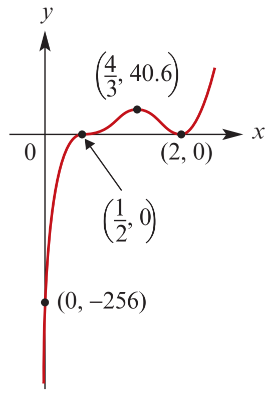
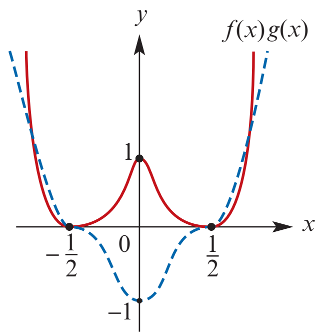
5 b $a = 3,\ b = 2,\ (0, 2)$ min, $(-2, 6)$ max

6 a $(0, -256),\ (\frac{1}{2}, 0),\ (2, 0)$

b $\left(\frac{1}{2}, 0\right)$ inflection, $\left(\frac{4}{3}, 40.6\right)$ max, $(2, 0)$ min



7 a



b i $x < -\frac{1}{\sqrt{2}}$ or $-\frac{1}{2} < x < \frac{1}{2}$ or $x > \frac{1}{\sqrt{2}}$

ii $-\frac{\sqrt{66}}{12} < x < -\frac{1}{2}$ or $-\frac{1}{2} < x < 0$ or $x > \frac{\sqrt{66}}{12}$

8 a $(-2, 0)$ max, $\left(\frac{4}{3}, -18\frac{14}{27}\right)$ min

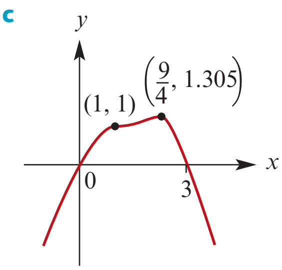
b No stationary points

9 a $(0, -1)$ stationary point of inflection, $(-1, -1)$ minimum

b $(0, -1)$ stationary point of inflection, $(-1.5, -2.6875)$ minimum

c No stationary points, gradient is always positive

10 b $x \leq \frac{9}{4}$ **c**



11 a $x = -1$ (infl), $x = 1$ (min), $x = 5$ (max)

b $x = 0$ (max), $x = 2$ (min)

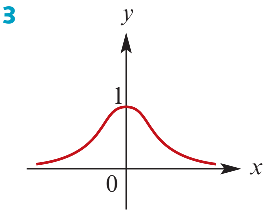
c $x = -4$ (min), $x = 0$ (max)

d $x = -3$ (min), $x = 2$ (infl)

12 a $(0, 0)$ local max; $(2\sqrt{2}, -64)$ and $(-2\sqrt{2}, -64)$ local min

b $(0, 0)$ local max; $\left(\pm 4\sqrt{\frac{m-1}{m}}, -\frac{16^m(m-1)^{m-1}}{m^m}\right)$ local min

13



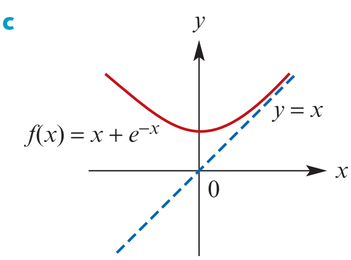
14 $-2 < x < 0$

15 $x < 1$; Max value $= \frac{100}{e^4} \approx 1.83$

16 a Min value $= f(0) = 0$

17 a $(0, 1)$ min **b** $y = x$

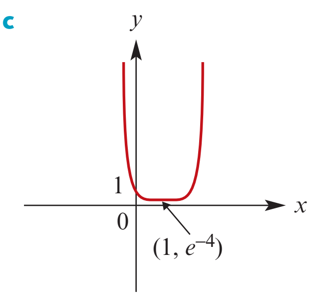
c



18 $p = 1,\ q = -6,\ r = 9$

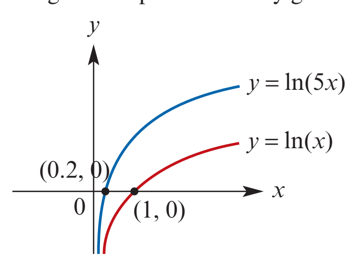
19 a $(8x - 8)e^{4x^2 - 8x}$ **b** $(1, e^{-4})$ min

c



d $y = -\frac{1}{8}x + \frac{5}{4}$

20 Tangents are parallel for any given value of x.



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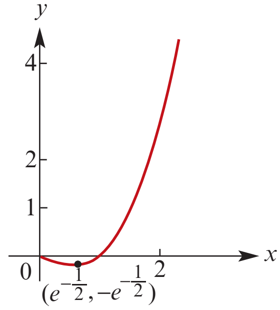
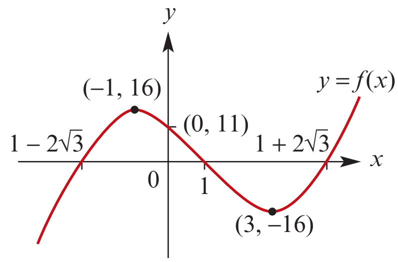
21 a $2x\ln(x) + x$ **b** $x = 1$ **c** $x = e^{-\frac{1}{2}}$

d

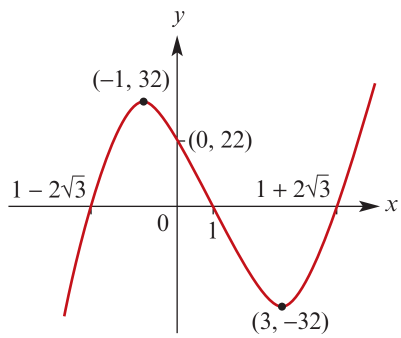


22 (e, e), local minimum

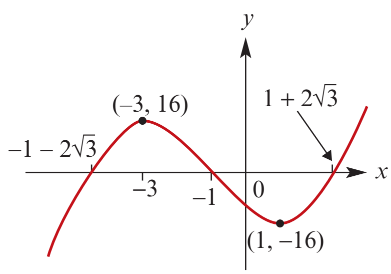
23 a



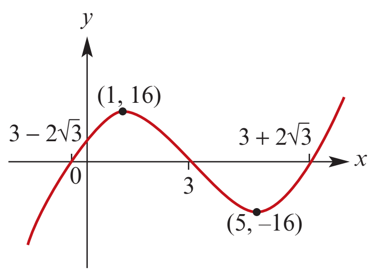
b Dilated by a factor of 2 from the x-axis:



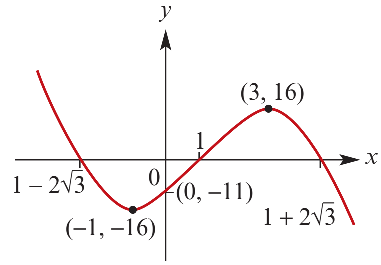
c Translated 2 units to the left:



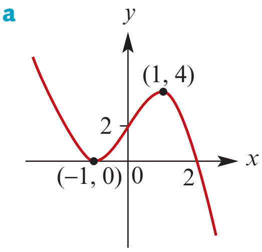
d Translated 2 units to the right:



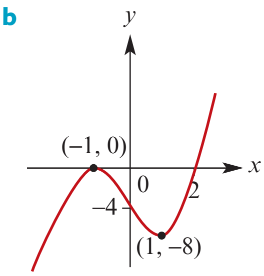
e Reflected in the x-axis:



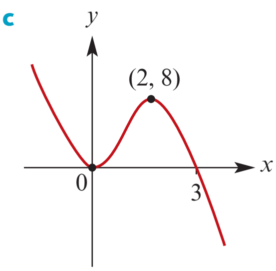
24 a



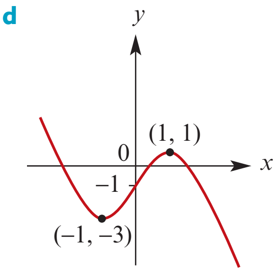
b



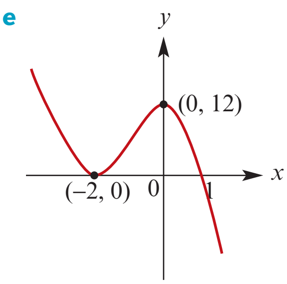
c



d



e



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25 a $(a+\ell, 0)$, $(b+\ell, 0)$ **b** $(h+\ell, kp)$

26 a Max $x = \frac{\pi}{3}, \frac{5\pi}{3}$; Min $x = 0, \pi, 2\pi$

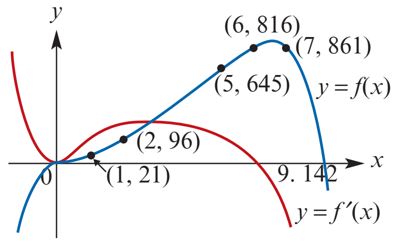
b Max $x = \frac{\pi}{6}$; Min $x = \frac{5\pi}{6}$; Infl $x = \frac{3\pi}{2}$

c Max $x = \frac{\pi}{2}, \frac{3\pi}{2}$; Min $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

d Max $x = \frac{\pi}{3}$; Infl $x = \pi$; Min $x = \frac{5\pi}{3}$

27 a $y = -x^4 + 8x^3 + 10x^2 + 4x$

b



Local max at (6.761, 867.07); No stationary point of inflection: $\frac{dy}{dx} = 4$ when $x = 0$

c -960

d $x = 4.317$ or $x = 8.404$

Chapter 8 review

Technology-free questions

1 a $1 - \frac{x}{\sqrt{1-x^2}}$ **b** $\frac{-4x^2 - 2x + 12}{(x^2+3)^2}$

c $\frac{3}{2\sqrt{1+3x}}$ **d** $\frac{-2}{x^2} - \frac{1}{2}x^{-\frac{3}{2}}$

e $\frac{3x-15}{2\sqrt{x-3}}$ **f** $\frac{1+2x^2}{\sqrt{1+x^2}}$ **g** $\frac{4x}{(x^2+1)^2}$

h $\frac{-x^2+1}{(x^2+1)^2}$ **i** $\frac{10x}{3}(2+5x^2)^{-\frac{2}{3}}$

j $\frac{-2x^2 - 2x + 4}{(x^2+2)^2}$ **k** $4x(3x^2+2)^{-\frac{1}{3}}$

2 a -6 **b** 1 **c** 5 **d** $\frac{1}{6}$

3 a $\frac{1}{x+2}$ **b** $3\cos(3x+2)$

c $-\frac{1}{2}\sin\left(\frac{x}{2}\right)$ **d** $(2x-2)e^{x^2-2x}$

e $\frac{1}{x-3}$ **f** $2\pi\cos(2\pi x)$

g $6\sin(3x+1)\cos(3x+1)$ **h** $\frac{1}{2x\sqrt{\ln x}}$

i $\frac{2 - 2\ln(2x)}{x^2}$

j $2x\sin(2\pi x) + 2\pi x^2\cos(2\pi x)$

4 a $e^x\sin(2x) + 2e^x\cos(2x)$

b $4x\ln x + 2x$ **c** $\frac{1 - 3\ln x}{x^4}$

d $2\cos(2x)\cos(3x) - 3\sin(2x)\sin(3x)$

e $\frac{2}{\cos^2(2x)}$

f $-9\cos^2(3x+2)\sin(3x+2)$

g $2x\sin^2(3x) + 6x^2\cos(3x)\sin(3x)$

5 a $2e^2 \approx 14.78$ **b** 0

c $15e^3 + 2 \approx 303.28$ **d** 1

6 a ae^{ax} **b** ae^{ax+b} **c** $-be^{a-bx}$

d $abe^{ax} - abe^{bx}$ **e** $(a-b)e^{(a-b)x}$

7 b $\frac{3}{2} < x < 4$

8 a $xf'(x) + f(x)$ **b** $\frac{-f'(x)}{[f(x)]^2}$

c $\frac{f(x) - xf'(x)}{[f(x)]^2}$ **d** $\frac{2xf(x) - 2x^2f'(x)}{[f(x)]^3}$

9 a $y = -x$ **b** $(0, 0)$

10 $y = 6ax - 3a^2$, $P(0, -3a^2)$

11 a $y = 3x - 3$ **b** $x = \frac{11}{3}$

12 a 5π square units/unit

b 6π square units/unit

13 a $x \neq 1$; $f'(x) = \frac{4}{5}(x-1)^{-\frac{1}{5}}$

b $y = \frac{4}{5}x - \frac{3}{5}$ and $y = -\frac{4}{5}x + 1$ **c** $\left(1, \frac{1}{5}\right)$

14 When $t = 0$: $v = 0.25$ m/s, $a = 0.25$ m/s^2
When $t = 1$: $v = 0.25e$ m/s, $a = 0.25e$ m/s^2
When $t = 4$: $v = 0.25e^4$ m/s, $a = 0.25e^4$ m/s^2

15 a $(25e^{100t})$°C/s **b** $(25e^5)$°C/s

16 $y = ex$

17 b 20 cm/year

18 a $y = \frac{1}{e}x$ **b** $y = \frac{x}{\sqrt{2}} - \frac{\pi}{2\sqrt{2}} + \sqrt{2}$

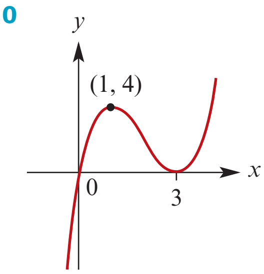
c $y = x - \frac{3\pi}{2}$ **d** $y = \frac{-2}{\sqrt{e}}x - 1$

19 a (1, 1) max; (0, 0) inflection

b (−1, 0) max; (1, −4) min

c $(-\sqrt{3}, 6\sqrt{3}+1)$ max; $(\sqrt{3}, -6\sqrt{3}+1)$ min

20



21 $x = 1$ (inflection); $x = 2$ (minimum)

22 2

Multiple-choice questions

1 A **2** C **3** A **4** A **5** B
6 E **7** C **8** B **9** A **10** E
11 B **12** E **13** A **14** E **15** B
16 B **17** D **18** C

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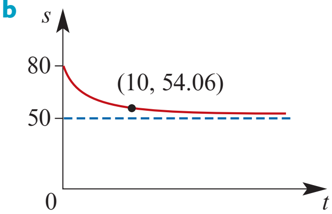
Extended-response questions

1 a 54.06 g **b**

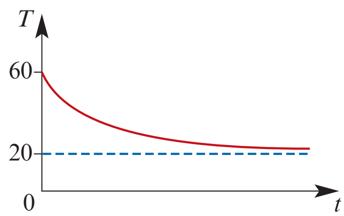


c $\frac{ds}{dt} = -6e^{-\frac{1}{5}t}$ **d** $\frac{ds}{dt} = -\frac{1}{5}(s - 50)$

e 0.8 g/L **f** 17 minutes

2 a 60°C

b



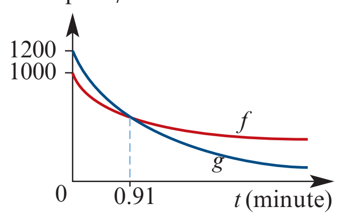
c $\frac{dT}{dt} = -14.4e^{-0.36t}$

d $\frac{dT}{dt} = -0.36(T - 20)$

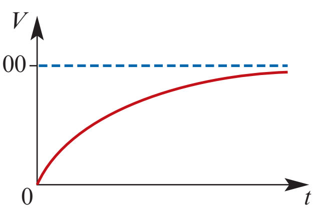
3 a 1.386 minutes **b** 2200, 5.38%

c 66.4 spores/minute **d** 0.9116 minutes

e



4 a



b i $20e^{-0.2t}$ m/s^2 **ii** $20 - \frac{V}{5}$ m/s^2

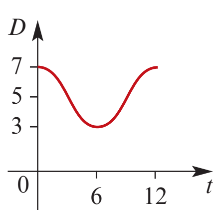
c 8.05 seconds

5 a $p = 5$

$q = 2$

$r = 30$

b



c $-\frac{2}{3}$ metres per hour

d $\frac{dD}{dt} = -\frac{\pi}{3}\sin(30t)°$

e A ship can enter 2 hours after low tide

6 a $(1, -6)$

b $3(x - 1)^2 + 3$

c $3(x - 1)^2 + 3 > 3$ for all $x \in \mathbb{R} \setminus \{1\}$

7 a $f'(x) = \frac{3\,000\,000e^{-0.3x}}{(1 + 100e^{-0.3x})^2}$

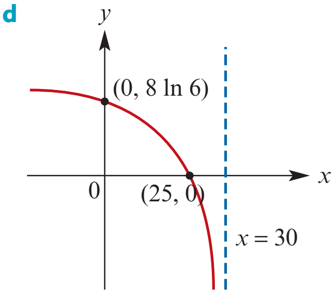
b i 294 kangaroos per year

ii 933 kangaroos per year

8 a $a = 30$ **b** $(0, 8\ln 6)$, $(25, 0)$

c $f'(20) = -0.8$

d



10 a $A = 1000$, $k = \frac{1}{5}\ln 10 \approx 0.46$

b $\frac{dN}{dt} = kAe^{kt}$ **c** $\frac{dN}{dt} = kN$

d i $\frac{dN}{dt} \approx 2905.7$ **ii** $\frac{dN}{dt} \approx 4.61 \times 10^{12}$

11 a $t \approx 34.66$ years **b** $t \approx 9.12$ years

12 a Max height 0.7 m first occurs at $t = \frac{1}{6}$ s

b $\frac{2}{3}$ s **c** 0.6π m/s, 0.6π m/s, 0 m/s

13 a i $r = \frac{1}{6}$ **ii** $p = 12$, $q = 8$

b ■ $T'(3) = -\frac{4\pi}{3}$, i.e. length of night is decreasing by $\frac{4\pi}{3}$ hours/month

■ $T'(9) = \frac{4\pi}{3}$, i.e. length of night is increasing by $\frac{4\pi}{3}$ hours/month

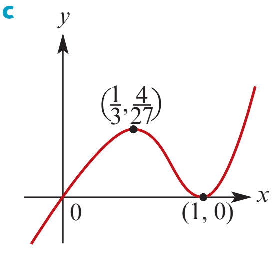
c $-\frac{8}{3}$ hours/month

d $t = 9$, i.e. after 9 months

14 a $a = 1$, $c = 1$, $b = -2$, $d = 0$

b $\frac{1}{3} < x < 1$

c



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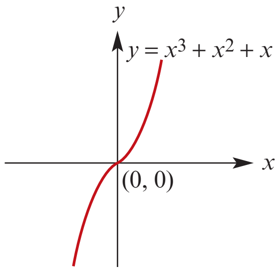
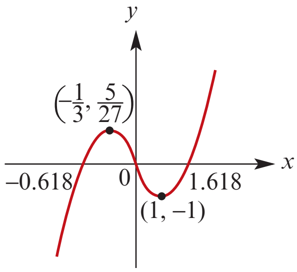
15 a i

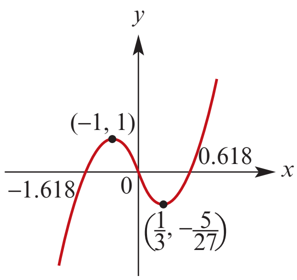


No stationary points

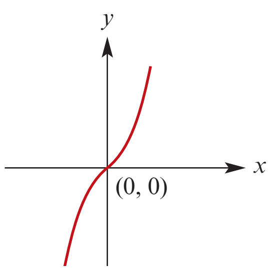
ii



iii



iv



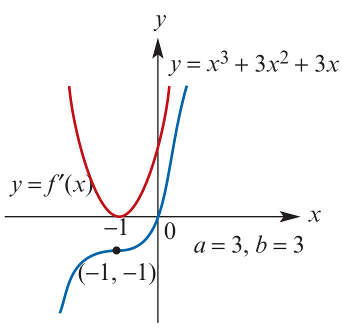
No stationary points

b i $f'(x) = 3x^2 + 2ax + b$

ii $x = \dfrac{-a \pm \sqrt{a^2 - 3b}}{3}$

c ii $a = -3$ or $a = 3$; $(-1, -1)$, $(1, 1)$ stationary points of inflection

iii



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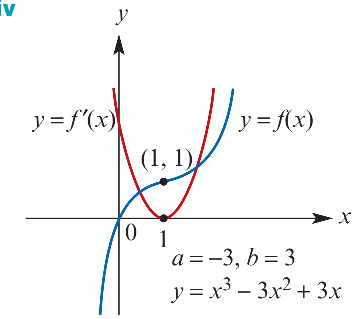


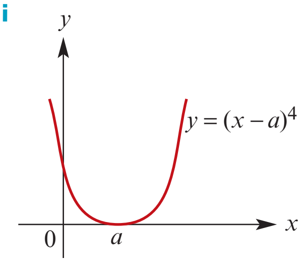
d $a^2 < 3b$

16 $x = e$

17 b i $a, b, \dfrac{a+b}{2}$ **ii** a, b

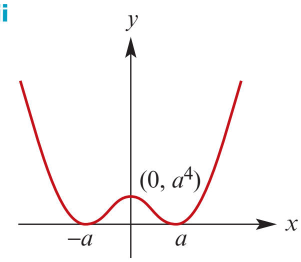
c $(a, 0)$, $(b, 0)$, $\left(\dfrac{a+b}{2}, \dfrac{(a-b)^4}{16}\right)$

e i



ii $(a, 0)$, $(-a, 0)$, $(0, a^4)$

iii



18 b i $a, \dfrac{3b+a}{4}$ **ii** a, b

c Local min at $\left(\dfrac{3b+a}{4}, \dfrac{-27}{256}(b-a)^4\right)$; Stationary point of inflection at $(a, 0)$

e $\left(-\dfrac{a}{2}, \dfrac{-27a^4}{16}\right)$ and $(a, 0)$

f i $b = -\dfrac{a}{3}$

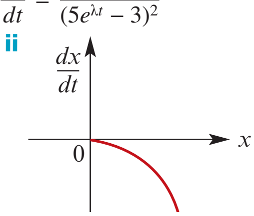
19 a $y = ex$ **b** $y = 2ex$ **c** $y = kex$

e i $k = \dfrac{1}{e}$ or $k \le 0$ **ii** $k > \dfrac{1}{e}$

20 a i 30 g **ii** 12.28 g

b $\dfrac{dx}{dt} = \dfrac{-300\lambda e^{\lambda t}}{(5e^{\lambda t} - 3)^2}$

c ii



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Chapter 9

Exercise 9A

1 a $\frac{x^4}{8}+c$ b x^3-2x+c c $\frac{5x^4}{4}-x^2+c$

d $\frac{x^4}{5}-\frac{2x^3}{3}+c$ e $\frac{x^3}{3}-x^2+x+c$

f $\frac{x^3}{3}+x+c$ g $\frac{z^4}{2}-\frac{2z^3}{3}+c$

h $\frac{4t^3}{3}-6t^2+9t+c$ i $\frac{t^4}{4}-t^3+\frac{3t^2}{2}-t+c$

2 $f(x)=x^4+2x^3+2x$

3 $y=2x^3+12$

4 a $y=x^2-x$ b $y=3x-\frac{x^2}{2}+1$

c $y=\frac{x^3}{3}+x^2+2$ d $y=3x-\frac{x^3}{3}+2$

e $y=\frac{2x^5}{5}+\frac{x^2}{2}$

5 a $V=\frac{t^3}{3}-\frac{t^2}{2}+\frac{9}{2}$ b $\frac{1727}{6}\approx 287.83$

6 $f(x)=x^3-x+2$

7 a B b $w=2000t-10t^2+100\,000$

8 $f(x)=5x-\frac{x^2}{2}+4$

9 $f(x)=\frac{x^4}{4}-x^3-2$

10 a $k=8$ b $(0,7)$

11 $8\frac{2}{3}$

12 a $k=-4$ b $y=x^2-4x+9$

13 a $k=-32$ b $f(7)=201$

14 $y=\frac{1}{3}(x^3-5)$

Exercise 9B

1 a $y=-\frac{1}{2x^2}+c$ b $y=3x^{\frac{4}{3}}+c$

c $y=\frac{4}{5}x^{\frac{5}{4}}+\frac{5}{2}x^{\frac{2}{5}}+c$

2 a $-\frac{3}{x}+c$ b $-\frac{2}{3x^3}+3x^2+c$

c $-\frac{2}{x}-\frac{3}{x^2}+c$ d $\frac{9}{4}x^{\frac{4}{3}}-\frac{20}{9}x^{\frac{9}{4}}+c$

e $\frac{12}{7}x^{\frac{7}{4}}-\frac{14}{3}x^{\frac{3}{2}}+c$ f $\frac{5}{2}x^{\frac{8}{5}}+\frac{9}{2}x^{\frac{8}{3}}+c$

3 a $f(x)=\frac{2}{7}x^{\frac{7}{2}}+\frac{5}{7}$ b $f(x)=\frac{3}{4}x^{\frac{4}{3}}+5$

c $f(x)=\frac{2}{3}x^{\frac{3}{2}}+\frac{1}{2}x^2-\frac{22}{3}$

4 a $\frac{4}{3}x^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}+c$ b $\frac{3z^3-4}{2z}+c$

c $\frac{5}{3}x^3+x^2+c$ d $\frac{4}{5}x^{\frac{5}{2}}+\frac{2}{7}x^{\frac{7}{2}}+c$

e $\frac{2x^3}{3}+\frac{3x^5}{5}+c$ f $\frac{3}{7}x^{\frac{7}{3}}+\frac{3}{16}x^{\frac{16}{3}}+c$

5 $f(x)=x^3+\frac{1}{x}-\frac{17}{2}$

6 $s=\frac{3}{2}t^2+\frac{8}{t}-8$

7 a $k=1$ b $f(4)=\frac{1}{4}$

Exercise 9C

1 a $\frac{1}{6}(2x-1)^3+c$ b $-\frac{1}{4}(t-2)^4+c$

c $\frac{1}{20}(5x-2)^4+c$ d $\frac{1}{24-16x}+c$

e $\frac{1}{8(6-4x)^2}+c$ f $\frac{-1}{8(3+4x)^2}+c$

g $\frac{2}{9}(3x+6)^{\frac{3}{2}}+c$ h $\frac{2}{3}(3x+6)^{\frac{1}{2}}+c$

i $\frac{1}{9}(2x-4)^{\frac{9}{2}}+c$ j $\frac{1}{7}(3x+11)^{\frac{7}{3}}+c$

k $-\frac{2}{9}(2-3x)^{\frac{3}{2}}+c$ l $-\frac{1}{10}(5-2x)^5+c$

2 $f(x)=-\frac{2}{3}(1-x)^{\frac{3}{2}}+\frac{5}{3}$

3 a $\frac{1}{2}\ln(x)+c$ b $\frac{1}{3}\ln(3x+2)+c$

c $\ln(1+4x)+c$ d $\frac{5}{3}\ln(3x-2)+c$

e $-\frac{3}{4}\ln(1-4x)+c$ f $-6\ln(4-x)+c$

4 a $5\ln(x)+c$ b $3\ln(x-4)+c$

c $5\ln(2x+1)+c$ d $-3\ln(5-2x)+c$

e $-3\ln(1-2x)+c$ f $-\frac{1}{3}\ln(4-3x)+c$

5 a $3x+\ln(x)+c$ b $x+\ln(x)+c$

c $-\frac{1}{x+1}+c$ d $2x+\frac{x^2}{2}+\ln(x)+c$

e $-\frac{3}{2(x-1)^2}+c$ f $-2x+\ln(x)+c$

6 a $y=\frac{1}{2}\ln(x)+1$ b $y=10-\ln(5-2x)$

7 $y=10\ln(x-5)$

8 a $x-\ln(x+1)+c$

b $-2(x+1)+3\ln(x+1)+c$

c $2(x+1)-\ln(x+1)+c$

9 $y=\frac{3}{2-x}-\frac{5}{2}$

10 $y=\frac{5}{4}\ln\left(\frac{5}{1-2x}\right)+10$

Exercise 9D

1 a $\frac{1}{6}e^{6x}+c$ b $\frac{1}{2}e^{2x}+\frac{3}{2}x^2+c$

c $-\frac{1}{3}e^{-3x}+x^2+c$ d $-\frac{1}{2}e^{-2x}+\frac{1}{2}e^{2x}+c$

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2 a $\frac{1}{2}e^{2x} - 2e^{\frac{x}{2}} + c$ **b** $e^x - e^{-x} + c$

c $\frac{2}{3}e^{3x} + e^{-x} + c$ **d** $15e^{\frac{x}{3}} - 10e^{\frac{x}{5}} + c$

e $\frac{9}{2}e^{\frac{2x}{3}} - \frac{15}{7}e^{\frac{7x}{5}} + c$ **f** $\frac{15}{4}e^{\frac{4x}{3}} - \frac{9}{2}e^{\frac{2x}{3}} + c$

3 a $y = \frac{1}{2}(e^{2x} - x^2 + 9)$ **b** $y = -\frac{3}{e^x} - e^x + 8$

4 $f(x) = \frac{1}{2}e^{2x} + 4 - \frac{1}{2}e^2$

5 $y = 9 - 2e^{-2}$

6 a $k = 2$ **b** $y = \frac{1}{2}e^{2x} + \frac{1}{2}e^2$

7 a $k = 3$ **b** $y = -\frac{1}{3}e^{3x} - \frac{2}{3}e^3$

Exercise 9E

1 a $\frac{1}{3}\sin(3x)$ **b** $-2\cos\left(\frac{1}{2}x\right)$ **c** $\sin(3x)$

d $-4\cos\left(\frac{1}{2}x\right)$ **e** $-\frac{1}{2}\cos\left(2x - \frac{\pi}{3}\right)$

f $\frac{1}{3}\sin(3x) - \frac{1}{2}\cos(2x)$

g $\frac{1}{4}\sin(4x) + \frac{1}{4}\cos(4x)$

h $\frac{1}{4}\cos(2x) + \frac{1}{3}\sin(3x)$

i $-\frac{1}{4}\sin\left(2x + \frac{\pi}{3}\right)$ **j** $-\frac{1}{\pi}\cos(\pi x)$

k $\sin(2\pi x)$ **l** $\sin(2x)$

2 a $f(x) = \frac{1}{2}\sin(2x) + \frac{1}{2}$

b $f(x) = -2\cos\left(\frac{x}{2}\right) + 1$

3 $y = \frac{1}{2}(x^2 - \cos(2x) + 3)$

4 $g(x) = -2\cos(2x) + 2$

5 a $y = \frac{1}{2}\sin(2x) - \frac{1}{3}\cos(3x)$

b $y = -2x + \pi$

6 $f(x) = \sin x - \frac{1}{2}\sin(2x) + 8$

7 a $f(x) = x + \cos(3x) + 1 - \pi$

b $y = x - \pi$

Exercise 9F

1 $\frac{6x}{3x^2 + 7}$, $\frac{1}{6}\ln(3x^2 + 7)$

2 a $\frac{1}{3}\ln(x^3 + 3)$ **b** $\frac{1}{2}\ln(x^2 + 4x)$

c $\frac{1}{2}\ln(3 + e^{2x})$ **d** $\frac{1}{3}\ln(x^3 + 3x)$

e $\frac{5}{3}\ln(3x - 2)$

3 a $\frac{1}{\cos^2 x}$, $\tan x$ **b** $-\frac{2}{\sin^2(2x)}$, $-\frac{\cos(2x)}{2\sin(2x)}$

4 $\frac{e^{2\sqrt{x}}}{\sqrt{x}}$, $e^{2\sqrt{x}} + c$

5 $-3\sin x \cdot \cos^2 x$, $-\frac{1}{3}\cos^3 x$

6 a $\ln(2x) + 1$, $x\ln(2x) - x + c$

b $2x\ln(2x) + x$, $\frac{x^2}{2}\ln(2x) - \frac{x^2}{4} + c$

7 $\sin x + x\cos x$, $\cos x + x\sin x + c$

8 $(2x + 1)e^{2x}$, $\frac{1}{4}(2x - 1)e^{2x} + c$

9 a $\frac{12x}{\sqrt{4x - a}}$ **b** $\frac{1}{12}(2x + a)\sqrt{4x - a}$

10 $\frac{4x}{\sqrt{1 + 4x^2}}$, $\frac{1}{4}\sqrt{1 + 4x^2}$

11 $\frac{1}{4}\ln\left(\frac{x + 2}{x - 2}\right)$ for $x > 2$

12 a $(x^2 + 1)^3 + c$ **b** $\sin(x^2) + c$

c $(x^2 + 1)^3 + \sin(x^2) + c$

d $-(x^2 + 1)^3 + c$ **e** $(x^2 + 1)^3 - 4x + c$

f $3\sin(x^2) + c$

13 $1 + \frac{x}{\sqrt{1 + x^2}}$, $\frac{1}{\sqrt{1 + x^2}}$, $\ln(x + \sqrt{1 + x^2}) + c$

Exercise 9G

1 a $x = t^2 - 3t$ **b** $x = 0$ **c** 0 m/s

d $\frac{9}{2}$ m **e** $\frac{3}{2}$ m/s

2 $x = t^3 - 2t + 5$

3 a $x = \frac{2t^3}{3} - 4t^2 + 6t + 4$, $a = 4t - 8$

b When $t = 1$, $x = \frac{20}{3}$ m;

When $t = 3$, $x = 4$ m

c When $t = 1$, $a = -4$ m/s^2;

When $t = 3$, $a = 4$ m/s^2

4 Velocity = 73 m/s; Position = $\frac{646}{3}$ m

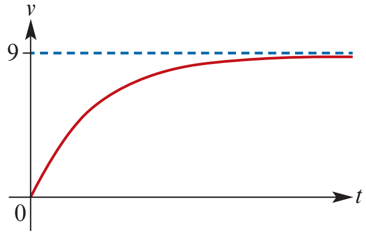
5 Initial position is 3 m to the left of O

6 a Velocity = $-10t + 25$

b Height = $-5t^2 + 25t$ **c** $\frac{5}{2}$ s

d $\frac{125}{4}$ m **e** 5 s

7 a



b $x = 9(t + 3e^{-\frac{t}{3}} - 3)$

c $27e^{-1} \approx 9.93$ m

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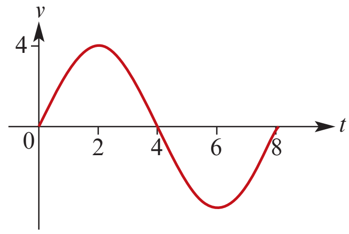
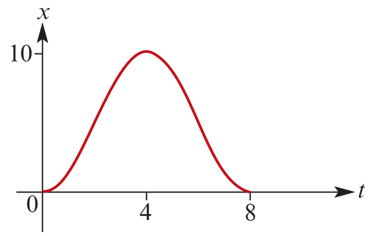
8 a



b $x = \frac{16}{\pi}\left(1 - \cos\left(\frac{\pi t}{4}\right)\right)$

c



d Max speed = 4 m/s; Position = $\frac{16}{\pi}$ m

9 The 29th floor

Chapter 9 review

Technology-free questions

1 a $\frac{x}{2} + c$ **b** $\frac{x^3}{6} + c$ **c** $\frac{x^3}{3} + \frac{3x^2}{2} + c$

d $\frac{4x^3}{3} + 6x^2 + 9x + c$ **e** $\frac{at^2}{2} + c$ **f** $\frac{t^4}{12} + c$

g $\frac{t^3}{3} - \frac{t^2}{2} - 2t + c$ **h** $\frac{-t^3}{3} + \frac{t^2}{2} + 2t + c$

2 $y = 4x^{\frac{3}{2}} + 2$

3 $y = -\frac{24}{x} + 18$

4 $y = x^2 + 5x - 25$

5 a $f(x) = \frac{1}{2}\sin(2x) + 1$

b $f(x) = 3\ln x + 6$ **c** $f(x) = 2e^{\frac{x}{2}} - 1$

6 a $y = x^3 - 4x^2 + 3x$ **b** 0, 1, 3

7 a $y = x^3 - 6x^2 + 8x$ **b** 0, 2, 4

8 a $k = \sqrt{17}$

b $y = 4x - \frac{\sqrt{17}}{2}x^2 + 18\sqrt{17} - 24$

9 a $\frac{15}{2}$ m/s **b** $\frac{128}{3}$ m **c** 0 m **d** $\frac{64}{9}$ m/s

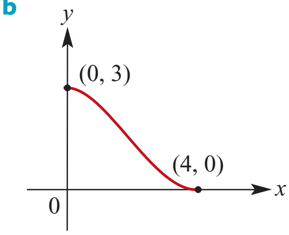
Multiple-choice questions

1 C	**2** D	**3** A	**4** D	**5** B
6 B	**7** A	**8** C	**9** A	**10** D
11 C	**12** B	**13** A	**14** E	**15** B

Extended-response questions

1 a $y = \frac{9}{32}\left(\frac{x^3}{3} - 2x^2\right) + 3$

b



c Yes, for $\frac{4}{3} < x < \frac{8}{3}$

2 a $\frac{d}{dx}\left(e^{-3x}\sin(2x)\right) =$
$-3e^{-3x}\sin(2x) + 2e^{-3x}\cos(2x)$

$\frac{d}{dx}\left(e^{-3x}\cos(2x)\right) =$
$-3e^{-3x}\cos(2x) - 2e^{-3x}\sin(2x)$

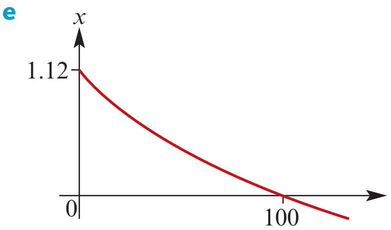
c $\int e^{-3x}\sin(2x)\,dx =$
$\frac{-1}{13}\left(3e^{-3x}\sin(2x) + 2e^{-3x}\cos(2x)\right)$

3 a $\ln(t + k) + 1$, $(t + k)\ln(t + k) - t + c$

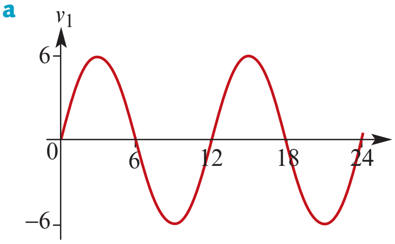
b $x = 6t - (t + k)\ln(t + k) + k\ln k$

c $k = 48.41$ **d** 45.77 m

e

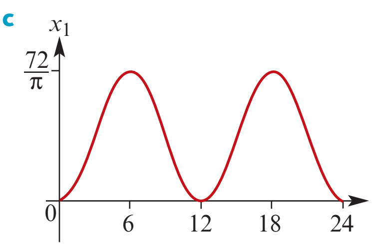


4 a



b $x_1 = -\frac{36}{\pi}\cos\left(\frac{\pi t}{6}\right) + \frac{36}{\pi}$

c



d Max speed = 6 m/s; Position = $\frac{36}{\pi}$ m

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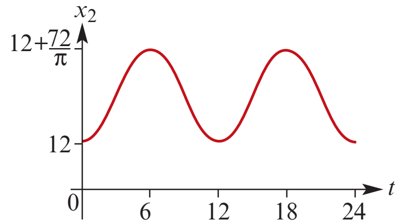
e $x_2 = -\dfrac{36}{\pi}\cos\left(\dfrac{\pi t}{6}\right) + \dfrac{36}{\pi} + 12$

f



g $\dfrac{72}{\pi}$ m

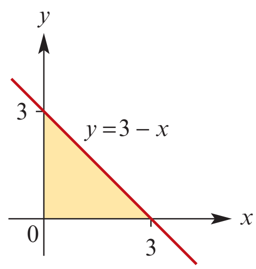
Chapter 10

Exercise 10A

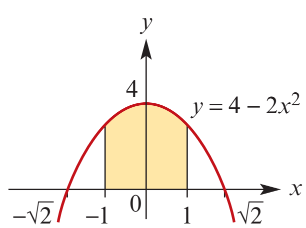
1 62 square units
2 80 square units
3 60.90623 square units
4 68 square units
5 **a** 13.2 **b** 10.2 **c** 11.7
6 **a** 4.375 **b** 4.536
7 **a** 36.8 **b** 36.75
8 $\pi \approx 3.13$
9 **a** 4.371 **b** 1.128
10 109.5 m^2

Exercise 10B

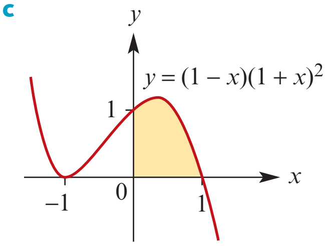
1 **a** $\dfrac{9}{2}$ **b** 9 **c** 4
2 **a** $\dfrac{7}{3}$ **b** $16\frac{1}{4}$ **c** $\dfrac{9}{4}$ **d** 9
e $\dfrac{15}{4}$ **f** $49\frac{1}{2}$ **g** $15\frac{1}{3}$ **h** 30
3 4 **4** 9 **5** $\dfrac{4}{3}$ **6** 4
7 **a**



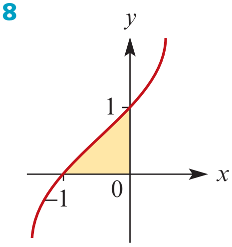
b



c

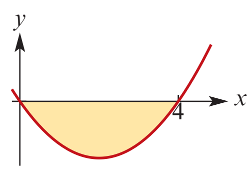


8 Area $= \dfrac{3}{4}$ square units

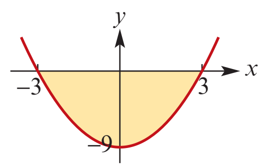


Exercise 10C

1 $-\dfrac{32}{3}$



2 -36



3 $\dfrac{37}{12}$ square units
4 **a** $\dfrac{5}{6}$ **b** $8\frac{1}{6}$ square units
5 **a** 8 **b** 16 **c** -4 **d** -8
6 **a** -12 **b** 36 **c** 20
7 **a** 24, 21, 45 **b** 4, -1, 3
8 4.5 square units
9 $166\frac{2}{3}$ square units
10 $\dfrac{37}{12}$ square units
11 **a** $\dfrac{4}{3}$ **b** $\dfrac{1}{6}$ **c** $121\frac{1}{2}$ **d** $\dfrac{1}{6}$ **e** $4\sqrt{3}$ **f** 108

Exercise 10D

1 **a** $\dfrac{1}{2}$ **b** $\dfrac{140}{3}$ **c** $343\frac{11}{20}$
2 **a** 1 **b** 1 **c** 14 **d** 31 **e** $2\frac{1}{4}$ **f** 0
3 **a** 10 **b** 1 **c** $\dfrac{13}{3}$ **d** $\dfrac{1}{3}$ **e** $\dfrac{10}{441}$
f 34 **g** $\dfrac{2}{3}\left(2^{\frac{3}{2}} - 1\right)$ **h** $2 - 2^{\frac{1}{2}}$ **i** $\dfrac{1}{15}$

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4 a $\frac{1}{2}(e^2 - 1)$ **b** $\frac{1}{2}(3 - e^{-2})$

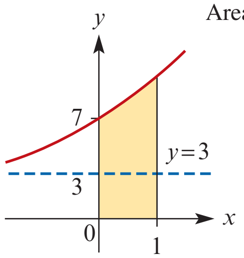
c $6e^{\frac{1}{3}} - 4$ **d** $e^2 - e^{-2}$

5 a $\ln 2$ **b** $\frac{1}{2}\ln 5$ **c** $\frac{3}{2}\ln\left(\frac{19}{17}\right)$

d $3\ln\left(\frac{8}{5}\right)$ **e** $3\ln\left(\frac{7}{4}\right)$ **f** $\frac{3}{2}\ln 3$

6 $\frac{321}{10}$ square units

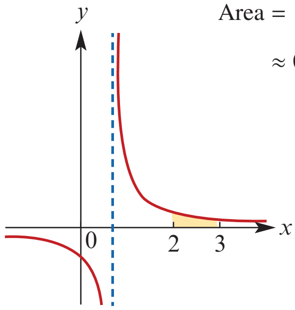
7



Area $= 2e^2 + 1$

≈ 15.78 square units

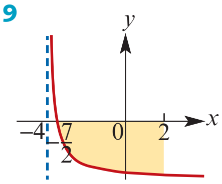
8



Area $= \frac{1}{3}\ln\left(\frac{7}{4}\right)$

≈ 0.19 square units

9



Area $= 11 - \ln 12$

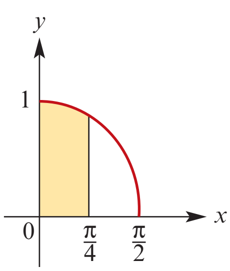
≈ 8.52 square units

10 a $1 - \frac{1}{\sqrt{2}}$ **b** $\frac{1}{2}$ **c** $1 + \frac{1}{\sqrt{2}}$ **d** 2 **e** 1

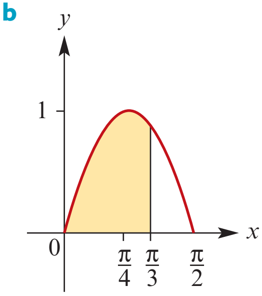
f $\frac{2}{3}$ **g** 4 **h** $\frac{1 - \sqrt{3}}{4}$ **i** -2

11 $-\sqrt{2} + 2$ square units

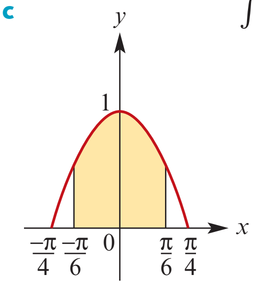
12 a $\int_0^{\frac{\pi}{4}} \cos x \, dx = \frac{1}{\sqrt{2}}$



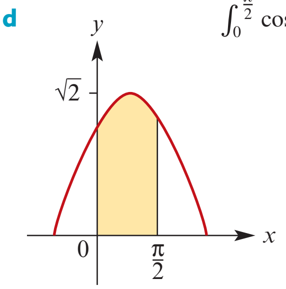
b $\int_0^{\frac{\pi}{3}} \sin(2x) \, dx = \frac{3}{4}$



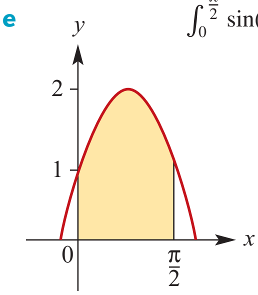
c $\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \cos(2x) \, dx = \frac{\sqrt{3}}{2}$



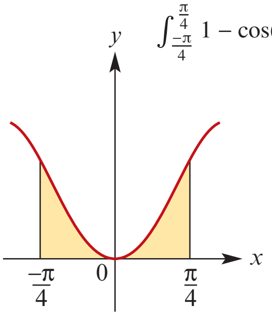
d $\int_0^{\frac{\pi}{2}} \cos\theta + \sin\theta \, d\theta = 2$



e $\int_0^{\frac{\pi}{2}} \sin(2\theta) + 1 \, d\theta = 1 + \frac{\pi}{2}$



f $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} 1 - \cos(2\theta) \, d\theta = \frac{\pi}{2} - 1$



13 a $\frac{\sqrt{2}}{2}$ **b** $-\frac{1}{3}$ **c** $-\frac{\sqrt{3}}{3}$ **d** $-\frac{1}{\sqrt{2}}$

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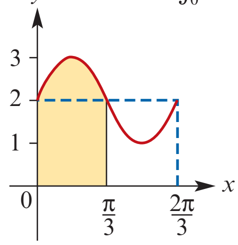
14 $\int_0^{\frac{\pi}{3}} 2 + \sin(3x)\, dx = \frac{2\pi}{3} + \frac{2}{3}$



15 **a** $4\frac{2}{3}$ **b** $2\frac{2}{3}$ **c** 12

d $\frac{5\sqrt{3}}{4} - 2$ **e** $\frac{e^4}{2} + 4\ln 2 - \frac{e^2}{2}$

f 4 **g** $\frac{5\pi^2}{8} + 1$ **h** $8\ln 2 + \frac{51}{4}$

i $\frac{1}{12}$

Exercise 10E

1 **a** $\frac{1}{2}\ln 2$ **b** 0 **c** $\ln\left(\frac{7}{6}\right)$

d $\ln\left(\frac{17}{7}\right)$ **e** 0 **f** $\ln 6$

g $\frac{1}{2}\ln\left(\frac{31}{16}\right)$ **h** $\ln\left(\frac{e^e + 1}{e + 1}\right)$ **i** $-\ln(\ln 2)$

2 $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$, $2e^{\sqrt{2}} - 2e$

3 $6\sin^2(2x)\cos(2x)$, $\frac{1}{6}$

4 **a** 139.68 **b** 18.50 **c** −0.66

5 $5\ln 3 + 4$

6 $5 + 6\ln 2$

7 **a** $(x+1)e^x$, 1

b $\sin(3x) + 3x\cos(3x)$, $\frac{\pi}{18} - \frac{1}{9}$

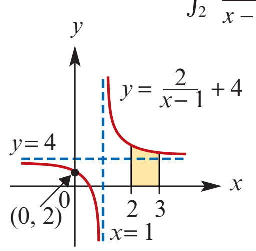
8 $a = 1$, $b = -2$; Area $= \frac{12\sqrt{3} - \pi - 12}{3\pi}$

9 **a** 1.450 square units

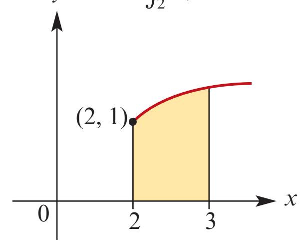
b 1.716 square units

10 0.1345

11 $\int_2^3 \frac{2}{x-1} + 4\, dx = 2\ln 2 + 4$



12 $\int_2^3 \sqrt{2x-4} + 1\, dx = \frac{1}{3} \times 2^{\frac{3}{2}} + 1$



13 **a** $A(0, 3)$, $B(1, 0)$ **b** 2 square units

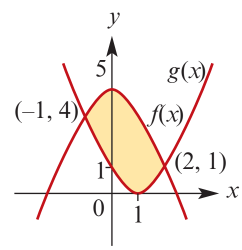
14 **b** Derivative: $(\ln a)e^{x\ln a} = (\ln a) \cdot a^x$

Anti-derivative: $\frac{e^{x\ln a}}{\ln a} = \frac{a^x}{\ln a}$

Exercise 10F

1 36 square units

2 Area = 9 square units



3 **a** 36 square units **b** $20\frac{5}{6}$ square units

c 4 square units **d** $4\frac{1}{2}$ square units

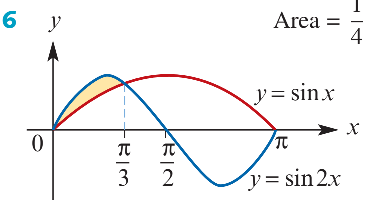
e $4\frac{1}{2}$ square units

4 **a** 2 square units

b $e + e^{-1} - 2 \approx 1.086$ square units

5 3.699 square units

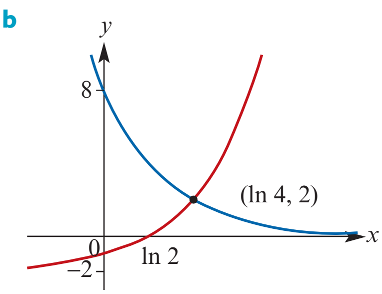
6 Area $= \frac{1}{4}$ square unit



7 $P(\ln 3, 3)$; Area ≈ 2.197 square units

8 **a** $(\ln 4, 2)$

b



Area $= 4\ln 2 + 3 \approx 5.77$ square units

9 $4 - 3\ln 3 \approx 0.704$ square units

10 $\frac{1}{3}$ square unit

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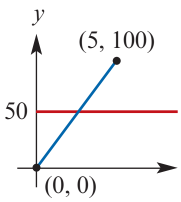
Exercise 10G

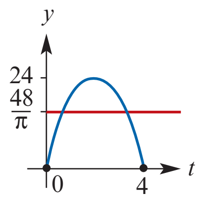
1 a $\frac{2}{3}$ **b** $\frac{2}{\pi}$ **c** $\frac{2}{\pi}$ **d** 0 **e** $\frac{1}{2}(e^2 - e^{-2})$

2 $10(e^5 - 1)e^{-5} \approx 9.93°\text{C}$

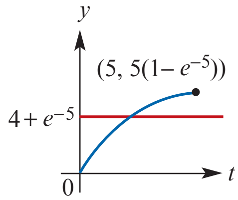
3 a



b



c

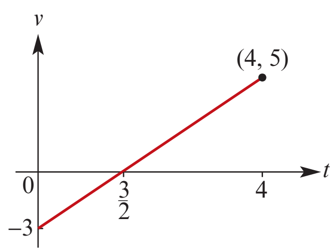


4 $\frac{147}{10}$ m/s

5 $\frac{a^2}{6}$

6 a $3000(2 - 2^{0.9})$ N/m^2
b $1000(4^{0.1} - 1)$ N/m^2

7 a



b 4 m **c** $\frac{17}{2}$ m

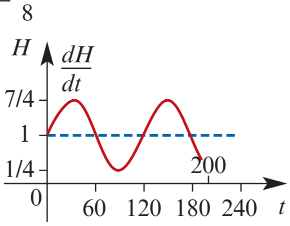
8 −3 m

9 $a = -\frac{20}{3}$

10 a 14 m **b** $\frac{7}{2}$ m/s

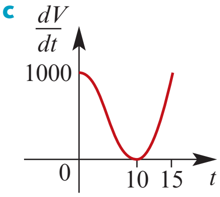
11 $k = \frac{3}{8}$

12 a



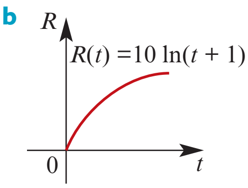
b $t \in (10, 50) \cup (130, 170)$
c $t = 30$ or $t = 150$
d i 120 kilojoules
ii 221.48 kilojoules

13 a When $t = 0$, 1000 million litres per hour;
When $t = 2$, 896 million litres per hour
b i $t = 0$ and $t = 15$
ii 1000 million litres per hour
c



d i 5000
ii 5000 million litres flowed out in the first 10 hours

14 a When $t = 5$, ≈ 17.9 penguins per year;
When $t = 10$, ≈ 23.97 penguins per year;
When $t = 100$, ≈ 46.15 penguins per year
b



c 3661; the growth in the penguin population over 100 years

15 $71\,466\frac{2}{3}$ m^3

16 1.26 m

17 a 6 metres
b 18π m^2
c i $y - 3 + 3\cos\left(\frac{a}{3}\right) = \frac{-1}{\sin\left(\frac{a}{3}\right)}(x - a)$
ii 5.409

18 a i 9 **ii** $\frac{3(\sqrt{2} + 2)}{2}$ **iii** 12
b Max value is 12; Min value is 0.834
c $\frac{48(\pi + 1)}{\pi}$ litres

Chapter 10 review

Technology-free questions

1 a $\frac{25}{12}$ **b** $\frac{77}{60}$ **c** $\frac{101}{60}$

2 a $\frac{14}{3}$ **b** $48\frac{3}{4}$ **c** $\frac{1}{2}$ **d** $\frac{15}{16}$ **e** $\frac{16}{15}$

3 a $B(1, 3)$, $C(3, 3)$ **b** 6 **c** $\frac{4}{3}$

4 $21\frac{1}{12}$ square units

5 a 0 **b** $-\frac{5}{3}a^{\frac{3}{2}}$ **c** $-\frac{55}{3}$ **d** $\frac{1}{2}$
e 1 **f** 0 **g** 0

6 $\frac{23}{2}$ **7** 3 **8** 4 **9** 820

10 $\frac{85}{4}$ **11** $\frac{5}{3}$ **12** $\frac{5}{3}$

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13 $\int_a^b f(x) - g(x)\,dx + \int_b^c g(x) - f(x)\,dx + \int_c^d f(x) - g(x)\,dx$

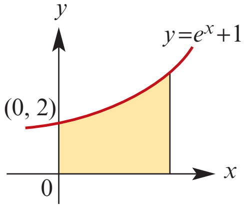
14 **a** $P(3, 9)$, $Q(7.5, 0)$ **b** 29.25 square units

15 **a** 5 **b** $p = \frac{20}{7}$

16 3.45 square units

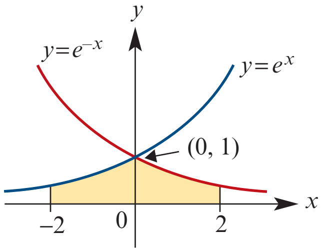
17 **a** $A(0, 6)$, $B(5, 5)$ **b** $15\frac{1}{6}$ **c** $20\frac{5}{6}$

18 **a**



b $e^2 + 1$

19 **a**



b $2 - 2e^{-2}$

20 **a** $e - 1$ **b** $2(e - 1)$ square units

21 $2 + e^2$ square units

22 **a** $3 - e^{-2}$ **b** $\ln 2 + \frac{5}{2}$ **c** $\frac{\pi^2}{8} + 1$

d $\frac{1}{2}\ln\left(\frac{5}{6}\right) - \frac{1}{e^4} + \frac{1}{e^5}$

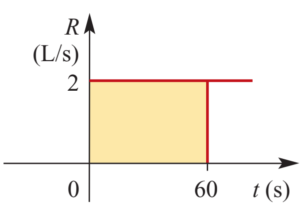
Multiple-choice questions

1 D **2** B **3** C **4** A **5** A
6 B **7** B **8** E **9** C **10** D

Extended-response questions

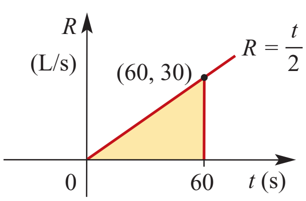
1 **a** **i** 120 L

ii



b **i** 900 L

ii



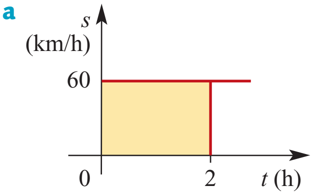
iii $900a^2$ L

c **i** 7200 square units

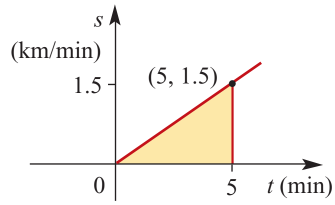
ii Volume of water which has flowed in

iii 66.94 s

2 **a**



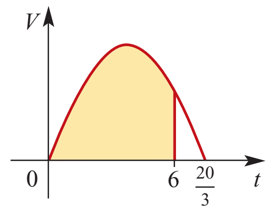
b **i**



ii 3.75 km

c **i** $20 - 6t$ m/s^2

ii



iii 144 metres

3 **a** 6 m^2

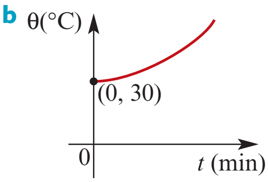
b **i** $y = x - \frac{1}{2}$ **ii** $\left(x^2 - \frac{1}{4}\right)$ m^2

c **i** $y = \frac{1}{2}x^2$, $P(-2, 2)$, $S(2, 2)$ **ii** $\frac{16}{3}$ m^2

4 **a** $4y - 5x = -3$ **b** $\left(\frac{3}{5}, 0\right)$ **c** $(1, 0)$

d $\frac{9}{40}$ **e** 9 : 49

5 **a** 968.3° **b**



c 2.7 minutes **d** 64.5°C/min

7 **a** $\frac{1}{3}$ square units

d $1 - \frac{n-1}{n+1} = \frac{2}{n+1}$ square units

e $\frac{9}{11}, \frac{99}{101}, \frac{999}{1001}$

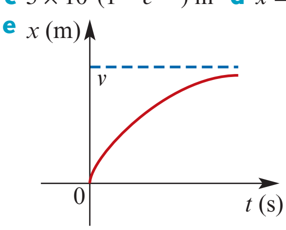
f Area between the curves approaches 1

8 **a** 5×10^4 m/s

b Magnitude of velocity becomes very small

c $5 \times 10^4(1 - e^{-20})$ m **d** $x = v(1 - e^{-t})$

e



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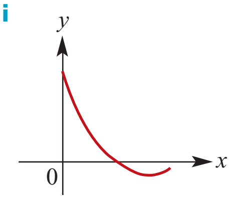
9 a $y = 7 \times 10^{-7}x^3 - 0.00116x^2 + 0.405x + 60$

b 100 m

c i



ii $(0, 60)$

d 51 307 m^2

10 a $\dfrac{dy}{dx} = -\dfrac{x}{10}e^{\frac{x}{10}}$, $\dfrac{dy}{dx} = -x(100 - x^2)^{-\frac{1}{2}}$

b When $x = 0$, $\dfrac{dy}{dx} = 0$ for both functions

c $-e$

d 6.71 square units

e 8.55%

f $100e - 250 \approx 21.828$ square units

g i $10(10e - 20)$

ii $(25\pi - 100e + 200)$ square units

Chapter 11

Technology-free questions

1 a i $f(g(x)) = \ln(3x^2)$;
Domain $= \mathbb{R} \setminus \{0\}$; Range $= \mathbb{R}$

ii $g(f(x)) = (\ln(3x))^2$;
Domain $= \mathbb{R}^+$; Range $= [0, \infty)$

b i $f(g(x)) = \ln(2 - x^2)$;
Domain $= (-\sqrt{2}, \sqrt{2})$;
Range $= (-\infty, \ln 2]$

ii $g(f(x)) = (\ln(2 - x))^2$;
Domain $= (-\infty, 2)$; Range $= [0, \infty)$

c i $f(g(x)) = -\ln(2x^2)$;
Domain $= \mathbb{R} \setminus \{0\}$; Range $= \mathbb{R}$

ii $g(f(x)) = (\ln(2x))^2$;
Domain $= (0, \infty)$; Range $= [0, \infty)$

2 1

3 $x = \sqrt[3]{12a}$

4 $x = -9$

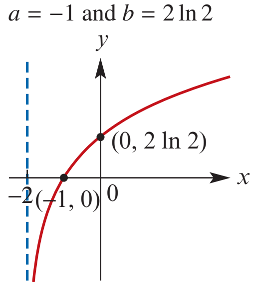
5 $x = \ln\left(\dfrac{3}{2}\right)$

6 $x = \dfrac{2e - 12}{e + 1}$

7 $\dfrac{1}{2}$

8 a $a = -1$ and $b = 2\ln 2$

b



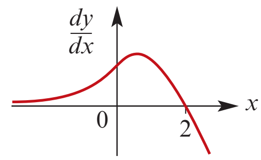
9 $x = 0$ or $x = 1$

10 $A = \dfrac{8}{5}$ and $k = \ln\left(\dfrac{5}{2}\right)$

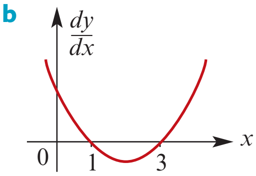
11 a $\dfrac{-2x}{(x^2 + 1)^2}$ **b** $x = 0$

12 $4(6x - 4)(3x^2 - 4x)^3$

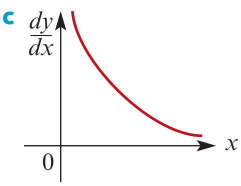
13 a



b



c



14 $2\left(4 - \dfrac{9}{x^2}\right)\left(4x + \dfrac{9}{x}\right)$, $x = \pm\dfrac{3}{2}$

15 $2x\ln(2x) + x$

16 a $b = \dfrac{1}{2}$ **b** $k = (2b - 1)e^{2b+1}$

17 $m = \dfrac{1}{12}$, $a = -\dfrac{22}{3}$, $c = -\dfrac{28}{3}$

18 a -7 **b** -14 **c** -20

19 a $v = 3e^{-0.5t}$ **b** $a = -1.5e^{-0.5t}$

c 1.5 m/s

20 $y = \dfrac{x}{2} + \ln 2 + 8$

21 $\left(\dfrac{1}{2}, \ln 2 + \dfrac{1}{2}\right)$, local minimum

22 $y = x$

23 $(0, 1)$, $(-3, e^{-81})$, $(3, e^{-81})$

24 a $\dfrac{3}{5}\ln(5x - 2)$ **b** $\dfrac{3}{10 - 25x}$

25 $y = \dfrac{1}{2}e^{2x} - \dfrac{1}{2}\cos(2x) - \dfrac{1}{2} - \dfrac{1}{2}e^{\pi}$

26 a $6\sin^2(2x)\cos(2x)$

b $\dfrac{1}{6}\sin(2x)\,(3 - \sin^2(2x))$

27 $y = -\dfrac{2}{x} + \dfrac{1}{2}\sin(2x) + \dfrac{4}{\pi}$

28 a $v = -6\cos(3t) + 6$ **b** $x = -2\sin(3t) + 6t$

c 6 m/s, $2 + 3\pi$ m

29 $f(x) = 5e^x - 4$

30 $m = \dfrac{1}{4}(-3 + \sqrt{105})$

31 a $(0, -4)$ and $(-2, 0)$ **b** 0 **c** 4 **d** $9\frac{1}{2}$

32 a $4 - 2\sqrt{2}$ **b** $2(e^{\frac{3}{4}} - 1)$ **c** $\dfrac{1}{2}\ln 2$

d $\ln\left(\dfrac{3}{2}\right)$ **e** $\dfrac{1}{4}$ **f** $\dfrac{1}{20}$

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33 a $C(-\frac{1}{2}\ln 6, 0)$, $D(10, 0)$

b $3\ln 6 + \dfrac{45}{2}$

34 a $C\left(\dfrac{7}{6}, 0\right)$, $D\left(\dfrac{11}{6}, 0\right)$ **b** $\dfrac{1}{2} + \dfrac{3\sqrt{3}+2}{\pi}$

35 a $(6, 2)$, $(2, 6)$ **b** $16 - 12\ln 3$

36 b $\dfrac{1}{5}$

37 a $8x\cos(4x) + 2\sin(4x)$ **b** 0

Multiple-choice questions

1 D	**2** B	**3** E	**4** C	**5** D
6 B	**7** C	**8** A	**9** B	**10** A
11 B	**12** E	**13** E	**14** D	**15** B
16 C	**17** D	**18** A	**19** E	**20** D
21 E	**22** E	**23** B	**24** C	**25** B
26 E	**27** D	**28** B	**29** B	**30** A
31 D	**32** D	**33** A	**34** C	**35** C
36 D	**37** B	**38** A	**39** E	**40** D
41 A	**42** A	**43** B	**44** B	**45** B
46 C	**47** D	**48** B	**49** D	**50** C
51 A	**52** B	**53** D	**54** E	**55** C
56 E	**57** B			

Extended-response questions

1 a $k = 0.0292$ **b** 150×10^6

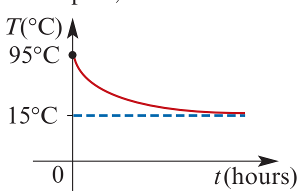
c 6.4494×10^8 **d** 23.738 years

2 a $A = 80$, $k = 0.3466$

b 17.5°C

c 6 hours 18 minutes and 14 seconds after 2 p.m., i.e. 20:18:14

d



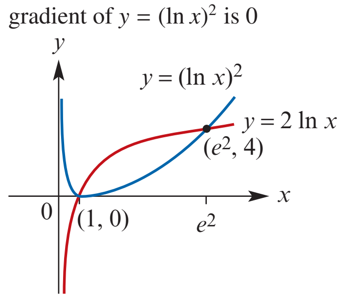
3 a $a = 45$, $b = 10$ **b** 104 dB

c Power setting 4

4 a $x = 1$ or $x = e^2$

b When $x = 1$, gradient of $y = 2\ln x$ is 2 and gradient of $y = (\ln x)^2$ is 0

c



d $1 < x < e^2$

5 b $k = 0.028$ **c** 0.846°C/min

6 a 27 square units **b** $y = \dfrac{3}{25}(x-4)^2$

c $\dfrac{189}{25}$ square units **d** $\dfrac{486}{25}$ square units

7 a $-5\frac{1}{3}$ **c** $a = 1$ or $a = -2$

8 a i $50e^{-1}$ litres/minute **ii** $t = 5$

iii 2 minutes 18 seconds

iv 3 minutes 48 seconds

b 14.74 litres

c 53 seconds

9 a $h = 250 + 25\sin\left(\dfrac{\pi t}{30}\right)$ **b** 275 cm

c 237.5 cm **d** After 5 seconds

e $\dfrac{5}{3}$ cm/s **f** $\dfrac{5\pi\sqrt{3}}{12}$ cm/s

10 a $\dfrac{3}{\ln 2}$

b i $A_1 = 5$ **ii** $E_1 = 0.67$

c i $A_2 = 4.5$ **ii** $E_2 = 0.17$

d $A_4 = 4.37$, $E_4 = 0.043$; $A_8 = 4.34$, $E_8 = 0.011$

11 a $f'(x) = 1 - \dfrac{1}{x}$ **b** $(1, 1)$

c $x = \dfrac{n}{n-1}$ **d** $a = e$

e $y = (1-e)x + 2$

f $y = (1 - e^{-n})x + 1 - n$, $(0, 1-n)$

g $\ln x + 1$, $-x\ln x + \dfrac{x^2}{2} + x$ **h** $\dfrac{e^2-3}{2}$

12 a $\dfrac{dt}{dN} = \dfrac{5000}{N(500-N)}$

b i 1.21 **ii** 2.34 **iii** 13.86 **iv** 35.84

c i 0.12 **ii** 0.11 **iii** 0.08 **iv** 0.22

d i $t = \dfrac{2}{25}N + 20(\ln 2 - 1)$

ii $t = \dfrac{1}{8}N - \dfrac{25}{2}$

e $N = \dfrac{500}{1 + 4e^{-\frac{t}{10}}}$

Degree-of-difficulty classified questions

Simple familiar questions

1 a $x = 5$ **b** $x = \log_3\left(\dfrac{5}{3}\right)$ **c** $x = \dfrac{2}{3}$

2 a $-\dfrac{\cos(2x) + 2x\sin(2x)}{x^2}$

b $2x(2x+1)e^{4x}$ **c** $6x\cos(3x^2)$

d $\dfrac{\cos x}{\sin x}$ **e** $x^2(3\ln x + 1)$

f $\dfrac{(2x+3)e^{2x}}{(x+2)^2}$

3 a -12 **b** 60 **c** 14

4 a $-\dfrac{1}{3}\cos(3x+5) + c$ **b** $-\dfrac{1}{4}e^{-4x} + c$

c $\dfrac{1}{2}\sin(2x-4) + c$

5 a $\dfrac{2}{5}\ln 2$ **b** $4\ln 2 + 15$ **c** $\dfrac{5\pi}{6}$

6 a $B(6, 12)$ **b** 36 square units

7 $(2e^{\frac{5}{2}} + 4\ln 2 - 14)$ square units

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8 **a** $\frac{1}{3}x^3 - 2x^2 + 6x$ **b** $3\ln x + \frac{x^3}{3} - \frac{3x^2}{2}$

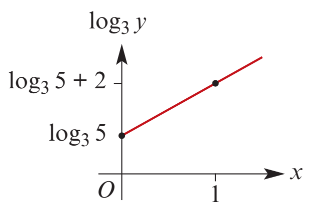
c $x^4 - x^2 + \frac{4}{x}$ **d** $\frac{2}{3}x^{\frac{3}{2}} + 2\sqrt{x}$

e $\frac{1}{4}\sin(4x) - \frac{1}{3}\cos(3x)$ **f** $\frac{1}{2}e^{2x-3}$

9 $f(x) = \frac{4x^3}{3} - 2x^2 - 24x + 12$

10 **a** $m = 2,\ c = \log_3 5$

b



Complex familiar questions

1 $A = e^2,\ b = -0.25$

2 **a** 12 s **b** 144 m **c** -2 m/s^2 **d** 12 m/s

3 **a** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), (0, 0), \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

b $(-\pi, -\pi), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), (0, 0), \left(\frac{\pi}{2}, -\frac{\pi}{2}\right), (\pi, \pi)$

4 $x = -1$ or $x = -\frac{3}{5}$

5 **a** pH 4.1 **b** pH 1

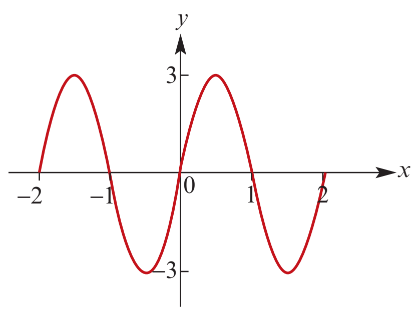
6 **a** 0 m/s **b** $a = 9.6e^{-0.2t}$

c 48 m/s^2 **d** 0 m/s^2

e As t becomes very large, the object approaches a terminal velocity of 48 m/s

f $x = 48t + 240e^{-0.2t} - 240$ **g** 9.21 s

7 **a**



b $y = 3$ **c** $\frac{3}{\pi} - \frac{1}{32}$

8 $4\ln 2$ square units

Complex unfamiliar questions

1 **a** **i** $\frac{4}{3}$ **ii** $\frac{4}{5}$ **iii** $\frac{3}{5}$

b 3 square units

2 **a** A reflection in the y-axis, then a translation of π units to the right

b $x = 0, \pi, 2\pi, 3\pi, 4\pi, \dots$

c **i** $y = e^{\pi}x$ **ii** $y = -x + \pi$

d $\left(\frac{\pi}{e^{\pi} + 1}, \frac{\pi e^{\pi}}{e^{\pi} + 1}\right)$

e $\left(\frac{(8n-7)\pi}{4}, \frac{1}{\sqrt{2}}e^{\frac{(11-8n)\pi}{4}}\right),\ n = 1, 2, 3, \dots$

f $\left(\frac{(8n-3)\pi}{4}, -\frac{1}{\sqrt{2}}e^{\frac{(7-8n)\pi}{4}}\right),\ n = 1, 2, 3, \dots$

g $x = \frac{\pi}{2} + 2n\pi,\ n = 0, 1, 2, \dots$

j $2e^{\pi - x}\sin x,\ \frac{e^{\pi} + 1}{2},\ -\frac{e^{-\pi} + 1}{2}$

3 **a** $(3, 1)$

b $\frac{1}{e} + 2 < x < \frac{1}{\sqrt{e}} + 2$

c **i** $y = -x + e + 2$ **ii** $y = x + e^{-1} - 2$

d $\left(\frac{1}{2}(e - e^{-1}) + 2, \frac{1}{2}(e + e^{-1})\right)$

e $b = 2 + \frac{a - 2}{\ln(a - 2)}$; Min value $= e + 2$

4 **a** 7100

b 11 months

c The population approaches 20 000

d $t = 5\ln\left(\frac{9P}{20\,000 - P}\right),\ 0 < P < 20\,000$

e $\frac{dP}{dt} = \frac{36\,000e^{-\frac{t}{5}}}{\left(1 + 9e^{-\frac{t}{5}}\right)^2}$

f **i** 712.64 **ii** 485.95 **iii** 85.38 **iv** 1.63

5 $s = \sqrt[3]{a^2b},\ r = \sqrt[3]{ab^2}$

6 **a** $\frac{dy}{dx} = \ln x + 1,\quad \int_1^e \ln x\, dx = 1$

b $\frac{dy}{dx} = (\ln x)^n + n(\ln x)^{n-1}$

d $\int_1^e (\ln x)^3\, dx = 6 - 2e$

Chapter 12

Exercise 12A

1 **a** 0 **b** $56x^6$ **c** $\frac{-1}{4\sqrt{x^3}}$

d $48(2x + 1)^2$ **e** $-\sin x$ **f** $-\cos x$

g e^x **h** $\frac{-1}{x^2}$ **i** $\frac{2}{(x + 1)^3}$

j $-4\sin\left(2x + \frac{\pi}{4}\right)$

2 **a** $\frac{15\sqrt{x}}{4}$ **b** $8(x^2 + 3)^2(7x^2 + 3)$

c $-\frac{1}{4}\sin\left(\frac{x}{2}\right)$ **d** $-48\cos(4x + 1)$

e $2e^{2x+1}$ **f** $\frac{-4}{(2x + 1)^2}$ **g** $12x^2 + 6$

h $(x^3 + 6x^2 + 6x)e^x$ **i** $\frac{1}{x}$

3 **a** $f''(x) = 24e^{3-2x}$

b $f''(x) = 8e^{-0.5x^2}(1 - x^2)$ **c** $f''(x) = 0$

d $f''(x) = \frac{-2(x^2 + 2x + 2)}{(x^2 + 2x)^2}$

e $f''(x) = 360(1 - 3x)^3$

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f $f''(x) = (4x^2 + 2)e^{x^2}$

g $f''(x) = \dfrac{-4}{(x+1)^3}$ **h** $f''(x) = \dfrac{3}{4\sqrt{(1-x)^5}}$

i $f''(x) = -5\sin(3 - x)$

j $f''(x) = -9\cos(1 - 3x)$

k $f''(x) = -\dfrac{1}{9}\sin\left(\dfrac{x}{3}\right)$ **l** $f''(x) = -\dfrac{1}{16}\cos\left(\dfrac{x}{4}\right)$

4 a 1 **b** −1 **c** −1 **d** 0

5 a $x = -\dfrac{2}{3}$ **b** $x = \dfrac{1}{15}$ **c** $x = \pm\dfrac{1}{\sqrt{2}}$

6 a $x = -2$ **b** $x = 0$ or $x = -3$

c $x = 0,\ x = -3 + \sqrt{3}$ or $x = -3 - \sqrt{3}$

d $x = e^2$ **e** $x = e^{\frac{3}{2}}$

7 a $t = 0, x = 9;\ t = 4, x = 5;\ t = 8, x = 9$

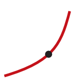
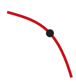
b $-\dfrac{\pi^2}{8}$ m/s², $\dfrac{\pi^2}{8}$ m/s², $-\dfrac{\pi^2}{8}$ m/s²

8 a $v = 2e^{-2t} - e^{-t},\ a = e^{-t} - 4e^{-2t}$

b $\dfrac{1}{4}$ m, $t = \ln 2$ **c** $-\dfrac{1}{2}$ m/s²

d $\dfrac{1}{8}$ m/s

Exercise 12B

1 a  **b**  **c**  **d** 

2 a Point of inflection (0, 0);
Concave up on $(0, \infty)$

b Point of inflection $\left(\dfrac{1}{3}, -\dfrac{2}{27}\right)$;
Concave up on $\left(\dfrac{1}{3}, \infty\right)$

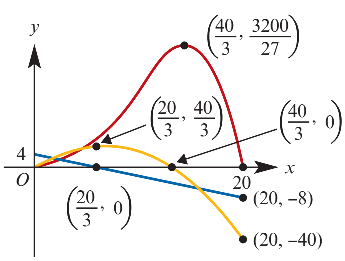
c Point of inflection $\left(\dfrac{1}{3}, \dfrac{2}{27}\right)$;
Concave up on $\left(-\infty, \dfrac{1}{3}\right)$

d Points of inflection (0, 0), $\left(\dfrac{1}{2}, -\dfrac{1}{16}\right)$;
Concave up on $(-\infty, 0) \cup \left(\dfrac{1}{2}, \infty\right)$

3 a Local min (0, 0); local max $\left(\dfrac{40}{3}, \dfrac{3200}{27}\right)$

b $\left(\dfrac{20}{3}, \dfrac{1600}{27}\right)$; gradient $= \dfrac{40}{3}$

c



4 a i $6x^2 + 12x$ **ii** $12x + 12$

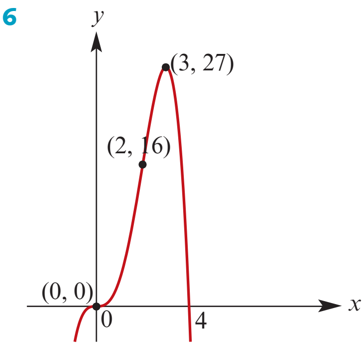
b Local min (0, −12); local max (−2, −4)

c (−1, −8)

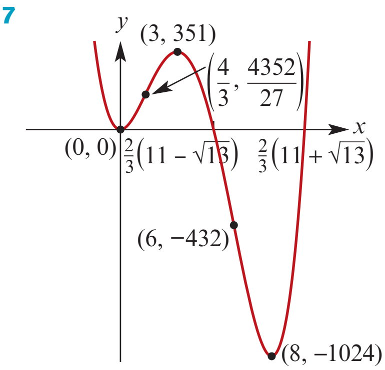
5 a $f'(x) = \cos x;\ f''(x) = -\sin x$;
Local max $\left(\dfrac{\pi}{2}, 1\right)$; Local min $\left(\dfrac{3\pi}{2}, -1\right)$;
Point of inflection $(\pi, 0)$

b $f'(x) = e^x(x + 1);\ f''(x) = e^x(x + 2)$;
Local min $(-1, -e^{-1})$;
Point of inflection $(-2, -2e^{-2})$

6

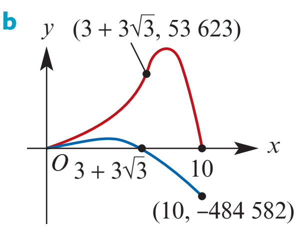


7



8 a $f'(x) = e^x(10 + 8x - x^2)$,
$f''(x) = e^x(18 + 6x - x^2)$

b



c $3 + 3\sqrt{3}$, $(3 + 3\sqrt{3}, 53\,623)$

9 $(0, 0), (\pi, \pi), (2\pi, 2\pi), (3\pi, 3\pi), (4\pi, 4\pi)$

10 a $0, \pi, 2\pi$ **b** $0, \pi, 2\pi$

c $0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi$

11 $\dfrac{d^2y}{dx^2} = 2a \neq 0$

12 a $\dfrac{3}{2} < x < 2$ **b** $1 < x < \dfrac{3}{2}$

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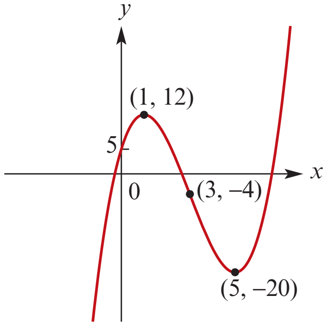
13 **a** $(0, 0), -6$ **b** $(-1, -1), 8;\ (1, -1), -8$
c $(0, 3), 0$ **d** No points of inflection
e No points of inflection
f $\left(-\sqrt{3}, \frac{-\sqrt{3}}{2}\right), \frac{-1}{4};\ (0, 0), 2;\ \left(\sqrt{3}, \frac{\sqrt{3}}{2}\right), \frac{-1}{4}$

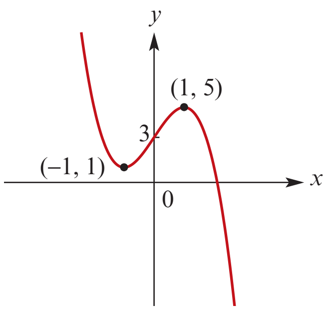
14 **a** $\frac{-7\pi}{4}, \frac{-3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$ **b** $\frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

16 **a** $f'(x) = 2x(1 + 2\ln x)$
b $f''(x) = 2(3 + 2\ln x)$
c Stationary point at $(e^{-\frac{1}{2}}, -e^{-1})$;
point of inflection at $(e^{-\frac{3}{2}}, -3e^{-3})$

17 **a** $a = -9,\ b = 15,\ c = 5$
b



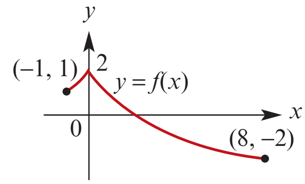
18 **a** $a = -1,\ b = 3,\ c = 3$ **b** $x = -1$
c



Exercise 12C

1 Absolute max = 2; Absolute min = –70
2 Absolute max = 15; Absolute min = –30
3 Absolute max = 0; Absolute min = –20.25
4 Absolute max = 2304; Absolute min = –8
5 **b** $\frac{dV}{dx} = 30x - 36x^2$
c Local max at $\left(\frac{5}{6}, \frac{125}{36}\right)$
d Absolute max value is 3.456 when $x = 0.8$
e Absolute max value is $\frac{125}{36}$ when $x = \frac{5}{6}$
6 **a** $25 \le y \le 28$
b Absolute max = 125; Absolute min = 56
7 **a** $\frac{1}{(x-4)^2} - \frac{1}{(x-1)^2}$ **b** $\left(\frac{5}{2}, \frac{4}{3}\right)$
c Absolute max = $\frac{3}{2}$; Absolute min = $\frac{4}{3}$

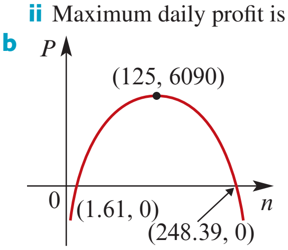
8 **b** $\frac{dA}{dx} = \frac{1}{4}(x - 5)$ **c** $x = 5$ **d** $\frac{61}{8}$ m^2
9 Absolute max = 12.1; Absolute min = 4
10 **a** $\frac{1}{(x-4)^2} - \frac{1}{(x+1)^2}$ **b** $\left(\frac{3}{2}, \frac{4}{5}\right)$
c Absolute max = $\frac{5}{4}$; Absolute min = $\frac{4}{5}$
11 Absolute max = $\frac{\sqrt{2}}{2}$; Absolute min = –1
12 Absolute max = 1; Absolute min = $\frac{\sqrt{2}}{2}$
13 Absolute max = 2; Absolute min = –2



14 Absolute max = $\frac{1}{e^2} + 2e^2$;
Absolute min = $2\sqrt{2}$
15 Absolute max = $-\ln 10$;
Absolute min = $-\frac{10}{e}$

Exercise 12D

1 625 m^2
2 First = $\frac{4}{3}$; Second = $\frac{8}{3}$
3 Max value of P is 2500
4 Max area is 2 km × 1 km = 2 km^2
5 $p = \frac{3}{2},\ q = \frac{8}{3}$
6 **b** $V = \frac{75x - x^3}{2}$ **c** 125 cm^3
7 **a** **i** $n = 125$
ii Maximum daily profit is \$6090
b



c $2 \le n \le 248$
d $n = 20$
8 12°C
9 8 mm for maximum; $\frac{4}{3}$ mm for minimum
10 **a** $8\cos\theta$
b Area = $16(1 + \cos\theta)\sin\theta$;
Max area = $12\sqrt{3}$ square units
11 (1, 1)

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12 a $\dfrac{75}{\cos\theta}$ seconds

b $220 - 60\tan\theta$ seconds

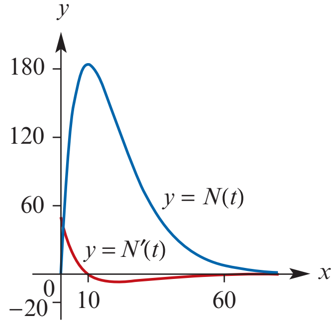
d $\dfrac{dT}{d\theta} = \dfrac{75\sin\theta - 60}{\cos^2\theta}$

e $\theta = \sin^{-1}\left(\dfrac{4}{5}\right) \approx 53.13°$

f Min time $T = 265$ seconds occurs when distance BP is 400 metres

13 Max population $\dfrac{500}{e}$ occurs when $t = 10$

14 a

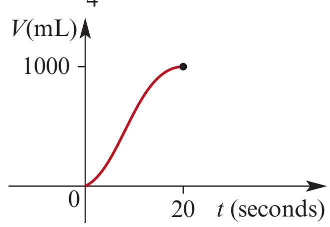


b Max rate of increase is 50, occurs at $t = 0$;
Max rate of decrease is $\dfrac{50}{e^2}$, occurs at $t = 20$

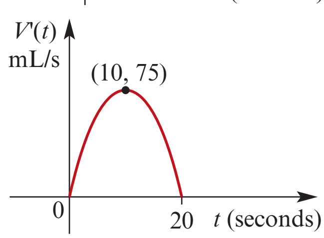
15 a i $V(0) = 0$ mL **ii** $V(20) = 1000$ mL

b $V'(t) = \dfrac{3}{4}(20t - t^2)$

c



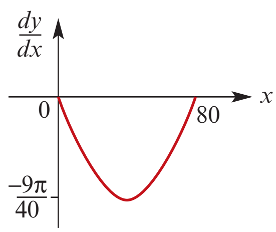
d



e $t = 10$ s, 75 mL/s

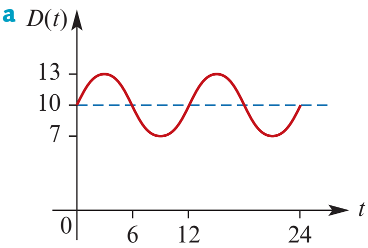
16 a $\dfrac{dy}{dx} = \dfrac{-9\pi}{40}\sin\left(\dfrac{\pi x}{80}\right)$

b



c $\left(40, \dfrac{-9\pi}{40}\right)$

17 a



b $t \in [0, 7] \cup [11, 19] \cup [23, 24]$

c i 0 m/h **ii** $-\dfrac{\pi}{2}$ m/h **iii** $\dfrac{\pi}{2}$ m/h

d i $t = 0, 12, 24$ **ii** $t = 6, 18$

Chapter 12 review

Technology-free questions

1 a $224(2x + 5)^6$ **b** $-4\sin(2x)$

c $-\dfrac{1}{9}\cos\left(\dfrac{x}{3}\right)$ **d** $16e^{-4x}$

e $-\dfrac{1}{x^2}$ **f** $-\dfrac{1}{\sin^2 x}$

2 a $\left(\dfrac{8}{3}, -\dfrac{1024}{27}\right)$ **b** $\left(\dfrac{\pi}{6}, 0\right), \left(\dfrac{7\pi}{6}, 0\right)$

c $\left(2, \ln 2 + \dfrac{1}{2}\right)$ **d** $\left(e^2, \dfrac{e^2}{2}\right)$

3 a $x = e^{-\frac{1}{2}}$ **b** $x = e^{-\frac{3}{2}}$

4 a $x = -3$ **b** $x = 0$

5 a $f'(x) = -2ckxe^{-kx^2}$,
$f''(x) = 2ck(2kx^2 - 1)e^{-kx^2}$

b $\left(\dfrac{1}{2}, e^{-\frac{1}{2}}\right), \left(-\dfrac{1}{2}, e^{-\frac{1}{2}}\right)$ **c** $k = 8$

6 a $v = te^{-3t}$ **b** $a = (1 - 3t)e^{-3t}$

c $\dfrac{1}{3e}$ m/s, $t = \dfrac{1}{3}$ s, $x = \dfrac{e - 2}{9e}$ m

7 a $W = xe^{-x^2}$

b $\dfrac{dW}{dx} = (1 - 2x^2)e^{-x^2}$, $\dfrac{d^2W}{dx^2} = 2x(2x^2 - 3)e^{-x^2}$

c $\left(\dfrac{\sqrt{6}}{2}, \dfrac{\sqrt{6}}{2}e^{-\frac{3}{2}}\right)$

d Max area $W = \dfrac{1}{\sqrt{2e}}$ occurs when $x = \dfrac{\sqrt{2}}{2}$

Multiple-choice questions

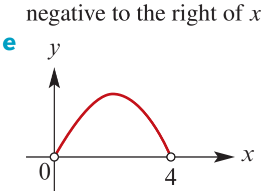
1 E **2** D **3** B **4** E **5** B
6 C **7** E **8** B **9** E **10** A

Extended-response questions

1 a $y = 4x - x^2$ **b** $0 < x < 4$ **c** $y = 4, x = 2$

d Gradient is positive to the left of $x = 2$, and negative to the right of $x = 2$

e



f $0 < y < 4$

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2 **a** $A = 4xy$ **b** $y = -\frac{2}{3}x + 8$

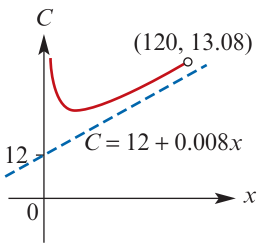
c $A = 32x - \frac{8}{3}x^2$ **d** $x = 6, y = 4$

e 96 m^2

3 **a** **i** \$12.68 **ii** \$12.74

b $C = 12 + 0.008x + \frac{14.40}{x}$

c

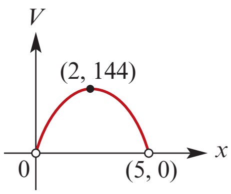


d 42.43 km/h

4 **b** $0 < x < 5$ **c** $x = 2$

d 2 cm, 12 cm, 6 cm **e** 144 cm^3

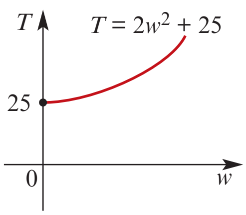
f



5 32 square units

6 **a** $k = 25$

b



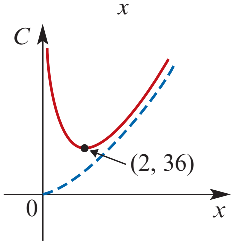
c $A = \frac{25}{w} + 2w$

d **i** $\frac{5\sqrt{2}}{2} \approx 3.54$ kg **ii** $10\sqrt{2} \approx 14.14$ s

7 10 m, 10 m, 5 m; Area 300 m^2

8 **b** $C = 3x^2 + \frac{48}{x}$

c



d **i** $x = 2, h = 3$; i.e. 2 m, 2 m, 3 m

ii 36 m^2

9 **a** $A = \frac{1}{2}a^2\theta$ **b** $A = \frac{1}{2}\left(\frac{100}{\theta + 2}\right)^2\theta$

c $\theta = 2$ **d** 625 cm^2

10 **b** **i** $r = \frac{L}{4}$ **ii** $\theta = 2$ **iii** Maximum

11 $t = 5,\ N(5) = \frac{120}{e}$

12 $t = 1.16$, 1.2 km apart

13 **b** $T = \frac{20 + 16\sqrt{2}}{15} \approx 2.84$ hours

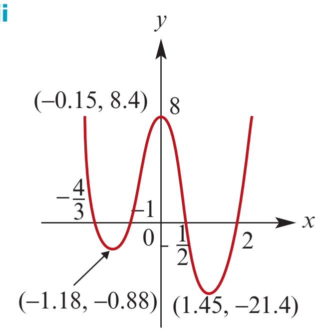
14 **b** $0 < x < 1$ **c** $x = \frac{1}{\sqrt{2}},\ y = \pm 1$

d $A = 2\sqrt{2}$

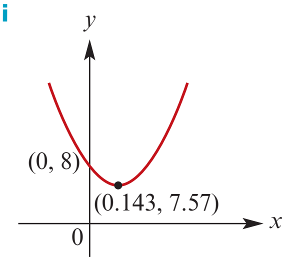
15 **c** **ii** $\frac{dA}{dx} = -3x^2 - 2ax + a^2$

16 **a** **i** $a = -21$

ii



b **i**



ii Min at (0.143, 7.57)

iii $g'(x) = 24x^3 - 3x^2 + 42x - 6$

iv $x = 0.1427$

v $g'(0) = -6,\ g'(10) = 24\,114$

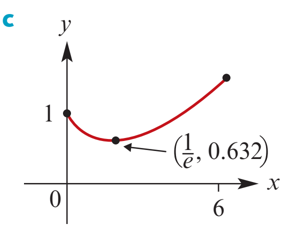
vi $g''(x) = 72x^2 - 6x + 42$

vii $g''(x) > 0$ for all x; thus $y = g'(x)$ has no turning points and crosses the x-axis only once

17 **a** $f'(x) = \ln x + 1$

b $x = \frac{1}{e} \approx 0.37$, i.e. during the first month

c



d When $x = 6$

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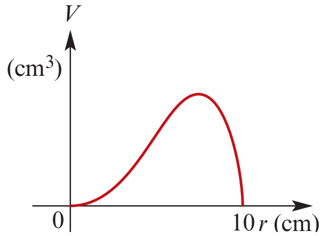
18 **a** **i** $y = \sqrt{100 - r^2}$, $h = 2\sqrt{100 - r^2}$
ii $V = 2\pi r^2 \sqrt{100 - r^2}$

b **i**



ii $V = 2418.4$, $r = 8.165$, $h = 11.55$
iii $r = 6.456$ or $r = 9.297$

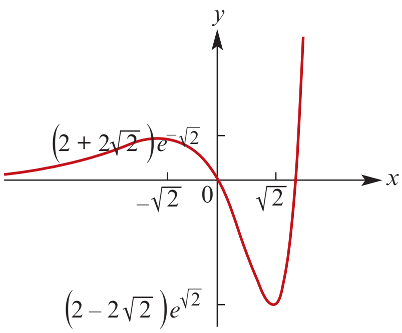
c **i** $\dfrac{dV}{dr} = \dfrac{400\pi r - 6\pi r^3}{\sqrt{100 - r^2}}$

ii $V_{\max} = \dfrac{4000\pi\sqrt{3}}{9}$ when $r = \dfrac{10\sqrt{6}}{3}$

d **ii** $\dfrac{dV}{dr} > 0$ for $r \in \left(0, \dfrac{20\sqrt{6}}{6}\right)$

iii $\dfrac{dV}{dr}$ is increasing for $r \in (0, 5.21)$

19 **a** $x = 0$ and $x = 2$ **b** $y = -ex$
c $y = 2e^2 x - 4e^2$ **d** $x = \pm\sqrt{2}$
e $x = -1 \pm \sqrt{3}$
f



20 **a** $x = 0$ and $x = \dfrac{5}{2}$

b $x = \dfrac{5a - 4 \pm \sqrt{25a^2 + 16}}{4a}$

c $x = \dfrac{5a - 8 \pm \sqrt{25a^2 + 32}}{4a}$

d $a = \dfrac{1}{3}\ln\left(\dfrac{10}{3}\right)$

21 **b** $MP = \dfrac{2}{\tan\theta}$ **c** $NQ = 8\tan\theta$

d $x = \dfrac{2}{\tan\theta} + 8\tan\theta + 10$

e $\dfrac{dx}{d\theta} = \dfrac{8}{\cos^2\theta} - \dfrac{2}{\sin^2\theta}$

f $x = 18$, $\theta = 26.6°$

22 **a** $h = a(1 + \cos\theta)$ **b** $r = a\sin\theta$

d $\dfrac{dV}{d\theta} = \dfrac{\pi a^3}{3}[2\sin\theta\cos\theta(1 + \cos\theta) - \sin^3\theta]$

$\theta = \cos^{-1}(\frac{1}{3}) \approx 70.53°$

e $V = \dfrac{32\pi a^3}{81}$ cm^3

23 **b** $\dfrac{dy}{dt} = \dfrac{bAe^{bt}}{(1 + Ae^{bt})^2}$

e After 7 hours (to the nearest hour)

24 **a** $f'(x) = \dfrac{xe^x - e^x}{x^2}$ **b** $x = 1$

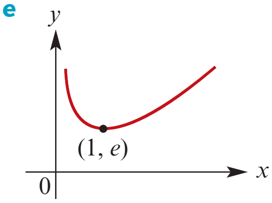
c $(1, e)$, local minimum

d **i** $\dfrac{f'(x)}{f(x)} = \dfrac{x - 1}{x}$

ii $\lim\limits_{x\to\infty} \dfrac{f'(x)}{f(x)} = 1$

i.e. $f'(x) \to f(x)$ as $x \to \infty$

e



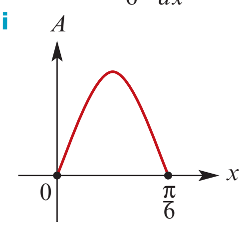
f $t = \dfrac{1}{k} \approx 45.27$ years, i.e. during 1945

25 **b** **i** $\dfrac{dA}{dx} = 2\cos(3x) - 6x\sin(3x)$

ii When $x = 0$, $\dfrac{dA}{dx} = 2$;

When $x = \dfrac{\pi}{6}$, $\dfrac{dA}{dx} = -\pi$

c **i**



ii $x = 0.105$ or $x = 0.449$
iii Max area 0.374, occurs when $x = 0.287$

Chapter 13

Exercise 13A

1 **a** A and C (SAS) **b** All of them (AAS)
c A and B (SSS)

2 **a** 4.10 **b** 0.87 **c** 2.94
d 4.08 **e** 33.69° **f** 11.92

3 $\dfrac{40\sqrt{3}}{3}$ cm

4 66.42°, 66.42°, 47.16°

5 23 m

6 **a** 9.59° **b** $\sqrt{35}$ m

7 **a** 60° **b** 17.32 m

8 **a** 6.84 m **b** 6.15 m

9 12.51°

10 182.7 m

11 1451 m

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12 a $5\sqrt{2}$ cm **b** 90°
13 3.07 cm **14** 37.8 cm
15 31.24 m **16** 4.38 m
17 57.74 m **18** $\dfrac{2\sqrt{3}}{2-\sqrt{3}} \approx 12.93$ m
19 $\dfrac{10}{1+\sqrt{3}} \approx 3.66$ **20** $\angle APB = 47.16°$

Exercise 13B

1 a 8.15 **b** 3.98 **c** 11.75 **d** 9.46
2 a 56.32° **b** 36.22° **c** 49.54°
d 98.16° or 5.84°
3 a $A = 48°$, $b = 13.84$, $c = 15.44$
b $a = 7.26$, $C = 56.45°$, $c = 6.26$
c $B = 19.8°$, $b = 4.66$, $c = 8.27$
d $C = 117°$, $b = 24.68$, $c = 34.21$
e $C = 30°$, $a = 5.40$, $c = 15.56$
4 a $B = 59.12°$, $A = 72.63°$, $a = 19.57$ or $B = 120.88°$, $A = 10.87°$, $a = 3.87$
b $C = 26.69°$, $A = 24.31°$, $a = 4.18$
c $B = 55.77°$, $C = 95.88°$, $c = 17.81$ or $B = 124.23°$, $C = 27.42°$, $c = 8.24$
5 554.26 m
6 35.64 m
7 1659.86 m
8 a 26.60 m **b** 75.12 m

Exercise 13C

1 5.93 cm
2 $\angle ABC = 97.90°$, $\angle ACB = 52.41°$
3 a 26 **b** 11.74 **c** 49.29° **d** 73
e 68.70 **f** 47.22° **g** 7.59 **h** 38.05°
4 3.23 km
5 2.626 km
6 55.93 cm
7 a 8.23 cm **b** 3.77 cm
8 a 7.326 cm **b** 5.53 cm
9 a 83.62° **b** 64.46°
10 a 87.61 m **b** 67.7 m

Exercise 13D

1 a 11.28 cm² **b** 15.10 cm²
c 10.99 cm² **d** 9.58 cm²
2 a 6.267 cm² **b** 15.754 cm²
c 19.015 cm² **d** 13.274 cm²
e 24.105 cm² or 29.401 cm²
f 2.069 cm²

Exercise 13E

1 400.10 m **2** 34.77 m **3** 575.18 m
4 109.90 m **5** 16.51 m **6** 056°
7 a 034° **b** 214°
8 a 3583.04 m **b** 353°
9 027° **10** 113° **11** 22.01°
12 a $\angle BAC = 49°$ **b** 264.24 km
13 10.63 km

Exercise 13F

1 a 13 cm **b** 15.26 cm
c 31.61° **d** 38.17°
2 a 4 cm **b** 71.57° **c** 12.65 cm
d 13.27 cm **e** 72.45° **f** 266.39 cm²
3 10.31° at B; 14.43° at A and C
4 a 85 m **b** 45.04 m
5 17.58°
6 1702.55 m
7 a 24.78° **b** 65.22° **c** 20.44°
8 42.40 m
9 1945.54 m
10 a 6.96 cm **b** 16.25 cm²
11 a 5 km **b** 215.65° **c** 6.55°

Exercise 13G

1 a $4a^2$, $3a^2$ and $12a^2$ square units respectively
b 14.04° **c** 18.43° **d** 11.31°
2 a 35.26° **b** 45°
3 a 0.28 **b** 15.78°
4 a 15.51 cm **b** 20 cm **c** 45.64°
5 a i 107 m **ii** 87 m **iii** 138 m
b 43.00°
6 a $5\sqrt{11}$ cm **b** 64.76° **c** 71.57° **d** 95.74°
7 26.57°
8 a 54.74° **b** 70.53°
9 1.67 km
10 a 141.42 m **b** 20.70°
11 16 cm
12 a $\dfrac{a\sqrt{3}}{2}$ cm **b** $\dfrac{a}{2}$ cm

Chapter 13 review

Technology-free questions

1 $\sin(\theta°) = \dfrac{5}{6}$
2 a $\angle ABC = 30°$, $\angle ACB = 120°$
b $40\sqrt{3}$ cm **c** 20 cm
3 $10\sqrt{3}$ cm **4** $4\sqrt{19}$ km **5** $\dfrac{1}{6}$
6 a $5\sqrt{3}$ cm **b** $\dfrac{25\sqrt{3}}{4}$ cm²
c $\dfrac{105}{4}$ cm² **d** $\dfrac{5(21+5\sqrt{3})}{4}$ cm²
7 143° **8** $\dfrac{17}{28}$ **9** $\dfrac{3\sqrt{93}}{31}$
10 $\sin(\angle PQR) = 0.9$

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11 a i 30° **ii** 15°
b $AT = 300(1+\sqrt{3})$ m, $BT = 150(\sqrt{6}+\sqrt{2})$ m

12 $\sqrt{181}$ km

13 a $AC = \frac{12\sqrt{3}}{5}$ km, $BC = 2.4$ km
b 57.6 km/h

14 11 m

Multiple-choice questions

1 D **2** C **3** C **4** B **5** D **6** B
7 C **8** A **9** C **10** A **11** A

Extended-response questions

1 a $\angle ACB = 12°$, $\angle CBO = 53°$, $\angle CBA = 127°$
b 189.33 m **c** 113.94 m

2 a $\angle TAB = 3°$, $\angle ABT = 97°$, $\angle ATB = 80°$
b 2069.87 m **c** 252.25 m

3 a 184.78 m **b** 199.71 m **c** 14.93 m

4 a 370.17 m **b** 287.94 m **c** 185.08 m

5 a 029.9°
b i $\angle XWY = 33.14°$ **ii** 105.86°

6 a $8\sqrt{2}$ cm **b** 10 cm **c** 10 cm **d** 68.90°

7 1942 m

8 a 26.57° **b** 39.81° **c** 38.66°

9 Area $= \dfrac{L^2 \sin\alpha \sin\beta \sin\gamma}{2(\sin\alpha + \sin\beta + \sin\gamma)^2}$

Chapter 14

Exercise 14A

1 1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T

2 HH1, HH2, HH3, HH4, HH5, HH6,
HT1, HT2, HT3, HT4, HT5, HT6,
TH1, TH2, TH3, TH4, TH5, TH6,
TT1, TT2, TT3, TT4, TT5, TT6

3 a $\frac{1}{13}$ **b** $\frac{3}{4}$ **c** $\frac{4}{13}$ **d** $\frac{2}{13}$

4 a $\frac{1}{2}$ **b** $\frac{2}{3}$

5 0.8

6 0.65

7 a 0.067 **b** 0.047

8 5%

9 $\frac{6}{7}$

10 a $\frac{17}{500}$ **b** $\frac{9}{250}$ **c** $\frac{41}{125}$ **d** $\frac{41}{500}$

11 a $\frac{13}{20}$ **b** $\frac{7}{20}$

12 a $\frac{57}{100}$ **b** $\frac{2}{19}$ **c** $\frac{27}{100}$ **d** $\frac{1}{250}$

13 $\frac{9}{25}$

14 a $\frac{1}{2}$ **b** $\frac{1}{6}$ **c** $\frac{5}{6}$

15 a 0.13 **b** 0.32

16 a 0.40 **b** 0.67 **c** 0.18

17 a 0.35 **b** 0.18 **c** 0.12 **d** 0.17

Exercise 14B

1 a 0.2 **b** 0.675 **c** 0.275

2 a $\frac{1}{6}$ **b** $\frac{1}{3}$

3 a 0.06 **b** $\frac{1}{5}$

4 $\frac{3}{5}$

5 $\frac{24}{59}$

6 a $\frac{1}{2}$ **b** $\frac{77}{200}$ **c** $\frac{40}{77}$ **d** 0.4

7 a $\frac{65}{224}$ **b** $\frac{115}{448}$ **c** $\frac{21}{65}$ **d** $\frac{61}{246}$

8 a 0.24 **b** 0.86

9 a Yes **b** No **c** No

10 a 0.5 **b** 0.2 **c** 0.7

11 0.39

12 $\frac{1}{7}$

13 0.0479

14 a 0.486 **b** 0.012 **c** 0.138

15 a $\frac{2}{5}$ **b** $\frac{1}{15}$ **c** $\frac{8}{15}$

16 a $\frac{2}{5}$ **b** $\frac{7}{40}$ **c** $\frac{7}{16}$ **d** $\frac{7}{15}$

17 a $\frac{5}{14}$ **b** $\frac{3}{5}$

18 a 0.735 **b** 0.453

19 $\frac{3}{44}$

Exercise 14C

1 a Discrete **b** Not discrete
c Discrete **d** Discrete

2 a Not discrete **b** Discrete
c Not discrete **d** Discrete

3 a $\{HHH, THH, HTH, HHT, HTT, THT, TTH, TTT\}$

b

x	Outcomes
0	TTT
1	HTT, THT, TTH
2	THH, HTH, HHT
3	HHH

c $\frac{1}{2}$

4 a Yes, as the sum of the probabilities is 1 and $p(x) \geq 0$ for all x
b 0.8

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5 a

x	0	1	2	3
$p(x)$	$\frac{125}{729}$	$\frac{300}{729}$	$\frac{240}{729}$	$\frac{64}{729}$

b $\frac{604}{729}$ **c** $\frac{304}{729}$

6 a $\{(1, 1), (1, 2), (1, 3), \ldots, (6, 4), (6, 5), (6, 6)\}$

b $Y = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

2nd die

1st die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

c i $\frac{1}{6}$ **ii** $\frac{1}{3}$ **iii** $\frac{1}{5}$ **iv** $\frac{7}{10}$ **v** $\frac{1}{5}$ **vi** $\frac{2}{7}$

7 a

2nd die

1st die	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

b 1, 2, 3, 4, 5, 6 **c** 0.19

8 a 0.288 **b** 0.064 **c** 0.352 **d** 0.182

9 a $\{(1, 1), (1, 2), (1, 3), \ldots, (6, 4), (6, 5), (6, 6)\}$

b $\Pr(A) = \frac{1}{6}$, $\Pr(B) = \frac{1}{6}$, $\Pr(C) = \frac{5}{12}$, $\Pr(D) = \frac{1}{6}$

c $\Pr(A \mid B) = \frac{1}{6}$, $\Pr(A \mid C) = \frac{1}{5}$, $\Pr(A \mid D) = \frac{1}{6}$

d i Independent **ii** Not independent **iii** Independent

10 a Yes **b** 0.5

11 a, c

12 a

x	0	1	2	3
$p(x)$	$\frac{27}{125}$	$\frac{54}{125}$	$\frac{36}{125}$	$\frac{8}{125}$

b

x	0	1	2	3
$p(x)$	$\frac{5}{30}$	$\frac{15}{30}$	$\frac{9}{30}$	$\frac{1}{30}$

13

x	0	1	2
$p(x)$	0.36	0.48	0.16

14 a

x	1	2	3	4	5
$p(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

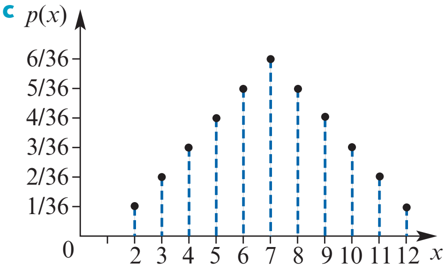
b $\frac{3}{5}$ **c** $\frac{1}{3}$

15 a $\{(1, 1), (1, 2), (1, 3), \ldots, (6, 4), (6, 5), (6, 6)\}$

b

x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

c



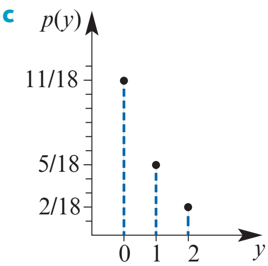
d $\frac{5}{18}$ **e** $\frac{7}{10}$

16 a $\{(1, 1), (1, 2), (1, 3), \ldots, (6, 4), (6, 5), (6, 6)\}$

b

y	0	1	2
$p(y)$	$\frac{11}{18}$	$\frac{5}{18}$	$\frac{2}{18}$

c



17 a

x	0	1	2
$p(x)$	$\frac{1}{3}$	$\frac{8}{15}$	$\frac{2}{15}$

b $\frac{7}{15}$

18 a

x	10	20	100
$p(x)$	$\frac{3}{4}$	$\frac{6}{25}$	$\frac{1}{100}$

b

y	20	30	40	110	120	200
$p(y)$	$\frac{9}{16}$	$\frac{9}{25}$	$\frac{36}{625}$	$\frac{3}{200}$	$\frac{3}{625}$	$\frac{1}{10\,000}$

19 a $\frac{1}{4}$

b $\{EENE, ENEE, ENNN, NEEE, NENN, NNEN\}$, $\Pr(X = 4) = \frac{3}{8}$

c $\Pr(X = 5) = \frac{3}{8}$

Exercise 14D

1 Lose \$150

2 a $E(X) = 6$ **b** $E(X) = 0$

c $E(X) = 2.33$

3 Expected profit = \$64 000

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4 A loss of 45c

5 a

x	0	1	2	3
$\Pr(X = x)$	0.14	0.39	0.36	0.11

b $E(X) = 1.44$

6 $E(X) = 4.083$

7 a $E(X) = 4.11$ **b** $E(5X - 4) = 16.55$

8 $5940

9 a $p = 0.28$ **b** $E(X) = 1.68$
c $\text{Var}(X) = 1.4976$

10 a $k = \frac{1}{36}$ **b** $E(X) = \frac{17}{3}$
c $\text{Var}(X) = \frac{35}{9}$

11 a $\frac{21}{4}$ **b** $\frac{7}{12}$ **c** $\frac{497}{48}$

12 a $E(X) = 2.5$
b $\text{Var}(X) = 2.25,\ \text{sd}(X) = 1.5$ **c** 1

13 a $k = \frac{1}{15}$ **b** $E(X) = 3.667$
c $\text{Var}(X) = 1.556$ **d** 0.9333

14 a 7 **b** 5.83 **c** 0.944

15 a 3 **b** 1.5 **c** 0.9688

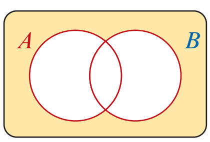
16 $c_1 = 40,\ c_2 = 60$

Chapter 14 review

Technology-free questions

1 Yes, as $\Pr(A \cap B) = 0$

2



$\Pr(A' \cap B') = \Pr(A \cup B)'$

3 a $\frac{40}{81}$ **b** $\frac{5}{9}$

4 0.4

6 a 0.1 **b** 1.3 **c** 2.01

7 a 21.5 **b** 630.75 **c** $\frac{29\sqrt{3}}{2}$

8 a

p	$x - 2$	-2
$\Pr(P = p)$	$\frac{4}{5}$	$\frac{1}{5}$

b $\frac{4}{5}x - 2$

c $x > \$2.50$

9 a 0.47 **b** $\frac{47}{70}$

10 21.5%

11 a $\frac{1}{24}$ **b** $\frac{17}{24}$ **c** $\frac{5}{6}$ **d** $\frac{11}{18}$

Multiple-choice questions

1 A **2** D **3** D **4** E **5** C
6 C **7** B

Extended-response questions

1 a 0.1 **b** 0.2 **c** 4

2 $14

3 a 0.5 **b** 0.05 **c** 0.033 **d** $\frac{25}{33}$

4 a i 1.21
ii $\text{Var}(P) = 1.6659,\ \text{sd}(P) = 1.2907$
iii 0.94

b i

t	1	0.40	0
$p(t)$	0.39	0.27	0.34

ii $E(T) = 0.498 \approx 0.50$ **iii** 1

5 a $E(Y) = 2.002$
b $\text{Var}(Y) = 2.014,\ \text{sd}(Y) = 1.419$
c i

b	0	100	200
$p(b)$	0.677	0.270	0.053

ii $E(B) = \$37.60$

6 b $x = \frac{1}{2}, \frac{49}{288}$

7 a i $\frac{1}{81}$ **ii** $\frac{8}{81}$ **iii** $\frac{4}{81}$ **iv** $\frac{4}{27}$ **v** $\frac{56}{81}$
b 4.4197 cents (as the lowest value coin is 5c, he can settle for that)

8 a i $E(X) = \frac{7}{2}$ **ii** $\text{Var}(X) = \frac{35}{12}$
b i $E(X) = \frac{n+1}{2}$ **ii** $\text{Var}(X) = \frac{n^2 - 1}{12}$
c i $\frac{1}{10}$ **ii** $\frac{2}{5}$ **iii** $\frac{11}{2}$ **iv** $\frac{33}{4}$

Chapter 15

Exercise 15A

1 A, B

2 0.2734

3 a 0.0256 **b** 0.0016

4 a 0.0778 **b** 0.2304 **c** 0.01024

5 a $\Pr(X = x) = \binom{3}{x}(0.5)^x(0.5)^{3-x},\ x = 0, 1, 2, 3$
b 0.375

6 a $\Pr(X = x) = \binom{6}{x}(0.48)^x(0.52)^{6-x}$,
$x = 0, 1, 2, 3, \dots, 6$
b 0.2527

7 a 0.0536 **b** 0.0087 **c** 0.2632

8 a $\Pr(X = x) = \binom{10}{x}(0.1)^x(0.9)^{10-x}$,
$x = 0, 1, 2, 3, \dots, 10$
b i 0.3487 **ii** 0.6513

9 a $\Pr(X = x) = \binom{11}{x}(0.2)^x(0.8)^{11-x}$,
$x = 0, 1, 2, 3, \dots, 11$
b i 0.2953 **ii** 0.0859 **iii** 0.9141

10 a $\Pr(X = x) = \binom{7}{x}(0.2)^x(0.8)^{7-x}$,
$x = 0, 1, 2, 3, \dots, 7$
b i 0.000013 **ii** 0.2097 **iii** 0.3899

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11 0.624

12 a $\left(\dfrac{x}{100}\right)^6$ **b** $\dfrac{6x^5(100-x)}{100^6}$

c $\dfrac{x^6}{100^6} + \dfrac{6x^5(100-x)}{100^6} + \dfrac{15x^4(100-x)^2}{100^6}$

13 0.6836

14 a 0.1156 **b** 0.7986 **c** 0.3170

15 0.6791

16 a 0.1123 **b** 0.5561 **c** 0.000 01 **d** 0.000 01

17 0.544

18 a $\left(\dfrac{1}{4}\right)^6 \approx 0.000\ 24$ **b** 0.1694

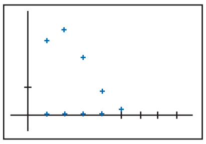
19 a 0.0138 **b** 0.2765 **c** 0.8208 **d** 0.3368

20 a $(0.8)^8 \approx 0.168$ **b** 0.001 23 **c** 0.0021

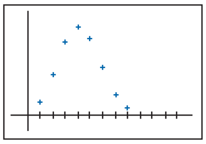
21 a $(0.15)^{10} \approx 0.000\ 000\ 006$ **b** $1-(0.85)^{10} \approx 0.8031$ **c** 0.5674

Exercise 15B

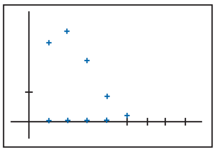
1



2



3 a b



c The distribution in part b is the reflection of the distribution in part a in the line $X = 5$.

4 a Mean = 5; Variance = 4

b Mean = 6; Variance = 2.4

c Mean = $\dfrac{500}{3}$; Variance = $\dfrac{1000}{9}$

d Mean = 8; Variance = 6.4

5 a 1 **b** 0.2632

6 37.5

7 Mean = 120; sd = $4\sqrt{3} \approx 6.93$

8 $n = 48,\ p = \dfrac{1}{4},\ \Pr(X = 7) = 0.0339$

9 $n = 100,\ p = \dfrac{3}{10},\ \Pr(X = 20) = 0.0076$

Exercise 15C

1 a i $(0.8)^5 \approx 0.3277$ **ii** 0.6723

b 14 **c** 22

2 a i 0.1937 **ii** $1-(0.9)^{10} \approx 0.6513$

b 12

3 7 **4** 7 **5** 10 **6** 42 **7** 86

Chapter 15 review

Technology-free questions

1 a $\dfrac{16}{81}$ **b** $\dfrac{32}{81}$ **c** $\dfrac{16}{27}$ **d** $\dfrac{65}{81}$

2 $\dfrac{54}{125}$ **3** $1-(0.9)^5$

4 a 2 **b** $\dfrac{3\sqrt{5}}{5}$

5 a $(1-p)^4$ **b** $4p(1-p)^3$ **c** $1-(1-p)^4$

d p^4 **e** $1-(1-p)^4-4p(1-p)^3$

6 120 **7** $\dfrac{5p(1-p)^4}{1-(1-p)^5}$ **8** $\dfrac{5}{16}$ **9** $\dfrac{32}{625}$

Multiple-choice questions

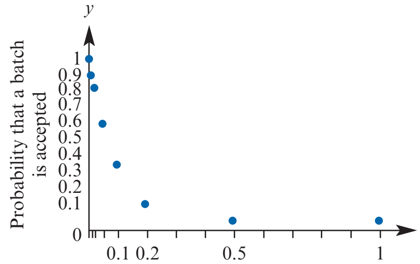
1 D **2** A **3** E **4** B **5** A
6 B **7** C **8** C **9** E **10** B

Extended-response questions

1 a 0.0173 **b** 0.2131

2

p	Probability that a batch is accepted
0	1
0.01	0.9044
0.02	0.8171
0.05	0.5987
0.1	0.3487
0.2	0.1074
0.5	0.00098
1	0



3 a 0.0582 = 5.82%

b $\mu = 0.4,\ \sigma = 0.6197,\ \mu \pm 2\sigma = 0.4 \pm 1.2394$

c Yes

4 0.0327

5 a i 0.0819 **ii** 0.9011

b i $P = 15p^2(1-p)^4$

ii $\dfrac{dP}{dp} = 30p(1-p)^3(1-3p)$

6 a 2 **b** $n = 6,\ p = \dfrac{1}{3}$

c

x	0	1	2	3	4	5	6
Freq	17.56	52.68	65.84	43.90	16.46	3.29	0.27

7 a 0.9139 **b** 0.041 45 **c** 10.702

8 a 0.0735 **b** 0.5015 **c** 27

9 $\dfrac{1}{3} \le q \le 1$

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Chapter 16

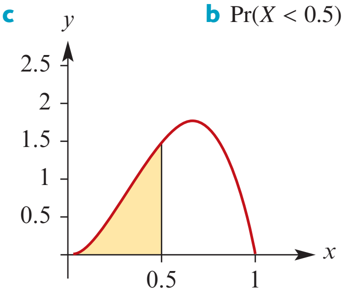
Exercise 16A

1 a 0.16 **b** 0.24

2 a 0.18 **b** 0.5

4 $k = -\frac{11}{6}$

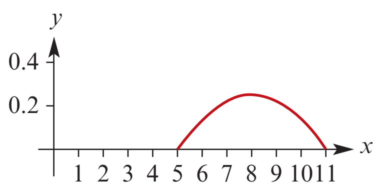
5 a c



b $\Pr(X < 0.5) = \frac{5}{16}$

6 a $k = 1$ **b** 0.865

7 a

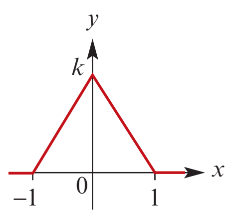


b 0.259 **c** 0.244 **d** 0.28

8 b i $0.024 = 2.4\%$ **ii** $0.155 = 15.5\%$

9 a $k = 0.005$ **b** 0.007

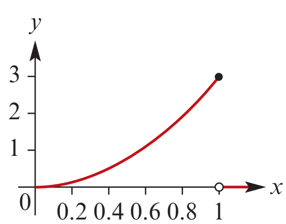
10 a



b $k = 1$

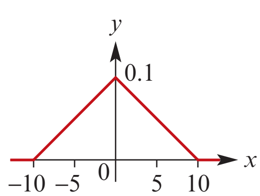
c $\frac{3}{4}$

11 a



b 0.406

12 a



b 0.190

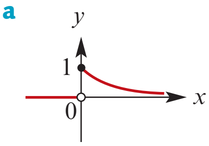
13 a $k = 1000$ **b** 0.5

14 a $\frac{2}{3}$ **b** $\frac{17}{30}$

15 a 0.202 **b** 0.449

16 a 0.45 **b** 0.711

17 a



b i $1 - e^{-\frac{1}{2}}$ **ii** e^{-1} **iii** $e^{-\frac{1}{2}}$

Exercise 16B

1 a $\frac{2}{3}$ **b** $\frac{1}{3}$ **c** $\frac{1}{2}$ **d** Does not exist

2 a 1 **b** 2.097 **c** 1.132 **d** 0.4444

3 a 0.567 **b** 0.458

4 0

5 $A = \frac{2}{9}$, $B = 3$

6 a 2 **b** 1.858

7 a 0.632 **b** 0.233 **c** 0.693

8 a 1 **b** 0.5

9 0.1294

10 2.773 minutes

11 a 1 **b** 1

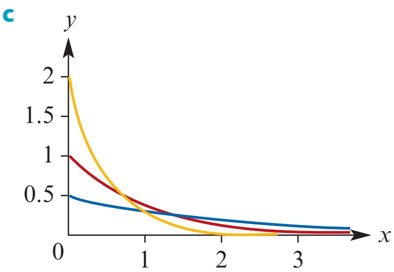
12 a 0.714 **b** 0.736

13 12

14 a 0.4 **b** $\frac{\sqrt{19} - 1}{6}$

15 a $ke^{-kx} - k^2xe^{-kx}$, $\frac{-(kx + 1)}{k}e^{-kx}$

c



d $y = e^{-x}$ is dilated by factor $\frac{1}{\lambda}$ from the x-axis and by factor λ from the y-axis

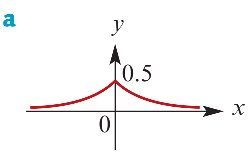
Exercise 16C

1 $\text{Var}(X) = \frac{1}{18}$, $\text{sd}(X) = \frac{\sqrt{2}}{6}$

2 (384, 416)

3 a 0.630 **b** 0.909 **c** 0.279

4 a



b 1.386

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5 a $\frac{1}{\ln 9}$ **b** $E(X) = 3.641$, $Var(X) = 4.948$

6 a 0.366 **b** $E(X) = 0.333$, $Var(X) = 0.056$

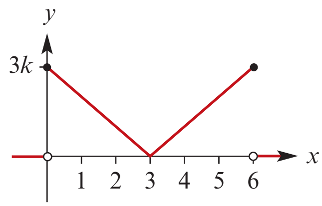
7 0.641

8 a 0.732 **b** $E(X) = \frac{4}{3}$, $Var(X) = \frac{2}{9}$

9 a 0.0004 **b** $\frac{16}{3}$ **c** 2.21

10 a $\frac{3}{4a^3}$ **b** $2\sqrt{5}$

11 a



b $k = \frac{1}{9}$ **d** 4.5

Exercise 16D

1 1300

2 $\mu = 1010$ g, $\sigma = 50$ g

3 a 0.708, 0.048 **b** \$98.94, \$0.33

4 a 0, 5.4 **b** 3, 0.6 **c** 1, 5.4

d $g(x) = \begin{cases} \frac{x^2}{18} & \text{if } -3 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$

e $h(x) = \begin{cases} \frac{(x-1)^2}{18} & \text{if } -2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$

5 a i $\frac{31}{40}$ **ii** $\frac{91}{20}$

b i $\frac{17}{12}, \frac{71}{144}$ **ii** $\frac{71}{36}$

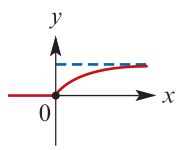
Exercise 16E

1 a $F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{5}x & \text{if } 0 < x < 5 \\ 1 & \text{if } x \ge 5 \end{cases}$ **b** $\frac{3}{5}$

2 a $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{4}x & \text{if } 0 \le x < 1 \\ \frac{1}{20}(x^4 + 4) & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$

b $\sqrt[4]{6}$

3 a



b e^{-4} **c** 0.0182

4 a $k = \frac{1}{36}$ **b** $\frac{1}{48}$

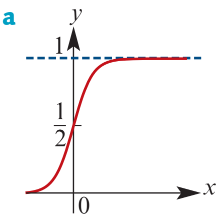
5 a $\frac{2}{3}$ **b** 20 **c** $\frac{80}{3}$

6 $f(x) = \begin{cases} 12x^2 - 12x^3 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

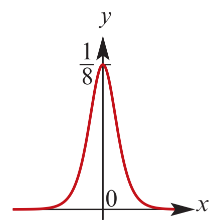
7 $f(x) = \begin{cases} 5(1-x)^4 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

8 $f(x) = \begin{cases} 0.5e^{x} & \text{if } x \le 0 \\ 0.5e^{-x} & \text{if } x > 0 \end{cases}$

9 a



b $f(x) = \frac{0.5e^{-0.5x}}{(1 + e^{-0.5x})^2}$



c 4.394°C

Chapter 16 review

Technology-free questions

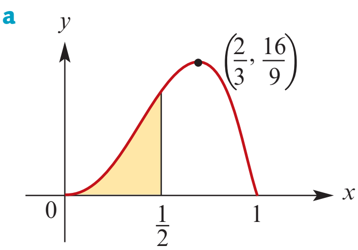
1 a 2 **b** 0.21 **c** 0.44

2 $a = \frac{1}{3}$, $b = 2$

3 $\frac{\pi}{2}$

4 a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{3}$

5 a



b $\Pr(X < 0.5) = \frac{5}{16}$

c $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^3(4 - 3x) & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

6 a $k = 12$ **b** $\Pr(X < \frac{2}{3}) = \frac{16}{27}$

c $\Pr(X < \frac{1}{3} \mid X < \frac{2}{3}) = \frac{3}{16}$

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7 a 0.008 **b** $\frac{8}{27}$

8 $\frac{2}{3}$

9 a $\frac{7}{3}$ **b** $a = 1$

10 a $c = \frac{3}{4}$ **b** 0

c $F(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{1}{4}(2 + 3x - x^3) & \text{if } -1 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

11 b $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - (1 - x)^n & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

12 a $F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \ln x & \text{if } 1 \le x \le e \\ 1 & \text{if } x > e \end{cases}$

b Median $= e^{\frac{1}{2}}$, IQR $= e^{\frac{3}{4}} - e^{\frac{1}{4}}$

13 $(320, 340)$

14 $(246, 254)$

Multiple-choice questions

1 B **2** D **3** D **4** A **5** E **6** B
7 C **8** D **9** A **10** A **11** B **12** C

Extended-response questions

1 a $a = \frac{-2}{81}$ **b** 700 hours **c** 736.4 hours

2 a $\frac{1}{4}$ **b** $\frac{5 - \sqrt{5}}{5}$ **c** $\frac{1}{3}, \frac{8}{15}$

3 a $F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2}\sin\left(\frac{\pi(x-6)}{10}\right) + \frac{1}{2} & 1 \le x \le 11 \\ 1 & x > 11 \end{cases}$

b Median = 6, IQR $= \frac{10}{3}$

c $E(X) = 6$, $Var(X) = 4.736$

4 a $F(y) = \begin{cases} 0 & \text{if } y < 1 \\ \frac{1}{25}(y - 1)^2 & \text{if } 1 \le y \le 6 \\ 1 & \text{if } y > 6 \end{cases}$

b $\frac{7}{25}$ **c** \$22.13

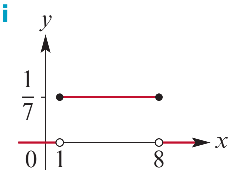
5 $c = \frac{8}{3}$ or $c = 4$

6 a $k = 25$ **b** $\frac{2}{3}$ **d** $\frac{2}{3}$

7 a $k = \frac{1}{4}$ **b** $\mu = 2$, $Var(X) = \frac{2}{3}$ **c** $\frac{3}{4}$

d $\frac{4\sqrt{5}}{5} \approx 1.8$

8 a i



ii $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{7}(x - 1) & \text{if } 1 \le x \le 8 \\ 1 & \text{if } x > 8 \end{cases}$

iii $E(X) = \frac{9}{2}$ **iv** $Var(X) = \frac{49}{12}$

b i $F(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x - a}{b - a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$

ii $E(X) = \frac{a + b}{2}$ **iii** $Var(X) = \frac{(b - a)^2}{12}$

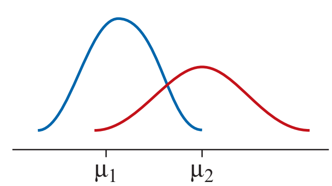
c i $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{15}x & \text{if } 0 \le x \le 15 \\ 1 & \text{if } x > 15 \end{cases}$

ii 7.5 minutes **iii** 18.75

Chapter 17

Exercise 17A

1



2 C

3 a 1

b i $E(X) = \frac{1}{3\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2} dx$ **ii** 2

c i $E(X^2) = \frac{1}{3\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2} dx$

ii 13 **iii** 3

4 a 1

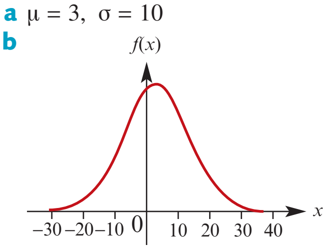
b i $E(X) = \frac{1}{5\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2} dx$ **ii** -4

c i $E(X^2) = \frac{1}{5\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2} dx$

ii 41 **iii** 5

5 a $\mu = 3$, $\sigma = 10$

b



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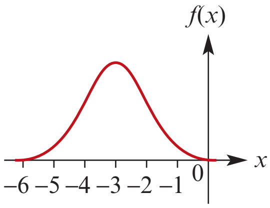
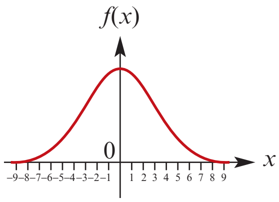
6 a $\mu = -3,\ \sigma = 1$

b



7 a $\mu = 0,\ \sigma = 3$

b



8 a Dilation of factor 2 from the y-axis and dilation of factor $\frac{1}{2}$ from the x-axis, then translation 3 units to the right

b Dilation of factor $\frac{1}{2}$ from the y-axis and dilation of factor 2 from the x-axis, then translation 3 units to the right

c Dilation of factor 2 from the y-axis and dilation of factor $\frac{1}{2}$ from the x-axis, then translation 3 units to the left

9 a Translation 3 units to the left, then dilation of factor $\frac{1}{2}$ from the y-axis and dilation of factor 2 from the x-axis

b Translation 3 units to the left, then dilation of factor 2 from the y-axis and dilation of factor $\frac{1}{2}$ from the x-axis

c Translation 3 units to the right, then dilation of factor $\frac{1}{2}$ from the y-axis and dilation of factor 2 from the x-axis

Exercise 17B

1 a 16% **b** 16% **c** 2.5% **d** 2.5%

2 a $\mu = 135,\ \sigma = 5$ **b** $\mu = 10,\ \sigma = \frac{4}{3}$

3 a 68% **b** 16% **c** 0.15%

4 21.1, 33.5

5 one, 95, 99.7, three

6 2.5%

7 a 16% **b** 16%

8 a 68% **b** 16% **c** 2.5%

9 a 95% **b** 16% **c** 50% **d** 99.7%

10 a 0 **b** $-\frac{5}{4}$ **c** 1.5

11 a −1.4 **b** 1.1 **c** 3.5

12 Michael 1.4, Cheryl 1.5; Cheryl

13 Biology 1.73, History 0.90; Biology

14 a

Student	French	English	Maths
Mary	1	0.875	0
Steve	−0.5	−1	1.25
Sue	0	0.7	−0.2

b i Mary **ii** Mary **iii** Steve

c Mary

Exercise 17C

1 a 0.9772 **b** 0.9938 **c** 0.9938 **d** 0.9943
e 0.0228 **f** 0.0668 **g** 0.3669 **h** 0.1562

2 a 0.9772 **b** 0.6915 **c** 0.9938 **d** 0.9003
e 0.0228 **f** 0.0099 **g** 0.0359 **h** 0.1711

3 a 0.6827 **b** 0.9545 **c** 0.9973

4 a 0.0214 **b** 0.9270 **c** 0.0441 **d** 0.1311

5 $c = 1.2816$ **6** $c = 0.6745$

7 $c = 1.96$ **8** $c = -1.6449$

9 $c = -0.8416$ **10** $c = -1.2816$

11 $c = -1.9600$

12 a 0.9522 **b** 0.7977 **c** 0.0478 **d** 0.1545

13 a 0.9452 **b** 0.2119 **c** 0.9452 **d** 0.1571

14 a $c = 9.2897$ **b** $k = 8.5631$

15 a $c = 10$ **b** $k = 15.88$

16 a $a = 0.994$ **b** $b = 1.96$ **c** $c = 2.968$

17 a 0.7161 **b** 0.0966 **c** 0.5204
d $c = 33.5143$ **e** $k = 13.02913$
f $c_1 = 8.28,\ c_2 = 35.72$

18 a 0.9772 **b** 0.9772 **c** $c = 10.822$
d $k = 9.5792$ **e** $c_1 = 9.02,\ c_2 = 10.98$

Exercise 17D

1 a i 0.2525 **ii** 0.0478 **iii** 0.0901
b 124.7

2 a i 0.7340 **ii** 0.8944 **iii** 0.5530
b 170.25 cm
c 153.267 cm

3 a i 0.0766 **ii** 0.9998 **iii** 0.153
b 57.3

4 a 10.56% **b** 78.51%

5 Mean = 1.55 kg; sd = 0.194 kg

6 a 36.9% **b** $c = 69$

7 a 0.0228 **b** 0.0005 **c** 0.0206

8 a 3.04 **b** 350.27

9 1003 mL

10 a Small 0.1587; Med 0.7745; Large 0.0668
b \$348.92

11 a i 0.1169 **ii** 17.7
b 0.0284

12 a A: 0.0228; B: 0.1587 **b** $c = \frac{34}{3}$

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Exercise 17E

1 0.9632

2 0.2442

3 **a** 0.0478 **b** 0.2521

4 **a** 0.7834 **b** 0.3208

5 0.2819

6 **a** 0.0416 **b** 0.0038

Chapter 17 review

Technology-free questions

1 **a** $1-p$ **b** $1-p$ **c** $2p-1$

2 **a** $a=-1$ **b** $b=1$ **c** 0.5

3 $(x, y) \to \left(\dfrac{x-8}{3}, 3y\right)$

4 **a** $\dfrac{q}{p}$ **b** $1-q$ **c** $\dfrac{1-p}{1-q}$

5 **a** $\Pr(Z < \frac{1}{2})$ **b** $\Pr(Z < -\frac{1}{2})$ **c** $\Pr(Z > \frac{1}{2})$
d $\Pr(-\frac{1}{2} < Z < \frac{1}{2})$ **e** $\Pr(-\frac{1}{2} < Z < 1)$

6 **a** 0.84 **b** 0.5 **c** 0.16 **d** 0.68

7 **a** 0.16 **b** 0.34 **c** 0.32 **d** 0.02

8 **a** 0.69 **b** 0.19 **c** 0.15 **d** 0.68

9 Best C, worst B

10 **a** 0.5 **b** $b=-1.5$

Multiple-choice questions

1 A **2** C **3** B **4** B **5** E **6** E
7 C **8** D **9** A **10** D **11** D **12** C

Extended-response questions

1

Category	Range
High	> 63
Moderate	$[56, 62]$
Average	$[45, 55]$
Little	$[37, 44]$
Low	< 37

2 $k = 3.92$

3 **a** **i** 0.1587 **ii** 0.9747 **iii** 0.0164
b $c = 53\,592$
c 3.7×10^{-11}

4 **a** 3.17×10^{-5} **b** False
c $c_1 = 13.53$, $c_2 = 16.47$

5 0.0803

6 0.92%

7 **a** $\mu = 60.027$, $\sigma = 0.2$ **b** 10%

8 **a** 0.9044 **b** 5.88 **c** 9.044
d 0.2650 **e** \$17.61, 54.0281

9 **a** $\mu = 0$, $\sigma = 2.658$ **b** 0.882

10 **a** 0.1056 **b** 0.0803 **c** 0.5944

Chapter 18

Exercise 18A

1 No; sample will be biased towards the type of movie being shown.

2 **a** No; biased towards shoppers.
b Randomly select a sample from telephone lists or an electoral roll.

3 No; only interested people will call, and they may call more than once.

4 **a** No; biased towards older, friendly or sick guinea pigs which may be easier to catch.
b Number guinea pigs and then generate random numbers to select a sample.

5 No; a student from a large school has less chance of being selected than a student from a small school.

7 **a** Unemployed will be under represented.
b Unemployed or employed may be under represented, depending on time of day.
c Unemployed will be over represented.
Use random sampling based on the whole population (e.g. electoral roll).

8 **a** Divide platform into a grid of 1 m^2 squares. Select squares using a random number generator to give two digits, one a vertical reference and one a horizontal reference.
b Yes, if crabs are fairly evenly distributed; otherwise, five squares may not be enough.

9 No; a parent's chance of selection depends on how many children they have at the school.

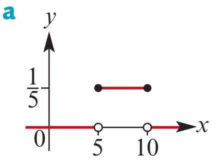
10 Not a random sample; only interested people will call, and they may call more than once.

11 People who go out in the evenings will not be included in the sample.

12 **a** All students at this school
b $p = 0.35$ **c** $\hat{p} = 0.42$

13 **a** 0.22 **b** $\hat{p}$

15 **a**



Exercise 18B

1 **a** 0.5 **b** $0, \frac{1}{3}, \frac{2}{3}, 1$

c

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

d $\frac{1}{2}$

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2 a $\frac{3}{5}$ **b** $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

c

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$\Pr(\hat{P} = \hat{p})$	0.0036	0.0542	0.2384
$\hat{p}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	0.3973	0.2554	0.0511

d 0.3065 **e** 0.6924

3 a 0.5 **b** $0, \frac{1}{3}, \frac{2}{3}, 1$

c

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	0.1	0.4	0.4	0.1

d 0.9

4 a 0.4 **b** $0, \frac{1}{3}, \frac{2}{3}, 1$

c

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

d $\frac{1}{3}$

e $\frac{3}{5}$

5 a 0.5 **b** $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

c

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

d $\frac{5}{16}$

6 a $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

b

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{32}$	$\frac{1}{32}$

c $\frac{3}{16}$ **d** $\frac{25}{26}$

7 a $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

b

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

c $\frac{17}{369}$

8 $\mu = 0.5,\ \sigma = 0.25$ **9** $\mu = 0.5,\ \sigma = 0.224$
10 $\mu = 0.2,\ \sigma = 0.2$ **11** $\mu = 0.3,\ \sigma = 0.084$
12 $\mu = 0.4,\ \sigma = 0.049$ **13** $\mu = 0.2,\ \sigma = 0.04$
14 a 0.1844 **b** 0.7600 **c** 0.9683

Exercise 18C

1 0.2858 **2** 0.8568 **3** 0.1587
4 0.0092 **5** 0.0614
6 a 1 **b** 0.5000 **c** 0.0412
7 0.9545
8 a 0.9650 **b** 0.9647
9 a 0.575 **b** 0.0139
10 a 0.848 **b** 0.2817 **c** Yes

Exercise 18D

1 a 0.08 **b** (0.0268, 0.1332)
2 a 0.192 **b** (0.1432, 0.2408)
3 a 0.2 **b** (0.1216, 0.2784)
4 (0.2839, 0.3761)
5 a (0.4761, 0.5739) **b** (0.5095, 0.5405)
c The second interval is narrower because the sample size is larger
6 1537
7 a 897 **b** 2017
c Reducing margin of error by 1% requires the sample size to be more than doubled
8 90%: (0.5194, 0.6801), 95%: (0.5034, 0.6940), 99%: (0.4738, 0.7262); Interval width increases as confidence level increases
9 90%: (0.5111, 0.5629), 95%: (0.5061, 0.5679), 99%: (0.4964, 0.5776); Interval width increases as confidence level increases
10 a (0.789, 0.907) **b** (0.830, 0.867)
c The second interval is narrower because the sample size is larger
11 173
12 a 2017 **b** 2401
c i $E = 1.8\%$ **ii** $E = 2.2\%$
d 2401, as this ensures that E is 2% or less, whoever is correct
13 d 8 **e** 40

Chapter 18 review

Technology-free questions

1 a All employees of the company
b $p = 0.35$ **c** $\hat{p} = 0.40$
2 a No; only people already interested in yoga
b Use electoral roll
3 a $\frac{k}{100}$ **b** $\frac{k}{100} \pm \frac{1.96\sqrt{k(100-k)}}{1000}$
4 a $\hat{p} = 0.9$ **b** $E = \frac{0.588}{\sqrt{n}}$
c Margin of error would decrease by a factor of $\sqrt{2}$
5 a 38 **b** $(0.95)^{40}$
6 a 45 **b** $5.9(0.9)^{49}$
7 a 0.60 **b** 0.10 **c** Increase sample size

Multiple-choice questions

1 B **2** C **3** D **4** E **5** C
6 D **7** B **8** E **9** C **10** E
11 A **12** B **13** C **14** D

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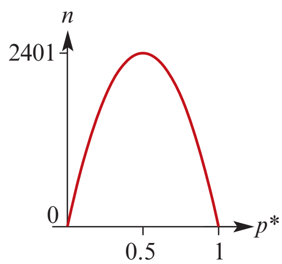
Extended-response questions

1 a



b 0.5 **c** 2401

2 a 0.1537 **b** 0.8257

3 a (0.4730, 0.6670)
b i 0.7738 **ii** 0.0000003
iii 0.2262 **iv** 4.75
c (0.5795, 0.6645)

4 a $p = \frac{500}{N}$ **b** $\hat{p} = 0.15$
c $N = 3333.33 \approx 3333$
e (2703, 4348)

Chapter 19

Technology-free questions

1 a $6x + 4$ **b** $\frac{1}{16}e^{-\frac{x}{4}}$
c $(25x - 10)e^{-5x}$ **d** $-\frac{1}{25}\sin\left(\frac{x}{5}\right)$
e $-\frac{\pi^2}{4}\cos\left(\frac{\pi x}{2}\right)$ **f** $4e^{2x} - 4e^{-2x}$

2 $v = 6t^2 - 24, \ a = 12t$

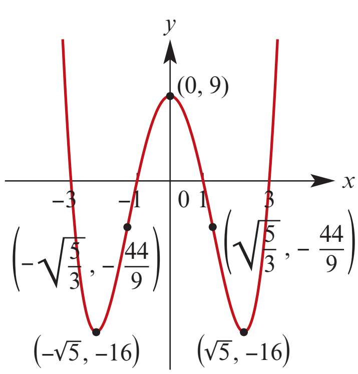
3 a $f'(x) = (4x^2 - 4x + 1)e^{2x^2 - 4x}$

4 $x > 1$

5 $(0, 0), \left(\frac{\pi}{2}, 2\pi\right), (\pi, 4\pi), \left(\frac{3\pi}{2}, 6\pi\right), (2\pi, 8\pi)$

6 a $x = 1, \ x = -1$ or $x = 3$
b Stationary point of inflection at $x = 1$; Local max at $x = -1$; Local min at $x = 3$
c $x = 1, \ x = 1 \pm \sqrt{2}$

7



8 $\frac{113}{140}$

9 $\frac{1}{3}$

10 $\frac{\sqrt{2} - \sqrt{6}}{4}$

11 a $2\sqrt{7}$ cm **b** $6\sqrt{3}$ cm^2

12 a $2\sqrt{19}$ cm **b** $6\sqrt{3}$ cm^2

13 a $\frac{1}{5}$ **b** $\frac{4}{9}$ **c** 1.7 **d** 2.01

14 $a = 0.34, \ b = 0.06$

15 a $(1 - p)^3$ **b** $p = \frac{1}{3}$

16 $n = 15, \ p = 0.4$

17 a $\frac{1}{216}$ **b** $\frac{2}{27}$

18 a p^8 **b** $(1 - p)^8$ **c** $1 - (1 - p)^8$

19 $\Pr(X = 1) = \frac{2^8}{3^7}$

20 a $\binom{20}{k}\left(\frac{1}{5}\right)^k\left(\frac{4}{5}\right)^{20-k}, \ \binom{20}{k+1}\left(\frac{1}{5}\right)^{k+1}\left(\frac{4}{5}\right)^{19-k}$
c $k = 4$

21 a $\frac{1}{36}$ **b** $\frac{20}{27}$ **c** 3 **d** 3 **e** $\frac{14}{27}$ **f** $\frac{13}{20}$

22 b $E(X) = \frac{3a}{2}$

23 a $F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x(2 - x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$ **b** $a = \frac{1}{2}$

24 $\Pr(X > 44) = \frac{1 - q}{2}$

25 $\Pr(40 < X < 60) = 2q - 1$

26 a 0.84 **b** 0.34

27 a 0.5 **b** $a = 2$

28 a $\frac{a + b}{2}$ **b** $\frac{b - a}{2}$

29 $n = 469$

30 a $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$
b i $\frac{1}{126}$ **ii** $\frac{125}{126}$ **iii** $\frac{10}{63}$

32 a 18 **b** $(0.9)^{20}$

33 $\frac{1}{90}$

Multiple-choice questions

1 C	**2** B	**3** B	**4** D	**5** D
6 B	**7** C	**8** C	**9** B	**10** A
11 B	**12** E	**13** E	**14** B	**15** E
16 D	**17** D	**18** A	**19** B	**20** C
21 D	**22** C	**23** B	**24** E	**25** E
26 A	**27** A	**28** D	**29** D	**30** C
31 E	**32** B	**33** C	**34** E	**35** B
36 E	**37** A	**38** B	**39** B	**40** B
41 B	**42** C	**43** B		

ISBN 978-1-108-45164-2

Extended-response questions

1 b $\frac{dT}{dx} = \frac{x}{\sqrt{x^2 + 900}} - \frac{3}{5}$

c i $x = 22.5$ **ii** 71 seconds

d 63 seconds

2 a $f'(x) = 1 + \cos x,\ \ f''(x) = -\sin x$

c $-4\pi, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi$

d $(-3\pi, -3\pi), (-\pi, -\pi), (\pi, \pi), (3\pi, 3\pi)$

e $-\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$

f $\left(-\frac{4\pi}{3}, \frac{\sqrt{3}}{2} - \frac{2\pi}{3}\right), \left(-\frac{2\pi}{3}, -\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right),$
$\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2} + \frac{\pi}{3}\right), \left(\frac{4\pi}{3}, -\frac{\sqrt{3}}{2} + \frac{2\pi}{3}\right)$

3 a i $\angle BCA = 138.19°, \angle ABC = 11.81°$

ii $\angle BC'A = 41.81°, \angle ABC' = 108.19°$

b i 24.56 **ii** 114.00 **iii** 89.44

c i 1788.85 **ii** 3027.87 **iii** 1239.01

4 $4 - \sqrt{\frac{12}{7}} \approx 2.7$ m

5 155 m

6 a 4.145 km

b i 078° **ii** 4.749 km

c i 8.803 km **ii** 101.229°

7 a 16.00 m **b** 29.04 m **c** 17°

8 b $\frac{9}{16}$

9 \$0.77

10 a $c = \frac{20}{49}$ **b** $E(X) = \frac{120}{49}$

c $\text{Var}(X) = \frac{6180}{2401}$

11 a 0.6915 **b** 0.1254

12 a i 0.1587 **ii** 511.63

b 0.1809

13 a i $\frac{1}{8}$

ii A: 0.6915, B: 0.5625

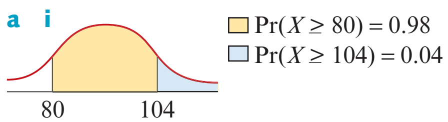
iii $E(X) = 10,\ E(Y) = 10.67$; Machine A

b $\frac{3}{4}$

14 a i $\frac{1}{2500}$ **ii** $\frac{16}{3}$ **iii** 0.8281 **iv** 0.7677

b 0.9971

15 a i



ii $\mu = 92.956,\ \sigma = 6.3084$

b i 16.73% of sensors **ii** 81°C

16 a i 0.2 **ii** 0.7 **iii** 0.125 **iv** $\frac{3}{160}$

b i 0.360 15 **ii** $\frac{128}{625}$

c 0.163 08

17 a i $\mu = 4.25$ **ii** $\sigma = 0.9421$ **iii** 0.94 **iv** 0.9

b i Binomial **ii** 18 **iii** 1.342 **iv** 0.3917

18 a (0.0814, 0.1186) **b** (0.0792, 0.1208)

c Larger sample of females

d 900 of each sex **e** 0.078 or 0.922

Degree-of-difficulty classified questions

Simple familiar questions

1 a 35° **b** 255°

2 a 12.038 cm **b** 18.685°

c 121.315° **d** 30.854 cm²

3 a $\cos\left(\frac{x}{2}\right) - \frac{x}{4}\sin\left(\frac{x}{2}\right)$ **b** $2\ln(2x) + 3$

c $8e^{-2x}\sin(2x)$

4 a (0, 0), (2, −16) **b** $\left(e^{-\frac{5}{6}}, \frac{5}{6}e^{-\frac{5}{2}}\right)$

c $\left(\frac{\pi}{24}, \frac{\pi^2}{144} + \frac{1}{2}\right), \left(\frac{5\pi}{24}, \frac{25\pi^2}{144} + \frac{1}{2}\right),$
$\left(\frac{13\pi}{24}, \frac{169\pi^2}{144} + \frac{1}{2}\right), \left(\frac{17\pi}{24}, \frac{289\pi^2}{144} + \frac{1}{2}\right)$

5 0.2965

6 a $\Pr(X = x) = \binom{4}{x}(0.55)^x(0.45)^{4-x}$,
$x = 0, 1, 2, 3, 4$

b 0.2005

7 a i 0.0000026 **ii** 0.1678 **iii** 0.0563

b 1.6

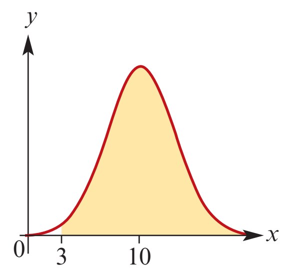
9 $E(X) = 0.6$

10 $5\ln 2 \approx 3.47$ minutes

11 $f(x) = \begin{cases} \frac{1}{4} & \text{if } 0 \le x < 1 \\ \frac{1}{5}x^3 & \text{if } 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$

12 a $\mu = 10,\ \sigma = 3$

b c



13 a 0.0228 **b** 0.3829

14 a 0.0228 **b** 0.2525 **c** 0.8176

15 a Maths 0.88, English 0.67, Chemistry 1.24, Psychology 0.92

b Chemistry, Psychology, Maths, English

16 a (0.712, 0.788) **b** (0.700, 0.800)

17 a 5649 **b** 8068

Complex familiar questions

1 a i $(-\infty, -2), (2, \infty)$ **ii** $(-2, 2)$

iii $(0, \infty)$ **iv** $(-\infty, 0)$

b i $(-\infty, 0), (8, \infty)$ **ii** $(0, 8)$

iii $(4, \infty)$ **iv** $(-\infty, 4)$

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c **i** $(\pi, 2\pi)$ **ii** $(0, \pi)$

iii $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ **iv** $\left(0, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, 2\pi\right)$

d **i** $\left(0, \frac{\pi}{2}\right)$ **ii** $\left(\frac{\pi}{2}, \pi\right)$

iii $\left(0, \frac{\pi}{4}\right), \left(\frac{3\pi}{4}, \pi\right)$ **iv** $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

e **i** $(1, \infty)$ **ii** $(-\infty, 1)$

iii $(-\infty, 0), \left(\frac{2}{3}, \infty\right)$ **iv** $\left(0, \frac{2}{3}\right)$

f **i** $\left(\frac{1}{2}e^{-\frac{1}{2}}, \infty\right)$ **ii** $\left(0, \frac{1}{2}e^{-\frac{1}{2}}\right)$

iii $\left(\frac{1}{2}e^{-\frac{3}{2}}, \infty\right)$ **iv** $\left(0, \frac{1}{2}e^{-\frac{3}{2}}\right)$

2 **a** $f'(x) = (x-a)^2(4x-a-3b)$,
$f''(x) = 6(x-a)(2x-a-b)$

b $(a, 0), \left(\frac{a+b}{2}, -\frac{(a-b)^4}{16}\right)$

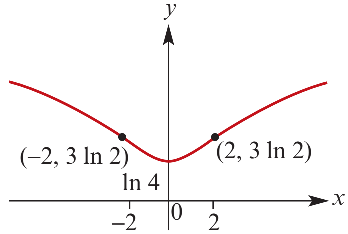
c $y = \frac{(a-b)^3}{4}\left(x - \frac{3a+b}{4}\right)$, $\left(\frac{3a+b}{4}, 0\right)$

3 **a** $(-2, 0)$ **b** $(1, 27)$ **c** $(-1, 11)$

4 **a** $f'(x) = \frac{2x}{x^2+4}$, $f''(x) = \frac{-2(x^2-4)}{(x^2+4)^2}$

b $(0, \ln 4)$ **c** $(2, 3\ln 2), (-2, 3\ln 2)$

d



5 $AC = 3$

6 **a** 2.11 km **b** 4.53 km
c 065° **d** 14.80 km

7 14.9°

8 3.76 cm, 8.24 cm

9 **a** $\Pr(X = x) = \binom{7}{x}(0.2)^x(0.8)^{7-x}$,
$x = 0, 1, 2, 3, \ldots, 7$

b **i** 0.1146 **ii** 0.3899 **iii** 0.2709

10 **a** 0.2592 **b** 10

11 **a** 0.4 **b** 0.5

12 **a** 70.5 **b** 5

13 0.284

Complex unfamiliar questions

1 **a** 72.97° **b** 63.51° **c** 56.30°

2 340 metres

3 $a = 2,\ b = -4,\ c = -2$

4 2.09 km

5 $\frac{24}{5}$ cm, 10 cm

6 **a** **i** 0.8424 **ii** 0.0143

b $p = 0.221$

c $p = 0.2$ or $p = 0.8$

7 **a** 0.3487 **b** 0.3231 **c** 40 **d** 29

8 $A = 3,\ B = 1$

9 **a** **i** $r = \frac{2\pi - \theta}{2\pi}$

ii $h = \sqrt{1 - \left(\frac{2\pi - \theta}{2\pi}\right)^2}$

b $\frac{49\sqrt{15}\pi}{1536}$

c 0.3281, 2.5271

d **i** $\theta = 1.153$ **ii** $V_{max} = 0.403\ \text{m}^3$

e $0.403\ \text{m}^3$

Chapter 20

Technology-free questions

1 **a** −0.3 **b** 0.3 **c** −0.6
d 0.6 **e** 0.6 **f** −0.3

2 **a** $y = -\cos x + x + 3$ **b** $y = 6e^{\frac{x}{3}} - ex - e$

c $y = \frac{1}{2}\ln(2x-3) + 4,\ x > \frac{3}{2}$

3 $f(g(x)) = (3x+1)^2 + 6 = 9x^2 + 6x + 7$

4 **a** $f'(x) = 21x^6(5x^3-3)^6(5x^2-1)$

b $f'(0) = 2$

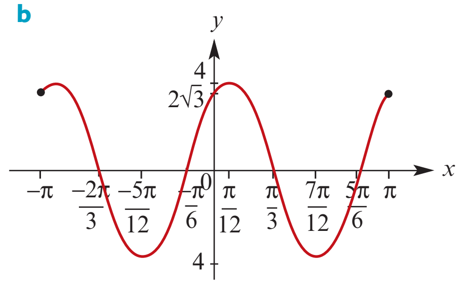
5 **a** $f'\left(\frac{\pi}{2}\right) = \frac{-2}{(\pi+1)^2}$ **b** $f'\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2} - \pi$

6 **a** $2\ln(2x) + 3$

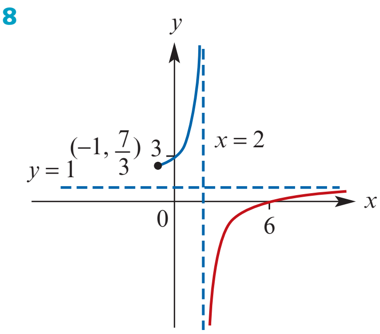
b $f''(x) = 4e^{\sin(2x)}(\cos^2(2x) - \sin(2x))$

7 **a** Amplitude = 4; Period = π

b



8



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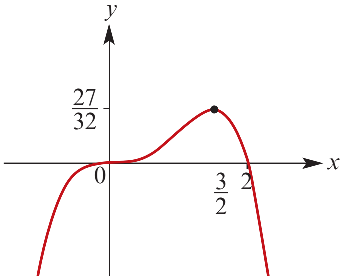
9 a 0, 2 **b** $(0, 0), \left(\frac{3}{2}, \frac{27}{32}\right)$ **c** $(0, 0), \left(1, \frac{1}{2}\right)$

d Stationary point of inflection at $(0, 0)$;
Local max at $\left(\frac{3}{2}, \frac{27}{32}\right)$

e



10 a $\frac{3}{8}$ **b** $\frac{7}{8}$ **c** $\frac{1}{2}$

11 a $a = 2,\ b = -5$ **b** $5\ln\left(\frac{25}{32}\right) + 2$

c $x = 1$ **d** $2\ln\left(\frac{27}{8}\right) - 2$

12 a $x = \ln\left(\frac{y+3}{5}\right) + 1$ **b** $x = 3 - e^{2y}$

13 $x = -\frac{2\pi}{15}$ or $x = \frac{2\pi}{15}$

14 $\frac{1}{4}(e^4 - 1)$

15 a $c = 6$

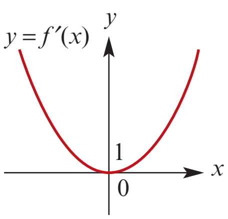
b $4a + b - 3 = 0,\ 3a + b = 0$

c $a = 3,\ b = -9$

16 a $10\sqrt{2 + \sqrt{3}} = 5(\sqrt{2} + \sqrt{6})$ cm **b** 25 cm^2

17 $10\sqrt{2}$ km, 15°

18 a



b $f'(x) = \begin{cases} -8x^3 & \text{if } x \le 0 \\ 8x^3 & \text{otherwise} \end{cases}$

19 $\left(\frac{\sqrt{6}}{3}, \frac{20}{9}\right), \left(-\frac{\sqrt{6}}{3}, \frac{20}{9}\right)$

20 $-\frac{1}{3}\ln(1 - 3x) + c$

21 a 0.5 **b** 0.68 **c** 0.32

22 a $\frac{1}{6}$ **b** $\sqrt{31}$

23 $a = \pm\frac{2}{3}$

24 a $\frac{5}{8}$

b $x^2 - 10\sqrt{3}x + 36 = 0$; There are two solutions as there are two such triangles.

25 a $A = 32a - 8a^3$

b Max value $A = \frac{128\sqrt{3}}{9}$ when $a = \frac{2\sqrt{3}}{3}$

26 $b = 3$

27 a \$0.65 **b** 0.425

28 a i $6p^2(1 - p)^2$

ii $4p^3(p - 1)$

b $p = \frac{3}{5}$

29 a $h = \frac{4000}{x^2}$ **c** $2000(2 + \sqrt{2})$

30 a No **b** Electoral roll

31 a 0.53 **b** $0.53 \pm 0.196\sqrt{0.53 \times 0.47}$

32 a 0.37 **b** $E = 1.96\sqrt{\frac{0.37 \times 0.63}{n}}$

c Increased by a factor of $\sqrt{2}$

33 a $0, \frac{1}{3}, \frac{2}{3}, 1$

b

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	0.512	0.384	0.096	0.008

c 0.488

Multiple-choice questions

1 B	**2** D	**3** D	**4** A	**5** C
6 D	**7** A	**8** A	**9** B	**10** D
11 E	**12** C	**13** B	**14** C	**15** A
16 E	**17** E	**18** D	**19** B	**20** D
21 A	**22** E	**23** D	**24** C	**25** A

Extended-response questions

1 a i $\left(\frac{1}{2}, 8\right)$ **ii** Minimum

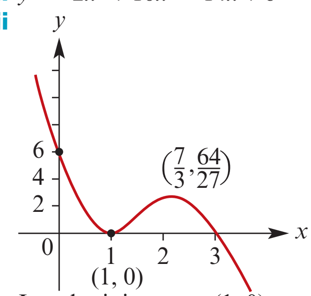
b ii $A = \frac{x}{12}(60 - 5x)$ **iii** Max area 15 cm^2

2 a $p = 1,\ q = 3,\ k = 2$

b i $m = -2$

ii $y = -2x^3 + 10x^2 - 14x + 6$

iii



Local minimum at (1, 0)
Local maximum at $\left(\frac{7}{3}, \frac{64}{27}\right)$

3 a 86°

b $AX = 418.304$ m, $BX = 538.97$ m

c 658.797 m

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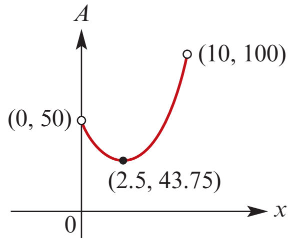
4 a i $A = x^2 - 5x + 50$ **ii** $(0, 10)$

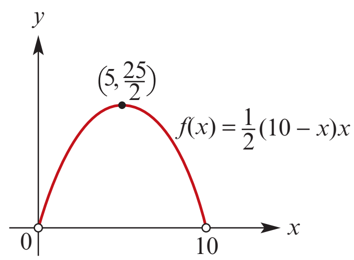
iii



iv Minimum area = 43.75 cm^2

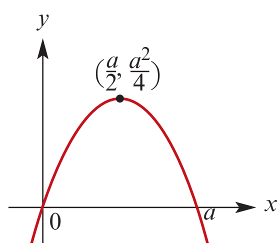
b i $f(x) = \frac{1}{2}(10 - x)x$

ii



c $AYX : OXYZ : ABY : CBYZ = 1 : 2 : 2 : 3$

5 a



b $\frac{a^3}{6}$ square units

c i $y = \frac{2a^2}{9}, y = \frac{2a^2}{9}$ **ii** $\frac{a^3}{162}$ square units

6 a i $f'(t) = -100e^{-\frac{t}{10}}(t^2 - 30t + 144)$

ii $f''(t) = 10e^{-\frac{t}{10}}(t^2 - 50t + 444)$

b i $6 < t < 24$

ii $11.546 < t \leq 35$
(exact: $25 - \sqrt{181} < t \leq 35$)

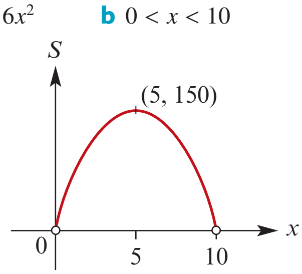
iii $11.546 < t < 24$

7 a i 0.995 **ii** $x = \pm 0.2$

b i $h(x) = \frac{1}{2}(x - \pi)^2 - 1$ **ii** −0.989 99

8 a $S = 60x - 6x^2$ **b** $0 < x < 10$

c $x = 5$ **d**

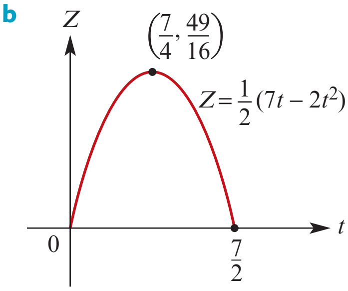


9 a i $OP = \frac{1}{\sin\theta}$ **ii** $BQ = \frac{1 - \cos\theta}{\sin\theta}$

d Min value $S = \frac{\sqrt{3}}{2}$ when $AP = \frac{2\sqrt{3} - 3}{3}$

10 a $Z = \frac{1}{2}(7t - 2t^2)$

b



c Max value $Z = \frac{49}{16}$ when $t = \frac{7}{4}$

11 a i $\frac{3}{8}$ **ii** $\frac{4}{15}$

b i $\frac{27}{125}$ **ii** $\frac{8}{125}$ **iii** $\frac{38}{125}$

12 a $k = \frac{b}{a^2}$

b i $y = \frac{b}{2a}x + \frac{b}{2}$

ii $\left(\frac{-a}{2}, \frac{b}{4}\right)$

d $S_1 : S_2 = 27 : 37$

13 a i 0.9332 **ii** 0.0668

iii 0.1151 **iv** 0.1151

b i 33.3% **ii** 866.4

iii 199.4

14 $90 - 8\sqrt{3}$ metres from A towards E

15 a i $y = -e^{-n}x + e^{-n}(n + 1)$

ii $x = n + 1$

b i $\frac{1}{e^n}\left(1 - \frac{1}{e}\right)$

ii $e : e - 2$

16 a 0.0023

b

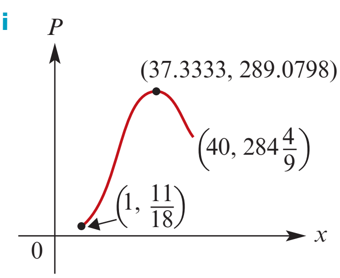
Q	$s - 1$	-1
$Pr(Q = q)$	$\frac{3}{4}$	$\frac{1}{4}$

c $E(Q) = \frac{3}{4}s - 1$, $sd(Q) = \frac{\sqrt{3}}{4}s$

17 a 0.091 21 **b** 0.2611 **c** 0.275

18 a $\frac{dP}{dx} = \frac{1}{90}(112x - 3x^2)$

b i



ii Max value of P is 289.0798 tonnes

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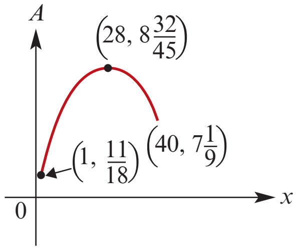
c $A = \frac{x}{90}(56 - x)$

i



ii Max value of A is $8\frac{32}{45}$ tonnes per miner, occurs when $x = 28$

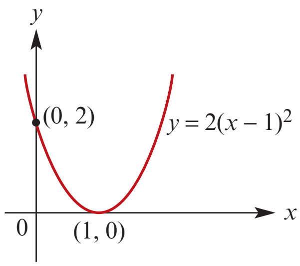
19 a $AX = BX = \dfrac{h}{\tan 12.5°}$, $CX = \dfrac{h}{\tan 9.5°}$

b $MX^2 = \left(\dfrac{h}{\tan 12.5°}\right)^2 - 600^2$

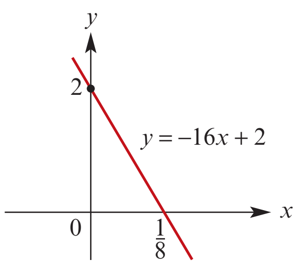
c 432.967 m **d** 1858.5 m

e $AX = BX = 1953$ m, $CX = 2587$ m

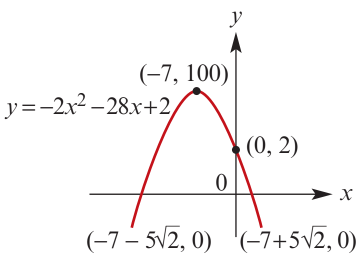
20 a i



ii



iii



b $\left(\dfrac{2 - 3k}{k + 2}, \dfrac{14k - 9k^2}{k + 2}\right)$

i $-2 < k < \frac{2}{3}$ **ii** $k = \frac{2}{3}$

iii $k < -2$ or $0 < k < \frac{14}{9}$

iv $-2 < k < 0$ or $k > \frac{14}{9}$

c $k < -2$

d i $k = 0$ or $k = \frac{14}{9}$ **ii** $0 < k < \frac{14}{9}$

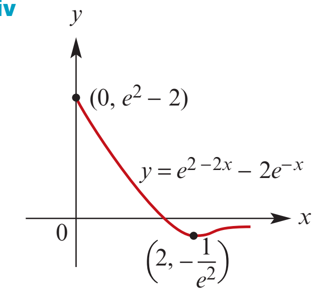
21 a $x = 2 - \ln 2$

b i $\dfrac{dy}{dx} = -2e^{2-2x} + 2e^{-x}$

ii $x = 2$

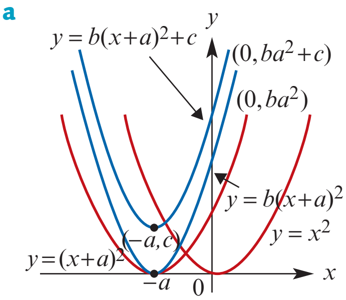
iii $\left(2, -\frac{1}{e^2}\right)$

iv



c $-\frac{1}{e^2} < k < 0$

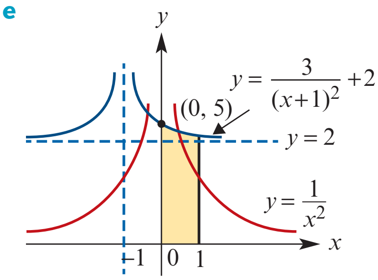
22 a



c Dilation of factor 3 from the x-axis, then translation 1 unit to the left and 2 units up

d $\frac{7}{2}$

e



23 a $\mu = 5.0290$, $\sigma = 0.0909$

b $409.27

24 a i $y = 50$

ii $y = x - 25$

b $a = -\frac{1}{15}$, $c = -\frac{25}{3}$

c i 1250 square units

ii $\frac{14\,375}{18}$ square units

iii $\frac{36\,875}{18}$ square units

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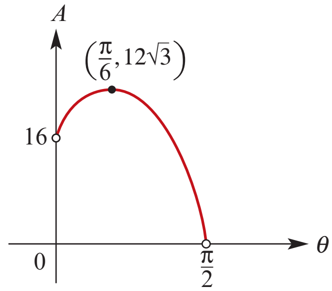
25 **c** $\theta = \frac{\pi}{6}$

d

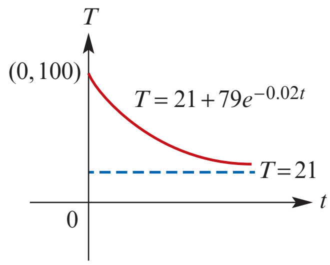


Maximum value of A is $12\sqrt{3}$

26 **a** $k = \frac{1}{10}\ln\left(\frac{79}{63}\right) \approx 0.02,\ A = 79$

b Approx 2:44 p.m.

c



d Average rate of change = −1.6°C/minute

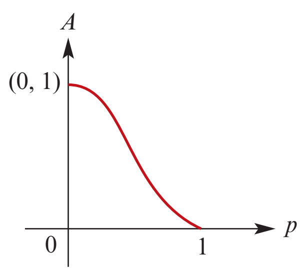
e **i** −1.5609°C/minute

ii −0.8826°C/minute

27 **a** $\frac{3}{16}$

b $b = 4$

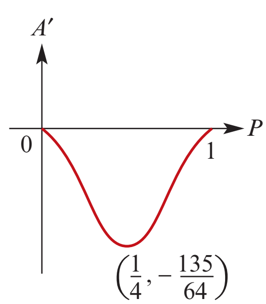
c



d **i** $p = 0.08$ **ii** $p = 0.66$

e **i** $A'(p) = -20p(1-p)^3$

ii



iii $p = \frac{1}{4}$

iv Most rapid rate of change of probability occurs when $p = \frac{1}{4}$

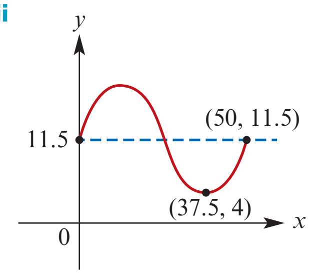
28 **a** 7

c $y = \frac{1}{10}x - \ln 10$

d **ii** 36.852

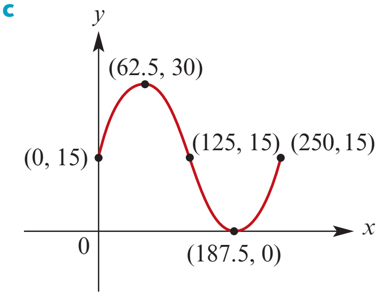
29 **a** **i** $m = 12.5,\ n = 15,\ d = 37.5,\ a = 7.5,\ b = 7.5$

ii



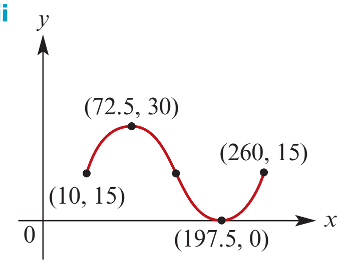
b (2.704, 10), (22.296, 10)

c



d **i** $h(x) = 15 + 15\sin\left(\frac{\pi(x-10)}{125}\right)$

ii



30 **a** 0.065 36

b **i** 0.6595 **ii** 0.198 14

c **i** 23.3% **ii** $c = 0.1075$

31 **a** **i** 0.32 **ii** 0.18 **iii** 0.5

b 0.64

c **i** 0.043 95 **ii** 0.999 **iii** $\frac{7}{128}$

32 **a** $k = 4$

b **i** $E(X) = \frac{13}{6}$ **ii** $\frac{10-\sqrt{2}}{4}$ **iii** $\frac{\sqrt{2}}{12}$

c 0.1857

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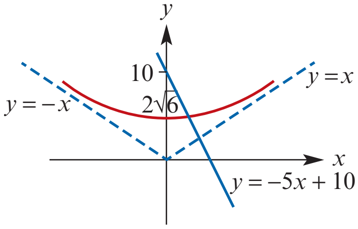
33 a $k = \frac{2}{a^2}$ **b** $\mu = \frac{a}{3},\ \sigma^2 = \frac{a^2}{18}$

c $\frac{6 - 4\sqrt{2}}{9}$ **d** $a = 1000(\sqrt{2} + 2)$

34 a $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 24}}$ **b** $(0, 2\sqrt{6})$

c Even

d



e $y = -5x + 10$

g $12\ln(\sqrt{7} - 1) - 2\sqrt{7} + \frac{35}{2}$

35 a i 0 **ii** −0.6745
iii 0.6745 **iv** 1.3490
v 0.9930 **vi** 0.7%

b i μ **ii** $\mu - 0.645\sigma$
iii $\mu + 0.645\sigma$ **iv** 1.3490σ
v 0.9930 **vi** 0.7%

36 a $\frac{527}{1000}$ **b** (0.4961, 0.5580)

c 67% **d** 99.9955%

37 a i $\frac{1}{1 - b}$ **ii** $1 + \sqrt{b - 1}$

b i 1 **ii** $1 + e^{-7}$

c i $\frac{b(b-2)}{2(b-1)}$ **ii** $9 + \sqrt{65}$

d e

38 a i $\frac{-(b+1)}{b^2}$ **ii** $\left(\frac{2b^2}{b+1}\right)^{\frac{1}{3}}$

b i $\frac{(b^2+1)(b-1)}{2b^2}$ **ii** $b = 3$

39 a Local maximum $\left(\frac{m}{n}, \left(\frac{m}{n}\right)^m e^{n-m}\right)$

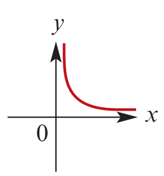
b $\left(\frac{m-1}{n}, \left(\frac{m-1}{n}\right)^m e^{n-m+1}\right)$

c i $\frac{4}{e^2}$ **ii** $1 - \frac{5}{e^2}$

40 a i q **ii** $\frac{1}{q}$ **iii** $\frac{1}{q^2}$

b $\frac{2}{3}$

c i



ii 0.37

iii 69.31

Degree-of-difficulty classified questions

Simple familiar questions

1 a $x^3g'(x) + 3x^2g(x)$

b $(2x + 3)g'(x) + 2g(x)$

c $3(g(x))^2g'(x)$ **d** $\frac{xg'(x) - 2g(x)}{x^3}$

2 a $\frac{11}{2}$ **b** $\frac{1}{8}$

3 $F'(x) = 3\sin\left(6x - \frac{\pi}{3}\right)$

4 a $f'(x) = 8x - 5$ **b** −5

c $x = \frac{3}{4}$

5 $\frac{f'(x)}{f(x)}$

6 $a = \frac{145}{144}$

7 a i $e^x(\cos x + \sin x)$ **ii** $-e^{\pi}$ **iii** 1

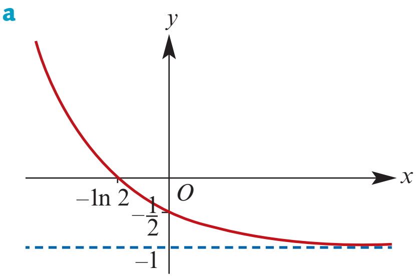
b i $y = x$ **ii** $y = -e^{\pi}x + \pi e^{\pi}$

c i $y = -e^{-\pi}x$ **ii** $y = x - \pi$

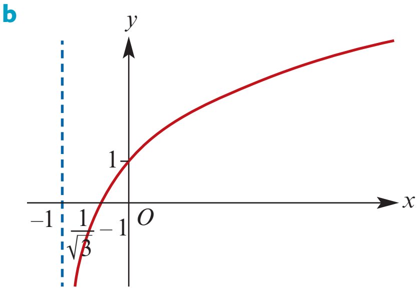
8 a $x = 0$ **b** $x = \frac{5}{3}$

c $x = \frac{1}{2}$ **d** $x = \frac{1}{4e}$

9 a



b



10 a $e^{2x}(2\cos(3x) - 3\sin(3x))$

b $(x - 2)(2\ln(x - 2) + 1)$

c $\frac{2 - \ln(x^2)}{x^2}$

d $3\cos(2x)\cos(3x) - 2\sin(2x)\sin(3x)$

e $\frac{-2\cos(2x)}{\sin^2(2x)}$

f $-15\sin^2(2 - 5x)\cos(2 - 5x)$

g $3x^2\cos^2(3x)(\cos(3x) - 3x\sin(3x))$

h $3x^2(\ln(x^3) + 1)$

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11 a $-2e^2$ **b** 0
c $8e^2+3$ **d** -1

12 a 100.81° **b** 59 m^2

13 a 3 square units
b $(1-2e^{-1}+e^{-2})$ square units

14 a i 0.06 **ii** 0.14
b $\frac{3}{7}$

15 a $k=0.1$
b i 0.7 **ii** $\frac{4}{7}$
c $\mu=1.1,\ \sigma=0.9434$
d 0.9

16 a i 0.954 **ii** 0.139
b i 35.8 g **ii** 41.3 g

Complex familiar questions

1 $[3,5]$

2 a $k=\frac{5\ln 2}{144}$ **b** 57.6 years

3 a $a=-16e^{2t}$ **b** $12-8\ln 2\approx 6.45$ m

4 a 0.322 **b** $(0.14, 0.26)$

5 a $x=-1$ or $x=3$
b $x=\frac{3}{2-e^2}$

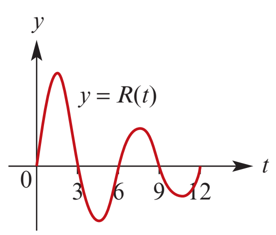
6 a $\angle ABC=131.4°,\ \angle BAC=18.6°$
b 1 m^2 **c** 6.3 m

7 97.5 m to the right of O

8 $A=2,\ b=1$

Complex unfamiliar questions

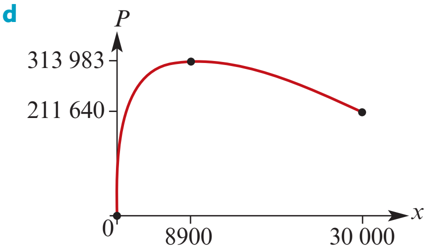
1 a i $R(0)=0$ **ii** $R(3)=0$
b $R'(t)=e^{-\frac{t}{10}}\left(\frac{10\pi}{3}\cos\left(\frac{\pi t}{3}\right)-\sin\left(\frac{\pi t}{3}\right)\right)$
c i 1.41, 4.41, 7.41, 10.41
ii Local max: (1.41, 8.65), (7.41, 4.75);
Local min: (4.41, −6.41) (10.41, −3.52)
d $t=0, 3, 6, 9$ or 12
e



f i 16.47 litres **ii** 12.20 litres **iii** 8.27 litres
g 12.99 litres

2 a Dilation of factor 2 from the x-axis, then translation 1 unit to the right and 1 unit up
b 4.5 m^2 **c** -4 **d** $y=2x-\frac{9}{2}$
e $\frac{3\sqrt{5}}{4}\approx 1.68$ m

3 a $P(x)=90\,000\ln\left(1+\frac{x}{100}\right)-10x-2000$
b $P'(x)=\frac{89\,000-10x}{x+100}$
c $x=8900$, \$313 983
d



4 a 16%
b i 0.933 **ii** 0.783
c 0.932 **d** 0.068 **e** $(0.72, 0.88)$
f 1537

Appendix A

Exercise A1

1 63
2 26
3 336
4 a 5040 **b** 210
5 a 120 **b** 120
6 18
7 a 5 852 925 **b** 1 744 200
8 100 386
9 a 792 **b** 336
10 a 200 **b** 75 **c** 6
d 462 **e** 81

Exercise A2

1 a $\sum_{i=1}^{4} i^3 = 1+8+27+64=100$
b $\sum_{k=1}^{5} k^3 = 1+8+27+64+125=225$
c $\sum_{i=1}^{5}(-1)^i\, i = -1+2-3+4-5=-3$

ISBN 978-1-108-45164-2

d $\frac{1}{5}\sum_{i=1}^{5} i = \frac{1}{5}(1+2+3+4+5) = 3$

e $\sum_{i=1}^{6} i = 1+2+3+4+5+6 = 21$

f $\sum_{k=1}^{4}(k-1)^2 = 0+1+4+9 = 14$

g $\frac{1}{3}\sum_{i=1}^{4}(i-2)^2 = \frac{1}{3}(1+0+1+4) = 2$

h $\sum_{i=1}^{6} i^2 = 1+4+9+16+25+36 = 91$

2 a $\sum_{i=1}^{n} i$ **b** $\sum_{i=1}^{11} x_i$ **c** $\frac{1}{10}\sum_{i=1}^{10} x_i$

d $\sum_{i=1}^{n+1} i^4$ **e** $\sum_{i=1}^{5} \frac{1}{i}$

3 a $x + x^2 + x^3 + \cdots + x^n$

b $32 + 16x + 8x^2 + 4x^3 + 2x^4 + x^5$

c $3^6 + (2x)\cdot 3^5 + (2x)^2\cdot 3^4 + (2x)^3\cdot 3^3 + (2x)^4\cdot 3^2 + (2x)^5\cdot 3 + (2x)^6$

d $(x-x_1) + (x-x_2)^2 + (x-x_3)^3 + (x-x_4)^4$

4 a $\sum_{i=0}^{5} x^{5-i}\cdot 3^i$ **b** $\sum_{i=0}^{5} x^{5-i}\cdot(-3)^i$

c $\sum_{i=0}^{2} x^{2-i}\cdot 2^{2-i}$ **d** $\sum_{i=0}^{3} (2x)^{3-i}\cdot 3^i$

Exercise A3

1 a $x^6 + 36x^5 + 540x^4 + 4320x^3 + 19\,440x^2 + 46\,656x + 46\,656$

b $32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1$

c $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$

d $64x^6 + 576x^5 + 2160x^4 + 4320x^3 + 4860x^2 + 2916x + 729$

e $64x^6 - 1152x^5 + 8640x^4 - 34\,560x^3 + 77\,760x^2 - 93\,312x + 46\,656$

f $16x^4 - 96x^3 + 216x^2 - 216x + 81$

g $x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$

h $x^{10} + 10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1$

2 a $-960x^3$ **b** $960x^3$

c $-960x^3$ **d** $192\,456x^5$

e $1732\,104x^5$ **f** $-25\,344b^7x^5$

3 $-\frac{16}{243}x^7$

4 $-336\,798x^6$

5 $(-x+1)^{11} = -x^{11} + 11x^{10} - 55x^9 + 165x^8 - 330x^7 + 462x^6 - 462x^5 + 330x^4 - 165x^3 + 55x^2 - 11x + 1$

6 a 40 **b** −160 **c** −80

d 181 440 **e** 432 **f** 1080

7 83 026 944

8 −768

ISBN 978-1-108-45164-2

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