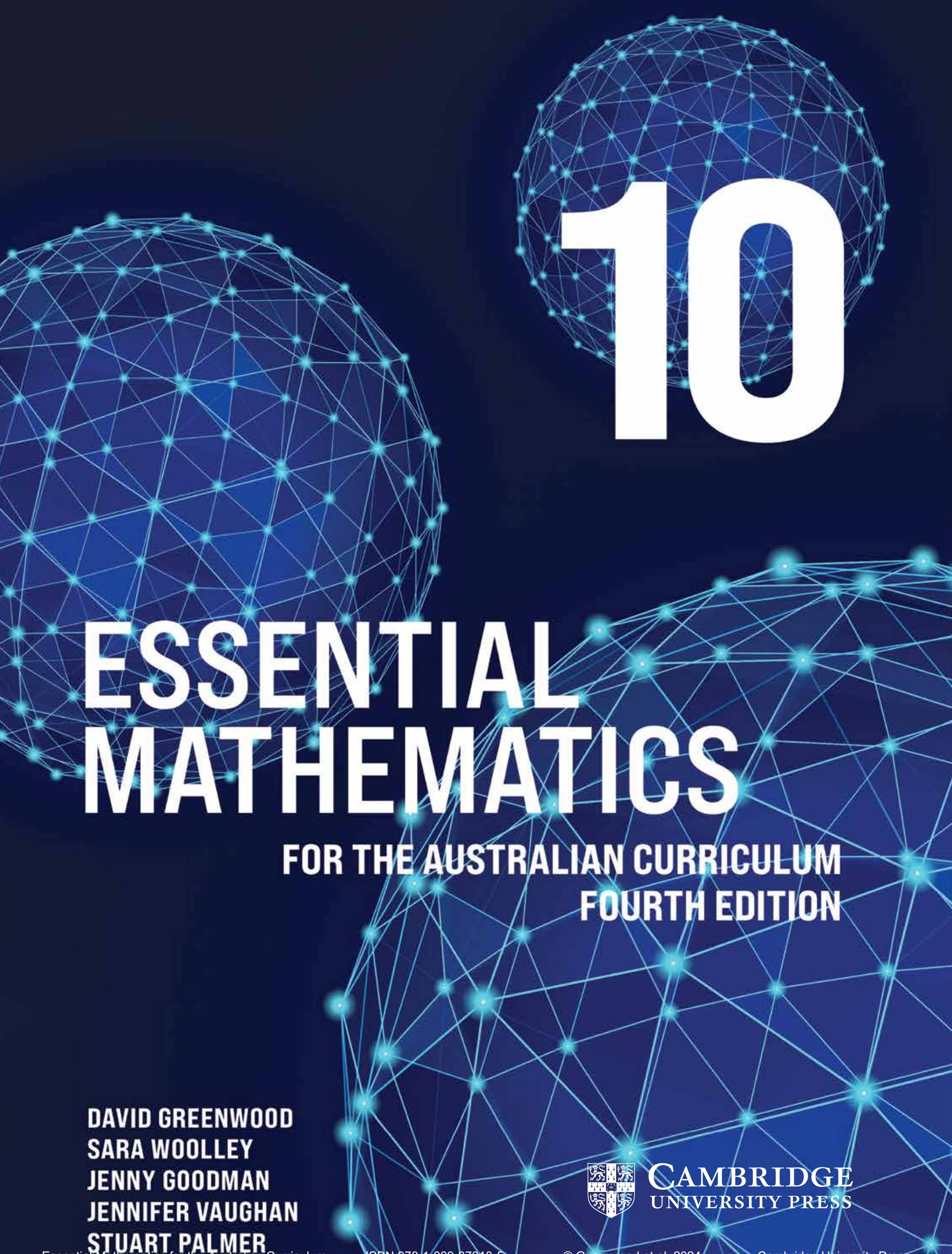

10

ESSENTIAL MATHEMATICS

FOR THE AUSTRALIAN CURRICULUM

FOURTH EDITION

DAVID GREENWOOD
SARA WOOLLEY
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About the authors

David Greenwood is the Head of Mathematics at Trinity Grammar School in Melbourne and has 30+ years teaching mathematics from Year 7 to 12. He is the lead author for the Cambridge Essential series and has authored more than 80 titles for the Australian Curriculum and for the syllabuses of the states and territories. He specialises in analysing curriculum and the sequencing of course content for school mathematics courses. He also has an interest in the use of technology for the teaching of mathematics.

Sara Woolley was born and educated in Tasmania. She completed an Honours degree in Mathematics at the University of Tasmania before completing her education training at the University of Melbourne. She has taught mathematics from Years 7 to 12 since 2006 and is currently a Head of Mathematics. She specialises in lesson design and creating resources that develop and build understanding of mathematics for all students.

Jennifer Vaughan has taught secondary mathematics for over 30 years in New South Wales, Western Australia, Queensland and New Zealand and has tutored and lectured in mathematics at Queensland University of Technology. She is passionate about providing students of all ability levels with opportunities to understand and to have success in using mathematics. She has had extensive experience in developing resources that make mathematical concepts more accessible; hence, facilitating student confidence, achievement and an enjoyment of maths.

Jenny Goodman has taught in schools for over 28 years and is currently teaching at a selective high school in Sydney. Jenny has an interest in the importance of literacy in mathematics education, and in teaching students of differing ability levels. She was awarded the Jones Medal for education at Sydney University and the Bourke Prize for Mathematics. She has written for *CambridgeMATHS NSW* and was involved in the *Spectrum* and *Spectrum Gold* series.

Stuart Palmer was born and educated in New South Wales. He is a fully qualified high school mathematics teacher with more than 25 years' experience teaching students from all walks of life in a variety of schools. He has been Head of Mathematics in two schools. He is very well known by teachers throughout the state for the professional learning workshops he delivers. Stuart also assists thousands of Year 12 students every year as they prepare for their HSC Examinations. At the University of Sydney, Stuart spent more than a decade running tutorials for pre-service mathematics teachers.

Introduction

The fourth edition of *Essential Mathematics for the Australian Curriculum* has been significantly revised and updated to suit the teaching and learning of Version 9.0 of the Australian Curriculum. Many of the established features of the series have been retained, but there have been some substantial revisions, improvements and new elements introduced for this edition across the print, digital and teacher resources.

New content and some restructuring

New topics have come in at all year levels. In **Year 7** there are new lessons on ratios, volume of triangular prisms, and measurement of circles, and all geometry topics are now contained in a single chapter (Chapter 4). In **Year 8**, there are new lessons on 3D-coordinates and techniques for collecting data. For **Year 9**, error in measurement is new in Chapter 5, and sampling and proportion is introduced in Chapter 9.

There are many changes in **Year 10**, including the inclusion of clearly marked 'Optional' content rather than 10A content, as it was previously. Among other changes, four lessons each on networks and combinatorics have been added, and there are new lessons on logarithmic scales, rates of change, two-way tables and cumulative frequency curves and percentiles. Although topics on polynomials, transformations of graphs, and graphs of circles and hyperbolas do not appear in Version 9.0, these are covered in an online-only Chapter 10, given the expected utility for students wanting to study a calculus-based course in Year 11 and 12. Some other 10A material from the previous curriculum that is not listed as part of Version 9.0 is also retained and labelled as Extension. For example: Measurement for cones, pyramids and spheres, and aspects of advanced Trigonometry.

Version 9.0 places increased emphasis on **investigations** and **modelling**, and this is covered with revised Investigations and Modelling activities at the end of chapters. There are also many new elaborations covering **First Nations Peoples' perspectives** on mathematics, ranging across all six content strands of the curriculum. These are covered in a suite of specialised investigations provided in the Online Teaching Suite.

Other new features

- **Technology and computational thinking** activities have been added to the end of every chapter to address the curriculum's increased focus on the use of technology and the understanding and application of algorithms.
- **Targeted Skillsheets** – downloadable and printable – have been written for every lesson in the series, with the intention of providing additional practice for students who need support at the basic skills covered in the lesson, with questions linked to worked examples in the book.
- **Editable PowerPoint lesson summaries** are also provided for each lesson in the series, with the intention of saving the time of teachers who were previously creating these themselves.

Diagnostic Assessment tool

Also new for this edition is a flexible, comprehensive Diagnostic Assessment tool, available through the Online Teaching Suite. This tool, featuring around 10,000 new questions, allows teachers to set diagnostic tests that are closely aligned with the textbook content, view student performance and growth via a range of reports, set follow-up work with a view to helping students improve, and export data as needed.

Guide to the working programs in exercises

The suggested working programs in the exercises in this book provide three pathways to allow differentiation for Foundation, Standard and Advanced students (schools will likely have their own names for these levels).

Each exercise is structured in subsections that match the mathematical proficiencies of Fluency, Problem-solving and Reasoning, as well as Enrichment (Challenge). (Note that Understanding is covered by 'Building understanding' in each lesson.) In the exercises, the questions suggested for each pathway are listed in three columns at the top of each subsection:

- The left column (lightest shaded colour) is the Foundation pathway
- The middle column (medium shaded colour) is the Standard pathway
- The right column (darkest shaded colour) is the Advanced pathway.

Foundation	Standard	Advanced
FLUENCY		
1, 2, 3(½), 4	2, 3(½), 4, 5(½)	3(½), 4, 5(½)
PROBLEM-SOLVING		
6, 7	6–8	7–9
REASONING		
10	10–12	12–14
ENRICHMENT		
–	–	15

The working program for Exercise 3A in Year 7. The questions recommended for a Foundation student are: 1, 2, 3(½), 4, 6 and 9. See note below.

Gradients within exercises and proficiency strands

The working programs make use of the two difficulty gradients contained within exercises. A gradient runs through the overall structure of each exercise – where there is an increasing level of mathematical sophistication required from Fluency to Problem-solving to Reasoning and Enrichment – but also within each proficiency; the first few questions in Fluency, for example, are easier than the last Fluency question.

The right mix of questions

Questions in the working programs are selected to give the most appropriate mix of *types* of questions for each learning pathway. Students going through the Foundation pathway should use the left tab, which includes all but the hardest Fluency questions as well as the easiest Problem-solving and Reasoning questions. An Advanced student can use the right tab, proceed through the Fluency questions (often half of each question), and have their main focus be on the Problem-solving and Reasoning questions, as well as the Enrichment questions. A Standard student would do a mix of everything using the middle tab.

Choosing a pathway

There are a variety of ways to determine the appropriate pathway for students through the course. Schools and individual teachers should follow the method that works for them. If required, the prior-knowledge pre-tests (now found online) can be used as a tool for helping students select a pathway. The following are recommended guidelines:

- A student who gets 40% or lower should complete the Foundation questions
- A student who gets above 40% and below 85% should complete the Standard questions
- A student who gets 85% or higher should complete the Advanced questions.

Note: The nomenclature used to list questions is as follows:

- 3, 4: complete all parts of questions 3 and 4
- 1-4: complete all parts of questions 1, 2, 3 and 4
- 10(½): complete half of the parts from question 10 (a, c, e, ... or b, d, f, ...)
- 2-4(½): complete half of the parts of questions 2, 3 and 4
- 4(½), 5: complete half of the parts of question 4 and all parts of question 5
- —: do not complete any of the questions in this section.

Guide to this resource

PRINT TEXTBOOK FEATURES

1 NEW **New lessons:** authoritative coverage of new topics in the Australian Curriculum 9.0 in the form of new, road-tested lessons throughout each book.

2 **Australian Curriculum 9.0:** content strands and content descriptions are listed at the beginning of the chapter (see the teaching program for more detailed curriculum documents)

3 **In this chapter:** an overview of the chapter contents

4 **Working with unfamiliar problems:** a set of problem-solving questions not tied to a specific topic

5 **Chapter introduction:** sets context for students about how the topic connects with the real world and the history of mathematics

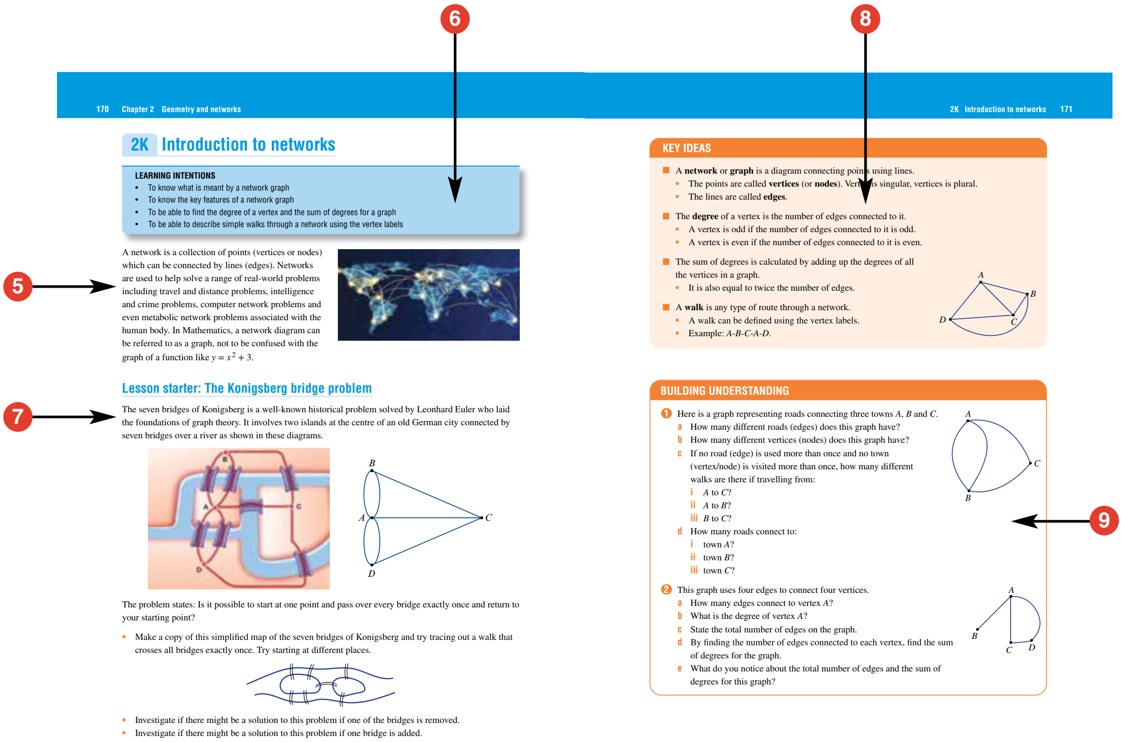
6 **Learning intentions:** sets out what a student will be expected to learn in the lesson

7 **Lesson starter:** an activity, which can often be done in groups, to start the lesson

8 **Key ideas:** summarises the knowledge and skills for the lesson

9 **Building understanding:** a small set of discussion questions to consolidate understanding of the Key ideas (replaces Understanding questions formerly inside the exercises)

10 **Worked examples:** solutions and explanations of each line of working, along with a description that clearly describes the mathematics covered by the example



170 Chapter 2 Geometry and networks

2K Introduction to networks

LEARNING INTENTIONS
- To know what is meant by a network graph
- To know the key features of a network graph
- To be able to find the degree of a vertex and the sum of degrees for a graph
- To be able to describe simple walks through a network using the vertex labels

A network is a collection of points (vertices or nodes) which can be connected by lines (edges). Networks are used to help solve a range of real-world problems including travel and distance problems, intelligence and crime problems, computer network problems and even metabolic network problems associated with the human body. In Mathematics, a network diagram can be referred to as a graph, not to be confused with the graph of a function like $y = x^2 + 3$.

Lesson starter: The Konigsberg bridge problem

The seven bridges of Konigsberg is a well-known historical problem solved by Leonhard Euler who laid the foundations of graph theory. It involves two islands at the centre of an old German city connected by seven bridges over a river as shown in these diagrams.

The problem states: Is it possible to start at one point and pass over every bridge exactly once and return to your starting point?

- Make a copy of this simplified map of the seven bridges of Konigsberg and try tracing out a walk that crosses all bridges exactly once. Try starting at different places.
- Investigate if there might be a solution to this problem if one of the bridges is removed.
- Investigate if there might be a solution to this problem if one bridge is added.

2K Introduction to networks 171

KEY IDEAS

- A **network** or **graph** is a diagram connecting points using lines.
  - The points are called **vertices** (or **nodes**). Vertex is singular, vertices is plural.
  - The lines are called **edges**.
- The **degree** of a vertex is the number of edges connected to it.
  - A vertex is odd if the number of edges connected to it is odd.
  - A vertex is even if the number of edges connected to it is even.
- The sum of degrees is calculated by adding up the degrees of all the vertices in a graph.
  - It is also equal to twice the number of edges.
- A **walk** is any type of route through a network.
  - A walk can be defined using the vertex labels.
  - Example: A-B-C-A-D.

BUILDING UNDERSTANDING

① Here is a graph representing roads connecting three towns A, B and C.
  a How many different roads (edges) does this graph have?
  b How many different vertices (nodes) does this graph have?
  c If no road (edge) is used more than once and no town (vertex/node) is visited more than once, how many different walks are there if travelling from:
    i A to C?
    ii A to B?
    iii B to C?
  d How many roads connect to:
    i town A?
    ii town B?
    iii town C?

② This graph uses four edges to connect four vertices.
  a How many edges connect to vertex A?
  b What is the degree of vertex A?
  c State the total number of edges on the graph.
  d By finding the number of edges connected to each vertex, find the sum of degrees for the graph.
  e What do you notice about the total number of edges and the sum of degrees for this graph?

11 Now you try: try-it-yourself questions provided after every worked example in exactly the same style as the worked example to give immediate practice

12 Gentle start to exercises: the exercise begins at Fluency, with the first question always linked to the first worked example in the lesson

13 Working programs: differentiated question sets for three ability levels in exercises

14 Example references: show where a question links to a relevant worked example – the first question is always linked to the first worked example in a lesson

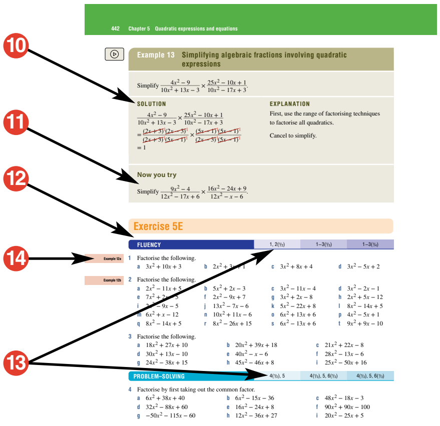
15 Problems and challenges: in each chapter provides practice with solving problems connected with the topic

16 Chapter checklist with success criteria: a checklist of the learning intentions for the chapter, with example questions

17 Applications and problem-solving: a set of three extended-response questions across two pages that give practice at applying the mathematics of the chapter to real-life contexts

18 NEW **Technology and computational thinking** activity in each chapter addresses the curriculum's increased focus on the use of different forms of technology, and the understanding and implementation of algorithms

19 Modelling activities: an activity in each chapter gives students the opportunity to learn and apply the mathematical modelling process to solve realistic problems



442 Chapter 5 Quadratic expressions and equations

Example 13 Simplifying algebraic fractions involving quadratic expressions

Simplify $\frac{4x^2-9}{10x^2+13x-3} \times \frac{25x^2-10x+1}{10x^2-17x+3}$.

SOLUTION

$\frac{4x^2-9}{10x^2+13x-3} \times \frac{25x^2-10x+1}{10x^2-17x+3}$

$= \frac{\cancel{(2x+3)}^1\cancel{(2x-3)}^1}{\cancel{(2x+3)}^1\cancel{(5x-1)}^1} \times \frac{\cancel{(5x-1)}^1\cancel{(5x-1)}^1}{\cancel{(2x-3)}^1\cancel{(5x-1)}^1}$

$= 1$

EXPLANATION

First, use the range of factorising techniques to factorise all quadratics.

Cancel to simplify.

Now you try

Simplify $\frac{9x^2-4}{12x^2-17x+6} \times \frac{16x^2-24x+9}{12x^2-x-6}$.

Exercise 5E

FLUENCY	1, 2(½)	1–3(⅓)	1–3(¼)

Example 12a

1 Factorise the following.

a $3x^2+10x+3$ b $2x^2+3x+1$ c $3x^2+8x+4$ d $3x^2-5x+2$

Example 12b

2 Factorise the following.

a $2x^2-11x+5$ b $5x^2+2x-3$ c $3x^2-11x-4$ d $3x^2-2x-1$
e $7x^2+2x-5$ f $2x^2-9x+7$ g $3x^2+2x-8$ h $2x^2+5x-12$
i $2x^2-9x-5$ j $13x^2-7x-6$ k $5x^2-22x+8$ l $8x^2-14x+5$
m $6x^2+x-12$ n $10x^2+11x-6$ o $6x^2+13x+6$ p $4x^2-5x+1$
q $8x^2-14x+5$ r $8x^2-26x+15$ s $6x^2-13x+6$ t $9x^2+9x-10$

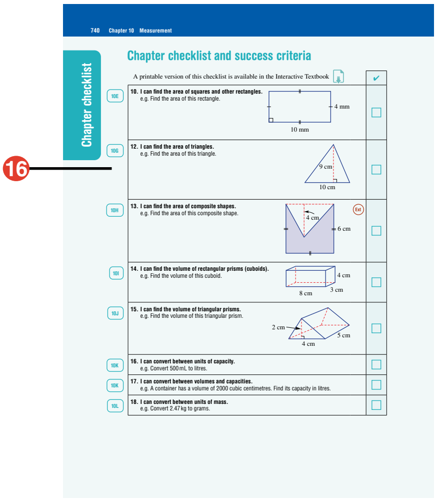
3 Factorise the following.

a $18x^2+27x+10$ b $20x^2+39x+18$ c $21x^2+22x-8$
d $30x^2+13x-10$ e $40x^2-x-6$ f $28x^2-13x-6$
g $24x^2-38x+15$ h $45x^2-46x+8$ i $25x^2-50x+16$

PROBLEM–SOLVING	4(½), 5	4(⅓), 5, 6(⅓)	4(⅓), 5, 6(⅓)

4 Factorise by first taking out the common factor.

a $6x^2+38x+40$ b $6x^2-15x-36$ c $48x^2-18x-3$
d $32x^2-88x+60$ e $16x^2-24x+8$ f $90x^2+90x-100$
g $-50x^2-115x-60$ h $12x^2-36x+27$ i $20x^2-25x+5$

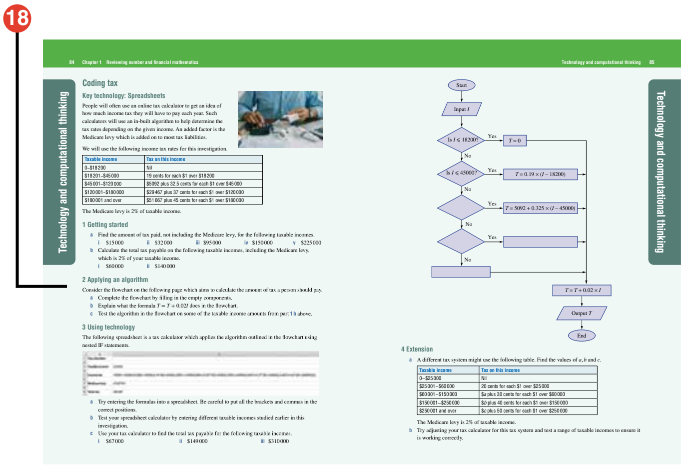


740 Chapter 10 Measurement

Chapter checklist and success criteria

A printable version of this checklist is available in the Interactive Textbook

		✓
10E	**10. I can find the area of squares and other rectangles.** e.g. Find the area of this rectangle.	☐
10G	**12. I can find the area of triangles.** e.g. Find the area of this triangle.	☐
10H	**13. I can find the area of composite shapes.** e.g. Find the area of this composite shape.	☐
10I	**14. I can find the volume of rectangular prisms (cuboids).** e.g. Find the volume of this cuboid.	☐
10J	**15. I can find the volume of triangular prisms.** e.g. Find the volume of this triangular prism.	☐
10K	**16. I can convert between units of capacity.** e.g. Convert 500 mL to litres.	☐
10K	**17. I can convert between volumes and capacities.** e.g. A container has a volume of 2000 cubic centimetres. Find its capacity in litres.	☐
10L	**18. I can convert between units of mass.** e.g. Convert 2.47 kg to grams.	☐



84 Chapter 1 Reviewing number and financial mathematics

Coding tax

Key technology: Spreadsheets

People will often use an online tax calculator to get an idea of how much income tax they will have to pay each year. Such calculators will use an in-built algorithm to help determine the tax rates depending on the given income. An added factor is the Medicare levy which is added on to most tax liabilities.

We will use the following income tax rates for this investigation.

Taxable income	Tax on this income
0–\$18 200	Nil
\$18 201–\$45 000	19 cents for each \$1 over \$18 200
\$45 001–\$120 000	\$5092 plus 32.5 cents for each \$1 over \$45 000
\$120 001–\$180 000	\$29 467 plus 37 cents for each \$1 over \$120 000
\$180 001 and over	\$51 667 plus 45 cents for each \$1 over \$180 000

The Medicare levy is 2% of taxable income.

1 Getting started

a Find the amount of tax paid, not including the Medicare levy, for the following taxable incomes.
i \$15 000 ii \$32 000 iii \$95 000 iv \$150 000 v \$225 000

b Calculate the total tax payable on the following taxable incomes, including the Medicare levy, which is 2% of your taxable income.
i \$60 000 ii \$140 000

2 Applying an algorithm

Consider the flowchart on the following page which aims to calculate the amount of tax a person should pay.

a Complete the flowchart by filling in the empty components.

b Explain what the formula $T = T + 0.02I$ does in the flowchart.

c Test the algorithm in the flowchart on some of the taxable income amounts from part 1 b above.

3 Using technology

The following spreadsheet is a tax calculator which applies the algorithm outlined in the flowchart using nested IF statements.

a Try entering the formulas into a spreadsheet. Be careful to put all the brackets and commas in the correct positions.

b Test your spreadsheet calculator by entering different taxable incomes studied earlier in this investigation.

c Use your tax calculator to find the total tax payable for the following taxable incomes.
i \$67 000 ii \$149 000 iii \$310 000

Technology and computational thinking 85

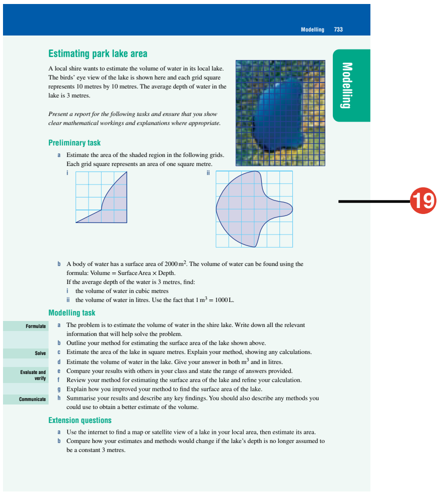
4 Extension

a A different tax system might use the following table. Find the values of a, b and c.

Taxable income	Tax on this income
0–\$25 000	Nil
\$25 001–\$60 000	20 cents for each \$1 over \$25 000
\$60 001–\$150 000	\$$a$ plus 30 cents for each \$1 over \$60 000
\$150 001–\$250 000	\$$b$ plus 40 cents for each \$1 over \$150 000
\$250 001 and over	\$$c$ plus 50 cents for each \$1 over \$250 000

The Medicare levy is 2% of taxable income.

b Try adjusting your tax calculator for this tax system and test a range of taxable incomes to ensure it is working correctly.



Modelling 733

Estimating park lake area

A local shire wants to estimate the volume of water in its local lake. The birds' eye view of the lake is shown here and each grid square represents 10 metres by 10 metres. The average depth of water in the lake is 3 metres.

Present a report for the following tasks and ensure that you show clear mathematical workings and explanations where appropriate.

Preliminary task

a Estimate the area of the shaded region in the following grids. Each grid square represents an area of one square metre.

b A body of water has a surface area of 2000 m^2. The volume of water can be found using the formula: Volume = Surface Area × Depth.
If the average depth of the water is 3 metres, find:
i the volume of water in cubic metres
ii the volume of water in litres. Use the fact that $1\ m^3 = 1000$ L.

Modelling task

a The problem is to estimate the volume of water in the shire lake. Write down all the relevant information that will help solve the problem.

b Outline your method for estimating the surface area of the lake shown above.

c Estimate the area of the lake in square metres. Explain your method, showing any calculations.

d Estimate the volume of water in the lake. Give your answer in both m^3 and in litres.

e Compare your results with others in your class and state the range of answers provided.

f Review your method for estimating the surface area of the lake and refine your calculation.

g Explain how you improved your method to find the surface area of the lake.

h Summarise your results and describe any key findings. You should also describe any methods you could use to obtain a better estimate of the volume.

Extension questions

a Use the internet to find a map or satellite view of a lake in your local area, then estimate its area.

b Compare how your estimates and methods would change if the lake's depth is no longer assumed to be a constant 3 metres.

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20 **Chapter reviews:** with short-answer, multiple-choice and extended-response questions; questions that are extension are clearly signposted

21 **Solving unfamiliar problems poster:** at the back of the book, outlines a strategy for solving any unfamiliar problem

INTERACTIVE TEXTBOOK FEATURES

22 NEW **Targeted Skillsheets**, one for each lesson, focus on a small set of related Fluency-style skills for students who need extra support, with questions linked to worked examples

23 **Workspaces:** almost every textbook question – including all working-out – can be completed inside the Interactive Textbook by using either a stylus, a keyboard and symbol palette, or uploading an image of the work

24 **Self-assessment:** students can then self-assess their own work and send alerts to the teacher. See the Introduction on page x for more information.

25 **Interactive working programs** can be clicked on so that only questions included in that working program are shown on the screen

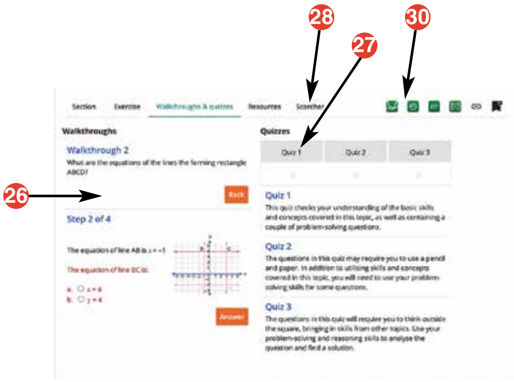
26 **HOTmaths resources:** a huge catered library of widgets, HOTsheets and walkthroughs seamlessly blended with the digital textbook

27 A revised set of **differentiated auto-marked practice quizzes** per lesson with saved scores

28 **Scorcher:** the popular competitive game

29 **Worked example videos:** every worked example is linked to a high-quality video demonstration, supporting both in-class learning and the flipped classroom

30 **Desmos graphing calculator**, scientific calculator and geometry tool are always available to open within every lesson



Walkthroughs

Walkthrough 2

What are the equations of the lines the forming rectangle ABCD?

Step 2 of 4

The equation of line AB is $x = -1$

The equation of line BC is:

a. ○ $x = 4$

b. ○ $y = 4$

Quizzes

Quiz 1

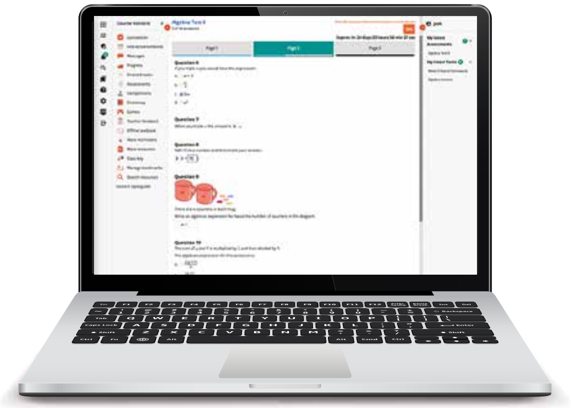
This quiz checks your understanding of the basic skills and concepts covered in this topic, as well as containing a couple of problem-solving questions.

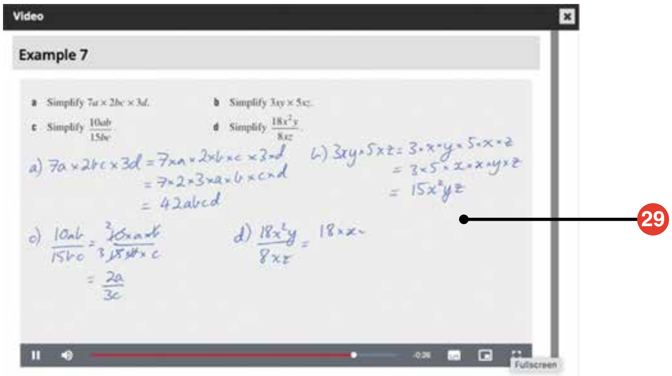
Quiz 2

The questions in this quiz may require you to use a pencil and paper. In addition to utilising skills and concepts covered in this topic, you will need to use your problem-solving skills for some questions.

Quiz 3

The questions in this quiz will require you to think outside the square, bringing in skills from other topics. Use your problem-solving and reasoning skills to analyse the question and find a solution.





a Simplify $7a \times 2bc \times 3d$.

b Simplify $3xy \times 5xz$.

c Simplify $\frac{10ab}{15bc}$.

d Simplify $\frac{18x^2y}{8xz}$.

a) $7a \times 2bc \times 3d = 7 \times a \times 2 \times b \times c \times 3 \times d$
$= 7 \times 2 \times 3 \times a \times b \times c \times d$
$= 42abcd$

b) $3xy \times 5xz = 3 \times x \times y \times 5 \times x \times z$
$= 3 \times 5 \times x \times x \times y \times z$
$= 15x^2yz$

c) $\frac{10ab}{15bc} = \frac{{}^{2}\cancel{10} \times a \times \cancel{b}}{{}^{3}\cancel{15} \times \cancel{b} \times c}$
$= \frac{2a}{3c}$

d) $\frac{18x^2y}{8xz} = 18 \times x$

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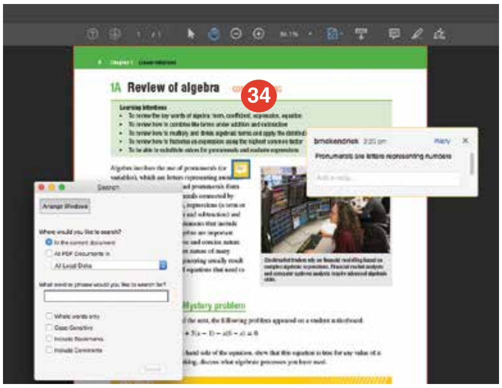
31 **Desmos interactives:** a set of Desmos activities written by the authors allow students to explore a key mathematical concept by using the Desmos graphing calculator or geometry tool

32 **Auto-marked prior knowledge pre-test** for testing the knowledge that students will need before starting the chapter

33 **Auto-marked progress quizzes and chapter review multiple-choice questions** in the chapter reviews can now be completed online

DOWNLOADABLE PDF TEXTBOOK

34 In addition to the Interactive Textbook, a **PDF version of the textbook** has been retained for times when users cannot go online. PDF search and commenting tools are enabled.



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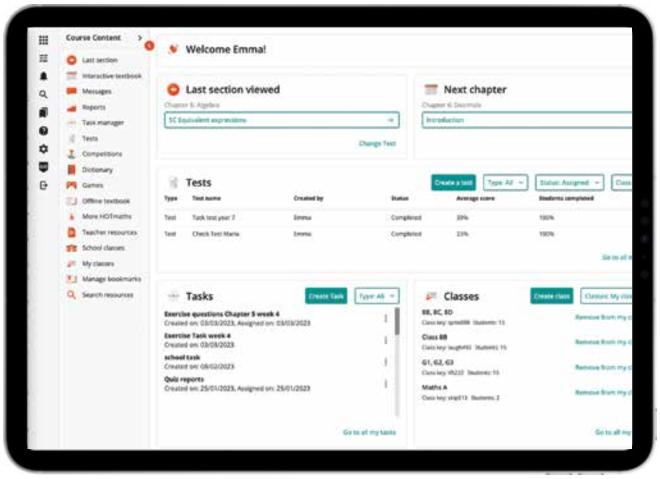

40 **Revamped task manager** allows teachers to incorporate many of the activities and tools listed above into teacher-controlled learning pathways that can be built for individual students, groups of students and whole classes

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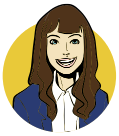
Working with unfamiliar problems: Part 1

The questions on the next four pages are designed to provide practice in solving unfamiliar problems. Use the 'Working with unfamiliar problems' poster at the back of this book to help you if you get stuck.

In Part 1, apply the suggested strategy to solve these problems, which are in no particular order. Clearly communicate your solution and final answer.

For Question 1, try looking for number patterns and algebraic patterns.



1 Discover the link between Pascal's triangle and expanded binomial products and use this pattern to help you expand $(x + y)^6$.

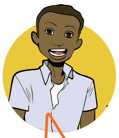
$$\begin{array}{lccccccc}
 & & & \text{Pascal's triangle} & & & & \\
(x+y)^0 & & & & 1 & & & \\
(x+y)^1 & & & 1 & & 1 & & \\
(x+y)^2 & & 1 & & 2 & & 1 & \\
(x+y)^3 & 1 & & 3 & & 3 & & 1
\end{array}$$

For Questions 2 and 3, try making a list or table.

2 How many palindromic numbers are there between 10^1 and 10^3?

3 Find the smallest positive integer values for x so that $60x$ is:

- **a** a perfect square
- **b** a perfect cube
- **c** divisible by both 8 and 9.



For Questions 4–8, try drawing a diagram to help you visualise the problem.

4 A Year 10 class raises money at a fete by charging players $1 to flip their dollar coin onto a red and white checked tablecloth with 50 mm squares. If the dollar coin lands fully inside a red square the player keeps their $1. What is the probability of keeping the $1?
How much cash is likely to be raised from 64 players?

5 The shortest side of a 60° set square is 12 cm. What is the length of the longest side of this set square?

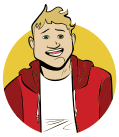
6 A Ferris wheel with diameter 24 metres rotates at a constant rate of 60 seconds per revolution.

- **a** Calculate the time taken for a rider to travel:
  - **i** from the bottom of the wheel to 8 m vertically above the bottom
  - **ii** from 8 m to 16 m vertically above the bottom of the wheel.
- **b** What fraction of the diameter is the vertical height increase after each one-third of the ride from the bottom to the top of the Ferris wheel?

7 *ABCD* is a rectangle with $AB = 16$ cm and $AD = 12$ cm. X and Y are points on BD such that AX and CY are each perpendicular to the diagonal BD. Find the length of the interval XY.

8 How many diagonal lines can be drawn inside a decagon (i.e. a 10-sided polygon)?

For Question 9, try to break up the numbers to help simplify.



9 The symbol ! means factorial.
e.g. $4! = 4 \times 3 \times 2 \times 1 = 24$.
Simplify $9! \div 7!$ without the use of a calculator.

10 In 2017 Charlie's age is the sum of the digits of his birth year 19*xy* and Bob's age is one less than triple the sum of the digits of his birth year 19*yx*. Find Charlie's age and Bob's age on their birthdays in 2017.

For Question 10, try to set up an equation

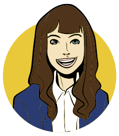
11 Let D be the difference between the squares of two consecutive positive integers. Find an expression for the average of the two integers in terms of D.

12 For what value of b is the expression $15ab + 6b - 20a - 8$ equal to zero for all values of a?

For Questions 11–13, try using algebra as a tool to work out the unknowns.

13 Find the value of k given $k > 0$ and that the area enclosed by the lines $y = x + 3,\ x + y + 5 = 0,\ x = k$ and the y-axis is 209 units^2.

14 The diagonal of a cube is $\sqrt{27}$ cm. Calculate the volume and surface area of this cube.



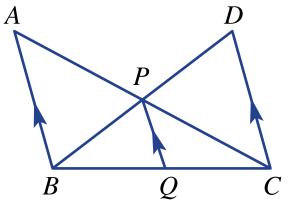
15 Two sides of a triangle have lengths 8 cm and 12 cm, respectively. Determine between which two values the length of the third side would fall. Give reasons for your answer.

16 When $10^{89} - 89$ is expressed as a single number, what is the sum of its digits?

17 Determine the reciprocal of this product $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{n+1}\right)$.

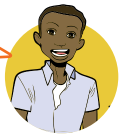
For Questions 14 and 15, try using concrete, everyday materials to help you understand the problem.

18 Find the value of $\dfrac{1002^2 - 998^2}{102^2 - 98^2}$, without using a calculator.



19 In the diagram at right, $AP = 9$ cm, $PC = 15$ cm, $BQ = 8.4$ cm and $QC = 14$ cm. Also, $CD \| QP \| BA$. Determine the ratio of the sides AB to DC.

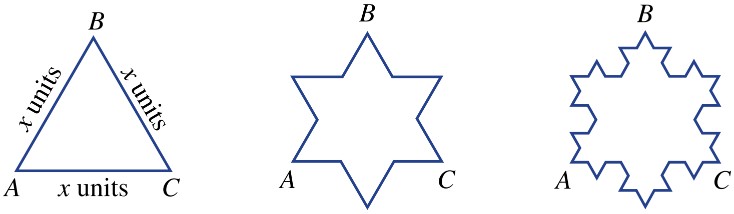
For Questions 16–19, try using a mathematical procedure to find a shortcut to the answer.



Working with unfamiliar problems: Part 2

For the questions in Part 2, again use the ‘Working with unfamiliar problems’ poster at the back of this book, but this time choose your own strategy (or strategies) to solve each problem. Clearly communicate your solution and final answer.

1 The Koch snowflake design starts with an equilateral triangle. A smaller equilateral triangle is built onto the middle third of each side and its base is erased. This procedure can be repeated indefinitely.



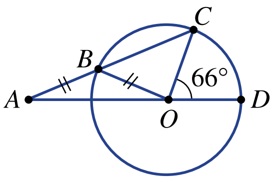
a For a Koch snowflake with initial triangle side length x units, determine expressions for the exact value of:
  i the perimeter after 5 procedure repeats and after n procedure repeats
  ii the sum of areas after 3 procedure repeats and the *change* in area after n procedure repeats.

b Comment on perimeter and area values as $n \to \infty$. Give reasons for your answers.

2 Two sides of a triangle have lengths in the ratio 3:5 and the third side has length 37 cm. If each side length has an integer value, find the smallest and largest possible perimeters, in cm.

3 The midpoints of each side of a regular hexagon are joined to form a smaller regular hexagon with side length k cm. Determine a simplified expression in terms of k for the exact difference in the perimeters of the two hexagons.

4 Angle COD is 66°. Find the size of angle CAD.



5 The graph of $y = ax^2 + 2x + 3$ has an axis of symmetry at $x = \frac{1}{4}$. Determine the maximum possible value of y.

6 Find the value of x and y given that $5^x = 125^{y-3}$ and $81^{x+1} = 9^y \times 3$.

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7 A rectangular prism has a surface area of 96 cm^2 and the sum of the lengths of all its edges is 64 cm. Determine the exact sum of the lengths of all its internal diagonals (i.e. diagonals not on a face).

8 In a Year 10 maths test, six students gained 100%, all students scored at least 75% and the mean mark was 82.85%. If the results were all whole numbers, what is the smallest possible number of students in this class? List the set of results for this class size.

9 Determine the exact maximum vertical height of the line $y = 2x$ above the parabola $y = 2x^2 - 5x - 3$.

10 $A + B = 6$ and $AB = 4$. Without solving for A and B, determine the values of:

a $(A + B)(B + 1)$

b $A^2 + B^2$

c $(A - B)^2$

d $\frac{1}{A} + \frac{1}{B}$

11 If $f(1) = 5$ and $f(x + 1) = 2f(x)$, determine the value of $f(8)$.

12 Four rogaining markers, $PQRS$, are in an area of bushland with level ground. Q is 1.4 km east of P, S is 1 km from P on a true bearing of 168° and R is 1.4 km from Q on a true bearing of 200°. To avoid swamps, Lucas runs the route $PRSQP$. Calculate the distances (in metres) and the true bearings from P to R, from R to S, from S to Q and from Q to P. Round your answers to the nearest whole number.

13 Consider all points (x, y) that are equidistant from the point (4, 1) and the line $y = -3$. Find the rule relating x and y and then sketch its graph, labelling all significant features. (Note: Use the distance formula.)

14 A 'rule of thumb' useful for 4WD beach driving is that the proportion of total tide height change after either high or low tide is $\frac{1}{12}$ in the first hour, $\frac{2}{12}$ in the second hour, $\frac{3}{12}$ in the third hour, $\frac{3}{12}$ in the fourth hour, $\frac{2}{12}$ in the fifth hour and $\frac{1}{12}$ in the sixth hour.

a Determine the accuracy of this 'rule of thumb' using the following equation for tide height: $h = 0.7\ \cos(30t) + 1$, where h is in metres and t is time in hours after high tide.

b Using $h = A\ \cos(30t) + D$, show that the proportion of total tide height change between any two given times, t_1 and t_2, is independent of the values of A and D.

15 All Golden Rectangles have the proportion $L : W = \Phi : 1$ where Φ (phi) is the Golden Number. Every Golden Rectangle can be subdivided into a square of side W and a smaller Golden Rectangle. Calculate phi as an exact number and also to six decimal places.

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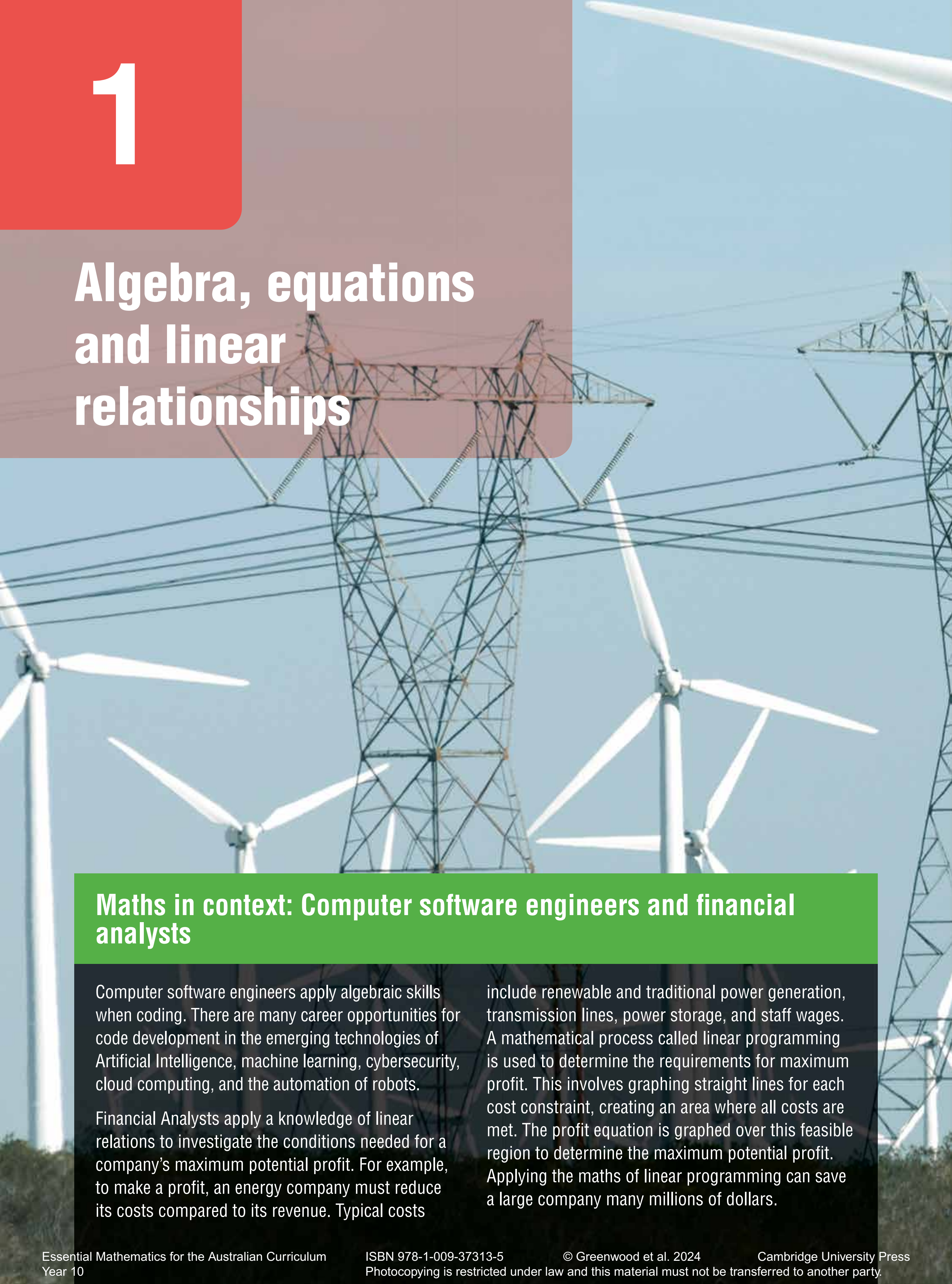

1

Algebra, equations and linear relationships

Maths in context: Computer software engineers and financial analysts

Computer software engineers apply algebraic skills when coding. There are many career opportunities for code development in the emerging technologies of Artificial Intelligence, machine learning, cybersecurity, cloud computing, and the automation of robots.

Financial Analysts apply a knowledge of linear relations to investigate the conditions needed for a company's maximum potential profit. For example, to make a profit, an energy company must reduce its costs compared to its revenue. Typical costs include renewable and traditional power generation, transmission lines, power storage, and staff wages. A mathematical process called linear programming is used to determine the requirements for maximum profit. This involves graphing straight lines for each cost constraint, creating an area where all costs are met. The profit equation is graphed over this feasible region to determine the maximum potential profit. Applying the maths of linear programming can save a large company many millions of dollars.

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ALGEBRA

Solve linear inequalities and simultaneous linear equations in 2 variables; interpret solutions graphically and communicate solutions in terms of the situation (AC9M10A02)

Use mathematical modelling to solve applied problems involving growth and decay, including financial contexts; formulate problems, choosing to apply linear, quadratic or exponential models; interpret solutions in terms of the situation; evaluate and modify models as necessary and report assumptions, methods and findings (AC9M10A04)

Simplification of combinations of linear expressions with rational coefficients and the solution of related equations (Year 10 optional content)

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1A Review of algebra CONSOLIDATING

LEARNING INTENTIONS

- To review the key words of algebra: term, coefficient, expression, equation
- To review how to combine like terms under addition and subtraction
- To review how to multiply and divide algebraic terms and apply the distributive law to expand brackets
- To review how to factorise an expression using the highest common factor
- To be able to substitute values for pronumerals and evaluate expressions

Algebra involves the use of pronumerals (or variables), which are letters representing numbers. Combinations of numbers and pronumerals form terms (numbers and pronumerals connected by multiplication and division), expressions (a term or terms connected by addition and subtraction) and equations (mathematical statements that include an equals sign). Skills in algebra are important when dealing with the precise and concise nature of mathematics. The complex nature of many problems in finance and engineering usually result in algebraic expressions and equations that need to be simplified and solved.

Stockmarket traders rely on financial modelling based on complex algebraic expressions. Financial market analysts and computer systems analysts require advanced algebraic skills.

Lesson starter: Mystery problem

Between one school day and the next, the following problem appeared on a student noticeboard.

Prove that $8 - x^2 + \frac{3x - 9}{3} + 5(x - 1) - x(6 - x) = 0$.

- By working with the left-hand side of the equation, show that this equation is true for any value of x.
- At each step of your working, discuss what algebraic processes you have used.

KEY IDEAS

- Key words in algebra:
  - **term**: $5x$, $7x^2y$, $\frac{2a}{3}$, 7 (a constant term)
  - **coefficient**: -3 is the coefficient of x^2 in $7 - 3x^2$; 1 is the coefficient of y in $y + 7x$.
  - **expression**: $7x$, $3x + 2xy$, $\frac{x + 3}{2}$, $\sqrt{2a^2 - b}$
  - **equation**: $x = 5$, $7x - 1 = 2$, $x^2 + 2x = -4$
- Expressions can be evaluated by substituting a value for each pronumeral (variable).
  - Order of operations are followed: First brackets, then indices, then multiplication and division, then addition and subtraction, working then from left to right.

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- **Like terms** have the same pronumeral part and, using addition and subtraction, can be collected to form a single term.
  For example, $3x - 7x + x = -3x$
  $6a^2b - ba^2 = 5a^2b$
  Note that $a^2b = ba^2$

- The symbols for multiplication (×) and division (÷) are usually not shown.
  $$7 \times x \div y = \frac{7x}{y}$$
  $$\begin{aligned} -6a^2b \div (ab) &= \frac{-6a^2b}{ab} \\ &= -6a \end{aligned}$$

- The **distributive law** is used to expand brackets.
  - $a(b + c) = ab + ac$ $\quad 2(x + 7) = 2x + 14$
  - $a(b - c) = ab - ac$ $\quad -x(3 - x) = -3x + x^2$

- **Factorisation** involves writing expressions as a product of factors.
  - Many expressions can be factorised by taking out the highest common factor (HCF).
    $15 = 3 \times 5$
    $3x - 12 = 3(x - 4)$
    $9x^2y - 6xy + 3x = 3x(3xy - 2y + 1)$

- Other general properties are:
  - **associative** $\quad a \times (b \times c) = (a \times b) \times c$ or $a + (b + c) = (a + b) + c$
  - **commutative** $\quad ab = ba$ or $a + b = b + a$ $\left(\text{Note: } \frac{a}{b} \neq \frac{b}{a} \text{ and } a - b \neq b - a\right)$
  - **identity** $\quad a \times 1 = a$ or $a + 0 = a$
  - **inverse** $\quad a \times \frac{1}{a} = 1$ or $a + (-a) = 0$

BUILDING UNDERSTANDING

1 Which of the following is an equation?

A $3x - 1$ **B** $\frac{x + 1}{4}$ **C** $7x + 2 = 5$ **D** $3x^2y$

2 Which expression contains a term with a coefficient of -9?

A $8 + 9x$ **B** $2x + 9x^2y$ **C** $9a - 2ab$ **D** $b - 9a^2$

3 State the coefficient of a^2 in these expressions.

a $a + a^2$ **b** $\frac{3}{2} - 4a^2$ **c** $1 - \frac{a^2}{5}$ **d** $-\frac{7a^2}{3} - 1$

4 Decide whether the following pairs of terms are like terms.

a xy and $2yx$ **b** $7a^2b$ and $-7ba^2$ **c** $-4abc^2$ and $8ab^2c$

5 Evaluate:

a $(-3)^2$ **b** $(-2)^3$ **c** -2^3 **d** -3^2

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Example 1 Collecting like terms

Simplify by collecting like terms.

a $7a + 3a$ **b** $3a^2b - 2a^2b$ **c** $5xy + 2xy^2 - 2xy + 3y^2x$

SOLUTION	EXPLANATION
a $7a + 3a = 10a$	Keep the pronumeral and add the coefficients.
b $3a^2b - 2a^2b = a^2b$	$3a^2b$ and $2a^2b$ have the same pronumeral part, so they are like terms. Subtract coefficients and recall that $1a^2b = a^2b$.
c $5xy + 2xy^2 - 2xy + 3y^2x = 3xy + 5xy^2$	Collect like terms, noting that $3y^2x = 3xy^2$. The + or − sign belongs to the term that directly follows it.

Now you try

Simplify by collecting like terms.

a $4a + 13a$ **b** $5ab^2 - 2ab^2$ **c** $3xy + 4x^2y - xy + 2yx^2$

Example 2 Multiplying and dividing expressions

Simplify the following.

a $2h \times 7l$ **b** $-3pr \times 2p$ **c** $-\frac{7xy}{14y}$

SOLUTION	EXPLANATION
a $2h \times 7l = 14hl$	Multiply the numbers and remove the × sign.
b $-3pr \times 2p = -6p^2r$	Multiply the numbers and recall that $p \times p$ is written as p^2.
c $-\frac{7xy}{14y} = -\frac{x}{2}$	Cancel the highest common factor of 7 and 14 and cancel the y.

Now you try

Simplify the following.

a $3a \times 6b$ **b** $-2xy \times 5x$ **c** $-\frac{4ab}{8a}$

Example 3 Expanding brackets

Expand the following using the distributive law. Simplify where possible.

a $2(x + 4)$ **b** $-3x(x - y)$ **c** $3(x + 2) - 4(2x - 4)$

SOLUTION	EXPLANATION
a $2(x + 4) = 2x + 8$	$2(x + 4) = 2 \times x + 2 \times 4$
b $-3x(x - y) = -3x^2 + 3xy$	Note that $x \times x = x^2$ and $-3 \times (-1) = 3$.
c $\begin{aligned} 3(x + 2) - 4(2x - 4) &= 3x + 6 - 8x + 16 \\ &= -5x + 22 \end{aligned}$	Expand each pair of brackets and simplify by collecting like terms.

Now you try

Expand the following using the distributive law. Simplify where possible.

a $3(x + 2)$ **b** $-2x(x - y)$ **c** $2(x + 3) - 3(2x - 1)$

Example 4 Factorising simple algebraic expressions

Factorise:

a $3x - 9$ **b** $2x^2 + 4x$

SOLUTION	EXPLANATION
a $3x - 9 = 3(x - 3)$	HCF of $3x$ and 9 is 3. Check that $3(x - 3) = 3x - 9$.
b $2x^2 + 4x = 2x(x + 2)$	HCF of $2x^2$ and $4x$ is $2x$. Check that $2x(x + 2) = 2x^2 + 4x$.

Now you try

Factorise:

a $2x - 10$ **b** $3x^2 + 9x$

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Example 5 Evaluating expressions

Evaluate $a^2 - 2bc$ if $a = -3$, $b = 5$ and $c = -1$.

SOLUTION

$$\begin{aligned} a^2 - 2bc &= (-3)^2 - 2(5)(-1) \\ &= 9 - (-10) \\ &= 19 \end{aligned}$$

EXPLANATION

Substitute for each pronumeral:
$(-3)^2 = (-3) \times (-3)$ and $2 \times 5 \times (-1) = -10$
To subtract a negative number, add its opposite.

Now you try

Evaluate $b^2 - 3ac$ if $a = 1$, $b = -2$ and $c = -3$.

Exercise 1A

FLUENCY 1–6(½) 1–6(½) 1–6(⅓)

1 Simplify by collecting like terms.

Example 1a

a $6a + 4a$ **b** $8d + 7d$ **c** $5y - 5y$
d $2xy + 3xy$ **e** $9ab - 5ab$ **f** $4t + 3t + 2t$

Example 1b

g $4a^2b - 2a^2b$ **h** $5x^2y - 4x^2y$ **i** $3st^2 - 4st^2$
j $4m^2n - 7nm^2$ **k** $0.3a^2b - ba^2$ **l** $0.2ab^2 - 2b^2a$

Example 1c

m $4gh + 5 - 2gh$ **n** $7xy + 5xy - 3y$ **o** $4a + 5b - a + 2b$
p $3jk - 4j + 5jk - 3j$ **q** $2ab^2 + 5a^2b - ab^2 + 5ba^2$ **r** $3mn - 7m^2n + 6nm^2 - mn$

2 Simplify the following.

Example 2a, b

a $4a \times 3b$ **b** $5a \times 5b$ **c** $-2a \times 3d$ **d** $5h \times (-2m)$
e $-6h \times (-5t)$ **f** $-5b \times (-6l)$ **g** $2s^2 \times 6t$ **h** $-3b^2 \times 7d^5$
i $4ab \times 2a$ **j** $-6p \times (-4pq)$ **k** $3ab \times (-5b)$ **l** $7mp \times 9mr$

Example 2c

m $\frac{7x}{7}$ **n** $\frac{6ab}{2}$ **o** $-\frac{3a}{9}$ **p** $-\frac{2ab}{8}$
q $\frac{4ab}{2a}$ **r** $-\frac{15xy}{5y}$ **s** $-\frac{4xy}{8x}$ **t** $-\frac{28ab}{56b}$

Example 3a, b

3 Expand the following, using the distributive law.

a $5(x + 1)$ **b** $2(x + 4)$ **c** $3(x - 5)$
d $-5(4 + b)$ **e** $-2(y - 3)$ **f** $-7(a + c)$
g $-6(-m - 3)$ **h** $4(m - 3n + 5)$ **i** $-2(p - 3q - 2)$
j $2x(x + 5)$ **k** $6a(a - 4)$ **l** $-4x(3x - 4y)$
m $3y(5y + z - 8)$ **n** $9g(4 - 2g - 5h)$ **o** $-2a(4b - 7a + 10)$
p $7y(2y - 2y^2 - 4)$ **q** $-3a(2a^2 - a - 1)$ **r** $-t(5t^3 + 6t^2 + 2)$
s $2m(3m^3 - m^2 + 5m)$ **t** $-x(1 - x^3)$ **u** $-3s(2t - s^3)$

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Example 3c

4 Expand and simplify the following, using the distributive law.

a $2(x + 4) + 3(x + 5)$
b $4(a + 2) + 6(a + 3)$
c $6(3y + 2) + 3(y - 3)$
d $3(2m + 3) + 3(3m - 1)$
e $2(2 + 6b) - 3(4b - 2)$
f $3(2t + 3) - 5(2 - t)$
g $2x(x + 4) + x(x + 7)$
h $4(6z - 4) - 3(3z - 3)$

Example 4

5 Factorise:

a $3x - 9$
b $4x - 8$
c $10y + 20$
d $6y + 30$
e $x^2 + 7x$
f $2a^2 + 8a$
g $5x^2 - 5x$
h $9y^2 - 63y$
i $xy - xy^2$
j $x^2y - 4x^2y^2$
k $8a^2b + 40a^2$
l $7a^2b + ab$
m $-5t^2 - 5t$
n $-6mn - 18mn^2$
o $-y^2 - 8yz$

Example 5

6 Evaluate these expressions if $a = -4, b = 3$ and $c = -5$.

a $-2a^2$
b $b - a$
c $abc + 1$
d $-ab$
e $\dfrac{a + b}{2}$
f $\dfrac{3b - a}{5}$
g $\dfrac{a^2 - b^2}{c}$
h $\dfrac{\sqrt{a^2 + b^2}}{\sqrt{c^2}}$

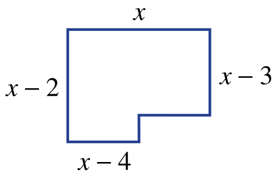
PROBLEM–SOLVING	7	7, 8	8

7 Find an expression for the area of a floor of a rectangular room with the following side lengths. Expand and simplify your answer.

a $x + 3$ and $2x$
b x and $x - 5$

8 Find expressions in simplest form for the perimeter (P) and area (A) of these shapes. (Note: All angles are right angles.)

a



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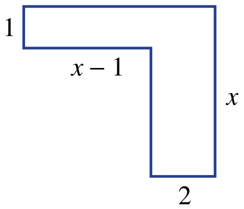

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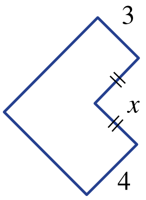


REASONING	9	9, 10	10, 11

9 When $a = -2$ give reasons why:

a $a^2 > 0$ **b** $-a^2 < 0$ **c** $a^3 < 0$

10 Decide whether the following are true or false for all values of a and b. If false, give an example to show that it is false.

a $a + b = b + a$

b $a - b = b - a$

c $ab = ba$

d $\frac{a}{b} = \frac{b}{a}$

e $a + (b + c) = (a + b) + c$

f $a - (b - c) = (a - b) - c$

g $a \times (b \times c) = (a \times b) \times c$

h $a \div (b \div c) = (a \div b) \div c$

11 **a** Write an expression for the statement 'the sum of x and y divided by 2'.

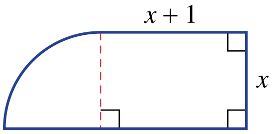
b Explain why the statement above is ambiguous.

c Write an unambiguous statement describing $\frac{a + b}{2}$.

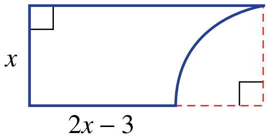
ENRICHMENT: Algebraic circular spaces	–	–	12

12 Find expressions in simplest form for the perimeter (P) and area (A) of these shapes. Your answers may contain π, for example 4π. Do not use decimals.

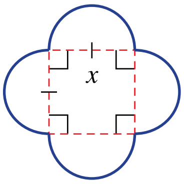
a



b



c



Architects, builders, carpenters and landscapers are among the many occupations that use algebraic formulas to calculate areas and perimeters in daily work.

1B Solving linear equations

LEARNING INTENTIONS

- To know the form of a linear equation
- To understand that an equivalent equation can be generated by applying the same operation to each side of the equation
- To be able to solve a linear equation involving two or more steps, including brackets and variables on both sides
- To be able to solve linear equations involving algebraic fractions
- To be able to combine simple algebraic fractions under addition or subtraction
- To understand that solutions can be checked by substituting into both sides of an equation

A linear equation is a statement that contains an equals sign and includes constants and pronumerals with a power of 1 only. Here are some examples:

$$\begin{aligned} 2x - 5 &= 7 \\ \frac{x+1}{3} &= x + 4 \\ -3(x+2) &= \frac{1}{2} \end{aligned}$$

We solve linear equations by operating on both sides of the equation until a solution is found.

A small business, such as a garden nursery, generates revenue from its sales. To calculate the number of employees (x) a business can afford, a linear revenue equation is solved for x:
Revenue (y) = pay (m) × employees (x) + costs (c)

Lesson starter: What's the best method?

Here are four linear equations.

- Discuss what you think is the best method to solve them using 'by hand' techniques.
- Discuss how it might be possible to check that a solution is correct.

a $\frac{7x - 2}{3} = 4$ **b** $3(x - 1) = 6$ **c** $4x + 1 = x - 2$

KEY IDEAS

- An equation is true for the given values of the pronumerals when the left-hand side equals the right-hand side.
  $2x - 4 = 6$ is true when $x = 5$ but false when $x \neq 5$.
- A **linear equation** contains pronumerals with a highest power of 1.
- Useful steps in solving linear equations are:
  - using inverse operations (backtracking)
  - collecting like terms
  - expanding brackets
  - multiplying by the denominator.
- Algebraic fractions can be added or subtracted by first finding the lowest common denominator (LCD) and then combining the numerators.

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BUILDING UNDERSTANDING

1. Decide whether the following are linear equations.
   a $x^2 - 1 = 0$ **b** $\sqrt{x} + x = 3$ **c** $\dfrac{x-1}{2} = 5$ **d** $\dfrac{3x}{4} = 2x - 1$
2. Decide whether these equations are true when $x = 2$.
   a $3x - 1 = 5$ **b** $4 - x = 1$ **c** $\dfrac{2x+1}{5} = x + 4$
3. Decide whether these equations are true when $x = -6$.
   a $-3x + 17 = x$ **b** $2(4 - x) = 20$ **c** $\dfrac{2-3x}{10} = \dfrac{-12}{x}$
4. Solve the following equations and check your solution using substitution.
   a $x + 8 = 13$ **b** $x - 5 = 3$ **c** $-x + 4 = 7$ **d** $-x - 5 = -9$
5. Simplify the following by firstly finding the lowest common denominator (LCD).
   a $\dfrac{1}{2} + \dfrac{1}{3}$ **b** $\dfrac{4}{3} - \dfrac{1}{5}$ **c** $\dfrac{3}{7} - \dfrac{1}{14}$ **d** $\dfrac{5}{3} + \dfrac{7}{6}$

Example 6 Solving linear equations

Solve the following equations and check your solution using substitution.

a $4x + 5 = 17$ **b** $3(2x + 5) = 4x$

SOLUTION	EXPLANATION
a $\begin{aligned} 4x + 5 &= 17 \\ 4x &= 12 \\ x &= 3 \end{aligned}$	Subtract 5 from both sides and then divide both sides by 4.
Check: LHS $= 4 \times 3 + 5 = 17$, RHS $= 17$	Check by seeing if $x = 3$ makes the equation true.
b $\begin{aligned} 3(2x + 5) &= 4x \\ 6x + 15 &= 4x \\ 2x + 15 &= 0 \\ 2x &= -15 \\ x &= -\frac{15}{2} \end{aligned}$	Expand the brackets. Gather like terms by subtracting $4x$ from both sides. Subtract 15 from both sides and then divide both sides by 2.
Check: $\begin{aligned} \text{LHS} &= 3\left(2 \times \left(-\frac{15}{2}\right) + 5\right) \\ &= -30 \end{aligned}$ $\begin{aligned} \text{RHS} &= 4 \times \left(-\frac{15}{2}\right) \\ &= -30 \end{aligned}$	Check by seeing if $x = -\frac{15}{2}$ makes the equation true by substituting into the equation's left-hand side (LHS) and right-hand side (RHS) and confirming they are equal.

Now you try

Solve the following equations and check your solution using substitution.

a $2x + 7 = 13$ **b** $4(2x + 1) = 2x$

Example 7 Solving equations involving fractions

Solve the following equations and check your solution using substitution.

a $\dfrac{x+3}{4}=2$ **b** $4-\dfrac{x}{2}=7$ **c** $\dfrac{5-x}{4}=x-3$

SOLUTION	EXPLANATION
a $\dfrac{x+3}{4}=2$ $x+3=8$ $x=5$	Multiply both sides by 4, since all of $x+3$ is divided by 4. Subtract 3 from both sides.
Check: LHS $=\dfrac{5+3}{4}=2$, RHS $=2$	Check by seeing if $x=5$ makes the equation true.
b $4-\dfrac{x}{2}=7$ $-\dfrac{x}{2}=3$ $x=-6$	Subtract 4 from both sides of the equation. The minus sign stays with $\frac{x}{2}$. Multiply both sides of the equation by -2.
Check: LHS $=4-\dfrac{(-6)}{2}=4+3=7$ RHS $=7$	Check by substituting $x=-6$ into the left-hand side.
c $\dfrac{5-x}{4}=x-3$ $5-x=4(x-3)$ $5-x=4x-12$ $5=5x-12$ $\therefore 5x=17$ $x=\dfrac{17}{5}$	Multiply both sides of the equation by 4 and include brackets. Expand the brackets and gather like terms by adding x to both sides. Add 12 to both sides and then divide both sides by 5.
Check: LHS $=\dfrac{5-\frac{17}{5}}{4}=\dfrac{8}{5}\div 4=\dfrac{2}{5}$ RHS $=\dfrac{17}{5}-3=\dfrac{2}{5}$	Check that LHS = RHS using substitution.

Now you try

Solve the following equations and check your solution using substitution.

a $\dfrac{x+1}{3}=4$ **b** $2-\dfrac{x}{5}=6$ **c** $\dfrac{4-x}{3}=x+2$

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Example 8 Solving equations by combining simple algebraic fractions

Consider these algebraic fractions.

a Simplify $\frac{a}{6} + \frac{2a}{9}$

b Hence, solve $\frac{a}{6} + \frac{2a}{9} = 2$

SOLUTION

a $$\begin{aligned}\frac{a}{6} + \frac{2a}{9} &= \frac{3a}{18} + \frac{4a}{18}\\ &= \frac{7a}{18}\end{aligned}$$

b From part **a**, $\frac{a}{6} + \frac{2a}{9} = \frac{7a}{18}$

Solve $$\begin{aligned}\frac{7a}{18} &= 2\\ 7a &= 36\\ \therefore a &= \frac{36}{7}\end{aligned}$$

EXPLANATION

The LCD of 6 and 9 is 18. Express each fraction as an equivalent fraction with a denominator of 18. Then add the numerators.

Combine the fractions on the left-hand side using the result from part **a**. Then multiply both sides by 18 and divide both sides by 7.

Now you try

Consider these algebraic fractions.

a Simplify $\frac{3a}{8} + \frac{a}{6}$

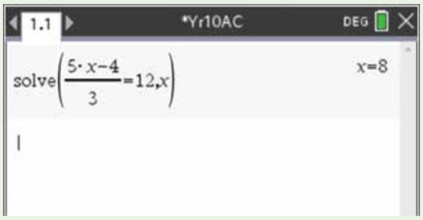
b Hence, solve $\frac{3a}{8} + \frac{a}{6} = 2$

Using calculators to solve equations

Solve the equation $\frac{5x - 4}{3} = 12$.

Using the TI-Nspire:

In a **Calculator** page use menu **>Algebra>Solve** and type the equation as shown.

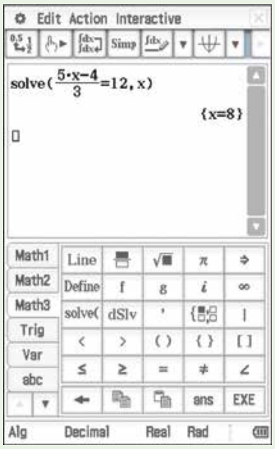


solve$\left(\frac{5\cdot x-4}{3}=12,x\right)$ x=8

Hint: use the fraction template (ctrl ÷)

Using the ClassPad:

Tap **solve(**, then and type the equation as shown.



solve$\left(\frac{5\cdot x-4}{3}=12, x\right)$

{x=8}

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Exercise 1B

FLUENCY	1–3(½), 5(½)	1–5(⅓)	1–5(⅓)

Example 6a

1 Solve the following equations and check your solution using substitution.

a $2x + 9 = 15$ **b** $4x + 3 = 15$ **c** $3x - 3 = -4$

d $6x + 5 = -6$ **e** $-3x + 5 = 17$ **f** $-2x + 7 = 4$

g $-4x - 9 = 9$ **h** $-3x - 7 = -3$ **i** $8 - x = 10$

j $5 - x = -2$ **k** $6 - 5x = 16$ **l** $4 - 9x = -7$

Example 6b

2 Solve the following equations and check your solution using substitution.

a $4(x + 3) = 16$ **b** $2(x - 3) = 12$ **c** $2(x - 4) = 15$

d $3(1 - 2x) = 8$ **e** $3(2x + 3) = -5x$ **f** $2(4x - 5) = -7x$

g $3(2x + 3) + 2(x + 4) = 25$ **h** $2(2x - 3) + 3(4x - 1) = 23$ **i** $2(3x - 2) - 3(x + 1) = 5$

j $5(2x + 1) - 3(x - 3) = 63$ **k** $5(x - 3) = 4(x - 6)$ **l** $4(2x + 5) = 3(x + 15)$

m $5(x + 2) = 3(2x - 3)$ **n** $3(4x - 1) = 7(2x - 7)$ **o** $7(2 - x) = 8 - x$

Example 7a, b

3 Solve the following equations and check your solution using substitution.

a $\frac{x - 4}{2} = 3$ **b** $\frac{x + 2}{3} = 5$ **c** $\frac{x + 4}{3} = -6$

d $\frac{2x + 7}{3} = 5$ **e** $\frac{2x + 1}{3} = -3$ **f** $\frac{3x - 2}{4} = 4$

g $\frac{x}{2} - 5 = 3$ **h** $\frac{3x}{2} + 2 = 8$ **i** $\frac{2x}{3} - 2 = -8$

j $5 - \frac{x}{2} = 1$ **k** $4 - \frac{2x}{3} = 0$ **l** $5 - \frac{4x}{7} = 9$

Example 7c

4 Solve the following equations.

a $\frac{2 - x}{3} = x + 1$ **b** $\frac{3 - x}{4} = x - 1$ **c** $\frac{x + 2}{5} = 2 - x$

d $\frac{x - 3}{5} - 2 = -6$ **e** $\frac{x + 1}{3} + 2 = 9$ **f** $\frac{x - 3}{2} - 4 = 2$

g $4 + \frac{x - 5}{2} = -3$ **h** $1 - \frac{2 - x}{3} = 2$ **i** $5 - \frac{1 - x}{2} = -1$

5 For each of the following statements, write an equation and solve it to find x.

a When 3 is added to x, the result is 7.

b When x is added to 8, the result is 5.

c When 4 is subtracted from x, the result is 5.

d When x is subtracted from 15, the result is 22.

e Twice the value of x is added to 5 and the result is 13.

f 5 less than x when doubled is -15.

g When 8 is added to 3 times x, the result is 23.

h 5 less than twice x is 3 less than x.

PROBLEM–SOLVING	6–9	6(½), 7, 9, 10	6–7(½), 10–12

Example 8

6 Consider the following algebraic fractions.

a **i** Simplify $\frac{x}{4} + \frac{x}{3}$ **ii** Hence, solve $\frac{x}{4} + \frac{x}{3} = 7$

b **i** Simplify $\frac{x}{6} + \frac{2x}{5}$ **ii** Hence, solve $\frac{x}{6} + \frac{2x}{5} = 2$

c **i** Simplify $\frac{3x}{8} - \frac{x}{4}$ **ii** Hence, solve $\frac{3x}{8} - \frac{x}{4} = 5$

d Solve:

  i $\frac{x}{2} + \frac{2x}{3} = 7$

  ii $\frac{3x}{5} - \frac{2x}{3} = 1$

  iii $\frac{2x}{5} - \frac{x}{4} = 3$

7 Substitute the given values and then solve for the unknown in each of the following common formulas.

a $v = u + at$ Solve for a given $v = 6$, $u = 2$ and $t = 4$.

b $s = ut + \frac{1}{2}at^2$ Solve for u given $s = 20$, $t = 2$ and $a = 4$.

c $A = h\left(\frac{a+b}{2}\right)$ Solve for b given $A = 10$, $h = 4$ and $a = 3$.

d $A = P\left(1 + \frac{r}{100}\right)$ Solve for r given $A = 1000$ and $P = 800$.

8 A service technician charges \$30 up front and \$46 for each hour that she works.

a What will a 4-hour job cost?

b If the technician works on a job for 2 days and averages 6 hours per day, what will be the overall cost?

c Find how many hours the technician worked if the cost is:

  i \$76

  ii \$513

  iii \$1000 (round to the nearest half hour).

9 The perimeter of a square is 68 cm. Use an equation to determine its side length.

10 The sum of two consecutive numbers is 35. What are the numbers?

11 I ride four times faster than I jog. If a trip took me 45 minutes and I spent 15 of these minutes jogging 3 km, use an equation to determine how far I rode.

12 The capacity of a petrol tank is 80 litres. If it initially contains 5 litres and a petrol pump fills it at 3 litres per 10 seconds, find:

a the amount of fuel in the tank after 2 minutes

b how long it will take to fill the tank to 32 litres

c how long it will take to fill the tank.

REASONING	13	13	14

13 Solve $2(x-5)=8$ using the following two methods and then decide which method you prefer. Give a reason.

a Method 1: First expand the brackets.

b Method 2: First divide both sides by 2.

14 A family of equations can be represented using other pronumerals (sometimes called parameters). For example, the solution to the family of equations $2x-a=4$ is $x=\frac{4+a}{2}$.

Find the solution for x in these equation families.

a $x+a=5$
b $6x+2a=3a$
c $ax+2=7$
d $ax-1=2a$
e $\frac{ax-1}{3}=a$
f $ax+b=c$

ENRICHMENT: Equations with more than one pronumeral	–	–	15, 16

15 Make a the subject in these equations.

a $a(b+1)=c$
b $ab+a=b$
c $\frac{1}{a}+b=c$
d $a-\frac{a}{b}=1$
e $\frac{1}{a}+\frac{1}{b}=0$
f $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$

16 Solve for x in terms of the other pronumerals.

a $\frac{x}{2}-\frac{x}{3}=a$
b $\frac{x}{a}+\frac{x}{b}=1$
c $\frac{x}{a}-\frac{x}{b}=c$

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1C Linear inequalities

LEARNING INTENTIONS

- To know the meaning of the term inequality
- To be able to use and interpret the symbols $>, \geqslant, \leqslant, <$
- To know how to interpret and represent an inequality on a number line
- To understand when to reverse the direction in an inequality
- To be able to solve a linear inequality

There are many situations in which a solution to the problem is best described using one of the symbols $<, \leqslant, >$ or $\geqslant$. For example, a pharmaceutical company might need to determine the possible number of packets of a particular drug that need to be sold so that the product is financially viable. This range of values may be expressed using inequality symbols.

An inequality is a mathematical statement that uses an *is less than* ($<$), an *is less than or equal to* ($\leqslant$), an *is greater than* ($>$) or an *is greater than or equal to* ($\geqslant$) symbol. Inequalities may result in an infinite number of solutions and these can be illustrated using a number line.

Doctors, nurses and pharmacists can use an inequality to express the dosage range of a medication from the lowest effective level to the highest safe level.

Lesson starter: What does it mean for *x*?

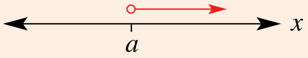
The following inequalities provide some information about the value of x.

a $2 \geqslant x$ **b** $-2x < 4$ **c** $3 - x \leqslant -1$

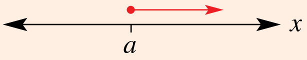
- Can you describe the possible values for x that satisfy each inequality?
- Test some values to check.
- How would you write the solution for x? Illustrate this on a number line.

KEY IDEAS

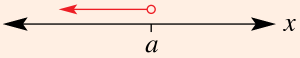
- The four **inequality symbols** are $<, \leqslant, >$ and $\geqslant$.
  - $x > a$ means x is greater than a.



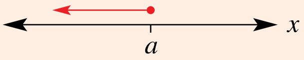
  - $x \geqslant a$ means x is greater than or equal to a.



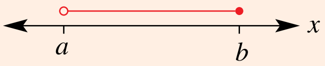
  - $x < a$ means x is less than a.



  - $x \leqslant a$ means x is less than or equal to a.



  - Also $a < x \leqslant b$ could be illustrated as shown.



- An open circle is used for $<$ or $>$ where the endpoint is not included.
- A closed circle is used for $\leqslant$ or $\geqslant$ where the endpoint is included.

■ Solving **linear inequalities** follows the same rules as solving linear equations, except:

- We reverse the inequality sign if we multiply or divide by a negative number.
  For example, $-5 < -3$ is equivalent to $5 > 3$ and if $-2x < 4$, then $x > -2$.
- We reverse the inequality symbol if the sides are switched.
  For example, if $2 \geqslant x$, then $x \leqslant 2$.

BUILDING UNDERSTANDING

1 State three numbers that satisfy each of these inequalities.

a $x \geqslant 3$

b $x < -1.5$

c $0 < x \leqslant 7$

d $-8.7 \leqslant x < -8.1$

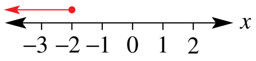
2 Match the graph **a**–**c** with the inequality **A**–**C**.

A $x > 2$

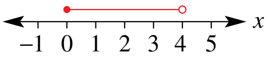
B $x \leqslant -2$

C $0 \leqslant x < 4$

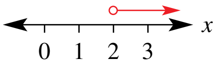
a



b



c



3 Phil houses x rabbits. If $10 < x \leqslant 13$, how many rabbits could Phil have?

4 Insert the correct inequality symbol.

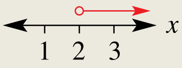
a If $-x < 2$ then x __ -2

b If $-a > -4$ then a __ 4

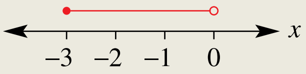
Example 9 Writing inequalities from number lines

Write as an inequality.

a



b

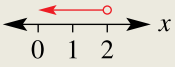


SOLUTION	EXPLANATION
a $x > 2$	An open circle means 2 is not included and x is greater than 2.
b $-3 \leqslant x < 0$	-3 is included but 0 is not, and x lies between -3 and 0.

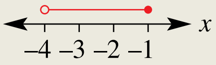
Now you try

Write as an inequality.

a



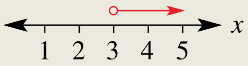
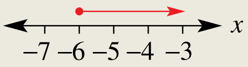
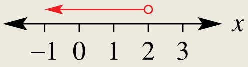
b



Example 10 Solving linear inequalities

Solve the following inequalities and graph their solutions on a number line.

a $3x + 4 > 13$ **b** $4 - \frac{x}{3} \leqslant 6$ **c** $3x + 2 > 6x - 4$

SOLUTION	EXPLANATION
a $3x + 4 > 13$ $3x > 9$ $\therefore x > 3$	Subtract 4 from both sides and then divide both sides by 3.
	Use an open circle since *x* does not include 3.
b $4 - \frac{x}{3} \leqslant 6$	Subtract 4 from both sides.
$-\frac{x}{3} \leqslant 2$	Multiply both sides by -3 and reverse the inequality symbol.
$\therefore\ x \geqslant -6$	
	Use a closed circle since *x* includes the number -6.
c $3x + 2 > 6x - 4$ $2 > 3x - 4$ $6 > 3x$ $2 > x$	Subtract $3x$ from both sides to gather the terms containing *x*. Add 4 to both sides and then divide both sides by 3.
$\therefore x < 2$	Make *x* the subject. Switching sides means the inequality symbol must be reversed.
	Use an open circle since *x* does not include 2.

Now you try

Solve the following inequalities and graph their solutions on a number line.

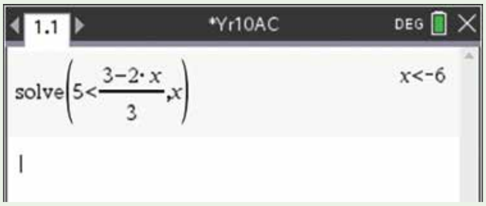
a $2x + 5 > 11$ **b** $2 - \frac{x}{3} \leqslant 4$ **c** $4x + 1 > 7x - 2$

Using calculators to solve inequalities

Solve the inequality $5 < \frac{3-2x}{3}$.

Using the TI-Nspire:

In a **Calculator** page use menu **>Algebra>Solve** and type the inequality as shown.



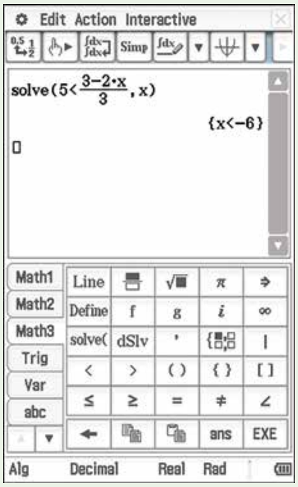
$\text{solve}\left(5<\frac{3-2\cdot x}{3},x\right)$ $x<-6$

Hint: the inequality symbols (e.g. <) are accessed using ctrl =

Hint: use the fraction template (ctrl ÷)

Using the ClassPad:

Tap **solve(** and type the inequality as shown.



$\text{solve}(5<\frac{3-2\cdot x}{3},x)$

$\{x<-6\}$

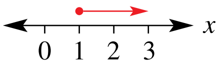
Exercise 1C

FLUENCY 1–4(½) 1–4(½) 1–4(⅓)

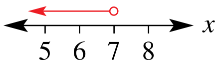
Example 9

1 Write each of the following as an inequality.

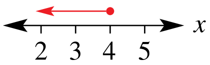
a



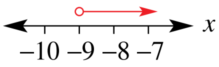
b



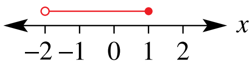
c



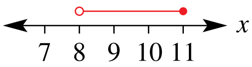
d



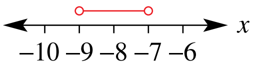
e



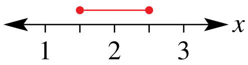
f



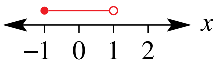
g



h



i



Example 10a

2 Solve the following inequalities and graph their solutions on a number line.

a $2x + 6 < 14$ **b** $3x + 5 \geqslant 20$ **c** $4x - 7 \geqslant 9$ **d** $\frac{x}{5} \leqslant 2$

e $\frac{x+4}{3} \leqslant 2$ **f** $\frac{5x-3}{2} > 6$ **g** $\frac{2x+3}{5} > 3$ **h** $\frac{x}{3} + 4 \leqslant 6$

i $\frac{x}{9} + 6 < 4$ **j** $-3 + \frac{x}{4} > 5$ **k** $3(3x - 1) \leqslant 7$ **l** $2(4x + 4) < 5$

Example 10b

3 Solve the following inequalities. Remember: if you multiply or divide by a negative number, you must reverse the inequality sign.

a $-5x + 7 \leqslant 12$ **b** $4 - 3x > -2$ **c** $-5x - 7 \geqslant 18$ **d** $\frac{3-x}{2} \geqslant 5$

e $\frac{5-2x}{3} > 7$ **f** $\frac{4-6x}{5} \leqslant -4$ **g** $3 - \frac{x}{2} \leqslant 8$ **h** $-\frac{x}{3} - 5 > 2$

Example 10c

4 Solve the following inequalities.

a $x + 1 < 2x - 5$ **b** $5x + 2 \geqslant 8x - 4$ **c** $7 - x > 2 + x$

d $3(x + 2) \leqslant 4(x - 1)$ **e** $7(1 - x) \geqslant 3(2 + 3x)$ **f** $-(2 - 3x) < 5(4 - x)$

PROBLEM–SOLVING	5, 6	5, 6	6, 7

5 For the following situations, write an inequality and solve it to find the possible values for x.

a 7 more than twice a number x is less than 12.

b Half of a number x subtracted from 4 is greater than or equal to -2.

c The product of 3 and one more than a number x is at least 2.

d The sum of two consecutive even integers, of which the smaller is x, is no more than 24.

e The sum of four consecutive even integers, of which x is the largest, is at most 148.

6 The cost of a satellite phone call is 30 cents plus 20 cents per minute.

a Find the possible cost of a call if it is:

- **i** shorter than 5 minutes
- **ii** longer than 10 minutes.

b For how many minutes can the phone be used if the cost per call is:

- **i** less than \$2.10?
- **ii** greater than or equal to \$3.50?

7 Solve these inequalities by first finding the lowest common denominator.

a $\frac{x}{3} - \frac{x}{2} > 1$ **b** $\frac{x}{4} - \frac{5x}{8} < -2$

c $\frac{x}{6} - \frac{3x}{4} > -4$ **d** $\frac{2x}{5} - \frac{9x}{20} < 0$

REASONING	8	8, 9	9, 10

8 How many whole numbers satisfy these inequalities? Give a reason.

a $x > 8$

b $2 < x \leqslant 3$

9 Solve these families of inequalities by writing x in terms of a. Consider cases where $a > 0$ and $a < 0$.

a $10x - 1 \geqslant a + 2$

b $\dfrac{2 - x}{a} > 4$

c $a(1 - x) > 7$

10 Describe the sets (in a form like $2 < x \leqslant 3$ or $-1 \leqslant x < 5$) that simultaneously satisfy these pairs of inequalities.

a $x < 5$
$x \geqslant -4$

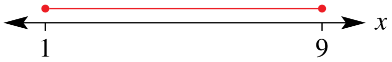
b $x \leqslant -7$
$x > -9.5$

c $x \leqslant 10$
$x \geqslant 10$

ENRICHMENT: Mixed inequalities	–	–	11–12(½)

11 Solve the inequalities and graph their solutions on a number line. Consider this example first.

Solve $-2 \leqslant x - 3 \leqslant 6$
$1 \leqslant x \leqslant 9$ (add 3 to both sides)



a $1 \leqslant x - 2 \leqslant 7$

b $-4 \leqslant x + 3 \leqslant 6$

c $-2 \leqslant x + 7 < 0$

d $0 \leqslant 2x + 3 \leqslant 7$

e $-5 \leqslant 3x + 4 \leqslant 11$

f $-16 \leqslant 3x - 4 \leqslant -10$

g $7 \leqslant 7x - 70 \leqslant 14$

h $-3 < \dfrac{x - 2}{4} < 0$

12 Solve these inequalities as per question **11** but noting the negative coefficient of x.

a $3 \leqslant 1 - x < 5$

b $-1 \leqslant 4 - x \leqslant 8$

c $-3 < 2 - x < 2$

d $1 < 5 - 2x < 11$

e $-4 < \dfrac{4 - x}{2} < -2$

f $-7 \leqslant \dfrac{1 - 3x}{2} \leqslant -2$

1D Linear equations involving more complex algebraic fractions OPTIONAL

LEARNING INTENTIONS

- To know how to find the lowest common denominator of algebraic fractions
- To be able to combine numerators using expansion and addition of like terms
- To be able to add and subtract algebraic fractions
- To be able to solve linear equations involving algebraic fractions

The sum or difference of two or more algebraic fractions can be simplified in a similar way to numerical fractions with the use of a common denominator. This is often the first step in solving a linear equation involving multiple algebraic fractions.

Electricians, electrical and electronic engineers work with algebraic fractions when modelling the flow of electric energy in circuits. The application of algebra when using electrical formulas is essential in these professions.

Lesson starter: Spot the difference

Here are two sets of simplification steps. One set has one critical error. Can you find and correct it?

$$\begin{aligned}\frac{2}{3}-\frac{5}{2}&=\frac{4}{6}-\frac{15}{6}\\&=\frac{-11}{6}\end{aligned}$$

$$\begin{aligned}\frac{x}{3}-\frac{x+1}{2}&=\frac{2x}{6}-\frac{3(x+1)}{6}\\&=\frac{2x-3x+3}{6}\\&=\frac{-x+3}{6}\end{aligned}$$

KEY IDEAS

- Add and subtract algebraic fractions by first finding the lowest common denominator (LCD) and then combining the numerators.
- Expand numerators correctly by taking into account addition and subtraction signs.
  For example, $-2(x+1)=-2x-2$ and $-5(2x-3)=-10x+15$.
- Solving linear equations can involve multiplying by the LCD of algebraic fractions.

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BUILDING UNDERSTANDING

1 Expand and simplify the following.

a $2(x-2)$
b $-(x+6)$
c $-6(x-2)$
d $5(x+1)-3(x+2)$

2 State the lowest common denominator for these pairs of fractions.

a $\frac{a}{3}, \frac{7a}{4}$
b $\frac{x}{2}, \frac{4xy}{6}$
c $\frac{3xy}{7}, \frac{-3x}{14}$
d $\frac{2}{x}, \frac{3}{2x}$

3 Solve these equations.

a $\frac{x+2}{3}=4$
b $\frac{x-3}{5}=1$
c $\frac{2x+1}{6}=2$

4 Consider the equation $\frac{x+1}{2}=\frac{x+5}{3}$.

a Multiply both sides by 2.
b Multiply both sides by 3.
c Solve the resulting equation.

Example 11 Adding and subtracting algebraic fractions

Simplify the following algebraic expressions.

a $\frac{x+3}{2}+\frac{x-2}{5}$
b $\frac{2x-1}{3}-\frac{x-1}{4}$

SOLUTION

a
$$\begin{aligned}\frac{x+3}{2}+\frac{x-2}{5} &= \frac{5(x+3)}{10}+\frac{2(x-2)}{10}\\ &= \frac{5(x+3)+2(x-2)}{10}\\ &= \frac{5x+15+2x-4}{10}\\ &= \frac{7x+11}{10}\end{aligned}$$

b
$$\begin{aligned}\frac{2x-1}{3}-\frac{x-1}{4} &= \frac{4(2x-1)}{12}-\frac{3(x-1)}{12}\\ &= \frac{4(2x-1)-3(x-1)}{12}\\ &= \frac{8x-4-3x+3}{12}\\ &= \frac{5x-1}{12}\end{aligned}$$

EXPLANATION

a LCD of 2 and 5 is 10. Express each fraction as an equivalent fraction with a denominator of 10.

Use brackets to ensure you retain equivalent fractions.

Combine the numerators, then expand the brackets and simplify.

b Express each fraction with the LCD of 12.

Combine the numerators.

Expand the brackets: $4(2x-1)=8x-4$ and $-3(x-1)=-3x+3$.

Simplify by collecting like terms.

Now you try

Simplify the following algebraic expressions.

a $\frac{x+1}{3}+\frac{x-2}{2}$
b $\frac{3x-2}{2}-\frac{x-2}{5}$

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Example 12 Solving more complex equations involving algebraic fractions

Solve the following equations and check your solution using substitution.

a $\frac{4x-2}{3} = \frac{3x-1}{2}$

b $\frac{x+2}{3} - \frac{2x-1}{2} = 4$

SOLUTION

a

$$\begin{aligned} \frac{4x-2}{3} &= \frac{3x-1}{2} \\ \frac{\cancel{6}^{2}(4x-2)}{\cancel{3}_{1}} &= \frac{\cancel{6}^{3}(3x-1)}{\cancel{2}_{1}} \\ 2(4x-2) &= 3(3x-1) \\ 8x-4 &= 9x-3 \\ -4 &= x-3 \\ -1 &= x \\ \therefore x &= -1 \end{aligned}$$

EXPLANATION

Multiply both sides by the LCD of 2 and 3, which is 6.

Cancel common factors. Alternatively, cross multiply for the same result.

Expand the brackets and gather terms containing x by subtracting $8x$ from both sides.

Rewrite with x as the subject. Check by seeing if $x = -1$ makes the equation true.

b

$$\begin{aligned} \frac{x+2}{3} - \frac{2x-1}{2} &= 4 \\ \frac{2(x+2)}{6} - \frac{3(2x-1)}{6} &= 4 \\ \frac{2x+4-6x+3}{6} &= 4 \\ \frac{-4x+7}{6} &= 4 \\ -4x+7 &= 24 \\ -4x &= 17 \\ x &= -\frac{17}{4} \end{aligned}$$

Express the algebraic fractions as a single fraction using the LCD of 6.

Alternatively, multiply both sides by the LCD of 2 and 3, which is 6.

Expand, noting that $-3 \times (-1) = 3$.

Simplify and solve for x.

Check your solution using substitution.

Now you try

Solve the following equations and check your solution using substitution.

a $\frac{2x-1}{3} = \frac{x-3}{4}$

b $\frac{x+1}{2} - \frac{2x-1}{3} = 2$

Exercise 1D

FLUENCY	1–3(½)	1–3(½)	1–3(⅓)

Example 11a

1 Simplify the following algebraic expressions.

a $\frac{x+3}{4}+\frac{x+2}{5}$ b $\frac{x+2}{3}+\frac{x+1}{4}$ c $\frac{x-3}{4}+\frac{x+2}{2}$

d $\frac{x+4}{3}+\frac{x-3}{9}$ e $\frac{2x+1}{2}+\frac{x-2}{3}$ f $\frac{3x+1}{5}+\frac{2x+1}{10}$

g $\frac{x-2}{8}+\frac{2x+4}{12}$ h $\frac{5x+3}{10}+\frac{2x-2}{4}$ i $\frac{3-x}{14}+\frac{x-1}{7}$

Example 11b

2 Simplify these algebraic fractions.

a $\frac{2x+1}{3}-\frac{x-1}{2}$ b $\frac{3x-1}{3}-\frac{2x-3}{4}$ c $\frac{x+6}{5}-\frac{x-4}{3}$

d $\frac{x-3}{2}-\frac{2x+1}{7}$ e $\frac{7x+2}{7}-\frac{x+2}{3}$ f $\frac{10x-4}{3}-\frac{2x+1}{6}$

g $\frac{4-x}{6}-\frac{1-x}{5}$ h $\frac{1-3x}{5}-\frac{x+2}{3}$ i $\frac{6-5x}{2}-\frac{2-7x}{4}$

Example 12a

3 Solve the following equations, which involve algebraic fractions.

a $\frac{2x+12}{7}=\frac{3x+5}{4}$ b $\frac{5x-4}{4}=\frac{x-5}{5}$ c $\frac{3x-5}{4}=\frac{2x-8}{3}$

d $\frac{1-x}{5}=\frac{2-x}{3}$ e $\frac{6-2x}{3}=\frac{5x-1}{4}$ f $\frac{10-x}{2}=\frac{x+1}{3}$

g $\frac{2(x+1)}{3}=\frac{3(2x-1)}{2}$ h $\frac{-2(x-1)}{3}=\frac{2-x}{4}$ i $\frac{3(6-x)}{2}=\frac{2(x+1)}{5}$

PROBLEM–SOLVING	4(½)	4(½), 5	4(⅓), 5–7

Example 12b

4 Solve the following equations.

a $\frac{x-1}{2}+\frac{x+2}{5}=2$ b $\frac{x+3}{3}+\frac{x-4}{2}=4$ c $\frac{x+2}{3}-\frac{x-1}{2}=1$

d $\frac{x-4}{5}-\frac{x+2}{3}=2$ e $\frac{7-2x}{3}-\frac{6-x}{2}=1$ f $\frac{2x+1}{3}-\frac{2-x}{4}=-1$

5 Matthew rides for x km at a speed of 12 km/h. Zoe rides for 4 more kilometres than Matthew and at a speed of 10 km/h. Their combined total riding time is 1.5 hours. Recalling that Time taken = Distance ÷ Speed, determine how far they each rode.

6 Solve these linear inequalities.

a $\frac{x+1}{2} + \frac{x-5}{3} > 2$

b $\frac{x-4}{5} - \frac{x+2}{3} < -4$

7 Solve these equations for x.

a $\frac{x+1}{2} - \frac{x}{3} + \frac{4x}{5} = -1$

b $\frac{4-2x}{3} - \frac{5x}{4} + \frac{3x+1}{2} = 2$

c $-\frac{x}{4} + \frac{3}{8} = \frac{1}{5}(3x-1)$

d $\frac{4-x}{2} + \frac{5}{3} = \frac{2}{5}(1-2x)$

REASONING

8	8, 9	8, 9

8 Describe the error in this working, then fix the solution.

$$\begin{aligned}\frac{x}{2} - \frac{x+1}{3} &= \frac{3x}{6} - \frac{2(x+1)}{6} \\ &= \frac{3x}{6} - \frac{2x+2}{6} \\ &= \frac{x+2}{6}\end{aligned}$$

9 **a** Explain why $2x - 3 = -(3 - 2x)$.

b Use this idea to help simplify these expressions.

i $\frac{1}{x-1} - \frac{1}{1-x}$

ii $\frac{3x}{3-x} + \frac{x}{x-3}$

iii $\frac{x+1}{7-x} - \frac{2}{x-7}$

ENRICHMENT: Binomial denominators

–	–	10, 11

10 To simplify the expression $\frac{3}{x-6} + \frac{2}{x+2}$, the LCD is $(x-6)(x+2)$.

So $\frac{3}{x-6} + \frac{2}{x+2} = \frac{3(x+2)}{(x-6)(x+2)} + \frac{2(x-6)}{(x-6)(x+2)}$. Combine the numerators and then simplify.

Simplify these algebraic expressions involving algebraic denominators.

a $\frac{5}{x+1} + \frac{2}{x+4}$

b $\frac{4}{x-7} + \frac{3}{x+2}$

c $\frac{1}{x-3} + \frac{2}{x+5}$

d $\frac{3}{x+3} - \frac{2}{x-4}$

e $\frac{6}{2x-1} - \frac{3}{x-4}$

f $\frac{4}{x-5} + \frac{2}{3x-4}$

g $\frac{5}{2x-1} - \frac{6}{x+7}$

h $\frac{2}{x-3} - \frac{3}{3x+4}$

i $\frac{8}{3x-2} - \frac{3}{1-x}$

11 By first simplifying the left-hand side of these equations, find the value of a.

a $\frac{a}{x-1} - \frac{2}{x+1} = \frac{4}{(x-1)(x+1)}$

b $\frac{3}{2x-1} + \frac{a}{x+1} = \frac{5x+2}{(2x-1)(x+1)}$

1E Graphing straight lines CONSOLIDATING

LEARNING INTENTIONS

- To understand what it means for a point to lie on a line: graphically and algebraically
- To understand that straight lines have a constant gradient that can be positive, negative, zero or undefined
- To know how to determine the gradient of a line from its equation and use it and a point to sketch its graph
- To be able to find the axis intercepts of a linear graph and use them to sketch the graph
- To be able to sketch straight lines with only one intercept

In two dimensions, a straight-line graph can be described by a linear equation. Common forms of such equations are $y = mx + c$ and $ax + by = d$, where a, b, c, d and m are constants. From a linear equation a graph can be drawn by considering such features as x- and y-intercepts and the gradient.

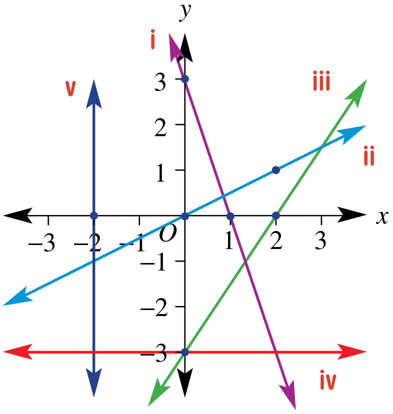
For any given straight-line graph the y-value changes by a fixed amount for each 1 unit change in the x-value. This change in y tells us the gradient of the line.

A financial analyst can use linear graphs to predict possible profit. The profit made by a lawn mower shop could be analysed with a straight-line graph of the equation: Profit (y) = mower price (m) × sales (x) − costs (c)

Lesson starter: Five graphs, five equations

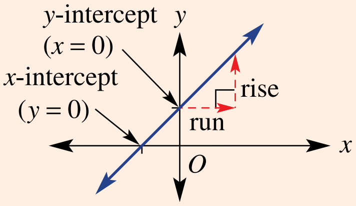
Here are five equations and five graphs. Match each equation with its graph. Give reasons for each of your choices.

a $y = -3$
b $x = -2$
c $y = \frac{1}{2}x$
d $y = -3x + 3$
e $3x - 2y = 6$



KEY IDEAS

- The **gradient**, m, is a number that describes the slope of a line.
  - Gradient $= \frac{\text{rise}}{\text{run}}$
  - The gradient is the change in y per 1 unit change in x. Gradient is also referred to as the 'rate of change of y'.



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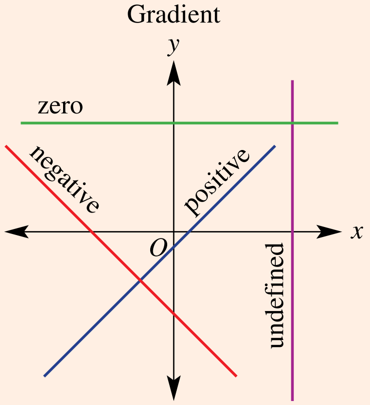
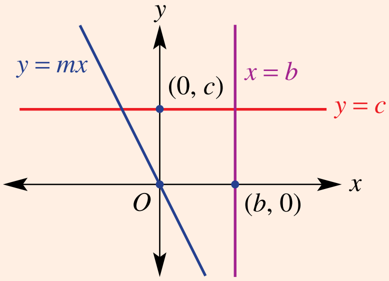
- The **intercepts** are the points where the line crosses the x-axis and the y-axis.
  - The y-intercept is the point where $x = 0$.
  - The x-intercept is the point where $y = 0$.
- The gradient of a line can be positive, negative, zero (i.e. a horizontal line) or undefined (i.e. a vertical line).
- The gradient–intercept form of a straight line is $y = mx + c$, where m is the gradient and $(0, c)$ are the coordinates of the y-intercept.



- Two points are needed to sketch most straight-line graphs.
- Special lines include those with only one axis intercept:
  - horizontal lines $y = c$
  - vertical lines $x = b$
  - lines passing through the origin $y = mx$.



BUILDING UNDERSTANDING

① Rearrange these equations into the form $y = mx + c$. Then state the gradient (m) and the y-coordinate of the y-intercept (c).

a $y + 2x = 5$ **b** $2y = 4x - 6$ **c** $x - y = 7$ **d** $-2x - 5y = 3$

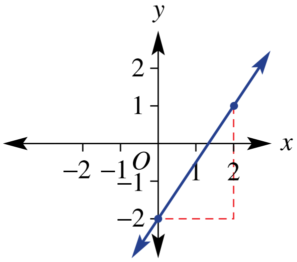
② The graph of $y = \frac{3x}{2} - 2$ is shown. Use the gradient to help answer the following.

a State the rise of the line if the run is:

i 2 **ii** 4 **iii** 7

b State the run in the line if the rise is:

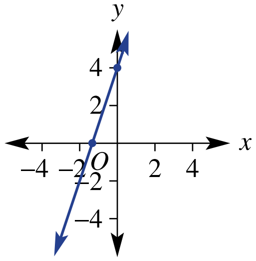
i 3 **ii** 9 **iii** 4



3 Match each of the following equations to one of the graphs shown.

a $y = 3x + 4$
b $y = -2x - 4$
c $y = 4$
d $y = x$
e $y = -2x + 4$
f $x = -3$

A



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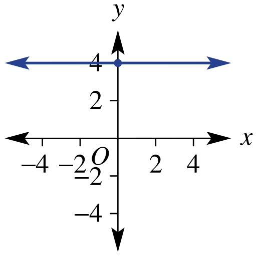
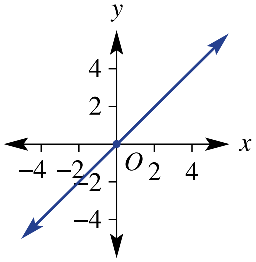


C



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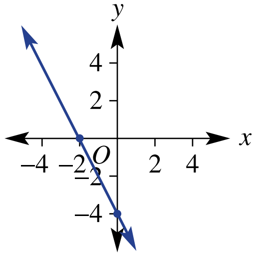
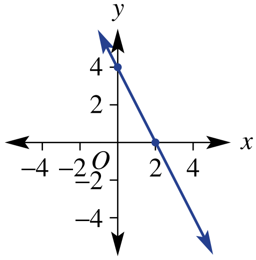
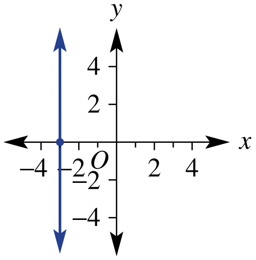


E



F



4 Find the value of the unknown in the following.

a $y = 2x - 4$ where $y = 0$
b $3x + 8y = 16$ where $x = 0$

Example 13 Deciding if a point is on a line

Decide if the point $(-2, 7)$ is on the line with the given equations.

a $y = -3x + 1$
b $2x + 2y = 1$

SOLUTION	EXPLANATION
a $y = -3x + 1$ $= -3(-2) + 1$ $= 7$ $\therefore (-2, 7)$ is on the line.	Substitute $x = -2$ into the equation of the line. Since $x = -2$ gives $y = 7$, then the point $(-2, 7)$ is on the line.
b $2x + 2y = 2(-2) + 2(7)$ $= -4 + 14$ $= 10 \neq 1$ $\therefore (-2, 7)$ is not on the line.	Since $x = -2$ and $y = 7$ does not give 1, then $(-2, 7)$ is not on the line, $2x + 2y = 1$.

Now you try

Decide if the point $(-1, 4)$ is on the line with the given equations.

a $y = -2x + 2$
b $3x + 3y = 2$

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Example 14 Sketching linear graphs using the gradient–intercept method

Find the gradient and y-intercept for these linear relations and sketch each graph.

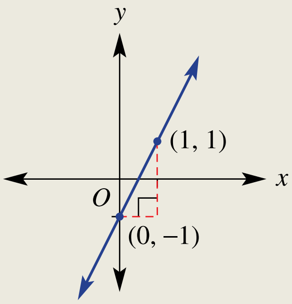
a $y = 2x - 1$ **b** $2x + 3y = 3$

SOLUTION

a $y = 2x - 1$

Gradient $= 2$

y-intercept has coordinates $(0, -1)$



EXPLANATION

In the form $y = mx + c$, the gradient is m (the coefficient of x) and $(0, c)$ are the coordinates of the y-intercept.

Start by plotting the y-intercept at $(0, -1)$ on the graph.

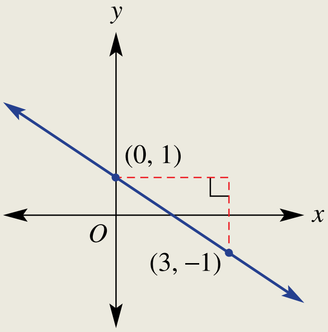
Gradient $= 2 = \frac{2}{1}$, thus rise $= 2$ and run $= 1$.

From the y-intercept move 1 unit right (run) and 2 units up (rise) to the point $(1, 1)$. Join the two points with a line.

b
$$\begin{aligned} 2x + 3y &= 3 \\ 3y &= -2x + 3 \\ y &= -\frac{2}{3}x + 1 \end{aligned}$$

Gradient $= -\frac{2}{3}$

y-intercept has coordinates $(0, 1)$



Rewrite in the form $y = mx + c$ by subtracting $2x$ from both sides and then dividing both sides by 3.

Note: $\frac{-2x}{3}$ can also be written as $-\frac{2x}{3}$, or $-\frac{2}{3}x$.

The gradient is the coefficient of x and the y-coordinate of the y-intercept is the constant term.

Start the graph by plotting the y-intercept at $(0, 1)$.

Gradient $= -\frac{2}{3}$ (run $= 3$ and fall $= 2$). From the point $(0, 1)$ move 3 units right (run) and 2 units down (fall) to $(3, -1)$.

Now you try

Find the gradient and y-intercept for these linear relations and sketch each graph.

a $y = 3x - 1$ **b** $3x + 4y = 4$

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Example 15 Sketching linear graphs using the x- and y-intercepts

Sketch the following by finding the x- and y-intercepts.

a $y = 2x - 8$ **b** $-3x - 2y = 6$

SOLUTION

a $y = 2x - 8$

y-intercept ($x = 0$):

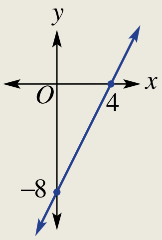
$$\begin{aligned} y &= 2(0) - 8 \\ y &= -8 \end{aligned}$$

The y-intercept is at $(0, -8)$.

x-intercept ($y = 0$):

$$\begin{aligned} 0 &= 2x - 8 \\ 8 &= 2x \\ x &= 4 \end{aligned}$$

The x-intercept is at $(4, 0)$.



b $-3x - 2y = 6$

y-intercept ($x = 0$):

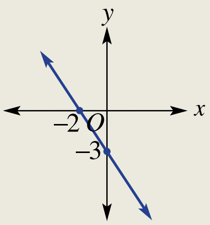
$$\begin{aligned} -3(0) - 2y &= 6 \\ -2y &= 6 \\ y &= -3 \end{aligned}$$

The y-intercept is at $(0, -3)$.

x-intercept ($y = 0$):

$$\begin{aligned} -3x - 2(0) &= 6 \\ -3x &= 6 \\ x &= -2 \end{aligned}$$

The x-intercept is at $(-2, 0)$.



EXPLANATION

The y-intercept is at $x = 0$. For $y = mx + c$, c is the y-coordinate of the y-intercept.

The x-intercept is on the x-axis, so $y = 0$. Solve the equation for x.

Plot and label the intercepts and join with a straight line.

The y-intercept is on the y-axis so substitute $x = 0$. Simplify and solve for y.

The x-intercept is on the x-axis so substitute $y = 0$. Simplify and solve for x.

Sketch by drawing a line passing through the two axes intercepts. Label the intercepts.

Now you try

Sketch the following by finding the x- and y-intercepts.

a $y = 2x - 4$ **b** $-2x - 5y = 10$

Example 16 Sketching lines with one intercept

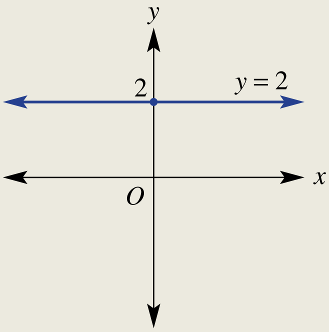
Sketch these special lines.

a $y = 2$ **b** $x = -3$ **c** $y = -\frac{1}{2}x$

SOLUTION

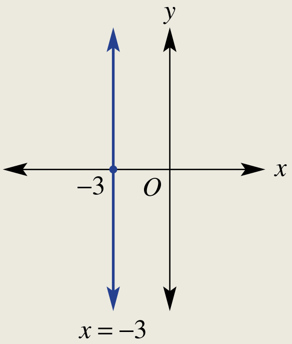
EXPLANATION

a



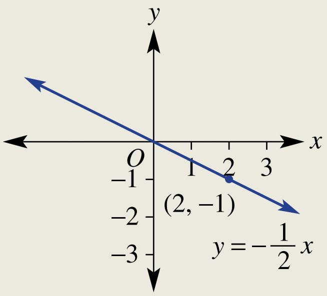
The y-coordinate of every point must be 2, hence $y = 2$ is a horizontal line passing through $(0, 2)$.

b



The x-coordinate of every point must be -3, hence $x = -3$ is a vertical line passing through $(-3, 0)$.

c



Both the x- and y-intercepts are $(0, 0)$, so the gradient can be used to find a second point.

The gradient $= -\frac{1}{2}$, hence use run $= 2$ and fall $= 1$.

Alternatively, substitute $x = 1$ to find a second point:

$$x = 1, y = -\frac{1}{2} \times (1)$$
$$= -\frac{1}{2}$$

Now you try

Sketch these special lines.

a $y = -1$ **b** $x = 2$ **c** $y = -\frac{1}{3}x$

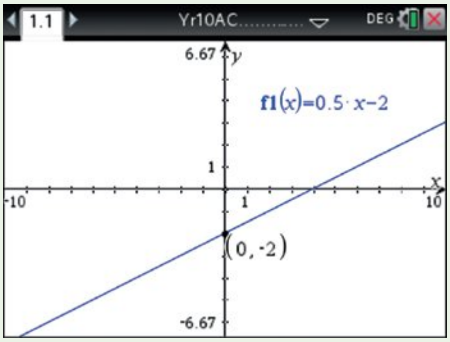
Using calculators to sketch straight lines

1 Sketch a graph of $y = 0.5x - 2$ and locate the x- and y-intercepts.

2 Construct a table of values for $y = 0.5x - 2$.

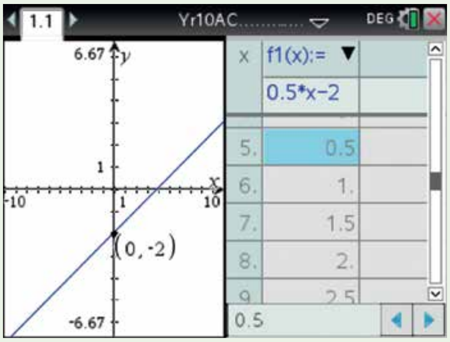
Using the TI-Nspire:

1 In a **Graphs** page, enter the rule $f1(x) = 0.5x - 2$. Use menu **>Trace>Graph Trace** and use the arrow keys to move left or right to observe intercepts. **Analyze Graph > Zero** can also be used for the x-intercept.



Hint: pressing enter will paste the intercept coordinates on the graph.

2 Press menu **>Table>Split-screen Table** to show the Table of Values.



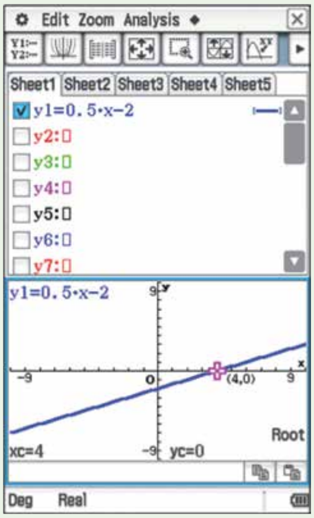
x	f1(x):=
	0.5*x−2
5.	0.5
6.	1.
7.	1.5
8.	2.
9.	2.5

0.5

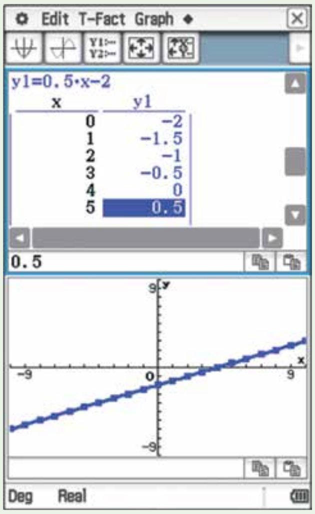
Hint: use ctrl T as a shortcut to show the Table of Values.

Using the ClassPad:

1 In the **Graph&table** application enter the rule $y1 = 0.5x - 2$ followed by **EXE**. Tap ⊞ to see the graph. Tap **Zoom**, **Quick Standard**. Tap **Analysis**, **G-Solve**, **Root** to locate the x-intercept. Tap **Analysis**, **G-Solve**, **y-intercept** to locate the y-intercept.



2 Tap ⊞ and set the table preferences to start at −10 and end at 10 with steps 1. Tap ⊞ to see the table.



x	y1
0	−2
1	−1.5
2	−1
3	−0.5
4	0
5	0.5

0.5

Exercise 1E

FLUENCY	1, 2–5($\frac{1}{2}$)	1, 2–5($\frac{1}{2}$)	1, 2–5($\frac{1}{3}$)

1 Decide if the point $(-1, 5)$ is on the line with the given equations.

Example 13a
a **i** $y = x + 4$ **ii** $y = -2x - 1$ **iii** $y = -3x + 2$

Example 13b
b **i** $2x + 2y = 5$ **ii** $3x + 3y = 12$ **iii** $-2x - y = -3$

Example 14a
2 Find the gradient and y-intercept for these linear relations and sketch a graph.

a $y = 5x - 3$ **b** $y = 2x + 3$ **c** $y = -2x - 1$
d $y = x - 4$ **e** $y = -\frac{3}{2}x + 1$ **f** $y = \frac{4}{3}x - 2$
g $y = 0.5x - 0.5$ **h** $y = 1 - x$ **i** $y = 3 + \frac{2}{3}x$

Example 14b
3 Find the gradient and y-intercept for these linear relations and sketch each graph.

a $3x + y = 12$ **b** $10x + 2y = 5$ **c** $x - y = 7$ **d** $3x - 3y = 6$
e $4x - 3y = 9$ **f** $-x - y = \frac{1}{3}$ **g** $-y - 4x = 8$ **h** $2y + x = \frac{1}{2}$

Example 15
4 Sketch the following by finding the x- and y-intercepts.

a $y = 3x - 6$ **b** $y = 2x + 4$ **c** $y = 4x + 10$
d $y = 3x - 4$ **e** $y = 7 - 2x$ **f** $y = 4 - \frac{x}{2}$
g $3x + 2y = 12$ **h** $2x + 5y = 10$ **i** $4y - 3x = 24$
j $x + 2y = 5$ **k** $3x + 4y = 7$ **l** $5y - 2x = 12$

Example 16
5 Sketch these special lines.

a $y = -4$ **b** $y = 1$ **c** $x = 2$ **d** $x = -\frac{5}{2}$
e $y = 0$ **f** $x = 0$ **g** $y = 4x$ **h** $y = -3x$
i $y = -\frac{1}{3}x$ **j** $y = \frac{5x}{2}$ **k** $x + y = 0$ **l** $4 - y = 0$

PROBLEM–SOLVING	6, 7	6–8	7–9

6 Sam is earning some money picking apples. He gets paid \$10 plus \$2 per kilogram of apples that he picks. If Sam earns \$$C$ for n kg of apples picked, complete the following.

a Write a rule for C in terms of n.
b Sketch a graph for $0 \leqslant n \leqslant 10$, labelling the endpoints.
c Use your rule to find:
  i the amount Sam earned after picking 9 kg of apples
  ii the number of kilograms of apples Sam picked if he earned \$57.

7 A 90 L tank full of water begins to leak at a rate of 1.5 litres per hour. If V litres is the volume of water in the tank after t hours, complete the following.

a Write a rule for V in terms of t.
b Sketch a graph for $0 \leqslant t \leqslant 60$, labelling the endpoints.
c Use your rule to find:
  i the volume of water after 5 hours
  ii the time taken to completely empty the tank.

8 It costs Jesse \$1600 to maintain and drive his car for 32 000 km.

a Find the cost in \$ per km.

b Write a formula for the cost, $\$C$, of driving Jesse's car for k kilometres.

c If Jesse also pays a total of \$1200 for registration and insurance, write the new formula for the cost to Jesse of owning and driving his car for k kilometres.

9 $D = 25t + 30$ is an equation for calculating the distance, D km, from home that a cyclist has travelled after t hours.

a What is the gradient of the graph of the given equation? What does it represent?

b What could the 30 represent?

c If a graph of D against t is drawn, what would be the intercept on the D-axis?

REASONING	10	10, 11	11, 12

10 A student with a poor understanding of straight-line graphs writes down some incorrect information next to each equation. Decide how the error might have been made and then correct the information.

a $y = \dfrac{2x+1}{2}$ (gradient = 2)

b $y = 0.5(x + 3)$ (y-intercept is at $(0, 3)$)

c $3x + y = 7$ (gradient = 3)

d $x - 2y = 4$ (gradient = 1)

11 Write expressions for the gradient and y-intercept coordinates of these equations.

a $ay = 3x + 7$

b $ax - y = b$

c $by = 3 - ax$

12 A straight line is written in the form $ax + by = d$. In terms of a, b and d, find:

a the coordinates of the x-intercept

b the coordinates of the y-intercept

c the gradient.

ENRICHMENT: Graphical areas	–	–	13

13 Find the area enclosed by these lines.

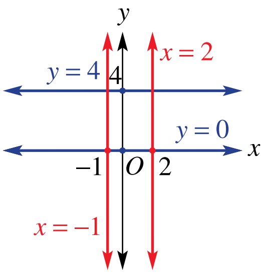
a $x = 2,\ x = -1,\ y = 0,\ y = 4$

b $x = 3,\ y = 2x,\ y = 0$

c $x = -3,\ y = -\frac{1}{2}x + 2,\ y = -2$

d $2x - 5y = -10,\ y = -2,\ x = 1$

e $y = 3x - 2,\ y = -3,\ y = 2 - x$



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1F Finding the equation of a line

LEARNING INTENTIONS

- To understand that the gradient is the same between any two points on a straight line
- To know how to find the gradient of a line using two points
- To understand the gradient–intercept form, $y = mx + c$, of a straight line equation
- To be able to find the equation of a line given two points on the line
- To know the form of the equation of horizontal and vertical lines

It is a common procedure in mathematics to find the equation (or rule) of a straight line. Once the equation of a line is determined, it can be used to find the exact coordinates of other points on the line. Mathematics such as this can be used, for example, to predict a future company share price or the water level in a dam after a period of time.

Business equipment, such as a parcel courier's van, must eventually be replaced. For tax purposes, accountants calculate annual depreciation using the straight-line method. This reduces the equipment's value by an equal amount each year.

Lesson starter: Fancy formula

Here is a proof of a rule for the equation of a straight line between any two given points.
Some of the steps are missing. See if you can fill them in.

$y = mx + c$

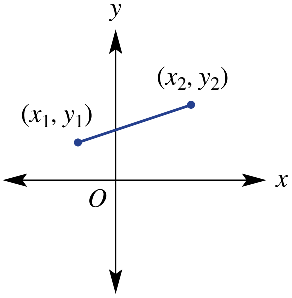
$y_1 = mx_1 + c$ (Substitute $(x, y) = (x_1, y_1)$.)

$\therefore c =$ ________

$\therefore y = mx +$ __________

$\therefore y - y_1 = m($________$)$

where $m = \frac{\qquad}{x_2 - x_1}$



KEY IDEAS

- Horizontal lines have the equation $y = c$, where c is the y-coordinate of the y-intercept.
- Vertical lines have the equation $x = k$, where k is the x-coordinate of the x-intercept.
- Given the gradient (m) and the y-intercept (c), use $y = mx + c$ to state the equation of the line.
- To find the equation of a line when given any two points, find the gradient (m), then:
  - substitute a point to find c in $y = mx + c$, or
  - use $y - y_1 = m(x - x_1)$, where $m = \frac{y_2 - y_1}{x_2 - x_1}$ and (x_1, y_1), (x_2, y_2) are points on the line.

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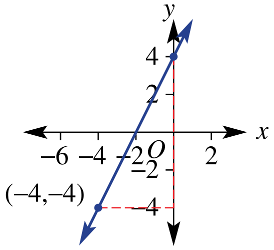
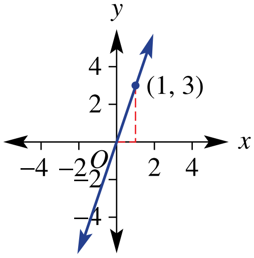
BUILDING UNDERSTANDING

① State the gradient of the following lines, using gradient $= \frac{\text{rise}}{\text{run}}$.

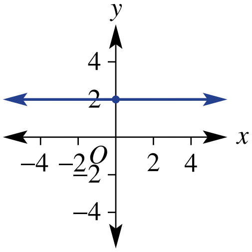
a



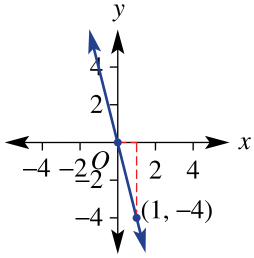
b



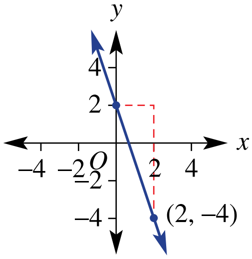
c



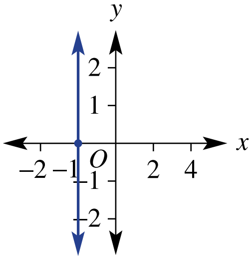
d



e



f



② Find the value of c in $y = -2x + c$ when:

a $x = 3$ and $y = 2$

b $x = -1$ and $y = -4$

c $x = \frac{5}{2}$ and $y = 7$

Example 17 Finding the gradient of a line joining two points

Find the gradient of the line joining the pair of points $(-3, 8)$ and $(5, -2)$.

SOLUTION

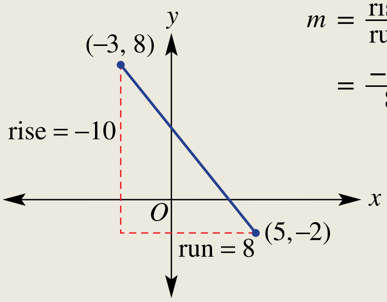
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-2 - 8}{5 - (-3)}$$
$$= -\frac{10}{8}$$
$$= -\frac{5}{4}$$

EXPLANATION

Use $(x_1, y_1) = (-3, 8)$ and $(x_2, y_2) = (5, -2)$.

Remember that $5 - (-3) = 5 + 3$.

Alternatively, plot the points and find the rise and the run.



$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{-10}{8} \text{ (from diagram)}$$

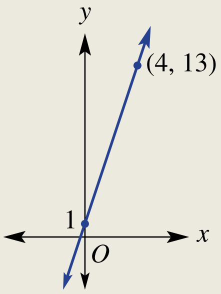
Note that run is taken from left to right.

Now you try

Find the gradient of the line joining the pair of points $(-2, 6)$ and $(3, -1)$.

Example 18 Finding the equation of a line given the *y*-intercept and a point

Find the equation of the straight line with the given *y*-intercept.



SOLUTION

The equation of a straight line is of the form $y = mx + c$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{13 - 1}{4 - 0} \\ &= \frac{12}{4} \\ &= 3 \end{aligned}$$

and $c = 1$

$\therefore y = 3x + 1$

EXPLANATION

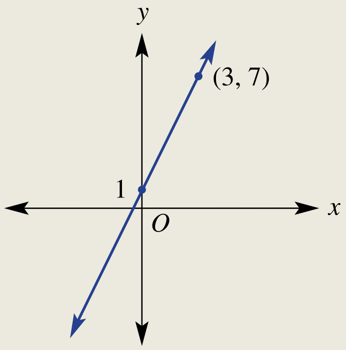
Find m using $(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = (4, 13)$, or using $m = \frac{\text{rise}}{\text{run}}$ from the graph.

The *y*-intercept is at (0, 1).

Substitute $m = 3$ and $c = 1$.

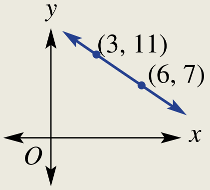
Now you try

Find the equation of the straight line with the given *y*-intercept.



Example 19 Finding the equation of a line given two points

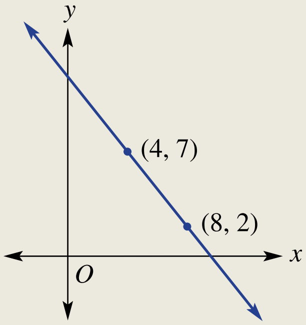
Find the equation of the straight line with the given two points.



SOLUTION	EXPLANATION
Method 1	
$y = mx + c$	
$m = \frac{y_2 - y_1}{x_2 - x_1}$	
$= \frac{7 - 11}{6 - 3}$	Use $(x_1, y_1) = (3, 11)$ and $(x_2, y_2) = (6, 7)$ in the gradient formula, or $m = \frac{\text{rise}}{\text{run}}$ from the graph where rise $= -4$ (fall).
$= -\frac{4}{3}$	
$y = -\frac{4}{3}x + c$	Substitute $m = -\frac{4}{3}$ into $y = mx + c$.
$7 = -\frac{4}{3} \times (6) + c$	Substitute the point (6, 7) or (3, 11) to find the value of c.
$7 = -8 + c$	
$15 = c$	
$\therefore y = -\frac{4}{3}x + 15$	Write the rule with both m and c.
Method 2	
$y - y_1 = m(x - x_1)$	Choose $(x_1, y_1) = (3, 11)$ or alternatively choose (6, 7).
$y - 11 = -\frac{4}{3}(x - 3)$	$m = -\frac{4}{3}$ was found using method 1.
$y = -\frac{4}{3}x + 4 + 11$	Expand brackets and make y the subject. $-\frac{4}{3} \times (-3) = 4$
$= -\frac{4}{3}x + 15$	

Now you try

Find the equation of the straight line with the given two points.



Exercise 1F

FLUENCY 1(½), 2, 3(½) 1–4(½) 1(⅓), 2–4(½)

Example 17

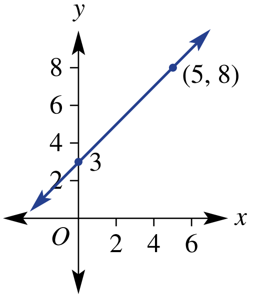
1 Determine the gradient of the line joining the following pairs of points.

a	(4, 2) and (12, 4)	**b**	(1, 4) and (3, 8)
c	(0, 2) and (2, 7)	**d**	(3, 4) and (6, 13)
e	(8, 4) and (5, 4)	**f**	(2, 7) and (4, 7)
g	(−1, 3) and (2, 0)	**h**	(−3, 2) and (−1, 7)
i	(−3, 4) and (4, −1)	**j**	(2, −3) and (2, −5)
k	(2, −3) and (−4, −12)	**l**	(−2, −5) and (−4, −2)

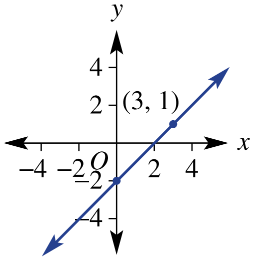
Example 18

2 Find the equation of the straight lines with the given y-intercepts.

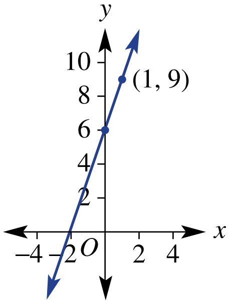
a



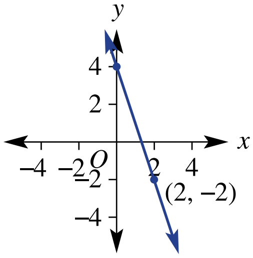
b



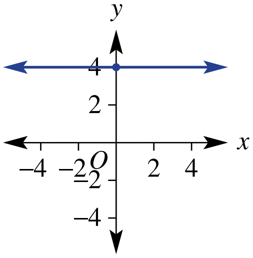
c



d



e



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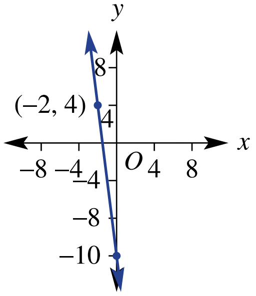
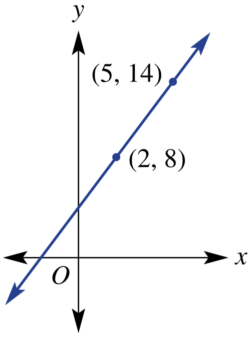


Example 19

3 Find the equation of the straight lines with the given points.

a


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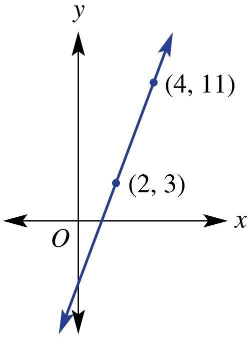

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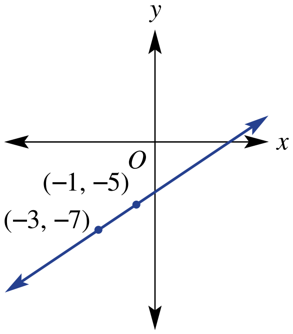

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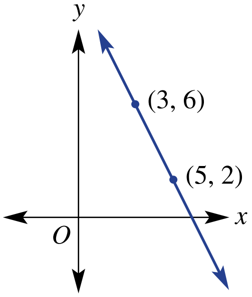
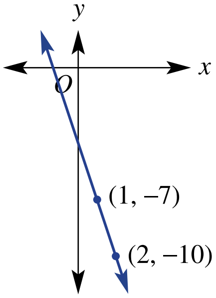


e


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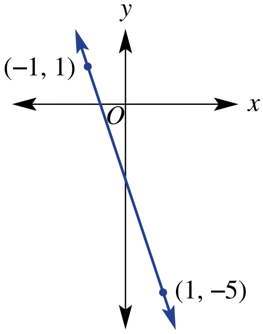


4 Given the following tables of values, determine the linear equation relating x and y in each case.

a

x	0	3
y	5	14

b

x	4	6
y	−4	−8

c

x	−1	3
y	−2	0

d

x	−2	1
y	2	−4

PROBLEM–SOLVING

5	5, 6	6, 7

5 The cost of hiring a surfboard involves an up-front fee plus an hourly rate. 3 hours of hire costs \$50 and 7 hours costs \$90.

a Sketch a graph of cost, $\$C$, for t hours of hire using the information given above.

b Find a rule linking $\$C$ in terms of t hours.

c i State the cost per hour.

ii State the up-front fee.

6 Kyle invests some money in a simple savings fund and the amount increases at a constant rate over time. He hopes to buy a boat when the investment amount reaches \$20 000.

After 3 years the amount is \$16 500 and after 6 years the amount is \$18 000.

a Find a rule linking the investment amount ($\$A$) and time ($t$ years).

b How much did Kyle invest initially (i.e. when $t = 0$)?

c How long does Kyle have to wait before he can buy his boat?

d What would be the value of the investment after $12\frac{1}{2}$ years?

7 **a** The following information applies to the filling of a flask with water, at a constant rate. In each case, find a rule for the volume, V litres, in terms of t minutes.
  - **i** Initially (i.e. at $t = 0$) the flask is empty (i.e. $V = 0$) and after 1 minute it contains 4 litres of water.
  - **ii** Initially (i.e. at $t = 0$) the flask is empty (i.e. $V = 0$) and after 3 minutes it contains 9 litres of water.
  - **iii** After 1 and 2 minutes, the flask has 2 and 3 litres of water, respectively.
  - **iv** After 1 and 2 minutes, the flask has 3.5 and 5 litres of water, respectively.

b For parts **iii** and **iv** above, find how much water was in the flask initially.

c Write your own information that would give the rule $V = -t + b$.

REASONING	8	8, 9	8, 9

8 A line joins the two points $(-1, 3)$ and $(4, -2)$.

a Calculate the gradient of the line using $m = \frac{y_2 - y_1}{x_2 - x_1}$ where $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (4, -2)$.

b Calculate the gradient of the line using $m = \frac{y_2 - y_1}{x_2 - x_1}$ where $(x_1, y_1) = (4, -2)$ and $(x_2, y_2) = (-1, 3)$.

c What conclusions can you draw from your results from parts **a** and **b** above? Give an explanation.

9 A line passes through the points $(1, 3)$ and $(4, -1)$.

a Calculate the gradient.

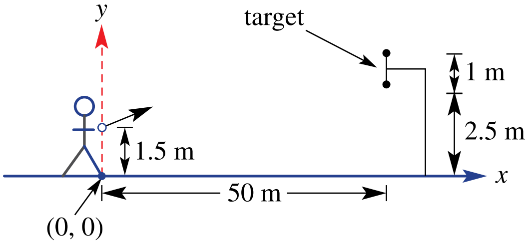
b Using $y - y_1 = m(x - x_1)$ and $(x_1, y_1) = (1, 3)$, find the rule for the line.

c Using $y - y_1 = m(x - x_1)$ and $(x_1, y_1) = (4, -1)$, find the rule for the line.

d What do you notice about your results from parts **b** and **c**? Can you explain why this is the case?

ENRICHMENT: Linear archery	–	–	10

10 An archer's target is 50 m away and is 2.5 m off the ground, as shown. Arrows are fired from a height of 1.5 m and the circular target has a diameter of 1 m.



a Find the gradient of the straight trajectory from the arrow (in firing position) to:
  - **i** the bottom of the target
  - **ii** the top of the target.

b If the position of the ground directly below the firing arrow is the point (0, 0) on a Cartesian plane, find the equation of the straight trajectory to:
  - **i** the bottom of the target
  - **ii** the top of the target.

c If $y = mx + c$ is the equation of the arrow's trajectory, what are the possible values of m if the arrow is to hit the target?

1A **1** Simplify the following.

a $15a^2b + 2ab - 6ba^2 + 8b$ **b** $-3xy \times 4x$ **c** $4(m + 5) + 3(3m - 2)$

1B **2** Solve the following equations and check your solution by substitution.

a $2x + 8 = 18$ **b** $2(3k - 4) = -17$ **c** $\frac{m + 5}{5} = 7$

1B **3** **a** Simplify $\frac{3}{4} + \frac{m}{8}$ **b** Hence, solve $\frac{3}{4} + \frac{m}{8} = 2$

1C **4** Solve the following inequalities and graph their solutions on a number line.

a $2a + 3 > 9$ **b** $8 - \frac{x}{2} \leqslant 10$

c $5m + 2 > 7m - 6$ **d** $-(a - 3) \leqslant 5(a + 3)$

1D Opt **5** Simplify the following algebraic fractions.

a $\frac{a + 4}{8} + \frac{1 - 3a}{12}$ **b** $\frac{3x + 2}{5} - \frac{x - 2}{3}$

1D Opt **6** Solve the following equations.

a $\frac{2y + 1}{2} - \frac{y + 4}{5} = 2$ **b** $\frac{2a - 3}{3} = \frac{4a + 2}{4}$

1E **7** Decide if the point $(-3, 2)$ is on the line with the given equations.

a $y = x + 2$ **b** $-2x + y = 8$

1E **8** Sketch the following linear relations. For parts **a** and **b**, use the method suggested.

a $y = -\frac{3}{2}x + 1$; use the gradient and y-intercept.

b $-2x - 3y = 6$; use the x- and y-intercepts.

c $y = 3$

d $x = -2$

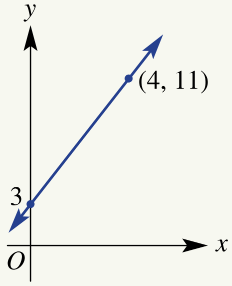
e $y = -\frac{3}{4}x$

1E **9** State the gradient and coordinates of the y-intercept of the following lines.

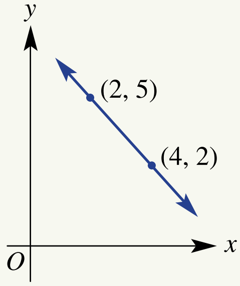
a $y = 3x - 2$ **b** $3x + 5y = 15$

1F **10** Find the equation of each straight line shown.

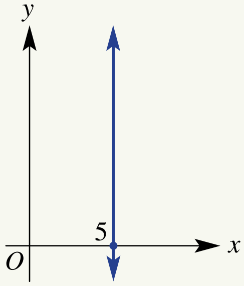
a



b



c



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1G Length and midpoint of a line segment

LEARNING INTENTIONS

- To know the meaning of the terms line segment and midpoint
- To understand that Pythagoras' theorem can be used to find the distance between two points
- To be able to find the length of a line segment (or distance between two points)
- To know how to find the midpoint of a line segment

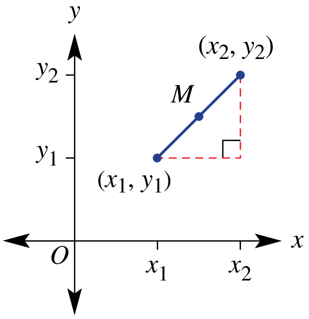
Two important features of a line segment (or line interval) are its length and midpoint. The length can be found using Pythagoras' theorem and the midpoint can be found by considering the midpoints of the horizontal and vertical components of the line segment.

Using coordinates to define locations and calculate distances are widely applied procedures, including by spatial engineers, geodetic engineers, surveyors, cartographers, navigators, geologists, archaeologists and biologists.

Lesson starter: Developing the rules

The line segment shown has endpoints (x_1, y_1) and (x_2, y_2).

- Length: Use your knowledge of Pythagoras' theorem to find the rule for the length of the segment.
- Midpoint: State the coordinates of M (the midpoint) in terms of x_1, y_1, x_2 and y_2. Give reasons for your answer.

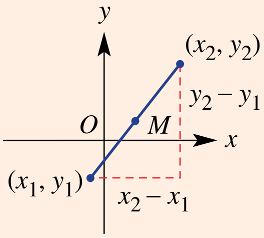


KEY IDEAS

- The **length of a line segment** d (or distance between two points (x_1, y_1) and (x_2, y_2)) is given by the rule:

  $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

  - This rule comes from Pythagoras' theorem where the distance d is the length of the hypotenuse of the right-angled triangle formed.

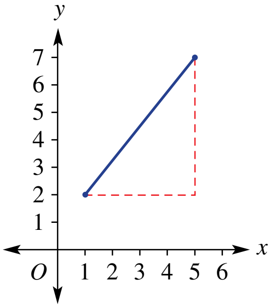


- The **midpoint** M of a line segment between (x_1, y_1) and (x_2, y_2) is given by:

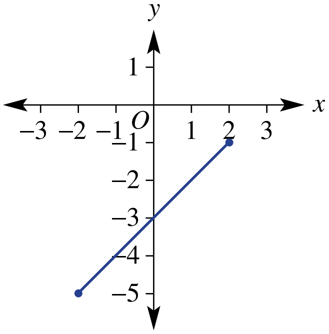
  $$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

  - This is the average of the x-coordinates and the average of the y-coordinates.

BUILDING UNDERSTANDING



1. The endpoints for the given line segment are (1, 2) and (5, 7).
   - **a** What is the horizontal distance between the two endpoints?
   - **b** What is the vertical distance between the two endpoints?
   - **c** Use Pythagoras' theorem ($c^2 = a^2 + b^2$) to find the exact length of the segment.
   - **d** State the midpoint of the segment.



2. The endpoints for the given line segment are (−2, −5) and (2, −1).
   - **a** What is the horizontal distance between the two endpoints?
   - **b** What is the vertical distance between the two endpoints?
   - **c** Use Pythagoras' theorem ($c^2 = a^2 + b^2$) to find the exact length of the segment.
   - **d** State the midpoint of the segment.

3. Simplify the following.
   - **a** $\dfrac{2+4}{2}$
   - **b** $\dfrac{3+8}{2}$
   - **c** $\dfrac{-4+10}{2}$
   - **d** $\dfrac{-6+(-2)}{2}$

Example 20 Finding the distance between two points

Find the exact distance between each pair of points.

a (0, 2) and (1, 7)

b (−3, 8) and (4, −1)

SOLUTION

a
$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1-0)^2 + (7-2)^2} \\ &= \sqrt{1^2 + 5^2} \\ &= \sqrt{26} \text{ units} \end{aligned}$$

b
$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4-(-3)^2)+(-1-8)^2} \\ &= \sqrt{7^2 + (-9)^2} \\ &= \sqrt{49 + 81} \\ &= \sqrt{130} \text{ units} \end{aligned}$$

EXPLANATION

a Let $(x_1, y_1) = (0, 2)$ and $(x_2, y_2) = (1, 7)$.
Alternatively, sketch the points and use Pythagoras' theorem, i.e. $d^2 = 1^2 + 5^2$.
Simplify and express your answer exactly, using a surd.

b Let $(x_1, y_1) = (-3, 8)$ and $(x_2, y_2) = (4, -1)$.
Alternatively, let $(x_1, y_1) = (4, -1)$ and $(x_2, y_2) = (-3, 8)$. Either way the answers will be the same.
When using Pythagoras' theorem, $d^2 = 7^2 + 9^2$.

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Now you try

Find the exact distance between each pair of points.

a (0, 3) and (1, 5)

b (−2, 7) and (3, −1)

Example 21 Finding the midpoint of a line segment joining two points

Find the midpoint of the line segment joining (−3, −5) and (2, 7).

SOLUTION

$$\begin{aligned} M &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{-3 + 2}{2}, \frac{-5 + 7}{2}\right) \\ &= \left(-\frac{1}{2}, 1\right) \end{aligned}$$

EXPLANATION

Let $(x_1, y_1) = (-3, -5)$ and $(x_2, y_2) = (2, 7)$.

This is equivalent to finding the average of the x-coordinates and the average of the y-coordinates of the two points.

Now you try

Find the midpoint of the line segment joining (−2, −6) and (3, −2).

Example 22 Using a given distance to find coordinates

Find the values of a if the distance between $(2, a)$ and $(4, 9)$ is $\sqrt{5}$.

SOLUTION

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ \sqrt{5} &= \sqrt{(4 - 2)^2 + (9 - a)^2} \\ \sqrt{5} &= \sqrt{2^2 + (9 - a)^2} \\ 5 &= 4 + (9 - a)^2 \\ 1 &= (9 - a)^2 \\ \pm 1 &= 9 - a \end{aligned}$$

So $9 - a = 1$ or $9 - a = -1$.

$\therefore a = 8$ or $a = 10$

EXPLANATION

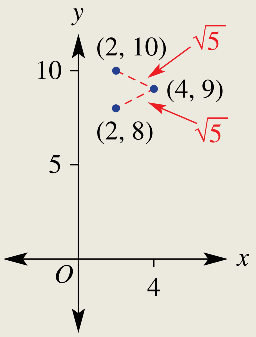
Substitute all the given information into the rule for the distance between two points.

Simplify and then square both sides to eliminate the square roots.

Subtract 4 from both sides and take the square root of each side.

Remember, if $x^2 = 1$ then $x = \pm 1$.

Solve for a. You can see there are two solutions as shown here.



Now you try

Find the values of a if the distance between $(1, a)$ and $(3, 7)$ is $\sqrt{13}$.

Exercise 1G

FLUENCY	1–2(½)	1–2(½)	1–2(⅓)

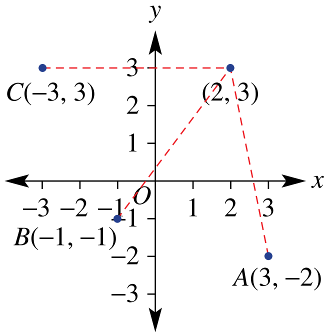
Example 20 **1** Find the exact distance between these pairs of points.

- **a** (0, 4) and (2, 9)
- **b** (0, 3) and (2, 10)
- **c** (2, 5) and (5, 7)
- **d** (−3, 8) and (1, 1)
- **e** (−2, −1) and (4, −2)
- **f** (−8, 9) and (1, −3)
- **g** (−8, −1) and (2, 0)
- **h** (−4, 6) and (8, −1)
- **i** (−10, 11) and (−4, 10)

Example 21 **2** Find the midpoint of the line segment joining the given points in Question **1**.

PROBLEM-SOLVING	3, 4–5(½)	3, 4–5(½)	4–5(½), 6

3 Which of the points A, B or C shown on these axes is closest to the point (2, 3)?



4 Find the value of a and b when:

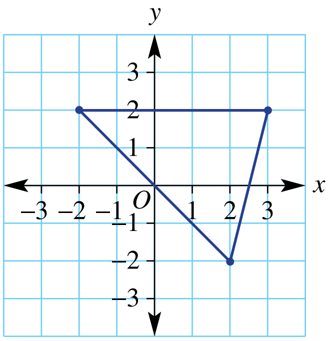
- **a** the midpoint of $(a, 3)$ and $(7, b)$ is (5, 4)
- **b** the midpoint of $(a, -1)$ and $(2, b)$ is $(-1, 2)$
- **c** the midpoint of $(-3, a)$ and $(b, 2)$ is $\left(-\frac{1}{2}, 0\right)$
- **d** the midpoint of $(-5, a)$ and $(b, -4)$ is $\left(-\frac{3}{2}, \frac{7}{2}\right)$.

Example 22 **5** Find the values of a when:

- **a** the distance between $(1, a)$ and (3, 5) is $\sqrt{8}$
- **b** the distance between $(2, a)$ and (5, 1) is $\sqrt{13}$
- **c** the distance between $(a, -1)$ and $(4, -3)$ is $\sqrt{29}$
- **d** the distance between $(-3, -5)$ and $(a, -9)$ is 5.

6 A block of land is illustrated on this simple map, which uses the ratio 1 : 100 (i.e. 1 unit represents 100 m).

- **a** Find the perimeter of the block, correct to the nearest metre.
- **b** The block is to be split up into four triangular areas by building three fences that join the three midpoints of the sides of the block. Find the perimeter of the inside triangular area.



REASONING	7	7, 8	8, 9(½)

7 A line segment has endpoints $(-2, 3)$ and $(1, -1)$.

a Find the midpoint using $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (1, -1)$.

b Find the midpoint using $(x_1, y_1) = (1, -1)$ and $(x_2, y_2) = (-2, 3)$.

c Give a reason why the answers to parts **a** and **b** are the same.

d Find the length of the segment using $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (1, -1)$.

e Find the length of the segment using $(x_1, y_1) = (1, -1)$ and $(x_2, y_2) = (-2, 3)$.

f What do you notice about your answers to parts **d** and **e**? Give an explanation for this.

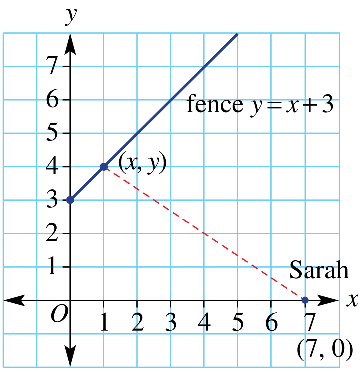
8 The distance between the points $(-2, -1)$ and $(a, 3)$ is $\sqrt{20}$. Find the values of a and use a Cartesian plane to illustrate why there are two solutions for a.

9 Find the coordinates of the point that divides the segment joining $(-2, 0)$ and $(3, 4)$ in the given ratio. Ratios are to be considered from left to right.

a $1:1$ **b** $1:2$ **c** $2:1$ **d** $4:1$ **e** $1:3$ **f** $2:3$

ENRICHMENT: Shortest distance	–	–	10

10 Sarah pinpoints her position on a map as $(7, 0)$ and wishes to hike towards a fence line that follows the path $y = x + 3$, as shown. (Note: 1 unit = 100 m).



a Using the points $(7, 0)$ and (x, y), write a rule in terms of x and y for the distance between Sarah and the fence.

b Use the equation of the fence line to write the rule in part **a** in terms of x only.

c Use your rule from part **b** to find the distance between Sarah and the fence line to the nearest metre when:

i $x = 1$ **ii** $x = 3$ **iii** $x = 2$ **iv** $x = 4$

d Which x-value from part **c** gives the shortest distance?

e Consider any point on the fence line and find the coordinates of the point such that the distance will be a minimum. Give reasons.

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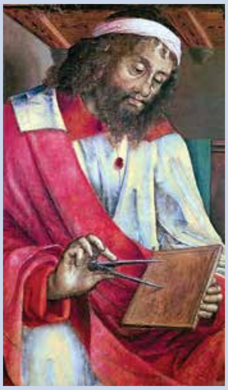
1H Parallel and perpendicular lines

LEARNING INTENTIONS

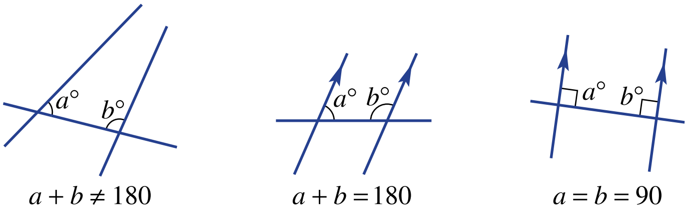
- To know what it means for lines to be parallel or perpendicular
- To know that parallel lines have the same gradient
- To know that the gradients of perpendicular lines multiply to −1
- To be able to determine if lines are parallel or perpendicular using their gradients
- To be able to find the equation of a parallel or perpendicular line given a point on the line

Euclid of Alexandria (300 BC) was a Greek mathematician and is known as the 'father of geometry'. In his texts, known as *Euclid's Elements*, his work is based on five simple axioms. The fifth axiom, called the 'Parallel Postulate', states: 'It is true that if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, intersect on that side on which are the angles less than the two right angles.'



The mathematician Euclid of Alexandria

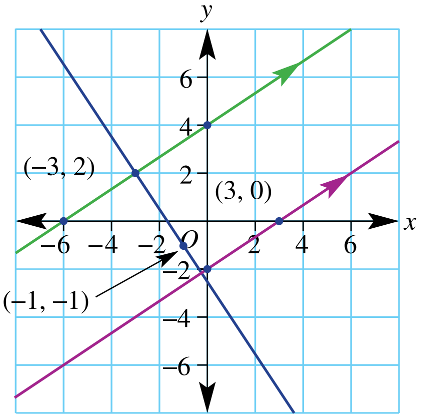
In simple terms, the Parallel Postulate says that if cointerior angles do not sum to 180°, then the two lines are not parallel. Furthermore, if the two interior angles are equal and also sum to 180°, then the third line must be perpendicular to the other two.



Lesson starter: Gradient connection

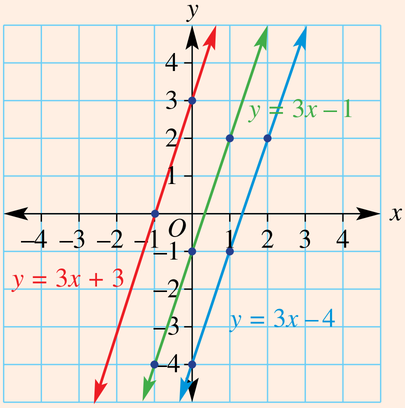
Shown here is a pair of parallel lines and a third line that is perpendicular to the other two lines.

- Find the equation of each line, using the coordinates shown on the graph.
- What is common about the rules for the two parallel lines?
- Is there any connection between the rules of the parallel lines and the perpendicular line? Can you write down this connection as a formula?

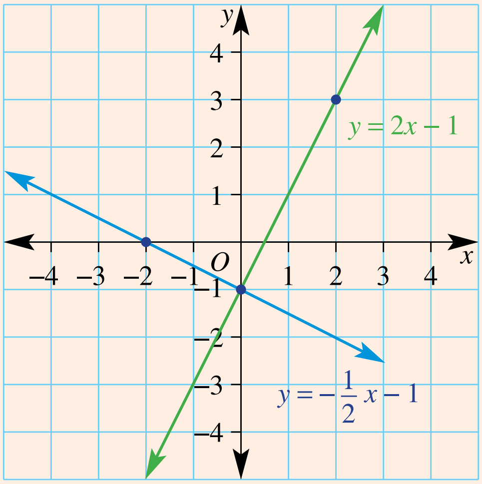


KEY IDEAS

- All **parallel lines** have the same gradient.
  For example, $y = 3x - 1$, $y = 3x + 3$ and $y = 3x - 4$ have the same gradient of 3.



- Two **perpendicular lines** (lines that meet at right angles) with gradients m_1 and m_2 satisfy the following rule:
  $m_1 \times m_2 = -1$ or $m_2 = -\frac{1}{m_1}$ (i.e. m_2 is the negative reciprocal of m_1).
  In the diagram, $m_1 \times m_2 = 2 \times \left(-\frac{1}{2}\right) = -1$.
- Equations of parallel or perpendicular lines can be found by:
  - first finding the gradient (m)
  - then substituting a point to find c in $y = mx + c$.



BUILDING UNDERSTANDING

1. What is the gradient of the line that is parallel to the graph of these equations?
   a $y = 4x - 6$ b $y = -7x - 1$ c $y = -\frac{3}{4}x + 2$ d $y = \frac{8}{7}x - \frac{1}{2}$
2. Use $m_2 = -\frac{1}{m_1}$ to find the gradient of the line that is perpendicular to the graphs of the following equations.
   a $y = 3x - 1$ b $y = -2x + 6$ c $y = \frac{7}{8}x - \frac{2}{3}$ d $y = -\frac{4}{9}x - \frac{4}{7}$
3. A line is parallel to the graph of the rule $y = 5x - 2$ and its y-intercept is at $(0, 4)$. The rule for the line is of the form $y = mx + c$.
   a State the value of m. b State the value of c. c Find the rule.
4. Answer true or false.
   a The lines $y = 2x$ and $y = 2x + 3$ are parallel.
   b The lines $y = 3x$ and $y = -3x + 2$ are perpendicular.

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Example 23 Deciding if lines are parallel or perpendicular

Decide if the graph of each pair of rules will be parallel, perpendicular or neither.

a $y = -3x - 8$ and $y = \frac{1}{3}x + 1$

b $y = \frac{1}{2}x + 2$ and $2y - x = 5$

c $3x + 2y = -5$ and $x - y = 2$

	SOLUTION		EXPLANATION
a	$y = -3x - 8,\ m = -3$	[1]	Both equations are in the form $y = mx + c$.
	$y = \frac{1}{3}x + 1,\ m = \frac{1}{3}$	[2]	
	$-3 \times \frac{1}{3} = -1$		Test: $m_1 \times m_2 = -1$.
	So the lines are perpendicular.		
b	$y = \frac{1}{2}x + 2,\ m = \frac{1}{2}$	[1]	Write both equations in the form $y = mx + c$.
	$2y - x = 5$ $2y = x + 5$ $y = \frac{1}{2}x + \frac{5}{2},\ m = \frac{1}{2}$	[2]	Both lines have a gradient of $\frac{1}{2}$, so the lines are parallel.
	So the lines are parallel.		
c	$3x + 2y = -5$ $2y = -3x - 5$ $y = -\frac{3}{2}x - \frac{5}{2},\ m = -\frac{3}{2}$	[1]	Write both equations in the form $y = mx + c$. Note: $\frac{-3x}{2} = -\frac{3x}{2} = -\frac{3}{2}x$
	$x - y = 2$ $-y = -x + 2$ $y = x - 2,\ m = 1$	[2]	Subtract x from both sides, then divide both sides by -1.
	$-\frac{3}{2} \times 1 \neq -1$		Test: $m_1 \times m_2 = -1$.
	So the lines are neither parallel nor perpendicular.		The gradients are not equal and $m_1 \times m_2 \neq -1$.

Now you try

Decide if the graph of each pair of rules will be parallel, perpendicular or neither.

a $y = -4x - 1$ and $y = \frac{1}{4}x + 3$

b $y = \frac{1}{2}x + 1$ and $2y + x = 7$

c $5x - 3y = -10$ and $5x = 3y + 2$

Example 24 Finding the equation of a parallel or perpendicular line

Find the equation of the line that is:

a parallel to $y = -2x - 7$ and passes through $(1, 9)$

b perpendicular to $y = \frac{3}{4}x - 1$ and passes through $(3, -2)$.

SOLUTION	EXPLANATION
a $y = mx + c$ $m = -2$ $y = -2x + c$	Write the general equation of a line. Since the line is parallel to $y = -2x - 7$, $m = -2$.
Substitute $(1, 9)$: $9 = -2(1) + c$ $11 = c$ $\therefore y = -2x + 11$	Substitute the given point $(1, 9)$, where $x = 1$ and $y = 9$, and solve for c.
b $y = mx + c$ $m = \dfrac{-1}{\left(\frac{3}{4}\right)}$ $= -\frac{4}{3}$ $y = -\frac{4}{3}x + c$	The perpendicular gradient is the negative reciprocal of $\frac{3}{4}$. $\dfrac{-1}{\left(\frac{3}{4}\right)} = -1 \div \frac{3}{4} = -1 \times \frac{4}{3}$.
Substitute $(3, -2)$: $-2 = -\frac{4}{3}(3) + c$ $-2 = -4 + c$ $c = 2$ $\therefore y = -\frac{4}{3}x + 2$	Substitute $(3, -2)$ and solve for c.

Now you try

Find the equation of the line that is:

a parallel to $y = -3x - 5$ and passes through $(1, 5)$

b perpendicular to $y = \frac{2}{3}x - 3$ and passes through $(2, -1)$.

Exercise 1H

FLUENCY	1–3(½)	1–3(½)	1(⅓), 2–3(½)

Example 23

1 Decide if the graph of each pair of linear rules will be parallel, perpendicular or neither.

a $y = 3x - 1$ and $y = 3x + 7$

b $y = \frac{1}{2}x - 6$ and $y = \frac{1}{2}x - 4$

c $y = -\frac{2}{3}x + 1$ and $y = \frac{2}{3}x - 3$

d $y = -4x - 2$ and $y = x - 7$

e $y = -\frac{3}{7}x - \frac{1}{2}$ and $y = \frac{7}{3}x + 2$

f $y = -8x + 4$ and $y = \frac{1}{8}x - 2$

g $2y + x = 2$ and $y = -\frac{1}{2}x - 3$

h $x - y = 4$ and $y = x + \frac{1}{2}$

i $8y + 2x = 3$ and $y = 4x + 1$

j $3x - y = 2$ and $x + 3y = 5$

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Example 24a

2 Find the equation of the line that is:

a parallel to $y = x + 3$ and passes through $(1, 5)$

b parallel to $y = -x - 5$ and passes through $(1, -7)$

c parallel to $y = -4x - 1$ and passes through $(-1, 3)$

d parallel to $y = \frac{2}{3}x + 1$ and passes through $(3, -4)$

e parallel to $y = -\frac{4}{5}x + \frac{1}{2}$ and passes through $(5, 3)$.

Example 24b

3 Find the equation of the line that is:

a perpendicular to $y = 2x + 3$ and passes through $(2, 5)$

b perpendicular to $y = -4x + 1$ and passes through $(-4, -3)$

c perpendicular to $y = \frac{2}{3}x - 4$ and passes through $(4, -1)$

d perpendicular to $y = \frac{4}{3}x + \frac{1}{2}$ and passes through $(-4, -2)$

e perpendicular to $y = -\frac{2}{7}x - \frac{3}{4}$ and passes through $(-8, 3)$.

PROBLEM–SOLVING

4(½)	4–6(½)	4–6(½), 7

4 This question involves vertical and horizontal lines. Find the equation of the line that is:

a parallel to $x = 3$ and passes through $(6, 1)$

b parallel to $x = -1$ and passes through $(0, 0)$

c parallel to $y = -3$ and passes through $(8, 11)$

d parallel to $y = 7.2$ and passes through $(1.5, 8.4)$

e perpendicular to $x = 7$ and passes through $(0, 3)$

f perpendicular to $x = -4.8$ and passes through $(2.7, -3)$

g perpendicular to $y = -\frac{3}{7}$ and passes through $\left(\frac{2}{3}, \frac{1}{2}\right)$

h perpendicular to $y = \frac{8}{13}$ and passes through $\left(-\frac{4}{11}, \frac{3}{7}\right)$.

5 Find the equation of the line that is parallel to these equations and passes through the given points.

a $y = \frac{2x - 1}{3}$, $(0, 5)$

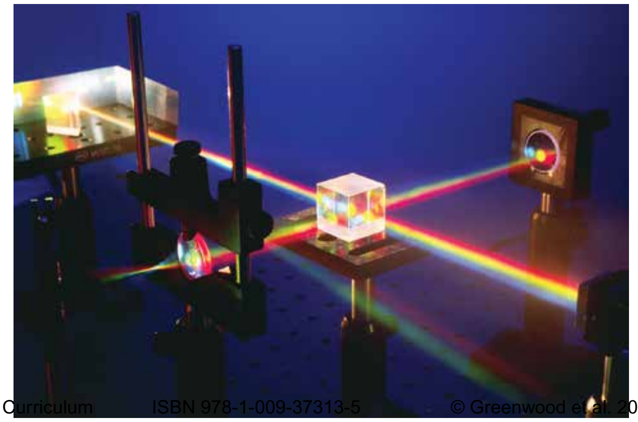
b $y = \frac{3 - 5x}{7}$, $(1, 7)$

c $3y - 2x = 3$, $(-2, 4)$

d $7x - y = -1$, $(-3, -1)$

6 Find the equation of the line that is perpendicular to the equations given in Question **5** and passes through the same given points.

7 A line with equation $3x - 2y = 12$ intersects a second line at the point where $x = 2$. If the second line is perpendicular to the first line, find where the second line cuts the x-axis.



REASONING	8	8, 9	8–10

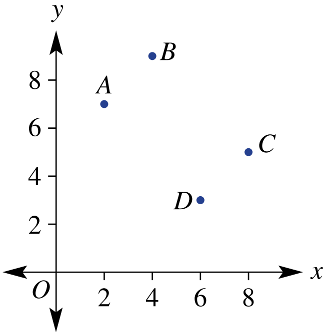
8 Find an expression for the gradient of a line if it is:
- **a** parallel to $y = mx + 8$
- **b** parallel to $ax + by = 4$
- **c** perpendicular to $y = mx - 1$
- **d** perpendicular to $ax + by = -3$.

9
- **a** Find the value of a if $y = \frac{a}{7}x + c$ is parallel to $y = 2x - 4$.
- **b** Find the value of a if $y = \left(\frac{2a+1}{3}\right)x + c$ is parallel to $y = -x - 3$.
- **c** Find the value of a if $y = \left(\frac{1-a}{2}\right)x + c$ is perpendicular to $y = \frac{1}{2}x - \frac{3}{5}$.
- **d** Find the value of a if $ay = 3x + c$ is perpendicular to $y = -\frac{3}{7}x - 1$.

10 Find the equation of a line that is:
- **a** parallel to $y = 2x + c$ and passes through (a, b)
- **b** parallel to $y = mx + c$ and passes through (a, b)
- **c** perpendicular to $y = -x + c$ and passes through (a, b)
- **d** perpendicular to $y = mx + c$ and passes through (a, b).

ENRICHMENT: Perpendicular and parallel geometry	–	–	11–13

11 A quadrilateral, $ABCD$, has vertex coordinates $A(2, 7)$, $B(4, 9)$, $C(8, 5)$ and $D(6, 3)$.
- **a** Find the gradient of these line segments.
  - **i** AB
  - **ii** BC
  - **iii** CD
  - **iv** DA
- **b** What do you notice about the gradients of opposite and adjacent sides?
- **c** What type of quadrilateral is $ABCD$?



12 The vertices of triangle ABC are $A(0, 0)$, $B(3, 4)$ and $C\left(\frac{25}{3}, 0\right)$.
- **a** Find the gradient of these line segments.
  - **i** AB
  - **ii** BC
  - **iii** CA
- **b** What type of triangle is ΔABC?
- **c** Find the perimeter of ΔABC.

13 Find the equation of the perpendicular bisector of the line segment joining $(1, 1)$ with $(3, 5)$ and find where this bisector cuts the x-axis.

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Business profit

1 Abby runs an online business making and selling Christmas stockings. Over time she has worked out that the cost to make and deliver 3 Christmas stockings is $125, while the cost to make and deliver 7 stockings is $185. The cost to produce each Christmas stocking is the same. Costs involved also include the purchase of the tools to make the stockings.

Abby is interested in exploring the relationship between her profit, costs and selling price. She wants to determine the 'break-even' point and look at how this is impacted if the selling price is adjusted.

- **a** Draw a graph of cost versus the number of stockings made using the information given and assuming a linear relationship.
- **b** From your graph, determine a rule for the cost, C dollars, of making and delivering s stockings.

Abby sells each stocking for $25.

- **c** Write a rule for the amount she would receive (revenue), R dollars, from selling s stockings and sketch its graph on the same axes, as in part **a**.
- **d** What are the coordinates of the point where the graphs intersect?
- **e** Profit is defined as revenue minus cost.
  - **i** Give a rule for the profit, P dollars, from selling s stockings.
  - **ii** Use your equation from part **i** to find how many stockings must be sold to break even.
- **f** In the lead up to Christmas each year, Abby finds that she sells on average t stockings. She considers adjusting the selling price of the stockings at this time of year.
  - **i** Determine the minimum price, p dollars, she should sell her stockings for, in terms of t, to break even.
  - **ii** Use your result from part **i** to determine the minimum selling price if $t = 5$ or if $t = 25$

Comparing speeds

2 To solve problems involving distance, speed and time we use the following well-known rule:

$$\text{distance} = \text{speed} \times \text{time}$$

We will explore how we can use this simple rule to solve common problems. These include problems where objects travel towards each other, follow behind or chase one another, and problems where speed is altered mid-journey.

- **a** Two cars travel towards each other on a 100 km stretch of road. One car travels at 80 km/h and the other at 70 km/h.
  - **i** How far does each car travel in 1 hour?
  - **ii** Complete the table below to determine how far each car has travelled after t hours.

	Speed (km/h)	Time (hours)	Distance (km)
Car A		t	
Car B		t	

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- **iii** Hence, if the cars set off at the same time, how long will it be before the cars meet (i.e. cover the 100 km between them)? Answer in minutes.
- **iv** If two cars travel at x km/h and y km/h respectively and there is d km between them, determine after how long they will meet in terms of x, y and d.

b Ed's younger brother leaves the house on his bicycle and rides at 2 km/h. Ed sets out after his brother on his bike x hours later, travelling at 7 km/h.

- **i** Use an approach like in part **a** to find a rule for the time taken for Ed to catch up to his younger brother in terms of x.
- **ii** What is this time if $x = 1$?

c Meanwhile, Sam is driving from city A to city B. After 2 hours of driving she noticed that she covered 80 km and calculated that, if she continued driving at the same speed, she would end up being 15 minutes late. She therefore increased her speed by 10 km/h and she arrived at city B 36 minutes earlier than she planned. Find the distance between cities A and B.

Crossing the road

3 Coordinate geometry provides a connection between geometry and algebra where points and lines can be explored precisely using coordinates and equations.

We will investigate the shortest path between sets of points positioned on parallel lines to find the shortest distance to cross the road.

Two parallel lines with equations $y = 2x + 2$ and $y = 2x + 12$ form the sides of a road. A chicken is positioned at (2, 6) along one of the sides of the road. Three bags of grain are positioned on the other side of the road at (0, 12), (−2, 8) and (3, 18).

a What is the shortest distance the chicken would have to cover to get to one of the bags of grain?

b If the chicken crosses the road to get directly to the closest bag of grain, give the equation of the direct line the chicken walks along.

c By considering your equation in part **b**, explain why this is the shortest possible distance the chicken could walk to cross the road.

d Using the idea from part **c**, find the distance between the parallel lines with equations $y = 3x + 1$ and $y = 3x + 11$ using the point (3, 10) on the first line.

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1I Simultaneous equations using substitution

LEARNING INTENTIONS

- To understand that a single linear equation in two variables has an infinite number of solutions
- To know that two different simultaneous linear equations (straight lines) can have 0 or 1 solution (points of intersection)
- To understand that the solution of two simultaneous equations satisfies both equations and lies on both straight line graphs
- To know how to substitute one algebraic expression for another to obtain an equation in one unknown
- To be able to solve simultaneous equations using the substitution method

When we try to find a solution to a set of equations rather than just a single equation, we say that we are solving simultaneous equations. For two linear simultaneous equations we are interested in finding the point where the graphs of the two equations meet. The point, for example, at the intersection of a company's cost equation and revenue equation is the 'break-even point'. This determines the point at which a company will start making a profit.

Simultaneous equations can solve personal finance questions such as: finding the best deal for renting a house or buying a car; the job where you will earn the most money over time; and the most profitable investment account.

Lesson starter: Give up and do the algebra

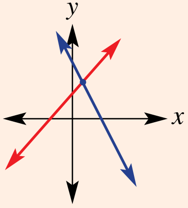
The two simultaneous equations $y = 2x - 3$ and $4x - y = 5\frac{1}{2}$ have a single solution (x, y).

- Use a guess and check (i.e. trial and error) technique to try to find the solution.
- Try a graphical technique to find the solution. Is this helpful?
- Now find the exact solution using the algebraic method of substitution.
- Which method do you prefer? Discuss.

KEY IDEAS

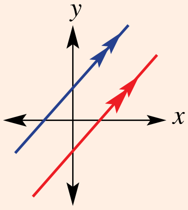
- Solving two **simultaneous equations** involves finding a solution that satisfies both equations.
  - When two straight lines are not parallel, there will be a single (**unique**) solution.

Non-parallel lines



1 point of intersection

Parallel lines (same gradient)



0 points of intersection

- The **substitution** method is usually used when at least one of the equations has one pronumeral as the subject. For example, $y = 3x + 2$ or $x = 3y - 1$.
  - By substituting one equation into the other, a single equation in terms of one pronumeral is formed and can then be solved.

    For example:

    $x + y = 8$ [1]

    $y = 3x + 4$ [2]

    $$\begin{aligned}\text{Substitute [2] in [1]: } x + (3x + 4) &= 8\\ 4x + 4 &= 8\\ \therefore x &= 1\end{aligned}$$

    Find y: $y = 3(1) + 4 = 7$

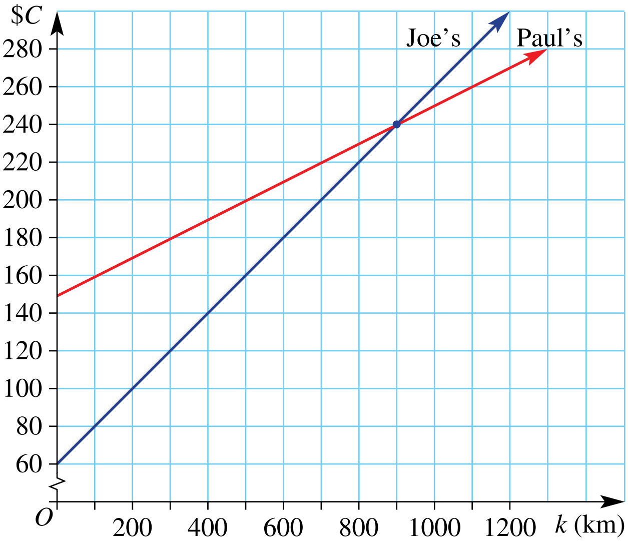
    So the solution is $x = 1$ and $y = 7$. The point $(1, 7)$ is the intersection point of the graphs of the two relations.

BUILDING UNDERSTANDING

① By substituting the given values of x and y into both equations, decide whether it is the solution to these simultaneous equations.

- **a** $x + y = 5$ and $x - y = -1$; $x = 2, y = 3$
- **b** $3x - y = 2$ and $x + 2y = 10$; $x = 2, y = 4$
- **c** $3x + y = -1$ and $x - y = 0$; $x = -1, y = 2$
- **d** $2(x + y) = -20$ and $3x - 2y = -20$; $x = -8, y = -2$

② This graph represents the rental cost, \$$C$, after k kilometres of a new car from two car rental firms called Paul's Motor Mart and Joe's Car Rental.



- **a** **i** Determine the initial rental cost from each company.
  - **ii** Find the cost per kilometre when renting from each company.
  - **iii** Find the linear equations for the total rental cost from each company.
  - **iv** Determine the number of kilometres for which the cost is the same from both rental firms.
- **b** If you had to travel 300 km, which company would you choose?
- **c** If you had \$260 to spend on travel, which firm would give you the most kilometres?

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Example 25 Solving simultaneous equations using substitution

Solve these pairs of simultaneous equations using the method of substitution.

a $2x + y = -7$ and $y = x + 2$ **b** $2x - 3y = -8$ and $y = x + 3$

SOLUTION

a $2x + y = -7$ [1]

$y = x + 2$ [2]

Substitute equation [2] into equation [1].

$$\begin{aligned} 2x + (x + 2) &= -7 \\ 3x + 2 &= -7 \\ 3x &= -9 \\ x &= -3 \end{aligned}$$

Substitute $x = -3$ into equation [2].

$$\begin{aligned} y &= (-3) + 2 \\ &= -1 \end{aligned}$$

Solution is $x = -3$, $y = -1$.

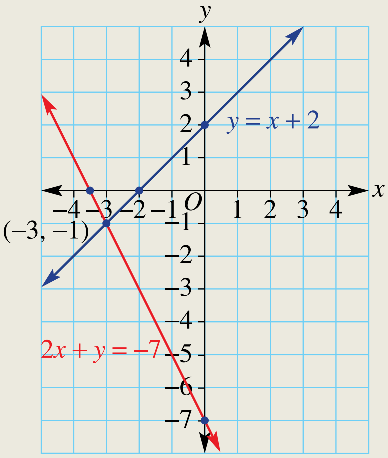
EXPLANATION

Label your equations.

Substitute equation [2] into equation [1] since equation [2] has a pronumeral as the subject.

Solve the resulting equation for x.

Substitute to find y.



b $2x - 3y = -8$ [1]

$y = x + 3$ [2]

Substitute equation [2] into equation [1].

$$\begin{aligned} 2x - 3(x + 3) &= -8 \\ 2x - 3x - 9 &= -8 \\ -x - 9 &= -8 \\ -x &= 1 \\ x &= -1 \end{aligned}$$

Substitute $x = -1$ into equation [2].

$$\begin{aligned} y &= (-1) + 3 \\ &= 2 \end{aligned}$$

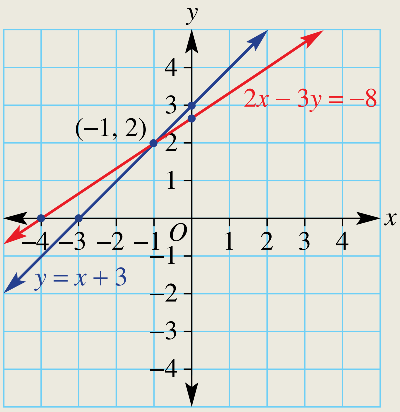
Solution is $x = -1$, $y = 2$.

Label your equations.

Substitute equation [2] into equation [1].

Expand and simplify then solve the equation for x.

Substitute $x = -1$ into either equation to find y.



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Now you try

Solve these pairs of simultaneous equations using the method of substitution.

a $3x + y = 4$ and $y = x - 4$ **b** $x - 2y = -7$ and $y = x + 4$

Example 26 Solving simultaneously with both equations in the form $y = mx + c$

Solve the pair of simultaneous equations: $y = -3x + 2$ and $y = 7x - 8$.

SOLUTION

$$\begin{aligned} y &= -3x + 2 \quad [1] \\ y &= 7x - 8 \quad [2] \end{aligned}$$

Substitute equation [2] into equation [1].

$$\begin{aligned} 7x - 8 &= -3x + 2 \\ 10x &= 10 \\ x &= 1 \end{aligned}$$

Substitute $x = 1$ into equation [1].

$$\begin{aligned} y &= -3(1) + 2 \\ &= -1 \end{aligned}$$

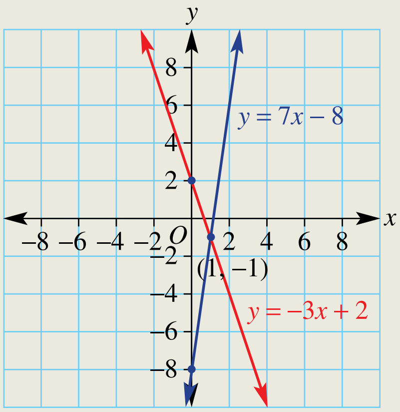
Solution is $x = 1$, $y = -1$.

EXPLANATION

Write down and label each equation.

Solve for x, then y.

Check the solution by substituting into equation [2] as well as graphically.



Now you try

Solve the pair of simultaneous equations: $y = -4x + 7$ and $y = 5x - 11$.

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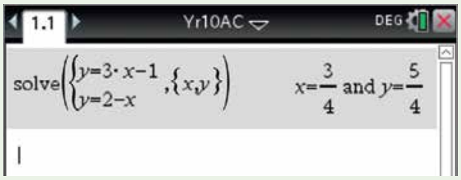
Using calculators to solve simultaneous equations

1 Solve $y = 3x - 1$ and $y = 2 - x$ simultaneously.

2 Solve $y = 3x - 1$ and $y = 2 - x$ by finding their point of intersection graphically.

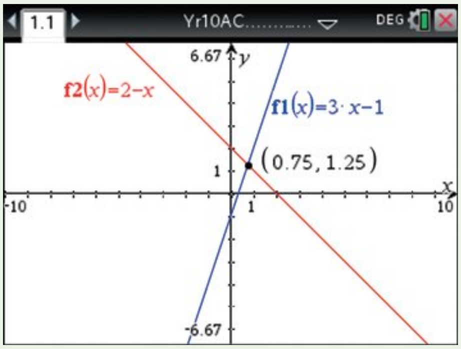
Using the TI-Nspire:

1 The equations can be solved simultaneously. Select menu **> Algebra > Solve System of Equations >** Solve System of Equations. Enter the number of equations and the variables, then type the equations as shown.



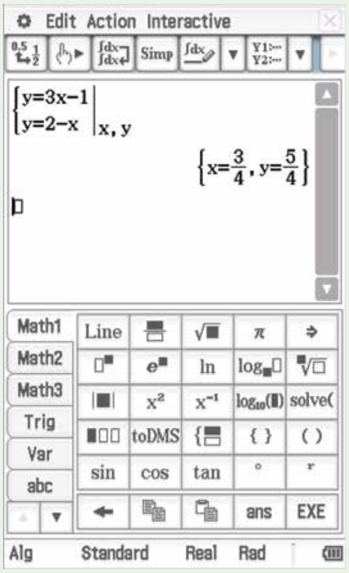
$\text{solve}\left(\begin{cases} y=3\cdot x-1 \\ y=2-x \end{cases}, \{x,y\}\right)$ $\quad x=\frac{3}{4}$ and $y=\frac{5}{4}$

2 In a **Graphs** page enter the rules $f1(x) = 3x - 1$ and $f2(x) = 2 - x$. Select menu **>Analyze Graph>Intersection** and select the lower and upper bounds containing the intersection point. Press enter to paste the coordinates to the graph.



Hint: if multiple graphs are being entered use the down arrow to enter subsequent graphs.
Hint: if the graph entry line is not showing, press tab or double click in an open area.

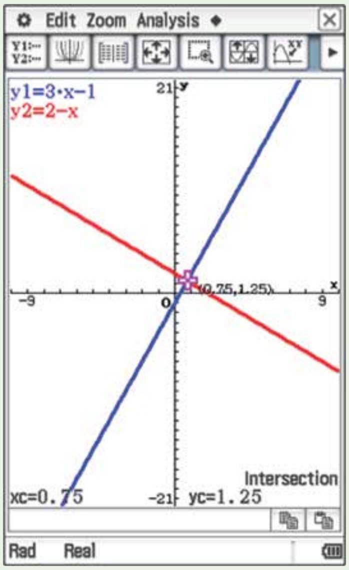
Using the ClassPad:



$\begin{cases} y=3x-1 \\ y=2-x \end{cases} \Big| x, y$

$\left\{x=\frac{3}{4}, y=\frac{5}{4}\right\}$

2 In the **Graph & Table** application enter the rules $y1 = 3x - 1$ and $y2 = 2 - x$ then tap ⊞. Tap **Zoom Quick, Quick Standard** to adjust the window. Tap **Analysis, G-Solve Intersect**.



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Exercise 1I

FLUENCY	1–2(½)	1–2(½)	1–2(⅓)

Example 25

1 Solve the following pairs of simultaneous equations, using the method of substitution. You can check your solution graphically by sketching the pair of graphs and locating the intersection point.

a $y = x + 5$ and $3x + y = 13$
b $y = x + 3$ and $6x + y = 17$
c $y = x$ and $4x + 3y = 7$
d $y = x$ and $7x + 3y = 10$
e $y = x - 1$ and $3x + 2y = 8$
f $y = x - 2$ and $3x - 2y = 7$
g $x = 2y + 3$ and $11y - 5x = -14$
h $x = 3y - 2$ and $7y - 2x = 8$
i $x = 3y - 5$ and $3y + 5x = 11$
j $x = 4y + 1$ and $2y - 3x = -23$

Example 26

2 Solve the following pairs of simultaneous equations, using the method of substitution. Check your solution graphically if you wish.

a $y = 4x + 2$ and $y = x + 8$
b $y = -2x - 3$ and $y = -x - 4$
c $x = y - 6$ and $x = -2y + 3$
d $x = -7y - 1$ and $x = -y + 11$
e $y = 4 - x$ and $y = x - 2$
f $y = 5 - 2x$ and $y = \frac{3}{2}x - 2$
g $y = 5x - 1$ and $y = \frac{11 - 3x}{2}$
h $y = 8x - 5$ and $y = \frac{5x + 13}{6}$

PROBLEM–SOLVING	3, 4	3–5	4–6

3 The salary structures for companies A and B are given by:
Company A: \$20 per hour
Company B: \$45 plus \$15 per hour

a Find a rule for \$$E$ earned for t hours for:
  i company A
  ii company B.
b Solve your two simultaneous equations from part **a**.
c **i** State the number of hours worked for which the earnings are the same for the two companies.
  ii State the amount earned when the earnings are the same for the two companies.

4 The sum of the ages of a boy and his mother is 48. If the mother is three more than twice the boy's age, find the difference in the ages of the boy and his mother.

5 The value of two cars is depreciating (i.e. decreasing) at a constant rate according to the information in this table.

Car	Initial value	Annual depreciation
Sports coupe	\$62 000	\$5000
Family sedan	\$40 000	\$3000

a Write rules for the value, \$$V$, after t years for:
  i the sports coupe
  ii the family sedan.
b Solve your two simultaneous equations from part **a**.
c **i** State the time taken for the cars to have the same value.
  ii State the value of the cars when they have the same value.

6 The perimeter of a rectangular farm is 1800 m and its length is 140 m longer than its width. Find the area of the farm.

REASONING	7(½)	7(½), 8	7(½), 8, 9

7 When two different lines have the same gradient, there will be no intersection point. Use this idea to decide if these pairs of simultaneous equations will have a solution.

a $y = 3x - 1$ and $y = 3x + 2$
b $y = 7x + 2$ and $y = 7x + 6$
c $y = 2x - 6$ and $y = -2x + 1$
d $y = -x + 7$ and $y = 2x - 1$
e $2x - y = 4$ and $y = 2x + 1$
f $7x - y = 1$ and $y = -7x + 2$
g $2y - 3x = 0$ and $y = -\frac{3x}{2} - 1$
h $\frac{x - 2y}{4} = 0$ and $y = \frac{x}{2} + 3$

8 For what value of k will these pairs of simultaneous equations have no solution?

a $y = -4x - 7$ and $y = kx + 2$
b $y = kx + 4$ and $3x - 2y = 5$
c $kx - 3y = k$ and $y = 4x + 1$

9 Solve these simple equations for x and y. Your solution should contain the pronumeral k.

a $x + y = k$ and $y = 2x$
b $x - y = k$ and $y = -x$
c $2x - y = -k$ and $y = x - 1$
d $y - 4x = 2k$ and $x = y + 1$

ENRICHMENT: Factorise to solve	–	–	10(½), 11

10 Factorisation can be used to help solve harder literal equations (i.e. equations including other pronumerals).

For example: $ax + y = b$ and $y = bx$

Substituting $y = bx$ into $ax + y = b$ gives:

$$\begin{aligned} ax + bx &= b \\ x(a + b) &= b \quad \text{(Factor out } x\text{.)} \\ x &= \frac{b}{a + b} \end{aligned}$$

Substitute to find y.

$$\therefore y = b \times \left(\frac{b}{a + b}\right) = \frac{b^2}{a + b}$$

Now solve these literal equations.

a $ax - y = b$ and $y = bx$
b $ax + by = 0$ and $y = x + 1$
c $x - by = a$ and $y = -x$
d $y = ax + b$ and $y = bx$
e $y = (a - b)x$ and $y = bx + 1$
f $ax + by = c$ and $y = ax + c$
g $\frac{x}{a} + \frac{y}{b} = 1$ and $y = ax$
h $ax + y = b$ and $y = \frac{x}{a}$

11 Make up your own literal equation like the ones in Question **10**. Solve it and then test it on a classmate.

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1J Simultaneous equations using elimination

LEARNING INTENTIONS

- To be able to identify and form equations involving a matching pair of terms
- To be able to use the process of elimination to solve simultaneous equations

The elimination method for the solution of simultaneous equations is commonly used when the equations are written in the same form. One pronumeral is eliminated using addition or subtraction and the value of the other pronumeral can then be found using one of the original equations.

Using the initial cost of machinery and the production cost per item, financial analysts working for manufacturing companies, e.g. biscuit makers, can use simultaneous equations to determine the most profitable equipment to invest in.

Lesson starter: Which operation?

Below are four sets of simultaneous equations.

- For each set discuss whether addition or subtraction would be used to eliminate one pronumeral.
- State which pronumeral might be eliminated first in each case.
- Describe how you would first deal with parts **c** and **d** so that elimination can be used.

a $x + y = 5$, $2x + y = 7$

b $-4x - 2y = -8$, $4x + 3y = 10$

c $5x - y = 1$, $3x - 2y = -5$

d $3x + 2y = -5$, $4x - 3y = -1$

KEY IDEAS

- The method of **elimination** is generally used when both equations are in the form $ax + by = d$.

  For example: $\begin{aligned} 2x - y &= 6 \\ 3x + y &= 10 \end{aligned}$ or $\begin{aligned} -5x + y &= -2 \\ 6x + 3y &= 5 \end{aligned}$

  - When there is no matching pair (as in the second example above) one or both of the equations can be multiplied by a chosen factor. This is shown in **Example 28a** and **b**.

BUILDING UNDERSTANDING

1. Find the answer.

   a $2x$ subtract $2x$ **b** $5y$ add $-5y$

   c $-2x$ add $2x$ **d** $-3y$ subtract $-3y$

2. Decide if you would add or subtract the two given terms to give a result of zero.

   a $7x, 7x$ **b** $-5y, 5y$ **c** $2y, -2y$ **d** $-7y, -7y$

3. What is the resulting equation when $2x - 3y = 4$ is multiplied on both sides by the following?

   a 2 **b** 3 **c** 4 **d** 10

Example 27 Using the elimination method with a matching pair of terms

Solve the following pair of simultaneous equations using the elimination method.
$x + y = 6$ and $3x - y = 10$

SOLUTION

$$\begin{aligned} x + y &= 6 \quad [1] \\ 3x - y &= 10 \quad [2] \\ \hline [1]+[2]: \quad 4x &= 16 \\ x &= 4 \end{aligned}$$

Substitute $x = 4$ into equation [1].

$$\begin{aligned} (4) + y &= 6 \\ \therefore y &= 2 \end{aligned}$$

Solution is $x = 4$, $y = 2$.

EXPLANATION

Label your equations to help you refer to them.

Add the two equations to eliminate y. Then solve for the remaining pronumeral x.

Substitute $x = 4$ into one of the equations to find y.

State the solution and check by substituting the solution into the original equations.

Now you try

Solve the following pair of simultaneous equations using the elimination method.
$x + y = 4$ and $5x - y = 14$

Example 28 Using the elimination method to solve simultaneous equations

Solve the following pairs of simultaneous equations using the elimination method.

a $y - 3x = 1$ and $2y + 5x = 13$

b $3x + 2y = 6$ and $5x + 3y = 11$

SOLUTION

a

$$\begin{aligned} y - 3x &= 1 \quad [1] \\ 2y + 5x &= 13 \quad [2] \\ [1]\times 2 \quad 2y - 6x &= 2 \quad [3] \\ \hline [2]-[3]: \quad 11x &= 11 \\ x &= 1 \end{aligned}$$

Substitute $x = 1$ into equation [1].

$$\begin{aligned} y - 3(1) &= 1 \\ \therefore y &= 4 \end{aligned}$$

Solution is $x = 1$, $y = 4$.

EXPLANATION

Label your equations.

Multiply equation [1] by 2 so that there is a matching pair ($2y$). Subtract equation [3] from [2]: $2y - 2y = 0$ and $5x - (-6x) = 11x$. Solve for x.

Substitute into one of the equations to find y.

State and check the solution.

Continued on next page

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b

$$\begin{aligned} 3x + 2y &= 6 \quad [1] \\ 5x + 3y &= 11 \quad [2] \\ [1]\times 3 \qquad 9x + 6y &= 18 \quad [3] \\ [2]\times 2 \qquad 10x + 6y &= 22 \quad [4] \\ [4]-[3]: \qquad x &= 4 \end{aligned}$$

Substitute $x = 4$ into equation [1].

$$\begin{aligned} 3(4) + 2y &= 6 \\ 2y &= -6 \\ y &= -3 \end{aligned}$$

Solution is $x = 4$, $y = -3$.

Multiply equation [1] by 3 and equation [2] by 2 to generate $6y$ in each equation. (Alternatively, multiply [1] by 5 and [2] by 3 to obtain matching x coefficients.)

Subtract to eliminate y.

Substitute $x = 4$ into one of the equations to find y.

State and check the solution.

Now you try

Solve the following pairs of simultaneous equations using the elimination method.

a $y - 2x = 1$ and $4y + 3x = 15$

b $5x + 2y = 11$ and $3x + 5y = -1$

Exercise 1J

FLUENCY	1–3(½)	1–3(½)	1–2(⅓), 3(½)

Example 27

1 Solve the following pairs of simultaneous equations using the elimination method.

a $x + y = 7$ and $5x - y = 5$

b $x + y = 5$ and $3x - y = 3$

c $x - y = 2$ and $2x + y = 10$

d $x - y = 0$ and $4x + y = 10$

e $3x + 4y = 7$ and $2x + 4y = 6$

f $x + 3y = 5$ and $4x + 3y = 11$

g $2x + 3y = 1$ and $2x + 5y = -1$

h $4x + y = 10$ and $4x + 4y = 16$

i $2x + 3y = 8$ and $2x - 4y = -6$

j $3x + 2y = 8$ and $3x - y = 5$

k $-3x + 2y = -4$ and $5x - 2y = 8$

l $-2x + 3y = 8$ and $-4x - 3y = -2$

Example 28a

2 Solve the following pairs of simultaneous equations using the elimination method.

a $3x + 5y = 8$ and $x - 2y = -1$

b $2x + y = 10$ and $3x - 2y = 8$

c $x + 2y = 4$ and $3x - y = 5$

d $3x - 4y = 24$ and $x - 2y = 10$

e $y - 3x = -\frac{1}{2}$ and $x + 2y = \frac{5}{2}$

f $7x - 2y = -\frac{5}{2}$ and $3x + y = -2$

Example 28b

3 Solve the following pairs of simultaneous equations using the elimination method.

a $3x + 2y = 6$ and $5x + 3y = 11$

b $3x + 2y = 5$ and $2x + 3y = 5$

c $4x - 3y = 0$ and $3x + 4y = 25$

d $2x + 3y = 10$ and $3x - 4y = -2$

e $-2y - 4x = 0$ and $3y + 2x = -2$

f $-7x + 3y = 22$ and $3x - 6y = -11$

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PROBLEM–SOLVING	4	4, 5	5, 6

4 The sum of two numbers is 1633 and their difference is 35. Find the two numbers.

5 The cost of one apple and one banana at the school canteen is \$1 and the cost of 3 apples and 2 bananas is \$2.40. Find the cost of a single banana.

6 A group of 5 adults and 3 children paid a total of \$108 for their concert tickets. Another group of 3 adults and 10 children paid \$155. Find the cost of an adult ticket and the cost of a child's ticket.

REASONING	7	7, 8(½)	8, 9

7 Describe the error made in this working and then correct the error to find the correct solution.

$$\begin{aligned}
3x - 2y &= 5 \qquad [1]\\
\underline{-4x - 2y} &\underline{= -2} \qquad [2]\\
[1]+[2]: \quad -x &= 3\\
\therefore x &= -3\\
3(-3) - 2y &= 5\\
-9 - 2y &= 5\\
-2y &= 14\\
y &= -7
\end{aligned}$$

Solution is $x = -3$ and $y = -7$.

8 Solve these literal simultaneous equations for x and y.

a $ax + y = 0$ and $ax - y = 2$

b $x - by = 4$ and $2x + by = 9$

c $ax + by = 0$ and $ax - by = -4$

d $ax + by = a$ and $ax - by = b$

e $ax + by = c$ and $bx + ay = c$

9 Explain why there is no solution to the set of equations $3x - 7y = 5$ and $3x - 7y = -4$.

ENRICHMENT: Partial fractions	–	–	10(½)

10 Writing $\dfrac{6}{(x-1)(x+1)}$ as a sum of two 'smaller' fractions $\dfrac{a}{x-1} + \dfrac{b}{x+1}$, known as partial fractions, involves a process of finding the values of a and b for which the two expressions are equal. Here is the process, which includes solving a pair of simultaneous equations.

$$\begin{aligned}
\frac{6}{(x-1)(x+1)} &= \frac{a}{x-1} + \frac{b}{x+1}\\
&= \frac{a(x+1) + b(x-1)}{(x-1)(x+1)}\\
\therefore a(x+1) + b(x-1) &= 6\\
ax + a + bx - b &= 6\\
ax + bx + a - b &= 6\\
x(a+b) + (a-b) &= 0x + 6
\end{aligned}$$

By equating coefficients: $a + b = 0$ [1]

$a - b = 6$ [2]

[1]+[2] gives $2a = 6$

$\therefore a = 3$

and so $b = -3$

$$\therefore \frac{6}{(x-1)(x+1)} = \frac{3}{x-1} - \frac{3}{x+1}$$

Use this technique to write the following as the sum of two fractions.

a $\dfrac{4}{(x-1)(x+1)}$

b $\dfrac{7}{(x+2)(2x-3)}$

c $\dfrac{-5}{(2x-1)(3x+1)}$

d $\dfrac{9x+4}{(3x-1)(x+2)}$

e $\dfrac{2x-1}{(x+3)(x-4)}$

f $\dfrac{1-x}{(2x-1)(4-x)}$

1K Further applications of simultaneous equations

LEARNING INTENTIONS

- To know the steps involved in solving a word problem with two unknowns
- To be able to form linear equations in two unknowns from a word problem
- To be able to choose an appropriate method to solve two equations simultaneously

When a problem involves two unknown variables, simultaneous equations can be used to find the solution to the problem, provided that the two pronumerals can be identified and two equations can be written from the problem description.

Simultaneous equations can be used by farmers, home gardeners, nurses and pharmacists to accurately calculate required volumes when mixing solutions of different concentrations to get a desired final concentration.

Lesson starter: 19 scores but how many goals?

Nathan heard on the news that his AFL team scored 19 times during a game and the total score was 79 points. He wondered how many goals (worth 6 points each) and how many behinds (worth 1 point each) were scored in the game. Nathan looked up simultaneous equations in his maths book and it said to follow these steps:

1. Define two variables.
2. Write two equations.
3. Solve the equations.
4. Answer the question in words.

Can you help Nathan with the four steps to find out what he wants to know?

KEY IDEAS

- When solving problems with two unknowns:
  - Define a variable for each unknown.
  - Write down two equations from the information given.
  - Solve the equations simultaneously to find the solution (using the method of substitution or elimination).
  - Interpret the solution and answer the question in words.

BUILDING UNDERSTANDING

① Let x and y be two numbers that satisfy the following statements. State two linear equations according to the information.

- **a** They sum to 16 but their difference is 2.
- **b** They sum to 7 and twice the larger number plus the smaller number is 12.
- **c** The sum of twice the first plus three times the second is 11 and the difference between four times the first and three times the second is 13.

② The perimeter of a rectangle is 56 cm. If the length, l, of the rectangle is three times its width, w, state two simultaneous equations that would allow you to solve to determine the dimensions.

③ State expressions for the following.

- **a** the cost of 5 tickets at \$$x$ each
- **b** the cost of y pizzas at \$15 each
- **c** the cost of 3 drinks at \$$d$ each and 4 pies at \$$p$ each

Example 29 Setting up and solving simultaneous equations

The sum of the ages of two children is 17 and the difference in their ages is 5. If Kara is the older sister of Ben, determine their ages.

SOLUTION	EXPLANATION
Let k be Kara's age and b be Ben's age. $k + b = 17 \quad [1]$ $k - b = 5 \quad [2]$ $[1]+[2]$: $2k = 22$ $\therefore k = 11$	Define the unknowns and use these to write two equations from the information in the question. • The sum of their ages is 17. • The difference in their ages is 5. Add the equations to eliminate b and then solve to find k.
Substitute $k = 11$ into equation [1]. $(11) + b = 17$ $b = 6$	Substitute $k = 11$ into one of the equations to find the value of b.
$\therefore$ Kara is 11 years old and Ben is 6 years old.	Answer the question in words.

Now you try

The sum of the ages of two children is 20 and the difference in their ages is 8. If Tim is the older brother of Tina, determine their ages.

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Example 30 Solving further applications of simultaneous equations involving two variables

John buys 3 daffodils and 5 petunias from the nursery and pays \$25. Julia buys 4 daffodils and 3 petunias for \$26. Determine the cost of each type of flower.

SOLUTION	EXPLANATION
Let \$$d$ be the cost of a daffodil and \$$p$ be the cost of a petunia. $3d + 5p = 25 \quad [1]$ $4d + 3p = 26 \quad [2]$	Define the unknowns and set up two equations from the question. If 1 daffodil costs d dollars then 3 will cost $3 \times d = 3d$. • 3 daffodils and 5 petunias cost \$25. • 4 daffodils and 3 petunias cost \$26.
$[1]\times 4 \quad 12d + 20p = 100 \quad [3]$ $[2]\times 3 \quad 12d + 9p = 78 \quad [4]$ $[3]-[4]\text{:} \quad 11p = 22$ $\therefore p = 2$	Multiply equation [1] by 4 and equation [2] by 3 to generate $12d$ in each equation. Subtract the equations to eliminate d and then solve for p.
Substitute $p = 2$ into equation [1]. $3d + 5(2) = 25$ $3d + 10 = 25$ $3d = 15$ $\therefore d = 5$	Substitute $p = 2$ into one of the equations to find the value of d.
A petunia costs \$2 and a daffodil costs \$5.	Answer the question in words.

Now you try

Georgie buys 2 coffees and 3 muffins for \$17 and Rick buys 4 coffees and 2 muffins for \$22 from the same shop. Determine the cost of each coffee and muffin.

Exercise 1K

FLUENCY	1–4	1, 3–5	2–5

Example 29

1 The sum of the ages of two children is 24 and the difference between their ages is 8. If Nikki is the older sister of Travis, determine their ages by setting up and solving a pair of simultaneous equations.

2 Cam is 3 years older than Lara. If their combined age is 63, determine their ages by solving an appropriate pair of equations.

Example 30

3 Luke buys 4 bolts and 6 washers for \$2.20 and Holly spends \$1.80 on 3 bolts and 5 washers at the same local hardware store. Determine the costs of a bolt and a washer.

4 It costs \$3 for children and \$7 for adults to attend a school basketball game. If 5000 people attended the game and the total takings at the door was \$25 000, determine the number of children and the number of adults that attended the game.

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5 A vanilla thickshake is \$2 more than a fruit juice. If 3 vanilla thickshakes and 5 fruit juices cost \$30, determine their individual prices.

PROBLEM–SOLVING	6, 7	6–8	7–9

6 A paddock contains ducks and sheep. There are a total of 42 heads and 96 feet in the paddock. How many ducks and how many sheep are in the paddock?

7 James has \$10 in 5-cent and 10-cent coins in his change jar and counts 157 coins in total. How many 10-cent coins does he have?

8 Connor the fruiterer sells two fruit packs.
Pack 1: 10 apples and 5 mangoes (\$12)
Pack 2: 15 apples and 4 mangoes (\$14.15)
Determine the cost of 1 apple and 5 mangoes.

9 Five years ago I was 5 times older than my son.
In 8 years' time I will be 3 times older than my son.
How old am I today?

REASONING	10	10, 11	11, 12

10 Erin goes off on a long bike ride, averaging 10 km/h. One hour later her brother Alistair starts chasing after her at 16 km/h. How long will it take Alistair to catch up to Erin? (*Hint:* Use the rule $d = s \times t$.)

11 Two ancient armies are 1 km apart and begin walking towards each other. The Vikons walk at a pace of 3 km/h and the Mohicas walk at a pace of 4 km/h. How long will they walk for before the battle begins?

12 A river is flowing downstream at a rate of 2 metres per second. Brendan, who has an average swimming speed of 3 metres per second, decides to go for a swim in the river. He dives into the river and swims downstream to a certain point, then swims back upstream to the starting point. The total time taken is 4 minutes. How far did Brendan swim downstream?

ENRICHMENT: Concentration time	–	–	13, 14

13 Molly has a bottle of 15% strength cordial and wants to make it stronger. She adds an amount of 100% strength cordial to her bottle to make a total volume of 2 litres of cordial drink. If the final strength of the drink is 25% cordial, find the amount of 100% strength cordial that Molly added.
(*Hint*: Use Concentration = Volume (cordial) ÷ Total volume.)

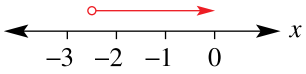
14 A fruit grower accidentally made a 5% strength chemical mixture to spray his grape vines. The strength of spray should be 8%. He then adds pure chemical until the strength reaches 8% by which time the volume is 350 litres. How much pure chemical did he have to add?

1L Regions on the Cartesian plane

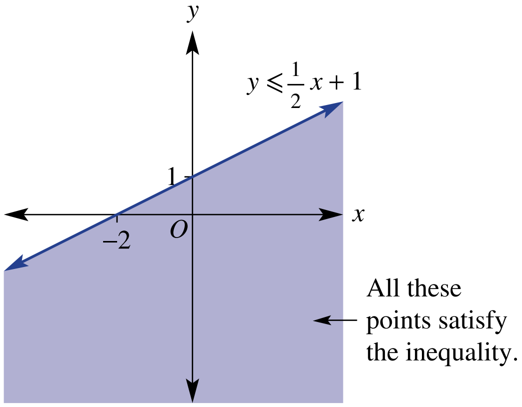
LEARNING INTENTIONS

- To know that a linear inequality can be represented as a region on the Cartesian plane
- To know how to determine which side of a line to shade to sketch a region
- To understand that if a pair of coordinates satisfy an inequality then the point is in the required region
- To be able to find the intersecting region of two or more linear inequalities

You will remember that an inequality is a mathematical statement that contains one of these symbols: $<$, $\leqslant$, $>$ or $\geqslant$. The linear inequality with one pronumeral, for example $2x - 5 > -10$, has the solution $x > -2.5$.



Linear inequalities can also have two variables: $2x - 3y \geqslant 5$ and $y < 3 - x$ are two examples. The solutions to such inequalities will be an infinite set of points on a plane sitting on one side of a line. This region is also called a half plane.



Operations research analysts use half-plane graph calculations to optimise profit within certain limitations. For example, for airline companies to find the most economical combination of flight routes, seat pricing and pilot scheduling that aligns with customer demand.

Lesson starter: Which side do I shade?

You are asked to shade all the points on a graph that satisfy the inequality $4x - 3y \geqslant 12$.

- First, graph the equation $4x - 3y = 12$.
- Substitute the point $(-2, 3)$ into the inequality $4x - 3y \geqslant 12$. Does the point satisfy the inequality? Plot the point on your graph.
- Now test these points:

  a $(3, -2)$ **b** $(3, -1)$ **c** $(3, 0)$ **d** $(3, 1)$
- Can you now decide which side of the line is to be shaded to represent all the solutions to the inequality? Should the line itself be included in the solution?

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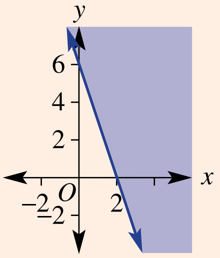
KEY IDEAS

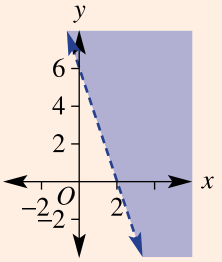
- The solution to a linear inequality with two variables is illustrated using a shaded region of the Cartesian plane.

- When y is the subject of the inequality, follow these simple rules.
  - $y \geqslant mx + c$ Draw a solid line (as it is included in the region) and shade above.
  - $y > mx + c$ Draw a broken line (as it is not included in the region) and shade above.
  - $y \leqslant mx + c$ Draw a solid line (as it is included in the region) and shade below.
  - $y < mx + c$ Draw a broken line (as it is not included in the region) and shade below.

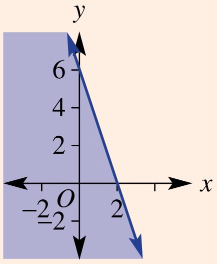
  Here are examples of each.



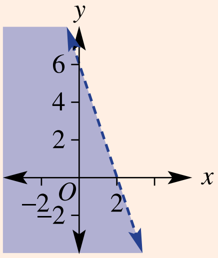
$y \geqslant -3x + 6$



$y > -3x + 6$

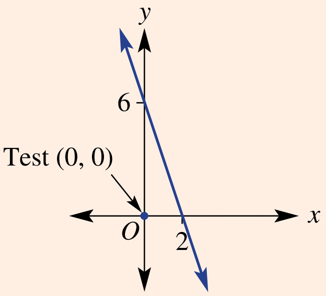


$y \leqslant -3x + 6$



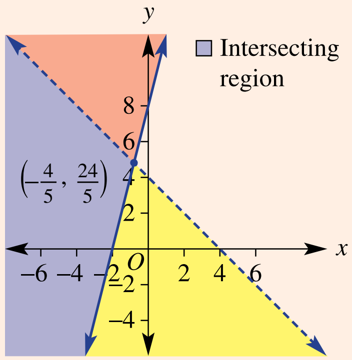
$y < -3x + 6$

- If the equation is of the form $ax + by = d$, it is usually simpler to test a point, for example (0, 0), to see which side of the line is to be included in the region.



- When two or more linear inequalities are sketched on the same set of axes, the regions overlap and form an **intersecting region**. The set of points inside the intersecting region will be the solution to the simultaneous inequalities.

  For example, $y \geqslant 4x + 8$ 🟧
  $y < -x + 4$ 🟨



  - To help define the intersecting region correctly, you should determine and label the point of intersection.

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BUILDING UNDERSTANDING

1 Substitute the point (0, 0) into these inequalities to decide if the point satisfies the inequality; i.e. is the inequality true for $x = 0$ and $y = 0$?

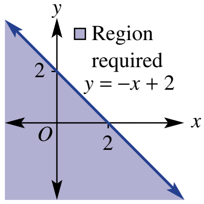
a $y < 3x - 1$ **b** $y > -\frac{x}{2} - 3$ **c** $y \geqslant 1 - 7x$

d $3x - 2y < -1$ **e** $x - y > 0$ **f** $2x - 3y \leqslant 0$

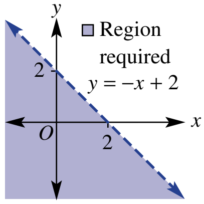
2 Match the rules with the graphs (**A**, **B** and **C**) below.

a $x + y < 2$ **b** $y \geqslant -x + 2$ **c** $y \leqslant -x + 2$

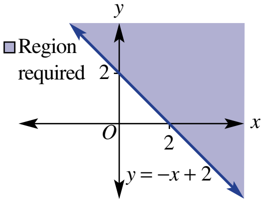
A



B



C



3 **a** Sketch the vertical line $x = -1$ and the horizontal line $y = 4$ on the same set of axes.

b Shade the region $x \geqslant -1$ (i.e. all points with an x-coordinate greater than or equal to -1).

c Shade the region $y \leqslant 4$ (i.e. all points with a y-coordinate less than or equal to 4).

d Now use a different colour to shade all the points that satisfy both $x \geqslant -1$ and $y \leqslant 4$ simultaneously.

Example 31 Sketching regions on the Cartesian plane

Sketch the region for the following linear inequalities.

a $y > 1.5x - 3$ **b** $y + 2x \leqslant 4$

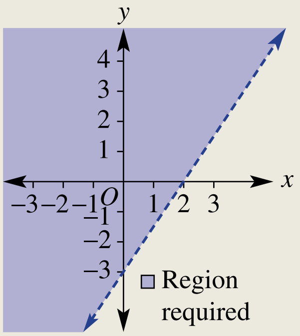
SOLUTION	EXPLANATION
a $y = 1.5x - 3$	First, sketch $y = 1.5x - 3$ by finding the x- and y-intercepts.

y-intercept ($x = 0$):

$y = -3$

x-intercept ($y = 0$):

$$\begin{aligned} 0 &= 1.5x - 3 \\ 1.5x &= 3 \\ \therefore x &= 2 \end{aligned}$$



Sketch a dotted line (since the sign is > not ⩾) joining the intercepts and shade above the line, since y is greater than $1.5x - 3$.

b $y + 2x = 4$

y-intercept ($x = 0$):

$y = 4$

x-intercept ($y = 0$):

$2x = 4$

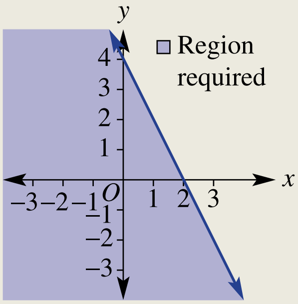
$\therefore x = 2$

Shading: Test (0, 0).

$0 + 2(0) \leqslant 4$

$0 \leqslant 4$ (True)

$\therefore$ (0, 0) is included.



First, sketch $y + 2x = 4$ by finding the x- and y-intercepts.

Decide which side to shade by testing the point (0, 0); i.e. substitute $x = 0$ and $y = 0$. Since $0 \leqslant 4$, the point (0, 0) should sit inside the shaded region.

Sketch a solid line since the inequality sign is $\leqslant$, and shade the region that includes (0, 0).

Now you try

Sketch the half planes for the following linear inequalities.

a $y > 2.5x - 5$

b $y + 3x \leqslant 6$

Example 32 Finding the intersecting region

Sketch both the inequalities $4x + y \leqslant 12$ and $3x - 2y < -2$ on the same set of axes, show the region of intersection and find the point of intersection of the two lines.

SOLUTION

$4x + y = 12$

y-intercept ($x = 0$):

$y = 12$

x-intercept ($y = 0$):

$4x = 12$

$\therefore x = 3$

Shading: Test (0, 0).

$4(0) + 0 \leqslant 12$

$0 \leqslant 12$ (True)

So (0, 0) is included.

EXPLANATION

First sketch $4x + y = 12$ by finding the x- and y-intercepts.

Test (0, 0) to see if it is in the included region.

Continued on next page

$3x - 2y = -2$

y-intercept ($x = 0$):

$-2y = -2$

$\therefore y = 1$

x-intercept ($y = 0$):

$3x = -2$

$\therefore x = -\frac{2}{3}$

Sketch $3x - 2y = -2$ by finding x- and y-intercepts.

Shading: Test (0, 0).

$3(0) + 2(0) < -2$

$0 < -2$ (False)

So (0, 0) is not included.

Test (0, 0) to see if it is in the included region.

Point of intersection:

$$\begin{array}{lrcll} & 4x + y &=& 12 & [1] \\ [1]\times 2 & 3x - 2y &=& -2 & [2] \\ [2]+[3]: & \underline{8x + 2y} &\underline{=}& \underline{24} & [3] \\ & 11x &=& 22 & \\ & x &=& 2 & \end{array}$$

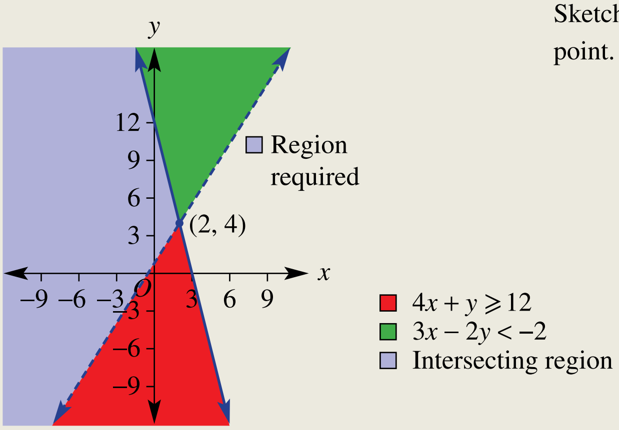
Substitute $x = 2$ into equation [1].

$4(2) + y = 12$

$y = 4$

The point of intersection is (2, 4).

Find the point of intersection by solving the equations simultaneously using the method of elimination.



Sketch both regions and label the intersection point. Also label the intersecting region.

Now you try

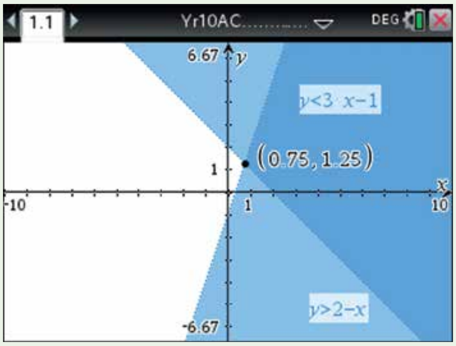
Sketch both the inequalities $3x + y \leqslant 6$ and $2x - 3y < -7$ on the same set of axes. Show the region of intersection and find the point of intersection of the two lines.

Using calculators to find the intersecting region

Sketch the intersecting region of $y < 3x - 1$ and $y > 2 - x$.

Using the TI-Nspire:

In a **Graphs** page press del and select the required inequality from the list and edit $f1(x)$ to $y < 3x - 1$ and $f2(x)$ to $y > 2 - x$.



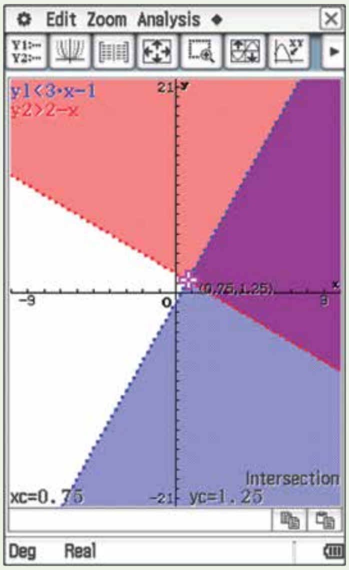
Find the intersection point as seen in 1I.

Hint: if multiple graphs are being entered use the down arrow to enter subsequent graphs.

Hint: if the graph entry line is not showing, press tab or double click in an open area.

Using the ClassPad:

Tap y1: y2: and clear all functions. With the cursor in $y1$ tap y=, select y<, enter the rule $3x - 1$ and press **EXE**. With the cursor in $y2$ tap y=, select y>, enter the rule $2 - x$ and press **EXE**. Tap .



Exercise 1L

FLUENCY	1–2(½), 3, 4(½)	1–4(½)	1–2(½), 4(½)

Example 31a

1 Sketch the region for the following linear inequalities.

a $y \geqslant x + 4$ **b** $y < 3x - 6$ **c** $y > 2x - 8$ **d** $y \leqslant 3x - 5$
e $y < 2 - 4x$ **f** $y \leqslant 2x + 7$ **g** $y < 4x$ **h** $y > 6 - 3x$
i $y \leqslant -x$ **j** $x > 3$ **k** $x < -2$ **l** $y \geqslant 2$

2 Decide whether the following points are in the region defined by $2x - 3y > 8$.

a $(5, 0)$ **b** $(2.5, -1)$ **c** $(0, -1)$ **d** $(2, -5)$

3 Decide if $(0, 0)$ lies in the region defined by the following.

a $x + y < 4$ **b** $x - 2y > -2$ **c** $3x - 2y < -4$ **d** $x - 5y < 2$

Example 31b

4 Sketch the region for the following linear inequalities.

a $x + 3y < 9$ **b** $3x - y \geqslant 3$ **c** $4x + 2y \geqslant 8$ **d** $2x - 3y > 18$
e $-2x + y \leqslant 5$ **f** $-2x + 4y \leqslant 6$ **g** $2x + 5y > -10$ **h** $4x + 9y < -36$

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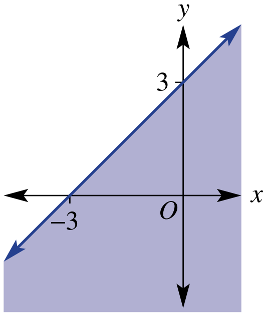
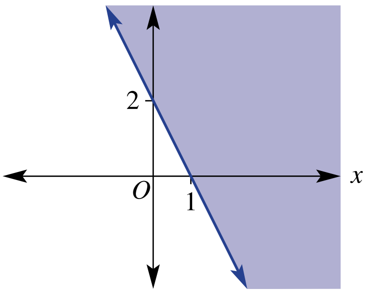
PROBLEM–SOLVING			
	5	5, 6(½)	5–6(½)

5 Write down the inequalities that give these regions.

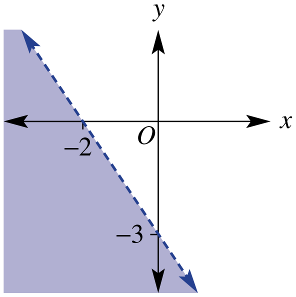
a



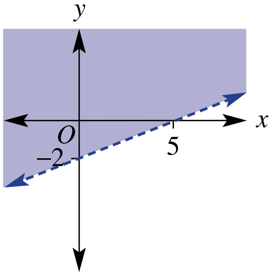
b



c



d



Example 32

6 Sketch both inequalities on the same set of axes, shade the region of intersection and find the point of intersection of the two lines.

a $x + 2y \geqslant 4$
$2x + 2y < 8$

b $3x + 4y \leqslant 12$
$3x + y > 3$

c $2x - 3y > 6$
$y < x - 2$

d $3x - 5y \leqslant 15$
$y - 3x > -3$

e $y \geqslant -x + 4$
$2x + 3y \geqslant 6$

f $2y - x \leqslant 5$
$y < 10 - x$

g $3x + 2y \leqslant 18$
$4y - x < 8$

h $2y \geqslant 5 + x$
$y < 6 - 3x$

REASONING			
	7(½)	7(½)	7, 8

7 Sketch the following systems of inequalities on the same axes. Show the intersecting region and label the points of intersection. The result should be a triangle in each case.

a $x \geqslant 0$
$y \geqslant 0$
$3x + 6y \leqslant 6$

b $x \geqslant 0$
$y \leqslant 0$
$2x - y \leqslant 4$

c $x \geqslant 0$
$5x + 2y \leqslant 30$
$4y - x \geqslant 16$

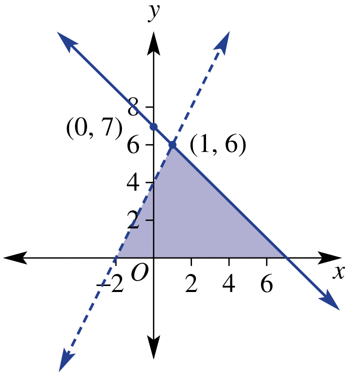
d $x < 2$
$y < 3$
$2x + 5y > 10$

e $x \leqslant 0$
$y < x + 7$
$2x + 3y \geqslant -6$

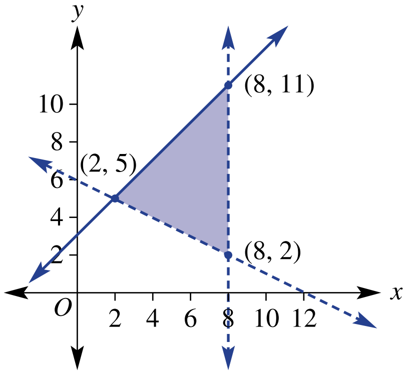
f $x + y \leqslant 9$
$2y - x \geqslant 6$
$3x + y \geqslant -2$

8 Determine the original inequalities that would give the following regions of intersection.

a



b



ENRICHMENT: Areas of regions – – 9, 10

9 Find the area of the triangles formed in Question **7** parts **a** to **d**.

10 **a** Find the exact area bound by:

i $x < 0$
$y > 0$
$x + 2y < 6$
$x - y > -7$

ii $y < 7$
$x + y > 5$
$3x - 2y < 14$

b Make up your own set of inequalities that gives an area of 6 square units.

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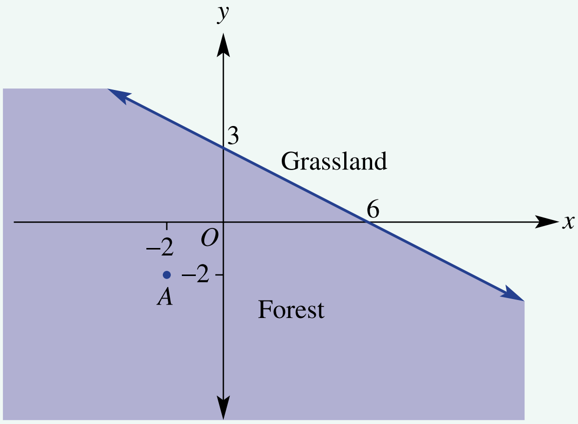
Daria's forest exit

Daria is currently located in a forest and aims to walk to a clear grassland area. If her map is placed on a Cartesian plane, then Daria is at point $A(-2, -2)$ and the edge of the forest is modelled by the line $y = -\frac{1}{2}x + 3$ as shown. All units are in kilometres.



Present a report for the following tasks and ensure that you show clear mathematical workings and explanations where appropriate.

Preliminary task

a If Daria walks on a path modelled by $y = x$ find:
   i at what point she will come to the edge of the forest
   ii the distance walked to get to the edge of the forest.

b Repeat part **a** for walking paths modelled by:
   i $x = -2$
   ii $y = 3x + 4$

c If the walking path is modelled by the rule $y = \frac{1}{2}x + c$, find the value of c.

Modelling task

Formulate

a The problem is to find the path so that Daria reaches the forest edge walking the minimum possible distance. Write down all the relevant information that will help solve this problem with the aid of a diagram.

Solve

b Investigate at least 4 possible paths for Daria of the form $y = mx + c$ by choosing values of m and c. For each path:
   i determine where Daria intersects the edge of the forest
   ii calculate how far she walks
   iii use a graph to illustrate your choices showing key features (use technology where appropriate).

Evaluate and verify

c Compare your choices of m and c which describe Daria's path and determine the values of m and c so that she reaches the forest edge walking the minimum distance.

d Determine the relationship between the slope of shortest path and the slope of the forest boundary.

Communicate

e Summarise your results and describe any key findings.

Extension questions

a If Daria's position (A) was altered, explore how this would impact your choices of m and c.

b If the forest boundary is modelled by a different straight-line equation, examine how this would impact on your conclusions regarding the minimum distance.

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Buying tickets with inequalities

Key technology: Graphing software

As an event organiser you have a \$300 budget to buy tickets for a number of students who will attend either of the following:

- Gallery tour at \$20 each
- Museum tour at \$15 each

Let:

- x be the number of students who go on the gallery tour
- y be the number of students who go on the museum tour
- T be the total number of students who can attend either tour

1 Getting started

a What would be the total cost of:

- **i** 9 gallery and 5 museum tickets?
- **ii** 8 gallery and 11 museum tickets?

b Given the \$300 budget limit decide if the following combinations of tickets are possible.

- **i** 10 gallery and 10 museum tickets
- **ii** 6 gallery and 9 museum tickets

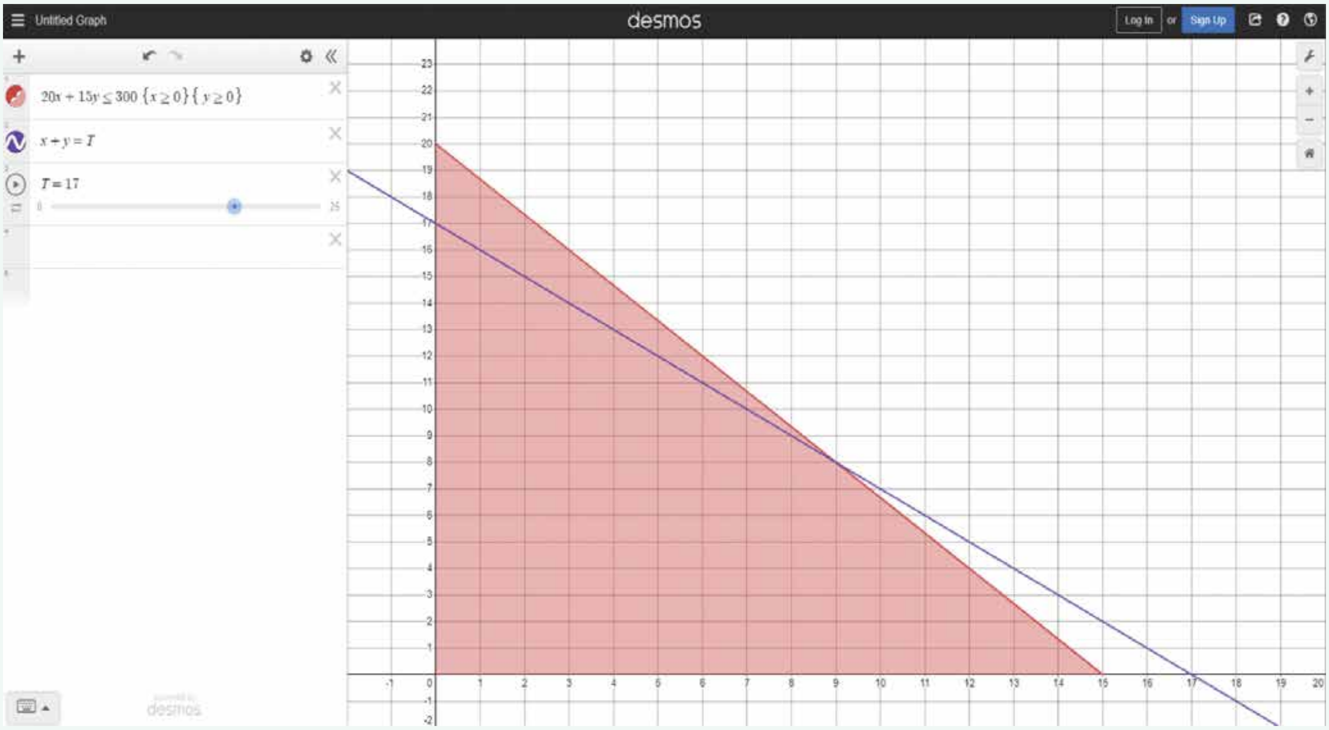
c Using the variables x and y:

- **i** give an expression for the total cost of x gallery and y museum tickets
- **ii** write an inequality using the fact that there is a \$300 budget limit
- **iii** write an equation connecting T with the number of gallery and museum tickets.

2 Using technology

a Use graphing software to show the inequality for question **1 c ii** above with the added natural conditions $x \geqslant 0$ and $y \geqslant 0$.

b Also sketch on the same set of axes the graph of $T = x + y$. Add a slider for T.



$20x + 15y \leq 300 \{x \geq 0\}\{y \geq 0\}$

$x + y = T$

$T = 17$

c Consider the point $(8, 7)$ representing the purchase of 8 gallery and 7 museum tickets. Note that this point sits inside the shaded region meaning that the \$300 budget limit is satisfied.
  i Drag the T slider so that the graph of $T = x + y$ passes through this point.
  ii What is the total number of student tickets represented by this point?
  iii What is the total cost of the tickets represented by this point?

d Drag the T slider to maximise the total number of tickets that could be purchased using the \$300 budget limit. What are the coordinates of the point where this occurs?

3 Applying an algorithm

A new condition from the school is that the number of tickets purchased for either the gallery or museum tours must be at least 4.

a Apply the following algorithm to test key points on the graph.
  - Step 1: Choose a point in the shaded region that also satisfies the new condition.
  - Step 2: Find the total cost of the tickets represented by your chosen point.
  - Step 3: Drag the slider so that the graph of $T = x + y$ passes through this point and note the total number of students T.
  - Step 4: Note if this value of T is higher than any other value of T found using the algorithm.
  - Repeat from step 1 until you have found the maximum total number of students T.

b State the maximum number of students that can participate in the tours and the number of students attending each tour. Also state the cost of this particular combination.

4 Extension

Explore the effect of adjusting either the ticket prices or the overall budget. Each time, find the maximum number of students that can participate in the tours and the number of students attending each tour. Also state the cost of each particular combination.

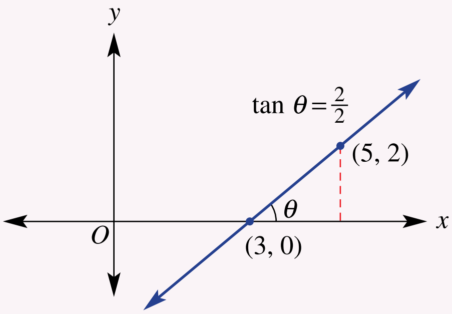
Angles between lines

Consider what happens when we pick up one end of a classroom desk and tilt the desk so that its surface has a non-zero gradient. The more we raise the desk the greater the slope of the surface of the desk – the gradient of the desk is changing as we raise it higher above the ground. We are also changing the angle that the desk makes with the horizontal as we do this. Here we will investigate the relationship between this angle and the gradient.

Angle and gradient

a Find the gradient of the line joining these points.
   i $(3, 0)$ and $(5, 2)$ **ii** $(2, 4)$ and $(5, 8)$ **iii** $(-1, 2)$ and $(4, 4)$

b Using your knowledge of trigonometry from Year 9, create a right-angled triangle using each pair of points in part **a**. Write a trigonometric ratio for the angle θ as shown. The first one is shown here.



c What do you notice about your trigonometric ratio and the gradient of each line?

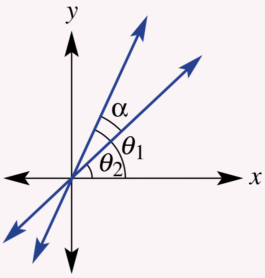
d Hence, write a statement defining the gradient (m) of a line in terms of the angle (θ) the line makes with the positive x-axis. i.e. Complete the rule $m =$ _____

e Using your rule from part **d**, determine the gradient of a line that makes the following angles with the x-axis. Give your answer to one decimal place.
   i $35°$ **ii** $54°$ **iii** $83°$

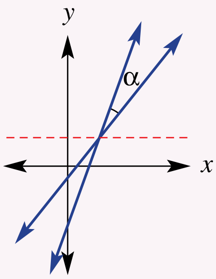
f Find the equation of a line that passes through the point $(-2, 5)$ and makes an angle of $45°$ with the positive part of the x-axis.

Acute angle between lines

Here we will use the result $m = \tan(\theta)$ to find the acute angle between two intersecting lines.



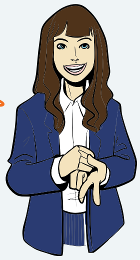
a Sketch the lines $y = 4x$ and $y = x$ on the same set of axes. Label the acute angle between the two lines α.
   i Use the gradient of each line to find the angles (θ_1 and θ_2) that each line makes with the x-axis. Round to one decimal place where necessary.
   ii How can θ_1 and θ_2 be used to find the acute angle α between the two lines?



b Sketch the lines $y = 5x - 12$ and $y = \sqrt{3}x - 2$ on the same set of axes.
   i Use the gradient of each line to find the angles (θ_1 and θ_2) that each line makes with the x-axis. Round to one decimal place where necessary.
   ii Insert a dashed line parallel to the x-axis and passing through the point of intersection of the two lines. Using this line, label the angles θ_1 and θ_2.
   iii Hence, use θ_1 and θ_2 to find the acute angle α between the two lines.

Problems and challenges

Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



1 Tom walks at 4 km/h and runs at 6 km/h. He can save $3\frac{3}{4}$ minutes by running from his house to the train station instead of walking. How many kilometres is it from his house to the station?

2 A fraction is such that when its numerator is increased by 1 and its denominator is decreased by 1, it equals 1 and when its numerator is doubled and its denominator increased by 4 it is also equal to 1. What is the fraction?

3 Show that the following sets of points are collinear (i.e. in a straight line).

a (2, 12), (−2, 0) and (−5, −9) **b** (a, $2b$), ($2a$, b) and ($-a$, $4b$)

4 Use two different methods from this chapter to prove that triangle ABC with vertices $A(1, 6)$, $B(4, 1)$ and $C(-4, 3)$ is a right-angled triangle.

5 Two missiles 2420 km apart are launched at the same time and are headed towards each other. They pass after 1.5 hours. The average speed of one missile is twice that of the other. What is the average speed of each missile?

6 Show that the points (7, 5) and (−1, 9) lie on a circle centred at (2, 5) with radius 5 units.

7 A quadrilateral whose diagonals bisect each other at right angles will always be a rhombus. Prove that the points $A(0, 0)$, $B(4, 3)$, $C(0, 6)$ and $D(-4, 3)$ are the vertices of a rhombus. Is it also a square?

8 Solve this set of simultaneous equations:

$$\begin{aligned} x - 2y - z &= 9 \\ 2x - 3y + 3z &= 10 \\ 3x + y - z &= 4 \end{aligned}$$

9 A triangle, PQR, has $P(8, 0)$, $Q(0, -8)$ and point R is on the line $y = x - 2$. Find the area of the triangle PQR.

10 The average age of players at a ten pin bowling alley increases by 1 when either four 10-year olds leave or, alternatively, if four 22-year olds arrive. How many players were there originally and what was their average age?

Algebra, equations and linear relationships

Midpoint and length of a line segment

Line segment joining (x_1, y_1) and (x_2, y_2) has:

Midpoint $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Length/distance

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Review of algebra

Add/subtract like terms only.

e.g. $2x + 4x + y = 6x + y$

Multiply: $3x \times 2y = 6xy$

Divide: $4ab \div (2b) = \frac{\overset{2}{\cancel{4}}a\cancel{b}}{\underset{1}{\cancel{2}}\cancel{b}}$

$= 2a$

Expand: $a(b + c) = ab + ac$

$a(b - c) = ab - ac$

Equations with algebraic fractions (Opt)

- More complex algebraic fractions

e.g. $\frac{x+2}{3} - \frac{3x-1}{2}$

$= \frac{2(x+2)}{6} - \frac{3(3x-1)}{6}$

$= \frac{2x + 4 - 9x + 3}{6}$

$= \frac{-7x + 7}{6}$

- To solve equations, combine fractions first using addition or subtraction

e.g. $\frac{x+3}{2} + \frac{x+1}{3} = 2$

$\frac{3(x+3)}{6} + \frac{2(x+1)}{6} = 2$

$\frac{5x + 11}{6} = 2$

Then multiply both sides of equation by the denominator.

Simultaneous equations

Solve two equations in two pronumerals using:

- substitution when one pronumeral is the subject; e.g. $y = x + 4$.
- elimination when adding or subtracting multiples of equations eliminates one variable.

Applications of simultaneous equations

- Define unknowns using pronumerals.
- Form two equations from the information in the problem.
- Solve simultaneously.
- Answer the question in words.

Parallel and perpendicular lines

Parallel lines have the same gradient; e.g. $y = 2x + 7$ and $y = 2x - 4$

For perpendicular lines with gradients m_1 and m_2:

$m_1 \times m_2 = -1$ or $m_2 = -\frac{1}{m_1}$

e.g. $y = 3x + 4$ and $y = -\frac{1}{3}x + 2$

Solving linear equations

Some steps include:

- using inverse operations
- collecting like terms on one side
- finding a common denominator.

Finding a rule for a graph

For points (x_1, y_1) and (x_2, y_2)

$m = \frac{y_2 - y_1}{x_2 - x_1}$

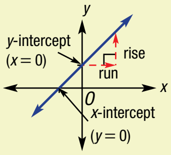
To find line equation $y = mx + c$:

- find gradient, m
- substitute a point if c is unknown using $y = mx + c$ or $y - y_1 = m(x - x_1)$

Graphs of linear relations

In the form $y = mx + c$, m is the gradient, c is the y-intercept.

$m = \frac{\text{rise}}{\text{run}}$



To graph:

a use gradient and y-intercept or

b locate x-intercept ($y = 0$) and y-intercept ($x = 0$)

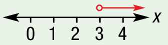
Linear inequalities

Equations contain an '=' sign.

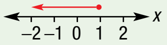
Inequalities have $>, \geqslant, <$ or $\leqslant$.

On a number line:

$x > 3$



$x \leqslant 1$



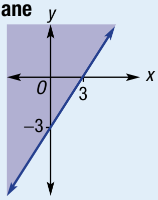
Solve as per linear equations except when multiplying or dividing by a negative, reverse the inequality symbol.

Regions on the Cartesian plane

Inequalities with two variables represent the region above or below the line.

For $y \geqslant x - 3$, shade above the line.

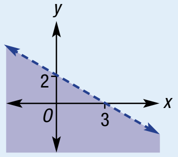
For $y > x - 3$, use a dashed line.



For $2x + 3y < 6$, test (0, 0) to decide which region to shade.

$2(0) + 3(0) < 6$ (true)

Include (0, 0); i.e. shade below dashed line.



Two or more half planes form an intersecting region.

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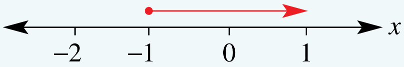
Chapter checklist with success criteria

A printable version of this checklist is available in the Interactive Textbook

		✔
1A	**1. I can simplify expressions using the four operations +, −, × , ÷.** e.g. Simplify $3x \times 2y - 4x^2y + 2xy + 3x^2y$.	☐
1A	**2. I can expand brackets using the distributive law.** e.g. Expand and simplify $-4y(2y + 3)$.	☐
1A	**3. I can factorise simple algebraic expressions with a common factor.** e.g. Factorise $6x^2 - 15x$.	☐
1A	**4. I can substitute numbers for pronumerals and evaluate.** e.g. Given $a = -3$ and $b = 4$, evaluate $ab + a^2$.	☐
1B	**5. I can solve linear equations including with brackets and variables on both sides.** e.g. Solve $4(3x - 5) = 7x$.	☐
1B	**6. I can solve linear equations involving simple fractions.** e.g. Solve $\frac{x+1}{3} = 4$ and $\frac{x}{2} - 3 = 7$.	☐
1B	**7. I can add / subtract simple algebraic fractions to solve an equation.** e.g. Simplify $\frac{x}{4} + \frac{2x}{3}$ and hence, solve $\frac{x}{4} + \frac{2x}{3} = 2$.	☐
1C	**8. I can interpret number lines to write inequalities.** e.g. Write as an inequality. 	☐
1C	**9. I can solve linear inequalities and graph the solution on a number line.** e.g. Solve $4 - \frac{x}{3} > 8$ and represent the solution on a number line.	☐
1D	**10. I can combine algebraic fractions under addition or subtraction.** Opt e.g. Simplify $\frac{x+2}{5} - \frac{x+4}{7}$.	☐
1D	**11. I can solve linear equations involving algebraic fractions.** Opt e.g. Solve $\frac{x+1}{3} + \frac{x-2}{5} = 2$ and $\frac{1-2x}{3} = \frac{x+1}{2}$.	☐
1E	**12. I can determine if a point is on a straight line.** e.g. Decide if the point $(3, -1)$ is on the line $3x + 2y = 7$.	☐
1E	**13. I can find the gradient and *y*-intercept from a straight line equation.** e.g. State the gradient and y-intercept of $3y - 2x = 6$.	☐
1E	**14. I can use the gradient and *y*-intercept to sketch a graph.** e.g. Find the gradient and y-intercept of $y = -2x + 7$ and sketch its graph.	☐
1E	**15. I can find the *x*- and *y*-intercepts of a linear graph.** e.g. Find the x- and y-intercepts and sketch the graph of $3x + y = 9$.	☐
1E	**16. I can sketch a horizontal or vertical line.** e.g. Sketch $y = 3$ and $x = -2$.	☐

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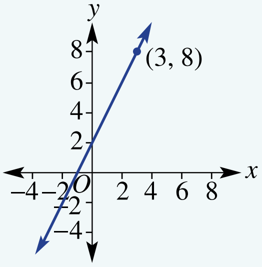
Chapter checklist with success criteria

		✔
1E	**17. I can sketch a line of the form $y = mx$.** e.g. Sketch $y = 2x$ labelling the axis intercept and one other point.	☐
1F	**18. I can find the gradient of a line joining two points.** e.g. Determine the gradient of the line joining the points $(-2, 4)$ and $(3, 1)$.	☐
1F	**19. I can find the equation of a line using a point and the y-intercept.** e.g. Find the equation of the straight line shown. 	☐
1F	**20. I can find the equation of a line given two points.** e.g. Find the equation of the straight line joining the points $(-2, -2)$ and $(2, 3)$.	☐
1G	**21. I can find the distance between two points.** e.g. Find the exact distance between the points $(2, 4)$ and $(5, 2)$.	☐
1G	**22. I can find the midpoint of a line segment joining two points.** e.g. Find the midpoint of the line segment joining $(-1, 5)$ and $(5, 2)$.	☐
1G	**23. I can use a given distance to find coordinates.** e.g. Find the values of a if the distance between $(3, a)$ and $(6, 10)$ is $\sqrt{34}$.	☐
1H	**24. I can decide if lines are parallel, perpendicular or neither.** e.g. Decide if the graph of the lines $y = 2x + 5$ and $2y + x = 3$ will be parallel, perpendicular or neither.	☐
1H	**25. I can find the equation of a parallel line.** e.g. Find the equation of the line that is parallel to $y = 3x - 4$ and passes through $(2, 4)$.	☐
1H	**26. I can find the equation of a perpendicular line.** e.g. Find the equation of a line that is perpendicular to $y = -2x + 5$ and passes through $(0, 8)$.	☐
1I	**27. I can solve simultaneous equations using substitution.** e.g. Solve the pair of simultaneous equations $x - 2y = 4$ and $y = x - 3$ using the method of substitution.	☐
1J	**28. I can solve simultaneous equations by adding or subtracting them.** e.g. Solve the simultaneous equations $x - 2y = 10$ and $x + y = 4$ using elimination.	☐
1J	**29. I can use the elimination method to solve simultaneous equations.** e.g. Solve the pair of simultaneous equations $2x + 3y = 5$ and $3x - 4y = -18$ using the elimination method.	☐

Chapter checklist with success criteria

		✔
1K	**30. I can set up and solve simultaneous equations.** e.g. A teacher buys 5 of the same chocolate bars and 2 of the same ice-creams for \$18 while another teacher buys 4 of the same chocolate bar and 5 of the same ice-creams for \$28. Determine the individual costs of these chocolate bars and ice-creams.	☐
1L	**31. I can sketch a region on the Cartesian plane.** e.g. Sketch the region $y > 2x - 5$.	☐
1L	**32. I can find the intersecting region of two inequalities.** e.g. Sketch both the inequalities $2x + y \geqslant -2$ and $2x - 3y < 6$ on the same set of axes, showing the point of intersection of the two lines and the intersecting region.	☐

Short-answer questions

1A 1 Simplify the following. You may need to expand the brackets first.

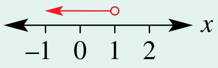
a $8xy + 5x - 3xy + x$
b $3a \times 4ab$
c $18xy \div (12y)$
d $3(b + 5) + 6$
e $-3m(2m - 4) + 4m^2$
f $3(2x + 4) - 5(x + 2)$

1B 2 Solve these linear equations.

a $3 - 2x = 9$
b $3(2x + 1) = 7 - 2(x + 5)$
c $\frac{x - 9}{4} = -2$

1C 3 Write each of the following as an inequality.

a



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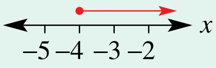

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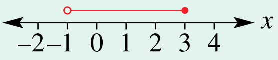


1C 4 Solve the following inequalities.

a $4x - 3 > 17$
b $3x + 2 \leqslant 4(x - 2)$
c $1 - \frac{x}{3} < 2$
d $-2x \geqslant -4(1 - 3x)$

1B 5 Marie's watering can is initially filled with 2 litres of water. However, the watering can has a small hole in the base and is leaking at a rate of 0.4 litres per minute.

a Write a rule for the volume of water, V litres, in the can after t minutes.
b What volume of water remains after 90 seconds?
c How long would it take for all the water to leak out?
d If Marie fills the can with 2 litres of water at her kitchen sink, what is the maximum amount of time she can take to get to her garden if she needs at least 600 mL to water her roses?

1D **Opt** 6 Simplify these algebraic fractions using the lowest common denominator.

a $\frac{2x}{3} + \frac{4}{9}$
b $\frac{x}{8} + \frac{x + 2}{4}$
c $\frac{x + 2}{3} - \frac{x - 4}{5}$

1D **Opt** 7 Solve the following equations.

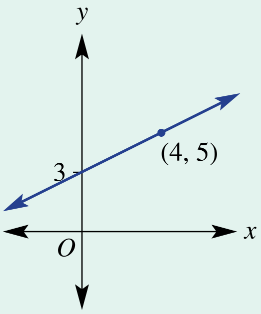
a $\frac{2x + 1}{5} + \frac{x - 3}{10} = 2$
b $\frac{4 - x}{6} - \frac{x + 2}{4} = 1$
c $\frac{x}{4} - 1 = \frac{3x}{5}$
d $\frac{x + 2}{3} = \frac{2x - 1}{4}$

1E 8 Sketch graphs of the following linear relations, labelling the axis intercepts.

a $y = 3x - 9$
b $y = 5 - 2x$
c $y = 3$
d $x = 5$
e $y = 2x$
f $y = -5x$
g $x + 2y = 8$
h $3x + 8y = 24$

1F 9 Find the equation of these straight lines.

a



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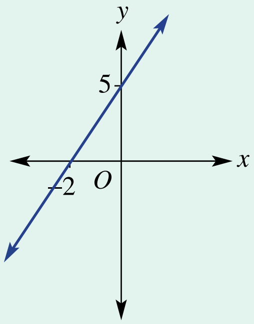

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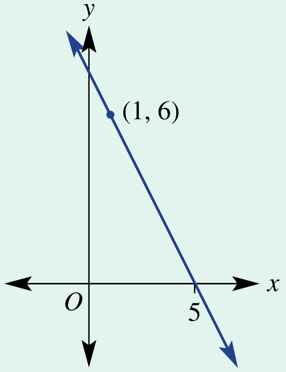
c



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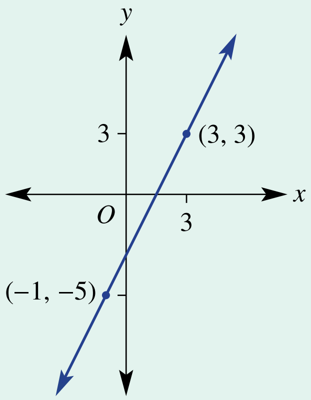


1F **10** For the line that passes through the points $(-2, 8)$ and $(3, 5)$, determine:

- **a** the gradient of the line
- **b** the equation of the line.

1G **11** Find the midpoint and the exact length of the line segment joining these points.

- **a** $(2, 5)$ and $(6, 11)$
- **b** $(3, -2)$ and $(8, 4)$
- **c** $(-1, -4)$ and $(2, -1)$

1H **12** Determine the equation of the line that is:

- **a** parallel to the line $y = 3x + 8$ and passes through the point $(2, 4)$
- **b** parallel to the line with equation $y = 4$ and passes through the point $(3, -1)$
- **c** perpendicular to the line $y = 2x - 4$ and has a y-intercept with coordinates $(0, 5)$
- **d** perpendicular to the line with equation $x + 3y = 5$ and passes through the point $(2, 5)$.

1G/H **13** Find the value(s) of the pronumeral in each situation below.

- **a** The gradient of the line joining the points $(2, -5)$ and $(6, a)$ is 3.
- **b** The line $bx + 2y = 7$ is parallel to the line $y = 4x + 3$.
- **c** The distance between $(c, -1)$ and $(2, 2)$ is $\sqrt{13}$.

1I **14** Solve the following simultaneous equations, using the substitution method.

- **a** $y = 5x + 14$
  $y = 2x + 5$
- **b** $3x - 2y = 18$
  $y - 2x = 5$

1J **15** Solve these simultaneous equations by elimination.

- **a** $3x + 2y = -11$
  $2x - y = -5$
- **b** $2y - 5x = 4$
  $3y - 2x = 6$

1K **16** At the movies Jodie buys three regular popcorns and five small drinks for her friends at a cost of \$24.50. Her friend Renee buys four regular popcorns and three small drinks for her friends at a cost of \$23.50. Find the individual costs of a regular popcorn and a small drink.

1L **17** Sketch these regions.

- **a** $y \geqslant 3x - 4$
- **b** $2x - 3y > -8$

1L **18** Shade the intersecting region of the inequalities $x + 2y \geqslant 4$ and $3x - 2y < 12$ by sketching their regions on the same axes and finding their point of intersection.

Multiple-choice questions

1A **1** The simplified form of $3x + 2x \times 7y - 3xy + 5x$ is:

A $10x + 4y$ **B** $35xy - 6x^2y$ **C** $\dfrac{35y - 6xy}{7}$

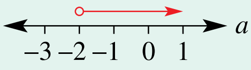
D $5xy - 6x^2$ **E** $8x + 11xy$

1B **2** The solution to $-4(2x - 6) = 10x$ is:

A $x = \dfrac{3}{2}$ **B** $x = 12$ **C** $x = \dfrac{4}{3}$ **D** $x = -12$ **E** $x = -\dfrac{4}{3}$

1C **3** The number line that represents the solution to the inequality $2 - \dfrac{a}{3} < 4$ is:

A



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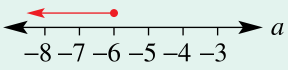

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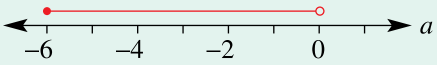
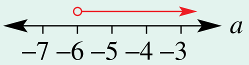
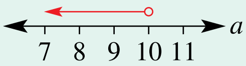


D



E



1B **Opt** **4** $\dfrac{x+1}{3} + \dfrac{2x}{7}$ simplifies to:

A $\dfrac{3x+1}{10}$ **B** $\dfrac{17x+3}{21}$ **C** $\dfrac{3x+1}{21}$ **D** $\dfrac{13x+7}{21}$ **E** $\dfrac{13x+1}{10}$

1E **5** If $(-1, 2)$ is a point on the line $ax - 4y + 11 = 0$, the value of a is:

A -19 **B** 3 **C** $-\dfrac{15}{2}$ **D** 5 **E** -1

1E **6** The graph shown has equation:

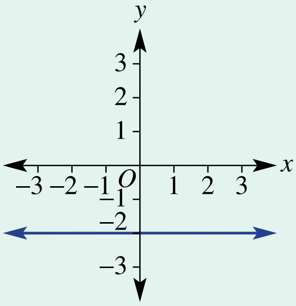
A $x = -2$

B $y = -2x$

C $y = -2$

D $x + y = -2$

E $y = x - 2$



1E **7** The gradient and the coordinates of the y-intercept, respectively, of the graph of $3x + 8y = 2$ are:

A $-\dfrac{3}{8}, \left(0, \dfrac{1}{4}\right)$ **B** $3, (0, 2)$ **C** $\dfrac{2}{3}, \left(0, \dfrac{1}{4}\right)$ **D** $-3, (2, 0)$ **E** $\dfrac{3}{8}, (2, 0)$

1F **8** The equation of the line joining the points $(-1, 3)$ and $(1, -1)$ is:

A $2y - x = 1$ **B** $y = 2x - 1$ **C** $y = -2x + 1$

D $y - 2x = 1$ **E** $y = \dfrac{1}{2}x + 1$

1G **9** The midpoint of the line segment joining the points $(a, -6)$ and $(7, b)$ is $(4.5, -1)$. The values of the pronumerals are:

A $a = 2, b = 8$ **B** $a = 3, b = -11$ **C** $a = 9, b = 5$

D $a = 2,\ b = 4$ **E** $a = 2.5, b = 5$

1H **10** The line that is perpendicular to the line with equation $y = -3x + 7$ is:

A $y = -3x + 2$ **B** $3x + y = -1$ **C** $y = 3x - 3$

D $3y = 4 - x$ **E** $3y - x = 4$

1H **11** The line that is parallel to the line with equation $y = 2x + 3$ and passes through the point $(-3, 2)$ has the equation:

A $2x + y = 5$ **B** $y = 2x + 8$ **C** $y = -\frac{1}{2}x + \frac{1}{2}$

D $y = 2x - 4$ **E** $y - 2x = -7$

1I **12** The solution to the simultaneous equations $2x - 3y = -1$ and $y = 2x + 3$ is:

A $x = -2, y = -1$ **B** $x = \frac{5}{2}, y = 8$ **C** $x = 2, y = 7$

D $x = -\frac{2}{3}, y = -\frac{1}{9}$ **E** $x = -3, y = 3$

1K **13** A community fundraising concert raises $3540 from ticket sales to 250 people. Children's tickets were sold for $12 and adult tickets sold for $18. If x adults and y children attended the concert, the two equations that represent this problem are:

A $\begin{aligned} x + y &= 250 \\ 18x + 12y &= 3540 \end{aligned}$ **B** $\begin{aligned} x + y &= 3540 \\ 216xy &= 3540 \end{aligned}$ **C** $\begin{aligned} x + y &= 250 \\ 12x + 18y &= 3540 \end{aligned}$

D $\begin{aligned} x + y &= 3540 \\ 18x + 12y &= 250 \end{aligned}$ **E** $\begin{aligned} 3x + 2y &= 3540 \\ x + y &= 250 \end{aligned}$

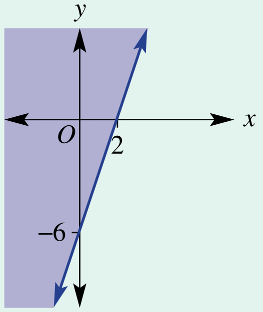
1L **14** The point that is *not* in the region defined by $2x - 3y \leqslant 5$ is:

A $(0, 0)$ **B** $(1, -1)$ **C** $(-3, 2)$

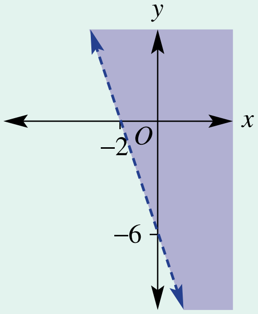
D $(2, -1)$ **E** $\left(\frac{5}{2}, 3\right)$

1L **15** The region that represents the inequality $y < 3x - 6$ is:

A



B



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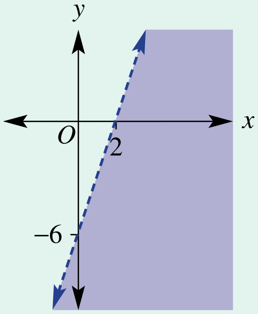

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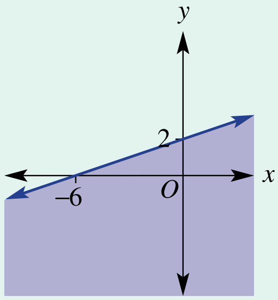
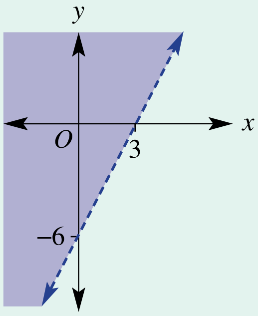


E

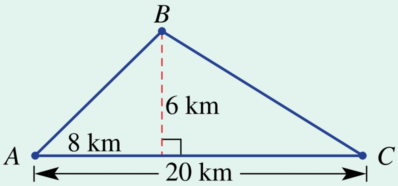


Extended-response questions

1 There are two shrubs in Chen's backyard that grow at a constant rate. Shrub A had an initial height of 25 cm and has grown to 33 cm after 2 months. Shrub B was 28 cm high after 2 months and 46 cm high after 5 months.

- **a** Plot the above points and find a rule for the height, h cm, after t months for:
  - **i** shrub A
  - **ii** shrub B.
- **b** What was the initial height (i.e. at $t = 0$) of shrub B?
- **c** Refer to your rules in part **a** to explain which shrub is growing at a faster rate.
- **d** Graph each of your rules from part **a** on the same set of axes for $0 \leqslant t \leqslant 12$.
- **e** Determine graphically after how many months the height of shrub B will overtake the height of shrub A.
- **f**
  - **i** Shrub B reaches its maximum height after 18 months. What is this height?
  - **ii** Shrub A has a maximum expected height of 1.3 m. After how many months will it reach this height?
  - **iii** Chen will prune shrub A when it is between 60 cm and 70 cm. Within what range of months after it is planted will Chen need to prune the shrub?

2 A triangular course has been roped off for a cross-country run. The run starts and ends at A and goes via checkpoints B and C, as shown.



- **a** Draw the area of land onto a set of axes, taking point A to be the origin (0, 0). Label the coordinates of B and C.
- **b** Find the length of the course, to one decimal place, by calculating the distance of legs AB, BC and CA.
- **c** A drink station is located at the midpoint of BC. Label the coordinates of the drink station on your axes.
- **d** Find the equation of each leg of the course:
  - **i** AB
  - **ii** BC
  - **iii** CA
- **e** Write a set of three inequalities that would overlap to form an intersecting region equal to the area occupied by the course.
- **f** A fence line runs beyond the course. The fence line passes through point C and would intersect AB at right angles if AB was extended to the fence line. Find the equation of the fence line.

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2

Geometry and networks

Maths in context: The Shortest Path Algorithm

Network geometry analyses the possible connections within a group of nodes. Examples of node groups include a selection of towns, or computers, or online friends.

The Dutch mathematician Edsger Dijkstra was one of the first computer programmers (code writer). In 1956, while relaxing at a café, he mentally invented code for 'The Shortest Path Algorithm' to find the shortest road distance between any two cities.

There are numerous applications in our world for this Shortest Path Algorithm (also named the Dijkstra Algorithm).

- Google Maps applies it to find the shortest distance and/or travel time between two locations.
- Social media companies suggest friend groups by first developing the shortest path between connected users measured through handshakes.
- Robots use the algorithm to efficiently deliver items from location A to location B in 3D, e.g., in Amazon's warehouses.
- Data packets use IP addresses to determine the shortest path between the source and destination router in a network.
- Flight companies solve the complex problem of flight scheduling, that is, to produce the most economical flight routes and in the shortest time, both for passenger planes and cargo aircraft.

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Chapter contents

Australian Curriculum 9.0

SPACE

Apply deductive reasoning to proofs involving shapes in the plane and use theorems to solve spatial problems (AC9M10SP01)

Interpret networks and network diagrams used to represent relationships in practical situations and describe connectedness (AC9M10SP02)

Design, test and refine solutions to spatial problems using algorithms and digital tools; communicate and justify solutions (AC9M10SP03)

Relationships between angles and various lines associated with circles (radii, diameters, chords, tangents) (Year 10 optional content)

MEASUREMENT

Use mathematical modelling to solve practical problems involving proportion and scaling of objects; formulate problems and interpret solutions in terms of the situation; evaluate and modify models as necessary, and report assumptions, methods and findings.

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

2A Review of geometry CONSOLIDATING

LEARNING INTENTIONS

- To review the names of types of angles and the names and properties of angles in parallel lines
- To review the properties of triangles and quadrilaterals and the names and the angle sum rule of polygons
- To know the meaning of the term regular polygon
- To be able to work with polygon angle sums to find missing angles including exterior angles

Based on just five simple axioms (i.e. known or self-evident facts) the famous Greek mathematician Euclid (about 300 BC) was able to deduce many hundreds of propositions (theorems and constructions) systematically presented in the 13-volume book collection called the *Elements*. All the basic components of school geometry are outlined in these books, including the topics *angle sums of polygons* and *angles in parallel lines*, which are studied in this section.

Trade workers who regularly use geometry include: sheet metal workers building a commercial kitchen; plumbers joining parallel but separated water pipes; carpenters building house roof frames; and builders of steps and wheelchair ramps.

Lesson starter: Exploring triangles with interactive geometry

Use a dynamic geometry software package to construct a triangle and illustrate the angle sum and the exterior angle theorem.

Interactive geometry instructions

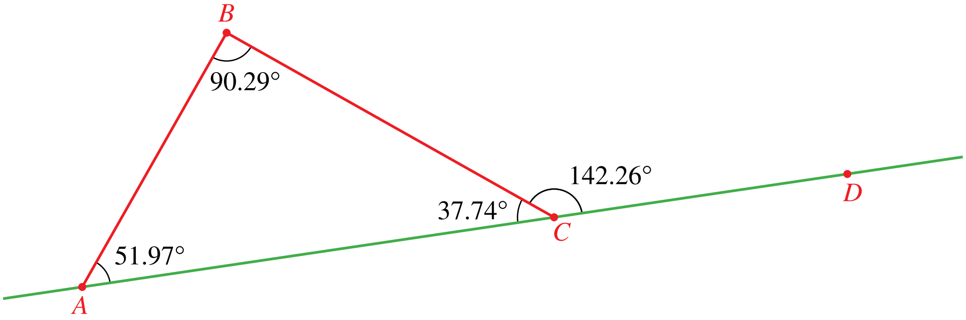
a Construct a line AD and triangle ABC, as shown.

b Measure all the interior angles and the exterior angle $\angle BCD$.

c Use the calculator to check that the angle sum is 180°.

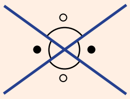
d Now use the calculator to find $\angle BAC + \angle ABC$.
What do you notice in comparison to the exterior angle $\angle BCD$?

e Drag one of the points to check that these properties are retained for all triangles.

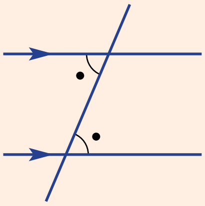


KEY IDEAS

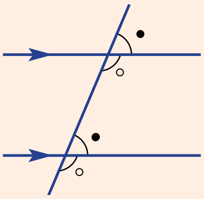
- Angles at a point
  - **Complementary** (sum to 90°)
  - **Supplementary** (sum to 180°)
  - **Revolution** (360°)
  - **Vertically opposite angles** (equal), as shown



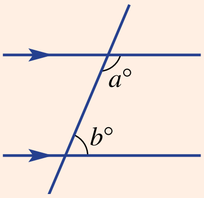
- Angles in **parallel lines**



Alternate angles are equal.



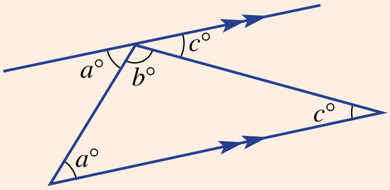
Corresponding angles are equal.



Cointerior angles are supplementary.
$a + b = 180$

- If two lines, AB and CD, are parallel, we write $AB \parallel CD$.

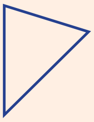
- **Triangles**
  - Angle sum is 180°.
    To prove this, draw a line parallel to a base and then mark the alternate angles in parallel lines. Note that angles on a straight line are supplementary.



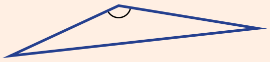
$a + b + c = 180$

  - Triangles classified by angles.

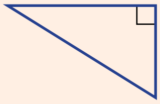
Acute: all angles acute



Obtuse: one angle obtuse

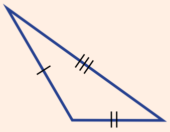


Right: one right angle

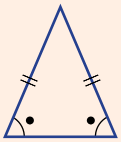


  - Triangles classified by side lengths.

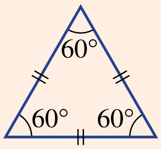
Scalene (3 different sides)



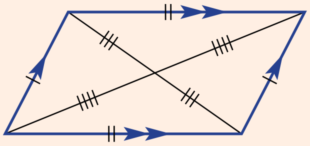
Isosceles (2 equal sides)



Equilateral (3 equal sides)



- **Quadrilaterals** (Refer to **Section 2C** for more details on quadrilaterals.)
  - **Parallelograms** are quadrilaterals with two pairs of parallel sides.
  - **Rectangles** are parallelograms with all angles 90°.
  - **Rhombuses** are parallelograms with sides of equal length.
  - **Squares** are parallelograms that are both rectangles and rhombuses.
  - **Kites** are quadrilaterals with two pairs of equal adjacent sides.
  - **Trapeziums** are quadrilaterals with at least one pair of parallel sides.



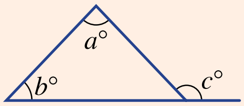
- **Polygons** have an angle sum given by $S = (n - 2) \times 180^\circ$, where n is the number of sides.
  - **Regular polygons** have equal sides and equal angles.

    A single interior angle $= \dfrac{(n-2) \times 180^\circ}{n}$

n	Name
3	triangle
4	quadrilateral
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	nonagon
10	decagon

- An **exterior angle** is supplementary to an interior angle.
  - For a triangle, the **exterior angle theorem** states that the exterior angle is equal to the sum of the two opposite interior angles.

    $c = a + b$



BUILDING UNDERSTANDING

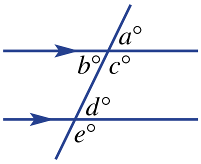
① State the names of the polygons with 3 to 10 sides, inclusive.

② Decide whether each of the following is true or false.

a The angle sum of a quadrilateral is 300°.
b A square has 4 lines of symmetry.
c An isosceles triangle has two equal sides.
d An exterior angle on an equilateral triangle is 120°.
e A kite has two pairs of equal opposite angles.
f A parallelogram is a rhombus.
g A square is a rectangle.
h Vertically opposite angles are supplementary.
i Cointerior angles in parallel lines are supplementary.

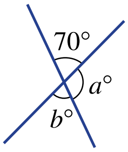
③ State the pronumeral in the diagram that matches the following descriptions.

a alternate to d°
b corresponding to e°
c cointerior to c°
d vertically opposite to b°

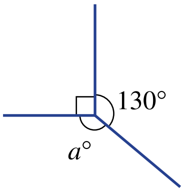


4 Find the values of the pronumerals, giving reasons.

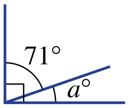
a



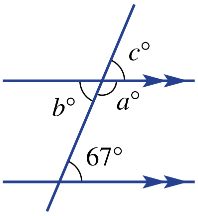
b



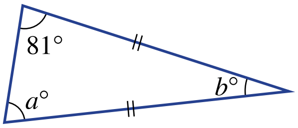
c



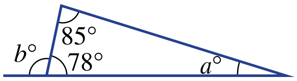
d



e



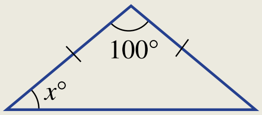
f



Example 1 Using the angle sum and exterior angles of triangles

Find the value of x in the following, giving reasons.

a



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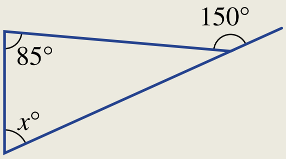
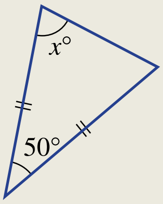


SOLUTION	EXPLANATION
a $x + x + 100 = 180$ (Angle sum is 180° and two angles are $x°$ since triangle is isosceles)	Use triangle angle sum (180°) and isosceles triangle.
$2x + 100 = 180$ $2x = 80$ $x = 40$	Collect like terms and solve for x.
b $x + 85 = 150$ (exterior angle theorem) $x = 65$	Use the exterior angle theorem for a triangle.

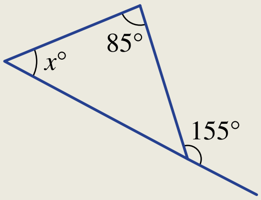
Now you try

Find the value of x in the following, giving reasons.

a



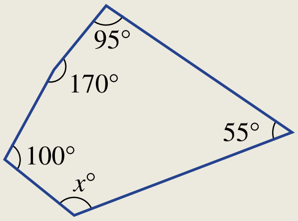
b



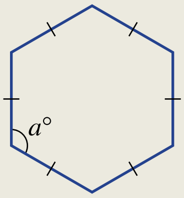
Example 2 Working with the angle sum of polygons

Find the value of the pronumeral.

a



b a regular hexagon



SOLUTION

a $S = (n - 2) \times 180°$
$= (5 - 2) \times 180°$
$= 540°$

$x + 100 + 170 + 95 + 55 = 540$
$x + 420 = 540$
$\therefore x = 120$

b $S = (n - 2) \times 180°$
$= (6 - 2) \times 180°$
$= 720°$
$a = 720 \div 6$
$= 120$

EXPLANATION

Use the rule for the angle sum of a polygon (5 sides, so $n = 5$).

The sum of all the angles is 540°.

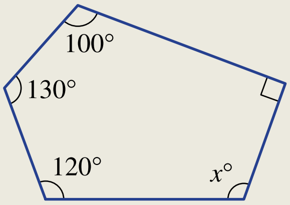
Use the angle sum rule for a polygon with $n = 6$.

In a regular hexagon there are 6 equal angles.

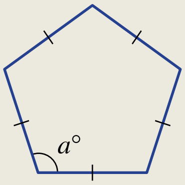
Now you try

Find the value of the pronumeral.

a



b a regular pentagon



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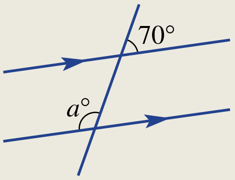
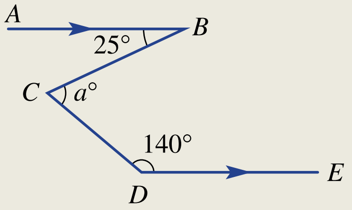
Example 3 Working with angles in parallel lines

Find the value of the pronumeral, giving reasons.

a



b



SOLUTION

a Label another angle as 70°.
(due to vertically opposite angles)

$\therefore a + 70 = 180$ (cointerior angles in parallel lines)

$a = 110$

b Construct a third parallel line, CF.

$\angle BCF = 25°$ (alternate angles in || lines)

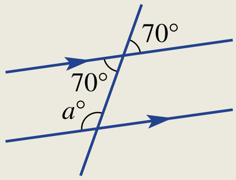
$\angle FCD = 180° - 140°$

$= 40°$ (cointerior angles in || lines)

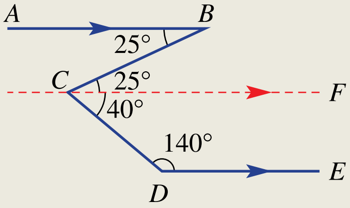
$\therefore a = 25 + 40$

$= 65$

EXPLANATION



Mark angle 70° (vertically opposite)
a and 70 are supplementary
i.e. cointerior angles in parallel lines.
Alternate methods are possible.

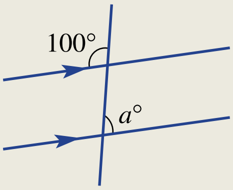


$\angle ABC$ and $\angle FCB$ are alternate angles in parallel lines. $\angle FCD$ and $\angle EDC$ are cointerior angles in parallel lines.

Now you try

Find the value of the pronumeral, giving reasons.

a



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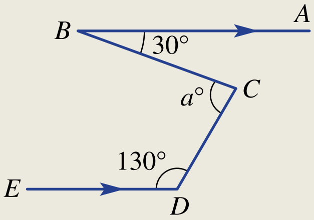

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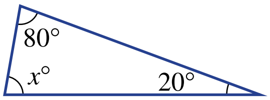
Exercise 2A

FLUENCY 1, 2(½), 3, 4 1–2(½), 3, 4 1–2(½), 3, 4(½)

Example 1 **1** Find the value of the pronumeral, using the exterior angle theorem in parts **d**–**f**.

a



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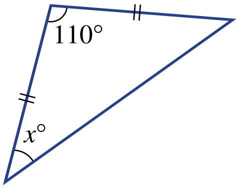
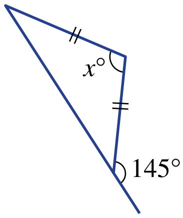
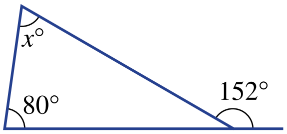


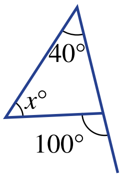
c



d



e



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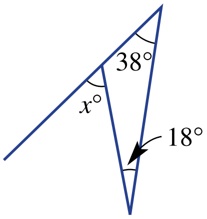
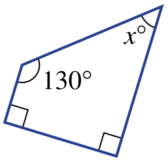
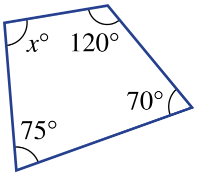


Example 2a **2** Find the value of x in the following, giving reasons.

a



b



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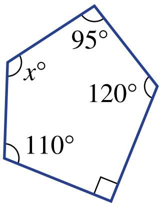

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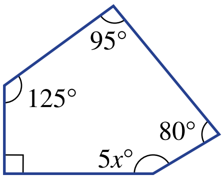

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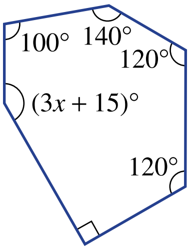
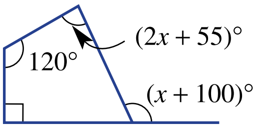


f

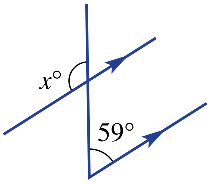


Example 2b **3** Find the size of an interior angle of these polygons if they are regular.

a pentagon **b** octagon **c** decagon

Example 3a **4** Find the value of the pronumeral, giving reasons.

a



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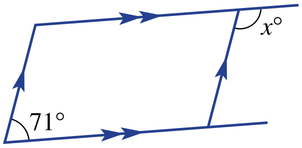

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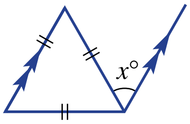
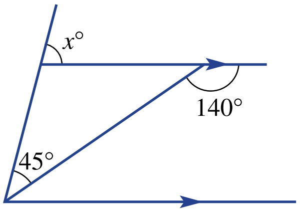


d



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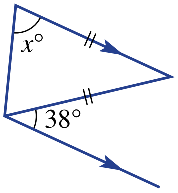
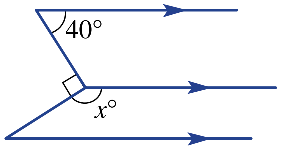


f

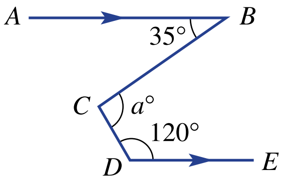


PROBLEM–SOLVING	5(½), 6	5(½), 6, 7	5(½), 7–9

Example 3b

5 Find the value of the pronumeral a, giving reasons.

a



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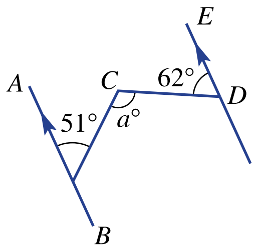

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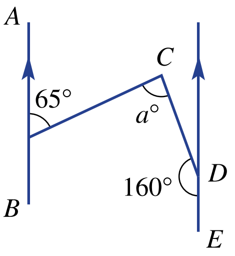
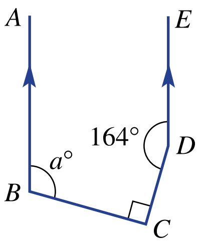


d



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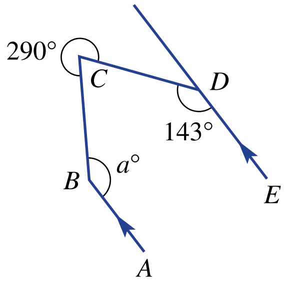
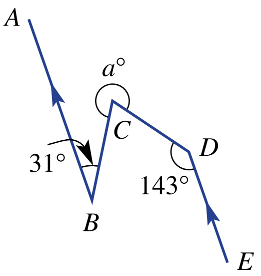


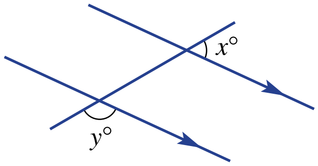
f



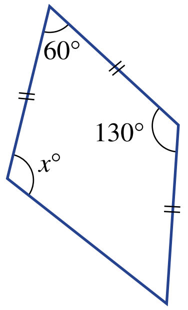
6 **a** Find the size of an interior angle of a regular polygon with 100 sides.
b What is the size of an exterior angle of a 100-sided regular polygon?

7 Find the number of sides of a regular polygon that has the following interior angles.
a 150° **b** 162° **c** 172.5°

8 In this diagram, $y = 4x$. Find the values of x and y.



9 Find the value of x in this diagram, giving reasons.
(*Hint*: Form isosceles and/or equilateral triangles.)



REASONING	10, 11	10–12	11–14

10 The rule for the sum of the interior angles of a polygon, S, is given by $S = (n - 2) \times 180°$.
a Show that $S = 180n - 360$.
b Find a rule for the number of sides n of a polygon with an angle sum S; i.e. write n in terms of S.
c Write the rule for the size of an interior angle I of a regular polygon with n sides.
d Write the rule for the size of an exterior angle E of a regular polygon with n sides.

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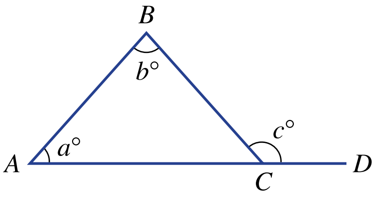
11 Prove that the exterior angle of a triangle is equal to the sum of the two opposite interior angles by following these steps.

a Write $\angle BCA$ in terms of a and b and give a reason.

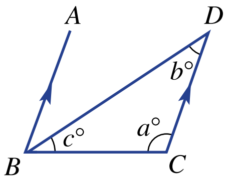
b Find c in terms of a and b using $\angle BCA$ and give a reason.



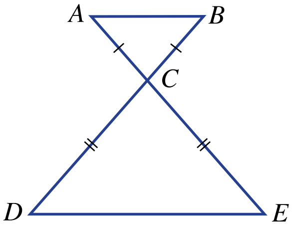
12 **a** Explain why in this diagram $\angle ABC$ is equal to $b°$.

b Using $\angle ABC$ and $\angle BCD$, what can be said about a, b and c?

c What does your answer to part **b** show?



13 Give reasons why AB and DE in this diagram are parallel; i.e. $AB \parallel DE$.

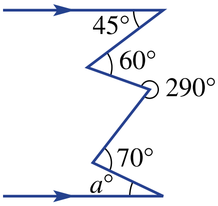


14 Each point on Earth's surface can be described by a line of longitude (degrees east or west from Greenwich, England) and a line of latitude (degrees north or south from the equator). Investigate and write a brief report (providing examples) describing how places on Earth can be located with the use of longitude and latitude.

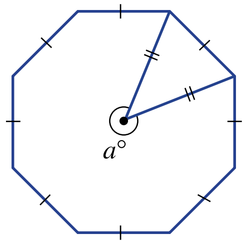
ENRICHMENT: Multilayered reasoning – – 15–17

15 Find the value of the pronumerals, giving reasons.

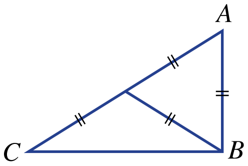
a



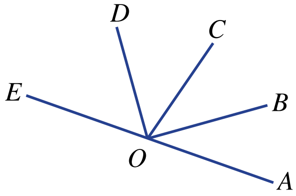
b



16 Give reasons why $\angle ABC = 90°$.



17 In this diagram $\angle AOB = \angle BOC$ and $\angle COD = \angle DOE$. Give reasons why $\angle BOD = 90°$.



2B Congruent triangles

LEARNING INTENTIONS

- To know the meaning of the term congruent
- To be able to match corresponding sides and angles in congruent figures
- To know the four tests for congruence of triangles
- To know how to prove that two triangles are congruent using one of the tests
- To be able to use congruence of triangles to prove other properties

In geometry it is important to know whether or not two objects are in fact identical in shape and size. If two objects are identical, then we say they are congruent. Two shapes that are congruent will have corresponding (i.e. matching) sides equal in length and corresponding angles also equal. For two triangles it is not necessary to know every side and angle to determine if they are congruent. Instead, a set of minimum conditions is enough. There are four sets of minimum conditions for triangles and these are known as the tests for congruence of triangles.

Civil engineers apply congruent triangle geometry in the design and construction of buildings, bridges, cranes and electricity pylons. Triangles are the strongest form of support and congruent triangles evenly distribute the weight of the construction.

Lesson starter: Which are congruent?

Consider these four triangles.

1 ΔABC with $\angle A = 37^\circ$, $\angle B = 112^\circ$ and $AC = 5$ cm.
2 ΔDEF with $\angle D = 37^\circ$, $DF = 5$ cm and $\angle E = 112^\circ$.
3 ΔGHI with $\angle G = 45^\circ$, $GH = 7$ cm and $HI = 5$ cm.
4 ΔJKL with $\angle J = 45^\circ$, $JK = 7$ cm and $KL = 5$ cm.

Sarah says that only ΔABC and ΔDEF are congruent. George says that only ΔGHI and ΔJKL are congruent and Tobias says that both pairs (ΔABC, ΔDEF and ΔGHI, ΔJKL) are congruent.

- Discuss which pairs of triangles might be congruent, giving reasons.
- What drawings can be made to support your argument?
- Who is correct: Sarah, George or Tobias? Explain why.

KEY IDEAS

- Two objects are said to be **congruent** when they are exactly the same size and shape. For two congruent triangles ΔABC and ΔDEF, we write $\Delta ABC \equiv \Delta DEF$.
  - When comparing two triangles, corresponding sides are equal in length and corresponding angles are equal.
  - When we prove congruence in triangles, we usually write vertices in matching order.

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- Two triangles can be tested for **congruence** using the following conditions.
  - Corresponding sides are equal (SSS).

  - Two corresponding sides and the included angle are equal (SAS).

  - Two corresponding angles and a side are equal (AAS).

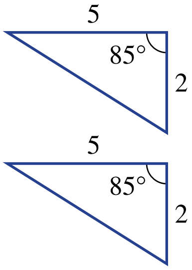
  - A right angle, the hypotenuse and one other pair of corresponding sides are equal (RHS).

- $AB \parallel CD$ means AB is parallel to CD.
- $AB \perp CD$ means AB is perpendicular to CD.

BUILDING UNDERSTANDING

① Which of the tests (SSS, SAS, AAS or RHS) would be used to decide whether the following pairs of triangles are congruent?

a



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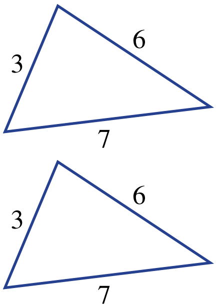

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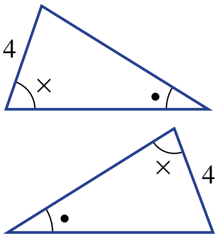

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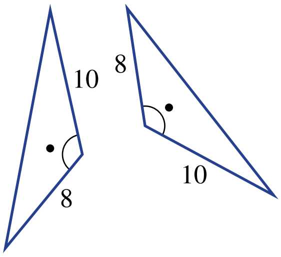

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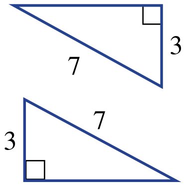

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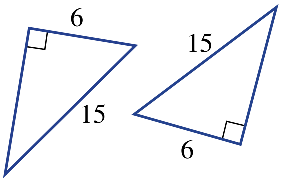
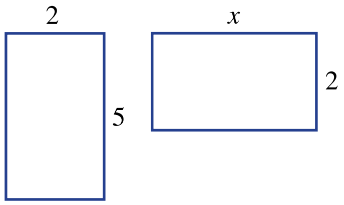


② Assume these pairs of figures are congruent and state the value of the pronumeral in each case.

a



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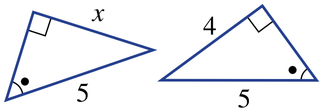

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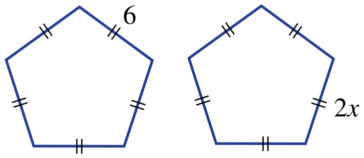

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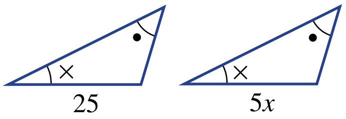
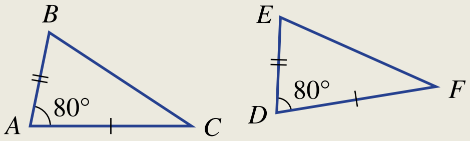


Example 4 Proving congruence in triangles

Prove that these pairs of triangles are congruent.

a



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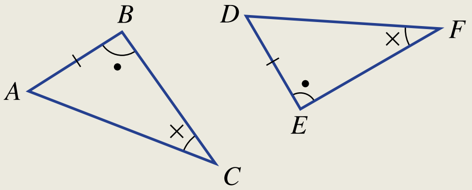
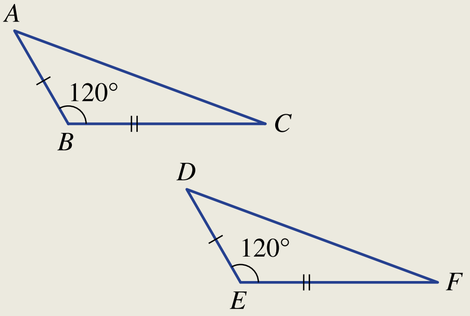


SOLUTION	EXPLANATION
a $AB = DE$ (given) S $\angle BAC = \angle EDF = 80°$ (given) A $AC = DF$ (given) S So, $\Delta ABC \equiv \Delta DEF$ (SAS)	List all pairs of corresponding sides and angles. The two triangles are therefore congruent, with two pairs of corresponding sides and the included angle equal.
b $\angle ABC = \angle DEF$ (given) A $\angle ACB = \angle DFE$ (given) A $AB = DE$ (given) S So, $\Delta ABC \equiv \Delta DEF$ (AAS)	List all pairs of corresponding sides and angles. The two triangles are therefore congruent with two pairs of corresponding angles and a corresponding side equal.

Now you try

Prove that these pairs of triangles are congruent.

a



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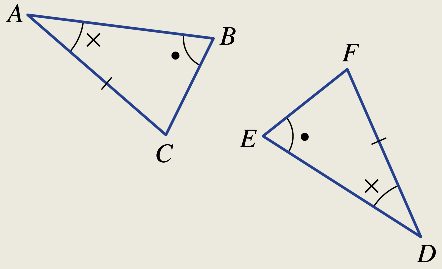


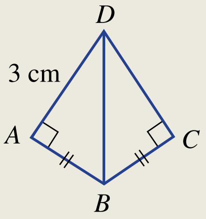
Example 5 Using congruence in proof

In this diagram, $\angle A = \angle C = 90^\circ$ and $AB = CB$.

a Prove $\Delta ABD \equiv \Delta CBD$.

b Prove $AD = CD$.

c State the length of CD.



SOLUTION

a $\angle A = \angle C = 90^\circ$ (given) R

BD is common H

$AB = CB$ (given) S

$\therefore \Delta ABD \equiv \Delta CBD$ (RHS)

b $\Delta ABD \equiv \Delta CBD$ so $AD = CD$

(corresponding sides in congruent triangles)

c $CD = 3$ cm

EXPLANATION

Systematically list corresponding pairs of equal angles and lengths.

Since ΔABD and ΔCBD are congruent, the matching sides AD and CD are equal.

$AD = CD$ from part **b** above.

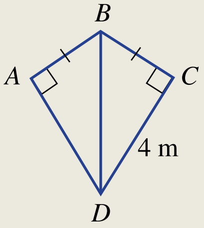
Now you try

In this diagram, $\angle A = \angle C = 90^\circ$ and $AB = CB$.

a Prove $\Delta ABD \equiv \Delta CBD$.

b Prove $AD = CD$.

c State the length of AD.



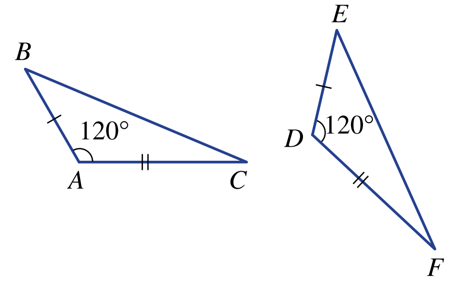
Exercise 2B

FLUENCY	1, 2	1–3(½)	1–3(½)

Example 4

1 Prove that these pairs of triangles are congruent.

a



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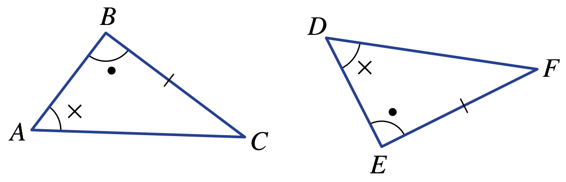

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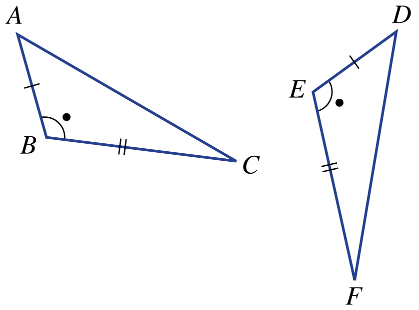

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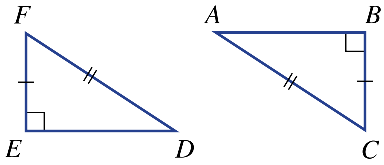

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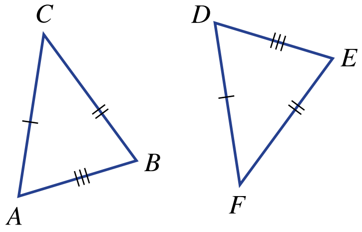

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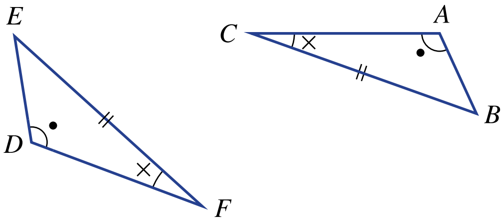
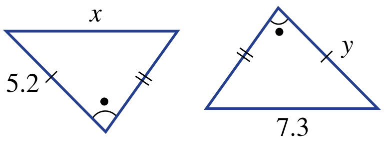


2 Find the value of the pronumerals in these diagrams, which include congruent triangles. Recall that corresponding sides in congruent triangles are equal.

a



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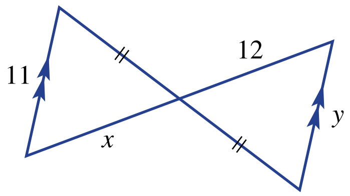

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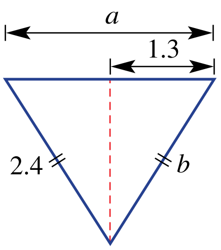

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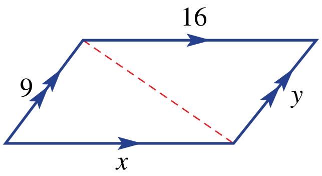

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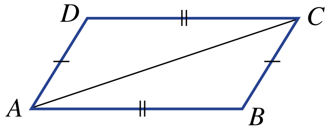
Example 5a

3 Prove that each pair of triangles in the following diagrams is congruent, giving reasons. Write the vertices in matching order.

a



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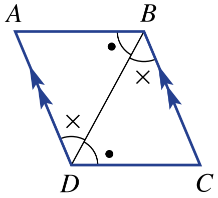

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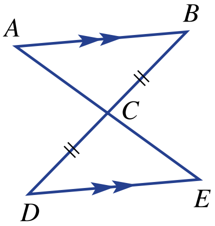

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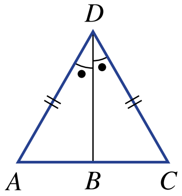

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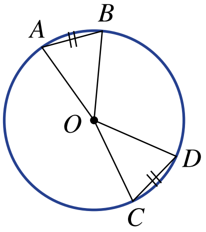

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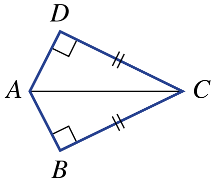


PROBLEM–SOLVING	4	4, 5	4–6

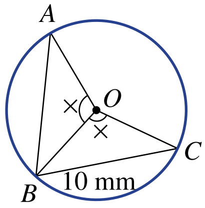
Example 5

4 In this diagram, O is the centre of the circle and $\angle AOB = \angle COB$.

a Prove $\Delta AOB \equiv \Delta COB$.

b Prove $AB = BC$.

c State the length of AB.



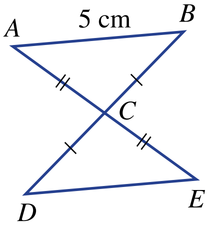
5 In this diagram, $BC = DC$ and $AC = EC$.

a Prove $\Delta ABC \equiv \Delta EDC$.

b Prove $AB = DE$.

c Prove $AB \parallel DE$.

d State the length of DE.

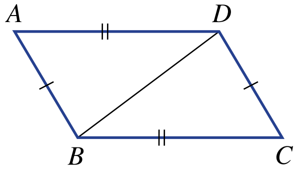


6 In this diagram, $AB = CD$ and $AD = CB$.

a Prove $\Delta ABD \equiv \Delta CDB$.

b Prove $\angle DBC = \angle BDA$.

c Prove $AD \parallel BC$.



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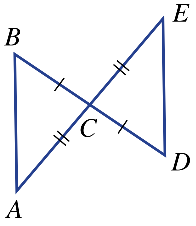
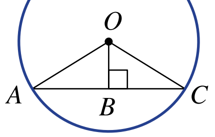
REASONING	7(½)	7(½)	7

7 Prove the following for the given diagrams. Give reasons.

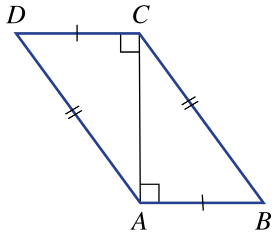
a $AB \parallel DE$



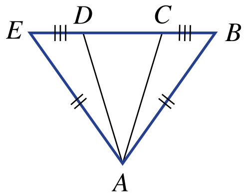
b OB bisects AC;
i.e. $AB = BC$



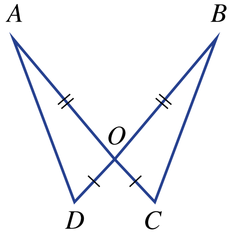
c $AD \parallel BC$



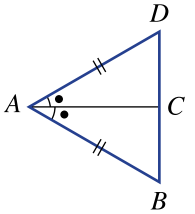
d $AD = AC$



e $\angle OAD = \angle OBC$



f $AC \perp BD$;
i.e. $\angle ACD = \angle ACB$

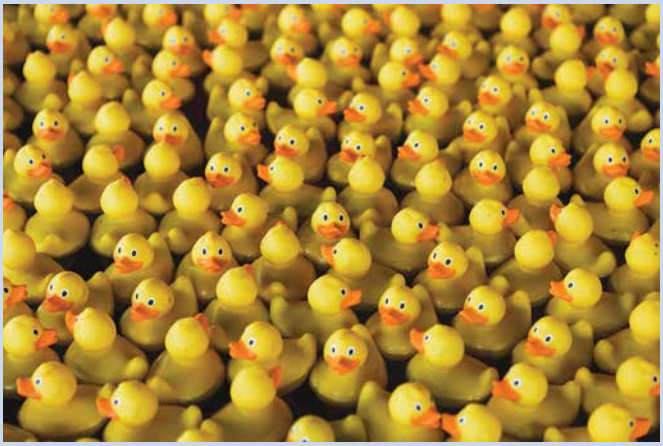


ENRICHMENT: Draw your own diagram	–	–	8

8 **a** A circle with centre O has a chord AB. M is the midpoint of the chord AB. Prove $OM \perp AB$.

b Two overlapping circles with centres O and C intersect at A and B. Prove $\angle AOC = \angle BOC$.

c ΔABC is isosceles with $AC = BC$, D is a point on AC such that $\angle ABD = \angle CBD$, and E is a point on BC such that $\angle BAE = \angle CAE$. AE and BD intersect at F. Prove $AF = BF$.



Congruent objects are identical: same size and shape.

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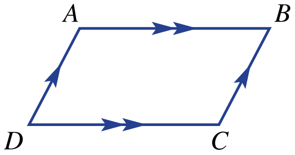
2C Using congruence to investigate quadrilaterals

LEARNING INTENTIONS

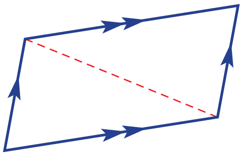
- To know the properties of different types of parallelograms
- To know what is required to prove various properties of parallelograms
- To understand that showing two triangles are congruent can help prove that other angles and lengths are equal or that lines must be parallel or perpendicular
- To be able to use congruence to prove properties of quadrilaterals or test for types of quadrilaterals

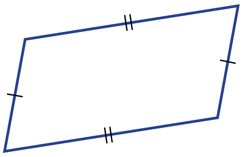
Recall that parallelograms are quadrilaterals with two pairs of parallel sides. We therefore classify rectangles, squares and rhombuses as special types of parallelograms, the properties of which can be explored using congruence. By dividing a parallelogram into 'smaller' triangles, we can prove congruence for pairs of these triangles and use this to deduce the properties of the shape.

The parallelogram law of forces is widely applied in engineering, architecture and navigation. In a parallelogram *ABCD*, a boat aiming in direction *AB* against a tide or wind in direction *AD* results in the boat moving in the direction of the diagonal *AC*.



Lesson starter: Aren't they the same proof?





Here are two statements involving the properties of a parallelogram.

1. *A parallelogram (a quadrilateral with parallel opposite sides) has opposite sides of equal length.*
2. *A quadrilateral with opposite sides of equal length is a parallelogram.*

- Are the two statements saying the same thing?
- Discuss how congruence can be used to help prove each statement.
- Formulate a proof for each statement.

KEY IDEAS

- Some vocabulary and symbols:
  - If *AB* is parallel to *CD*, then we write $AB \parallel CD$.
  - If *AB* is perpendicular to *BC*, then we write $AB \perp BC$.
  - To bisect means to cut in half.

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- **Parallelogram** properties and tests:
  - Parallelogram – a quadrilateral with opposite sides parallel.

Properties of a parallelogram	Tests for a parallelogram
• opposite sides are equal in length • opposite angles are equal • diagonals bisect each other	• if opposite sides are equal in length • if opposite angles are equal • if one pair of opposite sides are equal and parallel • if diagonals bisect each other

  - **Rhombus** – a parallelogram with all sides of equal length.

Properties of a rhombus	Tests for a rhombus
• all sides are equal length • opposite angles are equal • diagonals bisect each other at right angles • diagonals bisect the interior angles	• if all sides are equal length • if diagonals bisect each other at right angles

  - **Rectangle** – a parallelogram with all angles 90°.

Properties of a rectangle	Tests for a rectangle
• opposite sides are of equal length • all angles equal 90° • diagonals are equal in length and bisect each other	• if all angles are 90° • if diagonals are equal in length and bisect each other

  - **Square** – a parallelogram that is a rectangle and a rhombus.

Properties of a square	Tests for a square
• all sides are equal in length • all angles equal 90° • diagonals are equal in length and bisect each other at right angles • diagonals bisect the interior angles	• if all sides are equal in length and at least one angle is 90° • if diagonals are equal in length and bisect each other at right angles

BUILDING UNDERSTANDING

1 Name the special quadrilateral given by these descriptions.
  a a parallelogram with all angles 90°
  b a quadrilateral with opposite sides parallel
  c a parallelogram that is a rhombus and a rectangle
  d a parallelogram with all sides of equal length

2 Name all the special quadrilaterals that have these properties.
  a All angles 90°.
  b Diagonals are equal in length.
  c Diagonals bisect each other.
  d Diagonals bisect each other at 90°.
  e Diagonals bisect the interior angles.

3 Give a reason why:
  a a trapezium is not a parallelogram
  b a kite is not a parallelogram

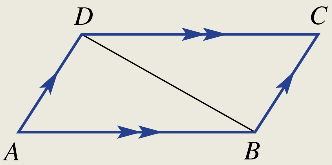
Example 6 Proving properties of quadrilaterals

a Prove that a parallelogram (with opposite sides parallel) has equal opposite angles.

b Use the property that opposite sides of a parallelogram are equal to prove that a rectangle (with all angles 90°) has diagonals of equal length.

SOLUTION

a



$\angle ABD = \angle CDB$ (alternate angles in $\parallel$ lines) A

$\angle ADB = \angle CBD$ (alternate angles in $\parallel$ lines) A

BD is common S

$\therefore \Delta ABD \equiv \Delta CDB$ (AAS)

$\therefore \angle DAB = \angle BCD$ (corresponding angles in congruent triangles)

Also since $\angle ADB = \angle CBD$ and $\angle ABD = \angle CDB$

then $\angle ADC = \angle CBA$

So opposite angles are equal.

EXPLANATION

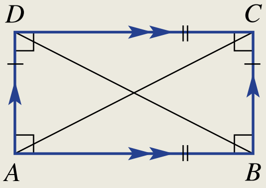
Draw a parallelogram with parallel sides and the segment BD.

Prove congruence of ΔABD and ΔCDB, noting alternate angles in parallel lines.

Note also that BD is common to both triangles.

Corresponding angles in congruent triangles.

b



Consider ΔABC and ΔBAD.

AB is common S

$\angle ABC = \angle BAD = 90^\circ$ A

$BC = AD$ (opposite sides of a parallelogram are equal in length) S

$\therefore \Delta ABC \equiv \Delta BAD$ (SAS)

$\therefore AC = BD$, so diagonals are of equal length.

First, draw a rectangle with the given properties.

Choose ΔABC and ΔBAD, which each contain one diagonal.

Prove congruent triangles using SAS.

Corresponding sides in congruent triangles.

Now you try

Prove that a kite (with two pairs of equal adjacent sides) has one pair of equal angles.

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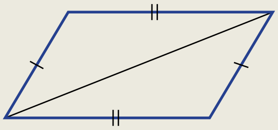
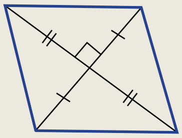
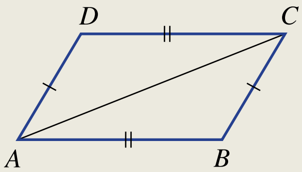
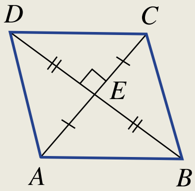
Example 7 Testing for a type of quadrilateral

a Prove that if opposite sides of a quadrilateral are equal in length, then it is a parallelogram.



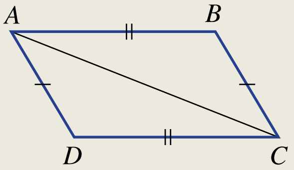
b Prove that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.



SOLUTION	EXPLANATION
a 	First, label your quadrilateral and choose two triangles, ΔABC and ΔCDA.
$AB = CD$ (given) S $BC = DA$ (given) S AC is common S $\therefore \Delta ABC \equiv \Delta CDA$ (SSS)	Prove that they are congruent using SSS.
$\therefore \angle BAC = \angle DCA$ So $AB \parallel DC$ (since alternate angles are equal). Also $\angle ACB = \angle CAD$. $\therefore AD \parallel BC$ (since alternate angles are equal). $\therefore ABCD$ is a parallelogram.	Choose corresponding angles in the congruent triangles to show that opposite sides are parallel. If alternate angles between lines are equal then the lines must be parallel.
b 	All angles at the point E are 90°, so it is easy to prove that all four smaller triangles are congruent using SAS.
$\Delta ABE \equiv \Delta CBE \equiv \Delta ADE \equiv \Delta CDE$ by SAS	
$\therefore AB = CB = CD = DA$ $\therefore ABCD$ is a rhombus.	Corresponding sides in congruent triangles. Every quadrilateral with four equal sides is a rhombus.

Now you try

Prove, using this diagram, that if opposite sides of a quadrilateral are equal in length, then it is a parallelogram.

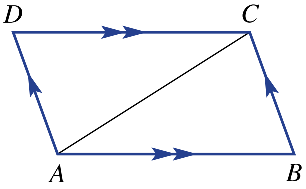


Exercise 2C

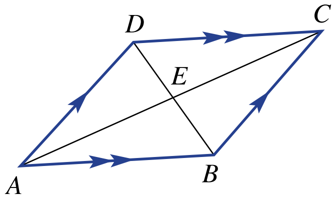
FLUENCY	1, 2	1–3	1–3

Example 6

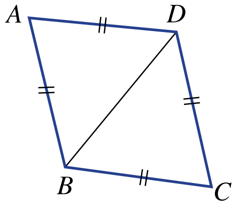
1 Complete these steps to prove that a parallelogram (with opposite parallel sides) has equal opposite sides.
   a Prove $\Delta ABC \equiv \Delta CDA$.
   b Hence, prove opposite sides are equal.



2 Complete these steps to prove that a parallelogram (with opposite equal parallel sides) has diagonals that bisect each other.
   a Prove $\Delta ABE \equiv \Delta CDE$.
   b Hence, prove $AE = CE$ and $BE = DE$.



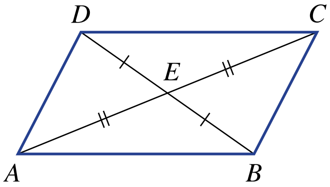
3 Complete these steps to prove that a rhombus (with sides of equal length) has diagonals that bisect the interior angles.
   a Prove $\Delta ABD \equiv \Delta CDB$.
   b Hence, prove BD bisects both $\angle ABC$ and $\angle CDA$.



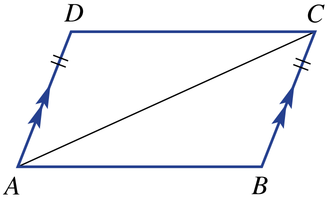
PROBLEM-SOLVING	4	4, 5	5, 6

Example 7

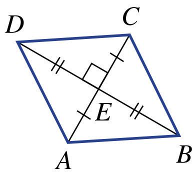
4 Complete these steps to prove that if the diagonals in a quadrilateral bisect each other, then it is a parallelogram.
   a Prove $\Delta ABE \equiv \Delta CDE$.
   b Hence, prove $AB \parallel DC$ and $AD \parallel BC$.



5 Complete these steps to prove that if one pair of opposite sides is equal and parallel in a quadrilateral, then it is a parallelogram.
   a Prove $\Delta ABC \equiv \Delta CDA$.
   b Hence, prove $AB \parallel DC$.



6 Complete these steps to prove that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.
   a Give a brief reason why $\Delta ABE \equiv \Delta CBE \equiv \Delta ADE \equiv \Delta CDE$.
   b Hence, prove $ABCD$ is a rhombus.



REASONING	7	7, 8	8, 9

7 Prove that the diagonals of a rhombus (i.e. a parallelogram with sides of equal length):

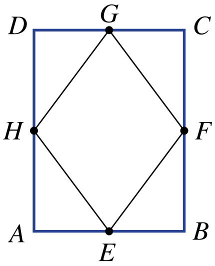
a intersect at right angles **b** bisect the interior angles.

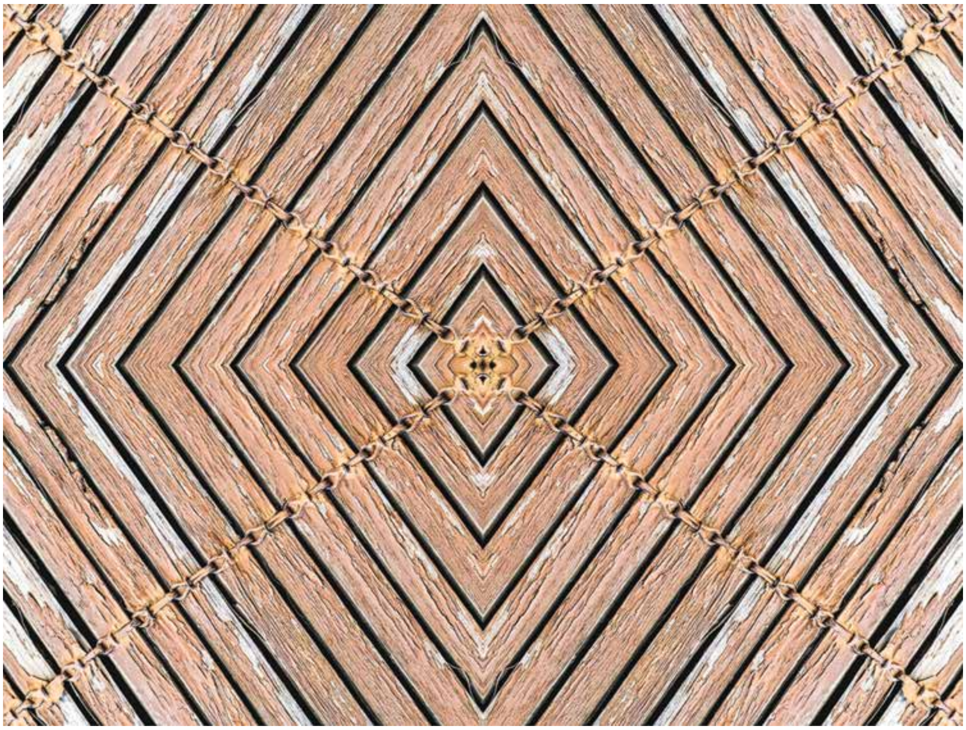
8 Prove that a parallelogram with one right angle is a rectangle.

9 Prove that if the diagonals of a quadrilateral bisect each other and are of equal length, then it is a rectangle.

ENRICHMENT: Rhombus in a rectangle	–	–	10

10 In this diagram, E, F, G and H are the midpoints of AB, BC, CD and DA, respectively, and $ABCD$ is a rectangle.
Prove that $EFGH$ is a rhombus.





2D Similar figures CONSOLIDATING

LEARNING INTENTIONS
- To know that similar figures have the same shape but a different size
- To understand that for figures to be similar their corresponding angles must be equal and their corresponding sides in the same ratio
- To know how to find the scale factor or ratio between two similar figures
- To be able to use the scale factor to determine side lengths on similar figures

You will recall that the transformation called enlargement involves reducing or enlarging the size of an object. The final image will be of the same shape but of different size. This means that matching pairs of angles will be equal and matching sides will be in the same ratio, just as in an accurate scale model.

Architects, engineers and governments use scale models to help find design errors and improvements. Businesses that specialise in making 3D scale models use physical construction, 3D printing and computer simulations.

Lesson starter: The Q1 tower

The Q1 tower, pictured below, is located on the Gold Coast and was the world's tallest residential tower up until 2011. It is 245 m tall.

- Measure the height and width of the Q1 tower in this photograph.
- Can a scale factor for the photograph and the actual Q1 tower be calculated? How?
- How can you calculate the actual width of the Q1 tower using this photograph? Discuss.

KEY IDEAS

- **Similar figures** have the same shape but are of different size.
  - Corresponding angles are equal.
  - Corresponding sides are in the same proportion (or ratio).
- **Scale factor** $= \dfrac{\text{image length}}{\text{original length}}$
- The symbols ||| or ~ are used to describe similarity and to write similarity statements. For example, $\Delta ABC \mathbin{|||} \Delta DEF$ or $\Delta ABC \sim \Delta DEF$.

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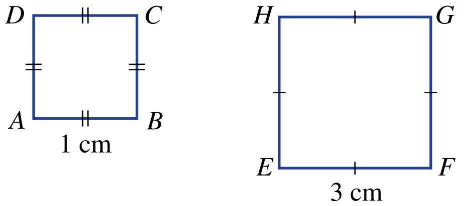
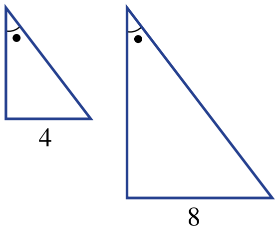
BUILDING UNDERSTANDING



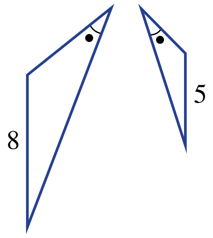
1. These two figures are squares.
   - **a** Would you say that the two squares are similar? Why?
   - **b** What is the scale factor when the smaller square is enlarged to the size of the larger square?
   - **c** If the larger square is enlarged by a factor of 5, what would be the side length of the image?

2. Find the scale factor for each shape and its larger similar image.

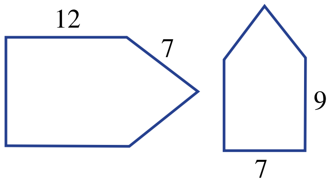
a



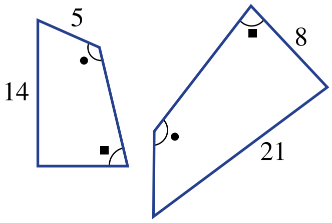
b

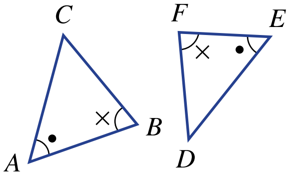


c



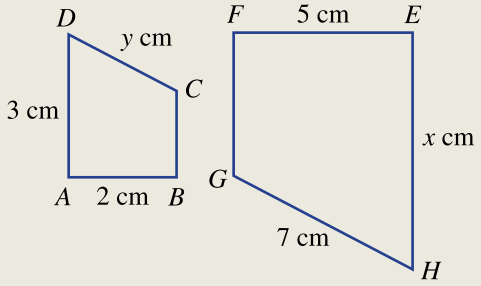
d





3. The two triangles shown opposite are similar.
   - **a** In ΔABC, which vertex corresponds to (matches) vertex E?
   - **b** In ΔABC, which angle corresponds to $\angle D$?
   - **c** In ΔDEF, which side corresponds to BC?
   - **d** Give the similarity statement for the two triangles with matching vertices in the same order.

Example 8 Finding and using scale factors



These two shapes are similar.

- **a** Write a similarity statement for the two shapes.
- **b** Complete the following: $\frac{EH}{\text{----}} = \frac{FG}{\text{----}}$.
- **c** Find the scale factor.
- **d** Find the value of x.
- **e** Find the value of y.

Continued on next page

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SOLUTION	**EXPLANATION**
a $ABCD \parallel\!\mid EFGH$ or $ABCD \sim EFGH$	Use the symbol $\parallel\!\mid$ or $\sim$ in similarity statements.
b $\frac{EH}{AD} = \frac{FG}{BC}$	Ensure you match corresponding vertices.
c $\frac{EF}{AB} = \frac{5}{2}$ or 2.5	EF and AB are matching sides and both lengths are given.
d $x = 3 \times 2.5$ $= 7.5$	EH corresponds to AD, which is 3 cm in length. Multiply by the scale factor.
e $y \times 2.5 = 7$ $y = 7 \div 2.5$ $= 2.8$	DC corresponds to HG, which is 7 cm in length. Multiply y by the scale factor to give 7. Solve for y.

Now you try

These two shapes are similar.

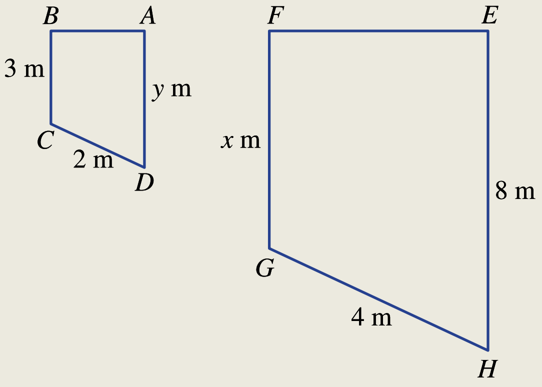
a Write a similarity statement for the two shapes.

b Complete the following: $\frac{EH}{----} = \frac{FG}{----}$.

c Find the scale factor.

d Find the value of x.

e Find the value of y.



Exercise 2D

FLUENCY	1, 2	1, 3, 4	3, 4

Example 8

1 These two shapes are similar.

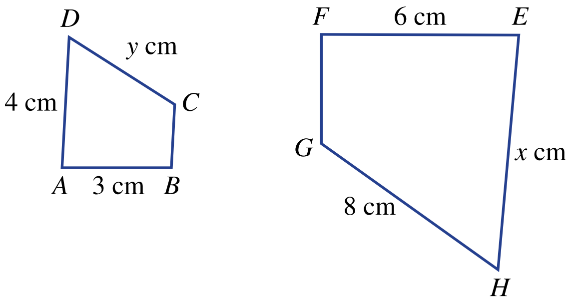
a Write a similarity statement for the two shapes.

b Complete the following: $\frac{EH}{----} = \frac{FG}{----}$.

c Find the scale factor.

d Find the value of x.

e Find the value of y.



2 These two shapes are similar.

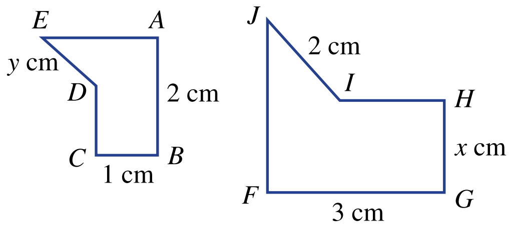
a Write a similarity statement for the two shapes.

b Complete the following: $\frac{AB}{----} = \frac{DE}{----}$.

c Find the scale factor.

d Find the value of x.

e Find the value of y.



3 These two shapes are similar.

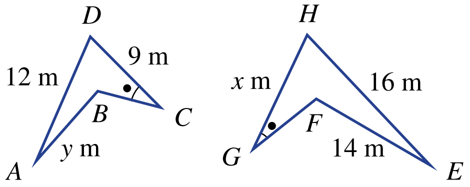
a Write a similarity statement for the two shapes.

b Complete the following: $\frac{EF}{----} = \frac{----}{CD}$.

c Find the scale factor.

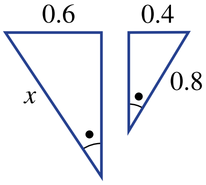
d Find the value of x.

e Find the value of y.

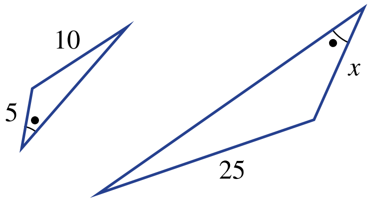


4 Find the value of the pronumeral in each pair of similar figures. Round to one decimal place where necessary.

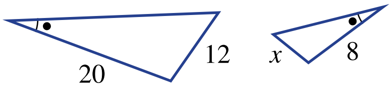
a



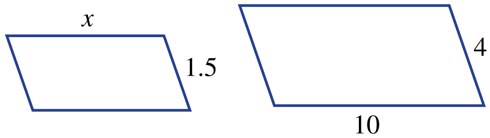
b



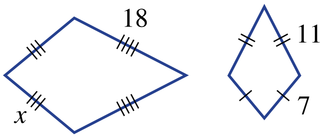
c



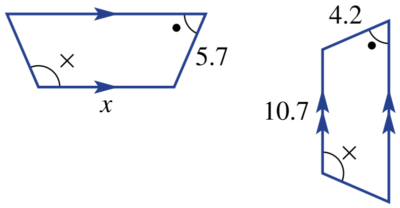
d



e



f



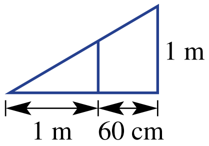
PROBLEM–SOLVING	5, 6	5, 6	6, 7

5 A 50 m tall structure casts a shadow 30 m in length. At the same time, a person casts a shadow of 1.02 m. Estimate the height of the person. (*Hint*: Draw a diagram of two triangles.)

6 A BMX ramp has two vertical supports, as shown.

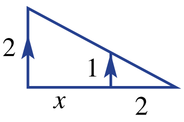
a Find the scale factor for the two triangles in the diagram.

b Find the length of the inner support.

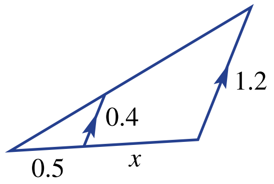


7 Find the value of the pronumeral if the pairs of triangles are similar. Round to one decimal place in part **d**.

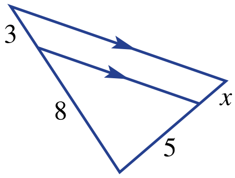
a



b



c



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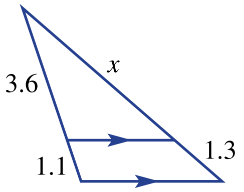

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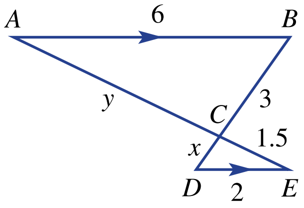
REASONING

8 | 8, 9 | 9, 10

8 In this diagram, the two triangles are similar and $AB \parallel DE$.

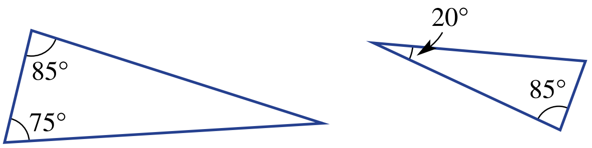


- **a** Which side in ΔABC corresponds to DC? Give a reason.
- **b** Write a similarity statement by matching the vertices.
- **c** Find the value of x.
- **d** Find the value of y.

9 Decide whether each statement is true or false.

- **a** All circles are similar.
- **b** All squares are similar.
- **c** All rectangles are similar.
- **d** All rhombuses are similar.
- **e** All parallelograms are similar.
- **f** All trapeziums are similar.
- **g** All kites are similar.
- **h** All isosceles triangles are similar.
- **i** All equilateral triangles are similar.
- **j** All regular hexagons are similar.

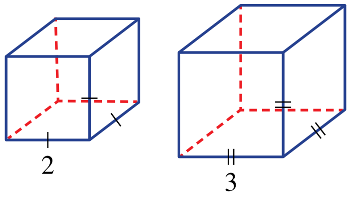
10 These two triangles each have two given angles. Decide whether they are similar and give reasons.



ENRICHMENT: Length, area, volume

– | – | 11

11 Shown here is a cube of side length 2 and its image after enlargement.



- **a** Write down the scale factor for the side lengths as an improper fraction.
- **b** Find the area of one face of:
  - **i** the smaller cube
  - **ii** the larger cube.
- **c** Find the volume of:
  - **i** the smaller cube
  - **ii** the larger cube.
- **d** Complete this table.

Cube	Length	Area	Volume
Small	2		
Large	3		
Scale factor (fraction)			

- **e** How do the scale factors for Area and Volume relate to the scale factor for side length?
- **f** If the side length scale factor was $\frac{b}{a}$, write down expressions for:
  - **i** the area scale factor
  - **ii** the volume scale factor.

2E Proving and applying similar triangles

LEARNING INTENTIONS
- To know the four tests for similarity of triangles
- To understand that two corresponding pairs of sides must be known to identify a common ratio
- To know how to prove that two triangles are similar using the tests
- To be able to use similarity of triangles to find unknown values

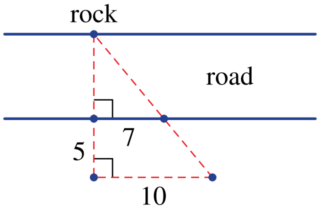
As with congruent triangles, there are four tests for proving that two triangles are similar. When solving problems involving similar triangles, it is important to recognise and be able to prove that they are in fact similar.

Optical engineers and optometrists use similar triangle geometry to model the path of light rays through lenses and to calculate the size of virtual images. The designs of spectacles, cameras, microscopes, telescopes and projectors all use similar triangle analysis.

Lesson starter: How far did the chicken travel?

A chicken is considering how far it is across a road, so it places four pebbles in certain positions on one side of the road. Two of the pebbles are directly opposite a rock on the other side of the road. The number of chicken paces between three pairs of the pebbles is shown in the diagram.

- Has the chicken constructed any similar triangles? If so, discuss why they are similar.
- What scale factor is associated with the two triangles?
- Is it possible to find how many paces the chicken must take to get across the road? If so, show a solution.
- Why did the chicken cross the road?
  Answer: To explore similar triangles.



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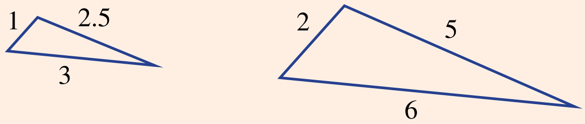
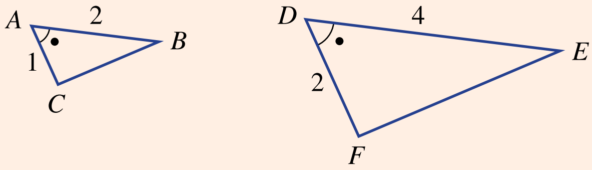
KEY IDEAS

- Two objects are said to be **similar** if they are of the same shape but of different size.
  - For two similar triangles ΔABC and ΔDEF, we write $\Delta ABC \,|||\, \Delta DEF$ or $\Delta ABC \sim \Delta DEF$.
  - When comparing two triangles, try to match up corresponding sides and angles, then look to see which similarity test can be used.
- Two triangles can be tested for **similarity** by considering the following conditions.
  - All pairs of corresponding sides are in the same ratio (or proportion) (SSS).



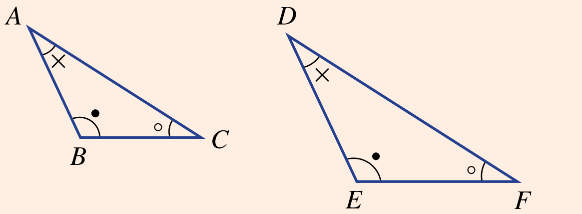
$$\frac{6}{3} = \frac{5}{2.5} = \frac{2}{1} = 2$$

  - Two pairs of corresponding sides are in the same ratio and the included corresponding angles are equal (SAS).



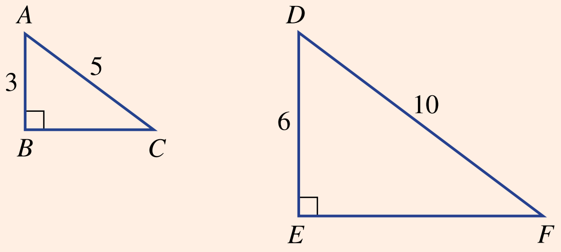
$$\frac{4}{2} = \frac{2}{1} = 2 \text{ and } \angle A = \angle D$$

  - Three corresponding angles are equal (AAA). (Remember that two pairs of corresponding equal angles implies that all three pairs of corresponding angles are equal.)



$$\angle A = \angle D$$
$$\angle B = \angle E$$
$$\angle C = \angle F$$

  - The hypotenuses of two right-angled triangles and another pair of corresponding sides are in the same ratio (RHS).

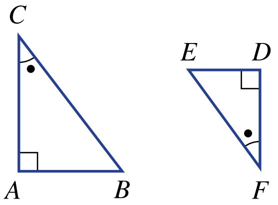


$$\angle B = \angle E = 90^\circ$$
$$\frac{10}{5} = \frac{6}{3} = 2$$

Note: If the test AAA is not used, then at least two pairs of corresponding sides in the same ratio are required for all the other three tests.

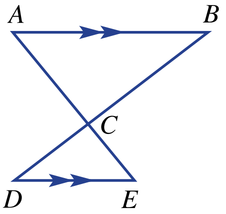
BUILDING UNDERSTANDING

1 This diagram includes two similar triangles.



a Which vertex in ΔDEF corresponds to vertex B?
b Which angle in ΔABC corresponds to $\angle F$?
c Which side in ΔABC corresponds to DE?
d Write a similarity statement for the two triangles.

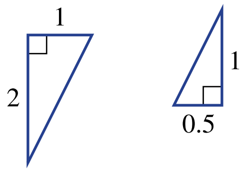
2 This diagram includes two similar triangles.



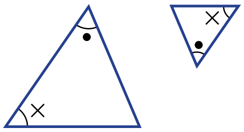
a Which angle in ΔCDE corresponds to $\angle B$ in ΔABC and why?
b Which angle in ΔABC corresponds to $\angle E$ in ΔCDE and why?
c Which angle is vertically opposite $\angle ACB$?
d Which side on ΔABC corresponds to side CE on ΔCDE?
e Write a similarity statement, being sure to write matching vertices in the same order.

3 Which similarity test (SSS, SAS, AAA or RHS) would be used to prove that these pairs of triangles are similar?

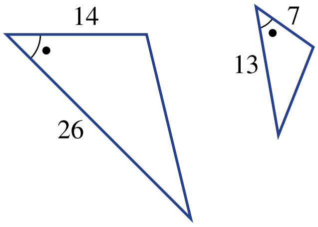
a



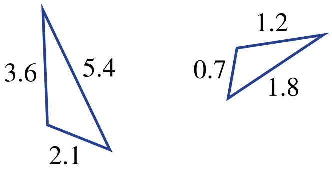
b



c



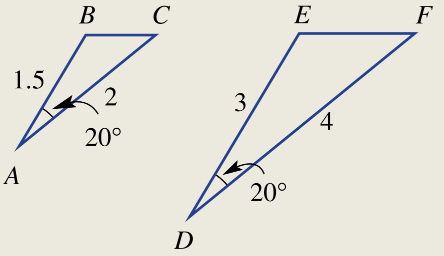
d



Example 9 Using similarity tests to prove similar triangles

Prove that the following pairs of triangles are similar.

a



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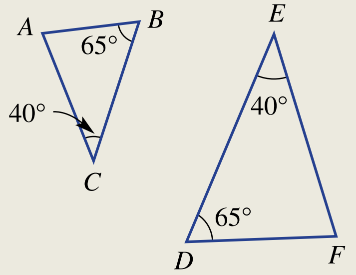


SOLUTION

a $\frac{DF}{AC} = \frac{4}{2} = 2$ (ratio of corresponding sides) S

$\frac{DE}{AB} = \frac{3}{1.5} = 2$ (ratio of corresponding sides) S

$\angle BAC = \angle EDF = 20°$ (given corresponding angles) A

$\therefore \Delta ABC \mathbin{|||} \Delta DEF$ (SAS)

b $\angle ABC = \angle FDE = 65°$ (given corresponding angles) A

$\angle ACB = \angle FED = 40°$ (given corresponding angles) A

$\therefore \Delta ABC \mathbin{|||} \Delta FDE$ (AAA)

EXPLANATION

DF and *AC* are corresponding sides and *DE* and *AB* are corresponding sides, and both pairs are in the same ratio.

The corresponding angle between the pair of corresponding sides in the same ratio is also equal.

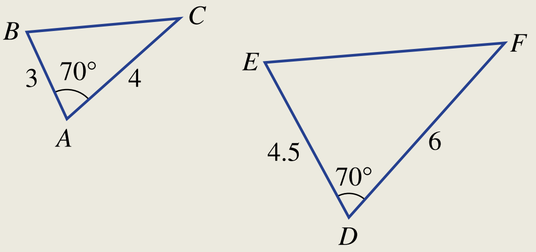
The two triangles are therefore similar.

There are two pairs of given corresponding angles. If two pairs of corresponding angles are equal, then the third pair must also be equal (due to angle sum).
The two triangles are therefore similar.

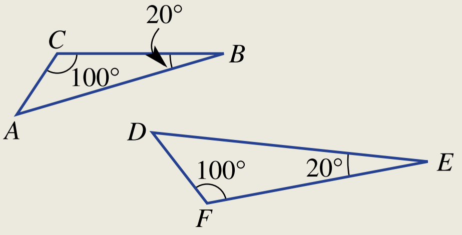
Now you try

Prove that the following pairs of triangles are similar.

a



b

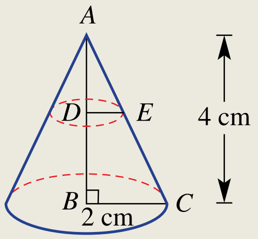


Example 10 Establishing and using similarity

A cone has radius 2 cm and height 4 cm. The top of the cone is cut horizontally through D.

a Prove $\Delta ADE \mid\mid\mid \Delta ABC$.

b If $AD = 1$ cm, find the radius DE.



SOLUTION

a $\angle BAC$ is common A

$\angle ABC = \angle ADE$ (corresponding angles in parallel lines) A

$\therefore \Delta ADE \mid\mid\mid \Delta ABC$ (AAA)

b
$$\frac{DE}{BC} = \frac{AD}{AB}$$
$$\frac{DE}{2} = \frac{1}{4}$$
$$\therefore DE = \frac{2}{4}$$
$$= 0.5 \text{ cm}$$

EXPLANATION

All three pairs of corresponding angles are equal.

Therefore, the two triangles are similar.

Given the triangles are similar, the ratio of corresponding sides must be equal.

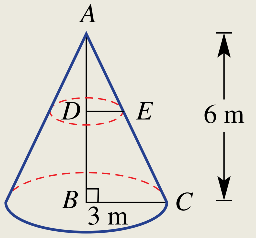
Solve for DE.

Now you try

A cone has radius 3 m and height 6 m. The top of the cone is cut horizontally through D.

a Prove $\Delta ADE \mid\mid\mid \Delta ABC$.

b If $AD = 2$ m, find the radius DE.



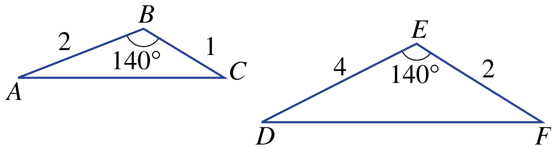
Exercise 2E

FLUENCY	1, 2(½)	1–3(½)	1–3(½)

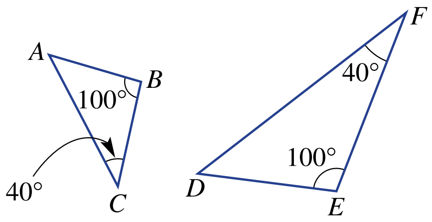
Example 9

1 Prove that the following pairs of triangles are similar.

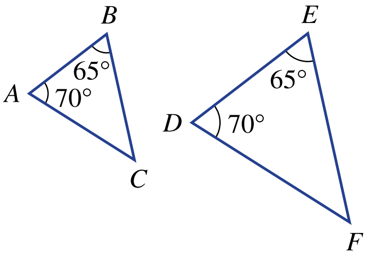
a



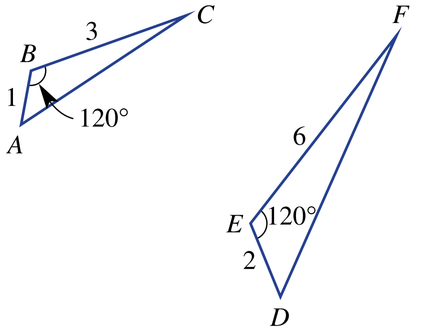
b



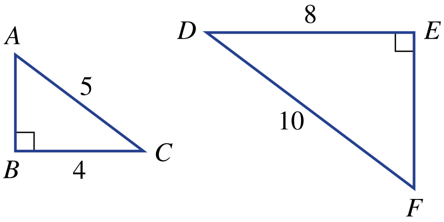
c



d



e



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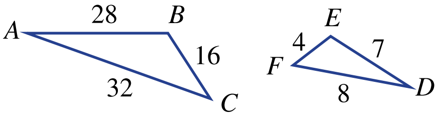
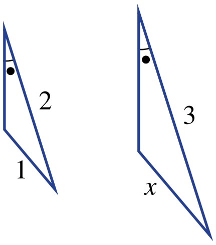


2 Find the value of the pronumerals in these pairs of similar triangles by first finding the scale factor.

a



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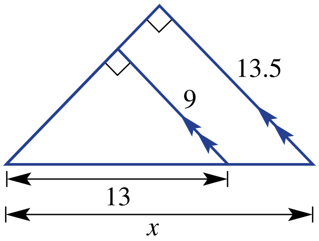

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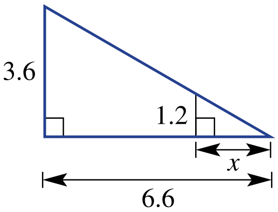

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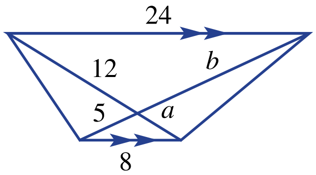
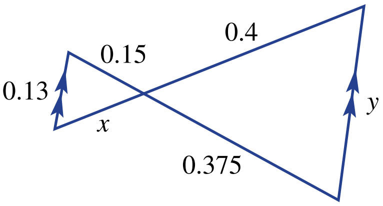


e



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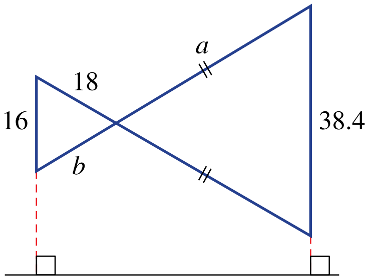
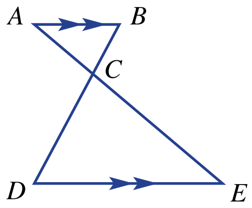
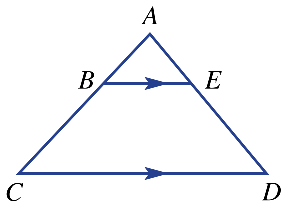


3 For the following proofs, give reasons at each step.

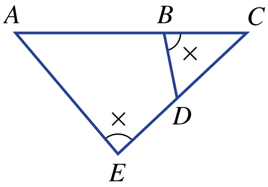
a Prove $\Delta ABC \;|||\; \Delta EDC$.



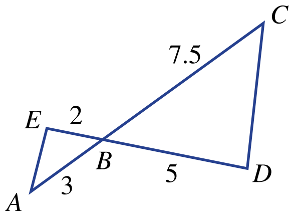
b Prove $\Delta ABE \;|||\; \Delta ACD$.



c Prove $\Delta BCD \;|||\; \Delta ECA$.



d Prove $\Delta AEB \;|||\; \Delta CDB$.



PROBLEM–SOLVING

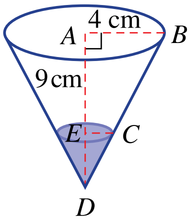
PROBLEM–SOLVING	4, 5	4, 6	6, 7

Example 10

4 A right cone with radius 4 cm has a total height of 9 cm. It contains an amount of water, as shown.

a Prove $\Delta EDC \;|||\; \Delta ADB$.

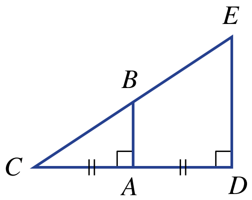
b If the depth of water in the cone is 3 cm, find the radius of the water surface in the cone.



5 A ramp is supported by a vertical stud AB, where A is the midpoint of CD. It is known that $CD = 4$ m and that the ramp is 2.5 m high; i.e. $DE = 2.5$ m.

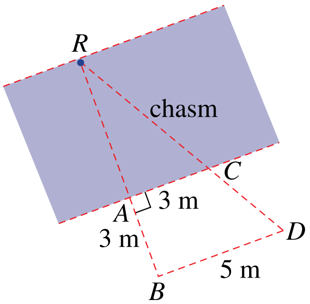
a Prove $\Delta BAC \;|||\; \Delta EDC$.

b Find the length of the stud AB.



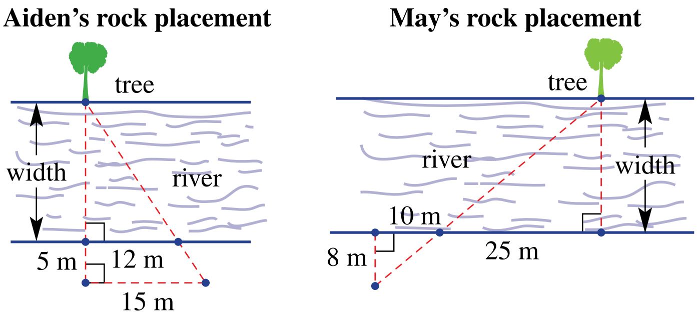
6 At a particular time in the day, Felix casts a shadow 1.3 m long and Curtis, who is 1.75 m tall, casts a shadow 1.2 m long. Find Felix's height, correct to two decimal places.

7 To determine the width of a chasm, a marker (A) is placed directly opposite a rock (R) on the other side. Point B is placed 3 m away from point A, as shown. Marker C is placed 3 m along the edge of the chasm, and marker D is placed so that BD is parallel to AC. Markers C and D and the rock are collinear (i.e. lie in a straight line). If BD measures 5 m, find the width of the chasm (AR).



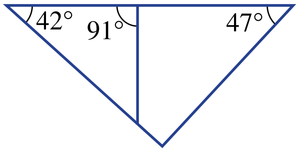
REASONING	8	8, 9	8–10

8 Aiden and May come to a river and notice a tree on the opposite bank. Separately they decide to place rocks (indicated with dots) on their side of the river to try to calculate the river's width.
They then measure the distances between some pairs of rocks, as shown.



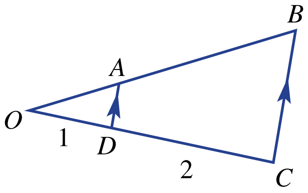
- **a** Have both Aiden and May constructed a pair of similar triangles? Give reasons.
- **b** Use May's triangles to calculate the width of the river.
- **c** Use Aiden's triangles to calculate the width of the river.
- **d** Which pair of triangles did you prefer to use? Give reasons.

9 There are two triangles in this diagram, each showing two given angles. Explain why they are similar.

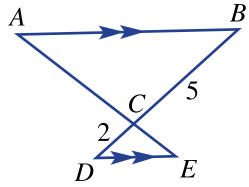


10 Prove the following, giving reasons.

a $OB = 3OA$



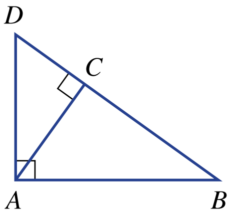
b $AE = \frac{7}{5}AC$



ENRICHMENT: Proving Pythagoras' theorem	–	–	11

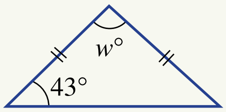
11 In this figure ΔABD, ΔCBA and ΔCAD are right angled.

- **a** Prove $\Delta ABD \,|||\, \Delta CBA$. Hence, prove $AB^2 = CB \times BD$.
- **b** Prove $\Delta ABD \,|||\, \Delta CAD$. Hence, prove $AD^2 = CD \times BD$.
- **c** Hence, prove Pythagoras' theorem $AB^2 + AD^2 = BD^2$.

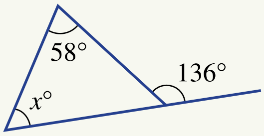


2A **1** Find the size of each pronumeral in the following diagrams, giving reasons.

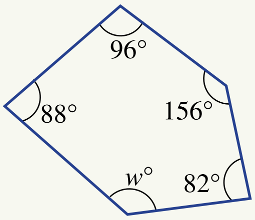
a



b



c



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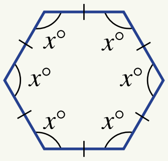
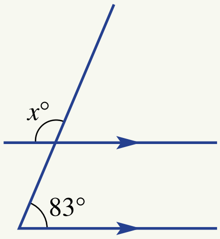


e



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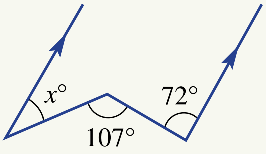
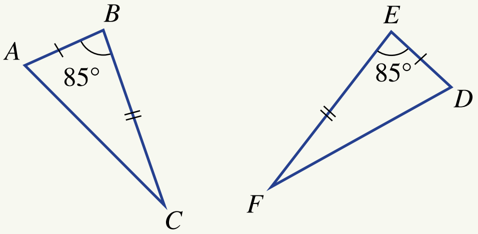
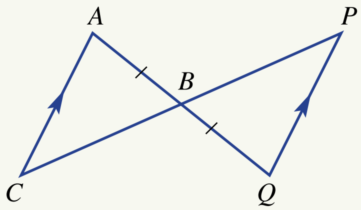


2B **2** Prove that this pair of triangles is congruent.

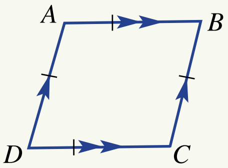


2B **3** **a** Prove that ΔABC is congruent to ΔQBP.



b Hence, prove that B is the midpoint of CP.

2C **4** Prove that a rhombus has its diagonals perpendicular to each other.



2C **5** Prove that if a quadrilateral is a parallelogram (opposite sides are parallel) the opposite sides are equal in length.

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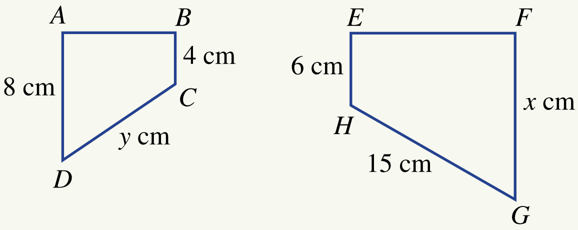
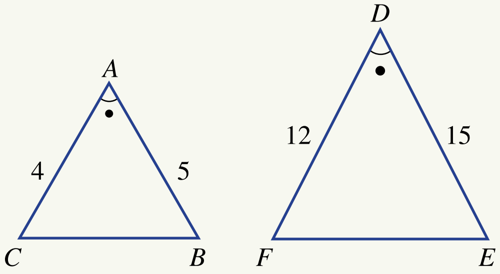
2D **6** For the two similar figures shown:

- **a** Write a similarity statement.
- **b** Find the scale factor.
- **c** Find the value of x.
- **d** Find the value of y.

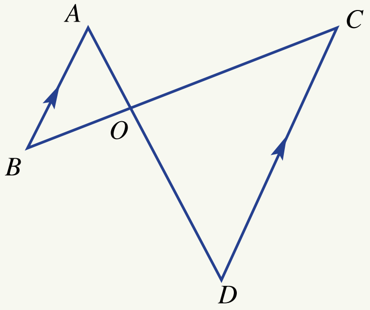


2E **7** Prove that the following pairs of triangles are similar.

a

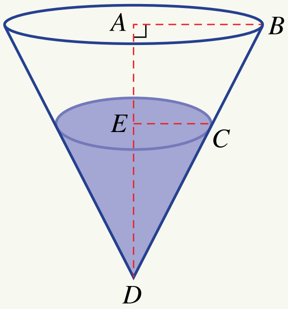


b

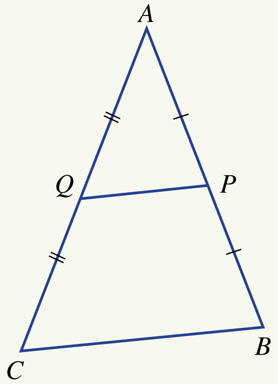


2E **8** A cone with radius 6 cm and height 10 cm is filled with water to a height of 5 cm.

- **a** Prove that ΔEDC is similar to ΔADB.
- **b** Find the radius of the water's surface (EC).



2E **9** Prove that if the midpoints, Q and P, of two sides of a triangle ABC are joined as shown, then QP is $\frac{1}{2}$ that of CB. (First prove similarity.)



2F Circle terminology and chord properties OPTIONAL

LEARNING INTENTIONS

- To know the meaning of the terms circle, chord, sector, arc and segment
- To understand what is meant by an angle that is subtended by an arc or chord
- To know the chord theorems and how to apply them to find certain lengths and angles
- To be able to prove the chord theorems using congruent triangles

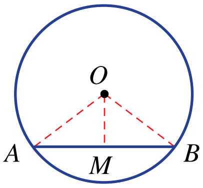
Although a circle appears to be a very simple object, it has many interesting geometrical properties. In this section we look at radii and chords in circles, and then explore and apply the properties of these objects. We use congruence to prove many of these properties.

Movie makers use circle geometry to help create the illusion of time passing slowly or of frozen motion, as in *The Matrix* (1999). Multiple cameras in a circle or arc sequentially or simultaneously photograph the actor and the images are stitched together.

Lesson starter: Dynamic chords

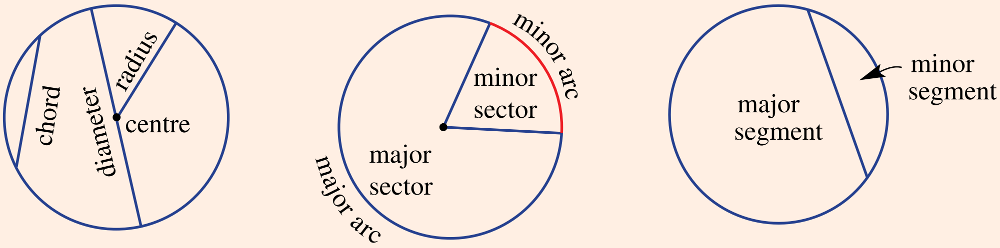
This activity would be enhanced with the use of interactive geometry. Chord AB sits on a circle with centre O. M is the midpoint of chord AB. Explore with interactive geometry software or discuss the following.

- Is ΔOAB isosceles and if so why?
- Is $\Delta OAM \equiv \Delta OBM$ and if so why?
- Is $AB \perp OM$ and if so why?
- Is $\angle AOM = \angle BOM$ and if so why?



KEY IDEAS

- Circle language



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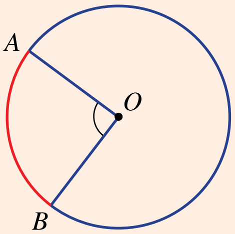
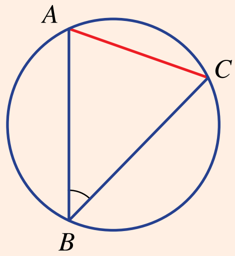
- An angle is **subtended** by an arc or chord if the arms of the angle meet the endpoints of the arc or chord.

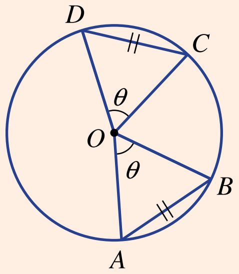


$\angle AOB$ is subtended at the centre by the minor arc AB.

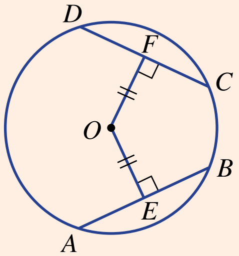


$\angle ABC$ is subtended at the circumference by the chord AC.

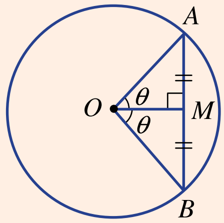
- **Chord theorem 1**: Chords of equal length subtend equal angles at the centre of the circle.
  - If $AB = CD$, then $\angle AOB = \angle COD$.
  - Conversely, if chords subtend equal angles at the centre of the circle, then the chords are of equal length.



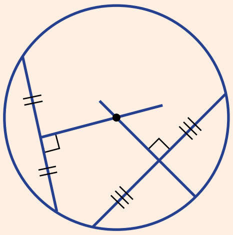
- **Chord theorem 2**: Chords of equal length are equidistant (i.e. of equal distance) from the centre of the circle.
  - If $AB = CD$, then $OE = OF$.
  - Conversely, if chords are equidistant from the centre of the circle, then the chords are of equal length.



- **Chord theorem 3**: The perpendicular from the centre of the circle to the chord bisects the chord and the angle at the centre subtended by the chord.
  - If $OM \perp AB$, then $AM = BM$ and $\angle AOM = \angle BOM$.
  - Conversely, if a radius bisects the chord (or angle at the centre subtended by the chord), then the radius is perpendicular to the chord.



- **Chord theorem 4**: The perpendicular bisectors of every chord of a circle intersect at the centre of the circle.
  - Constructing perpendicular bisectors of two chords will therefore locate the centre of a circle.



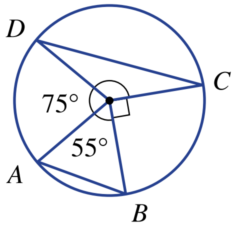
BUILDING UNDERSTANDING

1 Construct a large circle, then draw and label these features.

a chord
b radius
c centre
d minor sector
e major sector

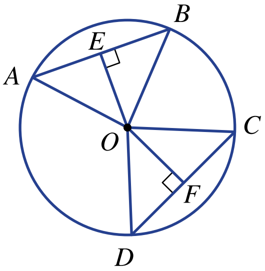
2 Give the size of the angle in this circle subtended by the following.

a chord AB
b minor arc BC
c minor arc AD
d chord DC



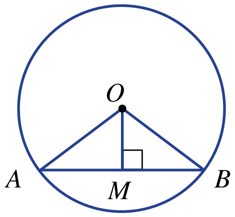
3 In this circle chords AB and CD are of equal length.

a Measure $\angle AOB$ and $\angle COD$.
b What do you notice about your answers from part **a**? Which chord theorem does this relate to?
c Measure OE and OF.
d What do you notice about your answers from part **c**? Which chord theorem does this relate to?



4 In this circle $OM \perp AB$.

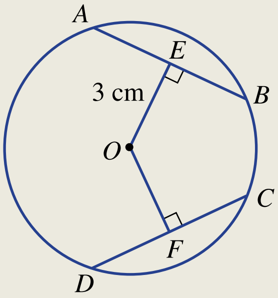
a Measure AM and BM.
b Measure $\angle AOM$ and $\angle BOM$.
c What do you notice about your answers from parts **a** and **b**? Which chord theorem does this relate to?



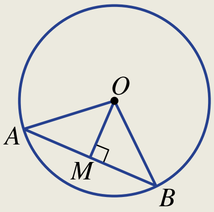
Example 11 Using chord theorems

For each part, use the given information and state which chord theorem is used.

a Given $AB = CD$ and $OE = 3$ cm, find OF.



b Given $OM \perp AB$, $AB = 10$ cm and $\angle AOB = 92°$, find AM and $\angle AOM$.



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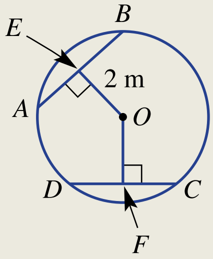
SOLUTION	EXPLANATION
a $OF = 3\text{ cm}$ (using chord theorem 2)	Chords of equal length are equidistant from the centre.
b Using chord theorem 3: $AM = 5\text{ cm}$ $\angle AOM = 46^\circ$	The perpendicular from the centre to the chord bisects the chord and the angle at the centre subtended by the chord. $10 \div 2 = 5$ and $92 \div 2 = 46$.

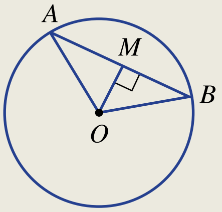
Now you try

For each part, use the given information and state which chord theorem is used.

a Given $AB = CD$ and $OE = 2\text{ m}$, find OF.



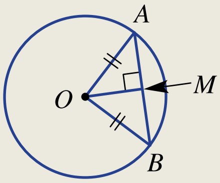
b Given $OM \perp AB$, $AB = 6\text{ m}$ and $\angle AOB = 120^\circ$, find AM and $\angle AOM$.



Example 12 Proving chord theorems

Prove chord theorem 3 in that the perpendicular from the centre of the circle to the chord bisects the chord and the angle at the centre subtended by the chord.

SOLUTION



$\angle OMA = \angle OMB = 90^\circ$ (given) R
$OA = OB$ (both radii) H
OM is common S
$\therefore \Delta OMA \equiv \Delta OMB$ (RHS)
$\therefore AM = BM$ and $\angle AOM = \angle BOM$

EXPLANATION

First, draw a diagram to represent the situation. The perpendicular forms a pair of congruent triangles.

Corresponding sides and angles in congruent triangles are equal.

Now you try

Prove chord theorem 2 in that chords of equal length are equidistant (of equal distance) from the centre of the circle.

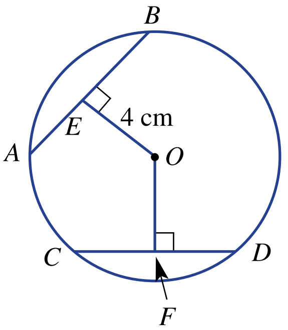
Exercise 2F

FLUENCY	1–3	1, 2, 4(½)	2, 3, 4(½)

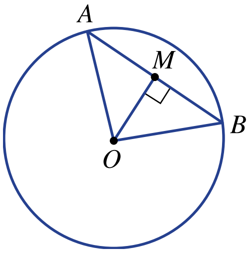
Example 11

1 For each part, use the given information and state which chord theorem is used.

a Given $AB = CD$ and $OE = 4\text{ cm}$, find OF.

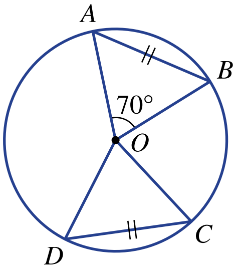


b Given $OM \perp AB$, $AB = 6\text{ m}$ and $\angle AOB = 100°$, find AM and $\angle AOM$.

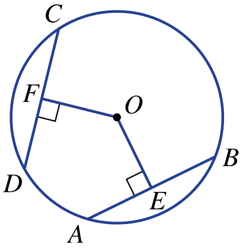


2 For each part, use the information given and state which chord theorem is used.

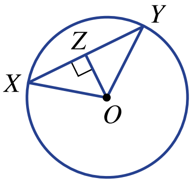
a Given $AB = CD$ and $\angle AOB = 70°$, find the value of $\angle DOC$.



b Given $AB = CD$ and $OF = 7.2\text{ cm}$, find the value of OE.



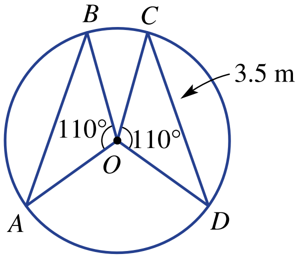
c Given $OZ \perp XY$, $XY = 8\text{ cm}$ and $\angle XOY = 102°$, find the value of XZ and $\angle XOZ$.



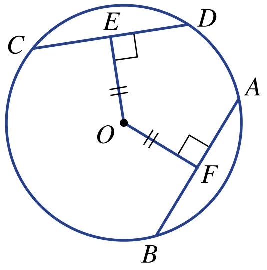
3 The perpendicular bisectors of two different chords of a circle are constructed. Describe where they intersect.

4 Use the information given to complete the following.

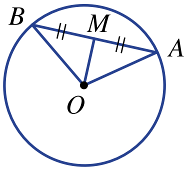
a Given $\angle AOB = \angle COD$ and $CD = 3.5$ m, find the value of AB.



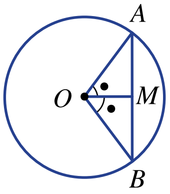
b Given $OE = OF$ and $AB = 9$ m, find the value of CD.



c Given M is the midpoint of AB, find the value of $\angle OMB$.



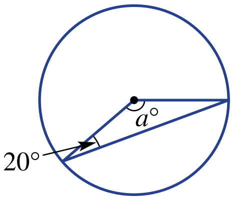
d Given $\angle AOM = \angle BOM$, find the value of $\angle OMB$.



PROBLEM–SOLVING	5	5, 6	5(½), 6, 7

5 Find the size of each unknown angle $a°$.

a



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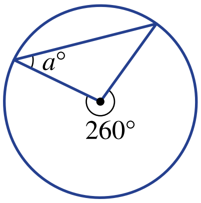

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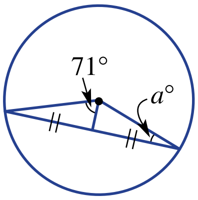

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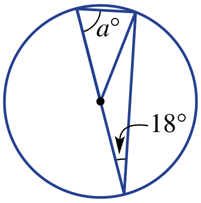

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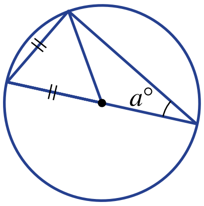

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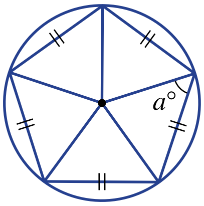
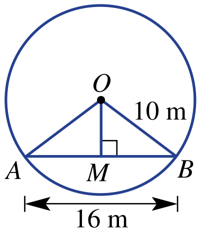
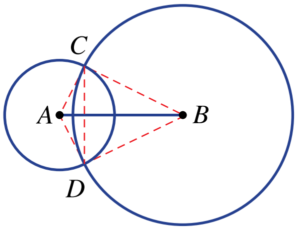


6 Find the length OM. (*Hint*: Use Pythagoras' theorem.)



7 In this diagram, radius $AD = 5$ mm, radius $BD = 12$ mm and chord $CD = 8$ mm. Find the exact length of AB, in surd form.



REASONING

8	8, 9	8–10

Example 12

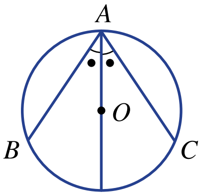
8 **a** Prove chord theorem 1 in that chords of equal length subtend equal angles at the centre of the circle.

b Prove the converse of chord theorem 1 in that if chords subtend equal angles at the centre of the circle then the chords are of equal length.

9 **a** Prove that if a radius bisects a chord of a circle then the radius is perpendicular to the chord.

b Prove that if a radius bisects the angle at the centre subtended by the chord, then the radius is perpendicular to the chord.

10 In this circle $\angle BAO = \angle CAO$. Prove $AB = AC$. (*Hint*: Construct two triangles.)



ENRICHMENT: Common chord proof

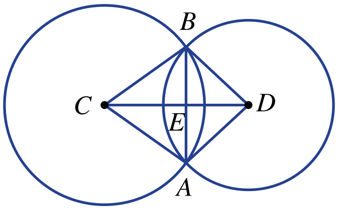
–	–	11

11 For this diagram, prove $CD \perp AB$ by following these steps.

a Prove $\Delta ACD \equiv \Delta BCD$.

b Hence, prove $\Delta ACE \equiv \Delta BCE$.

c Hence, prove $CD \perp AB$.



2G Angle properties of circles: Theorems 1 and 2 OPTIONAL

LEARNING INTENTIONS

- To know the relationship between angles at the centre of a circle and at the circumference subtended by the same arc
- To know that a triangle in a semicircle creates a right angle at the circumference
- To be able to combine these theorems with other properties of circles

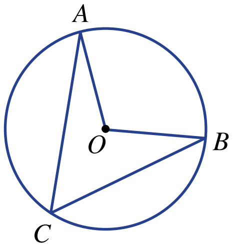
The special properties of circles extend to the pairs of angles formed by radii and chords intersecting at the circumference. In this section we explore the relationship between angles at the centre and at the circumference subtended by the same arc.

Road and railway tunnel design and construction are complex geological and technical processes. Civil engineers use geometry, including circle properties and theorems, to establish the geometrical requirements for structural stability.

Lesson starter: Discover angle properties – Theorems 1 and 2

This activity can be completed with the use of a protractor and pair of compasses, but would be enhanced by using interactive geometry software.

- First, construct a circle and include two radii and two chords, as shown. The size of the circle and position of points A, B and C on the circumference can vary.
- Measure $\angle ACB$ and $\angle AOB$. What do you notice?
- Now construct a new circle with points A, B and C at different points on the circumference. (If dynamic software is used simply drag the points.) Measure $\angle ACB$ and $\angle AOB$ once again. What do you notice?
- Construct a new circle with $\angle AOB = 180°$ so AB is a diameter. What do you notice about $\angle ACB$?



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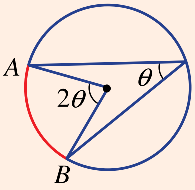
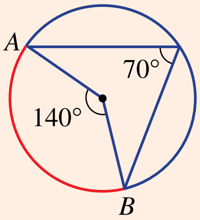
KEY IDEAS

- **Circle theorem 1**: Angles at the centre and circumference
  - The angle at the centre of a circle is twice the angle at a point on the circle subtended by the same arc.



For example:

1



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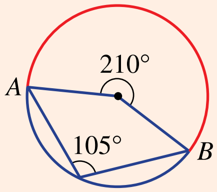

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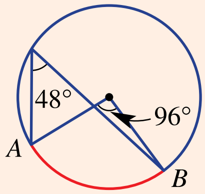
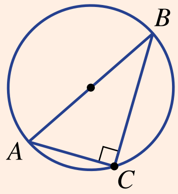


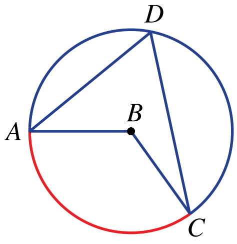
- **Circle theorem 2**: Angle in a semicircle
  - The angle in a semicircle is 90°.
  - This is a specific case of theorem 1, where $\angle ACB$ is known as the angle in a semicircle.



BUILDING UNDERSTANDING

① Name another angle that is subtended by the same arc as $\angle ABC$ in these circles.

a



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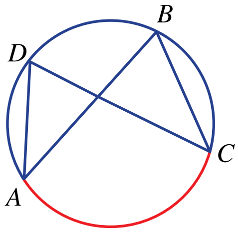

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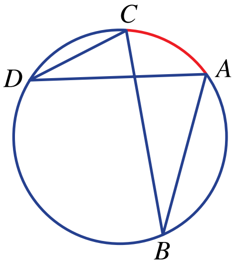

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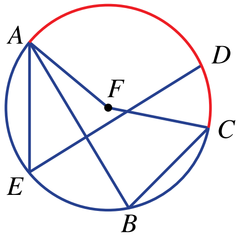

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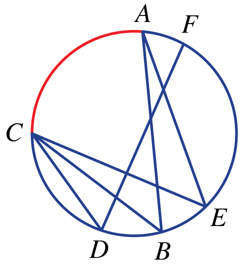

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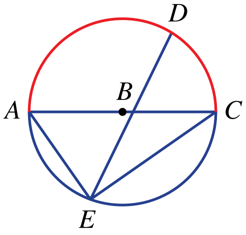
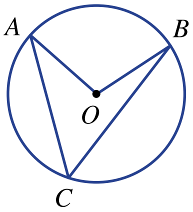
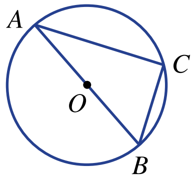


2 For this circle, O is the centre.
- a Name the angle at the centre of the circle.
- b Name the angle at the circumference of the circle.
- c If $\angle ACB = 40°$, find $\angle AOB$ using circle theorem 1.
- d If $\angle AOB = 122°$, find $\angle ACB$ using circle theorem 1.



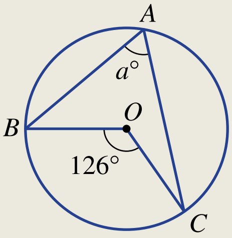
3 For this circle AB is a diameter.
- a What is the size of $\angle AOB$?
- b What is the size of $\angle ACB$ using circle theorem 2?
- c If $\angle CAB = 30°$, find $\angle ABC$.
- d If $\angle ABC = 83°$, find $\angle CAB$.



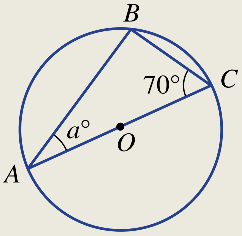
Example 13 Applying circle theorems 1 and 2

Find the value of the pronumerals in these circles.

a



b



SOLUTION

a
$$2a = 126$$
$$\therefore a = 63$$

From circle theorem 1, $\angle BOC = 2\angle BAC$.

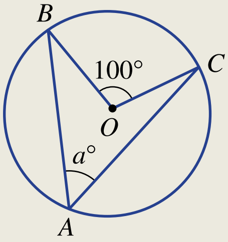
b $\angle ABC$ is 90°.
$$\therefore a + 90 + 70 = 180$$
$$\therefore a = 20$$

AC is a diameter, and from circle theorem 2 $\angle ABC = 90°$.

Now you try

Find the value of the pronumerals in these circles.

a



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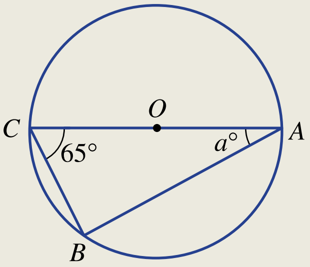
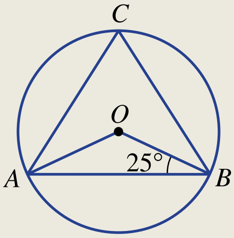


Example 14 Combining circle theorems with other circle properties

Find the size of ∠ACB.



SOLUTION

$$\begin{aligned} \angle OAB &= 25^\circ \\ \angle AOB &= 180^\circ - 2 \times 25^\circ \\ &= 130^\circ \\ \therefore \angle ACB &= 130^\circ \div 2 \\ &= 65^\circ \end{aligned}$$

EXPLANATION

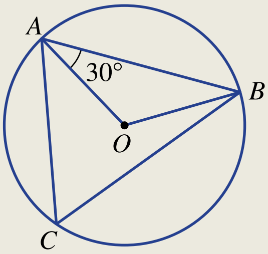
ΔAOB is isosceles.

Angle sum of a triangle is 180°.

The angle at the circumference is half the angle at the centre subtended by the same arc.

Now you try

Find the size of ∠ACB.



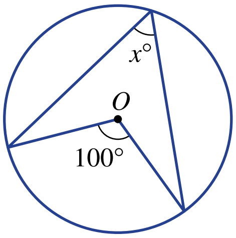
Exercise 2G

FLUENCY	1(½), 2	1(½), 2, 3	1(½), 2–4

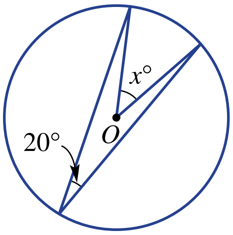
Example 13a

1 Find the value of x in these circles.

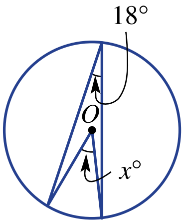
a



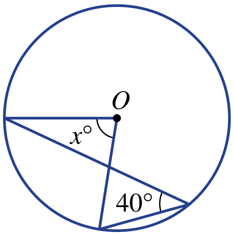
b



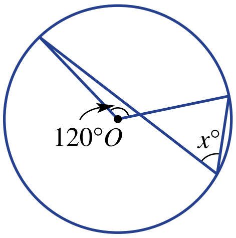
c



d



e



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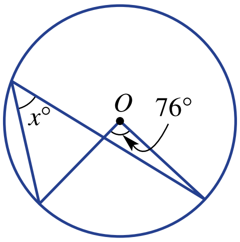
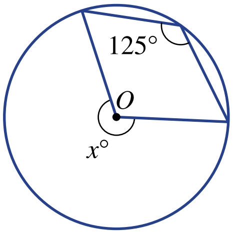
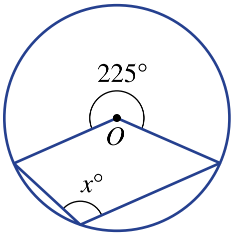


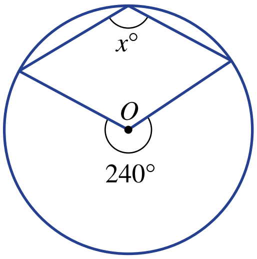
g



h



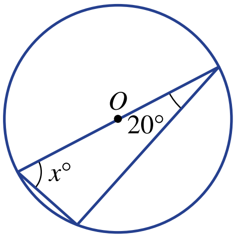
i



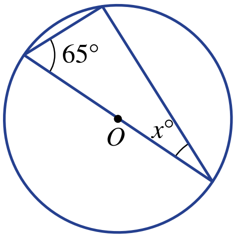
Example 13b

2 Find the value of x in these circles.

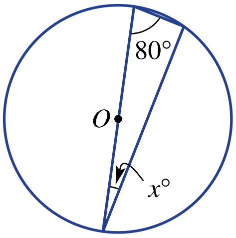
a



b

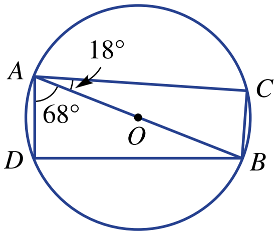


c

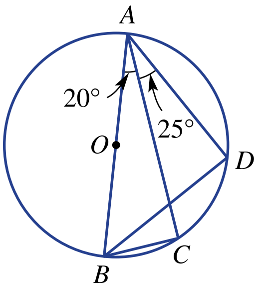


3 Find the size of both $\angle ABC$ and $\angle ABD$.

a



b



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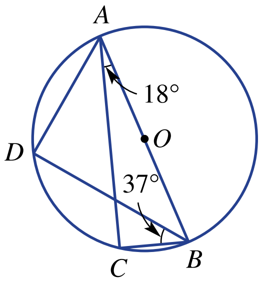
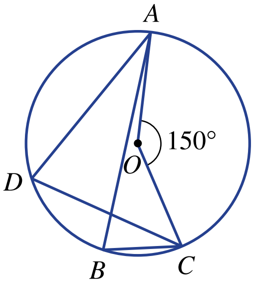
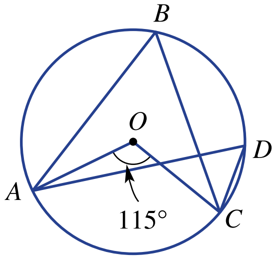


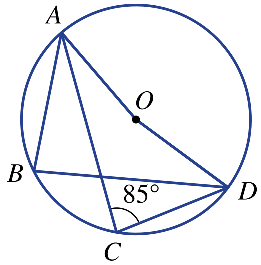
4 **a** Find the value of ∠*ADC* and ∠*ABC*.



b Find the value of ∠*ABC* and ∠*ADC*.



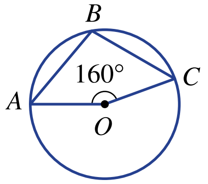
c Find the value of ∠*AOD* and ∠*ABD*.



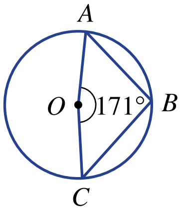
PROBLEM–SOLVING 5 5–6(½) 5–6(½)

5 Find the value of ∠*ABC*.

a



b



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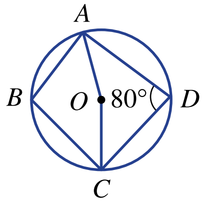

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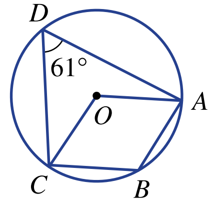

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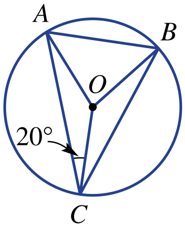

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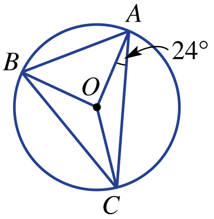
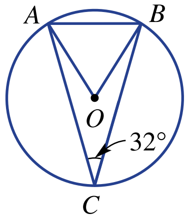


Example 14

6 Find the value of ∠*OAB*.

a



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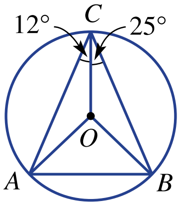

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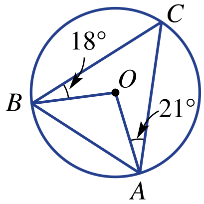

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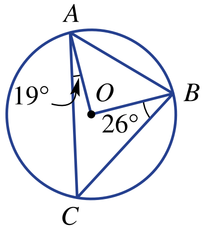

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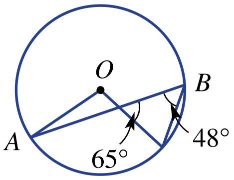

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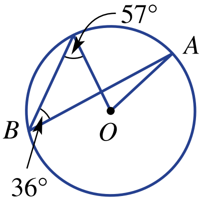
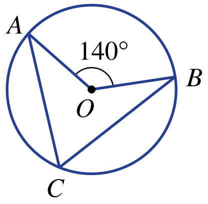


REASONING	7	7, 8	8–10

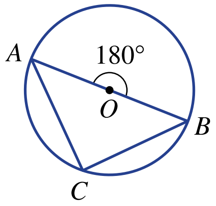
7 **a** For the first circle shown, use circle theorem 1 to find the value of $\angle ACB$.



b For the second circle shown, use circle theorem 1 to find the value of $\angle ACB$.

c For the second circle, what does circle theorem 2 say about $\angle ACB$?

d Explain why circle theorem 2 can be thought of as a special case of circle theorem 1.

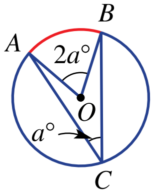


8 The two circles shown illustrate circle theorem 1 for both a minor arc and a major arc.

a When a minor arc is used, answer true or false.

- **i** $\angle AOB$ is always acute.
- **ii** $\angle AOB$ can be acute or obtuse.
- **iii** $\angle ACB$ is always acute.
- **iv** $\angle ACB$ can be acute or obtuse.

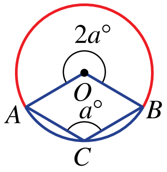
minor arc AB



b When a major arc is used, answer true or false.

- **i** $\angle ACB$ can be acute.
- **ii** $\angle ACB$ is always obtuse.
- **iii** The angle at the centre $(2a^\circ)$ is a reflex angle.
- **iv** The angle at the centre $(2a^\circ)$ can be obtuse.

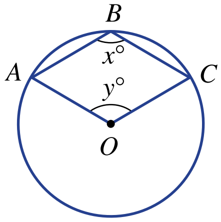
major arc AB



9 Consider this circle.

a Write reflex $\angle AOC$ in terms of x.

b Write y in terms of x.



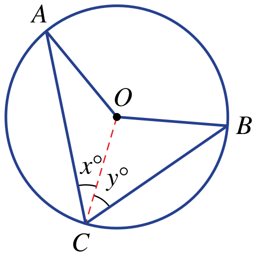
10 Prove circle theorem 1 for the case illustrated in this circle by following these steps and letting $\angle OCA = x^\circ$ and $\angle OCB = y^\circ$.

a Find $\angle AOC$ in terms of x, giving reasons.

b Find $\angle BOC$ in terms of y, giving reasons.

c Find $\angle AOB$ in terms of x and y.

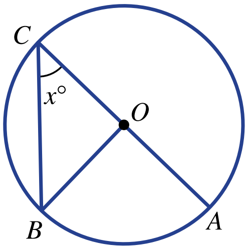
d Explain why $\angle AOB = 2\angle ACB$.



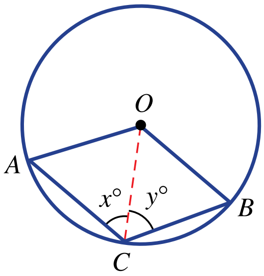
ENRICHMENT: Proving all cases	–	–	11, 12

11 Question **10** sets out a proof for circle theorem 1 using a given illustration. Now use a similar technique for these cases.

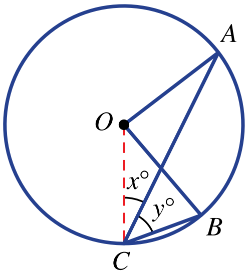
a Prove $\angle AOB = 2\angle ACB$; i.e. prove $\angle AOB = 2x^\circ$.



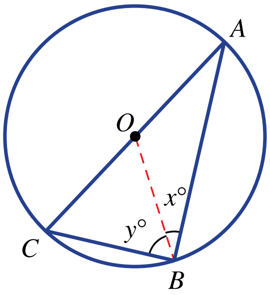
b Prove reflex $\angle AOB = 2\angle ACB$; i.e. prove reflex $\angle AOB = 2(x + y)^\circ$.



c Prove $\angle AOB = 2\angle ACB$; i.e. prove $\angle AOB = 2y^\circ$.



12 Prove circle theorem 2 by showing that $x + y = 90$.



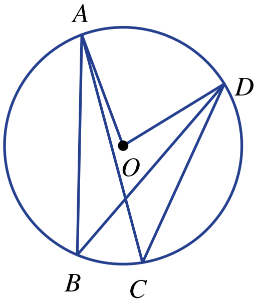
2H Angle properties of circles: Theorems 3 and 4 OPTIONAL

LEARNING INTENTIONS

- To know that angles at the circumference subtended by the same arc or chord are equal
- To know that a cyclic quadrilateral is one that has all four vertices on the circumference of a circle
- To know that opposite angles in a cyclic quadrilateral are supplementary
- To be able to apply the circle theorems to find unknown angles

When both angles are at the circumference, there are two important properties of pairs of angles in a circle to consider.

You will recall from circle theorem 1 that in this circle below, $\angle AOD = 2\angle ABD$ and also $\angle AOD = 2\angle ACD$. This implies that $\angle ABD = \angle ACD$, which is an illustration of circle theorem 3 – angles at the circumference subtended by the same arc are equal.



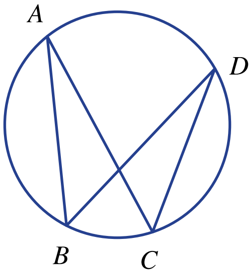
Agricultural engineers used circle geometry when designing the drive elements of the header, conveyer, separator, thresher and cutter units in the combine harvester.

The fourth theorem relates to cyclic quadrilaterals, which have all four vertices sitting on the same circle. This also will be explored in this section.

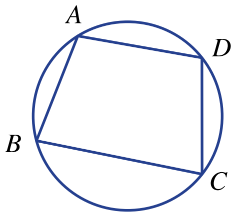
Lesson starter: Discover angle properties – Theorems 3 and 4

Use a protractor and a pair of compasses for this exercise or use interactive geometry software.

- Construct a circle with four points at the circumference, as shown.
- Measure $\angle ABD$ and $\angle ACD$. What do you notice? Drag A, B, C or D and compare the angles.

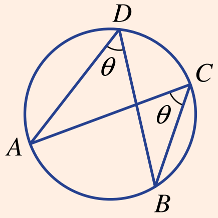


- Now construct this cyclic quadrilateral (or drag point C if using interactive geometry software).
- Measure $\angle ABC$, $\angle BCD$, $\angle CDA$ and $\angle DAB$. What do you notice? Drag A, B, C or D and compare angles.

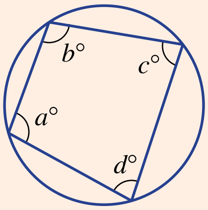


KEY IDEAS

- **Circle theorem 3**: Angles at the circumference
  - Angles at the circumference of a circle subtended by the same arc are equal.
    - As shown in the diagram, $\angle C = \angle D$ but note also that $\angle A = \angle B$.



- A **cyclic quadrilateral** has all four vertices sitting on the same circle.
- **Circle theorem 4**: Opposite angles in cyclic quadrilaterals
  - Opposite angles in a cyclic quadrilateral are supplementary (sum to 180°).

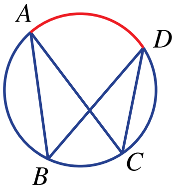


$a + c = 180$
$b + d = 180$

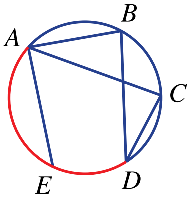
BUILDING UNDERSTANDING

① Name another angle that is subtended by the same arc as $\angle ABD$.

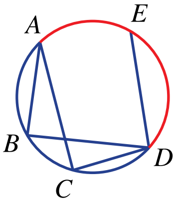
a



b



c



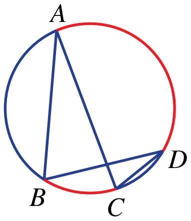
② For this circle, answer the following.

a Name two angles subtended by arc AD.

b Using circle theorem 3, state the size of $\angle ACD$ if $\angle ABD = 85°$.

c Name two angles subtended by arc BC.

d Using circle theorem 3, state the size of $\angle BAC$ if $\angle BDC = 17°$.



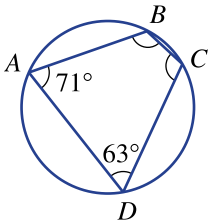
③ Circle theorem 4 states that opposite angles in a cyclic quadrilateral are supplementary.

a What does it mean when we say two angles are supplementary?

b Find the value of $\angle ABC$.

c Find the value of $\angle BCD$.

d Check that $\angle ABC + \angle BCD + \angle CDA + \angle DAB = 360°$.



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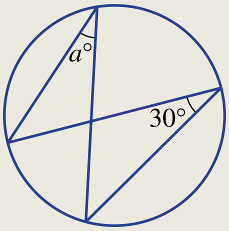
Cambridge University Press

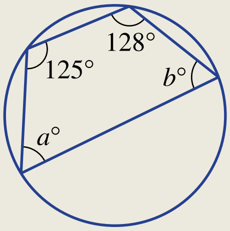
Example 15 Applying circle theorems 3 and 4

Find the value of the pronumerals in these circles.

a



b



SOLUTION

a $a = 30$

b
$$\begin{aligned} a + 128 &= 180 \\ \therefore a &= 52 \end{aligned}$$

$$\begin{aligned} b + 125 &= 180 \\ \therefore b &= 55 \end{aligned}$$

EXPLANATION

The $a°$ and 30° angles are subtended by the same arc. This is an illustration of circle theorem 3.

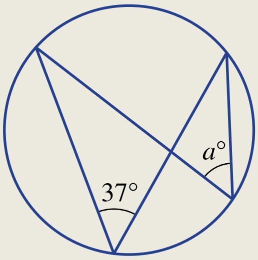
The quadrilateral is cyclic, so opposite angles sum to 180°.

This is an illustration of circle theorem 4.

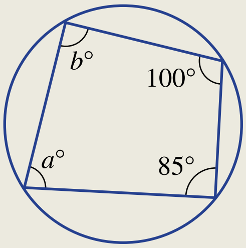
Now you try

Find the value of the pronumerals in these circles.

a



b



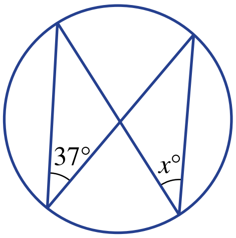
Exercise 2H

FLUENCY 1–2(½) 1–2(½) 1–2(⅓)

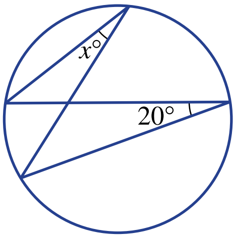
Example 15a

1 Find the value of x in these circles.

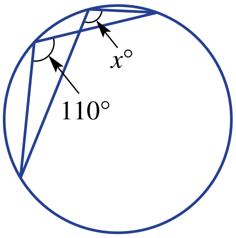
a



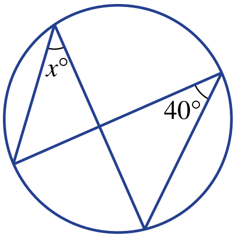
b



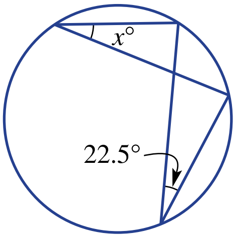
c



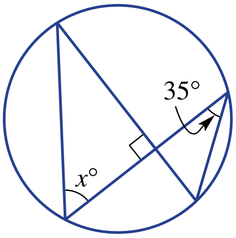
d



e



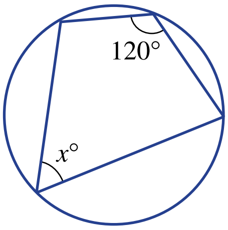
f



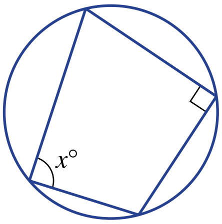
Example 15b

2 Find the value of the pronumerals in these circles.

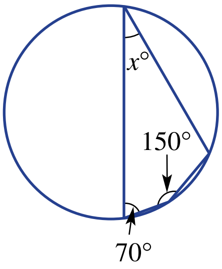
a



b



c



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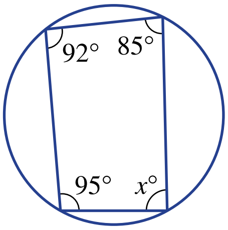

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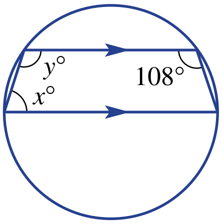

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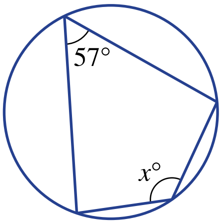
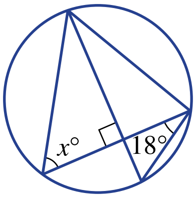


PROBLEM–SOLVING	3(1/2)	3–4(1/2)	3(1/3), 4(1/2)

3 Find the value of *x*.

a



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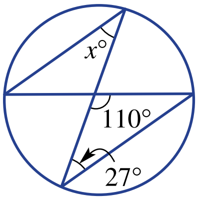

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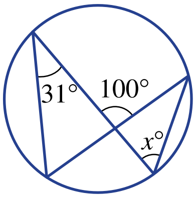

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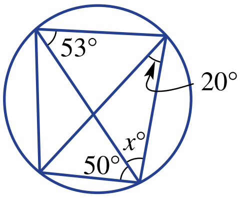

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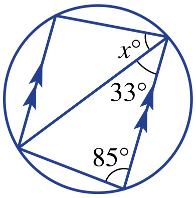

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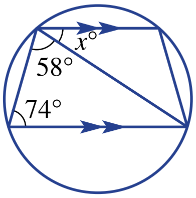
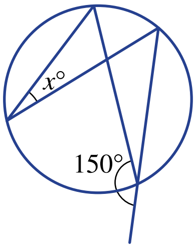
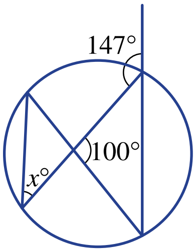


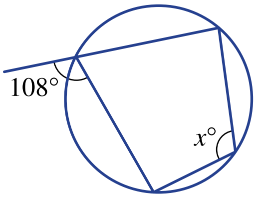
g



h

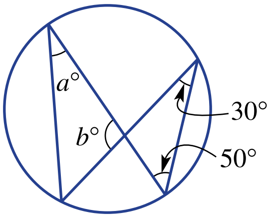


i

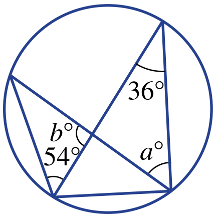


4 Find the values of the pronumerals in these circles.

a



b



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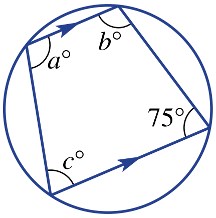

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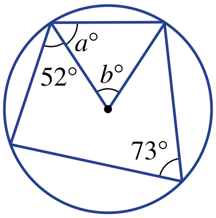
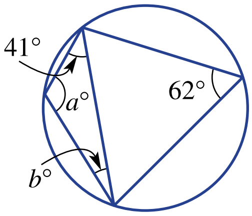


e



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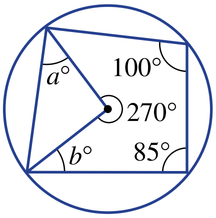


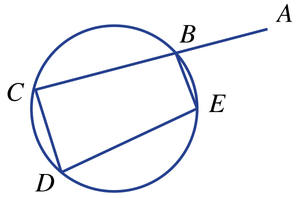
REASONING	5	5, 6	6, 7

5 $\angle ABE$ is an exterior angle to the cyclic quadrilateral $BCDE$.

a If $\angle ABE = 80°$, find $\angle CDE$.

b If $\angle ABE = 71°$, find $\angle CDE$.

c Prove that $\angle ABE = \angle CDE$ using circle theorem 4.

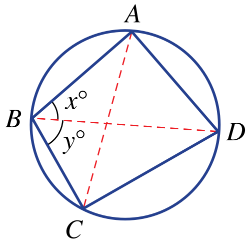


6 Prove that opposite angles in a cyclic quadrilateral are supplementary by following these steps.

a Explain why $\angle ACD = x°$ and $\angle DAC = y°$.

b Prove that $\angle ADC = 180° - (x + y)°$.

c What does this say about $\angle ABC$ and $\angle ADC$?

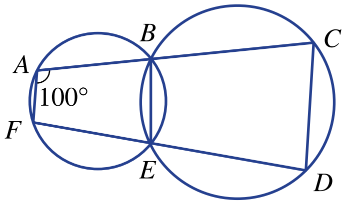


7 If $\angle BAF = 100°$, complete the following.

a Find:

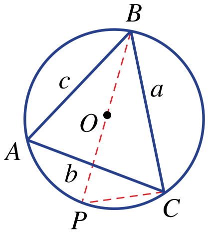
i $\angle FEB$ **ii** $\angle BED$ **iii** $\angle DCB$

b Explain why $AF \parallel CD$.



ENRICHMENT: A special property	–	–	8

8 Consider a triangle ABC inscribed in a circle. The construction line BP is a diameter and PC is a chord. If r is the radius, then $BP = 2r$.



a What can be said about $\angle PCB$? Give a reason.

b What can be said about $\angle A$ and $\angle P$? Give a reason.

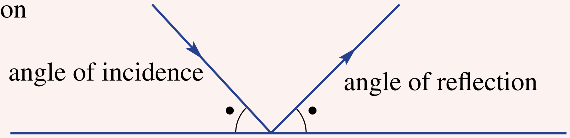
c If $BP = 2r$, use trigonometry with $\angle P$ to write an equation linking r and a.

d Prove that $2r = \dfrac{a}{\sin A}$, giving reasons.

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Mirror, mirror

1 The law of reflection says that the angle of reflection is equal to the angle of incidence as shown.



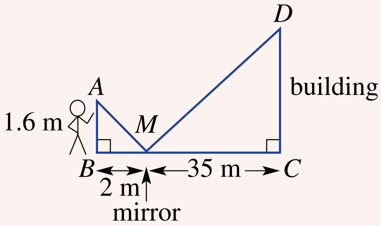
George, who is 1.6 m tall, places a mirror (M) on the ground in front of a building and then moves backwards away from the mirror until he can sight the top of the building in the centre of the mirror.

George is interested in how the height of the building can be calculated using the mirror. He wants to only use measurements that can be recorded from ground level and combine these with the similar triangles that are generated after positioning the mirror on the ground.

a The distance is measured between the mirror and the building (35 m) and the mirror and George's location (2 m) as shown below (not to scale).

i Prove that a pair of similar triangles has been formed.

ii Use similarity to find the height of this building.



b Another building is 24 m high. If the mirror is placed 1.5 m from George such that he can see the top of the building in the centre of the mirror, how far is George from the base of the building?

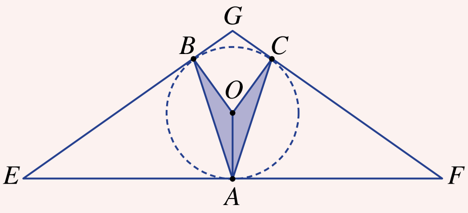
c George moves to 20 m from the base of another building.

i If the building is 11.2 m high, how far from George does the mirror need to be placed so that he can see the top of the building in the centre of the mirror?

ii Repeat part **i** to find an expression for how far from George the mirror needs to be placed for a building of height y m. Answer in terms of y and use your expression to check your answer to part **i**.

Airport terminal

2 Engineers are working on the design of a new airport terminal. A cross-section of an airport terminal design is illustrated in this diagram where the roof is held up by a V-shaped support as shown. Points A, B and C sit on a circle with centre O and OA, OB and OC are perpendicular to EF, EG and FG respectively. Also, $EG = FG$ and $EA = AF$.



The engineers are interested in the relationship between various angles within the design. Given the fixed geometric properties of the cross-section the engineers will explore how changing one angle affects the other angles so that they have a better understanding of the design limitations.

a If $\angle AEB$ is set at 40° find:

  i $\angle EGF$ **ii** $\angle BOC$ **iii** $\angle BAC$.

b If $\angle OCA$ is set at 10° find:

  i $\angle BOC$ **ii** $\angle AEB$.

c If $\angle AEB = a^\circ$ find $\angle BAC$ in terms of a.

d Use your rule in part **c** to verify your answer to part **a iii**.

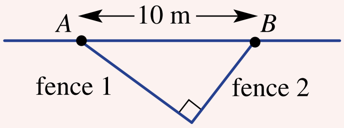
e As $\angle AEB$ increases, describe what happens to $\angle BAC$ and make a drawing to help explain your answer.

Circles on the farm

3 A circle is a locus defined as a set of points that are equidistant (the same distance) from a single point. While not always initially visible, such loci exist in many common situations.

A farmer wishes to investigate the existence of such circle loci in everyday situations in a farm environment. These include fencing an area using two fixed posts and training a horse around a given point.

a A triangular region is being fenced on a farm. It connects to posts, A and B, 10 m apart on an existing fence. The two new fences are to meet at right angles as shown.



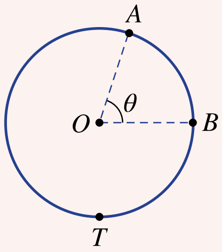
  i Describe and draw the location of all possible points where the two fence lines can meet.

  ii Which point from part **i** gives the maximum triangular area and what is this area?

  iii If the two fence posts are x m apart, give a rule for the maximum possible area of the triangular region in m^2.

  iv If fences 1 and 2 were replaced with a single fence in the shape of a semicircle, give an expression for the area gained in terms of x, where x metres is the distance between the fence posts.

b A horse is doing some training work around a circular paddock. The trainer stands on the edge of the paddock at T (as shown) watching the horse through binoculars as it moves from point A to point B.



  i Give an expression for $\angle ATB$.

  ii The horse is trotting at a constant rate of 1 lap (revolution) per minute. At what rate is the trainer moving his binoculars following the horse from A to B, in revolutions per minute?

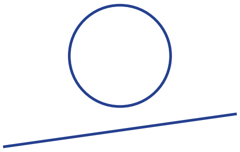
  iii If the horse completes a lap (revolution) in x minutes, at what rate is the trainer moving his binoculars in terms of x?

2I Theorems involving circles and tangents OPTIONAL

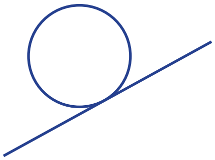
LEARNING INTENTIONS

- To know that a tangent is a line that touches a circle or curve at one point
- To know that a tangent is perpendicular to the radius at the point of contact
- To be able to find angles involving tangents
- To know that two different tangents drawn from an external point to the circle create line segments of equal length
- To know that the angle between a tangent and a chord is equal to the angle in the alternate segment
- To be able to apply the alternate segment theorem

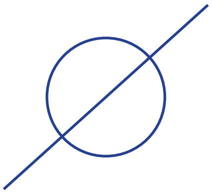
When a line and a circle are drawn, three possibilities arise: they could intersect 0, 1 or 2 times.



0 points



1 point



2 points

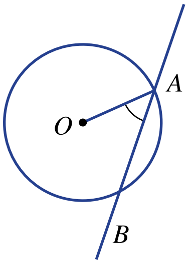
Where the 'pitch' circles meet, gears have a common tangent. Mechanical and auto engineers apply circle geometry when designing gears, including for vehicle engines, clocks, fuel pumps, automation machinery, printing presses and robots.

If the line intersects the circle once then it is called a tangent. If it intersects twice it is called a secant.

Lesson starter: From secant to tangent

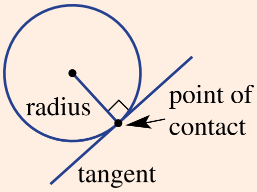
This activity is best completed using dynamic computer geometry software.

- Construct a circle with centre O and a secant line that intersects at A and B. Then measure $\angle BAO$.
- Drag B to alter $\angle BAO$. Can you place B so that line AB is a tangent? In this case, what is $\angle BAO$?

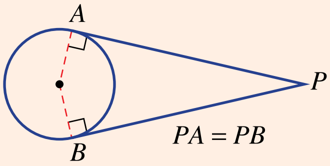


KEY IDEAS

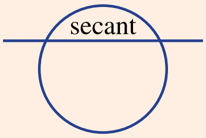
- A **tangent** is a line that touches a circle at a point called the **point of contact**.



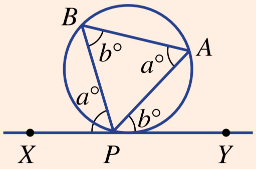
  - A tangent intersects the circle exactly once.
  - A tangent is perpendicular to the radius at the point of contact.
  - Two different tangents drawn from an external point to the circle create line segments of equal length.



- A **secant** is a line that cuts a circle twice.



- **Alternate segment theorem**: The angle between a tangent and a chord is equal to the angle in the alternate segment.
  $\angle APY = \angle ABP$ and
  $\angle BPX = \angle BAP$



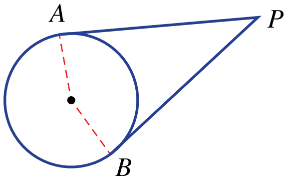
BUILDING UNDERSTANDING

① Answer true or false to the following statements.

a A tangent can intersect a circle more than once.

b A tangent makes an angle of 90° with a radius at the point of contact.

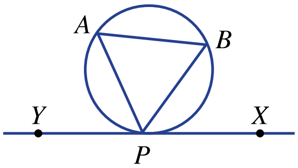
c AP is equal in length to BP in this diagram.



② For this diagram use the alternate segment theorem and name the angle that is:

a equal to $\angle BPX$ **b** equal to $\angle BAP$

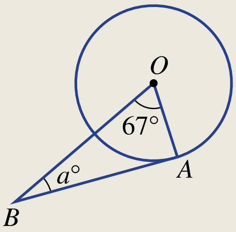
c equal to $\angle APY$ **d** equal to $\angle ABP$.



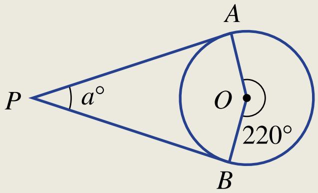
Example 16 Finding angles with tangents

Find the value of a in these diagrams that include tangents.

a



b



SOLUTION

a $\angle BAO = 90°$

$$a + 90 + 67 = 180$$
$$\therefore a = 23$$

b $\angle PAO = \angle PBO = 90°$

Obtuse $\angle AOB = 360° - 220° = 140°$

$$a + 90 + 90 + 140 = 360$$
$$\therefore a = 40$$

EXPLANATION

BA is a tangent, so $OA \perp BA$.

The sum of the angles in a triangle is 180°.

$PA \perp OA$ and $PB \perp OB$.

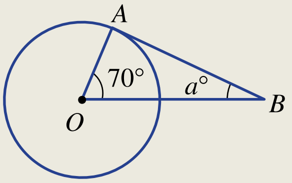
Angles in a revolution sum to 360°.

Angles in a quadrilateral sum to 360°.

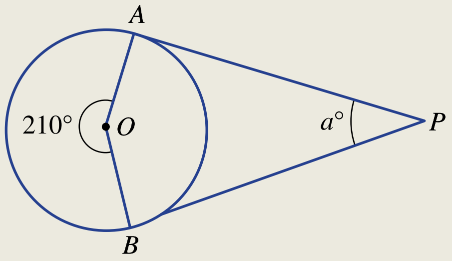
Now you try

Find the value of a in these diagrams that include tangents.

a



b

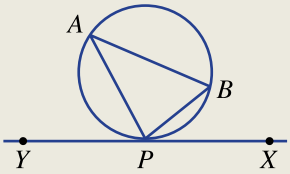


Example 17 Using the alternate segment theorem

In this diagram XY is a tangent to the circle.

a Find $\angle BPX$ if $\angle BAP = 38°$.

b Find $\angle ABP$ if $\angle APY = 71°$.



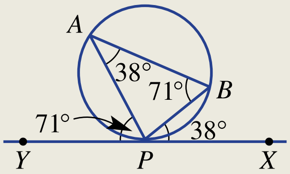
SOLUTION

a $\angle BPX = 38°$

b $\angle ABP = 71°$

EXPLANATION

The angle between a tangent and a chord is equal to the angle in the alternate segment.

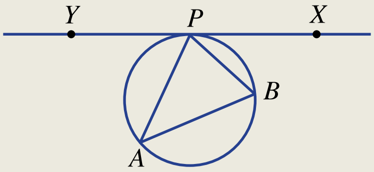


Now you try

In this diagram XY is a tangent to the circle.

a Find $\angle BPX$ if $\angle BAP = 50°$.

b Find $\angle ABP$ if $\angle APY = 70°$.



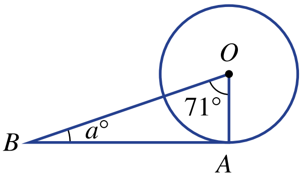
Exercise 2I

FLUENCY	1–4	1–5	2–5

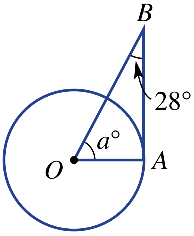
Example 16a

1 Find the value of a in these diagrams that include tangents.

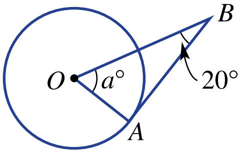
a



b



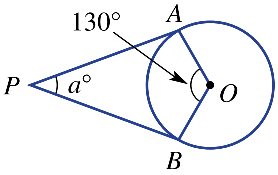
c



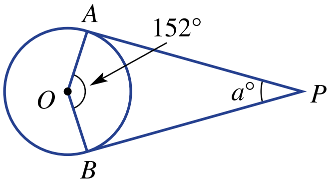
Example 16b

2 Find the value of a in these diagrams that include two tangents.

a



b



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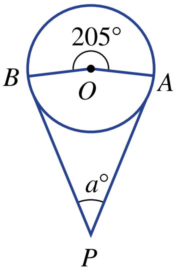
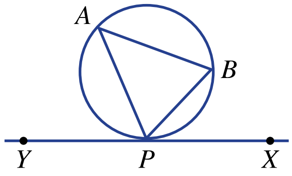


Example 17

3 In this diagram, XY is a tangent to the circle. Use the alternate segment theorem to find:

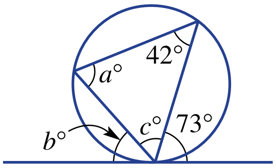
a $\angle PAB$ if $\angle BPX = 50°$

b $\angle APY$ if $\angle ABP = 59°$

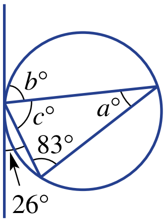


4 Find the value of a, b and c in these diagrams involving tangents.

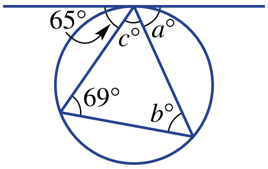
a



b

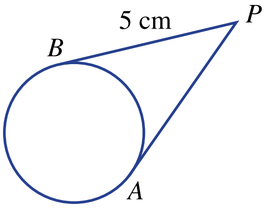


c

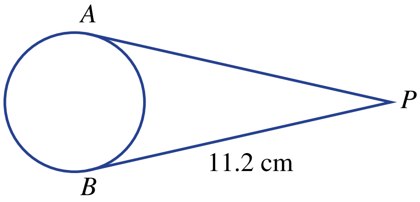


5 Find the length AP if AP and BP are both tangents.

a



b

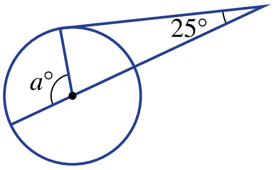


PROBLEM–SOLVING

6(½) 6–7(½) 6(⅓), 7(½), 8

6 Combine your knowledge of circles to find the value of a. All diagrams include one tangent line.

a



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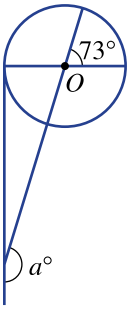
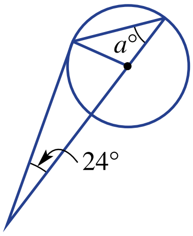
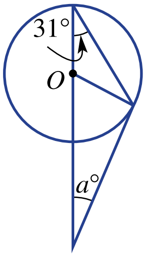


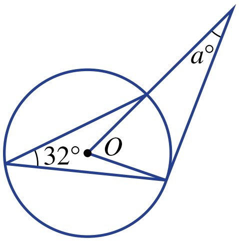
c



d



e



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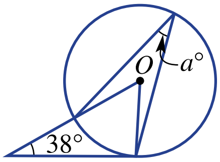
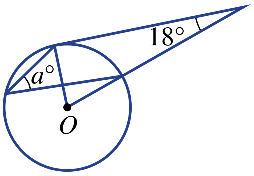
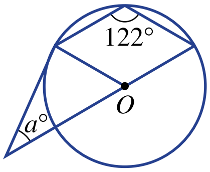


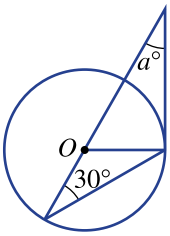
g



h

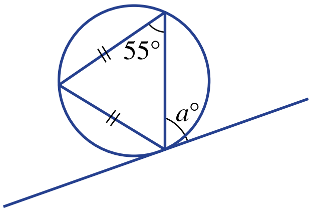


i

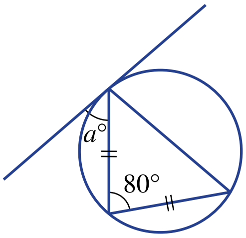


7 Find the value of a in these diagrams involving tangents.

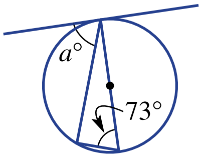
a



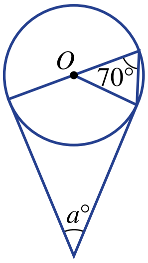
b



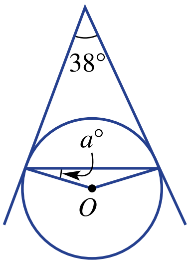
c



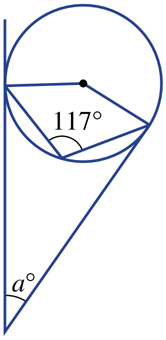
d



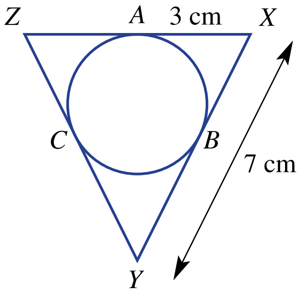
e



f



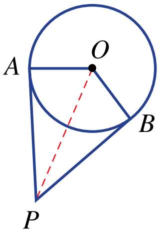
8 Find the length of CY in this diagram.



REASONING	9	9, 10	10–12

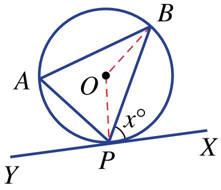
9 Prove that $AP = BP$ by following these steps.

- **a** Explain why $OA = OB$.
- **b** What is the size of $\angle OAP$ and $\angle OBP$?
- **c** Hence, prove that $\Delta OAP \equiv \Delta OBP$.
- **d** Explain why $AP = BP$.

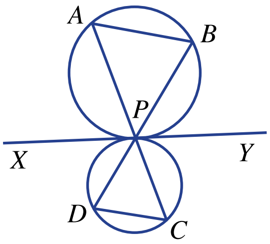


10 Prove the alternate angle theorem using these steps.
First, let $\angle BPX = x^\circ$, then give reasons at each step.

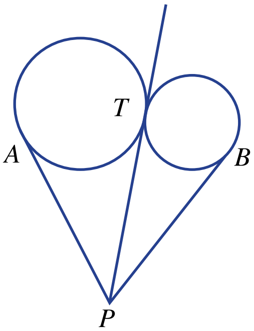
- **a** Write $\angle OPB$ in terms of x.
- **b** Write obtuse $\angle BOP$ in terms of x.
- **c** Use circle theorem 1 from angle properties of a circle to write $\angle BAP$ in terms of x.



11 These two circles touch with a common tangent XY.
Prove that $AB \parallel DC$. You may use the alternate segment theorem.



12 PT is a common tangent. Explain why $AP = BP$.



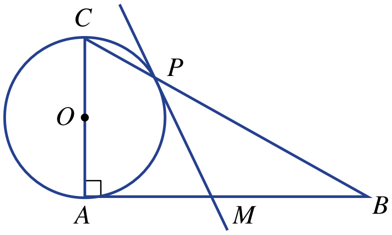
ENRICHMENT: Bisecting tangent

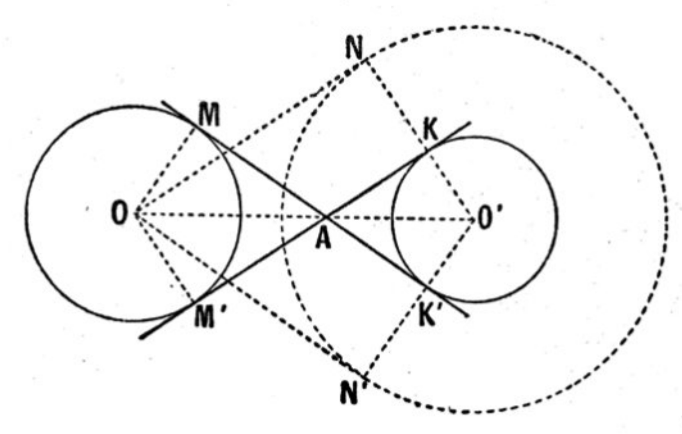
– – 13

13 In this diagram, ΔABC is right angled, AC is a diameter and PM is a tangent at P, where P is the point at which the circle intersects the hypotenuse.

a Prove that PM bisects AB; i.e. that $AM = MB$.

b Construct this figure using interactive geometry software and check the result. Drag A, B or C to check different cases.





2J Intersecting chords, secants and tangents OPTIONAL

LEARNING INTENTIONS

- To know the difference between a chord, a tangent and a secant
- To know the relationship between the lengths of intersecting chords
- To know the relationship between the lengths of secants that intersect at an external point
- To know the relationship between the lengths of an intersecting secant and tangent
- To be able to apply these relationships to find unknown lengths

In circle geometry, the lengths of the line segments (or intervals) formed by intersecting chords, secants or tangents are connected by special rules. There are three situations in which this occurs:

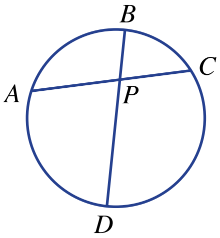
1. intersecting chords
2. intersecting secant and tangent
3. intersecting secants.

Architects use circle and chord geometry to calculate the dimensions of constructions, such as this glass structure.

Lesson starter: Equal products

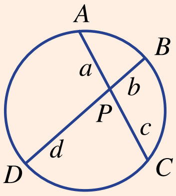
Use interactive geometry software to construct this figure and then measure AP, BP, CP and DP.

- Calculate $AP \times CP$ and $BP \times DP$. What do you notice?
- Drag A, B, C or D. What can be said about $AP \times CP$ and $BP \times DP$ for any pair of intersecting chords?

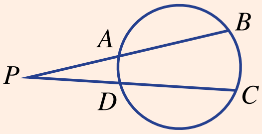


KEY IDEAS

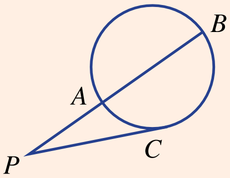
- When two chords intersect as shown, then $AP \times CP = BP \times DP$ or $ac = bd$.



- When two secants intersect at an external point P as shown, then $AP \times BP = DP \times CP$.



- When a secant intersects a tangent at an external point as shown, then $AP \times BP = CP^2$.



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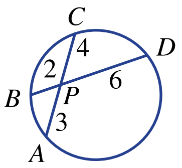
BUILDING UNDERSTANDING



1 State these lengths for the given diagram.

a AP **b** DP **c** AC **d** BD

2 Solve these equations for x.

a $x \times 2 = 7 \times 3$

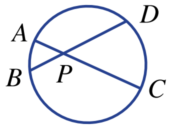
b $4x = 5 \times 2$

c $(x + 3) \times 7 = 6 \times 9$

d $7(x + 4) = 5 \times 11$

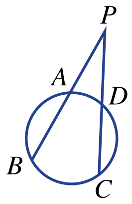
3 State the missing parts for the rules for each diagram.

a



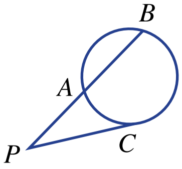
$AP \times CP =$ _____ $\times$ _____

b



$AP \times$ _____ $=$ _____ $\times$ _____

c

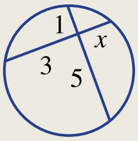


$AP \times$ _____ $=$ _____^{2}

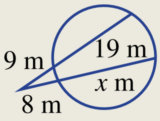
Example 18 Finding lengths using intersecting chords, secants and tangents

Find the value of x in each figure.

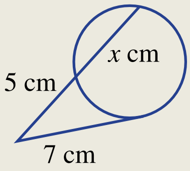
a



b



c



SOLUTION

a

$$\begin{aligned} x \times 3 &= 1 \times 5 \\ 3x &= 5 \\ x &= \frac{5}{3} \end{aligned}$$

EXPLANATION

Equate the products of each pair of line segments on each chord.

b

$$\begin{aligned} 8 \times (x + 8) &= 9 \times 28 \\ 8x + 64 &= 252 \\ 8x &= 188 \\ x &= \frac{188}{8} \\ &= \frac{47}{2} \end{aligned}$$

Multiply the entire length of the secant ($19 + 9 = 28$ and $x + 8$) by the length from the external point to the first intersection point with the circle. Then equate both products.
Expand brackets and solve for x.

c

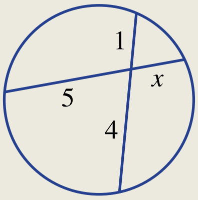
$$\begin{aligned} 5 \times (x + 5) &= 7^2 \\ 5x + 25 &= 49 \\ 5x &= 24 \\ x &= \frac{24}{5} \end{aligned}$$

Square the length of the tangent and then equate with the product from the other secant.

Now you try

Find the value of x in each figure.

a



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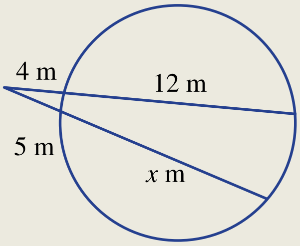

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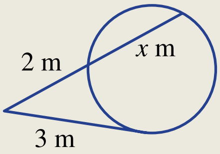


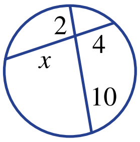
Exercise 2J

FLUENCY	1–3	1–3	1–3($\frac{1}{3}$), 4

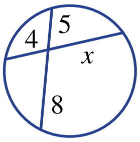
Example 18a

1 Find the value of x in each figure.

a



b



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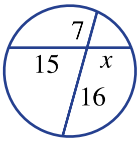
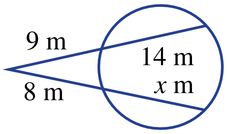


Example 18b

2 Find the value of x in each figure.

a



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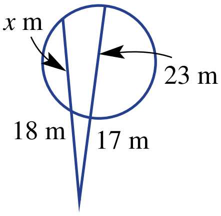
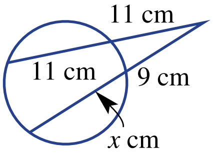


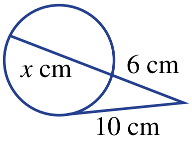
c



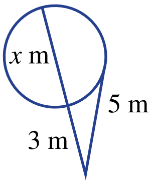
Example 18c

3 Find the value of x in each figure.

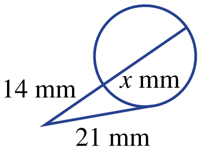
a



b

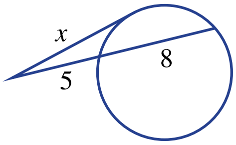


c

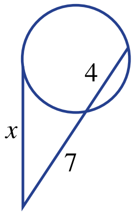


4 Find the exact value of x, in surd form. For example, $\sqrt{7}$.

a



b

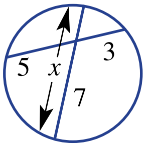


PROBLEM–SOLVING

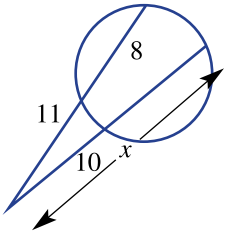
5(½) 5(½) 5(½), 6

5 Find the exact value of x.

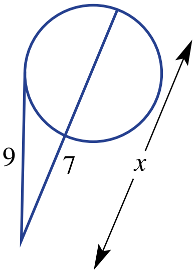
a



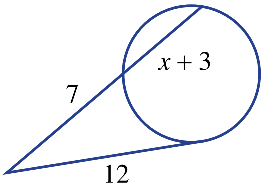
b



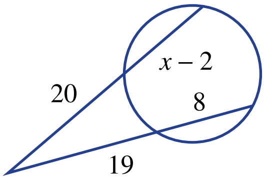
c



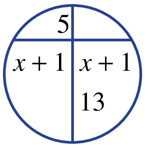
d



e

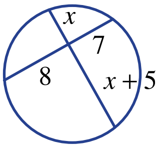


f

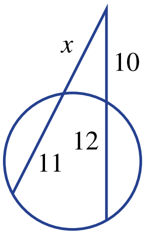


6 For each diagram, derive the given equations.

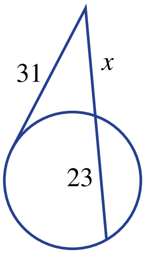
a $x^2 + 5x - 56 = 0$



b $x^2 + 11x - 220 = 0$



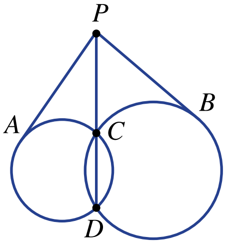
c $x^2 + 23x - 961 = 0$



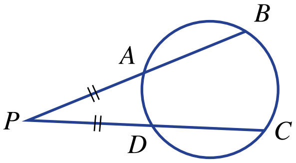
REASONING

7, 8 7–9 9–11

7 Explain why $AP = BP$ in this diagram, using your knowledge from this section.



8 In this diagram $AP = DP$. Explain why $AB = DC$.



9 Prove that $AP \times CP = BP \times DP$ by following these steps.

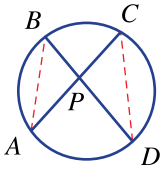
a What can be said about the pair of angles $\angle A$ and $\angle D$ in the given diagram and also about the pair of angles $\angle B$ and $\angle C$? Give a reason.

b Prove $\Delta ABP \mathbin{|||} \Delta DCP$.

c Complete:

$$\frac{AP}{\cdots\cdot} = \frac{\cdots\cdot}{CP}$$

d Prove $AP \times CP = BP \times DP$.

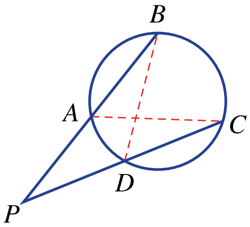


10 Prove that $AP \times BP = DP \times CP$ by following these steps.

a Consider ΔPBD and ΔPCA in the given diagram. What can be said about $\angle B$ and $\angle C$? Give a reason.

b Prove $\Delta PBD \mathbin{|||} \Delta PCA$.

c Prove $AP \times BP = DP \times CP$.



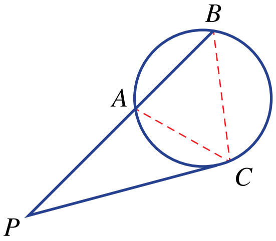
11 Prove that $AP \times BP = CP^2$ by following these steps.

a Consider ΔBPC and ΔCPA in the given diagram. Is $\angle P$ common to both triangles?

b Explain why $\angle ACP = \angle ABC$.

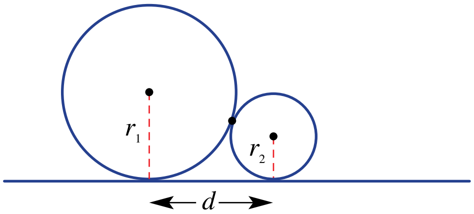
c Prove $\Delta BPC \mathbin{|||} \Delta CPA$.

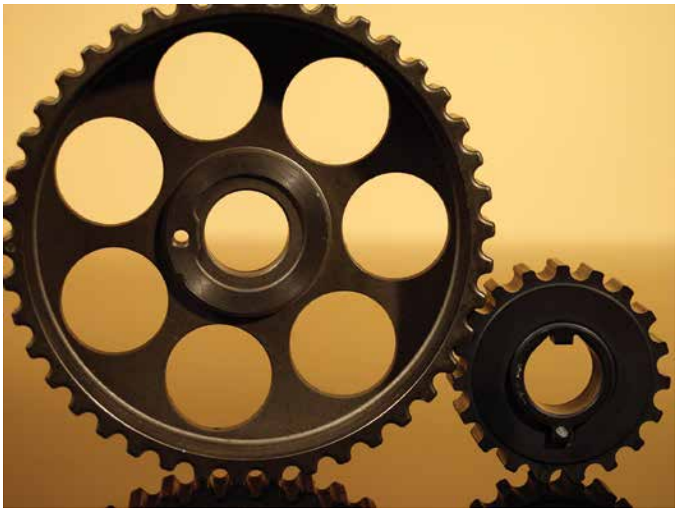
d Prove $AP \times BP = CP^2$.



ENRICHMENT: Horizontal wheel distance	–	–	12

12 Two touching circles have radii r_1 and r_2.
The horizontal distance between their centres is d.
Find a rule for d in terms of r_1 and r_2.





2K Introduction to networks

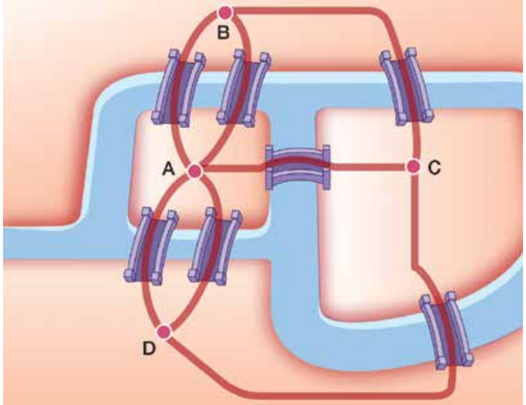
LEARNING INTENTIONS

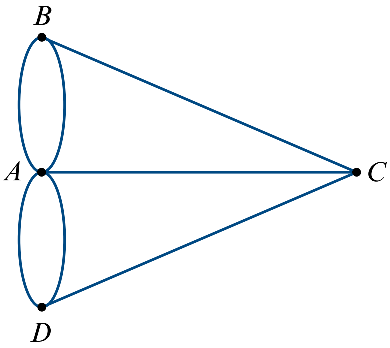
- To know what is meant by a network graph
- To know the key features of a network graph
- To be able to find the degree of a vertex and the sum of degrees for a graph
- To be able to describe simple walks through a network using the vertex labels

A network is a collection of points (vertices or nodes) which can be connected by lines (edges). Networks are used to help solve a range of real-world problems including travel and distance problems, intelligence and crime problems, computer network problems and even metabolic network problems associated with the human body. In Mathematics, a network diagram can be referred to as a graph, not to be confused with the graph of a function like $y = x^2 + 3$.

Lesson starter: The Konigsberg bridge problem

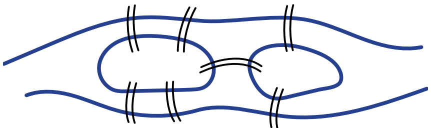
The seven bridges of Konigsberg is a well-known historical problem solved by Leonhard Euler who laid the foundations of graph theory. It involves two islands at the centre of an old German city connected by seven bridges over a river as shown in these diagrams.





The problem states: Is it possible to start at one point and pass over every bridge exactly once and return to your starting point?

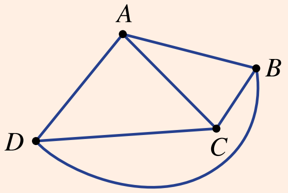
- Make a copy of this simplified map of the seven bridges of Konigsberg and try tracing out a walk that crosses all bridges exactly once. Try starting at different places.



- Investigate if there might be a solution to this problem if one of the bridges is removed.
- Investigate if there might be a solution to this problem if one bridge is added.

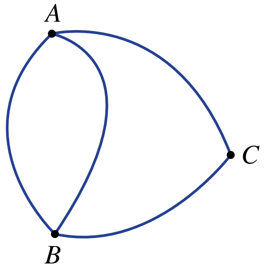
KEY IDEAS

- A **network** or **graph** is a diagram connecting points using lines.
  - The points are called **vertices** (or **nodes**). Vertex is singular, vertices is plural.
  - The lines are called **edges**.
- The **degree** of a vertex is the number of edges connected to it.
  - A vertex is odd if the number of edges connected to it is odd.
  - A vertex is even if the number of edges connected to it is even.
- The sum of degrees is calculated by adding up the degrees of all the vertices in a graph.
  - It is also equal to twice the number of edges.
- A **walk** is any type of route through a network.
  - A walk can be defined using the vertex labels.
  - Example: A-B-C-A-D.



BUILDING UNDERSTANDING

① Here is a graph representing roads connecting three towns A, B and C.



a How many different roads (edges) does this graph have?

b How many different vertices (nodes) does this graph have?

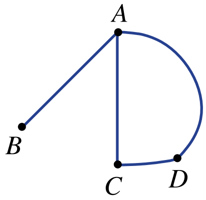
c If no road (edge) is used more than once and no town (vertex/node) is visited more than once, how many different walks are there if travelling from:

- **i** A to C?
- **ii** A to B?
- **iii** B to C?

d How many roads connect to:

- **i** town A?
- **ii** town B?
- **iii** town C?

② This graph uses four edges to connect four vertices.



a How many edges connect to vertex A?

b What is the degree of vertex A?

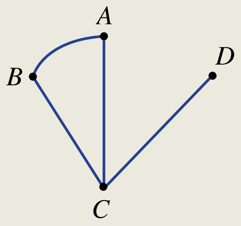
c State the total number of edges on the graph.

d By finding the number of edges connected to each vertex, find the sum of degrees for the graph.

e What do you notice about the total number of edges and the sum of degrees for this graph?

Example 19 Determining features of a graph

For the graph shown complete the following.



a State the total number of:
- **i** vertices (nodes)
- **ii** edges.

b State the degree of:
- **i** vertex *A*
- **ii** vertex *B*
- **iii** vertex *C*
- **iv** vertex *D*.

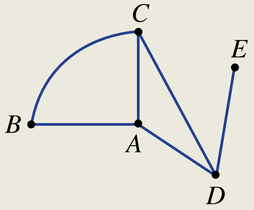
c Find the sum of degrees for the graph.

d State which vertices are:
- **i** odd
- **ii** even.

SOLUTION	EXPLANATION
a **i** 4	The four vertices are *A*, *B*, *C* and *D*.
ii 4	The four edges are *AB*, *AC*, *BC* and *CD*.
b **i** 2	Simply count how many edges connect to each vertex.
ii 2	
iii 3	
iv 1	
c $2 + 2 + 3 + 1 = 8$	Add up all the degrees from all vertices.
d **i** *C* and *D*	The degrees of vertices *C* and *D* are 3 and 1.
ii *A* and *B*	The degrees of vertices *A* and *B* are both 2.

Now you try

For the graph shown complete the following.



a State the total number of:
- **i** vertices (nodes)
- **ii** edges.

b State the degree of:
- **i** vertex *A*
- **ii** vertex *B*
- **iii** vertex *C*
- **iv** vertex *D*
- **v** vertex *E*.

c Find the sum of degrees for the graph.

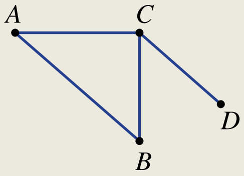
d State which vertices are:
- **i** odd
- **ii** even.

Example 20 Finding a walk

Consider this graph connecting the four vertices A, B, C and D. Without visiting a vertex (node) more than once or using an edge more than once, find how many walks there are connecting the following pairs of vertices.

a A and D

b B and D



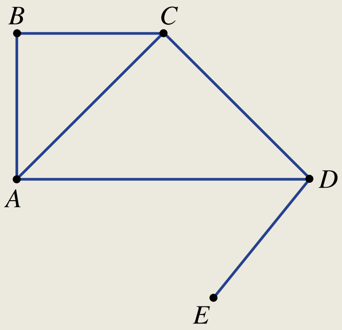
SOLUTION	EXPLANATION
a 2	The two walks are A-B-C-D and A-C-D.
b 2	The two walks are B-A-C-D and B-C-D.

Now you try

Consider this graph connecting the five vertices A, B, C, D and E. Without visiting a vertex (node) more than once or using an edge more than once, find how many walks there are connecting the following pairs of vertices.

a A and E

b B and D



Exercise 2K

FLUENCY	1, 2, 4–6	1, 3–6	3–6

Example 19

1 For the graph shown complete the following.

a State the total number of:

i vertices (nodes) **ii** edges.

b State the degree of:

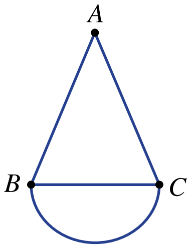
i vertex A **ii** vertex B

iii vertex C.

c Find the sum of degrees for the graph.

d State which vertices are:

i odd **ii** even.



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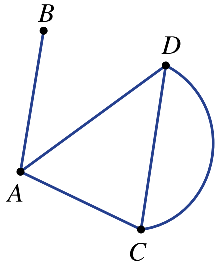
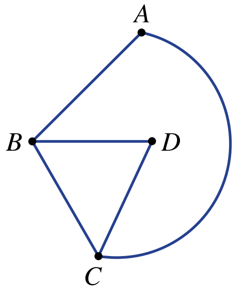
2 For the graph shown complete the following.



- **a** State the total number of:
  - **i** vertices (nodes)
  - **ii** edges.
- **b** State the degree of:
  - **i** vertex A
  - **ii** vertex B
  - **iii** vertex C
  - **iv** vertex D.
- **c** Find the sum of degrees for the graph.
- **d** State which vertices are:
  - **i** odd
  - **ii** even.

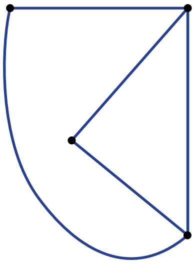
3 For the graph shown complete the following.



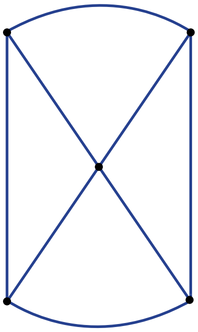
- **a** State the total number of:
  - **i** vertices (nodes)
  - **ii** edges.
- **b** State the degree of:
  - **i** vertex A
  - **ii** vertex B
  - **iii** vertex C
  - **iv** vertex D.
- **c** Find the sum of degrees for the graph.
- **d** State which vertices are:
  - **i** odd
  - **ii** even.

4 Which of the following graphs has vertices which are all odd?

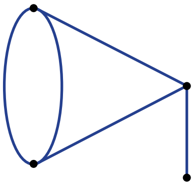
A



B

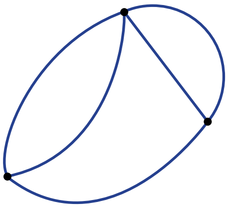


C

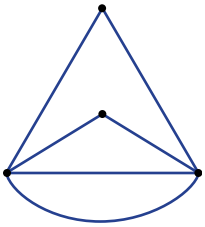


5 Which of the following graphs has vertices which are all even?

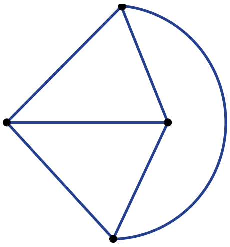
A



B



C



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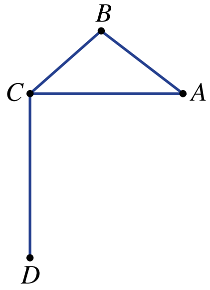
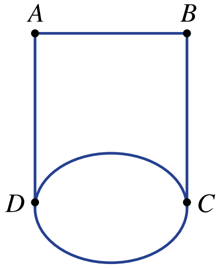
Example 20

6 Consider this graph connecting the four vertices A, B, C and D. Without visiting a vertex (node) more than once or using an edge more than once, find how many walks there are connecting vertex A with vertex D.

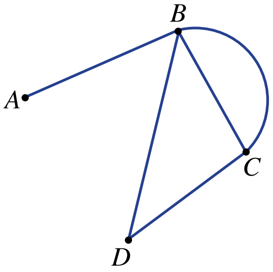
a



b



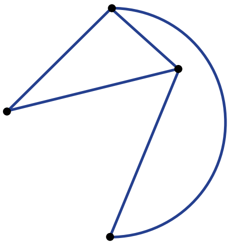
c



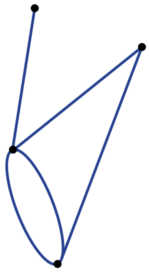
PROBLEM–SOLVING	7, 8	7–9	8–10

7 Which of the following graphs has the greatest sum of degrees?

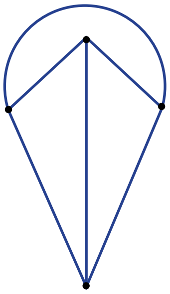
A



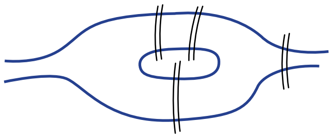
B



C



8 Similar to the Konigsberg bridge problem this diagram shows an island in the middle of a river and four bridges.



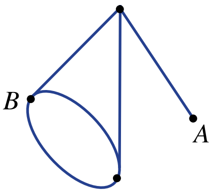
a Is it possible to start on the island and finish on the island by crossing each of the four bridges exactly once?

b Is it possible to start on one side of the river (not on the island) and finish on the same side of the river by crossing each of the four bridges exactly once?

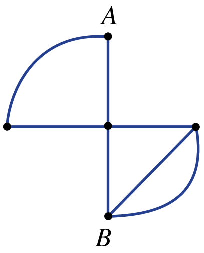
c Is it possible to start on one side of the river (not on the island) and finish on the opposite side of the river by crossing each of the four bridges exactly once?

9 Which of the following graphs has the greatest number of walks connecting vertices A and B if no edge or vertex can be used more than once?

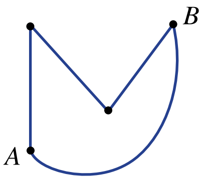
A



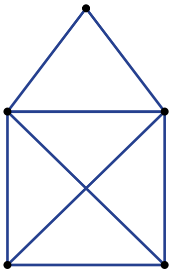
B



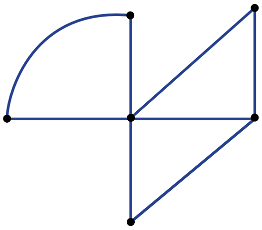
C



10 Is it possible to find a walk around this graph so that each edge is used exactly once? There is no restriction on the number of times vertices are visited or where you should start and finish. If the answer is 'yes', draw your walk. The place in the middle of the square is not a vertex, just an intersection of two edges. You cannot change direction at this position.



REASONING	11	11, 12	12, 13



11 Consider the given graph.

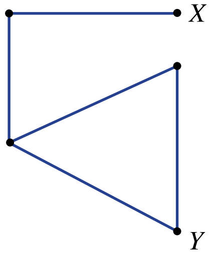
a How many edges does the graph have?

b Find the sum of degrees for the graph.

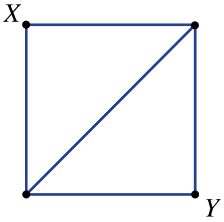
c What do you notice about your answers to part **a** and part **b** above? Decide if your conclusion is true for other graphs by drawing your own.

12 Decide if it is possible to find a walk from X to Y in these graphs by visiting each vertex exactly once and using each edge exactly once.

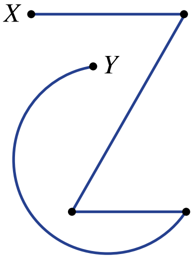
a



b

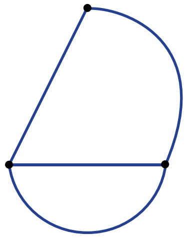


c

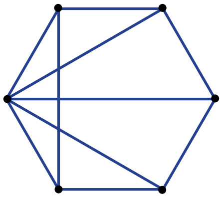


13 Sometimes edges can be removed from a graph with all remaining vertices still connected to the graph. Find the maximum number of edges that can be removed from these graphs so that all vertices remain connected to the graph.

a



b

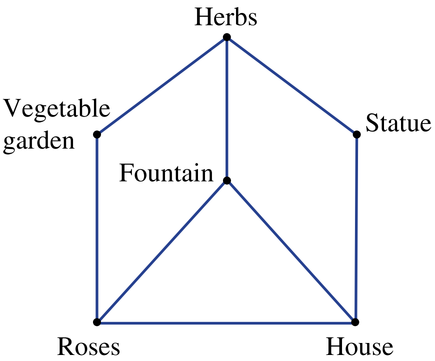


ENRICHMENT: Garden tours with conditions	–	–	14

14 Consider this graph representing a garden including six key features connected with garden paths.

a How many different ways are there of walking from the House to the Vegetable garden without visiting a feature more than once or using a garden path more than once?

b How many different tours are possible starting and ending at the House? Other garden features can only be visited once but the tour does not need to visit all features. Do not count tours which are the same if you ignore direction.



2L Isomorphic and planar graphs

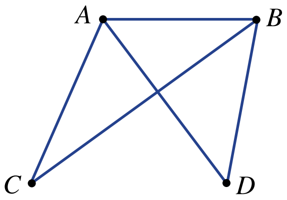
LEARNING INTENTIONS

- To know what isomorphic and planar graphs are
- To be able to identify isomorphic and planar graphs
- To be able to verify Euler's formula and use it to determine features of a graph

Many of the graphs considered so far may look different but in fact contain exactly the same information. Such graphs are said to be isomorphic and could be redrawn to look the same. Further, most of the graphs previously considered have been drawn with no intersecting edges. Such graphs are called planar graphs and often occur naturally in problems involving electronic circuits, railway networks and utility lines.

Lesson starter: Exploring Euler's formula with planar graphs

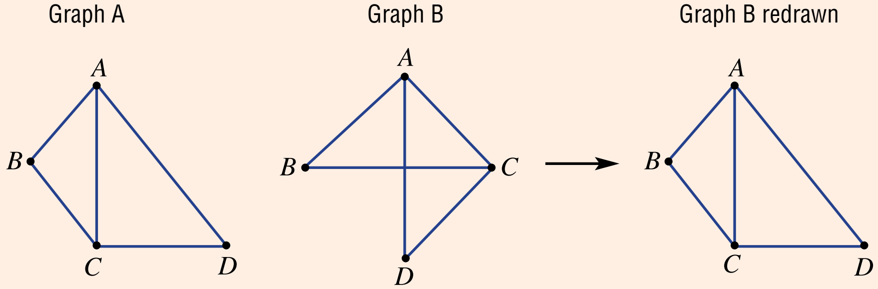
Consider this graph representing the connection between four vertices. There is no vertex at the intersection of the edges AD and BC.



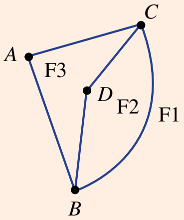
- Is it possible to redraw the graph so that all the vertices and edges are retained but edges do not intersect? Try this with a new drawing.
- Your new drawing should be a planar graph with no intersecting edges. How many different regions make up the graph including the outside region? These are called faces.
- Calculate the following.
  - $v + f$ (The sum of the number of vertices and the number of faces)
  - $e + 2$ (Two more than the number of edges)
- What do you notice about the two calculations above?
- Try drawing a different planar graph to verify your conclusion.

KEY IDEAS

- **Isomorphic graphs** contain the same information including:
  - the same number of vertices
  - the same number of edges
  - the same edge connections.
- Isomorphic graphs can be drawn to look the same such as Graph A and Graph B below.



- **A planar graph** can be drawn so that it has no intersecting edges.
  - Many graphs may not look planar initially but could be redrawn without any intersecting edges.
  - A non-planar graph cannot be drawn without some edges crossing.
- A **face** of a graph is a region bound by a set of edges and vertices. The region on the outside also counts as a face.



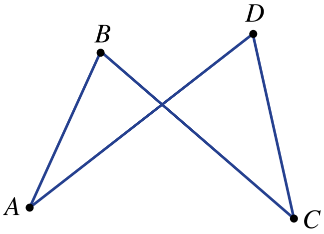
- **Euler's formula** applies to planar graphs and is such that $v + f = e + 2$ where:
  - v is the number of vertices
  - f is the number of faces including the outside face
  - e is the number of edges.
  - Euler's formula is often also written as $v - e + f = 2$.

BUILDING UNDERSTANDING

① Consider this graph connecting the four vertices A, B, C and D.

a Try redrawing the graph with no edges crossing.

b Would you therefore say that the graph is planar?



② Consider this planar graph.

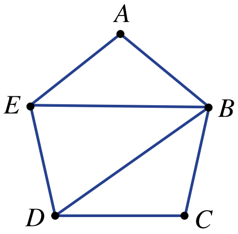
a State the number of:

- **i** vertices, v
- **ii** edges, e
- **iii** faces, f. (Don't forget to count the outside face)

b Calculate the following.

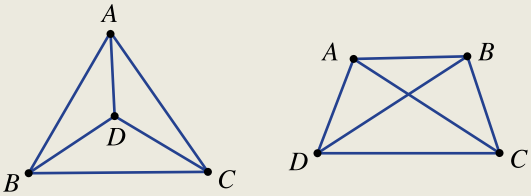
- **i** $v + f$
- **ii** $e + 2$

c What do you notice about your answers to part **b**?



Example 21 Deciding if graphs are isomorphic

Decide if the following two graphs are isomorphic.



SOLUTION

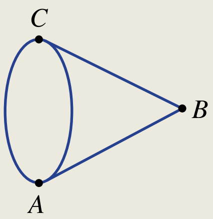
Yes, the graphs are isomorphic.

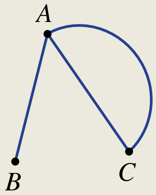
EXPLANATION

Both graphs have the same number of vertices and edges and the same connections. They could be redrawn to look identical.

Now you try

Decide if the following two graphs are isomorphic.

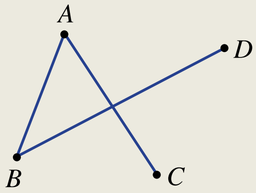




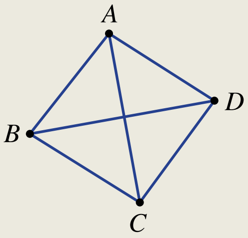
Example 22 Deciding if graphs are planar or non-planar

Decide if the following graphs are planar or non-planar.

a



b

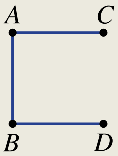


SOLUTION

a The graph is planar.

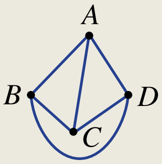
EXPLANATION

The graph can be drawn without any intersecting edges.



b The graph is planar.

The graph can be drawn without any intersecting edges.



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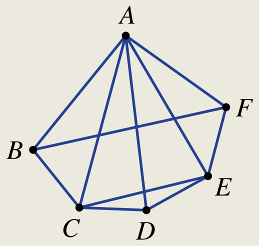
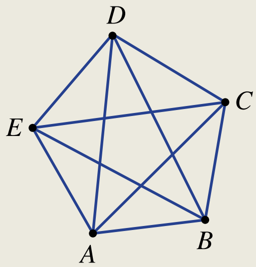
Now you try

Decide if the following graphs are planar or non-planar.

a



b



Example 23 Verifying Euler's formula

Consider this planar graph.

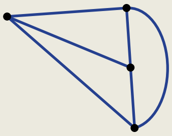
a Find the number of:

i vertices, v

ii edges, e

iii faces, f.

b Verify Euler's formula using the information in the given planar graph.



SOLUTION

a **i** 4

ii 6

iii 4

b $v + f = 4 + 4 = 8$

$e + 2 = 6 + 2 = 8$

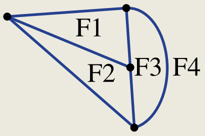
So, $v + f = e + 2$

OR $v - e + f = 4 - 6 + 4 = 2$

Therefore, Euler's formula is verified.

EXPLANATION

Simply count the number of vertices, edges and faces. Don't forget to count the infinite, outside face.



Substitute the values of v, e, and f into Euler's formula and check that the left-hand side equals the right-hand side. Using $v + f = e + 2$ is equivalent to using $v - e + f = 2$.

Now you try

Consider this planar graph.

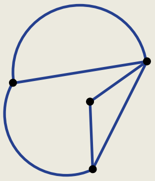
a Find the number of:

i vertices, v

ii edges, e

iii faces, f.

b Verify Euler's formula using the information in the given planar graph.



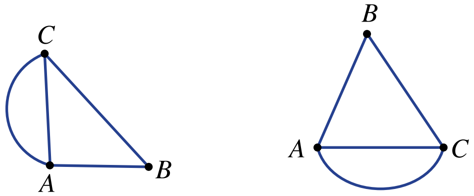
Exercise 2L

FLUENCY	1–3	1–4	1, 2, 4

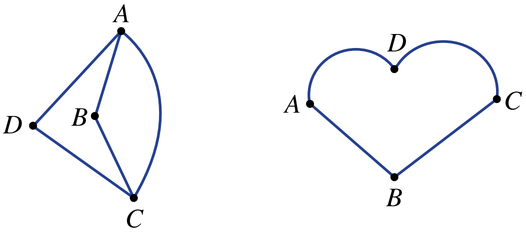
Example 21

1 Decide if the following pairs of graphs are isomorphic.

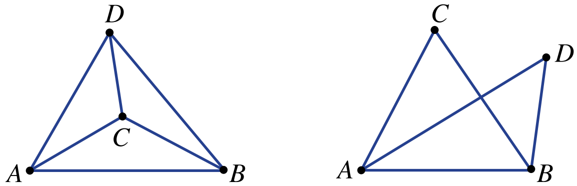
a



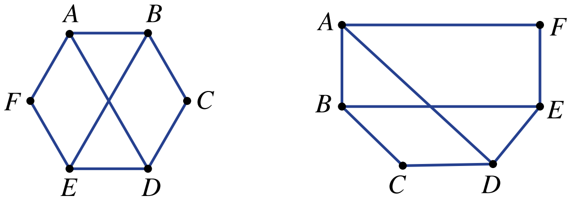
b



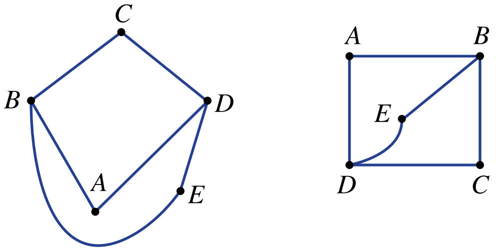
c



d



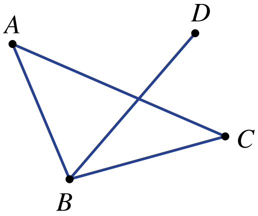
e



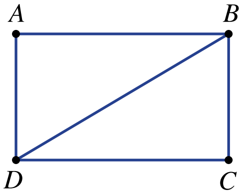
Example 22

2 Decide if the following graphs are planar or non-planar.

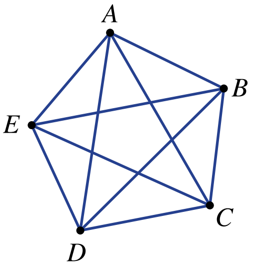
a



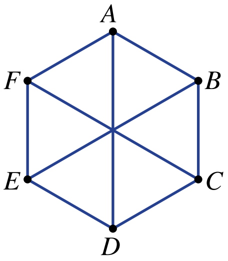
b



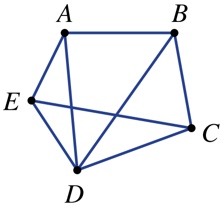
c



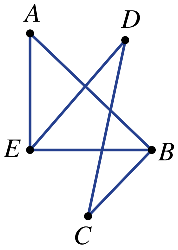
d



e

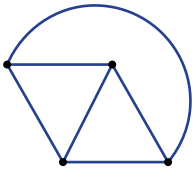


f



Example 23

3 Consider this planar graph.

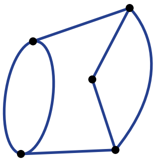


a Find the number of:

- **i** vertices, v
- **ii** edges, e
- **iii** faces, f.

b Verify Euler's formula using the information in the given planar graph.

4 Consider this planar graph.



a Find the number of:

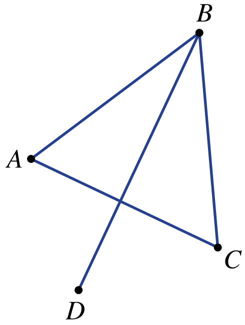
- **i** vertices, v
- **ii** edges, e
- **iii** faces, f.

b Verify Euler's formula using the information in the given planar graph.

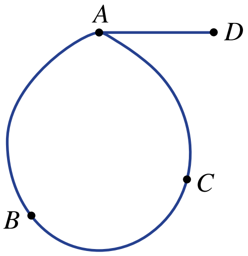
PROBLEM–SOLVING	5, 7	6, 7	7, 8

5 Which pair of the three graphs, A, B or C are isomorphic?

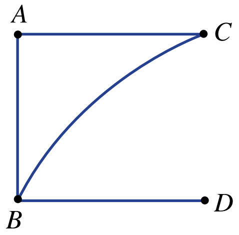
A



B

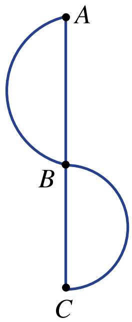


C

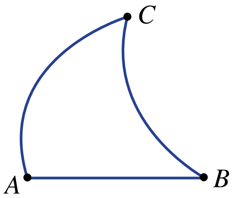


6 Which pair of the three graphs, A, B or C are isomorphic?

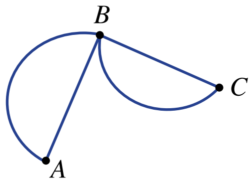
A



B

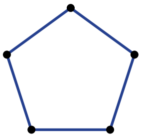


C

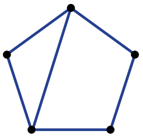


7 The following pentagons include some diagonal edges. Decide which ones are planar.

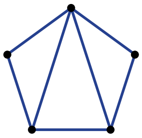
A



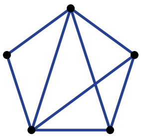
B



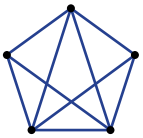
C



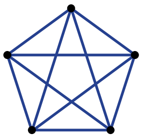
D



E



F



8 Use Euler's formula to answer the following which involve planar graphs.

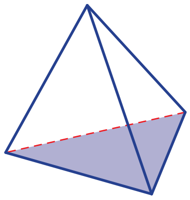
a If the number of vertices is 6 and the number of faces is 8, find the number of edges.

b If the number of vertices is 8 and the number of edges is 12, find the number of faces.

c If the number of edges is 10 and the number of faces is 8, find the number of vertices.

REASONING	9	9, 10	10, 11

9 Consider this four-sided, 3-dimensional polyhedron called a tetrahedron.



a State the number of vertices, edges and faces.

b Verify Euler's formula for the tetrahedron.

c Now draw the tetrahedron as a 2-dimensional planar graph. It does not need to look like a tetrahedron, just a planar graph with the same vertex, edge and face information.

d Verify Euler's formula using your planar graph from part **c**.

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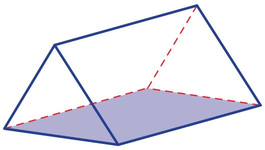
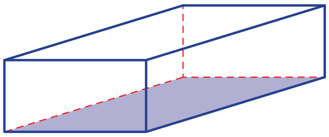
10 Complete parts **a**–**d** from question **9** above for the following 3-dimensional polyhedra.

a Pentahedron



b Hexahedron



11 An octahedron is a 3-dimensional polyhedron with eight faces. By drawing both a planar graph in two dimensions and a 3-dimensional representation of the solid, verify Euler's formula.

ENRICHMENT: Platonic solids rule	–	–	12

12 Earlier in the exercise we considered polyhedra as planar graphs which satisfy Euler's formula. Another interesting rule which only applies to regular polyhedra, the Platonic solids, is $\frac{1}{n} + \frac{1}{d} = \frac{1}{2} + \frac{1}{e}$ where:

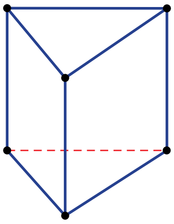
n is the number of sides on each face
e is the number of edges
d is the number of edges joining each vertex. This will be the same for every vertex.

Note that there only five Platonic solids including the regular tetrahedron (4 faces), cube (6 faces), octahedron (8 faces), dodecahedron (12 faces) and icosahedron (20 faces). Each face is a regular polygon.

a Consider the first Platonic solid, the regular tetrahedron.
  i Find the values of n, e and d.
  ii Verify the formula $\frac{1}{n} + \frac{1}{d} = \frac{1}{2} + \frac{1}{e}$.

b Repeat part **a** for one or more of the other Platonic solids.

c Consider this pentahedron. Show that Euler's formula can be verified and explain why the rule $\frac{1}{n} + \frac{1}{d} = \frac{1}{2} + \frac{1}{e}$ cannot.



2M Trails, paths and Eulerian circuits

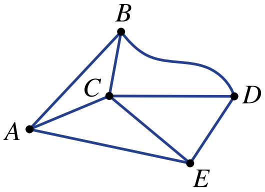
LEARNING INTENTIONS

- To know the difference between a walk, trail, circuit, path and cycle in a network
- To know what Eulerian trails and cycles are
- To be able to identify different types of walks within a network
- To be able to find Eulerian trails and circuits within a network

Depending on the application of graph theory, the desired walk through a network may involve a wide range of factors. The number of times that a vertex can be visited, or an edge is used, may or may not be important. In one application, it may be a requirement that a walk starts and ends at the same vertex. Alternatively, each edge might have to be used exactly once. In this section we will define and explore the different types of walks through a network and focus on a special group of them called Eulerian trails.

Lesson starter: Exploring Buxton

The small town of Buxton has five key tourist destinations labelled here with the letters A, B, C, D and E. They are connected by footpaths as shown in this network diagram.



A particular trail is defined by B-C-D-E-C.

- Are there any footpaths (edges) used more than once in this trail?
- Are there any destinations (vertices) visited more than once?
- What extensions to the trail could be added (at the end) so that it becomes a circuit where it ends at where it began? Is it possible to do this without using a particular footpath more than once?

A path is defined by A-B-C-E.

- Are there any footpaths (edges) used more than once in this trail?
- Are there any destinations (vertices) visited more than once?
- What extensions to the path could be added (at the end) so that it becomes a cycle where it ends at where it began? Is it possible to do this without using a particular footpath (edge) or destination (vertex) more than once?

Now imagine trying to visit every destination in Buxton.

- Is it possible to visit all the destinations on a given path so that all destinations are visited exactly once and no footpath is used more than once? If so, how?

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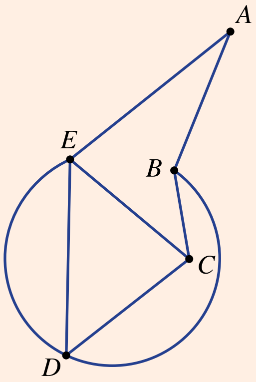
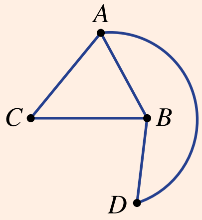
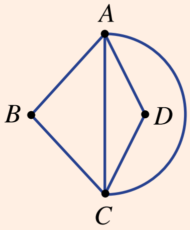
KEY IDEAS



- A **walk** is any type of route through a network. Examples:
  - *A-B-D-C-B*
  - *E-D-B-C-D-E*
- A **trail** is a walk where edges are not repeated. Examples:
  - *A-E-D-C-E*
  - *C-E-D-C-B*
- A **circuit** is a trail that begins and ends at the same vertex. Examples:
  - *E-D-B-C-D-E*
  - *B-C-D-E-D-B*
- A **path** is a walk where vertices and edges are not repeated. Examples:
  - *E-D-C-B*
  - *B-C-D-E-A*
- A **cycle** is a path that begins and ends at the same place. Examples:
  - *E-D-C-E*
  - *A-E-D-B-A*
- An **Eulerian trail** is a walk where every edge is included exactly once. Vertices are allowed to be revisited.
  - Eulerian trails exist if there are zero or exactly two vertices of odd degree.
  - If exactly two vertices are of odd degree, then Eulerian trails start at one of these vertices and end at the other.
  - If there are zero vertices of odd degree, then all Eulerian trails are circuits.
  - This graph has two vertices of odd degree and an example of an Eulerian trail is *B-D-A-C-B-A*.



- An **Eulerian circuit** is an Eulerian trail which starts and ends at the same vertex.
  - An Eulerian circuit exists if and only if zero vertices are of odd degree.
  - This graph has zero vertices of odd degree and an example of an Eulerian circuit is *A-B-C-A-D-C-A*.
  - An Eulerian circuit can start at any vertex.



BUILDING UNDERSTANDING

1 Give the formal name (walk, trail, circuit, path or cycle) of each of the following.

a Any type of route through a network.

b A trail that begins and ends at the same vertex.

c A path that begins and ends at the same vertex.

d A walk where edges are only used once.

e A walk where vertices and edges are only used once.

② Consider this simple graph.

a Which vertices are of odd degree?

b Find a trail so that all edges are used exactly once. Write down your trail using the letters A, B, C and/or D.

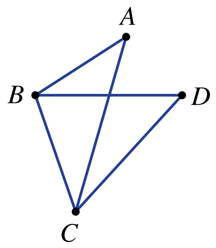
c What is the name of the type of walk you discovered in part **b**?

Assume now that the edge AD is added to the graph.

d Is it now possible to find an Eulerian trail through the network?

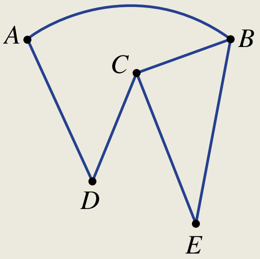
e How many vertices are of odd degree?

f How many vertices need to be of odd degree if a trail is to be Eulerian?



Example 24 Defining walks

Consider this graph.



a Decide if the following walks are trails. Answer Yes or No.

i A-D-C-B **ii** A-B-C-E-B

b Decide if the following walks are circuits. Answer Yes or No.

i A-B-C-D-A **ii** B-C-D-C-B

c Decide if the following walks are paths. Answer Yes or No.

i C-E-B-C-D **ii** E-B-C-D-A

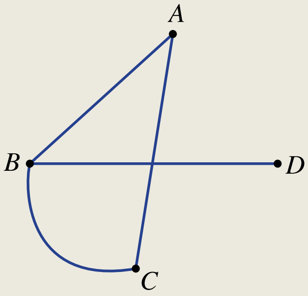
d Decide if the following walks are cycles. Answer Yes or No.

i C-D-A-B-C **ii** E-B-A-D-C-B-E

SOLUTION	EXPLANATION
a i Yes	Edges are only used once.
ii Yes	Edges are only used once.
b i Yes	Edges are only used once and it starts and ends at the same vertex.
ii No	The edge BC and CD are used twice.
c i No	The vertex C is visited twice.
ii Yes	Edges are used only once and vertices are visited once.
d i Yes	The cycle is a path that starts and ends at the same vertex.
ii No	The walk visits vertex B twice and uses the edge BE twice.

Now you try

Consider this graph.



a Decide if the following walks are trails. Answer Yes or No.
- **i** *D-B-A-C-A*
- **ii** *C-B-D*

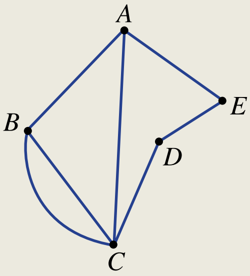
b Decide if the following walks are circuits. Answer Yes or No.
- **i** *D-B-C-B-D*
- **ii** *B-C-A-B*

c Decide if the following walks are paths. Answer Yes or No.
- **i** *D-B-A-C*
- **ii** *C-B-A-C-B*

d Decide if the following walks are cycles. Answer Yes or No.
- **i** *D-B-C-B*
- **ii** *B-C-A-B*

Example 25 Exploring Eulerian trails

Decide if this graph has an Eulerian trail. If so decide if all such trails will be circuits.



SOLUTION

Yes e.g. *A-E-D-C-A-B-C-B*

Eulerian trails will not be circuits.

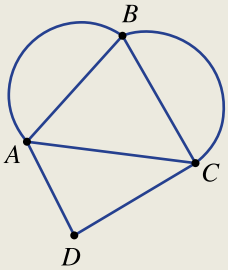
EXPLANATION

There are two vertices of odd degree and therefore an Eulerian trail exists.

For an Eulerian circuit to exist there must be zero vertices of odd degree.

Now you try

Decide if this graph has an Eulerian trail. If so decide if all such trails will be circuits.



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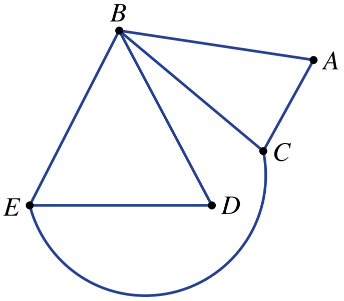
Exercise 2M

FLUENCY	1, 2, 4	1, 3–5	3–5

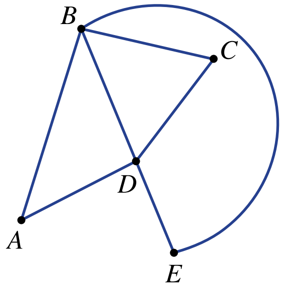
Example 24

1 Consider this graph.



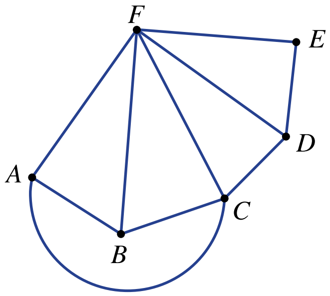
- **a** Decide if the following walks are trails. Answer Yes or No.
  - **i** $A\text{-}B\text{-}D\text{-}E\text{-}B$
  - **ii** $E\text{-}C\text{-}A\text{-}B\text{-}E\text{-}D$
- **b** Decide if the following walks are circuits. Answer Yes or No.
  - **i** $B\text{-}A\text{-}C\text{-}B\text{-}D\text{-}E\text{-}B$
  - **ii** $D\text{-}E\text{-}C\text{-}B\text{-}E\text{-}D$
- **c** Decide if the following walks are paths. Answer Yes or No.
  - **i** $A\text{-}B\text{-}E\text{-}D\text{-}B\text{-}C$
  - **ii** $A\text{-}C\text{-}B\text{-}E\text{-}D$
- **d** Decide if the following walks are cycles. Answer Yes or No.
  - **i** $D\text{-}E\text{-}B\text{-}D$
  - **ii** $C\text{-}A\text{-}B\text{-}E\text{-}D\text{-}B\text{-}C$

2 Consider this graph.



- **a** Decide if the following walks are trails. Answer Yes or No.
  - **i** $A\text{-}B\text{-}C\text{-}D\text{-}B\text{-}E$
  - **ii** $A\text{-}D\text{-}C\text{-}B\text{-}A\text{-}D$
- **b** Decide if the following walks are circuits. Answer Yes or No.
  - **i** $A\text{-}D\text{-}E\text{-}B\text{-}D\text{-}C\text{-}B\text{-}A$
  - **ii** $D\text{-}C\text{-}B\text{-}A\text{-}D\text{-}E$
- **c** Decide if the following walks are paths. Answer Yes or No.
  - **i** $D\text{-}E\text{-}B\text{-}D\text{-}C$
  - **ii** $C\text{-}D\text{-}E\text{-}B\text{-}A$
- **d** Decide if the following walks are cycles. Answer Yes or No.
  - **i** $D\text{-}E\text{-}B\text{-}C\text{-}D$
  - **ii** $A\text{-}D\text{-}E\text{-}B\text{-}D\text{-}A$

3 Consider this graph.

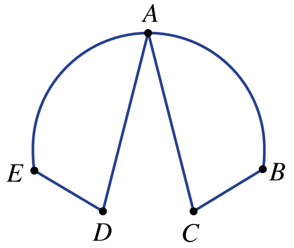


- **a** Decide if the following walks are trails. Answer Yes or No.
  - **i** $A\text{-}B\text{-}C\text{-}A\text{-}F\text{-}D$
  - **ii** $D\text{-}C\text{-}A\text{-}F\text{-}C\text{-}D$
- **b** Decide if the following walks are circuits. Answer Yes or No.
  - **i** $F\text{-}D\text{-}C\text{-}A\text{-}B\text{-}C\text{-}F$
  - **ii** $B\text{-}F\text{-}C\text{-}A\text{-}F\text{-}B$
- **c** Decide if the following walks are paths. Answer Yes or No.
  - **i** $A\text{-}B\text{-}C\text{-}D\text{-}E$
  - **ii** $F\text{-}C\text{-}B\text{-}F\text{-}A$
- **d** Decide if the following walks are cycles. Answer Yes or No.
  - **i** $D\text{-}F\text{-}A\text{-}C\text{-}D$
  - **ii** $C\text{-}F\text{-}B\text{-}A\text{-}F\text{-}C$

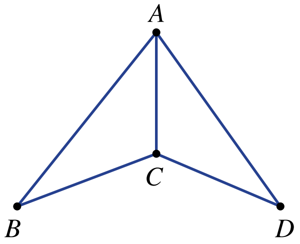
Example 25

4 Decide if the following graphs have an Eulerian trail. If so decide if all such trails will be circuits.

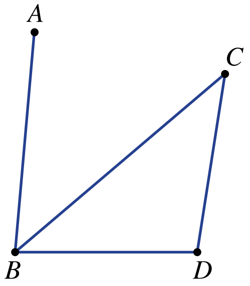
a



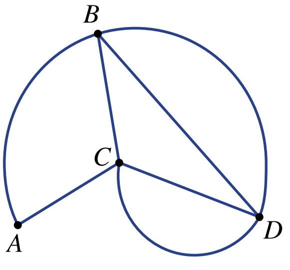
b



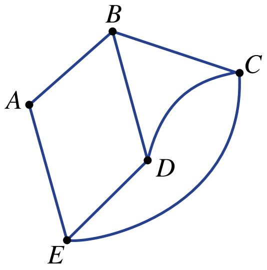
c



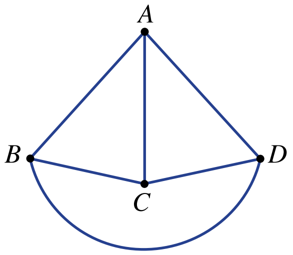
d



e

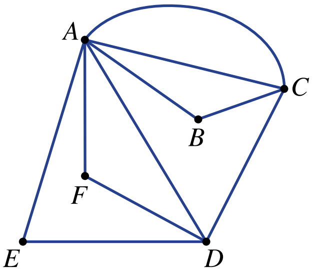


f

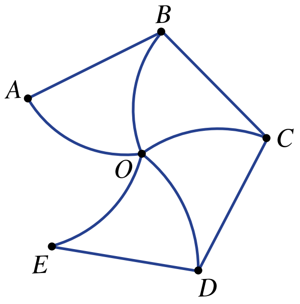


5 By determining the degree of each vertex decide if the following graphs have Eulerian trails.

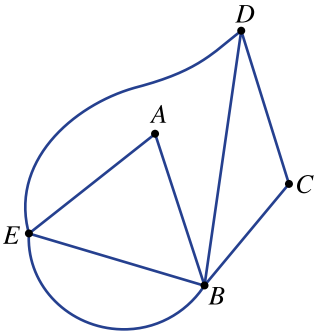
a



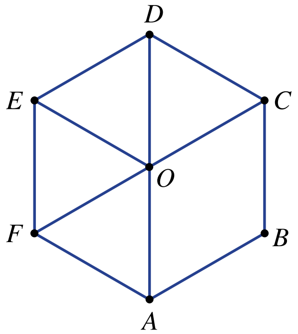
b



c



d

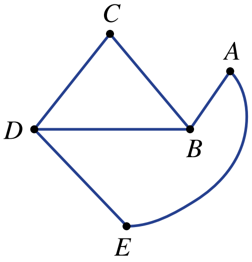


PROBLEM–SOLVING	6, 7	6–8	7–9

6 Consider the possible Eulerian trails in this graph.

a List the different Eulerian trails starting at vertex B and ending at vertex D.

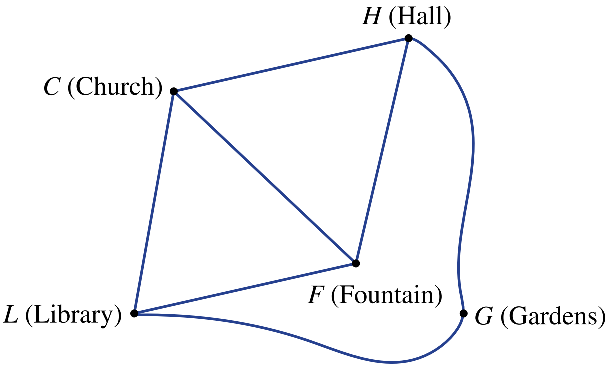
b List the different Eulerian trails starting at vertex D and ending at vertex B.



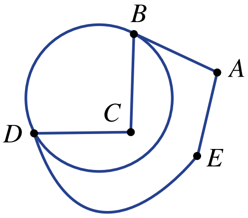
7 Margo wishes to visit every attraction in a village which are connected with walking paths as shown.

a Is it possible for Margo to find a walk which visits all the attractions starting and ending at the church without passing any given attraction twice? If so, how many ways can this be achieved?

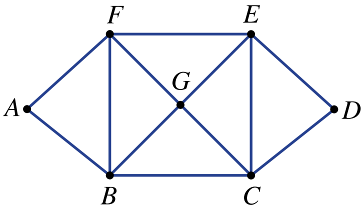
b Is it possible for Margo to use every footpath exactly once and visit every attraction at least once? If so, list the trail.



8 How many Eulerian circuits can you find through this network starting at vertex A? Do not count the same circuit in reverse.



9 Does this graph have an Eulerian circuit? If so find one.



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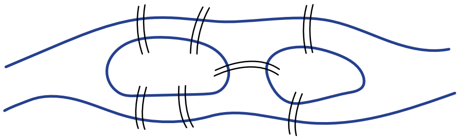
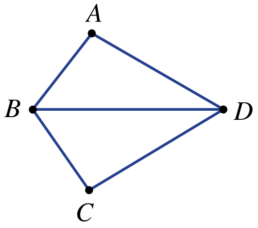
REASONING	10	10, 11	10–12

10 Let’s revisit the Konigsberg bridge problem using this diagram.

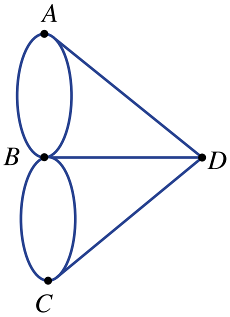


a Which of the Graphs 1 to 4 below is valid for the Konigsberg bridge problem?

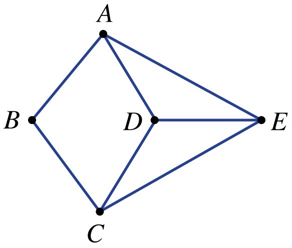
Graph 1



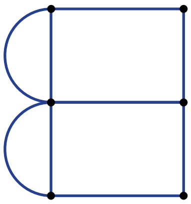
Graph 2



Graph 3



Graph 4



b What do the edges on the network represent in the Konigsberg bridge problem?

c How many vertices on the graph representing the problem are of odd degree?

d What does your answer to part **c** tell you about the Konigsberg bridge problem?

11 Answer true (T) or false (F).

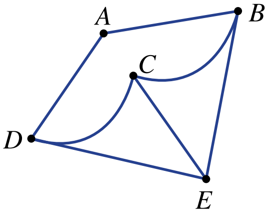
- **a** All paths are walks.
- **b** All cycles are paths.
- **c** All walks are trails.
- **d** All trails are circuits.
- **e** All paths are trails.
- **f** All circuits are cycles.

12 Semi-Eulerian graphs have an Eulerian trail but no Eulerian circuit. What can be said about the degree of the vertices in such a network? Try drawing an example.

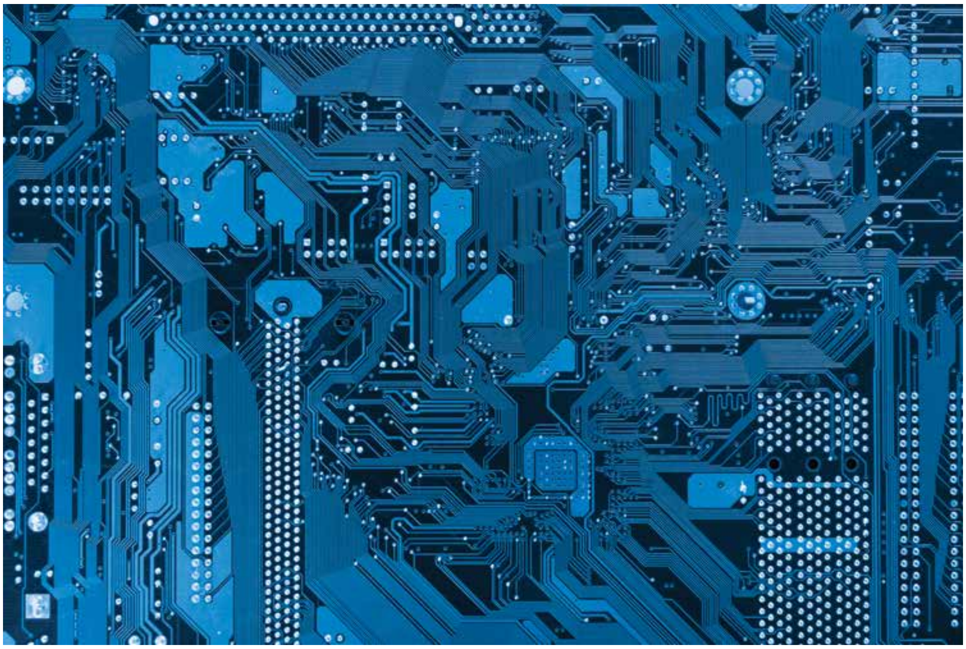
ENRICHMENT: Adding edges to form Eulerian trails and circuits

–	–	13

13 Consider the given graph.



- **a** By considering the degree of each vertex decide if the graph has the following:
  - **i** an Eulerian trail
  - **ii** an Eulerian circuit
- **b** Is it possible to add a single edge so that the graph will have an Eulerian trail? If so, give an example.
- **c** Is it possible to add a single edge so that the graph will have an Eulerian circuit? If so, give an example.
- **d** What is the minimum number of edges that need to be added so that the graph has an Eulerian circuit?



2N Shortest path problems

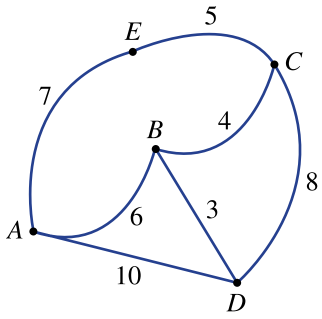
LEARNING INTENTIONS

- To understand that graphs can be weighted using distances
- To know how to interpret weighted graphs
- To be able to find a shortest path through a network

A weighted graph is a network where each edge is labelled with a number. These numbers could represent the cost to transport goods between points, the voltages in an electrical circuit or distances between towns on a map. In this section we will focus on networks including distances, namely, shortest path problems. It is common in network theory to try to minimise the distances between two points. An example is a bus network where we might be interested in stopping at a range of points across a town using the minimum possible distance.

Lesson starter: Village distances

This simplified map shows the distance, in kilometres, between five villages Almore (A), Bellan (B), Coldstom (C), Denont (D) and Elimono (E).



- Find the total distance between Denont and Elimono if travelling via:
  - Bellan and Coldstom
  - Bellan and Almore.
- Find the shortest path from Denont to Elimono and state this minimum distance.
- If the distance from Denont to Bellan was instead 4 km, would this change your mind when finding the minimum distance between villages Denont and Elimono? Discuss.

KEY IDEAS

- A **weighted graph** is a graph with numbers attached to each of the edges.
  - These numbers could represent for example, costs, volumes, times or distances.
- A shortest path problem involves finding a walk through a network which provides a minimum distance.

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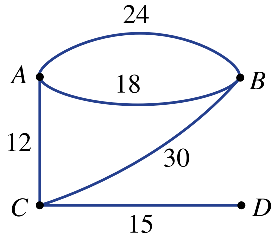
BUILDING UNDERSTANDING



1 This graph represents a rail map joining four towns A, B, C and D. Distances are in kilometres.

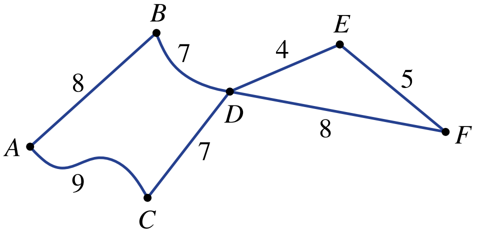
a How many edges are labelled representing distances greater than 20 km?

b How many different ways could you travel from town A to town C without using an edge more than once?

c How many different ways could you travel from town A to town C without visiting a town more than once?

d What is the length of the shortest path between towns A and C?

e What is the length of the shortest path between towns A and D?



2 This graph represents the distance in kilometres using trails between points on a bushwalking map.

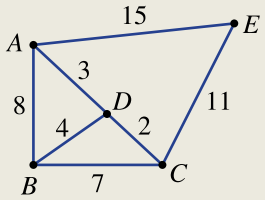
a Find the distance between points A and D via:

i B **ii** C.

b Find the distance between points D and F via point E.

c What points would be visited if the shortest path is taken from point A to point F?

Example 26 Interpreting a weighted graph



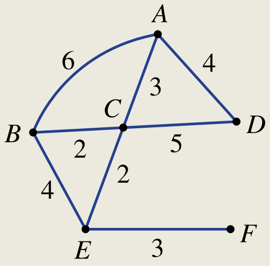
This graph shows the distance, in metres, between vertices in a pipe network.

a How far is it from point B to C via D on the network?

b Calculate the distance for the path A-D-B-C-E.

SOLUTION	**EXPLANATION**
a $4 + 2 = 6$ m	The distance from B to D is 4 m and from D to C is 2 m.
b $3 + 4 + 7 + 11 = 25$ m	Add all the distances connecting the points in the correct order.

Now you try



This graph shows the distance, in metres, between vertices on an electrical circuit.

a How far is it from point A to F via only B and E on the circuit?

b Calculate the distance for the path A-D-C-E-F.

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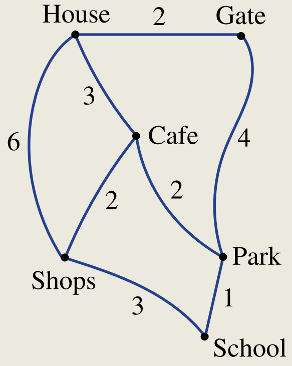
Example 27 Finding the shortest path

This graph represents a cycling network between a student's house and their school. Distances are in kilometres.



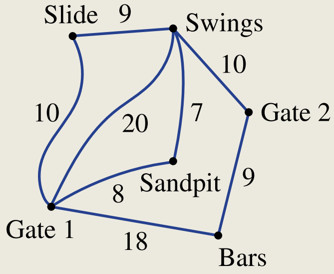
a How far is the walk from the house to the school via:
  i the shops without passing by the café?
  ii the gate and the park without passing by the café?

b Find the minimum distance from the house to the school via:
  i the shops
  ii any possible walk.

SOLUTION	**EXPLANATION**
a **i** $6 + 3 = 9$ km	The path House-Shops-School includes the distances 6 km and 3 km.
ii $2 + 4 + 1 = 7$ km	The path House-Gate-Park-School includes the distances 2 km, 4 km and 1 km.
b **i** $3 + 2 + 3 = 8$ km	It is shorter to get to the shops via the café.
ii $3 + 2 + 1 = 6$ km	Some possible walks are: • House-Gate-Park-School (7 km) • House-Shops-School (9 km) • House-Café-Shops-School (8 km) • House-Café-Park-School (6 km)

Now you try

This graph represents a path network connecting equipment in a playground. Distances are in metres.



a How far is the path from gate 1 to the swings via:
  i the slide?
  ii the sandpit?

b Find the minimum distance from gate 1 to gate 2 via:
  i the slide
  ii any possible walk.

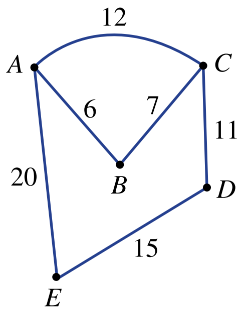
Exercise 2N

FLUENCY	1–4	1, 3, 4	3, 4

Example 26

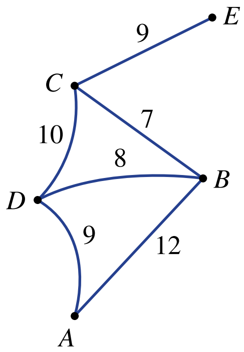
1 This graph shows the distance, in kilometres, for a simple road network between points A, B, C, D and E

- **a** How far is it from point E to C via D on the network?
- **b** Calculate the distance for the path A-B-C-D.



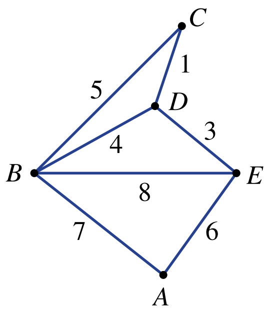
2 This graph shows the distance, in metres, between junctions on a cable network.

- **a** How far is it from point A to B via D and C on the network?
- **b** Calculate the distance for the path A-B-D-C-E.



3 This graph shows the distance, in centimetres, between points on an electrical circuit.

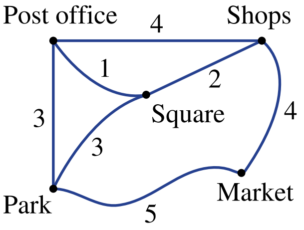
- **a** How far is it from point A to B via E, D and C on the network?
- **b** Calculate the distance for the path A-E-B-C.



Example 27

4 This graph represents a street map connecting a number of key places in a town. Distances are in kilometres.

- **a** How far is the walk from the post office to the park via:
  - **i** the square without passing by the shops?
  - **ii** the square, shops and market?
- **b** Find the minimum distance from the shops to the park via:
  - **i** the post office
  - **ii** any possible walk.



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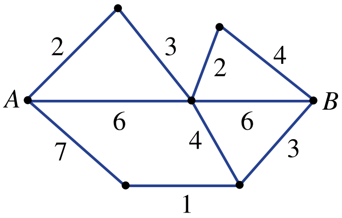
PROBLEM–SOLVING

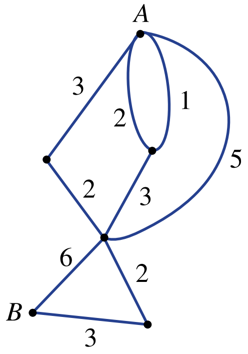
5, 6	5, 6	5(½), 6, 7

5 Find the shortest possible distance when travelling from point A to point B in these graphs. Distances are in kilometres.

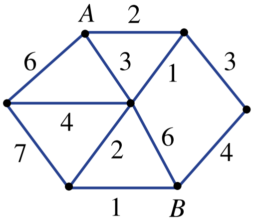
a



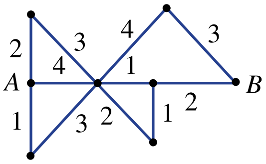
b



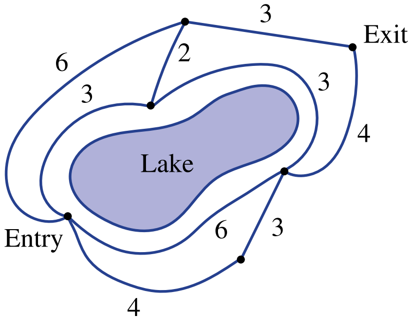
c



d

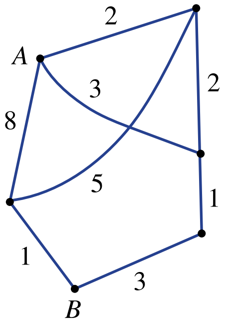


6 A network of walking paths around a lake are represented in this graph with distances in kilometres. Find the shortest distance between the entry and exit points.

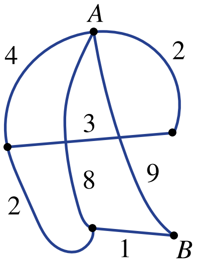


7 These graphs have intersecting edges but are in fact planar. Find the shortest distance from point A to point B. Distances are in centimetres.

a



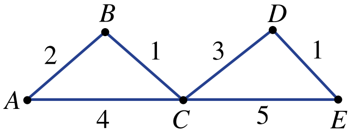
b



REASONING

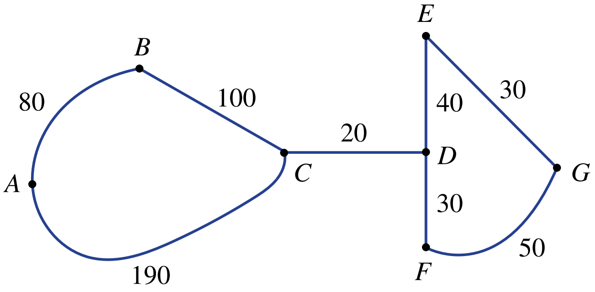
8	8, 9	8–10

8 Explain why the shortest distance between the points A and E in this graph involves going via points B and D rather than just via point C.



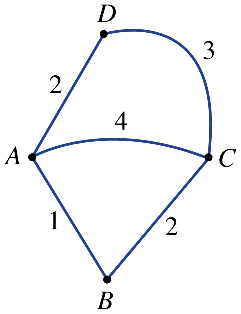
9 This weighted graph represents the cost of transporting goods via a network. The units are in dollars.

- **a** Which points should be visited to transport goods with minimum cost from point A to point G?
- **b** What is the minimum cost if transporting the goods from point A to point G?



10 An Eulerian trail follows every edge of a graph with no repeated edges. Consider this graph.

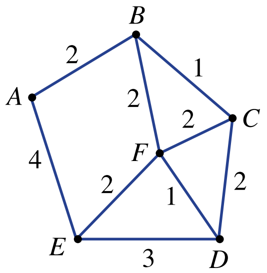
- **a** Decide if the following are Eulerian trails.
  - **i** A-B-C-D
  - **ii** A-D-C-A-B
  - **iii** A-B-C-A-D-C
- **b** True or False? Eulerian trails will have the same total distance.



ENRICHMENT: The travelling salesperson

–	–	11

11 A travelling salesperson problem involves completing a circuit of a network so that each vertex is visited exactly once using a minimum distance and returning to the starting point. This is also called a Hamiltonian circuit where the only vertex visited twice is the start and end point of the circuit. This weighted graph represents the distance between houses visited by a salesperson. Distances are in kilometres.



- **a** Decide if the following walks are Hamiltonian circuits.
  - **i** A-B-E-A
  - **ii** A-B-C-D-E-A
  - **iii** A-B-F-D-E-A
  - **iv** A-E-F-D-C-B-A
- **b** Find the distance of the Hamiltonian circuit A-B-F-C-D-E-A.
- **c** Find the length of the minimum Hamiltonian circuit; that is, find the shortest distance a salesperson can travel if visiting each point exactly once and starting and ending at point A.
- **d** If the salesperson started at a different point, is there a shorter total distance the salesperson can travel compared to your answer in part **c** above?

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The impassable object

Without the use of sophisticated equipment, it is possible to estimate the distance across an impassable object like a gorge or river, using similar triangles.

Present a report for the following tasks and ensure that you show clear mathematical workings and explanations where appropriate.

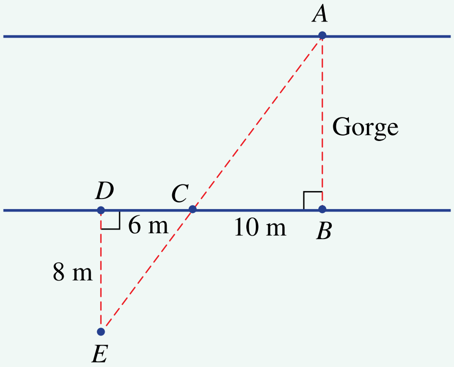
Preliminary task

On an adventure you come to a gorge and try to estimate the distance across it. You notice a boulder (A) directly across the other side of the gorge and then proceed to place rocks (B, C, D and E) on your side of the gorge in special positions as shown. You measure $BC = 10$ m, $CD = 6$ m and $DE = 8$ m.

a Prove the $\Delta ABC \;|||\; \Delta EDC$.

b Find the scale factor linking the two triangles.

c Find the distance across the gorge.



Modelling task

Formulate

a Choose an object near your school or house like a river, road or ravine.

b Consider the possible placements of pebbles or other objects (as per the Preliminary task) to create similar triangles.

c Assess your situation by taking measurements (without crossing your chosen object) and illustrate with a diagram.

Solve

d Prove that your triangles are similar.

e Determine a scale factor for your triangles.

f Estimate the distance across your chosen impassable object.

Evaluate and verify

g Construct other possible placements of the pebbles by adjusting their position.

h Assess each situation and recalculate your estimate for the distance across your impassable object.

i Compare your results from your constructions.

Communicate

j Summarise your results and describe any key findings.

Extension questions

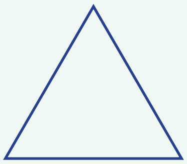
a Investigate other possible ways in which similar triangles can be constructed (resulting in a different type of diagram to the one above) to solve such a problem.

b Find a different type of problem where similar triangles can be identified. Prove that the triangles formed are similar and use them to solve the problem.

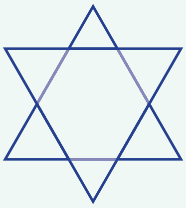
Koch snowflake

Key technology: Spreadsheets

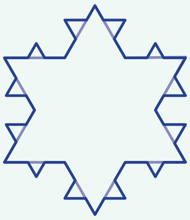
Self-similarity occurs when something can be decomposed into parts which are in themselves copies of the original. This can be seen in the natural world including in leaves, snowflakes and broccoli heads, for example. Applications are also visible in economic cycles, networks and in cybernetics. The Koch snowflake is a self-similar shape constructed by starting with an equilateral triangle and adding smaller and smaller equilateral triangles to its sides.
The first four iterations are shown right.



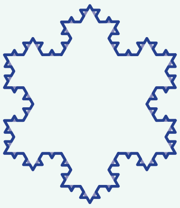
Iteration 1



Iteration 2



Iteration 3



Iteration 4

1 Getting started

Let the side length of the original equilateral triangle be 1 unit.

a Find the perimeter of the Koch snowflake after 1, 2, 3 and 4 iterations and add your results to this table.

Iteration	1	2	3	4
Perimeter	3			

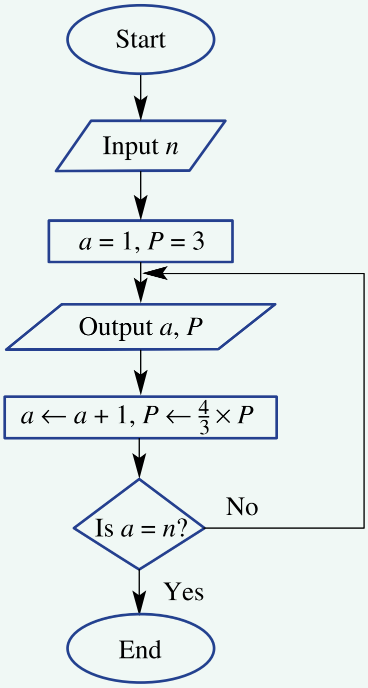
b Look at the pattern of numbers formed by the perimeters. What factor do you multiply by each time to calculate the next perimeter in the sequence?

c Use your answer from part **b** to find the perimeter for the 5th iteration.

2 Applying an algorithm

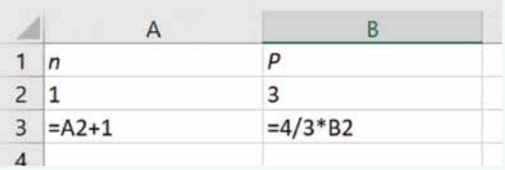
Here is a flowchart which uses an algorithm to generate the perimeter for n iterations. By choosing $n = 4$, run through the algorithm and complete this table for each pass.

a	P
1	3
2	



3 Using technology

a A spreadsheet can be used to carry out the above algorithm. Make a spreadsheet that finds the perimeter of a Koch snowflake for *n* iterations. Here are the key formulas to use.



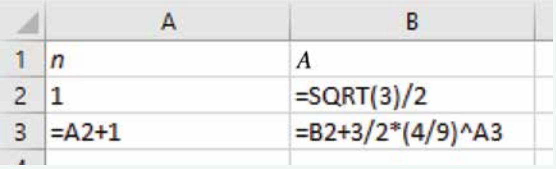
	A	B
1	*n*	*P*
2	1	3
3	=A2+1	=4/3*B2
4		

b After filling down from cells A3 and B3, find the perimeter of the Koch snowflake for:

- **i** 6 iterations
- **ii** 20 iterations.

c Will the perimeter of the Koch snowflake ever reach a maximum limit? Give reasons.

d Here is a spreadsheet which calculates the area of a Koch snowflake after *n* iterations. Copy this into a new sheet and fill down to $n = 20$.



	A	B
1	*n*	*A*
2	1	=SQRT(3)/2
3	=A2+1	=B2+3/2*(4/9)^A3

e What do you notice about the area values as the number of iterations increase? Estimate the limit of the area; that is, the value that the area approaches but never reaches.

4 Extension

a For the perimeter of the Koch snowflake adjust the flowchart and spreadsheet so that it can find the perimeter by starting with an equilateral triangle of any given side length, *s*.

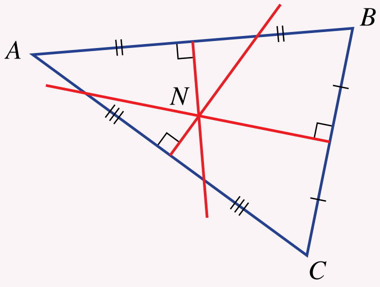
b Draw a flowchart for finding the area of a Koch snowflake for *n* iterations starting with an equilateral triangle of any given side length, *s*. Use the formulas given in the above spreadsheet to help.

c Adjust your area spreadsheet so that it can find the area by starting with an equilateral triangle of any given side length, *s*. What is the limiting area for a Koch snowflake that starts with a side length, of 5 units?

Some special points of triangles and Euler's line

This investigation explores three special points in a triangle: the circumcentre, the centroid and the orthocentre. This is best done using an interactive geometry package.

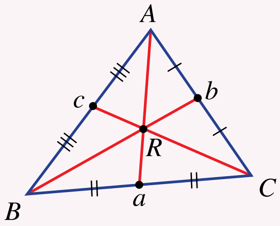
The circumcentre of a triangle



The perpendicular bisectors of each of the three sides of a triangle meet at a common point called the **circumcentre**.

- **a** Construct and label a triangle *ABC* and measure each of its angles.
- **b** Construct the perpendicular bisector of each side of the triangle.
- **c** Label the point of intersection of the lines *N*. This is the circumcentre.
- **d** By dragging the points of your triangle, observe what happens to the location of the circumcentre. Can you draw any conclusions about the location of the circumcentre for some of the different types of triangles; for example, equilateral, isosceles, right-angled or obtuse?
- **e** Construct a circle centred at *N* with radius *NA*. This is the **circumcircle**. Drag the vertex *A* to different locations. What do you notice about vertices *B* and *C* in relation to this circle?

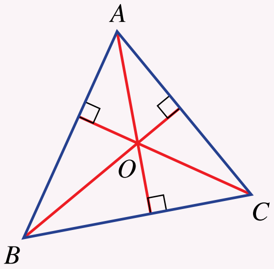
The centroid of a triangle



The three medians of a triangle intersect at a common point called the **centroid**. A **median** is the line drawn from a vertex to the midpoint of the opposite side.

- **a** Construct the centroid (*R*) for a triangle *ABC*.
- **b** Drag one of the vertices of the triangle and explore the properties of the centroid (*R*).

The orthocentre of a triangle



The three altitudes of a triangle intersect at a common point called the **orthocentre**. An **altitude** of a triangle is a line drawn from a vertex to the opposite side of the triangle, meeting it at right angles.

- **a** Construct the orthocentre of a triangle (*O*) for a triangle *ABC*.
- **b** Drag one of the vertices of the triangle and explore the properties of the orthocentre (*O*).

Euler's line

- **a** Construct a large triangle *ABC* and on this one triangle use the previous instructions to locate the circumcentre (*N*), the centroid (*R*) and the orthocentre (*O*).
- **b** Construct a line joining the points *N* and *R*. Drag the vertices of the triangle. What do you notice about the point *O* in relation to this line? This is called Euler's line.

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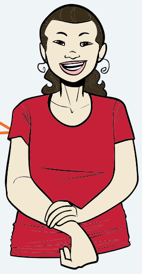
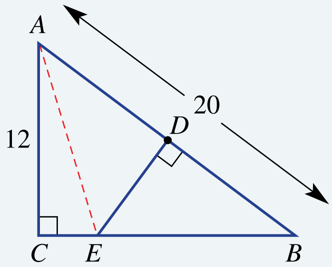
Problems and challenges

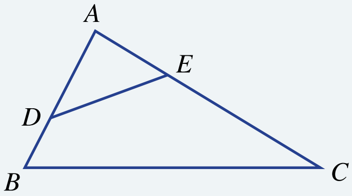
Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



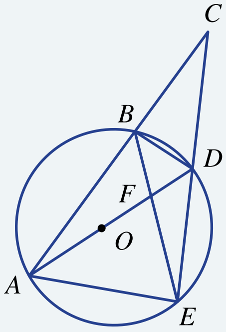
1 In a triangle ABC, angle C is a right angle, point D is the midpoint of AB and DE is perpendicular to AB. The length of AB is 20 units and the length of AC is 12 units. What is the area of triangle ACE?



2 In this diagram, $AB = 15$ cm, $AC = 25$ cm, $BC = 30$ cm and $\angle AED = \angle ABC$. If the perimeter of ΔADE is 28 cm, find the lengths of BD and CE.



3 Other than straight angles, name all the pairs of equal angles in the diagram shown.



4 A person stands in front of a cylindrical water tank and has a viewing angle of 27° to the sides of the tank. What percentage of the circumference of the tank can they see?

5 An isosceles triangle ABC is such that its vertices lie on the circumference of a circle. $AB = AC$ and the chord from point A to point D on the circle intersects BC at point E. Prove that $AB^2 - AE^2 = BE \times CE$.

6 D, E and F are the midpoints of the three sides of ΔABC. The straight line formed by joining two midpoints is parallel to the third side and half its length.

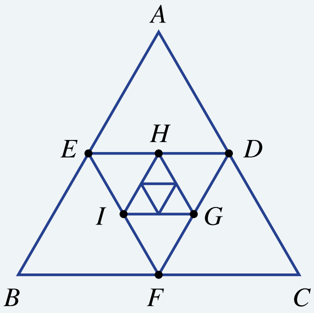
a Prove $\Delta ABC \mathbin{|||} \Delta FDE$.

ΔGHI is drawn in the same way such that G, H and I are the midpoints of the sides of ΔDEF.

b Find the ratio of the area of:

i ΔABC to ΔFDE **ii** ΔABC to ΔHGI.

c Hence, if ΔABC is the first triangle drawn, what is the ratio of the area of ΔABC to the area of the nth triangle drawn in this way?



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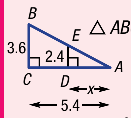
Geometry and networks

Similar triangles application



$\triangle ABC \mathrel{|||} \triangle AED$ (AAA)

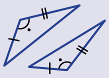
$$\text{Scale factor} = \frac{3.6}{2.4}$$
$$= 1.5$$
$$\therefore\ x \times 1.5 = 5.4$$
$$x = 5.4 \div 1.5$$
$$x = 3.6$$

Similar figures

All corresponding angles are equal, corresponding sides are in the same ratio; i.e. same shape but different in size.
Tests for similar triangles:
SSS, SAS, AAA, RHS.
Written as $\triangle ABC \mathrel{|||} \triangle DEF$ or $\triangle ABC \sim \triangle DEF$.

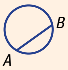
Congruent triangles

Such triangles are identical, written $\triangle ABC \equiv \triangle DEF$.
Tests for congruence are
SSS, SAS, AAS and RHS.



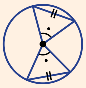
Congruent by SAS

Circles and chords (Opt)

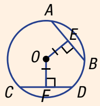


AB is a chord.

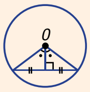
Chord theorem 1
Chords of equal length subtend equal angles.



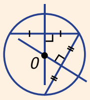
Chord theorem 2
If $AB = CD$,
then $OE = OF$.



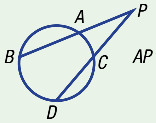
Chord theorem 3
Perpendicular from centre to chord bisects chord and angle at O.



Chord theorem 4
Perpendicular bisectors of every chord of a circle intersect at the centre.

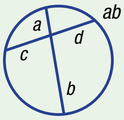


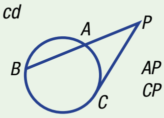
Intersecting chords, secants, tangents (Opt)



$AP \times BP = CP \times DP$

$ab = cd$



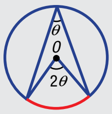


$AP \times BP = CP^2$
CP is a tangent.

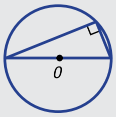
Polygons

Angle sum of polygon
$S = (n - 2) \times 180^\circ$, where n is the number of sides.

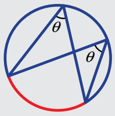
Angle properties of circles (Opt)



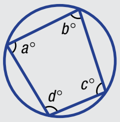
Angle at centre is twice angle at the circumference subtended by the same arc.



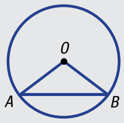
The angle in a semicircle is 90°.



Angles at circumference subtended by the same arc are equal.

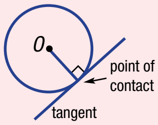


Opposite angles in cyclic quadrilaterals are supplementary.
$a + c = 180$
$b + d = 180$

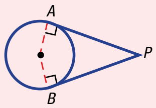


$\triangle OAB$ is isosceles given OA and OB are radii.

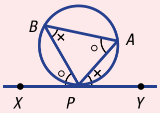
Tangents (Opt)



A tangent touches a circle once and is perpendicular to the radius at the point of contact.



Tangents PA and PB have equal length; i.e. $PA = PB$.



Alternate segment theorem: angle between tangent and chord is equal to the angle in the alternate segment.

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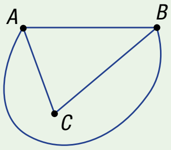
Chapter summary

Geometry and networks

Networks

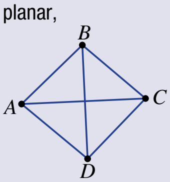
A network is a diagram connecting vertices (nodes) using lines (edges).
The degree of a vertex is the number of edges connected to it.



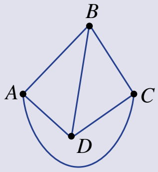
B is an odd vertex as it has an odd number of edges connected to it.

Planar graphs and Euler's fomula

Isomorphic graphs (same edges, vertices and connections) can be redrawn to look the same.
A planar graph can be drawn so it has no intersecting edges.
This graph is planar,



It can be redrawn as

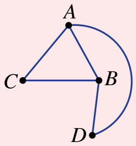


Euler's formula for planar graphs says:
$v + f = e + 2$ or alternatively $v - e + f = 2$
v = no. of vertices, f = no. of faces, e = no. of edges

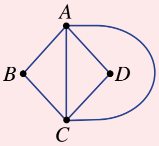
Walks

A walk is a sequence of edges through a network:

- trail – a walk with no repeated edges
- path – a walk with no repeated edges or vertices
- cycle – a path that starts and ends at the same vertex
- circuit – a trail that starts and ends at the same vertex
- Eulerian trail – every edge is used exactly once, needs 0 or 2 odd vertices to exist
  E.g. B – D – A – C – B – A



- Eulerian circuit – an Eulerian trail that starts and ends at the same vertex, requires no odd vertices.
  E.g. A – B – C – A – D – C – A



Weighted graphs

A weighted graph has a number attached to each edge which might represent distances, time, costs etc.

A shortest path problem involves finding a walk through the network of minimum distance.

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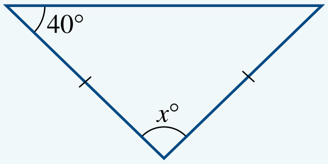
Chapter checklist with success criteria

A printable version of this checklist is available in the Interactive Textbook ✓

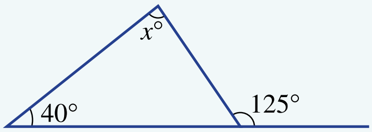
2A **1. I can use the angle sum of a triangle.** ☐

e.g. Find the value of *x*, giving reasons.



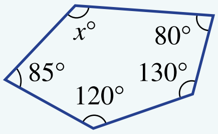
2A **2. I can apply the exterior angle theorem.** ☐

e.g. Find the value of *x*, giving reasons.



2A **3. I can find an unknown angle in a polygon.** ☐

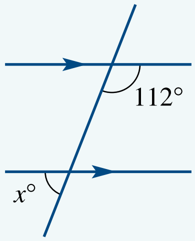
e.g. Find the value of *x* in the polygon shown.



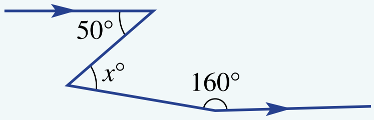
2A **4. I can work with angles in parallel lines.** ☐

e.g. Find the value of *x* in the following, giving reasons.

i

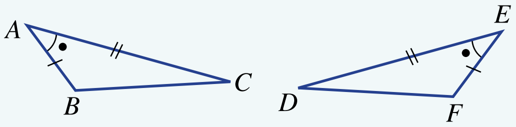


ii



2B **5. I can prove congruence of triangles.** ☐

e.g. Prove that this pair of triangles are congruent.



Chapter checklist with success criteria

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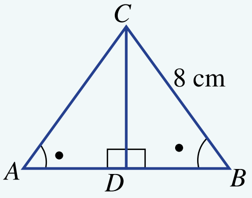
2B

6. I can use congruence in proof.
e.g. For the diagram shown, prove $\Delta ADC \equiv \Delta BDC$ and hence state the length of AC, giving a reason.



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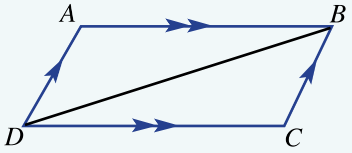
2C

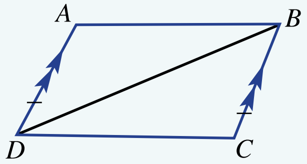
7. I can prove properties of quadrilaterals.
e.g. Prove that a parallelogram (with opposite parallel sides) has equal opposite sides.



☐

2C

8. I can test for a type of quadrilateral.
e.g. Prove that if one pair of opposite sides is equal and parallel in a quadrilateral then it is a parallelogram.



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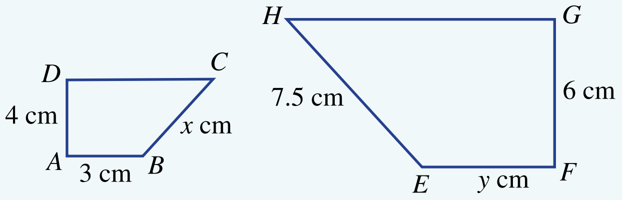
2D

9. I can find and use a scale factor in similar figures.
e.g. The two shapes shown are similar. Find the scale factor and use this to find the values of x and y.



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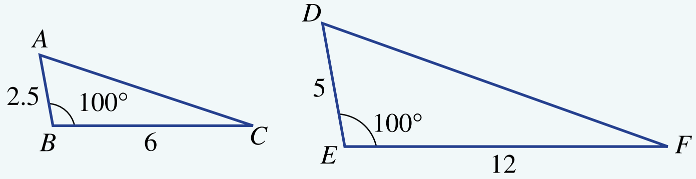
2E

10. I can prove similar triangles using similarity tests.
e.g. Prove that the following triangles are similar.



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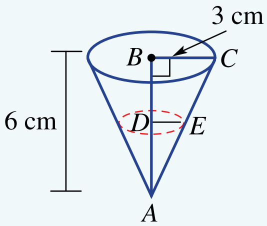
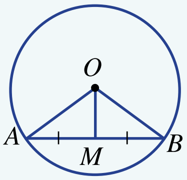
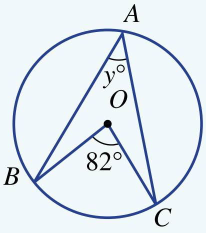
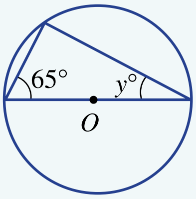
			✓
2E	**11. I can establish and use similarity.** e.g. A cone has radius 3 cm and height 6 cm. Prove that $\triangle ADE \mathbin{\vert\vert\vert} \triangle ABC$ and find the radius DE if $AD = 2$ cm. 		☐
2F	**12. I can use chord theorems.** e.g. Given $AM = BM$ and $\angle AOB = 100°$, find $\angle AOM$ and $\angle OMB$. 	Opt	☐
2F	**13. I can prove chord theorems.** e.g. Prove chord theorem 1 in that chords of equal length subtend equal angles at the centre of a circle.	Opt	☐
2G	**14. I can apply circle theorem 1.** e.g. Find the value of y in the circle shown. 	Opt	☐
2G	**15. I can apply circle theorem 2.** e.g. Find the value of y in the circle shown. 	Opt	☐

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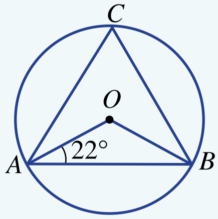
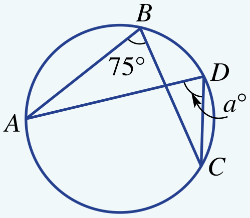
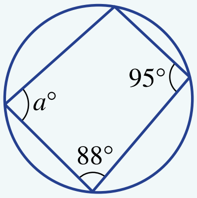
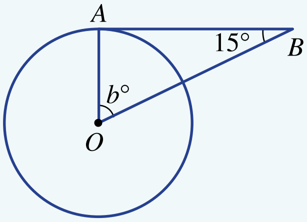
2G	**16. I can apply circle theorems with other circle properties.** Opt e.g. Find the size of ∠*ACB*. 	☐
2H	**17. I can apply circle theorem 3.** Opt e.g. Find the value of *a*. 	☐
2H	**18. I can apply circle theorem 4.** Opt e.g. Find the value of *a*. 	☐
2I	**19. I can find angles involving tangents.** Opt e.g. Find the value of *b* in this diagram involving a tangent. 	☐

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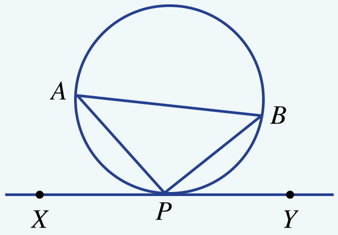
2I

20. I can use the alternate segment theorem. Opt ☐

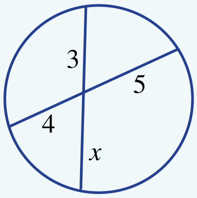
e.g. In the diagram *XY* is a tangent to the circle. Find $\angle ABP$ if $\angle BAP = 23^\circ$ and $\angle APX = 65^\circ$.



2J

21. I can find lengths using intersecting chords. Opt ☐

e.g. Find the value of *x* in the diagram.

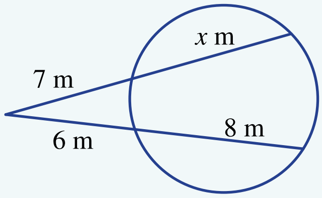


2J

22. I can find lengths using intersecting secants and tangents. Opt ☐

e.g. Find the value of *x* in the following diagrams.

i



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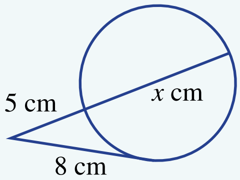
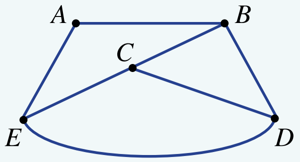


2K

23. I can determine the features of a network. ☐

e.g. For the graph shown, state:

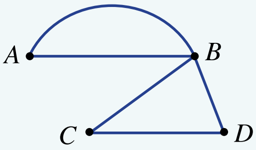
- i the number of vertices
- ii the degree of vertex *C*.



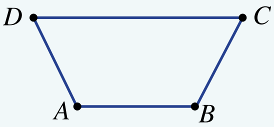
Chapter checklist with success criteria

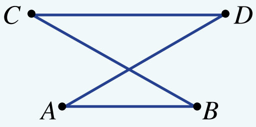
✓

2K **24. I can find walks between vertices.**
e.g. For the graph shown, find how many walks there are connecting A and C without visiting a vertex more than once or using an edge more than once. ☐

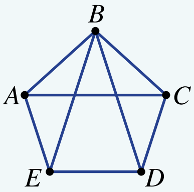


2L **25. I can decide if graphs are isomorphic.**
e.g. Decide if the following two graphs are isomorphic. ☐

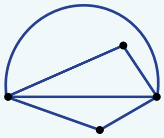




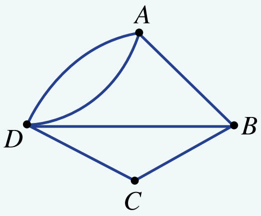
2L **26. I can decide if a graph is planar or non-planar.**
e.g. Decide if the following graph is planar or non-planar. ☐



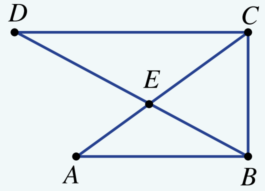
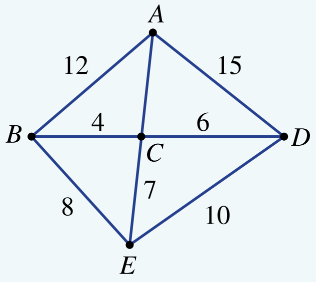
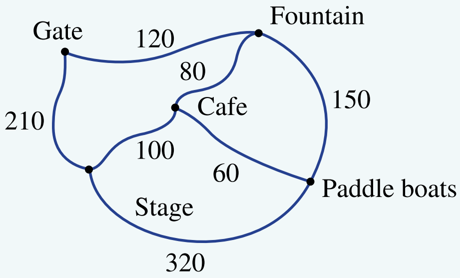
2L **27. I can verify Euler's formula.**
e.g. Find the number of vertices, edges and faces in this planar graph and verify Euler's formula. ☐



2M **28. I can define a type of walk.**
e.g. For the given graph, decide if:
i A-B-C-D-B is a trail
ii A-B-D-C-B-A is a cycle. ☐



Chapter checklist with success criteria

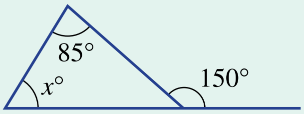
		✓
2M	**29. I can decide if a graph has an Eulerian trail or circuit.** e.g. Decide if this graph has an Eulerian trail and if so will all such trails be circuits. 	☐
2N	**30. I can interpret a weighted graph.** e.g. This graph shows the distances in kilometres along the roads connecting five towns, *A* to *E*. Calculate the distance for the path *A*-*B*-*C*-*E*-*D*. 	☐
2N	**31. I can find the shortest path on a network graph.** e.g. This graph represents a path network connecting landmarks in the Botanical gardens. Distances are in metres. Find the minimum distance from the Gate to the Paddle boats. 	☐

Short-answer questions

2A

1 Determine the value of each pronumeral.

a



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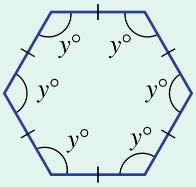
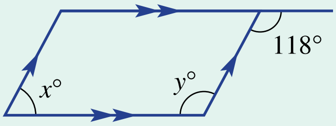
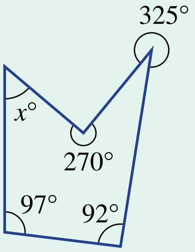


c



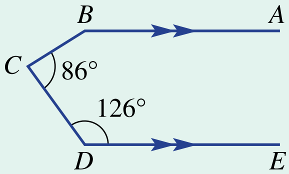
d



2A

2 Find the value of $\angle ABC$ by adding a third parallel line.

a



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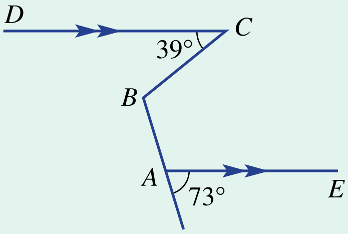
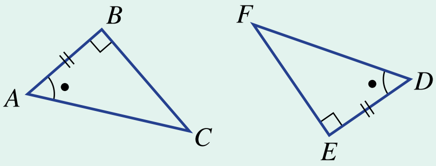


2B

3 Prove that each pair of triangles is congruent, giving reasons.

a



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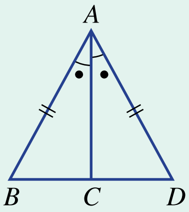

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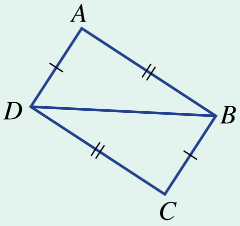
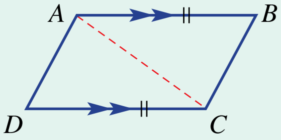


2C

4 Complete these steps to prove that if one pair of opposite sides is equal and parallel in a quadrilateral, then it is a parallelogram.

a Prove $\Delta ABC \equiv \Delta CDA$, giving reasons.

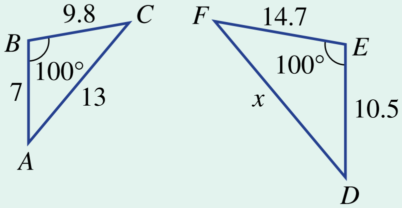
b Hence, prove $AD \parallel BC$.



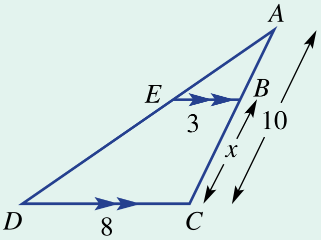
2E

5 In each of the following, identify pairs of similar triangles by proving similarity, giving reasons, and then use this to find the value of x.

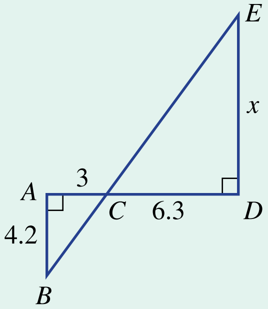
a



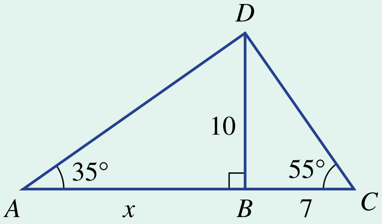
b



c



d

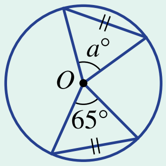


2F

Opt

6 Find the value of each pronumeral and state the chord theorem used.

a



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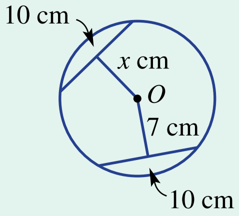
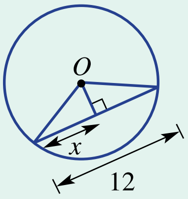


c

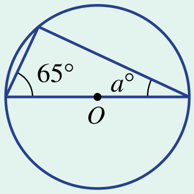


2G/H

Opt

7 Use the circle theorems to help find the values of the pronumerals.

a



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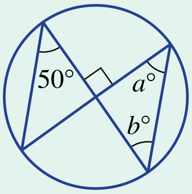
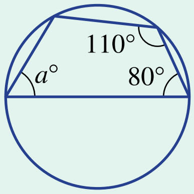


c



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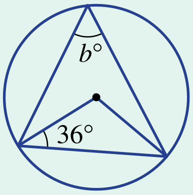
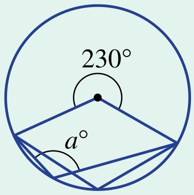
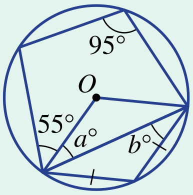


e



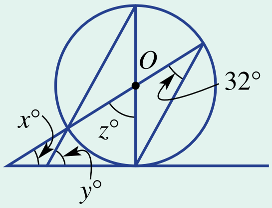
f



2I Opt

8 Find the value of the pronumerals in these diagrams involving tangents and circles.

a



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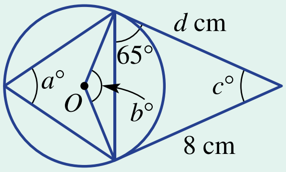
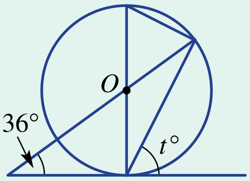


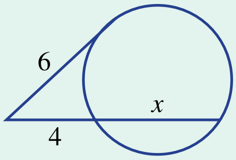
c



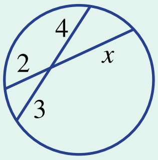
2J Opt

9 Find the value of x in each figure.

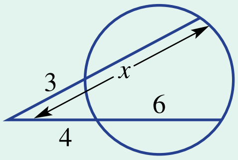
a



b

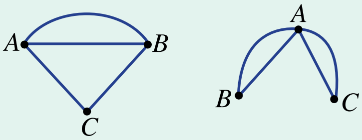


c



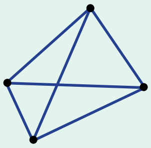
2L

10 **a** Decide if the following two graphs are isomorphic.

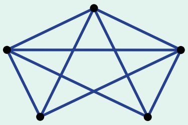


b Decide if the following graphs are planar or non-planar.

i



ii



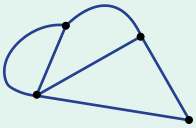
2K/L

11 Consider the planar graph shown.

a State the number of edges and the number of vertices.

b State the number of faces.

c Hence, verify Euler's formula for the graph.



2M

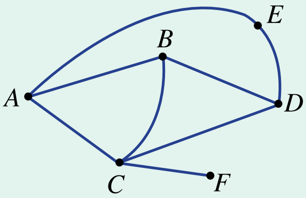
12 Consider the graph shown.

a Decide if the walk A-C-D-B-C-A is a trail.

b Explain why this graph does not have an Eulerian trail.

c Add one edge to the graph from D so that it has an Eulerian trail.

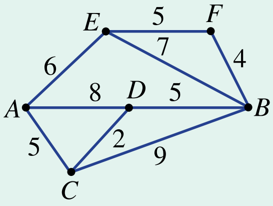
d State the Eulerian trail from part **c** and decide if it is a circuit.



2N **13** This graph shows the distances between points on an electrical circuit in centimetres.

a Calculate the distance for the path *A*-*E*-*F*-*B*.

b What is the shortest distance from *A* to *B*?

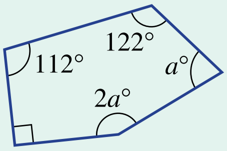


Multiple-choice questions

2A **1** The value of a in the polygon shown is:

A 46 **B** 64 **C** 72

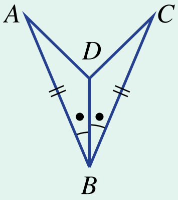
D 85 **E** 102



2B **2** The test that proves that $\Delta ABD \equiv \Delta CBD$ is:

A RHS **B** SAS **C** SSS

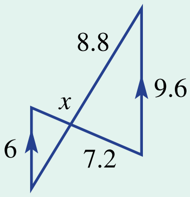
D AAA **E** AAS



2D **3** The value of x in the diagram shown is:

A 4.32 **B** 4.5 **C** 3.6

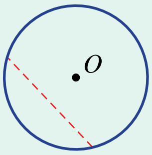
D 5.5 **E** 5.2



2F Opt **4** The name given to the dashed line in the circle with centre O is:

A a diameter **B** a minor arc **C** a chord

D a tangent **E** a secant



2F Opt **5** A circle of radius 5 cm has a chord 4 cm from the centre of the circle. The length of the chord is:

A 4.5 cm **B** 6 cm **C** 3 cm

D 8 cm **E** 7.2 cm

2G Opt **6** The values of the pronumerals in the diagram are:

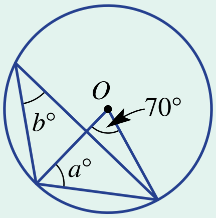
A $a = 55, b = 35$

B $a = 30, b = 70$

C $a = 70, b = 35$

D $a = 55, b = 70$

E $a = 40, b = 55$



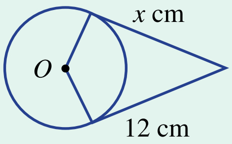
2H Opt **7** A cyclic quadrilateral has one angle measuring 63° and another angle measuring 108°. Another angle in the cyclic quadrilateral is:

A 63° **B** 108° **C** 122° **D** 75° **E** 117°

2I Opt

8 For the circle shown at right with radius 5 cm, the value of x is:

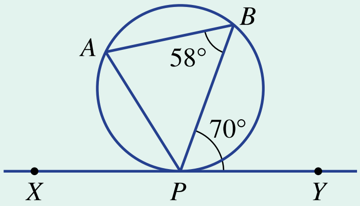
A 13 **B** 10.9 **C** 12
D 17 **E** 15.6



2I Opt

9 By making use of the alternate segment theorem, the value of $\angle APB$ is:

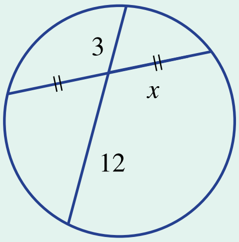
A 50° **B** 45° **C** 10°
D 52° **E** 25°



2J Opt

10 The value of x in the diagram is:

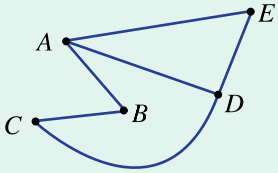
A 7.5 **B** 6 **C** 3.8
D 4 **E** 5



2K

11 How many odd vertices does the graph shown have?

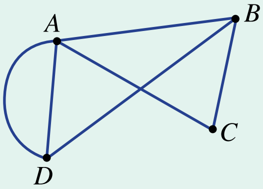
A 0 **B** 1 **C** 2
D 3 **E** 4



2M

12 So that the graph shown has an Eulerian circuit, which edge should be removed?

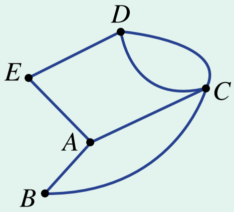
A A-D **B** A-C **C** B-D
D A-B **E** B-C



2M

13 Which of the following is not a path for the graph shown?

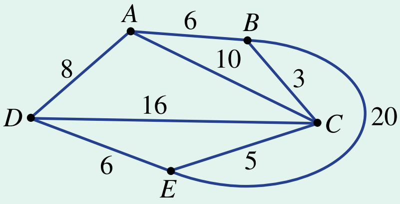
A A-B-C-D **B** A-C-D-E **C** E-A-B-C-D
D A-C-D-C-B **E** E-D-C-A-B



2N

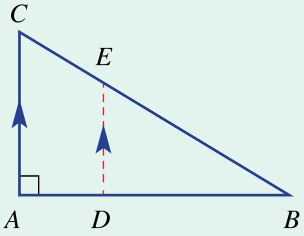
14 The shortest path through this network from A to D via B, where distance is in kilometres, is:

A 20 km **B** 25 km **C** 32 km
D 8 km **E** 19 km



Extended-response questions

1 The triangular area of land shown is to be divided into two areas such that $AC \parallel DE$. The land is to be divided so that $AC : DE = 3 : 2$.



a Prove that $\Delta ABC \mathbin{|||} \Delta DBE$.

b If $AC = 1.8$ km, find DE.

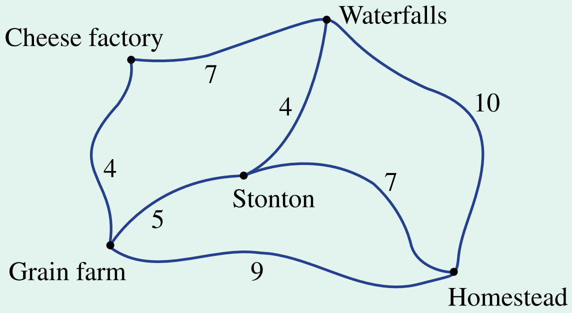
c If $AD = 1$ km and $DB = x$ km:

  i show that $2(x + 1) = 3x$

  ii solve for x.

d For the given ratio, what percentage of the land area does ΔDBE occupy? Answer to one decimal place.

2 This graph represents a map of the countryside near the village of Stonton. Distances are in kilometres.



a Find the sum of degrees of all the vertices.

b How far is the walk Stonton-Grain farm-Cheese factory?

c Confirm that the graph satisfies Euler's formula $v + f = e + 2$.

d Decide if the trail Stonton-Waterfalls-Homestead-Grain farm-Cheese factory is Eulerian. Give a reason.

e Find the length of the shortest path between the following places.

  i Grain farm and Waterfalls

  ii Homestead and Cheese factory if going via Stonton

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3

Indices, exponentials and logarithms

Maths in context: Using logs to calculate the light magnitude limit of a telescope

It is useful to know the magnitude of the faintest visible star that you can view through your telescope on a very dark night. This is called the telescope's light magnitude limit, L_{mag}. Light first enters a telescope's larger objective lens of diameter D_O mm. The light then passes through the eyepiece lens into your eye, with average pupil diameter $D_{eye} = 7$ mm.

G_{mag} is the telescope's brightness increase capacity, defined as:

$$G_{mag} = 2.5\log_{10}\left(\frac{D_O}{D_{eye}}\right)^2$$

$$G_{mag} = 5\log_{10}\left(D_O\right) - 5\log_{10}(7)$$

The Greek astronomers had only the 'naked eye' and defined the faintest visible light magnitude as $L_{mag} = 6$. Hence the approximate light magnitude limit $\left(L_{mag}\right)$ of a telescope, is defined as:

$$L_{mag} = G_{mag} + 6$$

$$L_{mag} = 5\log_{10}\left(D_O\right) + 2$$

For example:

If a telescope's objective lens has diameter, $D_O = 100$ mm, its light magnitude limit is:

$$L_{mag} = 5\log_{10}\left(D_O\right) + 2$$

$$L_{mag} = 5\log_{10}(100) + 2 = 12$$

Objects down to a magnitude of 12 should be visible through this telescope on a dark night. This is a useful 'rule of thumb' as not all potential variables are accounted for.

Chapter contents

Australian Curriculum 9.0

ALGEBRA

Expand, factorise and simplify expressions and solve equations algebraically, applying exponent laws involving products, quotients and powers of variables, and the distributive property (AC9M10A01)

Recognise the connection between algebraic and graphical representations of exponential relations and solve related exponential equations, using digital tools where appropriate (AC9M10A03)

Use mathematical modelling to solve applied problems involving growth and decay, including financial contexts; formulate problems, choosing to apply linear, quadratic or exponential models; interpret solutions in terms of the situation; evaluate and modify models as necessary and report assumptions, methods and findings (AC9M10A04)

Experiment with functions and relations using digital tools, making and testing conjectures and generalising emerging patterns (AC9M10A05)

The inverse relationship between exponential functions and logarithmic functions and the solution of related equations (Year 10 optional content)

MEASUREMENT

Interpret and use logarithmic scales in applied contexts involving small and large quantities and change (AC9M10M02)

NUMBER

Operations on numbers involving fractional exponents and surds (Year 10 optional content)

Online resources

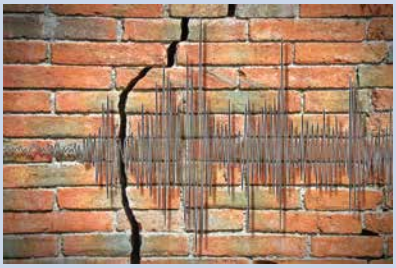
A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

3A Review of index laws CONSOLIDATING

LEARNING INTENTIONS

- To know that powers are used as a shorthand way of writing repeated multiplications
- To understand that index laws for multiplication and division apply only to common bases
- To know how to combine powers with the same base under multiplication and division
- To know how to apply powers where brackets are involved
- To know that any number (except 0) to the power of zero is equal to 1
- To be able to combine a number of index laws to simplify an expression

From your work in Year 9 you will recall that powers (i.e. numbers with indices) can be used to represent repeated multiplication of the same factor. For example, $2 \times 2 \times 2 = 2^3$ and $5 \times x \times x \times x \times x = 5x^4$. The five basic index laws and the zero power will be revised in this section.



Index laws efficiently simplify powers of a base. Powers of 2 calculate the size of digital data and bacterial populations, and powers of 10 are used when calculating earthquake and sound level intensities.

Lesson starter: Recall the laws

Try to recall how to simplify each expression and use words to describe the index law used.

- $5^3 \times 5^7$
- $x^4 \div x^2$
- $(a^7)^2$
- $(2a)^3$
- $\left(\frac{x}{3}\right)^4$
- $(4x^2)^0$

KEY IDEAS

- Recall that $a = a^1$ and $5a = 5^1 \times a^1$.
- The index laws
  - Index law for multiplication: $a^m \times a^n = a^{m+n}$ Retain the base and add the indices.
  - Index law for division: $a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$ Retain the base and subtract the indices.
  - Index law for power of a power: $(a^m)^n = a^{m \times n}$ Retain the base and multiply the indices.
  - Index law for brackets: $(a \times b)^m = a^m \times b^m$ Distribute the index number across the bases.
  - Index law for fractions: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ Distribute the index number across the bases.
- The zero index: $a^0 = 1$ Any number (except 0) to the power of zero is equal to 1.

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BUILDING UNDERSTANDING

1 Simplify, using index form.

a $3 \times 3 \times 3 \times 3$

b $7 \times 7 \times 7 \times 7 \times 7 \times 7$

c $2 \times x \times x \times 3 \times x$

d $2 \times b \times a \times 4 \times b \times a \times a$

2 State the missing parts to this table.

x	4	3	2	1	0
2^x		$2^3 = 8$			

3 State the missing components.

a $$\begin{aligned} 2^2 \times 2^3 &= 2 \times 2 \times \_\_\_\_\_\_ \\ &= 2^{--} \end{aligned}$$

b $$\begin{aligned} \frac{x^5}{x^3} &= \frac{x \times x \times x \times \_\_\_\_\_\_\_\_\_\_}{\_\_\_\_\_\_\_\_\_} \\ &= x^{--} \end{aligned}$$

c $$\begin{aligned} \left(a^2\right)^3 &= a \times a\_\_\_\_\_\_ \times \_\_\_\_\_\_ \\ &= a^{--} \end{aligned}$$

d $$\begin{aligned} (2x)^0 \times 2x^0 &= \_\_\_\_ \times \_\_\_\_ \\ &= 2 \end{aligned}$$

Example 1 Using index laws for multiplication and division

Simplify the following using the index laws for multiplication and division.

a $x^5 \times x^4$

b $3a^2b \times 4ab^3$

c $m^7 \div m^5$

d $4x^2y^5 \div (8xy^2)$

SOLUTION	EXPLANATION
a $x^5 \times x^4 = x^9$	There is a common base of x, so add the indices.
b $3a^2b \times 4ab^3 = 12a^3b^4$	Multiply coefficients and add indices for each base a and b. Recall that $a = a^1$.
c $m^7 \div m^5 = m^2$	Subtract the indices when dividing terms with the same base.
d $4x^2y^5 \div (8xy^2) = \frac{4x^2y^5}{8xy^2} = \frac{xy^3}{2} = \frac{1}{2}xy^3$	First, express as a fraction. Divide the coefficients and subtract the indices of x and y (i.e. $x^{2-1}y^{5-2}$).

Now you try

Simplify the following using the index laws for multiplication and division.

a $x^3 \times x^4$

b $2ab^2 \times 7a^2b^3$

c $m^5 \div m^3$

d $5x^2y^4 \div (10xy^2)$

Example 2 Using indices with brackets

Simplify the following using the index laws.

a $(a^3)^4$ **b** $(2y^5)^3$ **c** $\left(\frac{a^2}{2b}\right)^3$

SOLUTION

a $(a^3)^4 = a^{12}$

b $\begin{aligned}(2y^5)^3 &= 2^3y^{15}\\ &= 8y^{15}\end{aligned}$

c $\begin{aligned}\left(\frac{a^2}{2b}\right)^3 &= \frac{a^6}{2^3b^3}\\ &= \frac{a^6}{8b^3}\end{aligned}$

EXPLANATION

Use the index law for power of a power by multiplying the indices.

Use the index law for brackets and multiply the indices for each base 2 and y.
Note: $2 = 2^1$.

Use the index law for fractions and apply index to both 2 and b in the denominator.

Now you try

Simplify the following using the index laws.

a $(a^2)^3$ **b** $(3y^3)^3$ **c** $\left(\frac{x^3}{5y}\right)^2$

Example 3 Using the zero index

Evaluate, using the zero index.

a $4a^0$ **b** $2p^0 + (3p)^0$

SOLUTION

a $\begin{aligned}4a^0 &= 4 \times 1\\ &= 4\end{aligned}$

b $\begin{aligned}2p^0 + (3p)^0 &= 2 \times 1 + 1\\ &= 3\end{aligned}$

EXPLANATION

Any number to the power of zero is equal to 1.

Note: $(3p)^0$ is not the same as $3p^0$.

Now you try

Evaluate, using the zero index.

a $2a^0$ **b** $5p^0 + (7p)^0$

Example 4 Combining index laws

Simplify the following using index laws.

a $3x^2y^3 \times 6xy^4 \div (2y^2)$

b $\dfrac{3(xy^2)^3 \times 4x^4y^2}{8x^2y}$

SOLUTION	EXPLANATION
a $3x^2y^3 \times 6xy^4 \div (2y^2)$	
$= 18x^3y^7 \div (2y^2)$	Multiply first using the index law for multiplication: $x^2 \times x^1 = x^3$, $y^3 \times y^4 = y^7$.
$= \dfrac{18x^3y^7}{2y^2}$	Divide the coefficients and subtract the indices of y.
$= 9x^3y^5$	
b $\dfrac{3(xy^2)^3 \times 4x^4y^2}{8x^2y} = \dfrac{3x^3y^6 \times 4x^4y^2}{8x^2y}$	Remove brackets first by multiplying the indices for each base.
$= \dfrac{12x^7y^8}{8x^2y}$	Simplify the numerator using the index law for multiplication.
$= \dfrac{3x^5y^7}{2}$	Simplify the fraction using the index law for division, subtracting indices of the same base.

Now you try

Simplify the following using index laws.

a $4a^3b \times 3a^2b^4 \div (6a^4)$

b $\dfrac{4(x^2y)^3 \times 2xy^2}{2x^2y}$

Exercise 3A

FLUENCY 1–4(½) 1–4(⅓) 1–4(¼)

Example 1a, b

1 Simplify, using the index law for multiplication.

a $a^5 \times a^4$

b $x^3 \times x^2$

c $b \times b^5$

d $7m^2 \times 2m^3$

e $2s^4 \times 3s^3$

f $t^8 \times 2t^8$

g $\frac{1}{5}p^2 \times p$

h $\frac{1}{4}c^4 \times \frac{2}{3}c^3$

i $\frac{3}{5}s \times \frac{3s}{5}$

j $2x^2y \times 3xy^2$

k $3a^2b \times 5ab^5$

l $3v^7w \times 6v^2w$

m $3x^4 \times 5xy^2 \times 10y^4$

n $2rs^3 \times 3r^4s \times 2r^2s^2$

o $4m^6n^7 \times mn \times 5mn^2$

Example 1c, d

2 Simplify, using the index law for division.

a $x^5 \div x^2$
b $a^7 \div a^6$
c $q^9 \div q^6$
d $b^5 \div b$
e $\dfrac{y^8}{y^3}$
f $\dfrac{d^8}{d^3}$
g $\dfrac{j^7}{j^6}$
h $\dfrac{m^{15}}{m^9}$
i $2x^2y^3 \div x$
j $3r^5s^2 \div (r^3s)$
k $6p^4q^2 \div (3q^2p^2)$
l $16m^7x^5 \div (8m^3x^4)$
m $\dfrac{5a^2b^4}{a^2b}$
n $\dfrac{8st^4}{2t^3}$
o $\dfrac{2v^5}{8v^3}$
p $\dfrac{7a^2b}{14ab}$
q $\dfrac{-3x^4y}{9x^3y}$
r $\dfrac{-8x^2y^3}{16x^2y}$

Example 2

3 Simplify using the laws involving brackets.

a $\left(x^5\right)^2$
b $\left(t^3\right)^2$
c $4\left(a^2\right)^3$
d $5\left(y^5\right)^3$
e $\left(4t^2\right)^3$
f $\left(2u^2\right)^2$
g $\left(3r^3\right)^3$
h $\left(3p^4\right)^4$
i $\left(\dfrac{a^2}{b^3}\right)^2$
j $\left(\dfrac{x^3}{y^4}\right)^3$
k $\left(\dfrac{x^2y^3}{z^4}\right)^2$
l $\left(\dfrac{u^4w^2}{v^2}\right)^4$
m $\left(\dfrac{3f^2}{5g}\right)^3$
n $\left(\dfrac{3a^2b}{2pq^3}\right)^2$
o $\left(\dfrac{at^3}{3g^4}\right)^3$
p $\left(\dfrac{4p^2q^3}{3r}\right)^4$

Example 3

4 Evaluate the following using the zero index.

a $8x^0$
b $3t^0$
c $(5z)^0$
d $\left(10ab^2\right)^0$
e $5\left(g^3h^3\right)^0$
f $8x^0 - 5$
g $4b^0 - 9$
h $7x^0 - 4(2y)^0$

PROBLEM–SOLVING 5(½), 6 5(½), 6 5(⅓), 6, 7

Example 4

5 Use appropriate index laws to simplify the following.

a $x^6 \times x^5 \div x^3$
b $x^2y \div (xy) \times xy^2$
c $x^4n^7 \times x^3n^2 \div (xn)$
d $\dfrac{x^2y^3 \times x^2y^4}{x^3y^5}$
e $\dfrac{m^2w \times m^3w^2}{m^4w^3}$
f $\dfrac{r^4s^7 \times r^4s^7}{r^4s^7}$
g $\dfrac{9x^2y^3 \times 6x^7y^5}{12xy^6}$
h $\dfrac{4x^2y^3 \times 12x^2y^2}{24x^4y}$
i $\dfrac{16a^8b \times 4ab^7}{32a^7b^6}$
j $\left(3m^2n^4\right)^3 \times mn^2$
k $-5\left(a^2b\right)^3 \times (3ab)^2$
l $\left(4f^2g\right)^2 \times f^2g^4 \div \left(3\left(fg^2\right)^3\right)$
m $\dfrac{4m^2n \times 3\left(m^2n\right)^3}{6m^2n}$
n $\dfrac{\left(7y^2z\right)^2 \times 3yz^2}{7(yz)^2}$
o $\dfrac{2(ab)^2 \times \left(2a^2b\right)^3}{4ab^2 \times 4a^7b^3}$
p $\dfrac{\left(2m^3\right)^2}{3\left(mn^4\right)^0} \times \dfrac{\left(6n^5\right)^2}{(-2n)^3m^4}$

6 Simplify.

a $(-3)^3$ **b** $-(3)^3$ **c** $(-3)^4$ **d** -3^4

7 Simplify.

a $\left(\left(x^2\right)^3\right)^2$ **b** $\left(\left(a^5\right)^3\right)^7$ **c** $\left(\left(\frac{a^2}{b}\right)^3\right)^5$

REASONING	8(½)	8(½), 9	8(⅓), 9, 10

8 Evaluate without the use of a calculator.

a $\frac{13^3}{13^2}$ **b** $\frac{18^7}{18^6}$ **c** $\frac{9^8}{9^6}$ **d** $\frac{4^{10}}{4^7}$

e $\frac{25^2}{5^4}$ **f** $\frac{36^2}{6^4}$ **g** $\frac{27^2}{3^4}$ **h** $\frac{32^2}{2^7}$

9 When Billy uses a calculator to raise -2 to the power 4 he gets -16 when the answer is actually 16. What has he done wrong?

10 Find the value of a in these equations in which the index is unknown.

a $2^a = 8$ **b** $3^a = 81$

c $2^{a+1} = 4$ **d** $(-3)^a = -27$

e $(-5)^a = 625$ **f** $(-4)^{a-1} = 1$

ENRICHMENT: Indices in equations	–	–	11–13(½)

11 If $x^4 = 3$, find the value of:

a x^8 **b** $x^4 - 1$ **c** $2x^{16}$ **d** $3x^4 - 3x^8$

12 Find the value(s) of x.

a $x^4 = 16$ **b** $2^{x-1} = 16$ **c** $2^{2x} = 16$ **d** $2^{2x-3} = 16$

13 Find the possible pairs of positive integers for x and y when:

a $x^y = 16$ **b** $x^y = 64$ **c** $x^y = 81$ **d** $x^y = 1$

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3B Negative indices

LEARNING INTENTIONS

- To understand how a negative index relates to division
- To know how to rewrite expressions involving negative indices with positive indices
- To be able to apply index laws to expressions involving negative indices

The study of indices can be extended to include negative powers. Using the index law for division and the fact that $a^0 = 1$, we can establish rules for negative powers.

$$\begin{aligned} a^0 \div a^n &= a^{0-n} \text{ (index law for division)} \\ &= a^{-n} \end{aligned} \quad \text{also} \quad \begin{aligned} a^0 \div a^n &= 1 \div a^n \; \left(\text{as } a^0 = 1\right) \\ &= \frac{1}{a^n} \end{aligned}$$

Therefore: $a^{-n} = \dfrac{1}{a^n}$

$$\begin{aligned} \text{Also: } \frac{1}{a^{-n}} &= 1 \div a^{-n} \\ &= 1 \div \frac{1}{a^n} \\ &= 1 \times \frac{a^n}{1} \\ &= a^n \end{aligned}$$

Therefore: $\dfrac{1}{a^{-n}} = a^n$.

A half-life is the time taken for radioactive material to halve in size. Calculations of the quantity remaining after multiple halving use negative powers of 2. Applications include radioactive waste management and diagnostic medicine.

Lesson starter: The disappearing bank balance

Due to fees, an initial bank balance of $64 is halved every month.

Balance ($)	64	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
Positive indices only	2^6	2^5							$\frac{1}{2^2}$	
Positive and negative indices	2^6		2^4					2^{-1}		

- Copy and complete the table and continue each pattern.
- Discuss the differences in the way indices are used at the end of the rows.
- What would be a way of writing $\frac{1}{16}$ using positive indices?
- What would be a way of writing $\frac{1}{16}$ using negative indices?

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KEY IDEAS

- $a^{-m} = \frac{1}{a^m}$ For example, $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$.
- $\frac{1}{a^{-m}} = a^m$ For example, $\frac{1}{2^{-3}} = 2^3 = 8$.
- Note: $a^{-1} = \frac{1}{a}$ (which is the reciprocal of a) and $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$

BUILDING UNDERSTANDING

1 State the next three terms in these patterns.

a $2^3, 2^2, 2^1, 2^0, 2^{-1}, \_, \_, \_$ **b** $x^2, x^1, x^0, \_, \_, \_$

2 Recall that $\frac{1}{4} = \frac{1}{2^2}$. Similarly, state these fractions using positive indices for the denominator.

a $\frac{1}{9}$ **b** $\frac{1}{25}$ **c** $\frac{5}{16}$ **d** $-\frac{2}{27}$

3 State each rule for negative indices by completing each statement.

a $a^{-b} =$ ____ **b** $\frac{1}{a^{-b}} =$ ____

4 Express the following with positive indices using $a^{-m} = \frac{1}{a^m}$ and evaluate.

a 5^{-2} **b** 3^{-3} **c** 4×7^{-2}

Example 5 Writing expressions using positive indices

Express each of the following using positive indices.

a b^{-4} **b** $3x^{-4}y^2$ **c** $\frac{5}{x^{-3}}$

SOLUTION	EXPLANATION
a $b^{-4} = \frac{1}{b^4}$	Use $a^{-n} = \frac{1}{a^n}$.
b $3x^{-4}y^2 = \frac{3y^2}{x^4}$	x is the only base with a negative power. $\frac{3}{1} \times \frac{1}{x^4} \times \frac{y^2}{1} = \frac{3y^2}{x^4}$.
c $\frac{5}{x^{-3}} = 5 \times x^3$ $= 5x^3$	Use $\frac{1}{a^{-n}} = a^n$ and note that $\frac{5}{x^{-3}} = 5 \times \frac{1}{x^{-3}}$.

Now you try

Express each of the following using positive indices.

a b^{-3} **b** $2x^{-2}y^3$ **c** $\frac{2}{x^{-4}}$

Example 6 Using index laws with negative indices

Simplify the following expressing answers using positive indices.

a $\frac{2a^3b^2}{a^5b^3}$

b $\frac{4m^{-2}n^3}{8m^5n^{-4}}$

SOLUTION	EXPLANATION
a $\frac{2a^3b^2}{a^5b^3} = 2a^{-2}b^{-1}$	Use the index law for division to subtract powers with a common base: a^{3-5} and b^{2-3}.
$= \frac{2}{a^2b}$	Express with positive powers $\frac{2}{1} \times \frac{1}{a^2} \times \frac{1}{b}$.
b $\frac{4m^{-2}n^3}{8m^5n^{-4}} = \frac{{}^{1}\cancel{4}m^{-7}n^7}{\cancel{8}^{2}}$	Cancel the common factor of 4 and subtract powers m^{-2-5} and $n^{3-(-4)}$.
$= \frac{n^7}{2m^7}$	Express with positive powers $\frac{1}{2} \times \frac{1}{m^7} \times \frac{n^7}{1}$.

Now you try

Simplify the following expressing answers using positive indices.

a $\frac{5a^2b^3}{a^4b^7}$

b $\frac{3m^{-3}n^6}{9m^4n^{-2}}$

Example 7 Simplifying more complex expressions

Simplify the following and express your answers using positive indices.

a $\left(x^3y^{-1}\right)^3 \times \left(x^{-2}y^3\right)^{-5}$

b $\frac{\left(p^{-2}q\right)^4}{5pq^{-6}} \div \left(\frac{p^{-2}}{q^3}\right)^{-3}$

SOLUTION	EXPLANATION
a $\left(x^3y^{-1}\right)^3 \times \left(x^{-2}y^3\right)^{-5}$ $= x^9y^{-3} \times x^{10}y^{-15}$	Remove brackets first by using index laws for brackets e.g. $\left(x^3y^{-1}\right)^3 = \left(x^3\right)^3\left(y^{-1}\right)^3 = x^9y^{-3}$
$= x^{19}y^{-18}$	Use the index law for multiplication to add powers with a common base: $x^{9+10} = x^{19}$, $y^{-3+(-15)} = y^{-18}$
$= \frac{x^{19}}{y^{18}}$	Use $a^{-n} = \frac{1}{a^n}$ to express y^{-18} as $\frac{1}{y^{18}}$.

b $\dfrac{(p^{-2}q)^4}{5pq^{-6}} \div \left(\dfrac{p^{-2}}{q^3}\right)^{-3} = \dfrac{p^{-8}q^4}{5pq^{-6}} \div \dfrac{p^6}{q^{-9}}$

$= \dfrac{p^{-8}q^4}{5pq^{-6}} \times \dfrac{q^{-9}}{p^6}$

$= \dfrac{p^{-8}q^{-5}}{5p^7q^{-6}}$

$= \dfrac{p^{-15}q}{5}$

$= \dfrac{q}{5p^{15}}$

Deal with brackets first by multiplying the power to each of the indices within the brackets.
To divide, multiply by the reciprocal of the fraction after the ÷ sign.
Use the index laws for multiplication and division to combine indices of like bases.
Simplify each numerator and denominator first: $q^{4+(-9)} = q^{-5}$.

Then $p^{-8-7}q^{-5-(-6)} = p^{-15}q$.

Use $a^{-n} = \dfrac{1}{a^n}$ to express p^{-15} with a positive index.

Now you try

Simplify the following and express your answers using positive indices.

a $(x^{-2}y^4)^{-2} \times (x^3y^{-1})^3$

b $\dfrac{(p^{-1}q)^3}{2p^{-2}q^2} \div \left(\dfrac{p^{-2}}{q^2}\right)^{-1}$

Exercise 3B

FLUENCY	1–4(½)	1–5(⅓)	1–5(¼)

Example 5a, b

1 Express the following using positive indices.

a x^{-5} **b** a^{-4} **c** $2m^{-4}$ **d** $3y^{-7}$

e $3a^2b^{-3}$ **f** $4m^3n^{-3}$ **g** $10x^{-2}y^5z$ **h** $3x^{-4}y^{-2}z^3$

i $\frac{1}{3}p^{-2}q^3r$ **j** $\frac{1}{5}d^2e^{-4}f^5$ **k** $\frac{3}{8}u^2v^{-6}w^7$ **l** $\frac{2}{5}b^3c^{-5}d^{-2}$

Example 5c

2 Express the following using positive indices.

a $\dfrac{1}{x^{-2}}$ **b** $\dfrac{2}{y^{-3}}$ **c** $\dfrac{4}{m^{-7}}$ **d** $\dfrac{3}{b^{-5}}$

e $\dfrac{2b^4}{d^{-3}}$ **f** $\dfrac{3m^2}{n^{-4}}$ **g** $\dfrac{4b^4}{3a^{-3}}$ **h** $\dfrac{5h^3}{2g^{-3}}$

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Example 6a

3 Use the index laws for multiplication and division to simplify the following. Write your answers using positive indices.

a $x^3 \times x^{-2}$
b $a^7 \times a^{-4}$
c $2b^5 \times b^{-9}$
d $3y^{-6} \times y^3$
e $x^2y^3 \times x^{-3}y^{-4}$
f $4a^{-6}y^4 \times a^3y^{-2}$
g $2a^{-3}b \times 3a^{-2}b^{-3}$
h $6a^4b^3 \times 3a^{-6}b$
i $\dfrac{a^4b^3}{a^2b^5}$
j $\dfrac{m^3n^2}{mn^3}$
k $\dfrac{3x^2y}{6xy^2}$
l $\dfrac{4m^3n^4}{7m^2n^7}$
m $a^3b^4 \div (a^2b^7)$
n $p^2q^3 \div (p^7q^2)$
o $\dfrac{p^2q^2r^4}{pq^4r^5}$
p $\dfrac{12r^4s^6}{9rs^{-8}}$

Example 6b

4 Express the following in simplest form with positive indices.

a $\dfrac{2x^{-2}}{3x^{-3}}$
b $\dfrac{7d^{-3}}{10d^{-5}}$
c $\dfrac{5s^{-2}}{3s}$
d $\dfrac{4f^{-5}}{3f^{-3}}$
e $\dfrac{f^3g^{-2}}{f^{-2}g^3}$
f $\dfrac{r^{-3}s^{-4}}{r^3s^{-2}}$
g $\dfrac{3w^{-2}x^3}{6w^{-3}x^{-2}}$
h $\dfrac{15c^3d}{12c^{-2}d^{-3}}$

5 Express the following with positive indices.

a $\left(\dfrac{2x^2}{x^3}\right)^4$
b $\left(\dfrac{m^3}{4m^5}\right)^3$
c $2\left(x^{-7}\right)^3$
d $4\left(d^{-2}\right)^3$
e $\left(3t^{-4}\right)^2$
f $5\left(x^2\right)^{-2}$
g $\left(3x^{-5}\right)^4$
h $-8\left(x^5\right)^{-3}$
i $\left(4y^{-2}\right)^{-2}$
j $\left(3h^{-3}\right)^{-4}$
k $7\left(j^{-2}\right)^{-4}$
l $2\left(t^{-3}\right)^{-2}$

PROBLEM–SOLVING	6(½)	6–7(½)	6–7(⅓), 8

Example 7

6 Simplify the following and express your answers with positive indices.

a $\left(a^3b^2\right)^3 \times \left(a^2b^4\right)^{-1}$
b $\left(2p^2\right)^4 \times \left(3p^2q\right)^{-2}$
c $2\left(x^2y^{-1}\right)^2 \times \left(3xy^4\right)^3$
d $\dfrac{2a^3b^2}{a^{-3}} \times \dfrac{2a^2b^5}{b^4}$
e $\dfrac{\left(3rs^2\right)^4}{r^{-3}s^4} \times \dfrac{\left(2r^2s\right)^2}{s^7}$
f $\dfrac{4\left(x^{-2}y^4\right)^2}{x^2y^{-3}} \times \dfrac{xy^4}{2x^{-2}y}$
g $\left(\dfrac{a^2b^3}{b^{-2}}\right)^2 \div \left(\dfrac{ab^4}{a^2}\right)^{-2}$
h $\left(\dfrac{m^4n^{-2}}{r^3}\right)^2 \div \left(\dfrac{m^{-3}n^2}{r^3}\right)^2$
i $\dfrac{3\left(x^2y^{-4}\right)^2}{2\left(xy^2\right)^2} \div \dfrac{(xy)^{-3}}{\left(3x^{-2}y^4\right)^2}$

7 Evaluate without the use of a calculator.

a 5^{-2}
b 4^{-3}
c 2×7^{-2}
d $5 \times (-3^{-4})$
e $3^{10} \times \left(3^2\right)^{-6}$
f $\left(4^2\right)^{-5} \times 4\left(4^{-3}\right)^{-3}$
g $\dfrac{2}{7^{-2}}$
h $\dfrac{-3}{4^{-2}}$
i $\left(\dfrac{2}{3}\right)^{-2}$
j $\left(\dfrac{-5}{4}\right)^{-3}$
k $\dfrac{\left(4^{-2}\right)^3}{4^{-4}}$
l $\dfrac{\left(10^{-4}\right)^{-2}}{\left(10^{-2}\right)^{-3}}$

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8 The width of a hair on a spider is approximately 3^{-5} cm. How many centimetres is this, correct to four decimal places?

REASONING	9	9, 10	10–12

9 A student simplifies $2x^{-2}$ and writes $2x^{-2} = \frac{1}{2x^2}$. Explain the error made.

10 **a** Simplify the following.

i $\left(\frac{2}{3}\right)^{-1}$ **ii** $\left(\frac{5}{7}\right)^{-1}$ **iii** $\left(\frac{2x}{y}\right)^{-1}$

b What is $\left(\frac{a}{b}\right)^{-1}$ when expressed in simplest form? Explain.

11 Evaluate the following by combining fractions.

a $2^{-1} + 3^{-1}$ **b** $3^{-2} + 6^{-1}$ **c** $\left(\frac{3}{4}\right)^{-1} - \left(\frac{1}{2}\right)^{0}$

d $\left(\frac{3}{2}\right)^{-1} - 5\left(2^{-2}\right)$ **e** $\left(\frac{4}{5}\right)^{-2} - \left(\frac{2^{-2}}{3}\right)$ **f** $\left(\frac{3}{2^{-2}}\right) - \left(\frac{2^{-1}}{3^{-2}}\right)^{-1}$

12 Prove that $\left(\frac{1}{2}\right)^{x} = 2^{-x}$ giving reasons.

ENRICHMENT: Simple equations with negative indices	–	–	13(½)

13 Find the value of x.

a $2^x = \frac{1}{4}$ **b** $2^x = \frac{1}{32}$ **c** $3^x = \frac{1}{27}$

d $\left(\frac{3}{4}\right)^{x} = \frac{4}{3}$ **e** $\left(\frac{2}{3}\right)^{x} = \frac{9}{4}$ **f** $\left(\frac{2}{5}\right)^{x} = \frac{125}{8}$

g $\frac{1}{2^x} = 8$ **h** $\frac{1}{3^x} = 81$ **i** $\frac{1}{2^x} = 1$

j $5^{x-2} = \frac{1}{25}$ **k** $3^{x-3} = \frac{1}{9}$ **l** $10^{x-5} = \frac{1}{1000}$

m $\left(\frac{3}{4}\right)^{2x+1} = \frac{64}{27}$ **n** $\left(\frac{2}{5}\right)^{3x-5} = \frac{25}{4}$ **o** $\left(\frac{7}{4}\right)^{1-x} = \frac{4}{7}$

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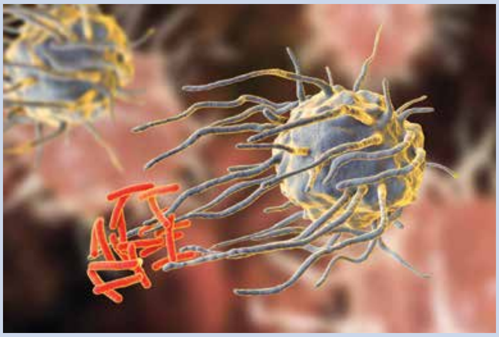
3C Scientific notation CONSOLIDATING

LEARNING INTENTIONS
- To understand that very large and very small numbers can be written in a shorthand form
- To know the general form of a number in scientific notation
- To be able to convert between scientific notation and basic numerals
- To know the meaning of the term significant figure
- To be able to round a number to a desired number of significant figures
- To know how to use technology in working with scientific notation

Scientific notation is useful when working with very large or very small numbers. Combined with the use of significant figures, numbers can be written down with an appropriate degree of accuracy and without the need to write all the zeros that define the position of the decimal place. The approximate distance between the Earth and the Sun is 150 million kilometres or 1.5×10^8 km when written in scientific notation using two significant figures. Negative indices can be used for very small numbers, such as $0.0000382 \text{ g} = 3.82 \times 10^{-5}$ g.



Everyday users of scientific notation include astronomers, space scientists, chemists, engineers, environmental scientists, physicists, biologists, lab technicians and medical researchers. This image shows white blood cells engulfing tuberculosis bacteria.

Lesson starter: Amazing facts large and small

Think of an object, place or living thing that is associated with a very large or small number.

- Give three examples using very large numbers.
- Give three examples using very small numbers.
- Can you remember how to write these numbers using scientific notation?
- How are significant figures used when writing numbers with scientific notation?

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KEY IDEAS

- A number written using **scientific notation** is of the form $a \times 10^m$, where $1 \leqslant a < 10$ or $-10 < a \leqslant -1$ and m is an integer.
  - Large numbers: $24\,800\,000 = 2.48 \times 10^7$
    $9\,020\,000\,000 = 9.02 \times 10^9$
  - Small numbers: $0.00307 = 3.07 \times 10^{-3}$
    $-0.0000012 = -1.2 \times 10^{-6}$
- **Significant figures** are counted from left to right, starting at the first non-zero digit.
  - When using scientific notation the digit to the left of the decimal point is the first significant figure.
    For example: $20\,190\,000 = 2.019 \times 10^7$ shows four significant figures.
  - The $\boxed{\times 10^n}$, $\boxed{\textbf{EE}}$ or $\boxed{\textbf{Exp}}$ keys can be used on calculators to enter numbers using scientific notation; e.g. 2.3E–4 means 2.3×10^{-4}.

BUILDING UNDERSTANDING

1 How many significant figures are showing in these numbers?
a 2.12×10^7 **b** 1.81×10^{-3} **c** 461 **d** 0.0000403

2 State these numbers as powers of 10.
a 1000 **b** 10 000 000 **c** 0.000001 **d** $\frac{1}{1000}$

3 Convert to numbers using scientific notation by stating the missing power.
a $43\,000 = 4.3 \times 10^{\square}$ **b** $712\,000 = 7.12 \times 10^{\square}$ **c** $9012 = 9.012 \times 10^{\square}$
d $0.00078 = 7.8 \times 10^{\square}$ **e** $0.00101 = 1.01 \times 10^{\square}$ **f** $0.00003 = 3 \times 10^{\square}$

Example 8 Converting from scientific notation to a basic numeral

Write these numbers as a basic numeral.
a 5.016×10^5 **b** 3.2×10^{-7}

SOLUTION	EXPLANATION
a $5.016 \times 10^5 = 501\,600$	Move the decimal point 5 places to the right.
b $3.2 \times 10^{-7} = 0.00000032$	Move the decimal point 7 places to the left.

Now you try

Write these numbers as a basic numeral.
a 2.048×10^4 **b** 4.7×10^{-5}

Example 9 Converting to scientific notation using significant figures

Write these numbers in scientific notation, using three significant figures.

a 5 218 300 **b** 0.0042031

SOLUTION

a $5\,218\,300 = 5.22 \times 10^6$ (to 3 significant figures)

b $0.0042031 = 4.20 \times 10^{-3}$ (to 3 significant figures)

EXPLANATION

Place the decimal point after the first non-zero digit. The digit following the third digit is at least 5, so round up.

Round down in this case, but retain the zero to show the value of the third significant figure.

Now you try

Write these numbers in scientific notation, using three significant figures.

a 7 937 200 **b** 0.00027103

Example 10 Using technology with scientific notation

Evaluate $\sqrt{2.61 \times 10^4} \div (3.2 \times 10^{-2})$, answering in scientific notation using three significant figures.

SOLUTION

$$\sqrt{2.61 \times 10^4} \div \left(3.2 \times 10^{-2}\right)$$
$$= 5048.59\ldots$$
$$= 5.05 \times 10^3 \text{ (to 3 significant figures)}$$

EXPLANATION

Enter the expression into a calculator. Scientific notation can be entered using $\boxed{\times 10^x}$, $\boxed{\text{EE}}$ or $\boxed{\text{Exp}}$ keys.

Express in scientific notation using three significant figures.

Now you try

Evaluate $\sqrt{4.62 \times 10^8} \times 5.24 \times 10^{-3}$, answering in scientific notation using three significant figures.

Exercise 3C

FLUENCY	1–4(½), 6(½)	1–4(½), 6(½)	1–6(⅓)

Example 8a

1 Write these numbers as a basic numeral.

a 3.12×10^{3}
b 5.4293×10^{4}
c 7.105×10^{5}
d 8.213×10^{6}
e 5.95×10^{4}
f -8.002×10^{5}
g -1.012×10^{4}
h 9.99×10^{6}
i 2.105×10^{8}
j -5.5×10^{4}
k 2.35×10^{9}
l 1.237×10^{12}

Example 8b

2 Write these numbers as a basic numeral.

a 4.5×10^{-3}
b 2.72×10^{-2}
c 3.085×10^{-4}
d 7.83×10^{-3}
e -9.2×10^{-5}
f 2.65×10^{-1}
g 1.002×10^{-4}
h -6.235×10^{-6}
i 9.8×10^{-1}
j -5.45×10^{-10}
k 3.285×10^{-12}
l 8.75×10^{-7}

Example 9a

3 Write these numbers in scientific notation, using three significant figures.

a 6241
b −572 644
c 30 248
d 423 578
e −10 089
f 34 971 863
g 72 477
h 356 088
i 110 438 523
j 909 325
k −4 555 678
l 9 826 100 005

Example 9b

4 Write these numbers in scientific notation, using three significant figures.

a 0.002423
b −0.018754
c 0.000125
d −0.0078663
e 0.0007082
f 0.11396
g 0.000006403
h 0.00007892
i 0.000129983
j 0.00000070084
k 0.000000009886
l −0.0004998

5 Write in scientific notation, using the number of significant figures given in the brackets.

a −23 900 (2)
b 5 707 159 (3)
c 703 780 030 (2)
d 4875 (3)
e 0.00192 (2)
f −0.00070507 (3)
g 0.000009782 (2)
h −0.35708 (4)
i 0.000050034 (3)

Example 10

6 Use a calculator to evaluate the following, giving the answers in scientific notation using three significant figures.

a $\left(2.83\times10^{2}\right)^{-3}$
b $5.1\div(8\times10^{2})$
c $(9.3\times10^{-2})\times(8.6\times10^{8})$
d $(3.27\times10^{4})\div(9\times10^{-5})$
e $\sqrt{3.23\times10^{-6}}$
f $\sqrt{5.73\times10^{-4}}\times4.28\times10^{3}$

PROBLEM–SOLVING	7	7(½), 8	7(⅓), 8

7 Write the following numerical facts using scientific notation.

a The area of Australia is about 7 700 000 km^{2}.
b The number of stones used to build the Pyramid of Khufu is about 2 500 000.
c The greatest distance of Pluto from the Sun is about 7 400 000 000 km.
d A human hair is about 0.01 cm wide.
e The mass of a neutron is about 0.000000000000000000000000001675 kg.
f The mass of a bacteria cell is about 0.00000000000095 g.

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8 The speed of light is approximately 3×10^5 km/s and the average distance between Pluto and the Sun is about 5.9×10^9 km. How long does it take for light from the Sun to reach Pluto? Answer correct to the nearest minute.

REASONING	9, 10(½)	9, 10–11(½)	10–11(⅓), 12

9 Explain why 38×10^7 is not written using scientific notation.

10 Write the following using scientific notation.

a 21×10^3 **b** 394×10^7 **c** 6004×10^{-2} **d** 179×10^{-6}

e 0.2×10^4 **f** 0.007×10^2 **g** 0.01×10^9 **h** 0.06×10^8

i 0.4×10^{-2} **j** 0.0031×10^{-11} **k** 210.3×10^{-6} **l** 9164×10^{-24}

11 Combine your knowledge of index laws with scientific notation to evaluate the following and express using scientific notation.

a $(3 \times 10^2)^2$ **b** $(2 \times 10^3)^3$ **c** $(8 \times 10^4)^2$

d $(12 \times 10^{-5})^2$ **e** $(5 \times 10^{-3})^{-2}$ **f** $(4 \times 10^5)^{-2}$

g $(1.5 \times 10^{-3})^2$ **h** $(8 \times 10^{-8})^{-1}$ **i** $(5 \times 10^{-2}) \times (2 \times 10^{-4})$

j $(3 \times 10^{-7}) \times (4.25 \times 10^2)$ **k** $(15 \times 10^8) \times (12 \times 10^{-11})$ **l** $(18 \times 10^5) \div (9 \times 10^3)$

m $(240 \times 10^{-4}) \div (3 \times 10^{-2})$ **n** $(2 \times 10^{-8}) \div (50 \times 10^4)$ **o** $(5 \times 10^2) \div (20 \times 10^{-3})$

12 Rewrite 3×10^{-4} with a positive index and use this to explain why, when expressing 3×10^{-4} as a basic numeral, the decimal point is moved four places to the left.

ENRICHMENT: $E = mc^2$	–	–	13

13 $E = mc^2$ is a formula derived by Albert Einstein (1879–1955). The formula relates the energy (E joules) of an object to its mass (m kg), where c is the speed of light (approximately 3×10^8 m/s).

Use $E = mc^2$ to answer these questions, using scientific notation.

a Find the energy, in joules, contained inside an object with these given masses.

i 10 kg **ii** 26 000 kg

iii 0.03 kg **iv** 0.00001 kg

b Find the mass, in kilograms, of an object that contains these given amounts of energy. Give your answer using three significant figures.

i 1×10^{25} J **ii** 3.8×10^{16} J **iii** 8.72×10^4 J **iv** 1.7×10^{-2} J

c The mass of Earth is about 6×10^{24} kg. How much energy does this convert to?

3D Fractional indices OPTIONAL

LEARNING INTENTIONS

- To understand how a rational index relates to the root of a number
- To know how to convert between bases with rational indices and surd form
- To be able to evaluate some numbers with rational indices without a calculator
- To be able to apply index laws to expressions involving rational indices

The square and cube roots of numbers, such as $\sqrt{81} = 9$ and $\sqrt[3]{64} = 4$, can be written using fractional powers.

The following shows that $\sqrt{9} = 9^{\frac{1}{2}}$ and $\sqrt[3]{8} = 8^{\frac{1}{3}}$.

Consider:

$$\begin{aligned} \sqrt{9} \times \sqrt{9} &= 3 \times 3 \\ &= 9 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 9^{\frac{1}{2}} \times 9^{\frac{1}{2}} &= 9^{\frac{1}{2}+\frac{1}{2}} \\ &= 9 \end{aligned}$$

$$\therefore \sqrt{9} = 9^{\frac{1}{2}}$$

Also:

$$\begin{aligned} \sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8} &= 2 \times 2 \times 2 \\ &= 8 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} &= 8^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} \\ &= 8 \end{aligned}$$

$$\therefore \sqrt[3]{8} = 8^{\frac{1}{3}}$$

A rational index is an index that can be expressed as a fraction.

Fractional indices are used in finance, electrical engineering, architecture, carpentry and for solving packing problems. Volume to the power of one-third (i.e. the cube root) finds a cube's side length and helps find a sphere's radius.

Lesson starter: Making the connection

For each part below use your knowledge of index laws and basic surds to simplify the numbers. Then discuss the connection that can be made between numbers that have a $\sqrt{}$ sign and numbers that have fractional powers.

- $\sqrt{5} \times \sqrt{5}$ and $5^{\frac{1}{2}} \times 5^{\frac{1}{2}}$
- $\sqrt[3]{27} \times \sqrt[3]{27} \times \sqrt[3]{27}$ and $27^{\frac{1}{3}} \times 27^{\frac{1}{3}} \times 27^{\frac{1}{3}}$
- $(\sqrt{5})^2$ and $\left(5^{\frac{1}{2}}\right)^2$
- $(\sqrt[3]{64})^3$ and $\left(64^{\frac{1}{3}}\right)^3$

KEY IDEAS

- $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $\sqrt[n]{a}$ is the nth root of a.

  For example: $3^{\frac{1}{2}} = \sqrt{3}$, $5^{\frac{1}{3}} = \sqrt[3]{5}$, $7^{\frac{1}{10}} = \sqrt[10]{7}$

- $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m$ or $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$

  For example: $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2$ or $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}}$

$$
\begin{aligned}
&= (\sqrt[3]{8})^2 \qquad && = (64)^{\frac{1}{3}} \\
&= 2^2 && = \sqrt[3]{64} \\
&= 4 && = 4
\end{aligned}
$$

- In most cases, the index laws apply to **rational indices** (i.e. fractional indices) just as they do for indices that are integers.

BUILDING UNDERSTANDING

① State the missing components for each statement.

a $2^{\square} = 8$ and $\sqrt[3]{8} = \_\_\_$

b $2^{\square} = 32$ and $\sqrt[\square]{32} = 2$

c $3^{\square} = 81$ and $\sqrt[\square]{81} = 3$

d $10^{\square} = 100\,000$ and $\sqrt[\square]{100\,000} = 10$

② Evaluate:

a $\sqrt{9}$

b $\sqrt{121}$

c $\sqrt[3]{27}$

d $\sqrt[3]{64}$

e $\sqrt[4]{16}$

f $\sqrt[4]{81}$

g $\sqrt[5]{32}$

h $\sqrt[5]{100\,000}$

③ Using a calculator, enter and evaluate each pair of numbers in their given form. Round your answer to two decimal places.

a $\sqrt[3]{7},\ 7^{\frac{1}{3}}$

b $\sqrt[5]{10},\ 10^{\frac{1}{5}}$

c $\sqrt[13]{100},\ 100^{\frac{1}{13}}$

Example 11 Writing roots in index form

Express the following in index form.

a $\sqrt{15}$

b $\sqrt{7x^5}$

c $3\sqrt[4]{x^7}$

d $10\sqrt{10}$

SOLUTION	EXPLANATION
a $\sqrt{15} = 15^{\frac{1}{2}}$	$\sqrt{}$ means to take the square root of the number beneath. Note: $\sqrt[n]{a} = a^{\frac{1}{n}}$.

b $\sqrt{7x^5} = (7x^5)^{\frac{1}{2}}$

$= 7^{\frac{1}{2}}x^{\frac{5}{2}}$

Rewrite $\sqrt{\ }$ as power $\frac{1}{2}$, then apply index laws to simplify: $5 \times \frac{1}{2} = \frac{5}{2}$.

c $3\sqrt[4]{x^7} = 3(x^7)^{\frac{1}{4}}$

$= 3x^{\frac{7}{4}}$

$\sqrt[4]{\ }$ means to the power of $\frac{1}{4}$.
Apply the index law for power of a power to multiply indices.

d $10\sqrt{10} = 10 \times 10^{\frac{1}{2}}$

$= 10^{\frac{3}{2}}$

Rewrite the square root as power $\frac{1}{2}$ and then add indices for the common base 10. Recall $10 = 10^1$, so $1 + \frac{1}{2} = \frac{3}{2}$.

An alternative answer is $\sqrt{100} \times \sqrt{10} = 1000^{\frac{1}{2}}$.

Now you try

Express the following in index form.

a $\sqrt{11}$

b $\sqrt{3x^7}$

c $2\sqrt[4]{x^9}$

d $7\sqrt{7}$

Example 12 Writing fractional indices in surd form

Express the following in surd form.

a $3^{\frac{1}{5}}$

b $5^{\frac{2}{3}}$

SOLUTION

a $3^{\frac{1}{5}} = \sqrt[5]{3}$

b $5^{\frac{2}{3}} = (5^2)^{\frac{1}{3}}$

$= \sqrt[3]{25}$

Alternatively:

$5^{\frac{2}{3}} = \left(5^{\frac{1}{3}}\right)^2$

$= (\sqrt[3]{5})^2$

EXPLANATION

$a^{\frac{1}{n}} = \sqrt[n]{a}$

Rewrite the fraction $\frac{2}{3}$ as $2 \times \frac{1}{3}$.

$25^{\frac{1}{3}} = \sqrt[3]{25}$

$2 \times \frac{1}{3}$ is the same as $\frac{1}{3} \times 2$.

$5^{\frac{1}{3}} = \sqrt[3]{5}$

Now you try

Express the following in surd form.

a $5^{\frac{1}{3}}$

b $11^{\frac{2}{3}}$

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Example 13 Evaluating numbers with fractional indices

Evaluate the following without a calculator.

a $16^{\frac{1}{2}}$ **b** $27^{-\frac{1}{3}}$ **c** $16^{\frac{3}{4}}$

SOLUTION	EXPLANATION
a $16^{\frac{1}{2}} = \sqrt{16}$ $= 4$	$16^{\frac{1}{2}}$ means $\sqrt{16}$.
b $27^{-\frac{1}{3}} = \dfrac{1}{27^{\frac{1}{3}}}$ $= \dfrac{1}{\sqrt[3]{27}}$ $= \dfrac{1}{3}$	Rewrite, using positive indices. Recall that $a^{-m} = \frac{1}{a^m}$. $27^{\frac{1}{3}}$ means $\sqrt[3]{27}$ and $3^3 = 27$.
c $16^{\frac{3}{4}} = \left(16^{\frac{1}{4}}\right)^3$ $= \left(\sqrt[4]{16}\right)^3$ $= 2^3$ $= 8$	Rewrite $\frac{3}{4}$ as $\frac{1}{4} \times 3$. It is simpler to find the root first as opposed to $(16^3)^{\frac{1}{4}}$. $\sqrt[4]{16} = 2$ since $2^4 = 16$. Then cube 2.

Now you try

Evaluate the following without a calculator.

a $25^{\frac{1}{2}}$ **b** $16^{-\frac{1}{4}}$ **c** $64^{\frac{2}{3}}$

Exercise 3D

FLUENCY	1–4(½)	1–4(½)	1–4(⅓)

Example 11a, b

1 Express the following in index form.

a $\sqrt{29}$ **b** $\sqrt[3]{35}$ **c** $\sqrt[5]{x^2}$ **d** $\sqrt[4]{b^3}$

e $\sqrt{2a}$ **f** $\sqrt[3]{4t^7}$ **g** $\sqrt[5]{10t^2}$ **h** $\sqrt[8]{8m^4}$

Example 11c, d

2 Express the following in index form.

a $7\sqrt{x^5}$ **b** $6\sqrt[3]{n^7}$ **c** $3\sqrt[4]{y^{12}}$

d $5\sqrt[3]{p^2r}$ **e** $2\sqrt[3]{a^4b^2}$ **f** $2\sqrt[4]{g^3h^5}$

g $5\sqrt{5}$ **h** $7\sqrt{7}$ **i** $4\sqrt[3]{4}$

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Example 12

3 Express the following in surd form.

a $2^{\frac{1}{5}}$ **b** $8^{\frac{1}{7}}$ **c** $6^{\frac{1}{3}}$ **d** $11^{\frac{1}{10}}$

e $3^{\frac{2}{3}}$ **f** $7^{\frac{2}{3}}$ **g** $2^{\frac{3}{5}}$ **h** $3^{\frac{4}{7}}$

Example 13a, b

4 Evaluate without using a calculator.

a $36^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** $64^{\frac{1}{3}}$

d $49^{\frac{1}{2}}$ **e** $16^{\frac{1}{4}}$ **f** $125^{\frac{1}{3}}$

g $9^{-\frac{1}{2}}$ **h** $32^{-\frac{1}{5}}$ **i** $81^{-\frac{1}{4}}$

j $1000^{-\frac{1}{3}}$ **k** $400^{-\frac{1}{2}}$ **l** $10\,000^{-\frac{1}{4}}$

PROBLEM–SOLVING 5(½) 5–6(½) 5–7(⅓)

Example 13c

5 Evaluate without using a calculator.

a $8^{\frac{2}{3}}$ **b** $32^{\frac{3}{5}}$ **c** $36^{\frac{3}{2}}$

d $16^{\frac{5}{4}}$ **e** $16^{-\frac{3}{4}}$ **f** $27^{-\frac{2}{3}}$

g $64^{-\frac{2}{3}}$ **h** $25^{-\frac{3}{2}}$ **i** $\dfrac{1}{25^{-\frac{3}{2}}}$

j $\dfrac{2}{4^{\frac{5}{2}}}$ **k** $\dfrac{3}{9^{\frac{5}{2}}}$ **l** $\dfrac{10}{100^{\frac{3}{2}}}$

6 Use index laws to simplify the following.

a $a^{\frac{1}{2}} \times a^{\frac{3}{2}}$ **b** $m^{\frac{3}{2}} \times m^{\frac{3}{2}}$ **c** $x^{\frac{7}{3}} \div x^{\frac{4}{3}}$ **d** $b^{\frac{5}{4}} \div b^{\frac{3}{4}}$

e $\left(s^{\frac{3}{2}}\right)^{\frac{4}{7}}$ **f** $\left(y^{\frac{1}{3}}\right)^{\frac{1}{3}}$ **g** $\left(\dfrac{2}{t^{11}}\right)^{0}$ **h** $\left(\dfrac{a^{\frac{2}{3}}}{b^{\frac{4}{3}}}\right)^{\frac{3}{4}}$

7 Simplify the following.

a $\sqrt{25s^4}$ **b** $\sqrt[3]{27t^6}$ **c** $\sqrt[4]{16t^8}$ **d** $\sqrt[3]{125t^{12}}$

e $(x^3)^{\frac{1}{3}}$ **f** $(b^{12})^{\frac{1}{3}}$ **g** $\left(t^{\frac{1}{4}}\right)^{12}$ **h** $\left(m^{\frac{1}{5}}\right)^{10}$

i $(16a^2b^8)^{\frac{1}{2}}$ **j** $(216m^6n^3)^{\frac{1}{3}}$ **k** $(32x^{10}y^{15})^{\frac{1}{5}}$ **l** $(343r^9t^6)^{\frac{1}{3}}$

m $\sqrt{\dfrac{25}{49}}$ **n** $\sqrt[3]{\dfrac{8x^3}{27}}$ **o** $\left(\dfrac{32}{x^{10}}\right)^{\frac{1}{5}}$ **p** $\left(\dfrac{10^2x^4}{0.01}\right)^{\frac{1}{4}}$

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REASONING	8	8	8, 9

8 As shown below, $16^{\frac{5}{4}}$ can be evaluated in two ways.

Method A

$$\begin{aligned} 16^{\frac{5}{4}} &= \left(16^5\right)^{\frac{1}{4}} \\ &= (1\,048\,576)^{\frac{1}{4}} \\ &= \sqrt[4]{1\,048\,576} \\ &= 32 \end{aligned}$$

Method B

$$\begin{aligned} 16^{\frac{5}{4}} &= \left(16^{\frac{1}{4}}\right)^5 \\ &= \left(\sqrt[4]{16}\right)^5 \\ &= 2^5 \\ &= 32 \end{aligned}$$

a If $16^{\frac{5}{4}}$ is to be evaluated without a calculator, which method above would be preferable?

b Use your preferred method to evaluate the following without a calculator.

i $8^{\frac{5}{3}}$ **ii** $36^{\frac{3}{2}}$ **iii** $16^{\frac{7}{4}}$ **iv** $27^{\frac{4}{3}}$

v $125^{\frac{4}{3}}$ **vi** $\left(\frac{1}{9}\right)^{\frac{3}{2}}$ **vii** $\left(\frac{4}{25}\right)^{\frac{5}{2}}$ **viii** $\left(\frac{27}{1000}\right)^{\frac{4}{3}}$

9 Explain why $\sqrt[6]{64}$ is not a surd.

ENRICHMENT: Does it exist?	–	–	10

10 We know that when $y = \sqrt{x}$, where $x < 0$, y is not a real number. This is because the square of y cannot be negative; i.e. $y^2 \neq x$ since y^2 is positive and x is negative.

But we know that $(-2)^3 = -8$ so $\sqrt[3]{-8} = -2$.

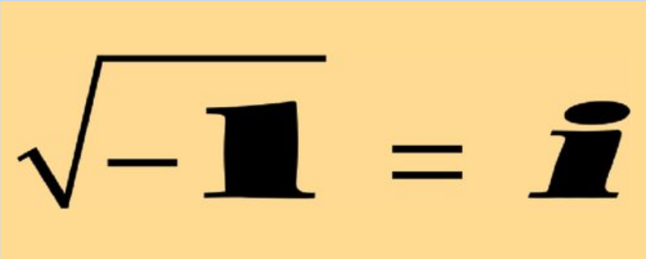
a Evaluate:

i $\sqrt[3]{-27}$ **ii** $\sqrt[3]{-1000}$ **iii** $\sqrt[5]{-32}$ **iv** $\sqrt[7]{-2187}$

b Decide if these are real numbers.

i $\sqrt{-5}$ **ii** $\sqrt[3]{-7}$ **iii** $\sqrt[5]{-16}$ **iv** $\sqrt[4]{-12}$

c If $y = \sqrt[n]{x}$ and $x < 0$, for what values of n is y a real number?



The square root of a negative value is not a real number. $\sqrt{-1} = i$ and is found in a special set of numbers called complex numbers.

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3E Exponential equations OPTIONAL

LEARNING INTENTIONS
- To know the form of an exponential equation
- To be able to rewrite an expression using its lowest base
- To be able to solve simple exponential equations using a common base

Equations can take many forms. For example, $2x - 1 = 5$ and $5(a - 3) = -3(3a + 7)$ are both linear equations; $x^2 = 9$ and $3x^2 - 4x - 9 = 0$ are quadratic equations; and $2^x = 8$ and $3^{2x} - 3^x - 6 = 0$ are exponential equations. Exponential equations contain a pronumeral within the index or indices of the terms in the equation. To solve for the unknown in exponential equations we use our knowledge of indices and surds and try to equate powers where possible.

Solving exponential equations can predict the timing of future outcomes. When will my new apartment double in value? When will Australia's population reach 30 million? How long before my coffee goes cold?

Lesson starter: 2 to the power of what number is 5?

We know that 2 to the power of 2 is 4 and 2 to the power of 3 is 8, but 2 to the power of what number is 5? That is, what is x when $2^x = 5$?

- Use a calculator and trial and error to estimate the value of x when $2^x = 5$ by completing this table.

x	2	3	2.5	2.1	
2^x	4	8	5.65 …		
Result	too small	too big	too big		

- Continue trying values until you find the answer, correct to three decimal places.

KEY IDEAS

- A simple **exponential equation** is of the form $a^x = b$, where $a > 0$, $b > 0$ and $a \neq 1$.
  - There is only one solution to exponential equations of this form.
- Many exponential equations can be solved by expressing both sides of the equation using the same base.
  - We use this fact: if $a^x = a^y$ then $x = y$.

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BUILDING UNDERSTANDING

1 **a** Evaluate the following.

- **i** 2^2
- **ii** 2^3
- **iii** 2^4
- **iv** 2^5

b Hence, state the value of x when:

- **i** $2^x = 8$
- **ii** $2^x = 32$
- **iii** $2^x = 64$

2 Complete these patterns, which involve powers.

- **a** 2, 4, 8, ___, ___, ___, ___, ___, ___
- **b** 3, 9, 27, ___, ___, ___, ___, ___
- **c** 4, 16, ___, ___, ___, ___
- **d** 5, 25, ___, ___, ___
- **e** 6, 36, ___, ___

3 State these numbers in index form. For example, $32 = 2^5$.

- **a** 9
- **b** 125
- **c** 243
- **d** 128
- **e** 729

Example 14 Solving exponential equations

Solve for x in each of the following.

a $2^x = 16$ **b** $3^x = \frac{1}{9}$ **c** $25^x = 125$

SOLUTION	EXPLANATION
a $2^x = 16$ $2^x = 2^4$ $\therefore x = 4$	Rewrite 16 as a power, using the base 2. Equate powers using the result: if $a^x = a^y$ then $x = y$.
b $3^x = \frac{1}{9}$ $3^x = \frac{1}{3^2}$	Rewrite 9 as a power of 3, then write using a negative index. Recall $\frac{1}{a^m} = a^{-m}$.
$3^x = 3^{-2}$ $\therefore x = -2$	Equate powers with the same base.
c $25^x = 125$ $(5^2)^x = 5^3$ $5^{2x} = 5^3$ $\therefore 2x = 3$ $x = \frac{3}{2}$	Since 25 and 125 are both powers of 5, rewrite both with a base of 5. Apply the index law to remove brackets by multiplying indices, then equate powers and solve for x.

Now you try

Solve for x in each of the following.

a $3^x = 27$ **b** $2^x = \frac{1}{8}$ **c** $16^x = 64$

Example 15 Solving exponential equations with a variable on both sides

Solve $3^{2x-1} = 27^x$.

SOLUTION

$$\begin{aligned} 3^{2x-1} &= 27^x \\ 3^{2x-1} &= (3^3)^x \\ 3^{2x-1} &= 3^{3x} \\ \therefore 2x - 1 &= 3x \\ -1 &= x \\ \therefore x &= -1 \end{aligned}$$

EXPLANATION

Rewrite 27 as a power of 3.

Apply the index law to remove brackets and then equate powers.

Subtract $2x$ from both sides and answer with x as the subject.

Now you try

Solve $5^{3x-1} = 25^x$.

Exercise 3E

FLUENCY 1–3(½) 1–3(½) 1–3(⅓)

Example 14a

1 Solve for x in each of the following.

a $3^x = 27$
b $2^x = 8$
c $6^x = 36$
d $9^x = 81$
e $5^x = 125$
f $4^x = 64$
g $3^x = 81$
h $6^x = 216$
i $5^x = 625$
j $2^x = 32$
k $10^x = 10\,000$
l $7^x = 343$

Example 14b

2 Solve for x in each of the following.

a $7^x = \frac{1}{49}$
b $9^x = \frac{1}{81}$
c $11^x = \frac{1}{121}$
d $4^x = \frac{1}{256}$
e $3^x = \frac{1}{243}$
f $5^{-x} = \frac{1}{125}$
g $3^{-x} = \frac{1}{9}$
h $2^{-x} = \frac{1}{64}$
i $7^{-x} = \frac{1}{343}$

Example 14c

3 Solve for x in each of the following.

a $9^x = 27$
b $8^x = 16$
c $25^x = 125$
d $16^x = 64$
e $81^x = 9$
f $216^x = 6$
g $32^x = 2$
h $10\,000^x = 10$
i $7^{-x} = 49$
j $4^{-x} = 256$
k $16^{-x} = 64$
l $25^{-x} = 125$

PROBLEM–SOLVING

4	4, 5(½)	5(⅓), 6

4 The population of bacteria in a dish is given by the rule $P = 2^t$, where P is the bacteria population and t is the time in minutes.

a What is the initial population of bacteria; i.e. when $t = 0$?

b What is the population of bacteria after:

i 1 minute? **ii** 5 minutes? **iii** 1 hour? **iv** 1 day?

c How long does it take for the population to reach:

i 8? **ii** 256? **iii** more than 1000?

Example 15

5 Solve for x in each of the following.

a $2^{x+1} = 8^x$ **b** $3^{2x+1} = 27^x$ **c** $7^{x+9} = 49^{2x}$

d $5^{x+3} = 25^{2x}$ **e** $6^{2x+3} = 216^{2x}$ **f** $9^{x+12} = 81^{x+5}$

g $27^{x+3} = 9^{2x}$ **h** $25^{x+3} = 125^{3x}$ **i** $32^{2x+3} = 128^{2x}$

j $27^{2x+3} = 9^{2x-1}$ **k** $9^{x-1} = 27^{2x-6}$ **l** $49^{2x-3} = 343^{2x-1}$

6 Would you prefer \$1 million now or 1 cent doubled every second for 30 seconds? Give reasons for your preference.

REASONING

7	7, 8(½)	8(½), 9, 10(½)

7 Consider a^x, where $a = 1$.

a Evaluate 1^x when:

i $x = 1$ **ii** $x = 3$ **iii** $x = 10\,000\,000$

b Are there any solutions to the equation $a^x = 2$ when $a = 1$? Give a reason.

8 Recall that $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x} = x^{\frac{1}{3}}$. Now solve the following.

a $3^x = \sqrt{81}$ **b** $5^x = \sqrt{25}$ **c** $6^x = \sqrt[3]{36}$ **d** $4^x = \sqrt[4]{64}$

e $2^x = \sqrt[4]{32}$ **f** $3^x = \sqrt[9]{27}$ **g** $25^x = \sqrt[5]{125}$ **h** $9^x = \dfrac{1}{\sqrt[3]{27}}$

9 a Write these numbers as decimals.

i $\dfrac{1}{2^2}$ **ii** 2^{-3} **iii** 10^{-3} **iv** $\left(\dfrac{1}{5}\right)^4$

b Write these decimal numbers as powers of prime numbers.

i 0.04 **ii** 0.0625 **iii** 0.5 **iv** 0.0016

10 Show how you can use techniques from this section to solve these equations involving decimals.

a $10^x = 0.0001$ **b** $2^x = 0.015625$ **c** $5^x = 0.00032$

d $(0.25)^x = 0.5$ **e** $(0.04)^x = 125$ **f** $(0.0625)^{x+1} = \dfrac{1}{2}$

ENRICHMENT: Mixing index laws with equations

–	–	11(½)

11 Solve for n in the following.

a $3^n \times 9^n = 27$ **b** $5^{3n} \times 25^{-2n+1} = 125$ **c** $2^{-3n} \times 4^{2n-2} = 16$

d $3^{2n-1} = \dfrac{1}{81}$ **e** $7^{2n+3} = \dfrac{1}{49}$ **f** $5^{3n+2} = \dfrac{1}{625}$

g $6^{2n-6} = 1$ **h** $11^{3n-1} = 11$ **i** $8^{5n-1} = 1$

j $\dfrac{3^{n-2}}{9^{1-n}} = 9$ **k** $\dfrac{5^{3n-3}}{25^{n-3}} = 125$ **l** $\dfrac{36^{3+2n}}{6^n} = 1$

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3F Exponential relations and their graphs

LEARNING INTENTIONS

- To know what defines an exponential relation
- To know the meaning of the term asymptote
- To know the basic features of an exponential graph
- To be able to sketch simple exponential graphs including those involving reflections
- To know how to find the point of intersection of an exponential graph and a horizontal line

We saw earlier that indices can be used to describe some special relations. The population of the world, for example, or the balance of an investment account can be described using exponential rules that include indices. The rule $A = 100\,000\,(1.05)^t$ describes the account balance of \$100 000 invested at 5% p.a. compound interest for t years.

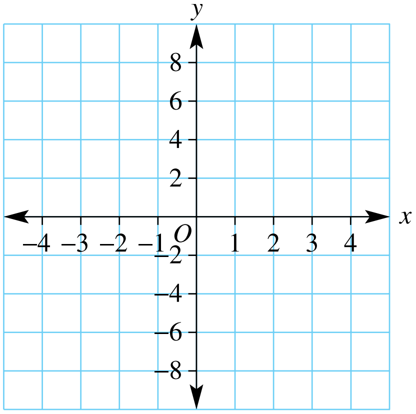
When a patient receives medication, the blood concentration decays exponentially as the body breaks it down. Exponential rules can determine the safe time between doses, from the highest safe level to the lowest effective level.

Lesson starter: What do $y = 2^x$, $y = -2^x$ and $y = 2^{-x}$ all have in common?

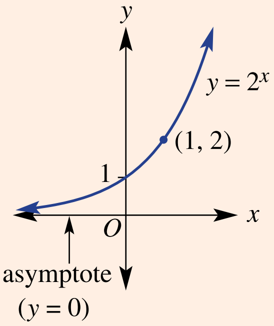
Complete this table and graph all three relations on the same set of axes before discussing the points below.

x	−3	−2	−1	0	1	2	3
$y_1 = 2^x$	$\frac{1}{8}$			1		4	
$y_2 = -2^x$							
$y_3 = 2^{-x}$							

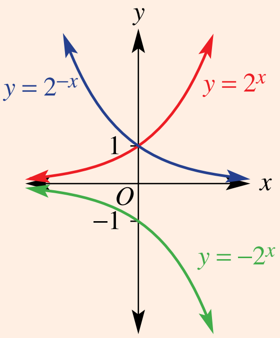
- Discuss the shape of each graph.
- Where does each graph cut the y-axis?
- Do the graphs have x-intercepts? Why not?
- What is the one feature they all have in common?



KEY IDEAS



- $y = 2^x$, $y = (0.4)^x$, $y = 3 \times (1.1)^x$ are examples of **exponential relations**.
- An **asymptote** is a line that a curve approaches, by getting closer and closer to it, but never reaching.
- A simple **exponential** rule is of the form $y = a^x$, where $a > 0$ and $a \neq 1$.
  - y-intercept has coordinates $(0, 1)$.
  - $y = 0$ is the equation of the asymptote.
- The graph of $y = -a^x$ is the reflection of the graph of $y = a^x$ in the x-axis. (Note: $y = -a^x$ means $y = -1 \times a^x$.)
- The graph of $y = a^{-x}$ is the reflection of the graph of $y = a^x$ in the y-axis.



- To find the intersection points of a simple exponential and a horizontal line draw accurate graphs and read off the coordinates of the point of intersection.
  - Alternatively, use the method of substitution and equate powers after expressing both sides of the equation using the same base. For example, for $y = 2^x$ and $y = 16$, solve

$$\begin{aligned} 2^x &= 16 \\ 2^x &= 2^4 \\ \therefore x &= 4 \end{aligned}$$

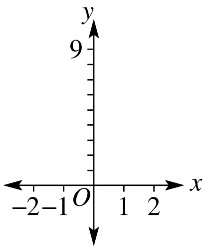
BUILDING UNDERSTANDING

① Consider the exponential rule $y = 3^x$.

a Complete this table.

x	−2	−1	0	1	2
y		$\frac{1}{3}$	1		

b Plot the points in the table to form the graph of $y = 3^x$.



② Complete the following.

a Graphs of the form $y = a^x$, $a > 0$ have an ______________ with equation $y = 0$ (the x-axis).

b The y-intercept of the graph $y = a^x$, $a > 0$ has coordinates ____________.

c The graph of $y = 4^{-x}$ is a reflection of the graph of $y = 4^x$ in the ______________.

d The graph of $y = -5^x$ is a reflection of the graph of $y = 5^x$ in the ______________.

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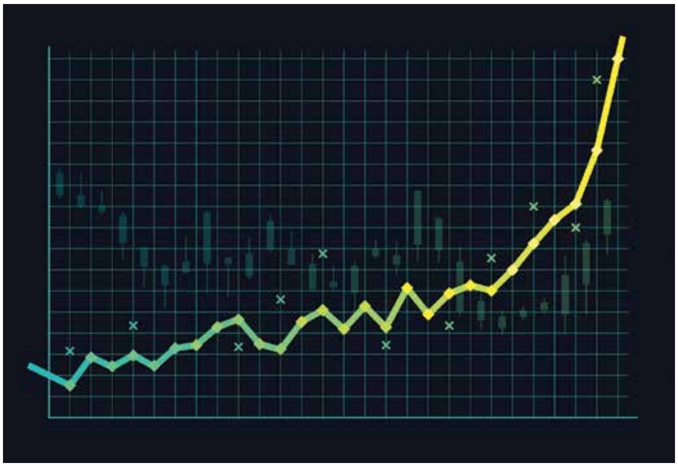
③ **a** Explain the difference between a^{-2} and $-a^2$.

b True or false: $-3^2 = \frac{1}{3^2}$? Explain why.

c Express with negative indices: $\frac{1}{5^3}, \frac{1}{3^2}, \frac{1}{2}$.

d Simplify: -3^2, -5^3, -2^{-2}.

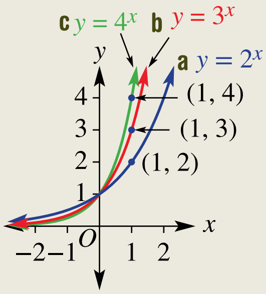


Example 16 Sketching graphs of exponentials

Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^x$ **b** $y = 3^x$ **c** $y = 4^x$

SOLUTION



EXPLANATION

$a^0 = 1$, so all y-intercepts are at (0, 1).

$y = 4^x$ is steeper than $y = 3^x$, which is steeper than $y = 2^x$.

Substitute $x = 1$ into each rule to obtain a second point to indicate the steepness of each curve.

Now you try

Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

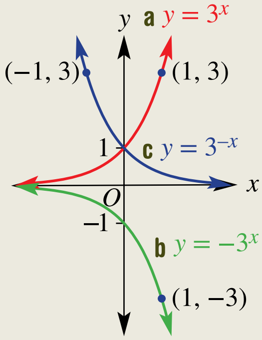
a $y = 2^x$ **b** $y = 5^x$

Example 17 Sketching with reflections

Sketch the graphs of these exponentials on the same set of axes.

a $y = 3^x$ **b** $y = -3^x$ **c** $y = 3^{-x}$

SOLUTION



EXPLANATION

The graph of $y = -3^x$ is a reflection of the graph of $y = 3^x$ in the x-axis.

Check: $x = 1, y = -3^1 = -3$

The graph of $y = 3^{-x}$ is a reflection of the graph of $y = 3^x$ in the y-axis.

Check: $x = 1,\ y = 3^{-1} = \frac{1}{3}$

$x = -1,\ y = 3^1 = 3$

Now you try

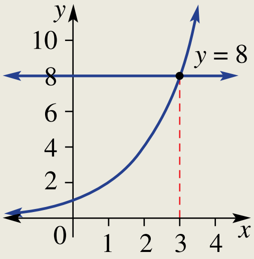
Sketch the graphs of these exponentials on the same set of axes.

a $y = 2^x$ **b** $y = -2^x$ **c** $y = 2^{-x}$

Example 18 Solving exponential equations graphically

Find the intersection of the graphs of $y = 2^x$ and $y = 8$, by sketching accurately.

SOLUTION



Intersection point is (3, 8).

Alternatively,

if $2^x = 8$ then

$x = 3$.

EXPLANATION

Sketch the graphs of $y = 2^x$ and $y = 8$.

Read off the coordinates of the point of intersection.

Consider powers of 2.

Now you try

Find the intersection of the graphs of $y = 3^x$ and $y = 27$, by sketching accurately.

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Exercise 3F

FLUENCY	1–4	1, 3, 4	1, 3, 4

Example 16

1 Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^x$ **b** $y = 4^x$ **c** $y = 5^x$

Example 17

2 Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^x$ **b** $y = -2^x$ **c** $y = 2^{-x}$

Example 17b

3 Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = -2^x$ **b** $y = -5^x$ **c** $y = -3^x$

Example 17c

4 Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^{-x}$ **b** $y = 3^{-x}$ **c** $y = 6^{-x}$

PROBLEM-SOLVING	5, 6	5(½), 6, 7	5(½), 6–8

Example 18

5 Find the intersection of the following graphs.

a $y = 2^x$ and $y = 4$

b $y = 3^x$ and $y = 9$

c $y = 4^x$ and $y = 16$

d $y = -4^x$ and $y = -4$

e $y = -3^x$ and $y = -1$

f $y = 2^{-x}$ and $y = 8$

6 A study shows that the population of a town is modelled by the rule $P = 2^t$, where t is in years and P is in thousands of people.

a State the number of people in the town at the start of the study.

b State the number of people in the town after:

i 1 year **ii** 3 years

c When is the town's population expected to reach:

i 4000 people? **ii** 16 000 people?

7 A single bacterium divides into two every second, so one cell becomes 2 in the first second and in the next second two cells become 4 and so on.

a Write a rule for the number of bacteria, N, after t seconds.

b How many bacteria will there be after 10 seconds?

c How long does it take for the population to exceed 10 000? Round to the nearest second.

8 Use trial and error to find x when $2^x = 5$. Give the answer correct to three decimal places.

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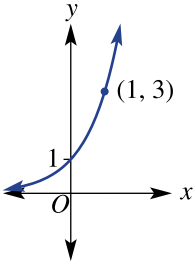
REASONING	9	9, 10	9–12

9 Match equations **a**–**f** with graphs **A**–**F** below.

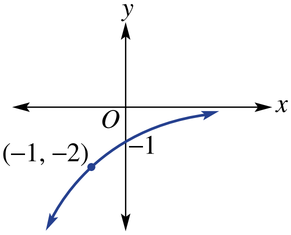
a $y = -x - 2$ **b** $y = 3^x$ **c** $y = 3^{-x}$

d $y = -2^x$ **e** $y = x$ **f** $y = -2^{-x}$

A



B



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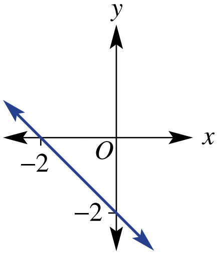

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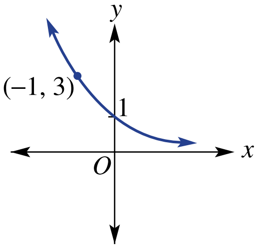
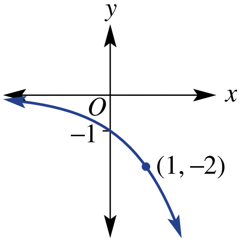
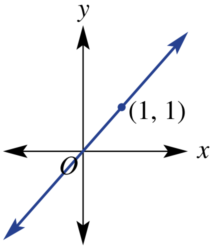


E



F



10 Explain why the point (2, 5) does not lie on the curve with equation $y = 2^x$.

11 Describe and draw the graph of the line with equation $y = a^x$ when $a = 1$.

12 Explain why $2^x = 0$ is never true for any value of x.

ENRICHMENT: $y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^x$	–	–	13

13 Consider the exponential rules $y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^x$.

a Using $-3 \leqslant x \leqslant 3$, sketch graphs of the rules on the same set of axes. What do you notice?

b Write the following rules in the form $y = a^x$, where $0 < a < 1$.

i $y = 3^{-x}$ **ii** $y = 5^{-x}$ **iii** $y = 10^{-x}$

c Write the following rules in the form $y = a^{-x}$, where $a > 1$.

i $y = \left(\frac{1}{4}\right)^x$

ii $y = \left(\frac{1}{7}\right)^x$

iii $y = \left(\frac{1}{11}\right)^x$

d Prove that $\left(\frac{1}{a}\right)^x = a^{-x}$, for $a > 0$.

3A **1** Simplify, using index laws.

a $a^3 \times a^2$

b $4x^2y \times 3xy^3$

c $h^6 \div h^2$

d $5m^9n^4 \div (10m^3n)$

e $(a^2)^3$

f $(3m^5)^2$

g $\left(\frac{m^2}{4}\right)^3$

h $(2ab)^0 + 5m^0$

3A **2** Combine index laws to simplify the following.

a $\frac{4x^2y \times 3x^3y^5}{6x^4y^2}$

b $\left(\frac{2p^4}{7qr^2}\right)^2$

3B **3** Simplify the following where possible and express your answers using positive indices.

a x^{-3}

b $2a^{-2}b^4c^{-3}$

c $\frac{7}{m^{-2}}$

d $\frac{4d^{-7}}{5d^{-5}}$

e $\left(\frac{4k^3}{k^7}\right)^2$

f $(2a^{-2})^{-3}$

g $6a^{-3}m^4 \times 2a^{-2}m^{-3}$

h $\frac{20c^{-3}d^2}{15c^{-1}d^{-3}}$

3B **4** Simplify the following and express your answers using positive indices.

a $(x^2y^{-3})^4 \times (x^{-1}y^3)^{-2}$

b $\frac{(a^3b)^5}{5a^3b^{-2}} \div \left(\frac{a^{-4}}{b^6}\right)^2$

3C **5** Write these numbers as a basic numeral.

a 3.204×10^7

b 4.7×10^{-4}

3C **6** Express the following in scientific notation using three significant figures.

a 34 721

b 0.00045681

c $5.21 \times 10^6 \div (4.8 \times 10^{-3})$

3D Opt **7** Express the following in index form.

a $\sqrt{10}$

b $\sqrt[3]{4x^2}$

c $6\sqrt{6}$

3D Opt **8** Express in surd form or evaluate where possible.

a $4^{\frac{1}{7}}$

b $25^{\frac{1}{2}}$

c $8^{-\frac{1}{3}}$

3E Opt **9** Solve for x in the following equations.

a $3^x = 27$

b $5^x = \frac{1}{125}$

c $9^{x+1} = 27^x$

3F **10** Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 4^x$

b $y = -4^x$

c $y = 4^{-x}$

3G Exponential growth and decay

LEARNING INTENTIONS

- To understand how percentage increase and decrease relate to exponential growth and decay
- To know the general form of the exponential growth and decay model
- To be able to write an exponential rule from a word problem and apply it

The population of a country increasing by 5% per year and an investment increasing, on average, by 12% per year are examples of exponential growth. When an investment grows exponentially, the increase per year is not constant. The annual increase is calculated on the value of the investment at that time, and this changes from year to year because of the added investment returns. The more money you have invested, the more interest you will make in a year.

In the same way, a population can grow exponentially. A growth of 5% in a large population represents many more babies born in a year than 5% of a small population.

Population growth can be modelled using exponential equations. Governments use projected population numbers when planning for future infrastructure, land use, and water, energy and food security.

Here we will focus on exponential growth and decay in general and compound interest will be studied in **Section 3H**.

Lesson starter: A compound rule

Imagine you have an antique car valued at $100 000 and you hope that it will increase in value at 10% p.a. The 10% increase is to be added to the value of the car each year.

- Discuss how to calculate the value of the car after 1 year.
- Discuss how to calculate the value of the car after 2 years.
- Complete this table.

Year	0	1	2	3
Value ($)	100 000	$100\,000 \times 1.1$ = ________	$100\,000 \times 1.1 \times$ ___ = ________	__________ = ________

- Discuss how indices can be used to calculate the value of the car after the second year.
- Discuss how indices can be used to calculate the value of the car after the tenth year.
- What might be the rule connecting the value of the car ($$A$) and the time n years?
- Repeat the steps above if the value of the car decreases by 10% p.a.

KEY IDEAS

- **Per annum** (p.a.) means ‘per year’.
- Exponential growth and decay can be modelled by the rule $A = ka^t$, where A is the amount, k is the initial amount and t is the time.
  - When $a > 1$, exponential growth occurs.
  - When $0 < a < 1$, exponential decay occurs.
- For a **growth** rate of $r\%$ p.a., the base ‘a’ is calculated using $a = 1 + \frac{r}{100}$.
- For a **decay** rate of $r\%$ p.a., the base ‘a’ is calculated using $a = 1 - \frac{r}{100}$.
- The basic **exponential formula** can be summarised as $A = A_0\left(1 \pm \frac{r}{100}\right)^n$.
  - The subscript zero is often used to indicate the initial value of a quantity (e.g. P_0 is initial population).

BUILDING UNDERSTANDING

1. An antique ring is purchased for \$1000 and is expected to grow in value by 5% per year. Round your answers to the nearest cent.
   - **a** Find the increase in value in the first year.
   - **b** Find the value of the ring at the end of the first year.
   - **c** Find the increase in value in the second year.
   - **d** Find the increase in value in the third year.
   - **e** Find the value of the ring at the end of the fifth year.

2. The mass of a limestone 5 kg rock exposed to the weather is decreasing at a rate of 2% per annum.
   - **a** Find the mass of the rock at the end of the first year.
   - **b** State the missing numbers for the mass of the rock (M kg) after t years.

     $$\begin{aligned} M &= 5(1 - \underline{\quad})^t \\ &= 5 \times \underline{\quad}^t \end{aligned}$$
   - **c** Use your rule to calculate the mass of the rock after 5 years, correct to two decimal places.

3. Decide if the following represent exponential *growth* or exponential *decay*.
   - **a** $A = 1000 \times 1.3^t$
   - **b** $A = 350 \times 0.9^t$
   - **c** $P = P_0\left(1 + \frac{3}{100}\right)^t$
   - **d** $T = T_0\left(1 - \frac{7}{100}\right)^t$

Example 19 Writing exponential rules

Form exponential rules for the following situations.

a John has a painting that is valued at \$100 000 and it is expected to increase in value by 14% per annum.

b A city's initial population of 50 000 is decreasing by 12% per year.

SOLUTION

a Let A = the value in \$ of the painting at any time

n = the number of years the painting is kept

$$\begin{aligned} r &= 14 \\ A_0 &= 100\,000 \\ A &= 100\,000\left(1+\frac{14}{100}\right)^n \\ \therefore A &= 100\,000\,(1.14)^n \end{aligned}$$

b Let P = the population at any time

n = the number of years the population decreases

$$\begin{aligned} r &= 12 \\ P_0 &= 50\,000 \\ P &= 50\,000\left(1-\frac{12}{100}\right)^n \\ \therefore P &= 50\,000\,(0.88)^n \end{aligned}$$

EXPLANATION

Define your variables.

$A = A_0\left(1 \pm \frac{r}{100}\right)^n$

Substitute $r = 14$ and $A_0 = 100\,000$ and use '+' since we have growth.

Define your variables.

$P = P_0\left(1 \pm \frac{r}{100}\right)^n$

Substitute $r = 12$ and $P_0 = 50\,000$ and use '−' since we have decay.

Now you try

Form exponential rules for the following situations.

a Caz has a vase that is valued at \$50 000 and it is expected to increase in value by 16% per annum.

b A town's initial population of 10 000 is decreasing by 9% per year.

Example 20 Applying exponential rules

House prices are rising at 9% per year and Zoe's flat is currently valued at \$600 000.

a Determine a rule for the value of Zoe's flat (\$$V$) in n years' time.

b What will be the value of her flat:

- **i** next year?
- **ii** in 3 years' time?

c Use trial and error to find when Zoe's flat will be valued at \$900 000, to one decimal place.

SOLUTION

a Let V = the value of Zoe's flat at any time

V_0 = starting value \$600 000

n = number of years from now

$$\begin{aligned} r &= 9 \\ V &= V_0\,(1.09)^n \\ \therefore V &= 600\,000\,(1.09)^n \end{aligned}$$

EXPLANATION

Define your variables.

$V = V_0\left(1 \pm \frac{r}{100}\right)^n$

Use '+' since we have growth.

b **i** When $n = 1$, $V = 600\,000\,(1.09)^1$
$= 654\,000$
Zoe's flat would be valued at \$654 000 next year.

Substitute $n = 1$ for next year.

ii When $n = 3$, $V = 600\,000\,(1.09)^3$
$= 777\,017.40$
In 3 years' time Zoe's flat will be valued at about \$777 017.

For 3 years, substitute $n = 3$.

c

n	4	5	4.6	4.8	4.7
V	846 949	923 174	891 894	907 399	899 613

Zoe's flat will be valued at \$900 000 in about 4.7 years' time.

Try a value of n in the rule. If V is too low, increase your n value. If V is too high, decrease your n value. Continue this process until you get close to 900 000.

Now you try

House prices are rising at 7% per year and Andrew's apartment is currently valued at \$400 000.

a Determine a rule for the value of Andrew's apartment (\$$V$) in n years' time.

b What will be the value of his apartment:

i next year? **ii** in 3 years' time?

c Use trial and error to find when Andrew's apartment will be valued at \$500 000, to one decimal place.

Exercise 3G

FLUENCY	1, 2–4	1, 2(½), 3–5	2(½), 3, 5, 6

Example 19

1 Form exponential rules for the following situations.

a Lara has a necklace that is valued at \$6000 and it is expected to increase in value by 12% per annum.

b A village's initial population of 2000 is decreasing by 8% per year.

2 Define variables and form exponential rules for the following situations.

a A flat is purchased for \$200 000 and is expected to grow in value by 17% per annum.

b A house initially valued at \$530 000 is losing value at 5% per annum.

c The value of a car, bought for \$14 200, is decreasing at 3% per annum.

d An oil spill, initially covering an area of 2 square metres, is increasing at 5% per minute.

e A tank with 1200 litres of water is leaking at a rate of 10% of the water in the tank every hour.

f A human cell of area 0.01 cm^2 doubles its area every minute.

g A population, which is initially 172 500, is increasing at 15% per year.

h A substance of mass 30 g is decaying at a rate of 8% per hour.

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Example 20

3 The value of a house purchased for \$500 000 is expected to grow by 10% per year. Let \$$A$ be the value of the house after t years.

a Write the missing number in the rule connecting A and t. $A = 500\,000 \times \_\_\_\_\_^{t}$

b Use your rule to find the expected value of the house after the following number of years. Round your answer to the nearest cent.

i 3 years **ii** 10 years **iii** 20 years

c Use trial and error to estimate when the house will be worth \$1 million. Round your answer to one decimal place.

4 A share portfolio, initially worth \$300 000, is reduced by 15% p.a. over a number of years. Let \$$A$ be the share portfolio value after t years.

a Write the missing number in the rule connecting A and t. $A = \_\_\_\_\_ \times 0.85^{t}$

b Use your rule to find the value of the shares after the following number of years. Round your answer to the nearest cent.

i 2 years **ii** 7 years **iii** 12 years

c Use trial and error to estimate when the share portfolio will be valued at \$180 000. Round your answer to one decimal place.

5 A water tank containing 15 000 L has a small hole that reduces the amount of water by 6% per hour.

a Determine a rule for the volume of water (V litres) left after t hours.

b Calculate (to the nearest litre) the amount of water left in the tank after:

i 3 hours **ii** 7 hours

c How much water is left after two days? Round your answer to two decimal places.

d Using trial and error, determine when the tank holds less than 500 L of water, to one decimal place.

6 Megan invests \$50 000 in a superannuation scheme that has an annual return of 11%.

a Determine the rule for the value of her investment (\$$V$) after n years.

b How much will Megan's investment be worth in:

i 4 years? **ii** 20 years?

c Find the approximate time before her investment is worth \$100 000. Round your answer to two decimal places.

PROBLEM–SOLVING	7, 8	7, 8	8, 9

7 A certain type of bacteria grows according to the equation $N = 3000\,(2.6)^{t}$, where N is the number of cells present after t hours.

a How many bacteria are there at the start?

b Determine the number of cells (round to the whole number) present after:

i 1 hour **ii** 2 hours **iii** 4.6 hours

c If 50 000 000 bacteria are needed to make a drop of serum, determine how long you will have to wait to make a drop (to the nearest minute).

8 A car tyre has 10 mm of tread when new. It is considered unroadworthy when there is only 3 mm left. The rubber wears at 12.5% every 10 000 km.

a Write an equation relating the depth of tread (D mm) for every 10 000 km travelled.

b Using trial and error, determine when the tyre becomes unroadworthy, to the nearest 10 000 km.

c If a tyre lasts 80 000 km, it is considered to be of good quality. Is this a good quality tyre?

9 A cup of coffee has an initial temperature of 90°C and the surrounding temperature is 0°C.

a If the temperature relative to surroundings reduces by 8% every minute, determine a rule for the temperature of the coffee ($T°$C) after t minutes.

b What is the temperature of the coffee (to one decimal place) after:

i 90 seconds? **ii** 2 minutes?

c When is the coffee suitable to drink if it is best consumed at a temperature of 68.8°C? Give your answer to the nearest second.

REASONING	10	10	10, 11

10 The monetary value of things can be calculated using different time periods. Consider a \$1000 collector's item that is expected to grow in value by 10% p.a. over 5 years.

- If the increase in value is added annually then $r = 10$ and $t = 5$, so $A = 1000(1.1)^5$.
- If the increase in value is added monthly then $r = \frac{10}{12}$ and $t = 5 \times 12 = 60$, so $A = 1000\left(1 + \frac{10}{1200}\right)^{60}$.

a If the increase in value is added annually, find the value of the collectors' item, to the nearest cent, after:

i 5 years **ii** 8 years **iii** 15 years

b If the increase in value is added monthly, find the value of the collectors' item, to the nearest cent, after:

i 5 years **ii** 8 years **iii** 15 years

11 You inherit a \$2000 necklace that is expected to grow in value by 7% p.a. What will the necklace be worth, to the nearest cent, after 5 years if the increase in value is added:

a annually? **b** monthly?

c weekly (assume 52 weeks in the year)?

ENRICHMENT: Half-life	–	–	12–14

Half-life is the period of time it takes for an object to decay by half. It is often used to compare the rate of decay for radioactive materials.

12 A 100 g mass of a radioactive material decays at a rate of 10% every 10 years.

a Find the mass of the material after the following time periods. Round your answer to one decimal place, where necessary.

i 10 years **ii** 30 years **iii** 60 years

b Estimate the half-life of the radioactive material (i.e. find how long it takes for the material to decay to 50 g). Use trial and error and round your answer to the nearest year.

13 An ice sculpture, initially containing 150 L of water, melts at a rate of 3% per minute.

a What will be the volume of the ice sculpture after half an hour? Round your answer to the nearest litre.

b Estimate the half-life of the ice sculpture. Give your answer in minutes, correct to one decimal place.

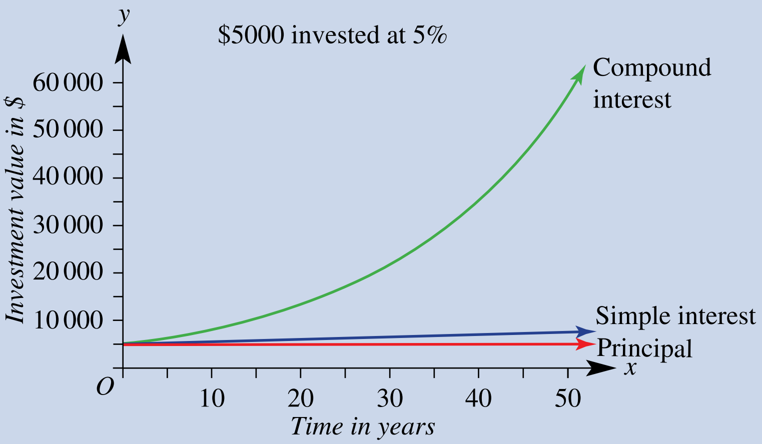
14 The half-life of a substance is 100 years. Find the rate of decay per annum, expressed as a percentage correct to one decimal place.

3H Compound interest

LEARNING INTENTIONS
- To know the meaning of the term compound interest
- To know how to apply the compound interest formula
- To know how compound interest is calculated for different time periods
- To be able to determine the total amount and the interest in a compound interest scenario

For simple interest, the interest is always calculated on the principal amount. Sometimes, however, interest is calculated on the actual amount present in an account at each time period that interest is calculated. This means that the interest is added to the amount, then the next lot of interest is calculated again using this new amount.

This process is called compound interest. Compound interest can be calculated using updated applications of the simple interest formula or by using the compound interest formula. It is a common example of exponential growth.



The ‘magic’ growth of compound interest comes from interest paid on previous interest. Retirement savings are especially suited to benefit from compound interest, as this type of investment grows at an increasingly faster rate over time, as you can see in the graph above.

Lesson starter: Investing using updated simple interest

Consider investing $400 at 12% per annum.

- Copy and complete the table below.

Time (n)	Amount (A)	Interest (I)	New amount
1st year	$400	$48	$448
2nd year	$448	$53.76	$501.76
3rd year	$501.76		
4th year			

- What is the balance at the end of 4 years if interest is added to the amount at the end of each year?
- Thinking about this as exponential growth, write a rule linking A with n.

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KEY IDEAS

- **Compound interest** is calculated using updated applications of the simple interest formula. For example, \$100 compounded at 10% p.a. for 2 years.
  Year 1: 100 + 10% of 100 = \$110
  Year 2: 110 + 10% of 110 = \$121, so compound interest = \$21
- The total amount in an account using compound interest for a given number of time periods is given by:
  $A = P\left(1 + \frac{r}{100}\right)^n$, where:
  - Principal (P) = the amount of money borrowed or invested.
  - Rate of interest (r) = the percentage applied to the principal per period of investment.
  - Periods (n) = the number of periods the principal is invested.
  - Amount (A) = the total amount of your investment.
- Interest = amount (A) − principal (P)

BUILDING UNDERSTANDING

1 Consider \$500 invested at 10% p.a., compounded annually.

a How much interest is earned in the first year?
b What is the balance of the account once the first year's interest is added?
c How much interest is earned in the second year?
d What is the balance of the account at the end of the second year?
e Use your calculator to work out $500(1.1)^2$.

2 By considering an investment of \$4000 at 5% p.a., compounded annually, calculate the missing values in the table below.

Year	Amount (\$)	Interest (\$)	New amount (\$)
1	4000	200	4200
2	4200		
3			
4			
5			

3 Find the value of the following, correct to two decimal places.

a $\$1000 \times 1.05 \times 1.05$
b $\$1000 \times 1.05^2$
c $\$1000 \times 1.05 \times 1.05 \times 1.05$
d $\$1000 \times 1.05^3$

4 State the missing numbers.

a \$700 invested at 8% p.a., compounded annually for 2 years.
$A = \square(1.08)^{\square}$

b \$1000 invested at 15% p.a., compounded annually for 6 years.
$A = 1000(\square)^6$

c \$850 invested at 6% p.a., compounded annually for 4 years.
$A = 850(\square)^{\square}$

Example 21 Using the compound interest formula

Determine the amount after 5 years if \$4000 is compounded annually at 8%. Round to the nearest cent.

SOLUTION	EXPLANATION
$P = 4000, n = 5, r = 8$	List the values for the terms
$A = P\left(1 + \frac{r}{100}\right)^n$ $= 4000\left(1 + \frac{8}{100}\right)^5$	Write the formula and then substitute the known values.
$= 4000(1.08)^5$	Simplify and evaluate.
$= \$5877.31$	Write your answer to two decimal places (the nearest cent).

Now you try

Determine the amount after 6 years if \$3000 is compounded annually at 7%. Round to the nearest cent.

Example 22 Converting rates and time periods

Calculate the number of periods and the rates of interest offered per period for the following.

a 6% p.a. over 4 years, paid monthly

b 18% p.a. over 3 years, paid quarterly

SOLUTION	EXPLANATION
a $n = 4 \times 12 = 48$ $r = 6 \div 12 = 0.5$	4 years is the same as 48 months, as 12 months = 1 year. 6% p. a. = 6% in 1 year. Divide by 12 to find the monthly rate.
b $n = 3 \times 4 = 12$ $r = 18 \div 4 = 4.5$	There are 4 quarters in 1 year.

Now you try

Calculate the number of periods and the rates of interest offered per period for the following.

a 5% p.a. over 5 years, paid monthly

b 14% p.a. over 3 years, paid quarterly

Example 23 Finding compounded amounts using months

Anthony's investment of $4000 is compounded at 8.4% p.a. over 5 years. Determine the amount he will have after 5 years if the interest is paid monthly. Round to the nearest cent.

SOLUTION	EXPLANATION
$P = 4000$ $n = 5 \times 12$ $= 60$ $r = 8.4 \div 12$ $= 0.7$	List the values of the terms you know. Convert the time in years to the number of periods (in this case, months); 60 months = 5 years. Convert the rate per year to the rate per period (months) by dividing by 12.
$A = P\left(1 + \frac{r}{100}\right)^n$	Write the formula.
$= 4000(1 + 0.007)^{60}$	Substitute the values, $0.7 \div 100 = 0.007$.
$= 4000(1.007)^{60}$ $= \$6078.95$	Simplify and evaluate, rounding to the nearest cent.

Now you try

Wendy's investment of $7000 is compounded at 6.2% p.a. over 4 years. Determine the amount she will have after 4 years if the interest is paid monthly. Round to the nearest cent.

Exercise 3H

FLUENCY	1–4(½)	1–4(½)	1(½), 3–4(½)

Example 21

1 Determine the amount after 5 years in the following, rounding to the nearest cent.

a $4000 is compounded annually at 5%
b $8000 is compounded annually at 8.35%
c $6500 is compounded annually at 16%
d $6500 is compounded annually at 8%

2 Determine the amount if $100 000 is compounded annually at 6% for the following time periods. Round to the nearest cent.

a 1 year
b 2 years
c 3 years
d 5 years
e 10 years
f 15 years

Example 22

3 Calculate the number of periods (n) and the rates of interest (r) offered per period for the following. (Round the interest rate to three decimal places where necessary.)

a 6% p.a. over 3 years, paid bi-annually
b 12% p.a. over 5 years, paid monthly
c 4.5% p.a. over 2 years, paid fortnightly
d 10.5% p.a. over 3.5 years, paid quarterly
e 15% p.a. over 8 years, paid quarterly
f 9.6% p.a. over 10 years, paid monthly

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Example 23

4 Calculate the value of the following investments if interest is compounded monthly.

a \$2000 at 6% p.a. for 2 years
b \$34 000 at 24% p.a. for 4 years
c \$350 at 18% p.a. for 8 years
d \$670 at 6.6% p.a. for $2\frac{1}{2}$ years
e \$250 at 7.2% p.a. for 12 years
f \$1200 at 4.8% p.a. for $3\frac{1}{3}$ years

PROBLEM–SOLVING	5	5, 6	5, 7

5 Darinia invests \$5000 compounded monthly at 18% p.a. Determine the value of the investment after:

a 1 month
b 3 months
c 5 months.

6 An investment of \$8000 is compounded at 12.6% over 3 years. Determine the amount the investor will have after 3 years if the interest is compounded monthly.

7 **a** For each rate below, calculate the amount of compound interest paid on \$8000 at the end of 3 years.

- **i** 12% compounded annually
- **ii** 12% compounded bi-annually (i.e. twice a year)
- **iii** 12% compounded monthly
- **iv** 12% compounded weekly
- **v** 12% compounded daily

b What is the interest difference between annual and daily compounding in this case?

REASONING	8	8	8(½), 9

8 The following are expressions relating to compound interest calculations. Determine the principal (P), number of periods (n), rate of interest per period (r%), annual rate of interest (R%) and the overall time (t).

a $300(1.07)^{12}$, bi-annually
b $5000(1.025)^{24}$, monthly
c $1000(1.00036)^{65}$, fortnightly
d $3500(1.000053)^{30}$, daily
e $10\,000(1.078)^{10}$, annually
f $6000(1.0022)^{91}$, fortnightly

9 Paula must decide whether to invest her \$13 500 for 6 years at 4.2% p.a. compounded monthly or 5.3% compounded bi-annually. Decide which investment would be the best choice for Paula.

ENRICHMENT: Double your money	–	–	10

10 You have \$100 000 to invest and wish to double that amount. Use trial and error in the following.

a Determine, to the nearest whole number of years, the length of time it will take to do this using the compound interest formula at rates of:

- **i** 12% p.a.
- **ii** 6% p.a.
- **iii** 8% p.a.
- **iv** 16% p.a.
- **v** 10% p.a.
- **vi** 20% p.a.

b If the amount of investment is \$200 000 and you wish to double it, determine the time it will take using the same interest rates as above.

c Are the lengths of time to double your investment the same in part **a** and part **b**?

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3I Introducing logarithms

LEARNING INTENTIONS

- To understand the form of a logarithm and its relationship with index form
- To be able to convert between equivalent index and logarithmic forms
- To be able to evaluate simple logarithms both with and without technology
- To be able to solve simple logarithmic equations

Logarithms ('logical arithmetic') are an important idea in mathematics and were invented by John Napier in the 17th century to simplify arithmetic calculations. Logarithms are linked directly to exponentials and can be used to solve a range of exponential equations.

Recall that $2^3 = 8$ (2 to the power 3 equals 8). We can also say that the logarithm of 8 to the base 2 equals 3 and we write $\log_2 8 = 3$. So for exponential equations such as $y = 2^x$, a logarithm finds x for a given value of y.

A logarithm can often be evaluated by hand but calculators can also be used.

Logarithms can also be used to create logarithmic scales, which are commonly used in science, economics and engineering. For example, the Richter scale, and the moment magnitude scale that replaced it, are logarithmic scales that illustrate the strength of an earthquake.

Seismologists calculate the magnitude of an earthquake using the logarithm of its intensity. The 2004 Sumatra earthquake of Richter magnitude 9.3 had 1000 times more intense shaking than the Richter magnitude 6.3 earthquake in Christchurch in 2011.

Lesson starter: Can you work out logarithms?

We know that $3^2 = 9$, so $\log_3 9 = 2$. This means that $\log_3 9$ is equal to the index that makes 3 to the power of that index equal 9. Similarly, $10^3 = 1000$ so $\log_{10} 1000 = 3$.

Now find the value of the following.

- $\log_{10} 100$
- $\log_2 64$
- $\log_{10} 10\,000$
- $\log_3 27$
- $\log_2 16$
- $\log_4 64$

KEY IDEAS

- A **logarithm** of a number to a given base is the power (or index) to which the base is raised to give the number.
  - For example: $\log_2 16 = 4$ since $2^4 = 16$.
  - The base a is written as a subscript to the operator word 'log'; i.e. $\log_a$.
- In general, if $a^x = y$ then $\log_a y = x$ with $a > 0$ and $y > 0$.
  - We say 'the logarithm of y to the base a is x'.

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BUILDING UNDERSTANDING

① State the missing values in this table.

x	0	1	2	3	4	5
2^x						
3^x						243
4^x					256	
5^x		5				
10^x			100			

② State the value of the unknown number for each statement.

a 2 to the power of what number gives 16?

b 3 to the power of what number gives 81?

c 7 to the power of what number gives 343?

d 10 to the power of what number gives 10 000?

③ Give these numbers as fractions.

a 0.0001 **b** 0.5 **c** 2^{-2} **d** 3^{-3}

Example 24 Writing equivalent statements involving logarithms

Write an equivalent statement to the following.

a $\log_{10} 1000 = 3$ **b** $2^5 = 32$

SOLUTION	EXPLANATION
a $10^3 = 1000$	$\log_a y = x$ is equivalent to $a^x = y$.
b $\log_2 32 = 5$	$a^x = y$ is equivalent to $\log_a y = x$.

Now you try

Write an equivalent statement to the following.

a $\log_{10} 100 = 2$ **b** $3^4 = 81$

Example 25 Evaluating logarithms

a Evaluate the following logarithms.

i $\log_2 8$ **ii** $\log_5 625$

b Evaluate the following.

i $\log_3 \frac{1}{9}$ **ii** $\log_{10} 0.001$

c Evaluate, correct to three decimal places, using a calculator.

i $\log_{10} 7$ **ii** $\log_{10} 0.5$

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SOLUTION	**EXPLANATION**
a **i** $\log_2 8 = 3$	Ask the question '2 to what power gives 8?' Note: $2^3 = 8$
ii $\log_5 625 = 4$	$5^4 = 5 \times 5 \times 5 \times 5 = 625$
b **i** $\log_3 \frac{1}{9} = -2$	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
ii $\log_{10} 0.001 = -3$	$10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = 0.001$
c **i** $\log_{10} 7 = 0.845$ (to 3 d.p.)	Use the log button on a calculator and use base 10. (Some calculators will give log base 10 by pressing the log button.)
ii $\log_{10} 0.5 = -0.301$ (to 3 d.p.)	Use the log button on a calculator.

Now you try

a Evaluate the following logarithms.

i $\log_2 16$ **ii** $\log_3 243$

b Evaluate the following.

i $\log_2 \frac{1}{8}$ **ii** $\log_{10} 0.01$

c Evaluate, correct to three decimal places, using a calculator.

i $\log_{10} 5$ **ii** $\log_{10} 0.45$

Example 26 Solving simple logarithmic equations

Find the value of x in these equations.

a $\log_4 64 = x$ **b** $\log_2 x = 6$

SOLUTION	**EXPLANATION**
a $\log_4 64 = x$ $4^x = 64$ $x = 3$	$\log_a y = x$ then $a^x = y$. $4^3 = 64$
b $\log_2 x = 6$ $2^6 = x$ $x = 64$	Write in index form: $2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$

Now you try

Find the value of x in these equations.

a $\log_3 81 = x$ **b** $\log_5 x = 3$

Exercise 3I

FLUENCY	1, 2, 3–5(½)	1–5(½)	1–5(⅓)

Example 24a

1 Write the following in index form.

a $\log_2 16 = 4$ b $\log_{10} 100 = 2$ c $\log_3 27 = 3$

d $\log_2 \frac{1}{4} = -2$ e $\log_{10} 0.1 = -1$ f $\log_3 \frac{1}{9} = -2$

Example 24b

2 Write the following in logarithmic form.

a $2^3 = 8$ b $3^4 = 81$ c $2^5 = 32$

d $4^2 = 16$ e $10^{-1} = \frac{1}{10}$ f $5^{-3} = \frac{1}{125}$

Example 25a

3 Evaluate the following logarithms.

a $\log_2 16$ b $\log_2 4$ c $\log_2 64$ d $\log_3 27$

e $\log_3 3$ f $\log_4 16$ g $\log_5 125$ h $\log_{10} 1000$

i $\log_7 49$ j $\log_{11} 121$ k $\log_{10} 100\,000$ l $\log_9 729$

m $\log_2 1$ n $\log_5 1$ o $\log_{37} 1$ p $\log_1 1$

Example 25b

4 Evaluate the following.

a $\log_2 \frac{1}{8}$ b $\log_2 \frac{1}{4}$ c $\log_3 \frac{1}{9}$ d $\log_{10} \frac{1}{1000}$

e $\log_7 \frac{1}{49}$ f $\log_3 \frac{1}{81}$ g $\log_5 \frac{1}{625}$ h $\log_8 \frac{1}{8}$

i $\log_{10} 0.1$ j $\log_{10} 0.001$ k $\log_{10} 0.00001$ l $\log_2 0.5$

m $\log_2 0.125$ n $\log_5 0.2$ o $\log_5 0.04$ p $\log_3 0.\dot{1}$

Example 25c

5 Evaluate, correct to three decimal places, using a calculator.

a $\log_{10} 5$ b $\log_{10} 47$ c $\log_{10} 162$

d $\log_{10} 0.8$ e $\log_{10} 0.17$ f $\log_{10} \frac{1}{27}$

PROBLEM–SOLVING	6(½), 7	6(⅓), 7	6(¼), 8

Example 26

6 Find the value of x in these equations.

a $\log_3 27 = x$ b $\log_2 32 = x$ c $\log_2 64 = x$ d $\log_5 625 = x$

e $\log_{10} 1000 = x$ f $\log_6 36 = x$ g $\log_2 x = 4$ h $\log_3 x = 4$

i $\log_{10} x = 3$ j $\log_3 x = -2$ k $\log_4 x = -1$ l $\log_7 x = -3$

m $\log_x 27 = 3$ n $\log_x 32 = 5$ o $\log_x 64 = 3$ p $\log_x 64 = 2$

q $\log_x 81 = 4$ r $\log_x 10\,000 = 4$ s $\log_x 0.5 = -1$ t $\log_4 0.25 = x$

7 A single bacterium cell divides into two every minute.

a Complete this cell population table.

Time (minutes)	0	1	2	3	4	5
Population	1	2				

b Write a rule for the population, P, after t minutes.

c Use your rule to find the population after 8 minutes.

d Use trial and error to find the time (correct to the nearest minute) for the population to rise to 10 000.

e Write the exact answer to part **d** as a logarithm.

8 Evaluate:

a $\log_2 4 \times \log_3 9 \times \log_4 16 \times \log_5 25$

b $2 \times \log_3 27 - 5 \times \log_8 64 + 10 \times \log_{10} 1000$

c $\dfrac{4 \times \log_5 125}{\log_2 64} + \dfrac{2 \times \log_3 9}{\log_{10} 10}$

REASONING	9	9, 10	10, 11

9 Consider a bacteria population growing such that the total increases 10-fold every hour.

a Complete this table for the population (P) and $\log_{10} P$ for 5 hours (h).

h	0	1	2	3	4	5
P	1	10	100			
$\log_{10} P$						

b Plot a graph of $\log_{10} P$ (y-axis) against hours (x-axis). What do you notice?

c Find a rule linking $\log_{10} P$ with h.

10 The Richter magnitude of an earthquake is determined from a logarithm of the amplitude of waves recorded by a seismograph. It uses log base 10. So for example, an earthquake of magnitude 3 is 10 times more powerful than one with magnitude 2 and an earthquake of magnitude 7 is 100 times more powerful than one with magnitude 5.

a Write the missing number. An earthquake of magnitude 6 is:

i ☐ times more powerful than one of magnitude 5.

ii ☐ times more powerful than one of magnitude 4.

iii ☐ times more powerful than one of magnitude 2.

b Write the missing number. An earthquake of magnitude 9 is:

i ☐ times more powerful than one of magnitude 8.

ii 1000 times more powerful than one of magnitude ☐.

iii 10^6 times more powerful than one of magnitude ☐.

11 Is it possible for a logarithm (of the form $\log_a b$) to give a negative result? If so, give an example and reasons.

ENRICHMENT: Fractional logarithms	–	–	12(½)

12 Observe that $\sqrt{2} = 2^{\frac{1}{2}}$, $\sqrt[3]{2} = 2^{\frac{1}{3}}$ and in general $\sqrt[n]{a} = a^{\frac{1}{n}}$, so $\log_2 \sqrt{2} = \frac{1}{2}$ and $\log_2 \sqrt[3]{2} = \frac{1}{3}$. Now evaluate the following without the use of a calculator.

a $\log_2 \sqrt[4]{2}$

b $\log_2 \sqrt[5]{2}$

c $\log_3 \sqrt{3}$

d $\log_3 \sqrt[3]{3}$

e $\log_7 \sqrt{7}$

f $\log_{10} \sqrt[3]{10}$

g $\log_{10} \sqrt[3]{100}$

h $\log_2 \sqrt[3]{16}$

i $\log_3 \sqrt[4]{9}$

j $\log_5 \sqrt[4]{25}$

k $\log_2 \sqrt[5]{64}$

l $\log_3 \sqrt[7]{81}$

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Air conditioner thermostat

1 An air conditioning unit inside a room has a thermostat that controls the temperature of the room. The temperature of the room, T°C, n hours after the air conditioning unit switches on is given by $T = 17 + \frac{8}{2^n}$.

The air conditioning unit is set to turn on when the room temperature reaches 25°C.

A technician wishes to investigate how exponential equations can model the change in air temperature and how thermostats can be used to control the use of air conditioners.

- **a** If the air conditioning unit remains on for 1 hour after it switches on, what will be the temperature in the room?
- **b** After how many hours of the unit being on would the temperature in the room reach 19°C?

The unit is programmed to switch off when the temperature in the room reaches 20°C.

- **c** Find the longest consecutive period of time that the unit could be on for, correct to one decimal place.
- **d** Sketch a graph of the temperature in the room, T, from when the unit switches on until when it switches off.
- **e** Express the rule for the temperature T in the form $T = 17 + 2^{k-n}$ where k is an integer.

The thermostat is adjusted so that it turns on at 24°C and so that the fan strength is decreased. This unit switches off when the room is cooled to 21°C, which occurs after it has been on for 2 hours.

- **f** Find the values of a and k, where a and k are integers, if the rule for the temperature, T°C, of the room n hours after this unit is turned on is given by $T = a + 2^{k-n}$.

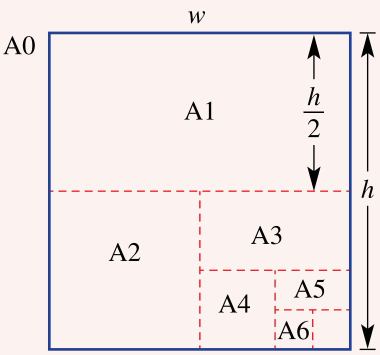
International paper sizes

2 The A series of paper sizes, e.g. A4, are based on international standards. The paper sizes are such that the ratio between the height and width of each paper size is the same. The height is taken to be the longer side length of each rectangle. Let an A0 piece of paper have width w mm and height h mm.

A paper company wants to explore the A series paper sizes and use ratios to connect the lengths and widths of successive sizes. It wishes to use these ratios to then determine various widths and heights and the rules that link these dimensions.

- **a** Complete the table below for the corresponding height and width of the A series paper in terms of h and w.

An	Width	Height
A0	w	h
A1	$\frac{h}{2}$	w
A2		
A3		
A4		
A5		
A6		



b Determine the ratio of the height to the width of A series paper if it is the same for each paper size An.

c From your result in part **b**, write a rule for the height, h, of A series paper in terms of its width, w.

d A0 paper has an area of 1 square metre (1000 mm × 1000 mm). Determine the dimensions, w and h, of A0 paper in exact form in mm.

e Use your values from part **d** and your table from part **a** to determine the dimensions of an A4 sheet to the nearest millimetre. Measure a sheet of A4 paper to compare.

f Consider the table in part **a** and paper sizes An.

- **i** Describe the changes to the values of the width and height as n increases when n is even and when n is odd.
- **ii** Use your table and dimensions from part **d** to come up with rules for w and h when n is even and when n is odd.
- **iii** Use your rule from part **ii** to find the length and width for A3 and A4 paper and check by measuring the paper.

Accumulating ants

3 When worker ants look for food they leave a scent along their path so that other ants can find the food source. This can lead to ants accumulating quickly in an area away from their nest, like around small crumbs they find in a household kitchen.

Scientists interested in the growth of the population of ants use exponential relations to describe this behaviour. They will use rules to predict ant numbers and model the population of ants by constructing suitable equations.

a A rule for a population, P, of ants which has found some food in a kitchen pantry is given by $P = 10 \times 2^{2t}$ where t is in hours after the food is first found.

- **i** What was the initial number of ants in the pantry when the food is first found?
- **ii** How many ants were in the pantry 2 hours after the food was found?
- **iii** After how many hours did the ant population reach 1000? Answer correct to one decimal place.
- **iv** By what factor does the population increase each hour according to this rule?

b Another group of ants has found the cat food in the laundry. The rule for the growth of this population of ants is given by $P = P_0 \times 3^{2t}$ where t is in hours and P_0 is the initial number of ants that found the cat food.

The rule for t can be expressed in the form

$$t = \frac{1}{2}\log_3\left(\frac{P}{P_o}\right)$$

- **i** Use your rule to find the number of hours it takes for the initial ant population to triple.
- **ii** If a general ant population model is given by $P = P_0 \times a^{bt}$, where a and b are constants, use the form of the rule for t above to express t in terms of P, P_0, a and b.

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3J Logarithmic scales

LEARNING INTENTIONS

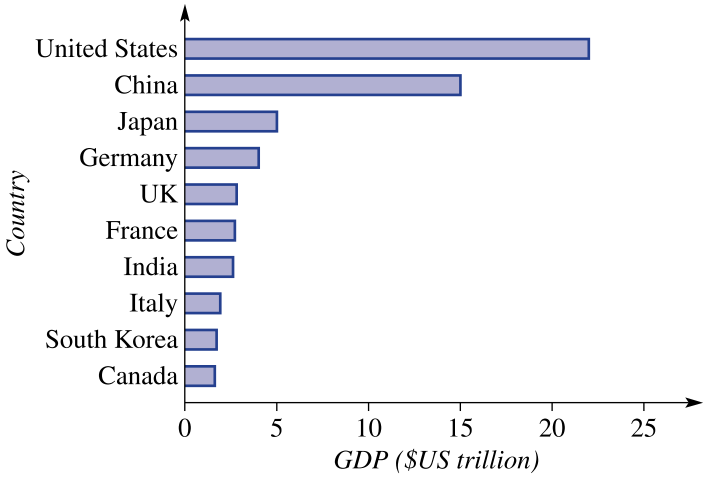
- To know what is meant by a logarithmic scale
- To understand why logarithmic scales are used
- To be able to interpret a logarithmic scale including on graphs and charts
- To be able to construct a logarithmic graph or chart

Logarithmic scales are commonly used when we are trying to display information involving exponential growth or decay. Charts and graphs using logarithmic scales can help to visualise very large or very small values in the given data and make it easier to describe percentage change.

Lesson starter: Exploring a logarithmic chart

The following table and chart show the top ten countries with the highest gross domestic product (GDP) using $US in 2020.

Country	GDP ($US trillion)
United States	22
China	15
Japan	5
Germany	4
UK	2.8
France	2.7
India	2.6
Italy	1.9
South Korea	1.7
Canada	1.6



- What do you notice about the spread of the data across the \$0 to \$25 trillion range for the ten countries?
- Is it easy to see the differences in the GDP values for the six lower countries on the chart?

We will now construct another chart using the logarithm of the GDP values, $\log_{10}(\text{GDP})$.

- Calculate the value of $\log_{10}(\text{GDP})$ for all the countries; e.g. $\log_{10}(2) = 1.3$ rounded to one decimal place.
- Construct a new chart using $\log_{10}(\text{GDP})$ on the horizontal axis. These values should range between 0 and 1.5.
- What do you notice about the differences in the chart bar lengths compared to the original chart?
- How might this new chart be more useful to the reader compared to the original?

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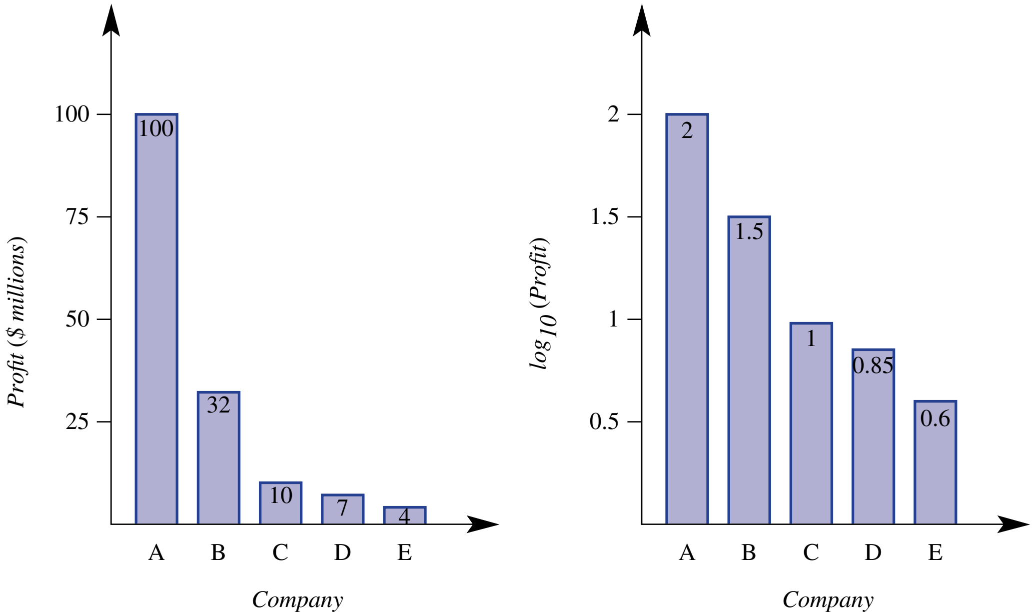
KEY IDEAS

- **Logarithmic scales** are used to help visualise the spread of data over a wide range of values including data arising from situations involving growth and decay. Some common fields where logarithmic scales are used include:
  - Seismology (the study of earthquakes)
  - Sound level and frequency
  - Timelines
  - Thermodynamics
  - Photographic exposure
  - pH and acidity
  - Finance
  - Population growth.
- The **order of magnitude** is the power of 10 used to express a number in scientific notation.
  - For example: $24\,000 = 2.4 \times 10^4$ and so the order of magnitude is 4.
  - To increase a number by an order of magnitude n we multiply by 10^n.
  - To decrease a number by an order of magnitude n we divide by 10^n.
- A logarithmic chart or graph uses the logarithm of a quantity on at least one of its axes.
  - Data including variables connected via an exponential relationship can be represented as a linear relationship using logarithms.
  - For example: If \$1000 is invested and compounded at 10% p.a. for t years then the amount \$$A$ is given by $A = 1000\,(1.1)^t$. If $\log_{10} A$ is plotted against t, the graph will be a straight line as $\log_{10} A = \log_{10}(1.1) \times t + 3$ which is in the form $y = mt + c$.
- Note: $\log_{10} x$ is sometimes written as $\log x$.

BUILDING UNDERSTANDING

1. Consider the numbers 1, 10, 100, 1000, 10 000 and 100 000.
   - **a** Write each of the numbers as powers of 10. For example: $1000 = 10^3$.
   - **b** How many times larger is:
     - **i** 1000 compared to 10?
     - **ii** 100 000 compared to 100?
   - **c** Find the values of $\log_{10} 1$, $\log_{10} 10$, $\log_{10} 100$, $\log_{10} 1000$, $\log_{10} 10\,000$ and $\log_{10} 100\,000$.
   - **d** State the order of magnitude of the numbers 1, 10, 100, 1000, 10 000 and 100 000.
     For example: $1000 = 10^3$ so the order of magnitude of 1000 is 3.
   - **e** What do you notice about your answers to parts **c** and **d**?

② These two charts show the annual profit of five companies for one year. The left chart uses the profit values in millions of dollars and the right chart uses the logarithm of the profit showing the order of magnitude.



a Use a calculator and $\log_{10}$ (Profit) to check the heights of the bars in the log chart.

b Which graph helps to visualise the differences between the companies with the smaller profits?

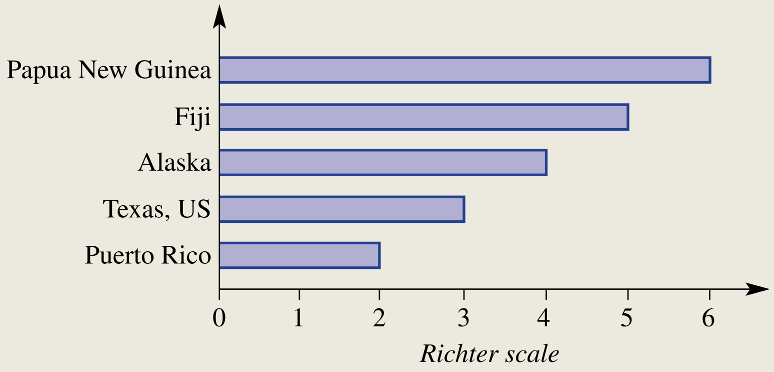
c Consider company A's profit compared to that of company C.

  i Company A's profit is how many times larger than company C?

  ii Use the log chart to find the difference in the order of magnitude of the profit for company A and company C.

Example 27 Interpreting a logarithmic chart

This logarithmic chart shows the Richter scale measurements rounded to the nearest integer for five different earthquakes recorded on one particular day in 2022.



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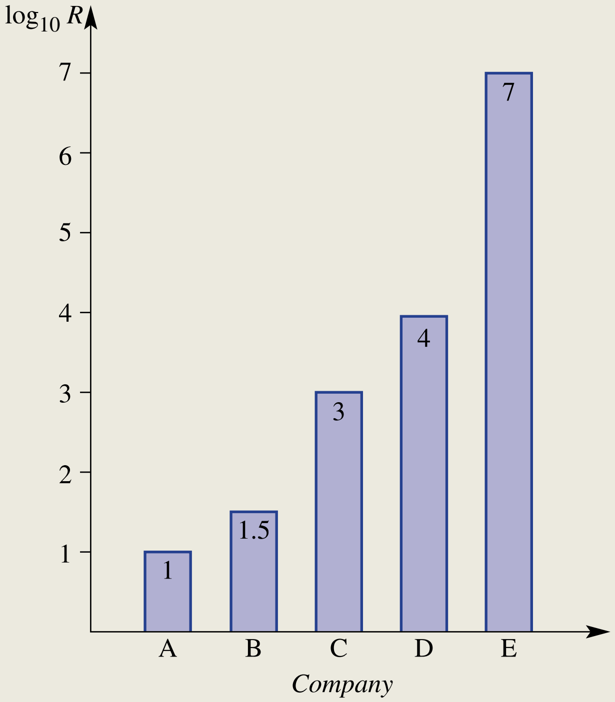
a What is the difference between the magnitude of the earthquakes for:
- **i** Papua New Guinea and Alaska?
- **ii** Fiji and Puerto Rico?

b Use your results from part **a** to state how many times more powerful the earthquake was in:
- **i** Papua New Guinea compared to Alaska
- **ii** Fiji compared to Puerto Rico.

SOLUTION	EXPLANATION
a **i** $6 - 4 = 2$ **ii** $5 - 2 = 3$	On a logarithm scale the values on the scale give the order of magnitude.
b **i** $10^2 = 100$ **ii** $10^3 = 1000$	The actual change in the strength of the earthquake is equal to 10^n where n is the difference in the order of the magnitude.

Now you try

This logarithmic chart shows the revenue, $\$R$, in millions of dollars for five Australian companies.



a What is the difference between the magnitude of the revenue for:
- **i** companies C and D?
- **ii** companies E and D?

b Use your results from part **a** to state how many times larger the revenue is for:
- **i** company C compared to company D
- **ii** company E compared to company D.

Example 28 Constructing a logarithmic graph

The following table includes data about the population of bacteria, P, in a dish over t hours.

t	0	1	2	3	4	5
P	20	200	2000	20 000	200 000	2 000 000

a Calculate the values of $\log_{10} P$ using the six different values of P given in the table. Round to one decimal place.

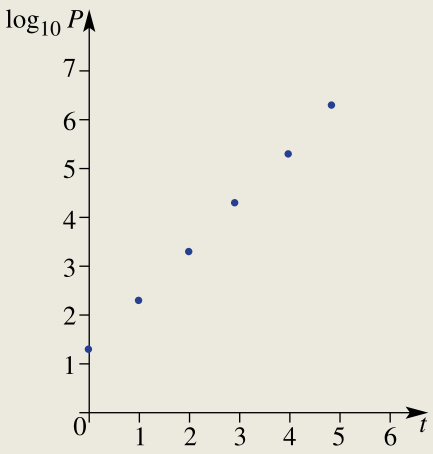
b Plot a graph of $\log_{10} P$ vs t, with t on the horizontal axis.

c Describe the shape of the graph of $\log_{10} P$ vs t. What does this say about the type of relationship between P and t?

SOLUTION

a 1.3, 2.3, 3.3, 4.3, 5.3, 6.3

b



c The graph is linear and therefore the relationship between P and t is exponential.

EXPLANATION

Use a calculator to find the value of $\log_{10} P$ for the six given values of P.

Plot $\log_{10} P$ vs t.

If the relationship between $\log_{10} P$ and t is linear, then the relationship between P and t is exponential.

Now you try

The following table includes data about the value of a company's share price, \$$S$, over t years.

t	0	1	2	3	4
S	1.2	1.3	1.45	1.675	2.0125

a Calculate the values of $\log_{10} S$ using the five different values of S given in the table. Round to two decimal places.

b Plot a graph of $\log_{10} S$ vs t, with t on the horizontal axis.

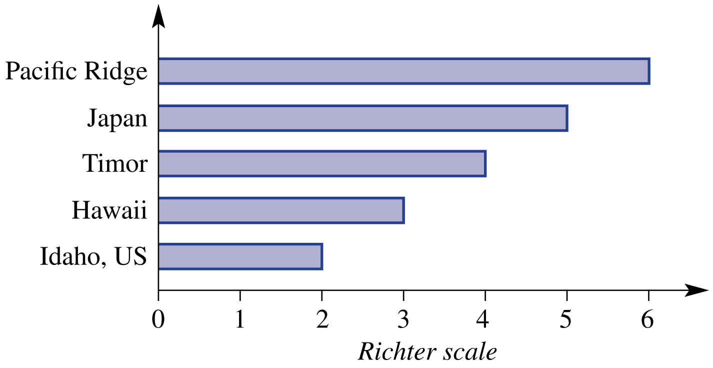
c Describe the shape of the graph of $\log_{10} S$ vs t. What does this say about the type of relationship between S and t?

Exercise 3J

FLUENCY	1–5	1, 3–5	3–5

Example 27

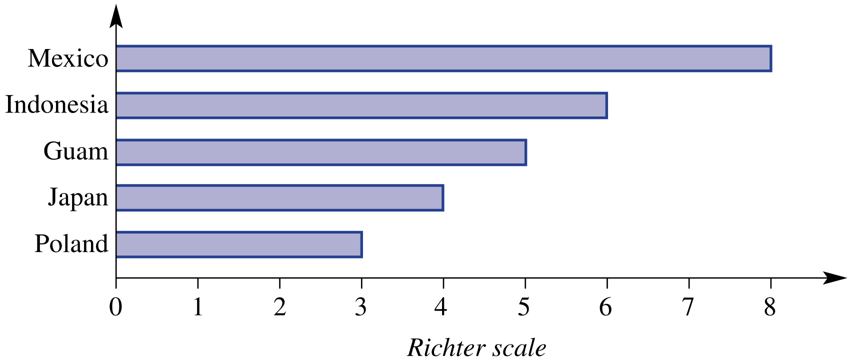
1 This logarithmic chart shows the Richter scale measurements rounded to the nearest integer for five different earthquakes recorded on one particular month in 2022.



a What is the difference between the magnitude of the earthquakes for:
  i Pacific Ridge and Timor?
  ii Japan and Idaho, US?

b Use your results from part **a** to state how many times more powerful the earthquake was in:
  i Pacific Ridge compared to Timor
  ii Japan compared to Idaho, US.

2 This logarithmic chart shows the Richter scale measurements rounded to the nearest integer for five different earthquakes recorded on one particular month in 1985.



a What is the difference between the magnitude of the earthquakes for:
  i Mexico and Guam?
  ii Indonesia and Japan?

b Use your results from part **a** to state how many times more powerful the earthquake was in:
  i Mexico compared to Guam
  ii Indonesia compared to Japan.

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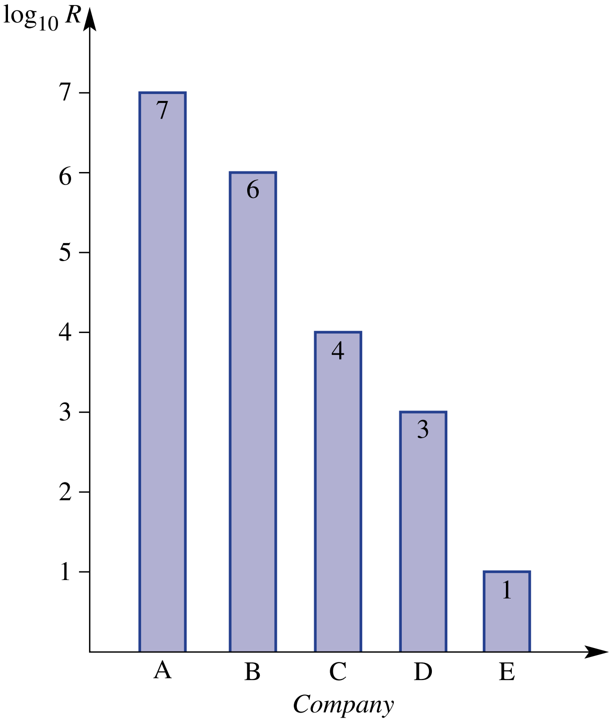
3 This logarithmic chart shows the revenue, $\$R$, in millions of dollars for five Australian companies.



a What is the difference between the magnitude of the revenue for:
  - **i** companies B and D?
  - **ii** companies A and E?

b Use your results from part **a** to state how many times larger the revenue is for:
  - **i** company B compared to company D
  - **ii** company A compared to company E.

Example 28

4 The following table includes data about the population of rabbits, P, in a district over t months.

t	0	1	2	3	4	5
P	100	400	1600	6400	25 600	102 400

a Calculate the values of $\log_{10} P$ using the six different values of P given in the table. Round to one decimal place.

b Plot a graph of $\log_{10} P$ vs t, with t on the horizontal axis.

c Describe the shape of the graph of $\log_{10} P$ vs t. What does this say about the type of relationship between P and t?

5 The following table includes data about the value of an investment, $\$A$, over t years.

t	0	1	2	3	4	5
A	10 000	12 000	14 400	17 280	20 736	24 883.2

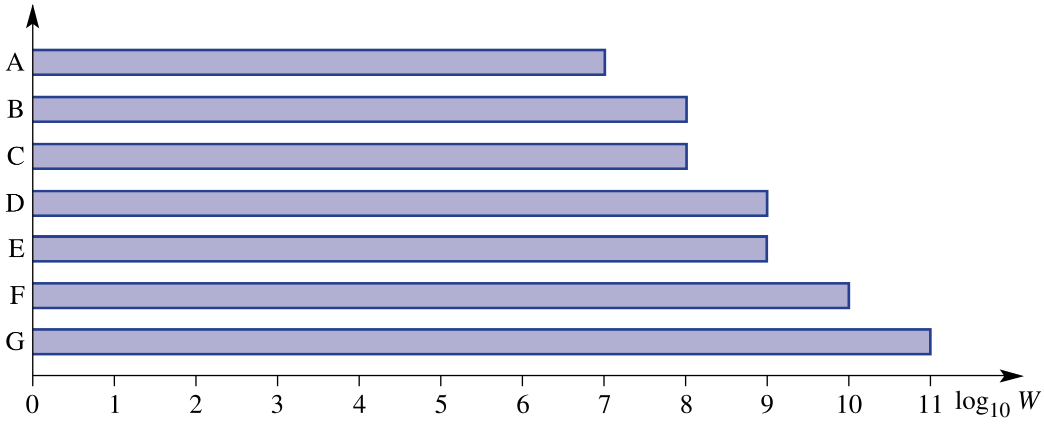
a Calculate the values of $\log_{10} A$ using the six different values of A given in the table. Round to two decimal places.

b Plot a graph of $\log_{10} A$ vs t, with t on the horizontal axis.

c Describe the shape of the graph of $\log_{10} A$ vs t. What does this say about the type of relationship between A and t?

PROBLEM–SOLVING	6	6, 7	7, 8

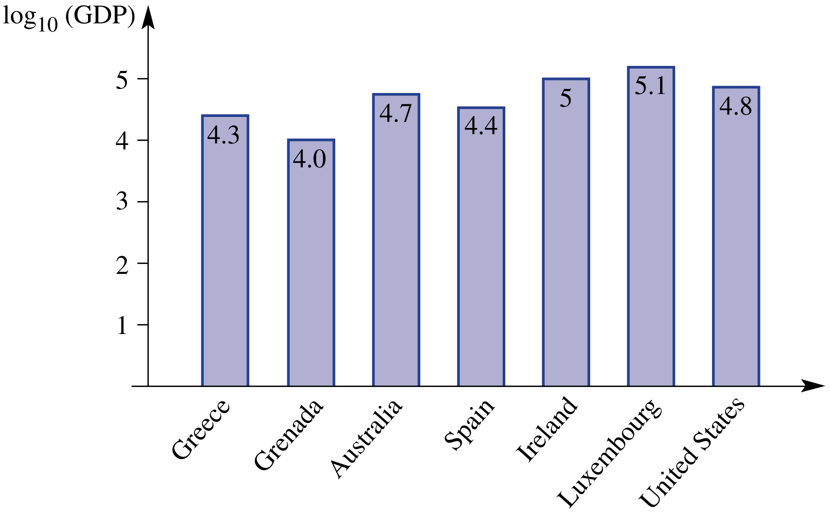
6 This graph shows the magnitude of the wealth of seven Australian families (A–G) that own a mining business as at June 2022. Let $\$W$ be the value of the wealth for the families.



a What is the difference in the magnitude of the wealth for:
- **i** family A and family E?
- **ii** family C and family G?

b What is the value of the wealth, $\$W$, for:
- **i** family A?
- **ii** family C?
- **iii** family E?
- **iv** family G?

7 The magnitude of the gross domestic product (GDP) per capita in 2021 for seven countries is given in this chart. Figures are approximate and are recorded in $US.



a What is the difference in the magnitude of the GDP per capita for:
- **i** Australia and Greece?
- **ii** Luxembourg and Grenada?

b What is the value in $US of the GDP per capita of the following countries? Round to the nearest ten thousand dollars.
- **i** Australia
- **ii** Luxembourg
- **iii** Greece
- **iv** Grenada

c What is the difference in the value of the GDP per capita in $US when comparing Australia and the United States? Round to the nearest $1000.

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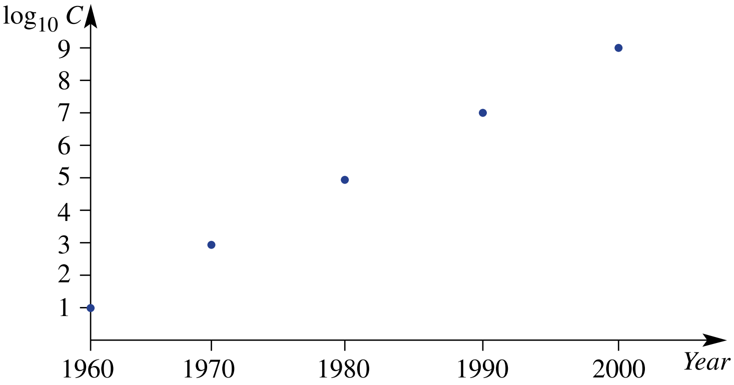
8 Using a log base 10 scale, the magnitude of the volume of water in Lake Victoria is 8 and the magnitude of the volume of water in Lake Anne is 5. If there is 500 000 ML of water in Lake Anne, how many ML are in Lake Victoria?

REASONING	9	9, 10	10, 11

9 This graph shows the magnitude of the average number of calculations per second, C, for computers for the given years starting at 1960 through to 2000.



a State the difference in the magnitude when comparing the years:
  i 1970 and 1990 **ii** 1960 and 2000

b Calculate the value of C, the average number of calculations per second, written as a power of 10 for the following years:
  i 1960 **ii** 1970 **iii** 1980 **iv** 1990 **v** 2000

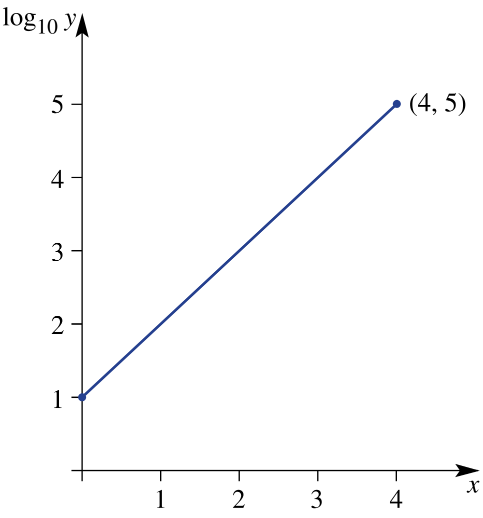
c Explain why the given graph is a straight line when in fact there is an exponential relationship between the average number of calculations per second and time.

d If this trend continued, what might be the value of $\log_{10} C$ for:
  i 2010? **ii** 2040?

10 This graph relates $\log_{10} y$ with x with rule $\log_{10} y = x + 1$.

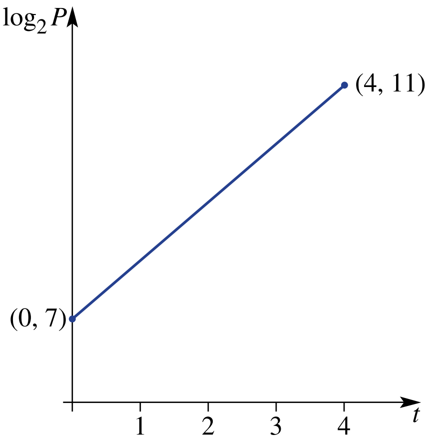
a Find the value of y if:
  i $x = 1$
  ii $x = 3$

b Find a rule for y in terms of x. Hint: If $\log_{10} a = b$ then $a = 10^b$.

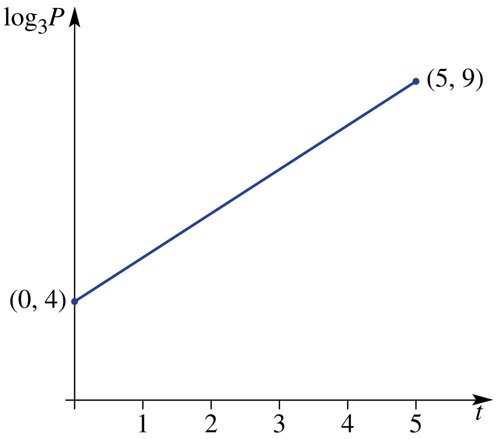


11 Find a rule for P in terms of t for these graphs which plot log P for a given base vs t.

a



b



ENRICHMENT: Exploring pH	–	–	12

12 pH is a measure of acidity and describes the concentration of hydrogen ions (H+) in a solution. The pH is calculated by taking the negative of log base 10 of H+. Higher levels of hydrogen ions lead to a lower pH (a higher level of acidity).

a Find the pH of a solution if the concentration H+ is equal to the following.

- **i** 0.001
- **ii** 0.000001
- **iii** 0.00000000001

b Pure water has a pH of 7. Find the concentration of hydrogen ions (H+) for water.

c How many more times acidic is a chemical with pH 4 compared to a chemical with pH 6?

d Research the pH of common liquids like lemon juice, vinegar, battery acid, milk and ammonia. Compare their acidity and comment on the differences in strength.

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3K Laws of logarithms OPTIONAL

LEARNING INTENTIONS

- To know how to combine logarithms with the same base using the logarithm laws for addition and subtraction
- To know properties of logarithms involving powers and the logarithm of 1
- To be able to use logarithm properties to simplify expressions

From the study of indices you will recall a number of index laws that can be used to manipulate expressions involving powers. Similarly, we have laws for logarithms and these can be derived using the index laws.

Audiologists measure the loudness of sound in decibels (dB), a logarithmic scale. Permanent hearing loss occurs after listening to 88 dB music 4 hours/day. Each 3 dB increase halves the safe time; at 100 dB hearing loss occurs in 15 minutes/day.

Recall the index law: $a^m \times a^n = a^{m+n}$

Now let $x = a^m$ and $y = a^n$ [1]

So $m = \log_a x$ and $n = \log_a y$ [2]

From equation [1] $xy = a^m \times a^n$

$= a^{m+n}$ (using the index law)

So: $m + n = \log_a (xy)$

From [2] $m + n = \log_a x + \log_a y$

So: $\log_a (xy) = \log_a x + \log_a y$

This is a proof for one of the logarithm laws and we will develop the others later in this section.

Lesson starter: Proving a logarithm law

In the introduction above there is a proof of the first logarithm law, which is considered in this section. It uses the index law for multiplication.

- Now complete a similar proof for the second logarithm law, $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$, using the index law for division.

KEY IDEAS

- $\log_a x + \log_a y = \log_a (xy)$
  - This relates to the index law: $a^m \times a^n = a^{m+n}$.
- $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$
  - This relates to the index law: $a^m \div a^n = a^{m-n}$.
- $\log_a (x^n) = n \log_a x$
  - This relates to the index law: $(a^m)^n = a^{m \times n}$.

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- Other properties of logarithms.
  - $\log_a 1 = 0$, $(a \neq 1)$ using $a^0 = 1$
  - $\log_a a = 1$, using $a^1 = a$
  - $\log_a \frac{1}{x} = \log_a x^{-1} = -\log_a x$

BUILDING UNDERSTANDING

1 Complete the rules for logarithms using the given pronumerals.

a $\log_b (xy) = \log_b x +$ ______

b $\log_b \left(\frac{x}{y}\right) =$ ______ $-$ ______

c $\log_a b^m = m \times$ ______

d $\log_a a =$ ______

e $\log_c 1 =$ ______

f $\log_a \frac{1}{b} =$ ______

2 State the missing numbers.

a $\log_2 \square = 1$

b $\log_3 \square = 0$

c $\log_a 2 + \log_a \square = \log_a 8$

d $\log_a 36 - \log_a \square = \log_a 3$

e $\log_a 3^4 = \square \times \log_a 3$

f $\square \log_a 3 = \log_a \frac{1}{3}$

3 Evaluate:

a $\log_{10} 100$

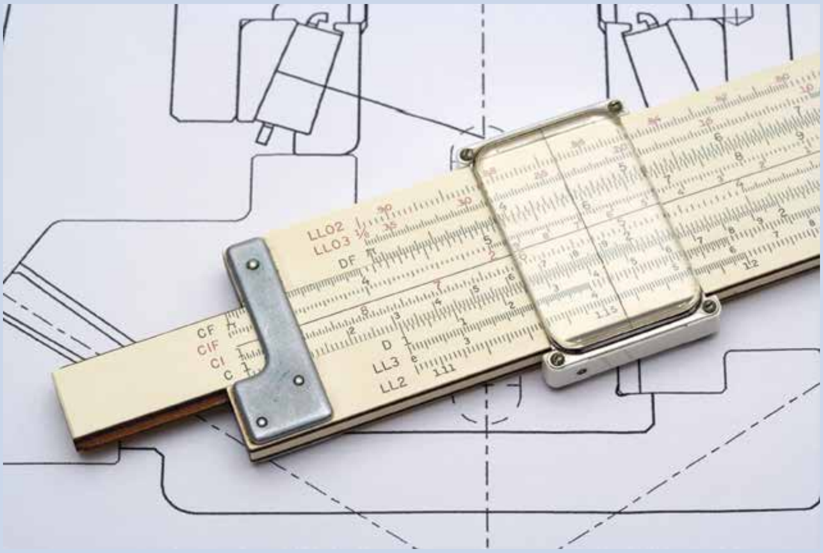
b $\log_2 32$

c $\log_3 27$

d $-2 \log_5 25$

e $4 \log_{10} 1000$

f $-6 \log_5 1$



Before the intervention of the electronic calculator, multiplication and division of numbers with many digits was done with tables of logarithms or slide rules with logarithmic scales.

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Example 29 Simplifying logarithmic expressions

Simplify the following expressing answers in the form $\log_a n$, where n is a positive integer.

a $\log_a 4 + \log_a 5$ **b** $\log_a 22 - \log_a 11$ **c** $3\log_a 2$

SOLUTION	EXPLANATION
a $\log_a 4 + \log_a 5 = \log_a 20$	This uses the law: $\log_a x + \log_a y = \log_a (xy)$
b $\log_a 22 - \log_a 11 = \log_a 2$	This uses the law: $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$ Note: $\log_a \frac{22}{11} = \log_a 2$
c $3\log_a 2 = \log_a 2^3$ $= \log_a 8$	This uses the law: $n\log_a x = \log_a x^n$

Now you try

Simplify the following expressing answers in the form $\log_a n$, where n is a positive integer.

a $\log_a 3 + \log_a 8$ **b** $\log_a 32 - \log_a 16$ **c** $2\log_a 4$

Example 30 Evaluating logarithmic expressions

Simplify and evaluate the following expressions.

a $\log_2 1$ **b** $\log_5 5$ **c** $\log_6 \frac{1}{36}$ **d** $\log_2 6 - \log_2 3$

SOLUTION	EXPLANATION
a $\log_2 1 = 0$	$2^0 = 1$
b $\log_5 5 = 1$	$5^1 = 5$
c $\log_6 \frac{1}{36} = \log_6 6^{-2}$ $= -2 \times \log_6 6$ $= -2 \times 1$ $= -2$	Alternatively, use the rule $\log_a \frac{1}{x} = -\log_a x$. So $\log_6 \frac{1}{36} = -\log_6 36$ $= -2$
d $\log_2 6 - \log_2 3 = \log_2 2$ $= 1$	$\log_2\left(\frac{6}{3}\right) = \log_2 2$ and $2^1 = 2$

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Now you try

Simplify and evaluate the following expressions.

a $\log_4 1$ **b** $\log_7 7$ **c** $\log_3 \frac{1}{9}$ **d** $\log_3 15 - \log_3 5$

Exercise 3K

FLUENCY	1–4($\frac{1}{2}$)	1–4($\frac{1}{2}$)	1–4($\frac{1}{3}$)

Example 29a

1 Simplify using the logarithm law for addition.

a $\log_a 3 + \log_a 2$ **b** $\log_a 5 + \log_a 3$ **c** $\log_a 7 + \log_a 4$

d $\log_b 6 + \log_b 3$ **e** $\log_b 15 + \log_b 1$ **f** $\log_b 1 + \log_b 17$

Example 29b

2 Simplify using the logarithm law for subtraction.

a $\log_a 10 - \log_a 5$ **b** $\log_a 36 - \log_a 12$ **c** $\log_a 100 - \log_a 10$

d $\log_b 28 - \log_b 14$ **e** $\log_b 3 - \log_b 2$ **f** $\log_b 7 - \log_b 5$

Example 29c

3 Express the following in the form $\log_a n$, where n is a positive integer.

a $2\log_a 3$ **b** $2\log_a 5$ **c** $3\log_a 3$

d $4\log_a 2$ **e** $5\log_a 2$ **f** $3\log_a 10$

Example 30a,b

4 Evaluate:

a $\log_3 1$ **b** $\log_7 1$ **c** $\log_x 1$

d $\log_4 4$ **e** $\log_{18} 18$ **f** $\log_a a$

g $5\log_2 1$ **h** $3\log_4 4$ **i** $\frac{1}{3}\log_7 7$

j $\frac{2}{3}\log_{10} 10$ **k** $\frac{\log_{15} 225}{2}$ **l** $\frac{\log_3 243}{10}$

PROBLEM–SOLVING	5–6($\frac{1}{2}$)	5–7($\frac{1}{2}$)	5–7($\frac{1}{3}$)

Example 30c

5 Simplify and evaluate.

a $\log_2 \frac{1}{4}$ **b** $\log_3 \frac{1}{27}$ **c** $\log_4 \frac{1}{64}$

d $\log_5 \frac{1}{5}$ **e** $\log_{10} \frac{1}{100}$ **f** $\log_{10} \frac{1}{100\,000}$

Example 30d

6 Simplify and evaluate.

a $\log_2 10 - \log_2 5$ **b** $\log_3 30 - \log_3 10$ **c** $\log_4 128 - \log_4 2$

d $\log_4 8 + \log_4 2$ **e** $\log_8 16 + \log_8 4$ **f** $\log_{10} 50 + \log_{10} 2$

7 Simplify using a combination of logarithmic laws.

a $2\log_3 2 + \log_3 5$ **b** $4\log_{10} 2 + \log_{10} 3$

c $3\log_{10} 2 - \log_{10} 4$ **d** $5\log_7 2 - \log_7 16$

e $\frac{1}{2}\log_3 4 + 2\log_3 2$ **f** $\log_5 3 - \frac{1}{2}\log_5 9$

g $\frac{1}{3}\log_2 27 - \frac{1}{3}\log_2 64$ **h** $\frac{1}{4}\log_5 16 + \frac{1}{5}\log_5 243$

REASONING	8	8, 9	8(½), 9, 10

8 Recall that $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x} = x^{\frac{1}{3}}$ and in general $\sqrt[n]{x} = x^{\frac{1}{n}}$. Use this to simplify the following.

a $\log_2 \sqrt{8}$ **b** $\log_2 \sqrt{32}$

c $\log_2 \sqrt[3]{16}$ **d** $\log_{10} \sqrt{1000}$

e $\log_7 \sqrt[3]{7}$ **f** $\log_5 \sqrt[5]{625}$

9 Prove that:

a $\log_a \frac{1}{x} = -\log_a x$ using the logarithm law for subtraction

b $\log_a \frac{1}{x} = -\log_a x$ using the logarithm law involving powers.

10 Prove that $\log_a \sqrt[n]{x} = \dfrac{\log_a x}{n}$.

ENRICHMENT: Proving the laws for logarithms	–	–	11

11 Read the proof for the logarithm law for addition in the introduction and then complete the following tasks.

a Complete a proof giving all reasons for the logarithm law: $\log_a (xy) = \log_a x + \log_a y$.

b Complete a proof for the logarithm law: $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.

c Complete a proof for the logarithm law: $\log_a x^n = n \log_a x$.

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3L Solving exponential equations using logarithms OPTIONAL

LEARNING INTENTIONS

- To know how to solve exponential equations by rewriting in logarithmic form using the given base
- To be able to solve an exponential equation using base 10
- To be able to use technology to evaluate logarithms

When solving a simple exponential equation like $2^x = 16$ we know that the solution is $x = 4$ because $2^4 = 16$. Solving $2^x = 10$, however, is trickier and requires the use of logarithms. Depending on what calculator functions you have, one of two different methods can be chosen. These methods can be used to solve many types of problems in science and finance.

The many applications of solving exponential equations include medical scientists calculating when a radioactive tracer has decayed; financiers determining when an investment doubles; and food scientists calculating the time for a bacteria population to reach food-poisoning levels.

Lesson starter: Trial and error versus logarithms

Consider the equation $10^x = 20$.

- First, use a calculator and trial and error to find a value of x (correct to three decimal places) that satisfies the equation.
- Now write $10^x = 20$ in logarithmic form and use the log function on your calculator to find the value of x.
- Check the accuracy of your value of x obtained by trial and error.

KEY IDEAS

- Solving for x if $a^x = y$
  - Using the given base: $x = \log_a y$
  - Using base 10:

$$\begin{aligned} a^x &= y \\ \log_{10} a^x &= \log_{10} y \quad \text{(taking } \log_{10} \text{ of both sides)} \\ x\log_{10} a &= \log_{10} y \quad \text{(using the } \log_a x^n = n\log_a x \text{ law)} \\ x &= \frac{\log_{10} y}{\log_{10} a} \quad \text{(dividing by } \log_{10} a) \end{aligned}$$

- Most calculators can evaluate using log base 10, but CAS calculators can work with any base.

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BUILDING UNDERSTANDING

1 Give the logarithmic form of these equations.

a $2^3 = 8$ **b** $4^{\frac{1}{2}} = 2$ **c** $3^x = 10$

2 State the missing number.

a $5^{\square} = 125$ **b** $10^{\square} = 10\,000$ **c** $\log_2 \square = 3$ **d** $\log_4 \square = \frac{1}{2}$

3 Use a calculator to evaluate the following, correct to three decimal places.

a $\log_{10} 7$ **b** $\log_{10} 0.6$ **c** $\log_{10} \frac{3}{4}$ **d** $\frac{\log_{10} 12}{\log_{10} 7}$

Example 31 Solving using the given base

Solve the following using the given base. Round your answer to three decimal places.

a $2^x = 7$ **b** $50 \times 1.1^x = 100$

SOLUTION	EXPLANATION
a $2^x = 7$ $x = \log_2 7$ $= 2.807$ (to 3 d.p.)	If $a^x = y$ then $x = \log_a y$. This method can be used on calculators that have a log function $\log_a y$, where both a and y can be entered.
b $50 \times 1.1^x = 100$ $1.1^x = 2$ $x = \log_{1.1} 2$ $= 7.273$ (to 3 d.p.)	Divide both sides by 50. Write in logarithmic form, then use a calculator for the approximation.

Now you try

Solve the following using the given base. Round your answer to three decimal places.

a $3^x = 10$ **b** $20 \times 1.2^x = 60$

Example 32 Solving using base 10

Solve using base 10 and evaluate, correct to three decimal places.

a $3^x = 5$ **b** $1000 \times 0.93^x = 100$

SOLUTION

a
$$\begin{aligned} 3^x &= 5 \\ \log_{10} 3^x &= \log_{10} 5 \\ x\log_{10} 3 &= \log_{10} 5 \\ x &= \frac{\log_{10} 5}{\log_{10} 3} \\ &= 1.465 \text{ (to 3 d.p.)} \end{aligned}$$

EXPLANATION

Take $\log_{10}$ of both sides.

Use the log law: $\log_a x^n = n\log_a x$.

Divide by $\log_{10} 3$.

Use the log function on a calculator.

b
$$\begin{aligned} 1000 \times 0.93^x &= 100 \\ 0.93^x &= 0.1 \\ \log_{10} 0.93^x &= \log_{10} 0.1 \\ x\log_{10} 0.93 &= \log_{10} 0.1 \\ x &= \frac{\log_{10} 0.1}{\log_{10} 0.93} \\ &= 31.729 \text{ (to 3 d.p.)} \end{aligned}$$

Divide both sides by 1000.

Take $\log_{10}$ of both sides.

Use the log law: $\log_a x^n = n\log_a x$ and solve for x by dividing both sides by $\log_{10} 0.93$.

Use the log function on a calculator.

Now you try

Solve using base 10 and evaluate, correct to three decimal places.

a $2^x = 11$ **b** $200 \times 0.85^x = 50$

Exercise 3L

FLUENCY	1–2(½)	1–3(½)	1–3(⅓)

Example 31a

1 Solve the following using the given base and round to three decimal places where necessary.

a $3^x = 5$ **b** $2^x = 11$ **c** $5^x = 13$

d $1.2^x = 3.5$ **e** $2.9^x = 3.5$ **f** $0.2^x = 0.04$

Example 31b

2 Solve the following using the given base and round to three decimal places where necessary.

a $10 \times 2^x = 20$ **b** $25 \times 3^x = 75$ **c** $4 \times 1.5^x = 20$

d $3.8 \times 1.7^x = 9.5$ **e** $300 \times 0.9^x = 150$ **f** $7.3 \times 0.4^x = 1.8$

Example 32

3 Solve using base 10 and evaluate, correct to three decimal places.

a $2^x = 6$ **b** $3^x = 8$ **c** $5^x = 7$ **d** $11^x = 15$

e $1.8^x = 2.5$ **f** $0.9^x = 0.5$ **g** $10 \times 2^x = 100$ **h** $7 \times 3^x = 28$

i $130 \times 7^x = 260$ **j** $4 \times 1.5^x = 20$ **k** $100 \times 0.8^x = 50$ **l** $30 \times 0.7^x = 20$

PROBLEM–SOLVING	4	4, 5	5, 6

4 The rule modelling a population (P) of mosquitoes is given by $P = 8^t$, where t is measured in days. Find the number of days, correct to three decimal places where necessary, required for the population to reach:

a 64 **b** 200 **c** 1000

5 An investment of \$10 000 is expected to grow by 5% p.a. so the balance \$$A$ is given by the rule $A = 10\,000 \times 1.05^n$, where n is the number of years. Find the time (to two decimal places) for the investment to grow to:

a \$20 000 **b** \$32 000 **c** \$100 000

6 50 kg of a radioactive isotope in a set of spent nuclear fuel rods is decaying at a rate of 1% per year. The mass of the isotope (m kg) is therefore given by $m = 50 \times 0.99^n$, where n is the number of years. Find the time (to two decimal places) when the mass of the isotope reduces to:

a 45 kg **b** 40 kg **c** 20 kg

REASONING	7	7, 8	8, 9

7 The value of a bank balance increases by 10% per year. The initial amount is \$2000.

a Write a rule connecting the balance \$$A$ with the time (n years).

b Find the time, correct to the nearest year, when the balance is double the original amount.

8 The value of a Ferrari is expected to reduce by 8% per year. The original cost is \$300 000.

a Find a rule linking the value of the Ferrari (\$$F$) and the time ($n$ years).

b Find the time it takes for the value of the Ferrari to reduce to \$150 000. Round your answer to one decimal place.

9 The half-life of a substance is the time it takes for the substance to reduce to half its original mass. Round answers to the nearest year.

a Find the half-life of a 10 kg rock if its mass reduces by 1% per year.

b Find the half-life of a 20 g crystal if its mass reduces by 0.05% per year.

ENRICHMENT: Change of base formula	–	–	10

10 If $a^x = y$ then we can write $x = \log_a y$. Alternatively, if $a^x = y$ we can find the logarithm of both sides, as shown here.

$$\begin{aligned} a^x &= y \\ \log_b a^x &= \log_b y \\ x\log_b a &= \log_b y \\ x &= \frac{\log_b y}{\log_b a} \\ \therefore \log_a y &= \frac{\log_b y}{\log_b a} \end{aligned}$$

This is the change of base formula.

a Use the change of base formula to write the following with base 10.

i $\log_2 7$ **ii** $\log_3 16$ **iii** $\log_5 1.3$

b Change to log base 10 and simplify.

i $\log_5 10$ **ii** $\log_2 1000$ **iii** $\log_3 0.1$

c Make x the subject and then change to base 10. Round your answer to three decimal places.

i $3^x = 6$ **ii** $9^x = 13$ **iii** $2 \times 1.3^x = 1.9$

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Luxury car investment

Mula and Will have different priorities when it comes to their luxury cars. At about the same time, Mula purchases an antique Rolls Royce for \$80 000 and Will purchases a brand-new Porsche for \$160 000.

Present a report for the following tasks and ensure that you show clear mathematical workings and explanations where appropriate.

Preliminary task

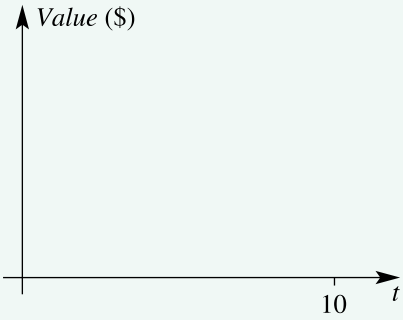
a Assuming Mula's car grows in value at 6% per year, find a rule for the value of the Rolls Royce (\$$M$) after t years.

b Find the value of Mula's car after 10 years.

c Assuming Will's car decays in value at 8% per year, find a rule for the value of the Porsche (\$$W$) after t years.

d Find the value of Will's car after 10 years.

e Plot a graph of the rules found above on the same set of axes. Use the axes as shown and prepare a table of values to help.



f Estimate when Mula's car and Will's car have the same value.

Modelling task

Formulate

a The problem is to find the time that it takes for Will's and Mula's cars to have the same value for given rates of growth and decay. Write down all the relevant information that will help solve this problem.

b Explain why Mula's car might grow in value and Will's car might decay in value.

Solve

c Choose a realistic value for the:
  - **i** growth rate of Mula's Rolls Royce
  - **ii** decay rate for Will's Porsche.

d Determine rules for the values of Mula's car and Will's car.

e Use tables and graphs to illustrate these rules for at least 10 years (use technology where appropriate).

f Estimate after how many years the cars will have the same value.

Evaluate and verify

g Calculate the value of each car using your rules at the estimated value of t found in part **f**.

h Examine your graphs and try to improve your estimate so that the values of the cars have the same value.

i Investigate how technology can be used to find a precise value of t for which the values of the cars are equal.

Communicate

j Summarise your results and describe any key findings.

Extension question

a Explore the effect on your results for different choices for the rates of growth and decay of the car's value.

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Comparing simple and compound interest

Key technology: Graphing and spreadsheets

In the world of finance, it is important to know the difference between simple and compound interest. The differences in the value of investments and loans can be very significant over the long term.

You will recall these rules for the amount A:

- Simple interest: $A = \dfrac{P \times r \times t}{100} + P$
- Compound interest: $A = P\left(1 + \dfrac{r}{100}\right)^t$

1 Getting started

Imagine investing \$100 000.

a Calculate the total value of the investment using the following simple interest terms.
  i 4% p.a. for 5 years
  ii 5% p.a. for 10 years

b Calculate the total value of the investment using the following compound interest terms.
  i 4% p.a. for 5 years
  ii 5% p.a. for 10 years

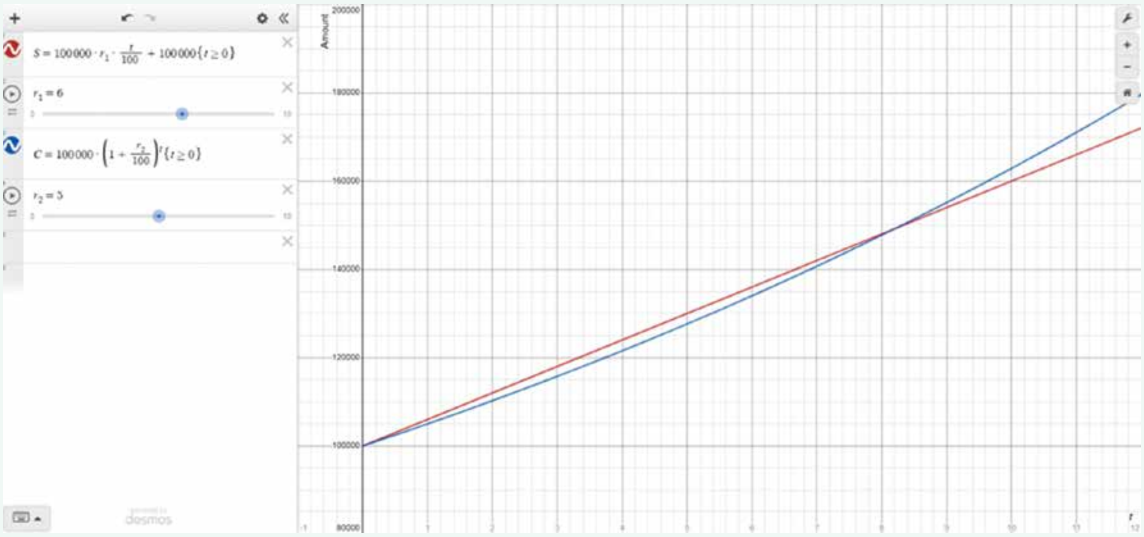
c Compare your answers from parts **a** and **b** above and describe what you notice. Can you explain why the compound interest returns are higher than simple interest returns?

2 Using technology

Two people invest \$100 000 in the following ways:

- A: Simple interest at r_1% for t years
- B: Compound interest at r_2% for t years

a Use graphing software like Desmos to construct a graph of the total value of the investments A and B on the same set of axes. Use sliders for r_1 and r_2 as shown.



$S = 100000 \cdot r_1 \cdot \frac{t}{100} + 100000\{t \geq 0\}$

$r_1 = 6$

$C = 100000 \cdot \left(1 + \frac{r_2}{100}\right)^t\{t \geq 0\}$

$r_2 = 5$

b Note in the previous example that r_1 is currently 6 and r_2 is currently 5. Drag the sliders to change the value of the interest rates and note the changes in the graphs.

c Choose a combination of r_1 and r_2 so that the values of the investments are roughly equal near the following number of years.

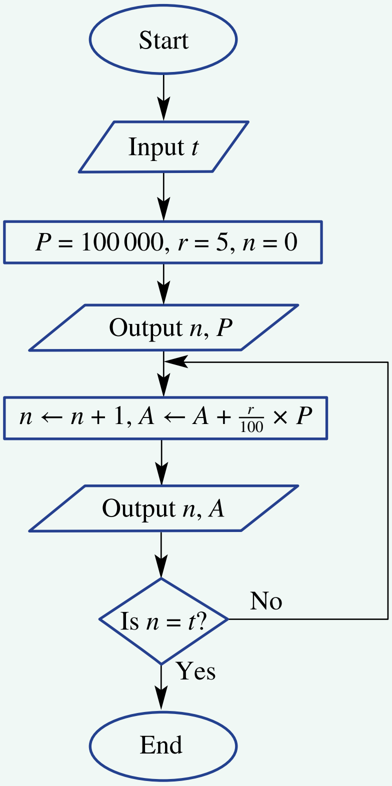
i 5 **ii** 10

d Set the compound interest rate r_2 at 4%. Drag the r_1 slider to find a simple interest rate so that the values of the investments are approximately equal after 10 years.

3 Applying an algorithm

A simple interest rate which is equivalent to a compound interest rate can be found using an algorithmic approach inside a spreadsheet.

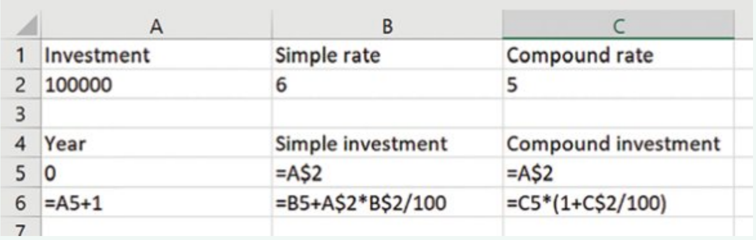
a Consider this flowchart which finds the value of a simple interest investment over t years. By choosing $t = 4$, run through the algorithm and complete this table for each pass.



n	*A*
0	100 000
1	

b Write a similar flowchart but this time for the compounding case.

c Apply these algorithms by setting up a spreadsheet like the following to compare the total value of a simple and compound interest investment of \$100 000 over t years.



	A	B	C
1	Investment	Simple rate	Compound rate
2	100000	6	5
3			
4	Year	Simple investment	Compound investment
5	0	=A\$2	=A\$2
6	=A5+1	=B5+A\$2*B\$2/100	=C5*(1+C\$2/100)
7			

d After filling down from cells in row 6 compare the values of the investments over a 12-year period. Experiment with the numbers in row 2 changing the initial investment amount and the interest rates.

e Using a \$100 000 investment and a compound interest rate of 5%, use your spreadsheet to find an equivalent simple interest rate that delivers an equal investment value after 10 years.

4 Extension

a Make modifications to your flowchart and spreadsheet so that it caters for investments where the interest is calculated on a monthly basis. Then repeat part **e** above.

b Use an algebraic method to answer part **e** above by setting up an equation and solving.

Generating wealth

Chances are that people who become wealthy have invested money in appreciating assets, such as property, shares and other businesses.

Appreciating or depreciating?

Imagine you have \$100 000 to invest or spend and you have these options.

Option 1: Invest in shares and expect a return of 8% p.a.

Option 2: Buy a car that depreciates at 8% p.a.

- **a** Find the value of the \$100 000 share investment after 10 years.
- **b** How long will it take for the share investment to double in value?
- **c** Find the value of the \$100 000 car after 10 years.
- **d** How long will it take for the value of the car to fall to half its original value?
- **e** Explain why people who want to create wealth might invest in appreciating assets.

Buying residential property

A common way to invest in Australia is to buy residential property. Imagine you have \$500 000 to buy an investment property that is expected to grow in value by 10% p.a. Stamp duty and other buying costs total \$30 000. Each year the property has costs of \$4000 (e.g. land tax, rates and insurance) and earns a rental income of \$1200 per month.

- **a** What is the initial amount you can spend on a residential property after taking into account the stamp duty and other buying costs?
- **b** What is the total net income from the property per year after annual expenses have been considered?
- **c** By considering only the property's initial capital value, find the expected value of the property after 10 years.
- **d** By taking into account the rise in value of the property and the net income, determine the total profit after 10 years.

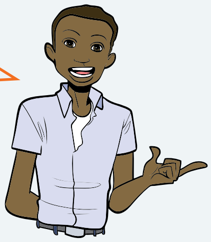
Borrowing to invest

Borrowing money to invest can increase returns but it can also increase risk. Interest has to be paid on the borrowed money, but this can be offset by the income of the investment. If there is a net loss at the end of the financial year, then negative gearing has occurred. This net loss can be used to reduce the amount of tax paid on other income, such as salary or other business income, under Australian taxation laws.

Imagine that you take out a loan of \$300 000 to add to your own \$200 000 so you can spend \$500 000 on the investment property. In summary:

- The property is expected to grow in value by 10% p.a.
- Your \$300 000 loan is interest only at 7% p.a., meaning that only interest is paid back each year and the balance remains the same.
- Property costs are \$4000 p.a.
- Rental income is \$1200 per month.
- Your taxable income tax rate is 30%.

  - **a** Find the net cash loss for the property per year. Include property costs, rent and loan interest.
  - **b** This loss reduces other income, so with a tax rate of 30% this loss is reduced by 30%. Now calculate the overall net loss, including this tax benefit.
  - **c** Now calculate the final net gain of the property investment for 10 years. You will need to find the value of the appreciating asset (which is initially \$470 000) and subtract the net loss for each year from part **b** above.



Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.

1 Write $3^{n-1} + 3^{n-1} + 3^{n-1}$ as a single term with base 3.

2 Simplify.

a $\dfrac{25^6 \times 5^4}{125^5}$ **b** $\dfrac{8^x \times 3^x}{6^x \times 9^x}$

3 Solve $3^{2x} \times 27^{x+1} = 81$.

4 Simplify.

a $\dfrac{2^{n+1} - 2^{n+2}}{2^{n-1} - 2^{n-2}}$ **b** $\dfrac{2^{a+3} - 4 \times 2^a}{2^{2a+1} - 4^a}$

5 Simplify.

a $\dfrac{x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}}}{\sqrt{xy}}$ **b** $\dfrac{x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}}}{x^{-1}y^{-1}}$

6 Given that $5^{x+1} - 5^{x-2} = 620\sqrt{5}$, find the value of x.

7 Simplify the following without the use of a calculator.

a $2\log_3 4 - \log_3 \frac{16}{9}$

b $-\log_2 \frac{1}{4} + 3\log_2 4$

c $\log_5 \sqrt{125} + \log_3 \frac{1}{3}$

d $2\log_2 27 \div \log_2 9$

8 Solve these equations using log base 10. Round your answers to two decimal places.

a $5^{x-1} = 2$ **b** $0.2^x = 10$ **c** $2^x = 3^{x+1}$

9 Solve for x: $2\log_{10} x = \log_{10}(5x + 6)$.

10 Given that $\log_a 3 = p$ and $\log_a 2 = q$, find an expression for $\log_a(4.5a^2)$.

11 Solve these inequalities using log base 10. Round your answers to two decimal places.

a $3^x > 10$ **b** $0.5^x \leqslant 7$

12 If $y = a \times 2^{bx}$ and the graph of y passes through $(-1, 2)$ and $(3, 6)$, find the exact values of a and b.

13 An amount of money is invested at 10% p.a., compound interest. How long will it take for the money to double? Give an exact value.

Chapter summary

Indices, exponentials and logarithms

Negative indices

$a^{-m} = \frac{1}{a^m}$

$\frac{1}{a^{-m}} = a^m$

Scientific notation

Numbers (large and small) in scientific notation are of the form $a \times 10^m$, where $1 \leqslant a < 10$ or $-10 < a \leqslant -1$ and m is an integer.

e.g. $3.45 \times 10^5 = 345\ 000$

$2.1 \times 10^{-3} = 0.0021$

e.g 1 475 231 in scientific notation, using three significant figures is 1.48×10^6.

Fractional indices (Opt)

$\sqrt{a} = a^{\frac{1}{2}}$

$\sqrt[n]{a} = a^{\frac{1}{n}}$

$\sqrt[n]{a^m} = a^{\frac{m}{n}}$

e.g. $27^{\frac{1}{3}} = \sqrt[3]{27}$

$= 3$

Index laws

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{mn}$

$(ab)^m = a^m b^m$

$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Zero index : $a^0 = 1,\ a \neq 0$

Logarithm definition

$\log_a y = x$ is equivalent to $a^x = y,\ a > 0$

- $\log_2 8 = 3$ since $2^3 = 8$
- $\log_{10} 100 = 2$ since $10^2 = 100$
- $\log_3 \frac{1}{27} = -3$ since $3^{-3} = \frac{1}{27}$
- $\log_{10} 0.001 = -3$ since $10^{-3} = 0.001$

Exponential equations (Opt)

If $a^x = a^y$, then $x = y$.

e.g. **1** $2^x = 16$

$2^x = 2^4$

$\therefore\ x = 4$

2 $27^{x-4} = 9^x$

$(3^3)^{x-4} = (3^2)^x$

$3^{3x-12} = 3^{2x}$

$\therefore\ 3x - 12 = 2x$

$x = 12$

Log law for addition (Opt)

$\log_a x + \log_a y = \log_a (xy)$

e.g. $\log_a 5 + \log_a 6 = \log_a 30$

Log law for subtraction (Opt)

$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

e.g. $\log_3 18 - \log_3 2 = \log_3 9$

$= 2$

Exponential growth/decay

Rule: amount $A = A_0\left(1 \pm \frac{r}{100}\right)^n$

A_0 = initial amount

r = rate of growth/decay, as a %

n = time period

Use + for growth.

Use – for decay.

Log law involving powers (Opt)

$\log_a x^n = n \log_a x$

e.g. $\log_2 8^{\frac{1}{2}} = \frac{1}{2} \log_2 8$

$= \frac{1}{2} \times 3$

$= \frac{3}{2}$

Compound interest

$A = P\left(1 + \frac{r}{100}\right)^n$

A (amount): total value

P (principal): initial amount

r (interest rate): per period

n (period): number of periods

Other properties (Opt)

- $\log_a 1 = 0 \quad (a^0 = 1)$
- $\log_a a = 1 \quad (a^1 = a)$
- $\log_a \frac{1}{x} = -\log_a x \quad \left(\frac{1}{x} = x^{-1}\right)$

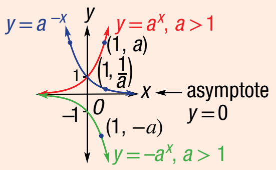
Logarithmic scales

These are used to help visualise the spread of data over a wide range of values.

A logarithmic graph uses the logarithm of a quantity on at least one axis.

Variables connected by an exponential relationship can be represented as a linear relationship using logarithms.

Exponential graphs



Solving exponential equations (Opt) using logarithms

$2^x = 5$

$x = \log_2 5$

or

$2^x = 5$

$\log_{10} 2^x = \log_{10} 5$

$x \log_{10} 2 = \log_{10} 5$

$x = \frac{\log_{10} 5}{\log_{10} 2}$

Chapter checklist with success criteria

A printable version of this checklist is available in the Interactive Textbook

		✓
3A	**1. I can apply index laws to multiply common bases.** e.g. Simplify $3x^2y^3 \times 2xy^4$.	☐
3A	**2. I can apply index laws in division.** e.g. Simplify $3x^5y^3 \div (12x^2y)$.	☐
3A	**3. I can apply index laws involving brackets.** e.g. Simplify $(2m^3)^2$ and $\left(\frac{x^2}{y}\right)^4$.	☐
3A	**4. I can simplify using a number of index laws.** e.g. Simplify $3(x^2y)^3 \times \left(\frac{2}{x}\right)^2$ using index laws.	☐
3A	**5. I can use the zero index.** e.g. Evaluate $(3a)^0 - 5a^0$.	☐
3B	**6. I can rewrite an expression using positive indices.** e.g. Express $3x^2y^{-3}$ using positive indices.	☐
3B	**7. I can rewrite an expression with a negative power in the denominator using positive indices.** e.g. Express $\frac{3}{y^{-4}}$ using positive indices.	☐
3B	**8. I can simplify expressions and apply index laws to negative indices.** e.g. Simplify $\frac{(x^{-1}y)^{-3}}{4x^{-2}y^3}$ and express using positive indices.	☐
3C	**9. I can convert from scientific notation to a basic numeral.** e.g. Write 3.07×10^4 and 4.1×10^{-3} as basic numerals.	☐
3C	**10. I can convert to scientific notation using significant figures.** e.g. Write 0.0035892 in scientific notation using three significant figures.	☐
3C	**11. I can use technology to perform calculations in scientific notation.** e.g. Evaluate $\sqrt{3.02 \times 10^{24}}$, answering in scientific notation using three significant figures.	☐
3D	**12. I can write roots in index form.** Opt e.g. Express $\sqrt[3]{x^6}$ in index form.	☐
3D	**13. I can write rational indices in surd form.** Opt e.g. Express $x^{\frac{3}{2}}$ in surd form.	☐
3D	**14. I can evaluate numbers with rational indices.** Opt e.g. Evaluate $25^{-\frac{1}{2}}$ without a calculator.	☐
3E	**15. I can solve exponential equations using a common base.** Opt e.g. Solve $3^x = 27$ for x.	☐

Chapter checklist with success criteria

		✓
3E	**16. I can solve exponential equations that require a common base to be found.** Opt e.g. Solve $25^x = 125^{x-2}$.	☐
3F	**17. I can sketch a graph of an exponential equation.** e.g. Sketch $y = 3^x$ labelling the y-intercept and one other point.	☐
3F	**18. I can sketch exponential graphs involving reflections.** e.g. Sketch $y = 2^x$, $y = -2^x$ and $y = 2^{-x}$ on the same axes.	☐
3F	**19. I can find the intersection of horizontal lines and exponential graphs.** e.g. Find the intersection of the graphs of $y = 3^x$ and $y = 27$.	☐
3G	**20. I can form an exponential rule for a situation.** e.g. Write an exponential rule for the value of Scott's car purchased for \$35 000 and decreasing in value by 15% per year.	☐
3G	**21. I can apply exponential rules.** e.g. The value of a house in n years' time is given by $V = 590\,000(1.06)^n$. Determine the value in 4 years' time and find when it will be valued at \$1 000 000.	☐
3H	**22. I can calculate compound interest using the formula.** e.g. Determine the amount after 3 years if \$6000 is compounded annually at 4%. Round to the nearest cent.	☐
3H	**23. I can convert rates and time periods.** e.g. Calculate the number of time periods and the rate of interest offered per period for 3% p.a. over 5 years paid monthly.	☐
3H	**24. I can find compounded amounts using different time periods.** e.g. An investment of \$2000 is compounded at 4.8% over 3 years. Determine the amount after 3 years if the interest is paid monthly.	☐
3I	**25. I can convert between index form and logarithmic form.** e.g. Write an equivalent statement in index form for $\log_2 8 = 3$ and in logarithm form for $3^4 = 81$.	☐
3I	**26. I can evaluate a logarithm.** e.g. Evaluate $\log_4 64$.	☐
3I	**27. I can evaluate a logarithm using a calculator.** e.g. Evaluate $\log_{10} 8$ correct to three decimal places.	☐
3I	**28. I can solve a simple logarithmic equation.** e.g. Find the value of x in the equation $\log_3 x = 5$.	☐

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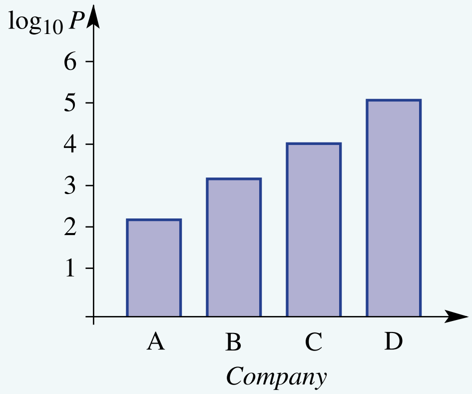
Chapter checklist with success criteria

✓

3J **29. I can interpret a logarithmic chart.** ☐

e.g. This logarithmic chart shows the profit, \$$P$, in millions of dollars for four companies.
Determine how many times larger the profit of company D compared to company B is by first finding the difference between the magnitude of the profit.



3J **30. I can construct a logarithmic graph.** ☐

e.g. The following table shows the value \$$A$ of an investment over t years.

t	0	1	2	3	4	5
A	1000	1100	1210	1331	1464.1	1610.51

Plot a graph of $\log_{10} A$ vs t, with t on the horizontal axis, and use the shape of the graph to comment on the type of relationship between A and t.

3K **31. I can apply laws of logarithms.** Opt ☐

e.g. Simplify $\log_2 5 + \log_2 3$ and $\log_2 18 - \log_2 6$.

3K **32. I can evaluate logarithmic expressions by simplifying.** Opt ☐

e.g. Simplify and evaluate $\log_5 10 - \log_5 2 + \log_7 1$.

3L **33. I can solve an exponential equation using logarithms with the given base.** Opt ☐

e.g. Solve $3^x = 20$ correct to three decimal places using the given base.

3L **34. I can solve exponential equations using base 10.** Opt ☐

e.g. Solve $2 \times 1.1^x = 6$ using base 10 and correct to three decimal places.

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Short-answer questions

3A/B

1 Simplify the following, expressing all answers with positive indices when required.

a $(5y^3)^2$ **b** $7m^0 - (5n)^0$ **c** $4x^2y^3 \times 5x^5y^7$

d $3x^2y^{-4}$ **e** $\left(\frac{3x}{y^{-3}}\right)^2 \times \frac{x^{-5}}{6y^2}$ **f** $\frac{3(a^2b^{-4})^2}{(2ab^2)^2} \div \frac{(ab)^{-2}}{(3a^{-2}b)^2}$

3C

2 **a** Write the following numbers as a basic numeral.

i 3.21×10^3 **ii** 4.024×10^6 **iii** 7.59×10^{-3} **iv** 9.81×10^{-5}

b Write the following numbers in scientific notation, using three significant figures.

i 0.0003084 **ii** 0.0000071753

iii 5 678 200 **iv** 119 830 000

3D Opt

3 Express in index form.

a $\sqrt{21}$ **b** $\sqrt[3]{x}$ **c** $\sqrt[3]{m^5}$

d $\sqrt[3]{2a^9b}$ **e** $7\sqrt{7}$

3D Opt

4 Evaluate these without using a calculator.

a $25^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$ **c** $49^{-\frac{1}{2}}$

d $100^{-\frac{1}{2}}$ **e** $125^{\frac{2}{3}}$ **f** $10000^{\frac{3}{4}}$

3E Opt

5 Solve the following exponential equations for x.

a $3^x = 27$ **b** $7^x = 49$ **c** $4^{2x+1} = 64$

d $9^x = \frac{1}{81}$ **e** $5^x = \frac{1}{125}$ **f** $36^x = 216$

g $7^{3x-4} = 49^x$ **h** $100^{x-2} = 1000^x$ **i** $9^{3-2x} = 27^{x+2}$

3F

6 Sketch the following graphs, labelling the y-intercept and the point where $x = 1$.

a $y = 4^x$ **b** $y = -3^x$ **c** $y = 5^{-x}$

3G

7 Form exponential rules for the following situations.

a An antique bought for \$800 is expected to grow in value by 7% per year.

b A balloon with volume 3000 cm^3 is leaking air at a rate of 18% per minute.

3H

8 Determine the final amount after 4 years if:

a \$1000 is compounded annually at 5%

b \$3000 is compounded monthly at 4%

c \$5000 is compounded daily at 3%.

3I

9 Write the following in logarithmic form.

a $2^4 = 16$ **b** $10^3 = 1000$ **c** $3^{-2} = \frac{1}{9}$

3I

10 Write the following in index form.

a $\log_3 81 = 4$ **b** $\log_4 \frac{1}{16} = -2$ **c** $\log_{10} 0.1 = -1$

3I

11 Evaluate the following.

a $\log_{10} 1000$ **b** $\log_3 81$ **c** $\log_7 1$

d $\log_3 \frac{1}{27}$ **e** $\log_5 \frac{1}{125}$ **f** $\log_4 0.25$

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3J

12 The following table includes data about the population of mice, P, on a farm after t months.

t	0	1	2	3	4	5
P	30	60	120	240	480	960

a Calculate the values of $\log_{10} P$ from the table correct to one decimal place and plot a graph of $\log_{10} P$ vs t, with t on the horizontal axis.

b Consider the shape of $\log_{10} P$ vs t and state what this says about the type of relationship between P and t.

3K **Opt**

13 Simplify using the laws for logarithms.

a $\log_a 4 + \log_a 2$
b $\log_b 7 + \log_b 3$
c $\log_a 1000 - \log_a 100$
d $2\log_a 2$
e $\log_3 60 - \log_3 20$
f $\log_2 \sqrt{8}$

3L **Opt**

14 Solve these equations using logarithms with the given base.

a $3^x = 6$
b $20 \times 1.2^x = 40$

3L **Opt**

15 Solve for x, in exact form, using base 10.

a $2^x = 13$
b $100 \times 0.8^x = 200$

Multiple-choice questions

3A

1 The simplified form of $4a^3b^2 \times 2b^{10}$ is:

A $8a^3b^{12}$
B $128a^3b^{12}$
C $8a^3b^9$
D $40a^3b^9$
E $8a^3b^{20}$

3A

2 The simplified form of $\dfrac{(6xy^3)^2}{3x^3y^2 \times 4x^4y^0}$ is:

A $\dfrac{y^4}{2x^6}$
B $\dfrac{3y^3}{x^{10}}$
C $\dfrac{y^6}{x^5}$
D $\dfrac{3y^4}{x^5}$
E $\dfrac{y^6}{2x^6}$

3B

3 $\dfrac{8a^{-1}b^{-2}}{12a^3b^{-5}}$ expressed with positive indices is:

A $\dfrac{2a^2}{3b^3}$
B $\dfrac{a^2b^3}{96}$
C $\dfrac{2b^3}{3a^4}$
D $-\dfrac{2b^7}{3a^2}$
E $\dfrac{3}{2}a^4b^7$

3C

4 The radius of Earth is approximately 6 378 137 m. In scientific notation, using three significant figures, this is:

A 6.378×10^6 m
B 6.38×10^6 m
C 6.4×10^5 m
D 6.37×10^6 m
E 6.36×10^6 m

3D **Opt**

5 $\sqrt{8x^6}$ in index form is:

A $8x^3$
B $8x^2$
C $4x^3$
D $8^{\frac{1}{2}}x^3$
E $8^{\frac{1}{2}}x^4$

3E **Opt**

6 The solution to $3^{2x-1} = 9^2$ is:

A $x = \dfrac{3}{2}$
B $x = 2$
C $x = \dfrac{5}{2}$
D $x = 6$
E $x = 3$

3F

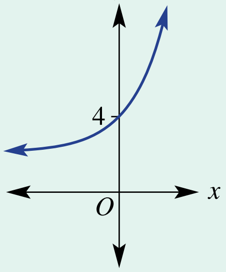
7 The graph of $y = 3^x$ intersects the y-axis at:

A $(0, 3)$
B $(3, 0)$
C $(0, 1)$
D $(1, 3)$
E $\left(0, \frac{1}{3}\right)$

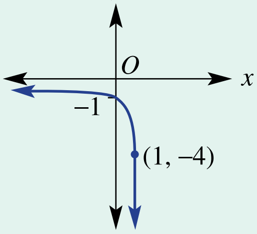
3F

8 The graph of $y = 4^{-x}$ is:

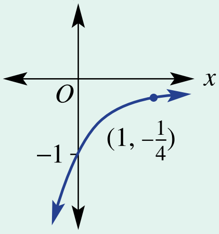
A



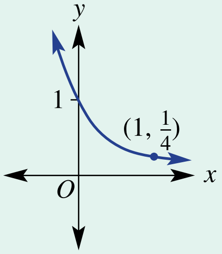
B



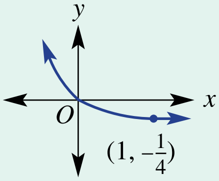
C



D



E



3F

9 The graph of $y = 3^x$ and $y = \frac{1}{3}$ intersect at the point:

A $(1, 3)$
B $\left(-1, \frac{1}{3}\right)$
C $(-1, 3)$
D $\left(\frac{1}{9}, \frac{1}{3}\right)$
E $\left(1, \frac{1}{3}\right)$

3G

10 A rule for the amount \$$A$ in an account after n years for an initial investment of \$5000 that is increasing at 7%, per annum is:

A $A = 5000(1.7)^n$
B $A = 5000(0.93)^n$
C $A = 5000(0.3)^n$
D $A = 5000(1.07)^n$
E $A = 5000(0.7)^n$

3I

11 Which of the following is equivalent to $5^3 = 125$?

A $\log_3 125 = 5$
B $\log_3 5 = 125$
C $\log_5 125 = 3$
D $125^3 = 5$
E $\log_{125} 3 = 5$

3I

12 If $\log_2 64 = x$, then x is equal to:

A 5
B 6
C 32
D 128
E 64^2

3J

13 A logarithmic chart shows the acidity level of two elements A and B to be 6 and 4. How many times more acidic is element B compared to element A?

A 2
B 20
C 100
D 40
E 200

3K
Opt

14 $\log_6 \frac{1}{6}$ simplifies to:

A -1
B 1
C 36
D 6
E 0

Extended-response questions

1 Georgia invests \$10 000 in shares in a new company. She has been told that their value is expected to increase at 6.5% per year.

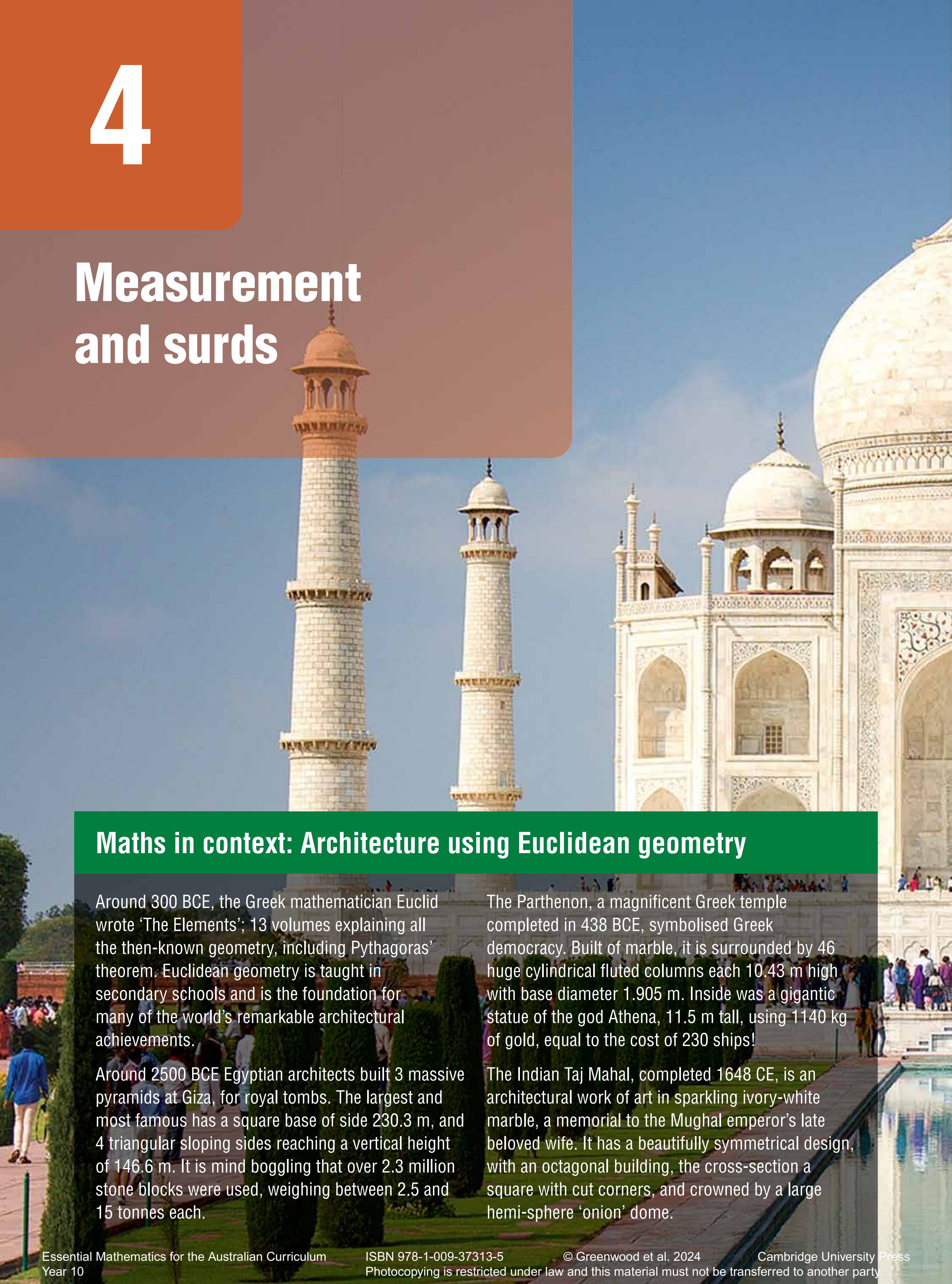
- **a** Write a rule for Georgia's expected value, V dollars, in shares after n years.
- **b** Use your rule to find the value she expects the shares to be after:
  - **i** 2 years
  - **ii** 5 years
- **c** When her shares are valued at \$20 000 Georgia plans to cash them in. According to this rule, how many years will it take to reach this amount? Give your answer to one decimal place.
- **d** After 6 years there is a downturn in the market and the shares start to drop, losing value at 3% per year.
  - **i** What is the value of Georgia's shares prior to the downturn in the market? Give your answer to the nearest dollar.
  - **ii** Using your answer from part **d i**, write a rule for the expected value, V dollars, of Georgia's shares t years after the market downturn.
  - **iii** Ten years after Georgia initially invested in the shares the market is still falling at this rate. She decides it's time to sell her shares. What is their value, to the nearest dollar? How does this compare with the original amount of \$10 000 she invested?

2 Sound is measured in decibels, dB, with rule given by $d = 10\log_{10}\frac{P}{P_0}$ where P is the power or intensity of the sound measured in watts/cm^2 and P_0 is the weakest sound that the human ear can hear in watts/cm^2.

- **a** Use the rule with $P_0 = 10^{-16}$ to find:
  - **i** the sound in decibels when P is 10^{-4} (the maximum intensity the human ear can tolerate before experiencing pain)
  - **ii** the intensity of sound at a rock concert when the sound is recorded as 100 decibels
  - **iii** what 0 decibels represents in terms of the power P.
- **b** A sound is being recorded at different time intervals in a suburban street. Over the course of the day the sound ranges from 50 dB to 70 dB. Use $P_0 = 10^{-16}$.
  - **i** Find the range of the intensity P throughout the day.
  - **ii** Describe the change in the intensity range compared to the change in decibel range.
- **c** Two speakers emit sound intensity power of P_1 and P_2 where $P_2 > P_1$.
  - **i** Give a simplified rule, c, for the difference in decibels between the two speakers.
  - **ii** If speaker 2 emits 100 times the power of speaker 1, what is the difference in decibels?

4

Measurement and surds



Maths in context: Architecture using Euclidean geometry

Around 300 BCE, the Greek mathematician Euclid wrote ‘The Elements’; 13 volumes explaining all the then-known geometry, including Pythagoras’ theorem. Euclidean geometry is taught in secondary schools and is the foundation for many of the world’s remarkable architectural achievements.

Around 2500 BCE Egyptian architects built 3 massive pyramids at Giza, for royal tombs. The largest and most famous has a square base of side 230.3 m, and 4 triangular sloping sides reaching a vertical height of 146.6 m. It is mind boggling that over 2.3 million stone blocks were used, weighing between 2.5 and 15 tonnes each.

The Parthenon, a magnificent Greek temple completed in 438 BCE, symbolised Greek democracy. Built of marble, it is surrounded by 46 huge cylindrical fluted columns each 10.43 m high with base diameter 1.905 m. Inside was a gigantic statue of the god Athena, 11.5 m tall, using 1140 kg of gold, equal to the cost of 230 ships!

The Indian Taj Mahal, completed 1648 CE, is an architectural work of art in sparkling ivory-white marble, a memorial to the Mughal emperor’s late beloved wife. It has a beautifully symmetrical design, with an octagonal building, the cross-section a square with cut corners, and crowned by a large hemi-sphere ‘onion’ dome.

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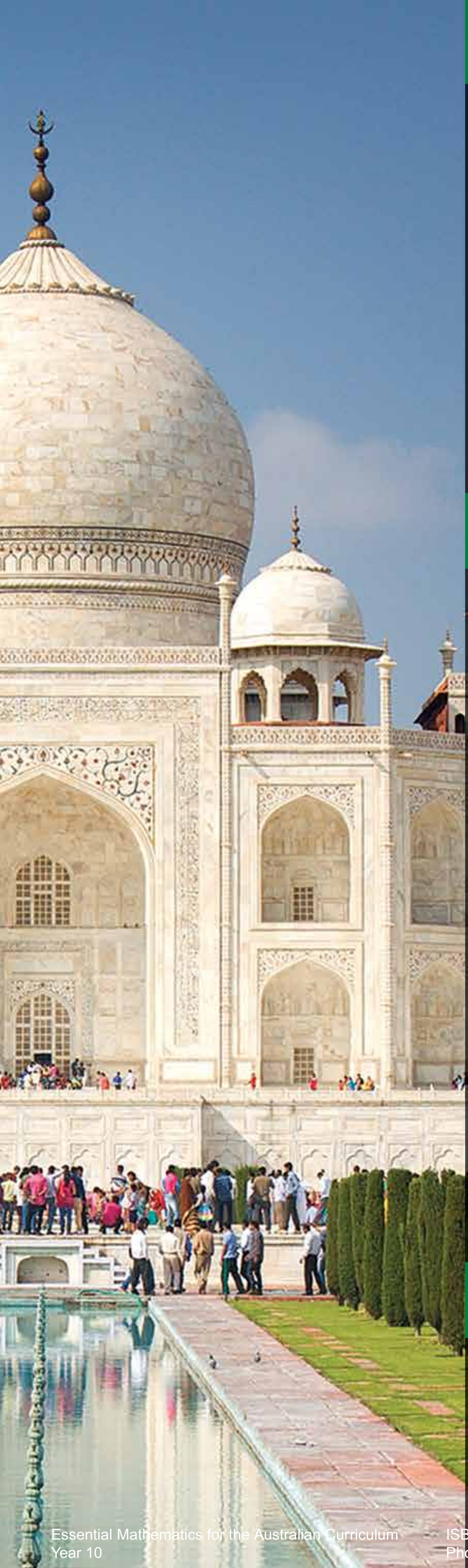


Chapter contents

Australian Curriculum 9.0

NUMBER

Recognise the effect of using approximations of real numbers in repeated calculations and compare the results when using exact representations (AC9M10N01)

Operations on numbers involving fractional exponents and surds (Year 10 optional content)

MEASUREMENT

Solve problems involving the surface area and volume of composite objects using appropriate units (AC9M10M01)

Solve practical problems applying Pythagoras' theorem and trigonometry of right-angled triangles, including problems involving direction and angles of elevation and depression (AC9M10M03)

Identify the impact of measurement errors on the accuracy of results in practical contexts (AC9M10M04)

Use mathematical modelling to solve practical problems involving proportion and scaling of objects; formulate problems and interpret solutions in terms of the situation; evaluate and modify models as necessary, and report assumptions, methods and findings (AC9M10M05)

Online resources

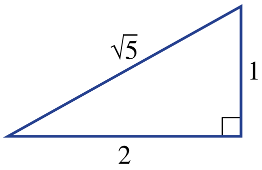
A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

4A Irrational numbers including surds OPTIONAL

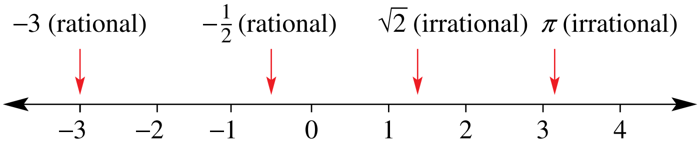
LEARNING INTENTIONS

- To know the meaning of the terms rational number, irrational number and surd
- To know how to identify a number as rational or irrational
- To know simple rules related to surds
- To be able to simplify surds using the highest square number factor

You will recall that when using Pythagoras' theorem to find unknown lengths in right-angled triangles, many answers expressed in exact form are surds. The length of the hypotenuse in this triangle, for example, is $\sqrt{5}$, which is a surd.



A surd is a number that uses a root sign $(\sqrt{})$, sometimes called a radical sign. They are irrational numbers, meaning that they cannot be expressed as a fraction in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Surds, together with other irrational numbers such as pi (π), and all rational numbers (fractions) make up the entire set of real numbers, which can be illustrated as points on a number line.



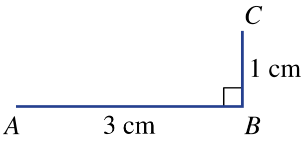
Many formulas contain numbers that are surds. The formulas for the speed of a rising weather balloon and the speed of its falling measuring device both include the surd $\sqrt{2}$.

Lesson starter: Constructing surds

Someone asks you: 'How do you construct a line that is $\sqrt{10}$ cm long?'

Use these steps to answer this question.

- First, draw a line segment AB that is 3 cm in length.
- Construct segment BC so that $BC = 1$ cm and $AB \perp BC$. You may wish to use a set square or pair of compasses.
- Now connect point A and point C and measure the length of the segment.
- Use Pythagoras' theorem to check the length of AC.



Use this idea to construct line segments with the following lengths. You may need more than one triangle for parts d to f.

a $\sqrt{2}$
b $\sqrt{17}$
c $\sqrt{20}$
d $\sqrt{3}$
e $\sqrt{6}$
f $\sqrt{22}$

KEY IDEAS

- All **real** numbers can be located as a point on a number line. Real numbers include:
  - **rational numbers** (i.e. numbers that can be expressed as fractions)
    For example: $\frac{3}{7}, -\frac{4}{39}, -3, 1.6, 2.\dot{7}, 0.\overline{19}$.
    The decimal representation of a rational number is either a **terminating** or **recurring decimal**.
  - **irrational numbers** (i.e. numbers that cannot be expressed as fractions)
    For example: $\sqrt{3}, -2\sqrt{7}, \sqrt{12}-1, \pi, 2\pi-3$
    The decimal representation of an irrational number is an **infinite non-recurring decimal**.
- **Surds** are irrational numbers that use a root sign $(\sqrt{\ })$.
  - For example: $\sqrt{2}, 5\sqrt{11}, -\sqrt{200}, 1+\sqrt{5}$
  - These numbers are not surds: $\sqrt{4}\,(=2), \sqrt[3]{125}\,(=5), -\sqrt[4]{16}\,(=-2)$.
- The nth root of a number x is written $\sqrt[n]{x}$.
  - If $\sqrt[n]{x}=y$ then $y^n=x$. For example: $\sqrt[5]{32}=2$ since $2^5=32$.
- The following rules apply to surds.
  - $(\sqrt{x})^2=x$ and $\sqrt{x^2}=x$ when $x \geqslant 0$.
  - $\sqrt{xy}=\sqrt{x}\times\sqrt{y}$ when $x \geqslant 0$ and $y \geqslant 0$.
  - $\sqrt{\frac{x}{y}}=\frac{\sqrt{x}}{\sqrt{y}}$ when $x \geqslant 0$ and $y > 0$.
- $\sqrt{x+y} \neq \sqrt{x}+\sqrt{y}$ unless x and/or y equal 0.
- When a factor of a number is a perfect square we call that factor a square factor. Examples of perfect squares are: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …
- When simplifying surds, look for square factors of the number under the root sign and then use $\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$.

BUILDING UNDERSTANDING

1 Choose the correct word(s) from the words given in orange to make the sentence true.

a A number that cannot be expressed as a fraction is a rational/irrational number.
b A surd is an irrational number that uses a root/square symbol.
c The decimal representation of a surd is a terminating/recurring/non-recurring decimal.
d $\sqrt{25}$ is a surd/rational number.

2 State the highest square factor of these numbers. For example, the highest square factor of 45 is 9.

a 20 **b** 125 **c** 48 **d** 72

Example 1 Defining and locating surds

Express each number as a decimal and decide if it is rational or irrational. Then locate all the numbers on the same number line.

a $-\sqrt{3}$ **b** 137% **c** $\frac{3}{7}$

SOLUTION

a $-\sqrt{3} = -1.732050807\ldots$

$-\sqrt{3}$ is irrational.

b $137\% = \frac{137}{100} = 1.37$

137% is rational.

c $\frac{3}{7} = 0.\overline{428571}$

$\frac{3}{7}$ is rational.

EXPLANATION

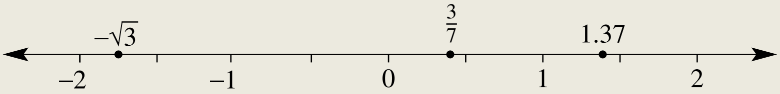
Use a calculator to express as a decimal.

The decimal does not terminate and there is no recurring pattern.

137% is a fraction and can be expressed using a terminating decimal.

$\frac{3}{7}$ is an infinitely recurring decimal.

Use the decimal equivalents to locate each number on the real number line.



Now you try

Express each number as a decimal and decide if they are rational or irrational. Then locate all the numbers on the same number line.

a $-\sqrt{5}$ **b** -40% **c** $\frac{2}{7}$

Example 2 Simplifying surds

Simplify the following.

a $\sqrt{32}$ **b** $3\sqrt{200}$ **c** $\frac{5\sqrt{40}}{6}$ **d** $\sqrt{\frac{75}{9}}$

SOLUTION

a
$$\begin{aligned}\sqrt{32} &= \sqrt{16 \times 2}\\ &= \sqrt{16} \times \sqrt{2}\\ &= 4\sqrt{2}\end{aligned}$$

EXPLANATION

When simplifying, choose the highest square factor of 32 (i.e. 16 rather than 4) as there is less work to do to arrive at the same answer.

Compare with

$\sqrt{32} = \sqrt{4 \times 8} = 2\sqrt{8} = 2\sqrt{4 \times 2} = 2 \times 2\sqrt{2} = 4\sqrt{2}$

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b $3\sqrt{200} = 3\sqrt{100 \times 2}$ $= 3 \times \sqrt{100} \times \sqrt{2}$ $= 3 \times 10 \times \sqrt{2}$ $= 30\sqrt{2}$	Select the appropriate factors of 200 by finding its highest square factor: 100. Use $\sqrt{x \times y} = \sqrt{x} \times \sqrt{y}$ and simplify.
c $\frac{5\sqrt{40}}{6} = \frac{5\sqrt{4 \times 10}}{6}$ $= \frac{5 \times \sqrt{4} \times \sqrt{10}}{6}$ $= \frac{\cancel{10}^{5}\sqrt{10}}{\cancel{6}^{3}}$ $= \frac{5\sqrt{10}}{3}$	Select the appropriate factors of 40. The highest square factor is 4. Cancel and simplify.
d $\sqrt{\frac{75}{9}} = \frac{\sqrt{75}}{\sqrt{9}}$ $= \frac{\sqrt{25 \times 3}}{\sqrt{9}}$ $= \frac{5\sqrt{3}}{3}$	Use $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$. Then select the factors of 75 that include a square number and simplify.

Now you try

Simplify the following.

a $\sqrt{20}$ **b** $2\sqrt{300}$ **c** $\frac{2\sqrt{27}}{3}$ **d** $\sqrt{\frac{125}{16}}$

Example 3 Expressing as a single square root of a positive integer

Express these surds as a square root of a positive integer.

a $2\sqrt{5}$ **b** $7\sqrt{2}$

SOLUTION	EXPLANATION
a $2\sqrt{5} = \sqrt{4} \times \sqrt{5}$ $= \sqrt{20}$	Write 2 as $\sqrt{4}$ and then combine the two surds using $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$.
b $7\sqrt{2} = \sqrt{49} \times \sqrt{2}$ $= \sqrt{98}$	Write 7 as $\sqrt{49}$ and combine.

Now you try

Express these surds as a square root of a positive integer.

a $3\sqrt{2}$ **b** $5\sqrt{3}$

Exercise 4A

FLUENCY 1–5(½) 1–6(⅓) 1–6(¼)

Example 1

1 Express each number as a decimal and decide if it is rational or irrational. Then locate all the numbers on the same number line.

a $\sqrt{5}$ **b** 18% **c** $\frac{2}{5}$ **d** -124%
e $1\frac{5}{7}$ **f** $-\sqrt{2}$ **g** $2\sqrt{3}$ **h** π

2 Decide if these numbers are surds.

a $\sqrt{7}$ **b** $2\sqrt{11}$ **c** $2\sqrt{25}$ **d** $-5\sqrt{144}$
e $\frac{3\sqrt{9}}{2}$ **f** $\frac{-5\sqrt{3}}{2}$ **g** $1 - \sqrt{3}$ **h** $2\sqrt{1} + \sqrt{4}$

Example 2a

3 Simplify the following surds.

a $\sqrt{12}$ **b** $\sqrt{45}$ **c** $\sqrt{24}$ **d** $\sqrt{48}$
e $\sqrt{75}$ **f** $\sqrt{500}$ **g** $\sqrt{98}$ **h** $\sqrt{90}$
i $\sqrt{128}$ **j** $\sqrt{360}$ **k** $\sqrt{162}$ **l** $\sqrt{80}$

Example 2b,c

4 Simplify the following.

a $2\sqrt{18}$ **b** $3\sqrt{20}$ **c** $4\sqrt{48}$ **d** $2\sqrt{63}$
e $3\sqrt{98}$ **f** $4\sqrt{125}$ **g** $\frac{\sqrt{45}}{3}$ **h** $\frac{\sqrt{28}}{2}$
i $\frac{\sqrt{24}}{4}$ **j** $\frac{\sqrt{54}}{12}$ **k** $\frac{\sqrt{80}}{20}$ **l** $\frac{\sqrt{99}}{18}$
m $\frac{3\sqrt{44}}{2}$ **n** $\frac{5\sqrt{200}}{25}$ **o** $\frac{2\sqrt{98}}{7}$ **p** $\frac{3\sqrt{68}}{21}$
q $\frac{6\sqrt{75}}{20}$ **r** $\frac{4\sqrt{150}}{5}$ **s** $\frac{2\sqrt{108}}{18}$ **t** $\frac{3\sqrt{147}}{14}$

Example 2d

5 Simplify the following.

a $\sqrt{\frac{8}{9}}$ **b** $\sqrt{\frac{12}{49}}$ **c** $\sqrt{\frac{18}{25}}$ **d** $\sqrt{\frac{11}{25}}$
e $\sqrt{\frac{10}{9}}$ **f** $\sqrt{\frac{12}{144}}$ **g** $\sqrt{\frac{26}{32}}$ **h** $\sqrt{\frac{25}{50}}$
i $\sqrt{\frac{15}{27}}$ **j** $\sqrt{\frac{27}{4}}$ **k** $\sqrt{\frac{45}{72}}$ **l** $\sqrt{\frac{56}{76}}$

Example 3

6 Express these surds as a square root of a positive integer.

a $2\sqrt{3}$ **b** $4\sqrt{2}$ **c** $5\sqrt{2}$ **d** $3\sqrt{3}$
e $3\sqrt{5}$ **f** $6\sqrt{3}$ **g** $8\sqrt{2}$ **h** $10\sqrt{7}$
i $9\sqrt{10}$ **j** $5\sqrt{5}$ **k** $7\sqrt{5}$ **l** $11\sqrt{3}$

PROBLEM–SOLVING	7, 8	7, 8, 10(½)	8, 9, 10(½)

7 Determine the exact side length, in simplest form, of a square with the given area.

a 32 m^2 **b** 120 cm^2 **c** 240 mm^2

8 Determine the exact radius and diameter of a circle, in simplest form, with the given area. Recall that the area of a circle is πr^2.

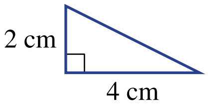
a 24π cm^2 **b** 54π m^2 **c** 128π m^2

9 Simplify by searching for the highest square factor.

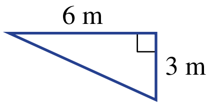
a $\sqrt{675}$
b $\sqrt{1183}$
c $\sqrt{1805}$
d $\sqrt{2883}$

10 Use Pythagoras' theorem to find the unknown length in these triangles, in simplest form.

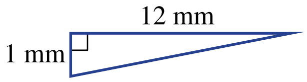
a



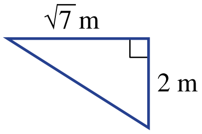
b



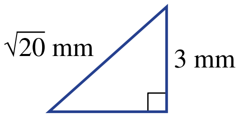
c



d



e



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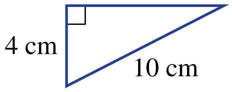

REASONING	11	11, 12	12, 13

11 Ricky uses the following working to simplify $\sqrt{72}$. Show how Ricky could have simplified $\sqrt{72}$ using fewer steps.

$$\begin{aligned}\sqrt{72} &= \sqrt{9\times 8}\\ &= 3\sqrt{8}\\ &= 3\sqrt{4\times 2}\\ &= 3\times 2\times\sqrt{2}\\ &= 6\sqrt{2}\end{aligned}$$

12 **a** List all the factors of 450 that are perfect squares.
b Now simplify $\sqrt{450}$ using the highest of these factors.

13 Use Pythagoras' theorem to construct a line segment with the given lengths. You can use only a ruler and a set square or compasses. Do not use a calculator.

a $\sqrt{10}$ cm
b $\sqrt{29}$ cm
c $\sqrt{6}$ cm
d $\sqrt{22}$ cm

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ENRICHMENT: Proving that $\sqrt{2}$ is irrational

– – 14

14 We will prove that $\sqrt{2}$ is irrational by the method called 'proof by contradiction'. Your job is to follow and understand the proof, then copy it out and try explaining it to a friend or teacher.

a Before we start, we first need to show that if a perfect square a^2 is even then a is even. We do this by showing that if a is even then a^2 is even and if a is odd then a^2 is odd.

If a is even then $a = 2k$, where k is an integer.

$$\begin{aligned} \text{So } a^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2 \times 2k^2, \text{ which must be even.} \end{aligned}$$

If a is odd then $a = 2k + 1$, where k is an integer.

$$\begin{aligned} \text{So } a^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2 \times (2k^2 + 2k) + 1, \text{ which must be odd.} \end{aligned}$$

$\therefore$ If a^2 is even then a is even.

b Now, to prove $\sqrt{2}$ is irrational let's suppose that $\sqrt{2}$ is instead rational and can be written in the form $\frac{a}{b}$ in simplest form, where a and b are integers $(b \neq 0)$ and at least one of a or b is odd.

$$\begin{aligned} \therefore \sqrt{2} &= \frac{a}{b} \\ \text{So} \quad 2 &= \frac{a^2}{b^2} \text{ (squaring both sides)} \\ a^2 &= 2b^2 \end{aligned}$$

$\therefore a^2$ is even and, from part **a** above, a must be even.

If a is even, then $a = 2k$, where k is an integer.

$$\begin{aligned} \therefore \text{ If } a^2 &= 2b^2 \\ \text{Then } (2k)^2 &= 2b^2 \\ 4k^2 &= 2b^2 \\ 2k^2 &= b^2 \end{aligned}$$

$\therefore b^2$ is even and therefore b is even.

This is a contradiction because at least one of a or b must be odd. (Recall that $\frac{a}{b}$ in simplest form will have at least one of a or b being odd.) Therefore, the assumption that $\sqrt{2}$ can be written in the form $\frac{a}{b}$ must be incorrect and so $\sqrt{2}$ is irrational.

4B Adding and subtracting surds OPTIONAL

LEARNING INTENTIONS

- To understand that only like surds can be combined under addition and subtraction
- To know how to add and subtract like surds
- To know that it is helpful to simplify all surds before determining if they can be added or subtracted

We can apply our knowledge of like terms in algebra to help simplify expressions involving the addition and subtraction of surds. Recall that $7x$ and $3x$ are like terms, so $7x + 3x = 10x$. The pronumeral x represents any number. When $x = 5$ then $7 \times 5 + 3 \times 5 = 10 \times 5$, and when $x = \sqrt{2}$ then $7\sqrt{2} + 3\sqrt{2} = 10\sqrt{2}$. Multiples of the same surd are called 'like surds' and can be collected (i.e. counted) in the same way as we collect like terms in algebra.

To design the Hearst Tower in New York, architects solved many equations, such a linear, quadratic and trigonometric. Where possible, architects use surds in mathematical solutions to achieve precise results.

Lesson starter: Can $3\sqrt{2} + \sqrt{8}$ be simplified?

To answer this question, first discuss these points.

- Are $3\sqrt{2}$ and $\sqrt{8}$ like surds?
- How can $\sqrt{8}$ be simplified?
- Now decide whether $3\sqrt{2} + \sqrt{8}$ can be simplified. Discuss why $3\sqrt{2} - \sqrt{7}$ cannot be simplified.

KEY IDEAS

- **Like surds** are multiples of the same surd.
  For example: $\sqrt{3}, -5\sqrt{3}, \sqrt{12} = 2\sqrt{3}, 2\sqrt{75} = 10\sqrt{3}$
- Like surds can be added and subtracted.
- Simplify all surds before attempting to add or subtract them.

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BUILDING UNDERSTANDING

① Decide if the following pairs of numbers are like surds.

a $\sqrt{3}, 2\sqrt{3}$ **b** $5, \sqrt{5}$ **c** $2\sqrt{2}, 2$ **d** $4\sqrt{6}, \sqrt{6}$

e $2\sqrt{3}, 5\sqrt{3}$ **f** $3\sqrt{7}, 3\sqrt{5}$ **g** $-2\sqrt{5}, 3\sqrt{5}$ **h** $-\sqrt{7}, -2\sqrt{7}$

② Recall your basic skills in algebra to simplify these expressions.

a $11x - 5x$ **b** $2x - 7x$ **c** $-4a + 21a$ **d** $4t - 5t + 2t$

③ **a** Simplify the surd $\sqrt{48}$.

b Hence, simplify the following.

i $\sqrt{3} + \sqrt{48}$ **ii** $\sqrt{48} - 7\sqrt{3}$ **iii** $5\sqrt{48} - 3\sqrt{3}$

Example 4 Adding and subtracting surds

Simplify the following.

a $2\sqrt{3} + 4\sqrt{3}$ **b** $4\sqrt{6} + 3\sqrt{2} - 3\sqrt{6} + 2\sqrt{2}$

SOLUTION	EXPLANATION
a $2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$	Collect the like surds by adding the coefficients: $2 + 4 = 6$.
b $4\sqrt{6} + 3\sqrt{2} - 3\sqrt{6} + 2\sqrt{2} = \sqrt{6} + 5\sqrt{2}$	Collect like surds involving $\sqrt{6}$: $4\sqrt{6} - 3\sqrt{6} = 1\sqrt{6} = \sqrt{6}$ Then collect those terms with $\sqrt{2}$.

Now you try

Simplify the following.

a $2\sqrt{5} + 3\sqrt{5}$ **b** $3\sqrt{7} + 2\sqrt{3} - 2\sqrt{7} + 5\sqrt{3}$

Example 5 Simplifying surds to add or subtract

Simplify these surds.

a $5\sqrt{2} - \sqrt{8}$ **b** $2\sqrt{5} - 3\sqrt{20} + 6\sqrt{45}$

SOLUTION	EXPLANATION
a $\begin{aligned} 5\sqrt{2} - \sqrt{8} &= 5\sqrt{2} - \sqrt{4 \times 2} \\ &= 5\sqrt{2} - 2\sqrt{2} \\ &= 3\sqrt{2} \end{aligned}$	First, look to simplify surds: $\sqrt{8}$ has a highest square factor of 4 and can be simplified to $2\sqrt{2}$. Then subtract like surds.

b $2\sqrt{5} - 3\sqrt{20} + 6\sqrt{45} = 2\sqrt{5} - 3\sqrt{4 \times 5} + 6\sqrt{9 \times 5}$
$= 2\sqrt{5} - 6\sqrt{5} + 18\sqrt{5}$
$= 14\sqrt{5}$

Simplify the surds and then collect like surds. Note that:
$3\sqrt{4 \times 5} = 3 \times \sqrt{4} \times \sqrt{5} = 6\sqrt{5}$.

Now you try

Simplify these surds.

a $7\sqrt{2} - \sqrt{8}$ **b** $2\sqrt{3} - 2\sqrt{27} + 3\sqrt{12}$

Exercise 4B

FLUENCY	1–3(½)	1–3(½)	1–3(⅓)

Example 4a

1 Simplify the following.

a $2\sqrt{5} + 4\sqrt{5}$ **b** $5\sqrt{3} + 2\sqrt{3}$ **c** $7\sqrt{2} - 3\sqrt{2}$
d $8\sqrt{2} - 5\sqrt{2}$ **e** $7\sqrt{5} + 4\sqrt{5}$ **f** $6\sqrt{3} - 5\sqrt{3}$
g $4\sqrt{10} + 3\sqrt{10} - \sqrt{10}$ **h** $6\sqrt{2} - 4\sqrt{2} + 3\sqrt{2}$ **i** $\sqrt{21} - 5\sqrt{21} + 2\sqrt{21}$
j $3\sqrt{11} - 8\sqrt{11} - \sqrt{11}$ **k** $-2\sqrt{13} + 5\sqrt{13} - 4\sqrt{13}$ **l** $10\sqrt{30} - 15\sqrt{30} - 2\sqrt{30}$

Example 4b

2 Simplify the following.

a $2\sqrt{3} + 3\sqrt{2} - \sqrt{3} + 2\sqrt{2}$ **b** $5\sqrt{6} + 4\sqrt{11} - 2\sqrt{6} + 3\sqrt{11}$
c $3\sqrt{5} - 4\sqrt{2} + \sqrt{5} - 3\sqrt{2}$ **d** $5\sqrt{2} + 2\sqrt{5} - 7\sqrt{2} - \sqrt{5}$
e $2\sqrt{3} + 2\sqrt{7} + 2\sqrt{3} - 2\sqrt{7}$ **f** $5\sqrt{11} + 3\sqrt{6} - 3\sqrt{6} - 5\sqrt{11}$
g $2\sqrt{2} - 4\sqrt{10} - 5\sqrt{2} + \sqrt{10}$ **h** $-4\sqrt{5} - 2\sqrt{15} + 5\sqrt{15} + 2\sqrt{5}$

Example 5a

3 Simplify the following.

a $\sqrt{8} - \sqrt{2}$ **b** $\sqrt{8} + 3\sqrt{2}$ **c** $\sqrt{27} + \sqrt{3}$
d $\sqrt{20} - \sqrt{5}$ **e** $4\sqrt{18} - 5\sqrt{2}$ **f** $2\sqrt{75} + 2\sqrt{3}$
g $3\sqrt{44} + 2\sqrt{11}$ **h** $3\sqrt{8} - \sqrt{18}$ **i** $\sqrt{24} + \sqrt{54}$
j $2\sqrt{125} - 3\sqrt{45}$ **k** $3\sqrt{72} + 2\sqrt{98}$ **l** $3\sqrt{800} - 4\sqrt{200}$

PROBLEM–SOLVING	4(½)	4(½), 6(½)	4–6(⅓)

Example 5b

4 Simplify the following.

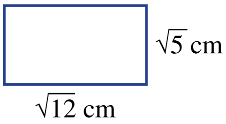
a $\sqrt{2} + \sqrt{50} + \sqrt{98}$ **b** $\sqrt{6} - 2\sqrt{24} + 3\sqrt{96}$
c $5\sqrt{7} + 2\sqrt{5} - 3\sqrt{28}$ **d** $2\sqrt{80} - \sqrt{45} + 2\sqrt{63}$
e $7\sqrt{3} - 2\sqrt{8} + \sqrt{12} + 3\sqrt{8}$ **f** $\sqrt{12} + \sqrt{125} - \sqrt{50} + \sqrt{180}$
g $\sqrt{150} - \sqrt{96} - \sqrt{162} + \sqrt{72}$ **h** $\sqrt{36} - \sqrt{108} + \sqrt{25} - 3\sqrt{3}$
i $3\sqrt{49} + 2\sqrt{288} - \sqrt{144} - 2\sqrt{18}$ **j** $2\sqrt{200} + 3\sqrt{125} + \sqrt{32} - 3\sqrt{242}$

5 Simplify these surds that involve fractions. Remember to use the LCD (lowest common denominator).

a $\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3}$ **b** $\frac{\sqrt{5}}{4} + \frac{\sqrt{5}}{3}$ **c** $\frac{\sqrt{2}}{5} - \frac{\sqrt{2}}{6}$
d $\frac{\sqrt{7}}{4} - \frac{\sqrt{7}}{12}$ **e** $\frac{2\sqrt{2}}{5} - \frac{\sqrt{2}}{2}$ **f** $\frac{3\sqrt{3}}{7} + \frac{\sqrt{3}}{2}$
g $\frac{7\sqrt{5}}{6} - \frac{4\sqrt{5}}{9}$ **h** $\frac{3\sqrt{3}}{10} - \frac{8\sqrt{3}}{15}$ **i** $\frac{-5\sqrt{10}}{6} + \frac{3\sqrt{10}}{8}$

6 Find the perimeter of these rectangles and triangles, in simplest form. Recall that $(\sqrt{x})^2 = x$.

a



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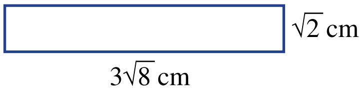

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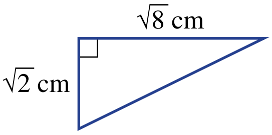

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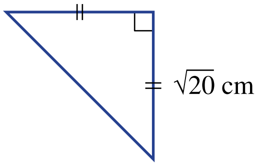

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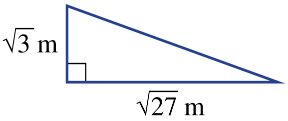
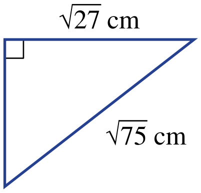


f



REASONING	7	7, 8(½)	8(½), 9

7 **a** Explain why $\sqrt{5}$ and $\sqrt{20}$ can be thought of as like surds.
b Explain why $3\sqrt{72}$ and $\sqrt{338}$ can be thought of as like surds.

8 Prove that each of the following simplifies to zero by showing all steps.

a $5\sqrt{3} - \sqrt{108} + \sqrt{3}$
b $\sqrt{6} + \sqrt{24} - 3\sqrt{6}$
c $6\sqrt{2} - 2\sqrt{32} + 2\sqrt{2}$
d $\sqrt{8} - \sqrt{18} + \sqrt{2}$
e $2\sqrt{20} - 7\sqrt{5} + \sqrt{45}$
f $3\sqrt{2} - 2\sqrt{27} - \sqrt{50} + 6\sqrt{3} + \sqrt{8}$

9 Prove that the surds in these expressions cannot be added or subtracted.

a $3\sqrt{12} - \sqrt{18}$
b $4\sqrt{8} + \sqrt{20}$
c $\sqrt{50} - 2\sqrt{45}$
d $5\sqrt{40} + 2\sqrt{75}$
e $2\sqrt{200} + 3\sqrt{300}$
f $\sqrt{80} - 2\sqrt{54}$

ENRICHMENT: Simplifying both surds and fractions	–	–	10(½)

10 To simplify the following, you will need to simplify surds and combine using a common denominator.

a $\frac{\sqrt{8}}{3} - \frac{\sqrt{2}}{5}$
b $\frac{\sqrt{12}}{4} + \frac{\sqrt{3}}{6}$
c $\frac{3\sqrt{5}}{4} - \frac{\sqrt{20}}{3}$
d $\frac{\sqrt{98}}{4} - \frac{5\sqrt{2}}{2}$
e $\frac{2\sqrt{75}}{5} - \frac{3\sqrt{3}}{2}$
f $\frac{\sqrt{63}}{9} - \frac{4\sqrt{7}}{5}$
g $\frac{2\sqrt{18}}{3} - \frac{\sqrt{72}}{2}$
h $\frac{\sqrt{54}}{4} + \frac{\sqrt{24}}{7}$
i $\frac{\sqrt{27}}{5} - \frac{\sqrt{108}}{10}$
j $\frac{5\sqrt{48}}{6} + \frac{2\sqrt{147}}{3}$
k $\frac{2\sqrt{96}}{5} - \frac{\sqrt{600}}{7}$
l $\frac{3\sqrt{125}}{14} - \frac{2\sqrt{80}}{21}$

4C Multiplying and dividing surds OPTIONAL

LEARNING INTENTIONS

- To know how to multiply and divide surds
- To understand that, by definition, $\sqrt{x} \times \sqrt{x}$ is equal to x and that this can be helpful in simplifying multiplications
- To be able to apply the distributive law to brackets involving surds

When simplifying surds such as $\sqrt{18}$, we write $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$, where we use the fact that $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$. This can be used in reverse to simplify the product of two surds. A similar process is used for division.

A surd represents an accurate value until approximated with a decimal. Surveyor training includes solving problems where the trigonometry ratios are expressed as surds because these are exact values and give accurate results.

Lesson starter: Exploring products and quotients

When adding and subtracting surds we can combine like surds only. Do you think this is true for multiplying and dividing surds?

- Use a calculator to find a decimal approximation for $\sqrt{5} \times \sqrt{3}$ and for $\sqrt{15}$.
- Use a calculator to find a decimal approximation for $2\sqrt{10} \div \sqrt{5}$ and for $2\sqrt{2}$.
- What do you notice about the results from above? Try other pairs of surds to see if your observations are consistent.

KEY IDEAS

- When multiplying surds, use the following result.
  - $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$
  - More generally: $a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$
- When dividing surds, use the following result.
  - $\dfrac{\sqrt{x}}{\sqrt{y}} = \sqrt{\dfrac{x}{y}}$
  - More generally: $\dfrac{a\sqrt{x}}{b\sqrt{y}} = \dfrac{a}{b}\sqrt{\dfrac{x}{y}}$
- Use the distributive law to expand brackets.
  - $\overset{\frown}{a(b + c)} = ab + ac$

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BUILDING UNDERSTANDING

1 State the missing parts.

a $\sqrt{15} \div \sqrt{3} = \sqrt{\dfrac{15}{\_\_\_}}$
$= \sqrt{\_\_\_}$

b $\sqrt{42} \div \sqrt{7} = \sqrt{\dfrac{42}{\_\_\_}}$
$= \sqrt{\_\_\_}$

c $\sqrt{6} \times \sqrt{5} = \sqrt{6 \times \_\_\_}$
$= \sqrt{\_\_\_}$

d $\sqrt{11} \times \sqrt{2} = \sqrt{11 \times \_\_\_}$
$= \sqrt{\_\_\_}$

2 Use the definition of squares and square roots to simplify the following.

a $\sqrt{6} \times \sqrt{6}$ **b** $\sqrt{7^2}$ **c** $(\sqrt{5})^2$

3 Expand the brackets.

a $2(x + 3)$ **b** $5(2x - 1)$ **c** $6(5 - 4x)$

Example 6 Simplifying a product of two surds

Simplify the following.

a $\sqrt{2} \times \sqrt{3}$ **b** $2\sqrt{3} \times 3\sqrt{15}$ **c** $(2\sqrt{5})^2$

SOLUTION	EXPLANATION
a $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3}$ $= \sqrt{6}$	Use $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$.
b $2\sqrt{3} \times 3\sqrt{15} = 2 \times 3 \times \sqrt{3 \times 15}$ $= 6\sqrt{45}$ $= 6\sqrt{9 \times 5}$ $= 6 \times \sqrt{9} \times \sqrt{5}$ $= 18\sqrt{5}$	Use $a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$. Then simplify the surd $\sqrt{45}$, which has a highest square factor of 9, using $\sqrt{9} = 3$. Alternatively, using $\sqrt{15} = \sqrt{3} \times \sqrt{5}$: $2\sqrt{3} \times 3\sqrt{15} = 2 \times 3 \times \sqrt{3} \times \sqrt{3} \times \sqrt{5}$ $= 2 \times 3 \times 3 \times \sqrt{5}$ $= 18\sqrt{5}$
c $(2\sqrt{5})^2 = 2\sqrt{5} \times 2\sqrt{5}$ $= 4 \times 5$ $= 20$	Recall that $a^2 = a \times a$. Combine the whole numbers and surd components by multiplying $2 \times 2 = 4$ and $\sqrt{5} \times \sqrt{5} = 5$.

Now you try

Simplify the following.

a $\sqrt{5} \times \sqrt{3}$ **b** $3\sqrt{2} \times 4\sqrt{6}$ **c** $(3\sqrt{7})^2$

Example 7 Simplifying surds using division

Simplify these surds.

a $-\sqrt{10} \div \sqrt{2}$

b $\dfrac{12\sqrt{18}}{3\sqrt{3}}$

SOLUTION

a $$\begin{aligned}-\sqrt{10} \div \sqrt{2} &= -\sqrt{\frac{10}{2}}\\ &= -\sqrt{5}\end{aligned}$$

b $$\begin{aligned}\frac{12\sqrt{18}}{3\sqrt{3}} &= \frac{12}{3}\sqrt{\frac{18}{3}}\\ &= 4\sqrt{6}\end{aligned}$$

EXPLANATION

Use $\sqrt{x} \div \sqrt{y} = \sqrt{\frac{x}{y}}$.

Use $\frac{a\sqrt{x}}{b\sqrt{y}} = \frac{a}{b}\sqrt{\frac{x}{y}}$.

Now you try

Simplify these surds.

a $-\sqrt{15} \div \sqrt{5}$

b $\dfrac{14\sqrt{22}}{7\sqrt{11}}$

Example 8 Using the distributive law

Use the distributive law to expand the following and then simplify the surds where necessary.

a $\sqrt{3}(3\sqrt{5} - \sqrt{6})$

b $3\sqrt{6}(2\sqrt{10} - 4\sqrt{6})$

SOLUTION

a $$\begin{aligned}\sqrt{3}(3\sqrt{5} - \sqrt{6}) &= 3\sqrt{15} - \sqrt{18}\\ &= 3\sqrt{15} - \sqrt{9 \times 2}\\ &= 3\sqrt{15} - 3\sqrt{2}\end{aligned}$$

b $$\begin{aligned}3\sqrt{6}(2\sqrt{10} - 4\sqrt{6}) &= 6\sqrt{60} - 12 \times 6\\ &= 6\sqrt{4 \times 15} - 72\\ &= 12\sqrt{15} - 72\end{aligned}$$

EXPLANATION

Expand the brackets $\sqrt{3} \times 3\sqrt{5} = 3\sqrt{15}$ and $\sqrt{3} \times \sqrt{6} = \sqrt{18}$. Simplify $\sqrt{18}$. (or $\sqrt{3} \times \sqrt{6} = \sqrt{3} \times \sqrt{3} \times \sqrt{2} = 3\sqrt{2}$).

Expand the brackets and simplify the surds. Recall that $\sqrt{6} \times \sqrt{6} = 6$ and $\sqrt{4 \times 15} = 2\sqrt{15}$.

Now you try

Use the distributive law to expand the following and then simplify the surds where necessary.

a $\sqrt{2}(5\sqrt{3} - \sqrt{7})$

b $5\sqrt{3}(2\sqrt{6} - 3\sqrt{3})$

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Exercise 4C

FLUENCY 1–7(½) 1–7(½) 1–7(⅓)

Example 6a

1 Simplify the following.

a $\sqrt{3} \times \sqrt{5}$ **b** $\sqrt{7} \times \sqrt{3}$ **c** $\sqrt{2} \times \sqrt{13}$
d $\sqrt{5} \times \sqrt{7}$ **e** $\sqrt{2} \times (-\sqrt{15})$ **f** $-\sqrt{6} \times \sqrt{5}$
g $-\sqrt{6} \times (-\sqrt{11})$ **h** $-\sqrt{3} \times (-\sqrt{2})$ **i** $\sqrt{10} \times \sqrt{7}$

Example 7a

2 Simplify the following.

a $\sqrt{20} \div \sqrt{2}$ **b** $\sqrt{18} \div \sqrt{3}$ **c** $\sqrt{33} \div (-\sqrt{11})$
d $-\sqrt{30} \div (-\sqrt{6})$ **e** $\dfrac{\sqrt{15}}{\sqrt{5}}$ **f** $\dfrac{\sqrt{30}}{\sqrt{3}}$
g $\dfrac{\sqrt{40}}{\sqrt{8}}$ **h** $-\dfrac{\sqrt{26}}{\sqrt{2}}$ **i** $-\dfrac{\sqrt{50}}{\sqrt{10}}$

3 Simplify the following, making use of $\sqrt{x} \times \sqrt{x} = x, x \geqslant 0$, in each part.

a $\sqrt{3} \times \sqrt{3}$ **b** $\sqrt{5} \times \sqrt{5}$ **c** $\sqrt{9} \times \sqrt{9}$
d $\sqrt{14} \times \sqrt{7}$ **e** $\sqrt{2} \times \sqrt{22}$ **f** $\sqrt{3} \times \sqrt{18}$
g $\sqrt{10} \times \sqrt{5}$ **h** $\sqrt{12} \times \sqrt{8}$ **i** $\sqrt{5} \times \sqrt{20}$

Example 6b

4 Simplify the following.

a $2\sqrt{5} \times 3\sqrt{7}$ **b** $3\sqrt{7} \times \sqrt{14}$ **c** $4\sqrt{6} \times 2\sqrt{5}$
d $-5\sqrt{10} \times \sqrt{30}$ **e** $3\sqrt{6} \times (-\sqrt{18})$ **f** $5\sqrt{3} \times \sqrt{15}$
g $3\sqrt{14} \times 2\sqrt{21}$ **h** $-4\sqrt{6} \times 5\sqrt{15}$ **i** $2\sqrt{10} \times (-2\sqrt{25})$
j $-2\sqrt{7} \times (-3\sqrt{14})$ **k** $4\sqrt{15} \times 2\sqrt{18}$ **l** $9\sqrt{12} \times 4\sqrt{21}$

Example 6c

5 Simplify the following.

a $(\sqrt{11})^2$ **b** $(\sqrt{13})^2$ **c** $(2\sqrt{3})^2$
d $(5\sqrt{5})^2$ **e** $(7\sqrt{3})^2$ **f** $(9\sqrt{2})^2$

Example 7b

6 Simplify the following.

a $\dfrac{6\sqrt{14}}{3\sqrt{7}}$ **b** $\dfrac{15\sqrt{12}}{5\sqrt{2}}$ **c** $\dfrac{4\sqrt{30}}{8\sqrt{6}}$
d $-\dfrac{8\sqrt{2}}{2\sqrt{26}}$ **e** $-\dfrac{3\sqrt{3}}{9\sqrt{21}}$ **f** $\dfrac{12\sqrt{70}}{18\sqrt{14}}$

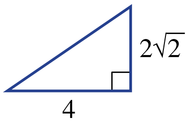
Example 8

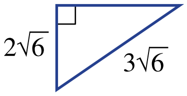
7 Use the distributive law to expand the following and then simplify the surds where necessary.

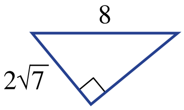
a $\sqrt{3}(\sqrt{2} + \sqrt{5})$ **b** $\sqrt{2}(\sqrt{7} - \sqrt{5})$ **c** $-\sqrt{5}(\sqrt{11} + \sqrt{13})$
d $-2\sqrt{3}(\sqrt{5} + \sqrt{7})$ **e** $3\sqrt{2}(2\sqrt{13} - \sqrt{11})$ **f** $4\sqrt{5}(\sqrt{5} - \sqrt{10})$
g $5\sqrt{3}(2\sqrt{6} + 3\sqrt{10})$ **h** $-2\sqrt{6}(3\sqrt{2} - 2\sqrt{3})$ **i** $3\sqrt{7}(2\sqrt{7} + 3\sqrt{14})$
j $6\sqrt{5}(3\sqrt{15} - 2\sqrt{8})$ **k** $-2\sqrt{8}(2\sqrt{2} - 3\sqrt{20})$ **l** $2\sqrt{3}(7\sqrt{6} + 5\sqrt{3})$

PROBLEM–SOLVING	8	8, 9(½)	8, 9(½), 10

8 Determine the unknown side of the following right-angled triangles. Recall that $a^2 + b^2 = c^2$ for right-angled triangles.

a


b


c


9 Simplify the following.

a $(2\sqrt{7})^2$ **b** $(-3\sqrt{2})^2$ **c** $(-5\sqrt{3})^2$

d $\sqrt{2}(3 - \sqrt{3}) - \sqrt{8}$ **e** $\sqrt{8}(\sqrt{6} + \sqrt{2}) - \sqrt{3}$ **f** $\sqrt{5}(\sqrt{2} + 1) - \sqrt{40}$

g $\sqrt{44} - 2(\sqrt{11} - 1)$ **h** $\sqrt{24} - 2\sqrt{2}(\sqrt{3} - 4)$ **i** $2\sqrt{3}(\sqrt{6} - \sqrt{3}) - \sqrt{50}$

10 **a** The perimeter of a square is $2\sqrt{3}$ cm. Find its area.

b Find the length of a diagonal of a square that has an area of 12 cm^2.

REASONING	11	11, 12	12, 13

11 Use $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$ to prove the following results.

a $\sqrt{6} \times \sqrt{6} = 6$ **b** $-\sqrt{8} \times \sqrt{8} = -8$ **c** $-\sqrt{5} \times (-\sqrt{5}) = 5$

12 $\sqrt{8} \times \sqrt{27}$ could be simplified in two ways, as shown.

Method A

$$\begin{aligned}\sqrt{8} \times \sqrt{27} &= \sqrt{4 \times 2} \times \sqrt{9 \times 3}\\ &= 2\sqrt{2} \times 3\sqrt{3}\\ &= 2 \times 3 \times \sqrt{2 \times 3}\\ &= 6\sqrt{6}\end{aligned}$$

Method B

$$\begin{aligned}\sqrt{8} \times \sqrt{27} &= \sqrt{8 \times 27}\\ &= \sqrt{216}\\ &= \sqrt{36 \times 6}\\ &= 6\sqrt{6}\end{aligned}$$

a Describe the first step in method A.

b Why is it useful to simplify surds before multiplying, as in method A?

c Multiply by first simplifying each surd.

i $\sqrt{18} \times \sqrt{27}$ **ii** $\sqrt{24} \times \sqrt{20}$ **iii** $\sqrt{50} \times \sqrt{45}$

iv $\sqrt{54} \times \sqrt{75}$ **v** $2\sqrt{18} \times \sqrt{48}$ **vi** $\sqrt{108} \times (-2\sqrt{125})$

vii $-4\sqrt{27} \times (-\sqrt{28})$ **viii** $\sqrt{98} \times \sqrt{300}$ **ix** $2\sqrt{72} \times 3\sqrt{80}$

13 $\dfrac{\sqrt{12}}{\sqrt{3}}$ could be simplified in two ways.

Method A

$$\begin{aligned}\frac{\sqrt{12}}{\sqrt{3}} &= \sqrt{\frac{12}{3}}\\ &= \sqrt{4}\\ &= 2\end{aligned}$$

Method B

$$\begin{aligned}\frac{\sqrt{12}}{\sqrt{3}} &= \frac{2\cancel{\sqrt{3}}^{1}}{\cancel{\sqrt{3}}^{1}}\\ &= 2\end{aligned}$$

Choose a method to simplify these surds. Compare your method with that of another student.

a $\dfrac{\sqrt{27}}{\sqrt{3}}$ **b** $\dfrac{\sqrt{20}}{\sqrt{5}}$ **c** $-\dfrac{\sqrt{162}}{\sqrt{2}}$

d $-\dfrac{2\sqrt{2}}{5\sqrt{8}}$ **e** $\dfrac{2\sqrt{45}}{15\sqrt{5}}$ **f** $\dfrac{5\sqrt{27}}{\sqrt{75}}$

ENRICHMENT: Higher powers

–	–	14–15(½)

14 Look at this example before simplifying the following.

$$\begin{aligned}(2\sqrt{3})^3 &= 2^3(\sqrt{3})^3\\ &= 2\times 2\times 2\times\sqrt{3}\times\sqrt{3}\times\sqrt{3}\\ &= 8\times 3\times\sqrt{3}\\ &= 24\sqrt{3}\end{aligned}$$

a $(3\sqrt{2})^3$

b $(5\sqrt{3})^3$

c $2(3\sqrt{3})^3$

d $(\sqrt{5})^4$

e $(-\sqrt{3})^4$

f $(2\sqrt{2})^5$

g $-3(2\sqrt{5})^3$

h $2(-3\sqrt{2})^3$

i $5(2\sqrt{3})^4$

j $\dfrac{(2\sqrt{7})^3}{4}$

k $\dfrac{(3\sqrt{2})^3}{4}$

l $\dfrac{(3\sqrt{2})^4}{4}$

m $\dfrac{(5\sqrt{2})^2}{4}\times\dfrac{(2\sqrt{3})^3}{3}$

n $\dfrac{(2\sqrt{3})^2}{9}\times\dfrac{(-3\sqrt{2})^4}{3}$

o $\dfrac{(2\sqrt{5})^3}{5}\times\dfrac{(-2\sqrt{3})^5}{24}$

p $\dfrac{(3\sqrt{3})^3}{2}\div\dfrac{(5\sqrt{2})^2}{4}$

q $\dfrac{(2\sqrt{5})^4}{50}\div\dfrac{(2\sqrt{3})^3}{5}$

r $\dfrac{(2\sqrt{2})^3}{9}\div\dfrac{(2\sqrt{8})^2}{(\sqrt{27})^3}$

15 Fully expand and simplify these surds.

a $(2\sqrt{3}-\sqrt{2})^2+(\sqrt{3}+\sqrt{2})^2$

b $(\sqrt{5}-\sqrt{3})^2+(\sqrt{5}+\sqrt{3})^2$

c $(\sqrt{3}-4\sqrt{5})(\sqrt{3}+4\sqrt{5})-(\sqrt{3}-\sqrt{5})^2$

d $-10\sqrt{3}-(2\sqrt{3}-5)^2$

e $(\sqrt{3}-2\sqrt{6})^2+(1+\sqrt{2})^2$

f $(2\sqrt{7}-3)^2-(3-2\sqrt{7})^2$

g $(2\sqrt{3}-3\sqrt{2})(2\sqrt{3}+3\sqrt{2})-(\sqrt{6}-\sqrt{2})^2$

h $\sqrt{2}(2\sqrt{5}-3\sqrt{3})^2+(\sqrt{6}+\sqrt{5})^2$

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4D Rationalising the denominator OPTIONAL

LEARNING INTENTIONS

- To understand that a surd multiplied by itself gives a whole number
- To know that rationalising the denominator refers to converting an irrational denominator to one that is rational
- To be able to rationalise the denominator

As you know, it is easier to add or subtract fractions when the fractions are expressed with the same denominator. In a similar way, it is easier to work with surds such as $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{3}-1}{\sqrt{5}}$ when they are expressed using a whole number in the denominator. The process that removes a surd from the denominator is called 'rationalising the denominator' because the denominator is being converted from an irrational number to a rational number.

Working through a problem using surds provides exact value solutions. Navigation training uses surd manipulation to solve problems of speed and direction, applying Pythagoras' theorem and trigonometry.

Lesson starter: What do I multiply by?

When trying to rationalise the denominator in a surd like $\frac{1}{\sqrt{2}}$, you must multiply the surd by a chosen number so that the denominator is converted to a whole number.

- First, decide what each of the following is equivalent to.

  a $\frac{\sqrt{3}}{\sqrt{3}}$ **b** $\frac{\sqrt{2}}{\sqrt{2}}$ **c** $\frac{\sqrt{21}}{\sqrt{21}}$

- Recall that $\sqrt{x} \times \sqrt{x} = x$ and simplify the following.

  a $\sqrt{5} \times \sqrt{5}$ **b** $2\sqrt{3} \times \sqrt{3}$ **c** $4\sqrt{7} \times \sqrt{7}$

- Now, decide what you can multiply $\frac{1}{\sqrt{2}}$ by so that:
  - the value of $\frac{1}{\sqrt{2}}$ does not change, and
  - the denominator becomes a whole number.
- Repeat this for:

  a $\frac{1}{\sqrt{5}}$ **b** $\frac{3}{2\sqrt{3}}$

KEY IDEAS

- **Rationalising a denominator** involves multiplying by a number equivalent to 1, which changes the denominator to a whole number.

$$\frac{x}{\sqrt{y}} = \frac{x}{\sqrt{y}} \times \frac{\sqrt{y}}{\sqrt{y}} = \frac{x\sqrt{y}}{y}$$

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BUILDING UNDERSTANDING

1 Simplify.

a $\dfrac{\sqrt{6}}{\sqrt{6}}$ **b** $\dfrac{2\sqrt{5}}{4\sqrt{5}}$ **c** $-\dfrac{\sqrt{8}}{\sqrt{2}}$ **d** $\dfrac{\sqrt{72}}{\sqrt{2}}$

2 State the missing number.

a $\sqrt{3} \times \_\_ = 3$ **b** $\sqrt{10} \times \sqrt{10} = \_\_$ **c** $2\sqrt{5} \times \_\_ = 10$ **d** $\_\_ \times 3\sqrt{7} = 21$

3 Use a calculator to find a decimal approximation to each number in the following pairs of numbers. What do you notice?

a $\dfrac{1}{\sqrt{7}}, \dfrac{\sqrt{7}}{7}$ **b** $\dfrac{5}{\sqrt{3}}, \dfrac{5\sqrt{3}}{3}$ **c** $\dfrac{11\sqrt{11}}{\sqrt{5}}, \dfrac{11\sqrt{55}}{5}$

Example 9 Rationalising the denominator

Rationalise the denominator in the following.

a $\dfrac{2}{\sqrt{3}}$ **b** $\dfrac{3\sqrt{2}}{\sqrt{5}}$ **c** $\dfrac{2\sqrt{7}}{5\sqrt{2}}$ **d** $\dfrac{1-\sqrt{3}}{\sqrt{3}}$

SOLUTION	EXPLANATION
a $\dfrac{2}{\sqrt{3}} = \dfrac{2}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$ $= \dfrac{2\sqrt{3}}{3}$	Choose the appropriate fraction equivalent to 1 to multiply by. In this case, choose $\dfrac{\sqrt{3}}{\sqrt{3}}$ since $\sqrt{3} \times \sqrt{3} = 3$.
b $\dfrac{3\sqrt{2}}{\sqrt{5}} = \dfrac{3\sqrt{2}}{\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}}$ $= \dfrac{3\sqrt{10}}{5}$	Choose the appropriate fraction. In this case, use $\dfrac{\sqrt{5}}{\sqrt{5}}$ since $\sqrt{5} \times \sqrt{5} = 5$. Recall $\sqrt{2} \times \sqrt{5} = \sqrt{2 \times 5} = \sqrt{10}$.
c $\dfrac{2\sqrt{7}}{5\sqrt{2}} = \dfrac{2\sqrt{7}}{5\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$ $= \dfrac{\cancel{2}^{1}\sqrt{14}}{\cancel{10}^{5}}$ $= \dfrac{\sqrt{14}}{5}$	Choose the appropriate fraction; i.e. $\dfrac{\sqrt{2}}{\sqrt{2}}$. $5 \times \sqrt{2} \times \sqrt{2} = 5 \times 2 = 10$ Cancel the common factor of 2.
d $\dfrac{1-\sqrt{3}}{\sqrt{3}} = \dfrac{1-\sqrt{3}}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$ $= \dfrac{\sqrt{3}-3}{3}$	Expand using the distributive law: $(1-\sqrt{3}) \times \sqrt{3} = 1 \times \sqrt{3} - \sqrt{3} \times \sqrt{3} = \sqrt{3} - 3$

Now you try

Rationalise the denominator in the following.

a $\dfrac{3}{\sqrt{2}}$ **b** $\dfrac{4\sqrt{3}}{\sqrt{7}}$ **c** $\dfrac{2\sqrt{5}}{3\sqrt{2}}$ **d** $\dfrac{2-\sqrt{7}}{\sqrt{7}}$

Exercise 4D

FLUENCY

1–4($\frac{1}{2}$) 1–4($\frac{1}{2}$) 1–4($\frac{1}{3}$)

Example 9a

1 Rationalise the denominators.

a $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{\sqrt{7}}$ **c** $\frac{3}{\sqrt{11}}$ **d** $\frac{4}{\sqrt{5}}$

e $\frac{5}{\sqrt{3}}$ **f** $\frac{8}{\sqrt{2}}$ **g** $\frac{\sqrt{5}}{\sqrt{3}}$ **h** $\frac{\sqrt{2}}{\sqrt{7}}$

2 Rewrite each of the following in the form $\frac{\sqrt{a}}{\sqrt{b}}$ and then rationalise the denominators.

a $\sqrt{\frac{2}{3}}$ **b** $\sqrt{\frac{5}{7}}$ **c** $\sqrt{\frac{6}{11}}$ **d** $\sqrt{\frac{2}{5}}$

e $\sqrt{\frac{7}{3}}$ **f** $\sqrt{\frac{6}{7}}$ **g** $\sqrt{\frac{10}{3}}$ **h** $\sqrt{\frac{17}{2}}$

Example 9b

3 Rationalise the denominators.

a $\frac{4\sqrt{2}}{\sqrt{7}}$ **b** $\frac{5\sqrt{2}}{\sqrt{3}}$ **c** $\frac{3\sqrt{5}}{\sqrt{2}}$

d $\frac{3\sqrt{6}}{\sqrt{7}}$ **e** $\frac{7\sqrt{3}}{\sqrt{10}}$ **f** $\frac{2\sqrt{7}}{\sqrt{15}}$

Example 9c

4 Rationalise the denominators.

a $\frac{4\sqrt{7}}{5\sqrt{3}}$ **b** $\frac{2\sqrt{3}}{3\sqrt{2}}$ **c** $\frac{5\sqrt{7}}{3\sqrt{5}}$ **d** $\frac{4\sqrt{5}}{5\sqrt{10}}$

e $\frac{2\sqrt{7}}{3\sqrt{35}}$ **f** $\frac{5\sqrt{12}}{3\sqrt{27}}$ **g** $\frac{9\sqrt{6}}{2\sqrt{3}}$ **h** $\frac{7\sqrt{90}}{2\sqrt{70}}$

PROBLEM-SOLVING

6 5($\frac{1}{2}$), 6 5–7($\frac{1}{3}$)

Example 9d

5 Rationalise the denominators.

a $\frac{1+\sqrt{2}}{\sqrt{3}}$ **b** $\frac{3+\sqrt{5}}{\sqrt{7}}$ **c** $\frac{2-\sqrt{3}}{\sqrt{5}}$

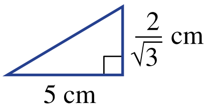
d $\frac{\sqrt{3}-\sqrt{5}}{\sqrt{2}}$ **e** $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{7}}$ **f** $\frac{\sqrt{10}-\sqrt{7}}{\sqrt{3}}$

g $\frac{\sqrt{2}+\sqrt{7}}{\sqrt{6}}$ **h** $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{10}}$ **i** $\frac{\sqrt{6}-\sqrt{10}}{\sqrt{5}}$

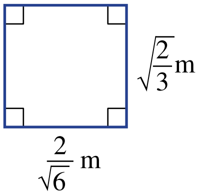
j $\frac{4\sqrt{2}-5\sqrt{3}}{\sqrt{6}}$ **k** $\frac{3\sqrt{5}+5\sqrt{2}}{\sqrt{10}}$ **l** $\frac{3\sqrt{10}+5\sqrt{3}}{\sqrt{2}}$

6 Determine the exact value of the area of the following shapes. Express your answers using a rational denominator. Recall the area of a trapezium $= \frac{1}{2}h(a+b)$ or $\frac{h}{2}(a+b)$.

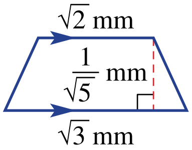
a



b



c



7 Simplify the following by first rationalising denominators and then using a common denominator.

a $\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{2}}$ **b** $\frac{3}{\sqrt{5}}+\frac{1}{\sqrt{2}}$ **c** $\frac{3}{\sqrt{7}}-\frac{2}{\sqrt{3}}$

d $\frac{5}{2\sqrt{3}}-\frac{2}{3\sqrt{2}}$ **e** $\frac{1}{3\sqrt{2}}+\frac{5}{4\sqrt{3}}$ **f** $\frac{3}{2\sqrt{5}}+\frac{2}{5\sqrt{3}}$

g $\frac{7\sqrt{2}}{5\sqrt{7}}-\frac{2\sqrt{7}}{3\sqrt{2}}$ **h** $\frac{10\sqrt{6}}{3\sqrt{5}}+\frac{4\sqrt{2}}{3\sqrt{3}}$ **i** $\frac{5\sqrt{2}}{3\sqrt{5}}-\frac{4\sqrt{7}}{3\sqrt{6}}$

REASONING

8	8, 9(½)	9(⅓), 10

8 Explain why multiplying a number by $\frac{\sqrt{x}}{\sqrt{x}}$ does not change its value.

9 Rationalise the denominators and simplify the following.

a $\frac{\sqrt{3}+a}{\sqrt{7}}$ **b** $\frac{\sqrt{6}+a}{\sqrt{5}}$ **c** $\frac{\sqrt{2}+a}{\sqrt{6}}$

d $\frac{\sqrt{3}-3a}{\sqrt{3}}$ **e** $\frac{\sqrt{5}-5a}{\sqrt{5}}$ **f** $\frac{\sqrt{7}-7a}{\sqrt{7}}$

g $\frac{4a+\sqrt{5}}{\sqrt{10}}$ **h** $\frac{3a+\sqrt{3}}{\sqrt{6}}$ **i** $\frac{2a+\sqrt{7}}{\sqrt{14}}$

10 To explore how to simplify a number such as $\frac{3}{4-\sqrt{2}}$, first answer these questions.

a Simplify.

i $(4-\sqrt{2})(4+\sqrt{2})$ **ii** $(3-\sqrt{7})(3+\sqrt{7})$ **iii** $(5\sqrt{2}-\sqrt{3})(5\sqrt{2}+\sqrt{3})$

b What do you notice about each question and answer in part **a** above?

c Now decide what to multiply $\frac{3}{4-\sqrt{2}}$ by to rationalise the denominator.

d Rationalise the denominator in these expressions.

i $\frac{3}{4-\sqrt{2}}$ **ii** $\frac{-3}{\sqrt{3}-1}$ **iii** $\frac{\sqrt{2}}{\sqrt{4}-\sqrt{3}}$ **iv** $\frac{2\sqrt{6}}{\sqrt{6}-2\sqrt{5}}$

ENRICHMENT: Binomial denominators

–	–	11(½)

11 Rationalise the denominators in the following by forming a 'difference of two perfect squares'.

For example:
$$\begin{aligned}\frac{2}{\sqrt{2}+1}&=\frac{2}{\sqrt{2}+1}\times\frac{\sqrt{2}-1}{\sqrt{2}-1}\\&=\frac{2(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}\\&=\frac{2\sqrt{2}-2}{2-1}\\&=2\sqrt{2}-2\end{aligned}$$

a $\frac{5}{\sqrt{3}+1}$ **b** $\frac{4}{\sqrt{3}-1}$ **c** $\frac{3}{\sqrt{5}-2}$ **d** $\frac{4}{1-\sqrt{2}}$

e $\frac{3}{1-\sqrt{3}}$ **f** $\frac{7}{6-\sqrt{7}}$ **g** $\frac{4}{3-\sqrt{10}}$ **h** $\frac{7}{2-\sqrt{5}}$

i $\frac{2}{\sqrt{11}-\sqrt{2}}$ **j** $\frac{6}{\sqrt{2}+\sqrt{5}}$ **k** $\frac{4}{\sqrt{3}+\sqrt{7}}$ **l** $\frac{\sqrt{2}}{\sqrt{7}+1}$

m $\frac{\sqrt{6}}{\sqrt{6}-1}$ **n** $\frac{3\sqrt{2}}{\sqrt{7}-2}$ **o** $\frac{2\sqrt{5}}{\sqrt{5}+2}$ **p** $\frac{b}{\sqrt{a}+\sqrt{b}}$

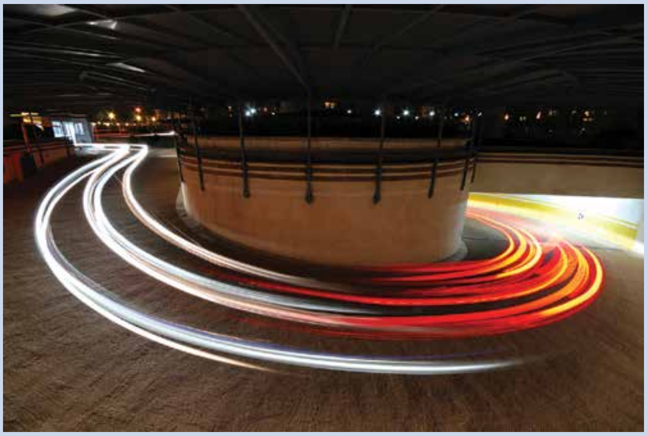
q $\frac{a}{\sqrt{a}-\sqrt{b}}$ **r** $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ **s** $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$ **t** $\frac{\sqrt{ab}}{\sqrt{a}-\sqrt{b}}$

4E Review of length CONSOLIDATING

LEARNING INTENTIONS

- To know how to convert between metric units of length
- To know the meaning of the terms perimeter, circumference and sector
- To review how to find the perimeter of a closed shape
- To be able to find the circumference of a circle and the perimeter of a sector
- To be able to find both exact and rounded answers to problems involving perimeters

Length measurements are common in many areas of mathematics, science and engineering, and are clearly associated with the basic measures of perimeter and circumference, which will be studied here.

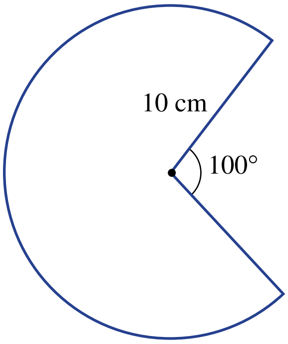


Auto engineers apply arc geometry in vehicle steering design. When turning, the outside and inside wheels follow arcs of differing radii and length; hence these wheels rotate at different rates and are steered at slightly different angles.

Lesson starter: The simple sector

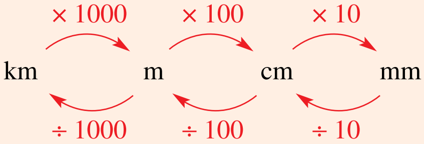
This sector looks simple enough but can you describe how to find its perimeter? Discuss these points to help.

- Recall the rule for the circumference of a circle.
- What is a definition of perimeter?
- What fraction of a circle is this sector?
- Find the perimeter using both exact and rounded numbers.

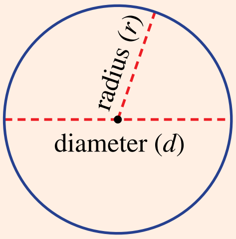


KEY IDEAS

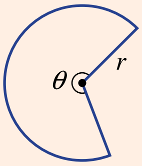
- Converting between metric units of length



- **Perimeter** is the distance around the outside of a closed shape.
- The **circumference** of a circle is the distance around the circle.
  - $C = 2\pi r = \pi d$, where $d = 2r$.



- A **sector** of a circle is a portion of a circle enclosed by two radii and the arc between them.
- Perimeter of a sector
  - $P = 2r + \frac{\theta}{360} \times 2\pi r$



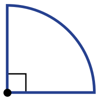
BUILDING UNDERSTANDING

① Convert the following length measurements to the units given in brackets.

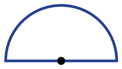
a 4 cm (mm)
b 0.096 m (cm)
c 0.001 km (m)
d 800 cm (m)
e 297 m (km)
f 5102 mm (cm)

② What fraction of a circle (in simplest form) is shown in these sectors?

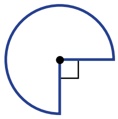
a



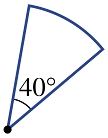
b



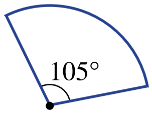
c



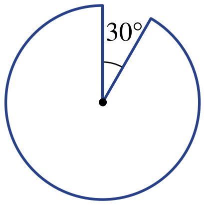
d



e

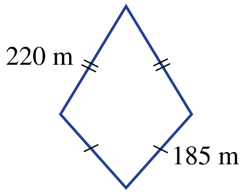


f

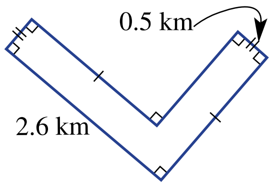


③ Find the perimeter of these shapes.

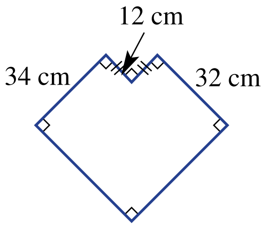
a



b



c



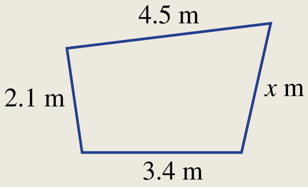
Example 10 Finding the perimeter of polygons

Consider the given two-dimensional shape.

a Find the perimeter of the shape when $x = 2.6$.

b Find x when the perimeter is 11.9 m.

c Write an expression for x in terms of the perimeter, P.



SOLUTION

a $$\begin{aligned}\text{Perimeter} &= 4.5 + 2.1 + 3.4 + 2.6 \\ &= 12.6\text{ m}\end{aligned}$$

b $$\begin{aligned}11.9 &= 4.5 + 2.1 + 3.4 + x \\ &= 10 + x \\ \therefore x &= 1.9\end{aligned}$$

c $$\begin{aligned}P &= 4.5 + 2.1 + 3.4 + x \\ &= 10 + x \\ \therefore x &= P - 10\end{aligned}$$

EXPLANATION

Simply add the lengths of all four sides using $x = 2.6$.

Add all four sides and set equal to the perimeter 11.9. Simplify and solve for x by subtracting 10 from both sides.

Use P for the perimeter and add all four sides. Simplify and rearrange to make x the subject.

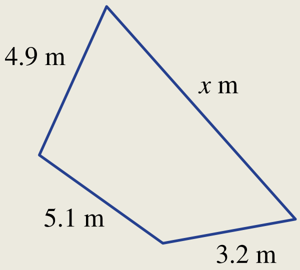
Now you try

Consider the given two-dimensional shape.

a Find the perimeter of the shape when $x = 7$.

b Find x when the perimeter is 21.3 m.

c Write an expression for x in terms of the perimeter, P.



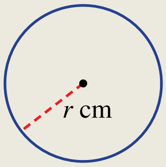
Example 11 Using the formula for the circumference of a circle

If a circle has radius r cm, find the following, rounding the answer to two decimal places where necessary.

a the circumference of a circle when $r = 2.5$

b a rule for r in terms of the circumference, C

c the radius of a circle with a circumference of 10 cm



SOLUTION

a $$\begin{aligned}\text{Circumference} &= 2\pi r \\ &= 2\pi(2.5) \\ &= 15.71\text{ cm (to 2 d.p.)}\end{aligned}$$

EXPLANATION

Write the rule for circumference and substitute $r = 2.5$, then evaluate and round as required.

Continued on next page

b $C = 2\pi r$

$\therefore r = \dfrac{C}{2\pi}$

Write the rule for circumference, then divide both sides by 2π to make r the subject.

c $r = \dfrac{C}{2\pi}$

$= \dfrac{10}{2\pi}$

$= 1.59$ cm (to 2 d.p.)

Substitute $C = 10$ into the rule from part **b** and evaluate.

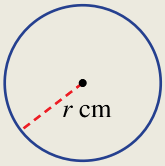
Now you try

If a circle has radius r cm, find the following, rounding the answer to two decimal places where necessary.

a the circumference of a circle when $r = 3.5$

b a rule for r in terms of the circumference, C

c the radius of a circle with a circumference of 12 cm

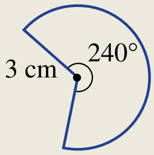


Example 12 Finding perimeters of sectors

This sector has a radius of 3 cm.

a Find the sector's exact perimeter.

b Find the perimeter, correct to one decimal place.



SOLUTION

a $P = 2r + \dfrac{\theta}{360} \times 2\pi r$

$= 2 \times 3 + \dfrac{240}{360} \times 2 \times \pi \times 3$

$= 6 + \dfrac{2}{3} \times 2 \times \pi \times 3$

$= 6 + 4\pi$ cm

b $P = 6 + 4\pi$

$= 18.6$ cm (to 1 d.p.)

EXPLANATION

The perimeter of a sector consists of two radii and a fraction $\left(\dfrac{240}{360} = \dfrac{2}{3}\right)$ of the circumference of a circle.

$6 + 4\pi$ is the exact value.

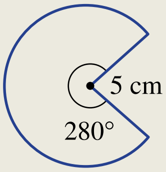
Round to the required one decimal place, using a calculator.

Now you try

This sector has a radius of 5 cm.

a Find the sector's exact perimeter.

b Find the perimeter, correct to one decimal place.



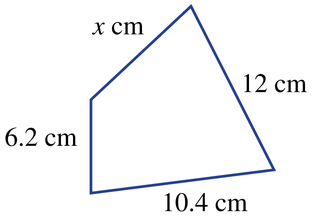
Exercise 4E

FLUENCY	1, 2, 3(½), 4	1, 3(½), 4, 5(½)	2, 3(½), 4, 5(½)

Example 10

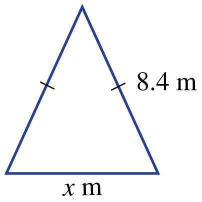
1 Consider the given two-dimensional shape.

- **a** Find the perimeter of the shape when $x = 8$.
- **b** Find x when the perimeter is 33.7 cm.
- **c** Write an expression for x in terms of the perimeter, P.



2 Consider the given two-dimensional shape.

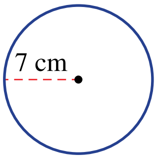
- **a** Find the perimeter of the shape when $x = 5$.
- **b** Find x when the perimeter is 20 m.
- **c** Write an expression for x in terms of the perimeter, P.



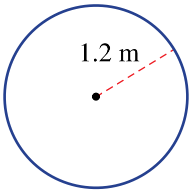
Example 11a

3 Find the circumference of these circles, correct to two decimal places.

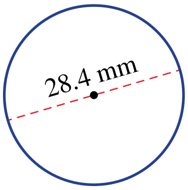
a



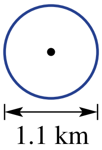
b



c



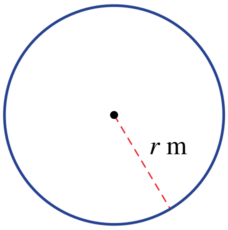
d



Example 11

4 If a circle has radius r m, find the following, rounding to two decimal places, where necessary.

- **a** The circumference of a circle when $r = 12$.
- **b** A rule for r in terms of the circumference, C.
- **c** The radius of a circle with a circumference of 35 m.

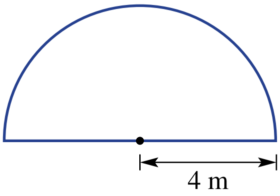


Example 12

5 Find the perimeter of these sectors:

- **i** using exact values
- **ii** by rounding the answer to one decimal place.

a



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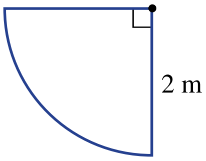

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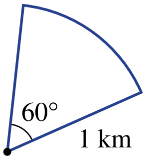

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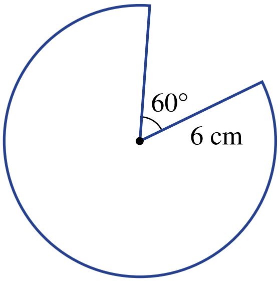

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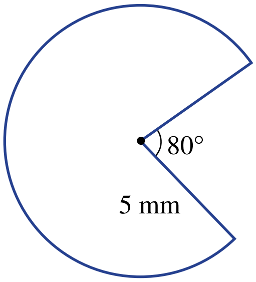
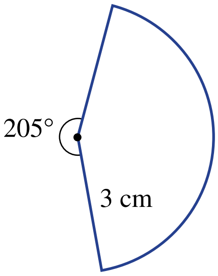


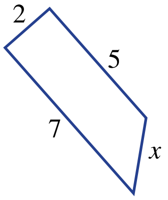
f



PROBLEM–SOLVING 6(½), 7 6(½), 7, 8(½) 6(⅓), 8, 9

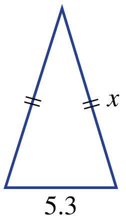
6 Find the value of x for these shapes with the given perimeters.

a



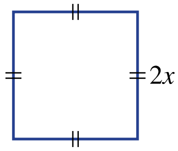
Perimeter = 17

b



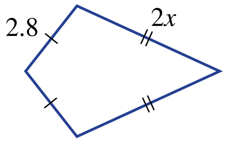
Perimeter = 22.9

c



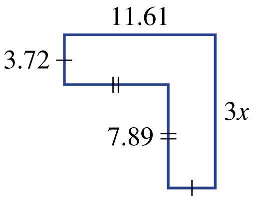
Perimeter = 0.072

d



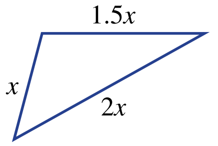
Perimeter = 16.2

e



Perimeter = 46.44

f

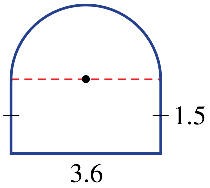


Perimeter = 10.8

7 A rectangular rose garden of length 15 m and width 9 m is surrounded by a path of width 1.2 m. Find the distance around the outside of the path.

8 Find the perimeter of these composite shapes, correct to two decimal places.

a



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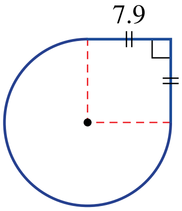

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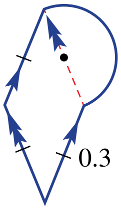

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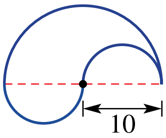

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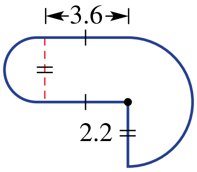

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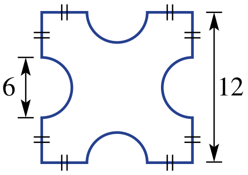
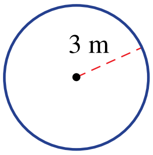


9 Find the perimeter of these shapes, giving your answers as exact values.

a



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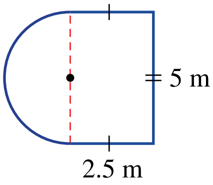

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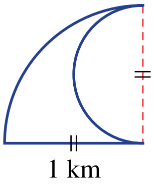


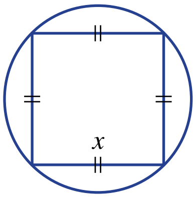
REASONING	10	10, 11	11–13

10 A bicycle has wheels with diameter 64 cm.

- **a** Find how far, correct to the nearest centimetre, the bicycle moves when the wheels turn:
  - **i** one rotation
  - **ii** five rotations.
- **b** How many rotations are required for the bike to travel 10 km? Round your answer to the nearest whole number.
- **c** Find an expression for the number of rotations required to cover 10 km if the wheel has a diameter of d cm.

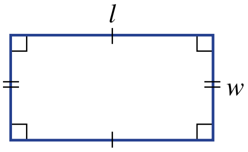
11 A square of side length n has the same perimeter as a circle. What is the radius of the circle? Give an expression in terms of n.

12 A square of side length x just fits inside a circle. Find the exact circumference of the circle in terms of x.



13 Consider a rectangle with perimeter P, length l and width w.

- **a** Express l in terms of w and P.
- **b** Express l in terms of w when $P = 10$.
- **c** If $P = 10$, state the range of all possible values of w.
- **d** If $P = 10$, state the range of all possible values of l.

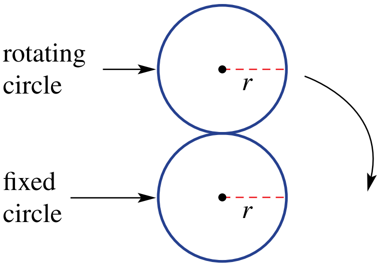


ENRICHMENT: Rotating circles	–	–	14

14 When a circle rolls around the outside of another circle it will rotate by a certain angle.

For these problems the fixed circle will have radius r. Given the following conditions, by how many degrees will the moving circle rotate if it rolls around the fixed circle once?

- **a** Assume the rotating circle has radius r (shown).
- **b** Assume the rotating circle has radius $\frac{1}{2}r$.
- **c** Assume the rotating circle has radius $2r$.
- **d** Assume the rotating circle has radius $\frac{1}{3}r$.

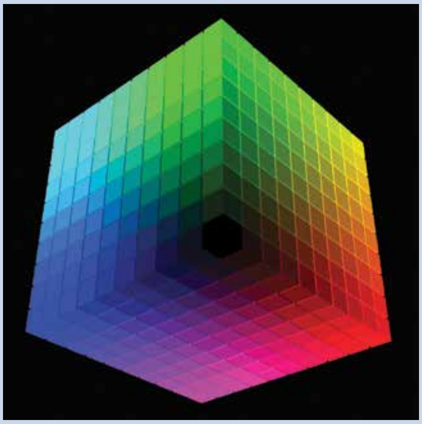


4F Pythagoras' theorem including three-dimensional problems

LEARNING INTENTIONS

- To know the relationship between the square of the hypotenuse of a right-angled triangle and the sum of the squares of the other two side lengths
- To be able to apply Pythagoras' theorem to find a missing side length of a right-angled triangle
- To be able to identify right-angled triangles in 3D objects and apply Pythagoras' theorem

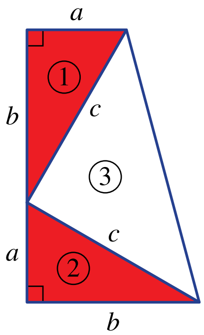
You will recall that for any right-angled triangle we can connect the length of the three sides using Pythagoras' theorem. When given two of the sides, we can work out the length of the remaining side. This has applications in all sorts of two- and three-dimensional problems.



In a colour cube, each colour has coordinates (x, y, z). Colour specialists use Pythagoras' theorem in 3D to find the shortest distance between any two colours. Applications include print and digital advertising, web page design and image editing.

Lesson starter: President Garfield's proof

Five years before he became president of the United States of America in 1881, James Garfield discovered a proof of Pythagoras' theorem. It involves arranging two identical right-angled triangles (① and ②) to form a trapezium, as shown.



- Use the formula for the area of a trapezium $\left(\frac{1}{2}(a+b)h \text{ or } \frac{h}{2}(a+b)\right)$ to find an expression for the area of the entire shape.
- Explain why the third triangle ③ is right-angled.
- Find an expression for the sum of the areas of the three triangles.
- Hence, prove that $c^2 = a^2 + b^2$.

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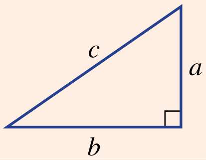
KEY IDEAS

- **Pythagoras' theorem** states that:
  The sum of the squares of the two shorter sides of a right-angled triangle equals the square of the hypotenuse.

$$a^2 + b^2 = c^2$$



- To write an answer using an exact value, use a square root sign where possible (e.g. $\sqrt{3}$).
- Pythagoras' theorem can also be applied to right-angled triangles identified in 3D objects.

BUILDING UNDERSTANDING

① Find the value of a in these equations. Express your answer in exact form using a square root sign. Assume $a > 0$.

a $a^2 + 3^2 = 8^2$

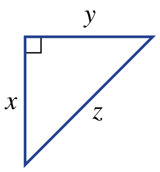
b $2^2 + a^2 = 9^2$

c $a^2 + a^2 = 2^2$

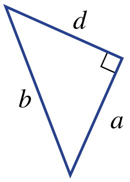
d $a^2 + a^2 = 10^2$

② State an equation connecting the pronumerals in these right-angled triangles.

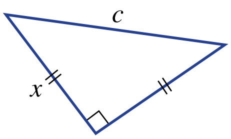
a



b



c



For centuries builders, carpenters and landscapers have used Pythagoras' theorem to construct right angles for their foundations and plots. The ancient Egyptians used three stakes joined by a rope to make a triangular shape with side lengths of 3, 4 and 5 units, which form a right angle when the rope is taut.

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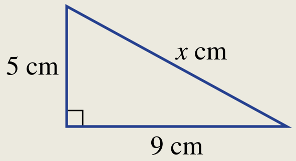
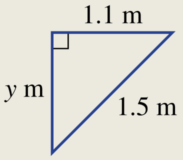
Example 13 Finding side lengths using Pythagoras' theorem

Find the length of the unknown side in these right-angled triangles, correct to two decimal places.

a



b

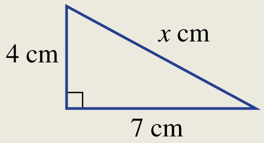


SOLUTION	**EXPLANATION**
a $c^2 = a^2 + b^2$ $\therefore x^2 = 5^2 + 9^2$ $= 106$ $\therefore x = \sqrt{106}$ $= 10.30$ (to 2 d.p.) The length of the unknown side is 10.30 cm.	x cm is the length of the hypotenuse. Substitute the two shorter sides $a = 5$ and $b = 9$ (or $a = 9$ and $b = 5$). Find the square root of both sides and round your answer as required.
b $a^2 + b^2 = c^2$ $y^2 + 1.1^2 = 1.5^2$ $y^2 = 1.5^2 - 1.1^2$ $= 2.25 - 1.21$ $= 1.04$	Substitute the shorter side $b = 1.1$ and the hypotenuse $c = 1.5$. Subtract 1.1^2 from both sides.
$\therefore y = \sqrt{1.04}$ $= 1.02$ (to 2 d.p.) The length of the unknown side is 1.02 m.	Find the square root of both sides and evaluate.

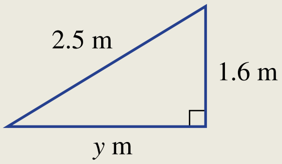
Now you try

Find the length of the unknown side in these right-angled triangles, correct to two decimal places.

a



b



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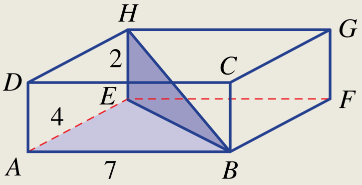
Example 14 Using Pythagoras' theorem in 3D

Consider a rectangular prism *ABCDEFGH* with the side lengths $AB = 7$, $AE = 4$ and $EH = 2$. Find:

a *BE*, leaving your answer in exact form

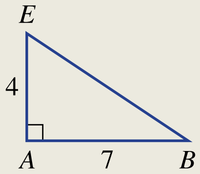
b *BH*, correct to two decimal places.



SOLUTION

EXPLANATION

a



Draw the appropriate right-angled triangle with two known sides.

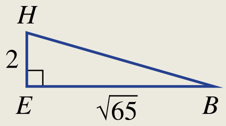
$$\begin{aligned} c^2 &= a^2 + b^2 \\ \therefore BE^2 &= 4^2 + 7^2 \\ &= 65 \\ \therefore BE &= \sqrt{65} \end{aligned}$$

Substitute $a = 4$ and $b = 7$.

Solve for *BE* exactly.

Leave intermediate answers in surd form to reduce the chance of accumulating errors in further calculations.

b



Draw the appropriate triangle.

$$\begin{aligned} BH^2 &= HE^2 + EB^2 \\ &= 2^2 + \left(\sqrt{65}\right)^2 \\ &= 4 + 65 \\ &= 69 \\ \therefore BE &= \sqrt{69} \\ &= 8.31 \text{ (to 2 d.p.)} \end{aligned}$$

Substitute $HE = 2$ and $EB = \sqrt{65}$.

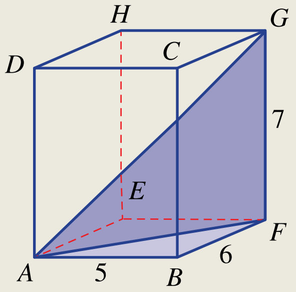
Note: $(\sqrt{65})^2 = \sqrt{65} \times \sqrt{65} = 65$.

Now you try

Consider a rectangular prism *ABCDEFGH* with the side lengths $AB = 5$, $BF = 6$ and $FG = 7$. Find:

a *AF*, leaving your answer in exact form

b *AG*, correct to two decimal places.



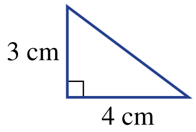
Exercise 4F

FLUENCY 1–3(½) 1–4(½) 1–2(⅓), 3–4(½)

Example 13a

1 Use Pythagoras’ theorem to find the length of the hypotenuse for these right-angled triangles. Round your answers to two decimal places where necessary.

a



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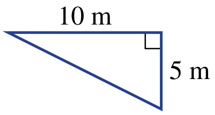

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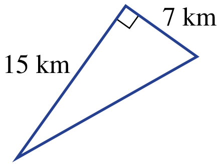

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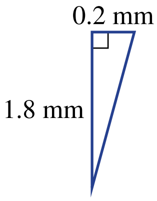

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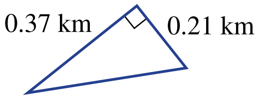

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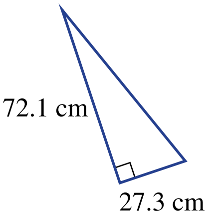
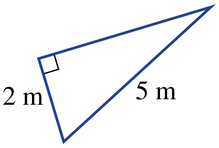


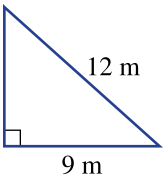
Example 13b

2 Find the length of the unknown side in these right-angled triangles, correct to two decimal places.

a



b



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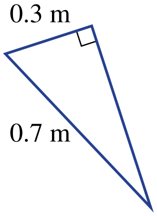

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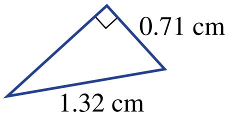

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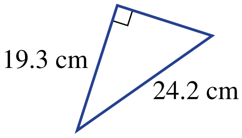

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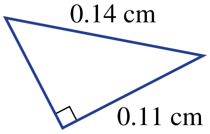
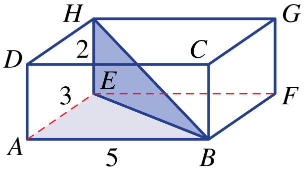


Example 14

3 For each of these rectangular prisms, find:

- **i** BE, leaving your answer in exact form
- **ii** BH, correct to two decimal places.

a



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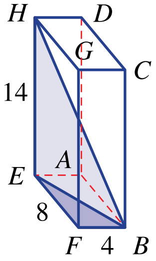

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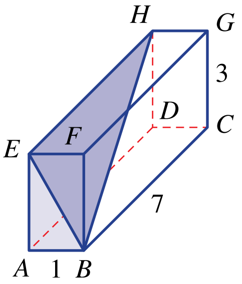
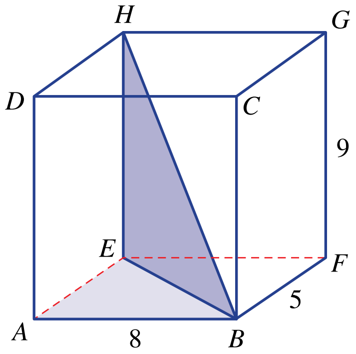
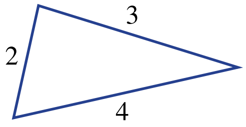


d



4 Use Pythagoras' theorem to help decide whether these triangles are right-angled. They may not be drawn to scale.

a



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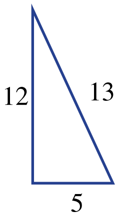

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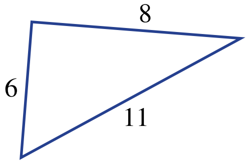

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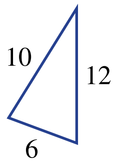
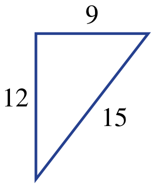


e



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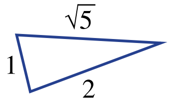
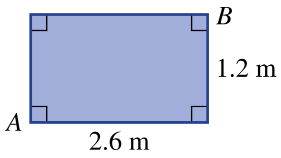


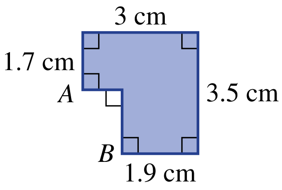
PROBLEM–SOLVING	5(½), 6	5(½), 6, 7(½)	7–9

5 Use Pythagoras' theorem to find the distance between points A and B in these diagrams, correct to two decimal places.

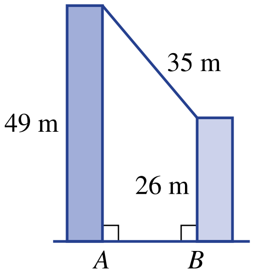
a



b



c



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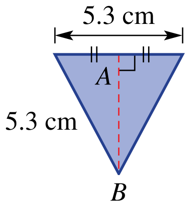

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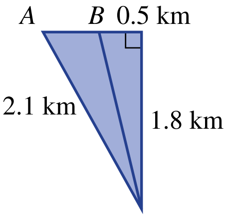

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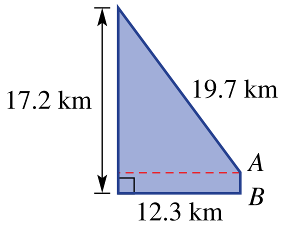

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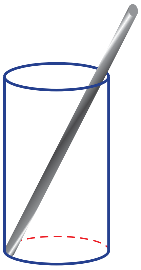
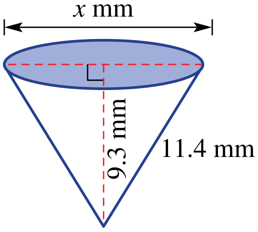
6 A 20 cm drinking straw sits diagonally in a glass of radius 3 cm and height 10 cm. What length of straw protrudes from the glass? Round your answer to one decimal place.



7 Find the value of x, correct to two decimal places, in these three-dimensional solids.

a



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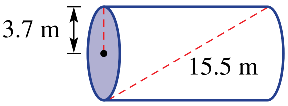

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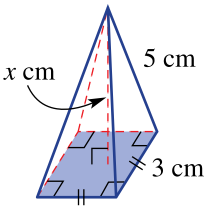

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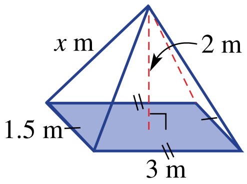

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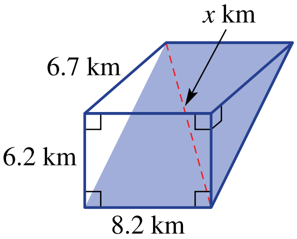

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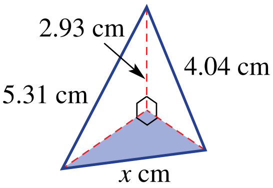
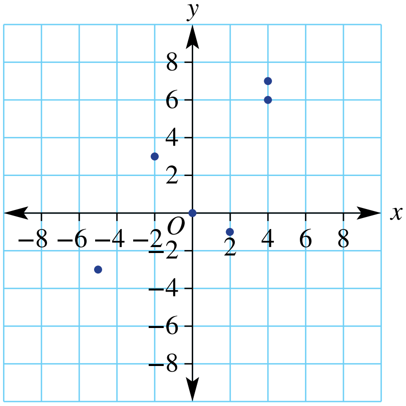


8 Find the exact distance between these pairs of points on a number plane.

a (0, 0) and (4, 6)

b (−2, 3) and (2, −1)

c (−5, −3) and (4, 7)



9 **a** Find the length of the longest rod that will fit inside these objects. Give your answer correct to one decimal place.

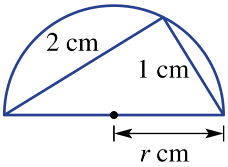
i a cylinder with diameter 10 cm and height 20 cm

ii a rectangular prism with side lengths 10 cm, 20 cm and 10 cm

b Investigate the length of the longest rod that will fit in other solids, such as triangular prisms, pentagonal prisms, hexagonal prisms and truncated rectangular pyramids. Include some three-dimensional diagrams.

REASONING	10	10, 11	11, 12

10 Two joining chords in a semicircle have lengths 1 cm and 2 cm, as shown. Find the exact radius, r cm, of the semicircle. Give reasons.



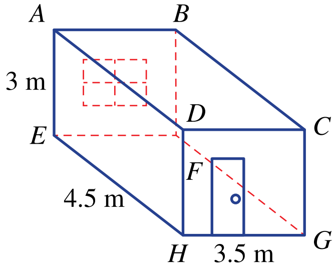
11 The diagonals of a rectangle are 10 cm long. Find the exact dimensions of the rectangle if:

a the length is twice the width

b the length is three times the width

c the length is ten times the width.

12 Streamers are used to decorate the interior of a rectangular room that is 4.5 m long, 3.5 m wide and 3 m high, as shown.



a Find the length of streamer, correct to two decimal places, required to connect from:

i A to H **ii** E to B

iii A to C **iv** A to G via C

v E to C via D **vi** E to C directly.

b Find the shortest length of streamer required, correct to two decimal places, to reach from A to G if the streamer is not allowed to reach across open space. (*Hint*: Consider a net of the prism.)

ENRICHMENT: How many proofs?	–	–	13

13 There are hundreds of proofs of Pythagoras' theorem.

a Research some of these proofs using the internet and pick one you understand clearly.

b Write up the proof, giving full reasons.

c Present your proof to a friend or the class. Show all diagrams, algebra and reasons.

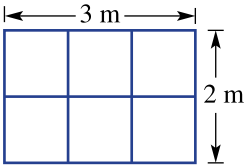
4G Review of area CONSOLIDATING

LEARNING INTENTIONS

- To understand the meaning of square units and the definition of area
- To know how to convert between metric units of area
- To know how to find the area of a square, rectangle, triangle, rhombus, parallelogram, trapezium, kite, circle and sector
- To be able to use the formulas of regular shapes to find areas of composite shapes using addition or subtraction

Area is a measure of surface and is expressed as a number of square units.

By the inspection of a simple diagram like the one shown, a rectangle with side lengths 2 m and 3 m has an area of 6 square metres or 6 m^2.



Area = 6 m^2

For rectangles and other basic shapes, we can use area formulas to help us calculate the number of square units.

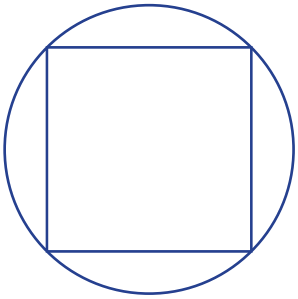
Some common metric units for area include square kilometres (km^2), square metres (m^2), square centimetres (cm^2) and square millimetres (mm^2).

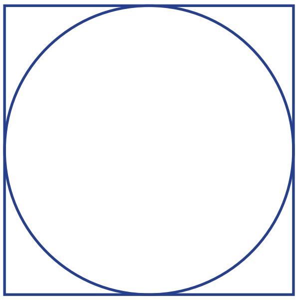
Architects apply circle sector geometry to design spiral staircases. A circle sector with the stairwell's diameter and arc length equal to the spiral's outer length is used. This sector is divided into equal smaller sectors for the steps.

Lesson starter: Pegs in holes

Discuss, with reasons relating to the area of the shapes, which is the better fit:

- a square peg in a round hole or
- a round peg in a square hole.





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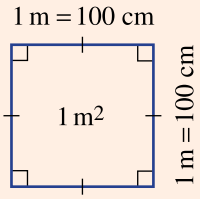
KEY IDEAS

- Conversion of units of area

$$\begin{tikzcd} \text{km}^2 \arrow[r, bend left, "\times 1000^2"] & \text{m}^2 \arrow[l, bend left, "\div 1000^2"] \arrow[r, bend left, "\times 100^2"] & \text{cm}^2 \arrow[l, bend left, "\div 100^2"] \arrow[r, bend left, "\times 10^2"] & \text{mm}^2 \arrow[l, bend left, "\div 10^2"] \end{tikzcd}$$

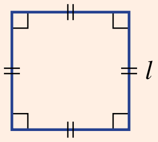


$$\begin{aligned} 1 \text{ m}^2 &= 100 \text{ cm} \times 100 \text{ cm} \\ &= 100^2 \text{ cm}^2 \end{aligned}$$

- 1 hectare (1 ha) = $10\,000 \text{ m}^2$

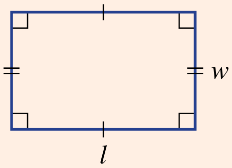
- The area of a two-dimensional shape can be defined as the number of square units contained within its boundaries. Some common area formulas are given.

square



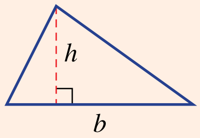
Area = l^2

rectangle



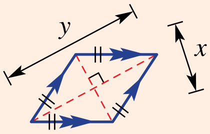
Area = lw

triangle



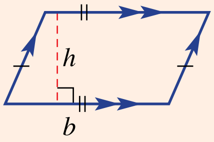
Area = $\frac{1}{2}bh$

rhombus



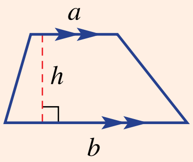
Area = $\frac{1}{2}xy$

parallelogram



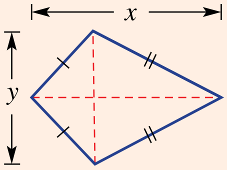
Area = bh

trapezium



Area = $\frac{1}{2}(a + b)h$

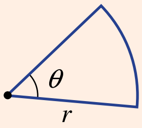
kite



Area = $\frac{1}{2}xy$

- The rule for the area of a circle is:
  Area = πr^2, where r is the radius.

- The rule for the area of a sector is $A = \frac{\theta}{360}\pi r^2$.



BUILDING UNDERSTANDING

① State the formula for the area of these shapes.

a circle
b sector
c square
d rectangle
e kite
f trapezium
g triangle
h rhombus
i parallelogram
j semicircle
k quadrant (quarter circle)

② Decide how many:

a **i** mm in 1 cm **ii** mm^2 in 1 cm^2
b **i** cm in 1 m **ii** cm^2 in 1 m^2
c **i** m in 1 km **ii** m^2 in 1 km^2
d m^2 in 1 ha

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Example 15 Converting between units of area

Convert these areas to the units shown in the brackets.

a $2.5\ \text{cm}^2\ (\text{mm}^2)$

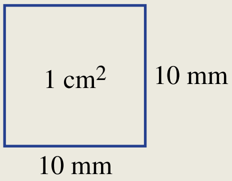
b $2\,000\,000\ \text{cm}^2\ (\text{km}^2)$

SOLUTION

a
$$\begin{aligned} 2.5\ \text{cm}^2 &= 2.5 \times 10^2\ \text{mm}^2 \\ &= 2.5 \times 100\ \text{mm}^2 \\ &= 250\ \text{mm}^2 \end{aligned}$$

b
$$\begin{aligned} 2\,000\,000\ \text{cm}^2 &= 2\,000\,000 \div 100^2\ \text{m}^2 \\ &= 200\ \text{m}^2 \\ &= 200 \div 1000^2\ \text{km}^2 \\ &= 0.0002\ \text{km}^2 \end{aligned}$$

EXPLANATION



$1\ \text{cm}^2 = 10 \times 10\ \text{mm}^2 = 100\ \text{mm}^2$

$$\begin{tikzcd} \text{km}^2 & \text{m}^2 \arrow[l, bend left, "\div 1000^2"] & \text{cm}^2 \arrow[l, bend left, "\div 100^2"] \end{tikzcd}$$

$100^2 = 10\,000$ and $1000^2 = 1\,000\,000$

Note: squaring doubles the number of zeros.

Now you try

Convert these areas to the units shown in the brackets.

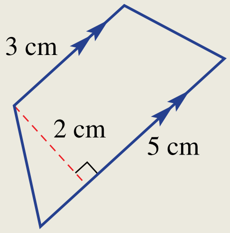
a $3.5\ \text{m}^2\ (\text{cm}^2)$

b $50\,000\ \text{mm}^2\ (\text{m}^2)$

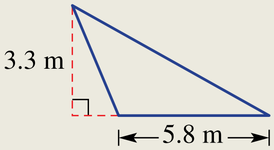
Example 16 Finding the area of basic shapes

Find the area of these basic shapes, correct to two decimal places where necessary.

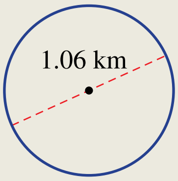
a



b



c



SOLUTION

a
$$\begin{aligned} A &= \frac{1}{2}(a+b)h \\ &= \frac{1}{2}(3+5)2 \\ &= 8\ \text{cm}^2 \end{aligned}$$

b
$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(5.8)(3.3) \\ &= 9.57\ \text{m}^2 \end{aligned}$$

EXPLANATION

The shape is a trapezium, so use this formula.

Substitute $a = 3$, $b = 5$ and $h = 2$.

Simplify and include the correct units.

The shape is a triangle.

Substitute $b = 5.8$ and $h = 3.3$.

Simplify and include the correct units.

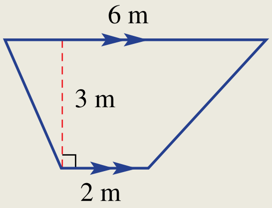
c $A = \pi r^2$
$= \pi(0.53)^2$
$= 0.88\ \text{km}^2$ (to 2 d.p.)

The shape is a circle.
The radius, r, is half the diameter;
i.e. $1.06 \div 2 = 0.53$
Evaluate using a calculator and round your answer to the required number of decimal places.

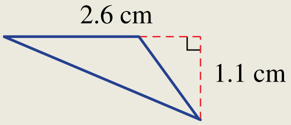
Now you try

Find the area of these basic shapes, correct to two decimal places where necessary.

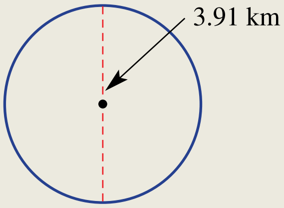
a



b



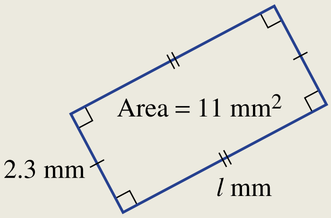
c



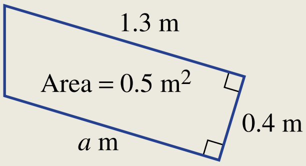
Example 17 Using area to find unknown lengths

Find the value of the pronumeral for these basic shapes, rounding to two decimal places where necessary.

a



b



SOLUTION

a

$$\begin{aligned} A &= lw \\ 11 &= l \times 2.3 \\ \therefore l &= \frac{11}{2.3} \\ &= 4.78 \text{ (to 2 d.p.)} \end{aligned}$$

b

$$\begin{aligned} A &= \frac{1}{2}(a+b)h \\ 0.5 &= \frac{1}{2}(a+1.3) \times 0.4 \\ 0.5 &= 0.2(a+1.3) \\ 2.5 &= a + 1.3 \\ \therefore a &= 1.2 \end{aligned}$$

EXPLANATION

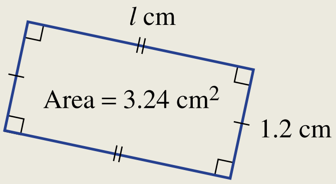
a Use the rectangle area formula.
Substitute $A = 11$ and $w = 2.3$.
Divide both sides by 2.3 to solve for l.

b Use the trapezium area formula.
Substitute $A = 0.5$, $b = 1.3$ and $h = 0.4$.
Simplify, noting that multiplication can be done in any order $\left(\frac{1}{2} \times 0.4 = 0.2\right)$, then divide both sides by 0.2 and solve for a.

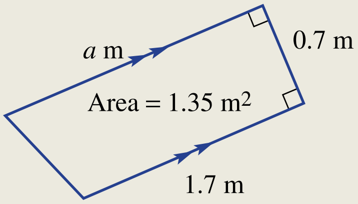
Now you try

Find the value of the pronumeral for these basic shapes, rounding to two decimal places where necessary.

a



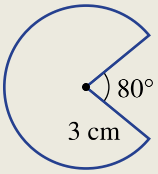
b



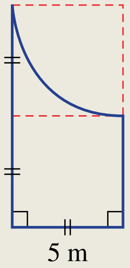
Example 18 Finding areas of sectors and composite shapes

Find the area of this sector and composite shape. Write your answer as an exact value and as a decimal, correct to two decimal places.

a



b



SOLUTION

a $A = \frac{\theta}{360} \times \pi r^2$

$= \frac{280}{360} \times \pi \times 3^2$

$= 7\pi$

$= 21.99$ cm^2 (to 2 d.p.)

b $A = 2 \times 5^2 - \frac{1}{4} \times \pi \times 5^2$

$= 50 - \frac{25\pi}{4}$

$= 30.37$ m^2 (to 2 d.p.)

EXPLANATION

Write the formula for the area of a sector.

Sector angle = 360° − 80° = 280°.

Simplify to express as an exact value (7π), then round as required.

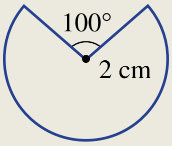
The area consists of two squares minus a quarter circle with radius 5 m.

$50 - \frac{25\pi}{4}$ is the exact value.

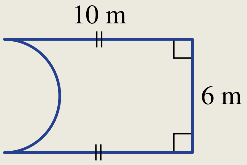
Now you try

Find the area of this sector and composite shape. Write your answer as an exact value and as a decimal, correct to two decimal places.

a



b



Exercise 4G

FLUENCY	1, 2–4(½)	1–4(½)	1–4(⅓)

Example 15

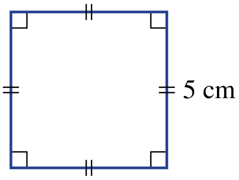
1 Convert the following area measurements to the units given in brackets.

- a $1.5\ \text{cm}^2\ (\text{mm}^2)$
- b $5\ \text{m}^2\ (\text{cm}^2)$
- c $0.2\ \text{km}^2\ (\text{m}^2)$
- d $7\,000\,000\ \text{cm}^2\ (\text{km}^2)$
- e $450\,000\ \text{mm}^2\ (\text{m}^2)$
- f $6\,000\,000\,000\ \text{mm}^2\ (\text{km}^2)$
- g $0.023\ \text{m}^2\ (\text{cm}^2)$
- h $537\ \text{cm}^2\ (\text{mm}^2)$
- i $0.0027\ \text{km}^2\ (\text{m}^2)$
- j $10\ \text{m}^2\ (\text{mm}^2)$
- k $0.00022\ \text{km}^2\ (\text{cm}^2)$
- l $145\,000\,000\ \text{mm}^2\ (\text{km}^2)$

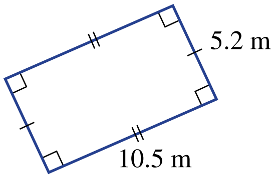
Example 16

2 Find the area of these basic shapes, rounding to two decimal places where necessary.

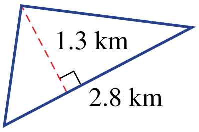
a



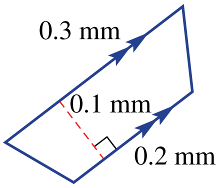
b



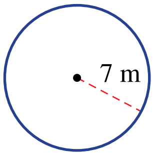
c



d



e



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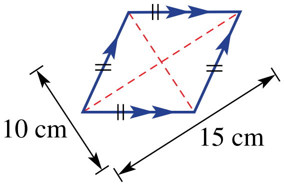
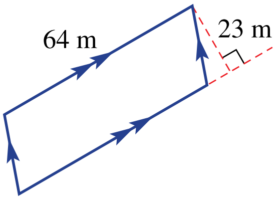
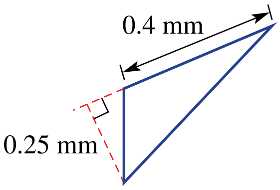


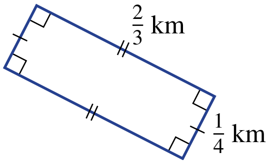
g



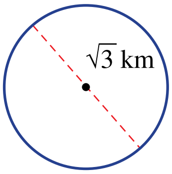
h



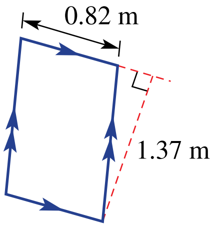
i



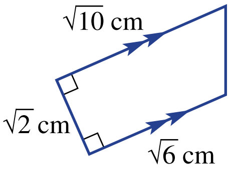
j



k



l



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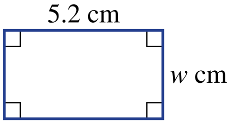
Cambridge University Press

Example 17

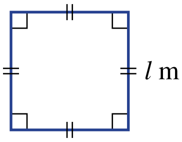
3 Find the value of the pronumeral for these basic shapes with given areas, rounding to two decimal places where necessary.

a



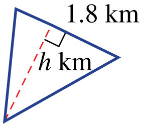
Area = 15 cm^2

b



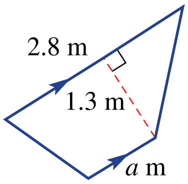
Area = 206 m^2

c



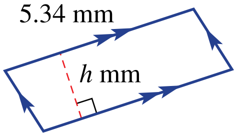
Area = 1.3 km^2

d



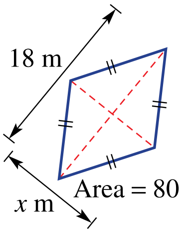
Area = 2.5 m^2

e



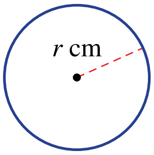
Area = 10.21 mm^2

f



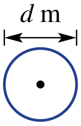
Area = 80 m^2

g



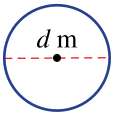
Area = 5 cm^2

h



Area = 0.21 m^2

i

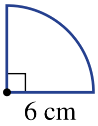


Area = 26 km^2

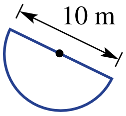
Example 18a

4 Find the area of each sector. Write your answer as an exact value and as a decimal rounded to two decimal places.

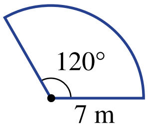
a



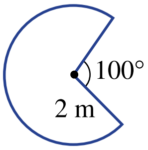
b



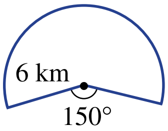
c



d



e



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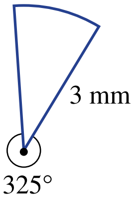
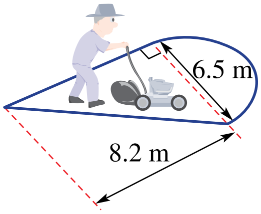


PROBLEM–SOLVING 5, 6 5, 6(½) 6(½), 7

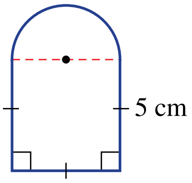
5 A lawn area is made up of a semicircular region with diameter 6.5 metres and a triangular region of length 8.2 metres, as shown. Find the total area of lawn, to one decimal place.



Example 18b

6 Find the area of these composite shapes. Write your answers as exact values and as decimals, correct to two decimal places.

a



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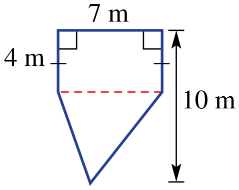
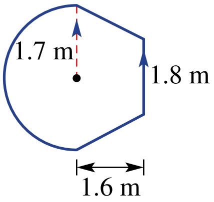
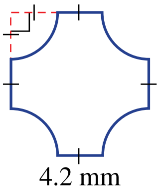


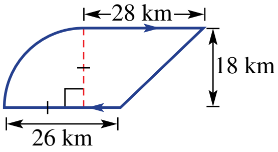
c



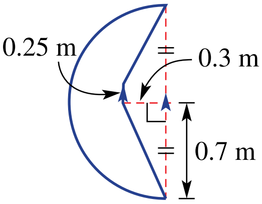
d



e



f



7 An L-shaped concrete slab being prepared for the foundation of a new house is made up of two rectangles with dimensions 3 m by 2 m and 10 m by 6 m.

a Find the total area of the concrete slab.

b If two bags of cement are required for every 5 m^2 of concrete, how many whole bags of cement will need to be purchased for the job?

REASONING	8	8, 9	9, 10

8 1 hectare (1 ha) is 10 000 m^2 and an acre is $\frac{1}{640}$ square miles (1 mile ≈ 1.61 km).

Find how many:

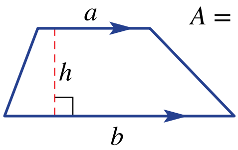
a hectares in 1 km^2

b square metres in 20 hectares

c hectares in 1 acre (round to one decimal place)

d acres in 1 hectare (round to one decimal place).

9 Consider a trapezium with area A, parallel side lengths a and b and height h.



$$A = \frac{1}{2}(a+b)h \;= \frac{h}{2}(a+b)$$

a Rearrange the area formula to express a in terms of A, b and h.

b Hence, find the value of a for these given values of A, b and h.

i $A = 10, b = 10, h = 1.5$ **ii** $A = 0.6, b = 1.3, h = 0.2$ **iii** $A = 10, b = 5, h = 4$

c Sketch the trapezium with the dimensions found in part **b iii** above. What shape have you drawn?

10 Provide a proof of the following area formulas, using only the area formulas for rectangles and triangles.

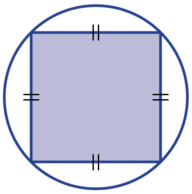
a parallelogram **b** kite **c** trapezium

ENRICHMENT: Percentage areas

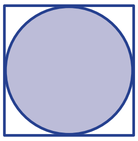
–	–	11

11 Find, correct to one decimal place, the percentage areas for these situations.

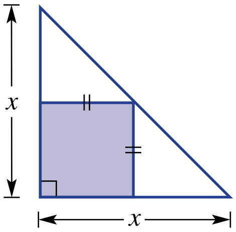
a The largest square inside a circle.



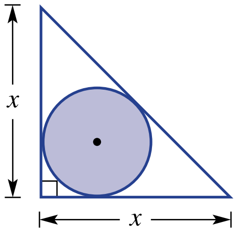
b The largest circle inside a square.

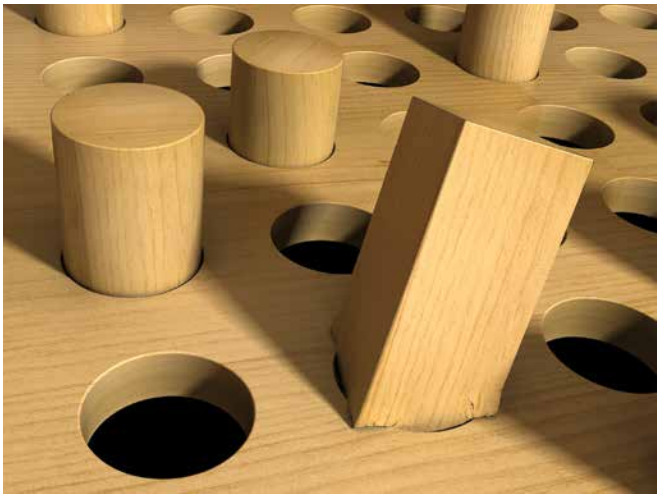


c The largest square inside a right isosceles triangle.



d The largest circle inside a right isosceles triangle.





4A Opt

1 Express each number as a decimal and decide if it is rational or irrational. Then locate all the numbers on the same number line.

a $\sqrt{10}$ **b** $\frac{22}{7}$ **c** π **d** 315%

4A Opt

2 Simplify the following.

a $\sqrt{98}$ **b** $2\sqrt{75}$ **c** $\frac{5\sqrt{32}}{8}$ **d** $\sqrt{\frac{125}{16}}$

4A Opt

3 Express $8\sqrt{3}$ as the square root of a positive integer.

4B Opt

4 Simplify the following.

a $7\sqrt{3} - 5\sqrt{3} + \sqrt{3}$ **b** $4\sqrt{2} - 3\sqrt{5} + 2\sqrt{2} + 5\sqrt{5}$

c $5\sqrt{48} - 2\sqrt{12}$ **d** $7\sqrt{45} - \sqrt{243} - 2\sqrt{20} + \sqrt{27}$

4C Opt

5 Simplify the following.

a $-\sqrt{3} \times \sqrt{5}$ **b** $-5\sqrt{2} \times (-2\sqrt{14})$ **c** $14\sqrt{35} \div (21\sqrt{7})$

d $\sqrt{6} \times \sqrt{6}$ **e** $(2\sqrt{13})^2$

4C Opt

6 Use the distributive law to expand $2\sqrt{3}(\sqrt{6} + 5\sqrt{24})$ and simplify the surds where necessary.

4D Opt

7 Rationalise the denominators.

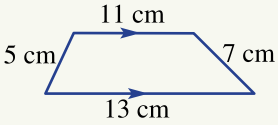
a $\frac{3}{\sqrt{7}}$ **b** $\frac{2\sqrt{3}}{\sqrt{5}}$ **c** $\frac{6 - 3\sqrt{5}}{\sqrt{2}}$

In questions 8–12 and 14, round answers to two decimal places, where necessary.

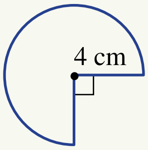
4E

8 Find the perimeter of these shapes. (Note: In part **c**, all angles are right angles.)

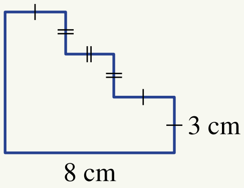
a



b



c



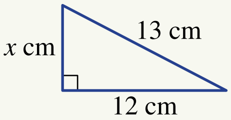
4F/G

9 For each of these shapes, find:

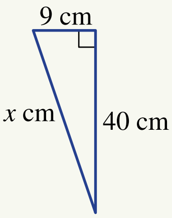
i the value of x, correct to 2 decimal places where necessary

ii the area.

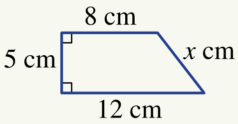
a



b



c



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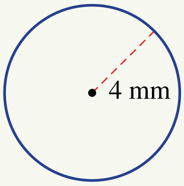
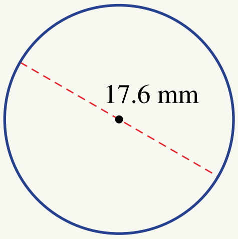
4E/G

10 Calculate the circumference and area of these circles, correct to 2 decimal places.

a

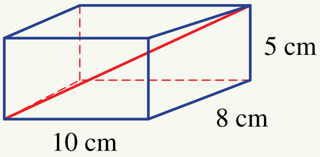


b



4F

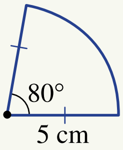
11 Find the diagonal length of a rectangular prism with dimensions 10 cm by 8 cm by 5 cm, correct to 2 decimal places.



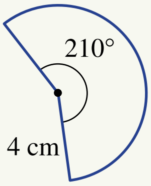
4G

12 Find the area of each of these sectors, correct to 2 decimal places.

a



b



4G

13 Convert 4.5 cm^2 to the following units.

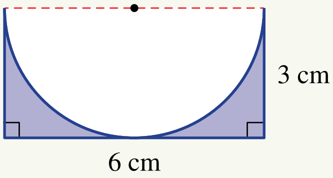
a mm^2

b m^2

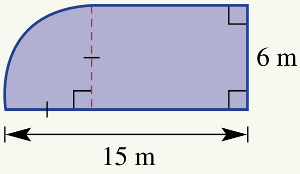
4G

14 Find the shaded (purple) area of these shapes, correct to 2 decimal places.

a



b



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4H Measurement errors and accuracy

LEARNING INTENTIONS
- To understand the difficulty in obtaining exact measurements
- To know how to find the upper and lower boundaries (limits of accuracy) for the true measurement
- To understand that rounding off in intermediate calculations leads to an accumulated error

Humans and machines measure many different things, such as the time taken to swim a race, the length of timber needed for a building and the volume of cement needed to lay a concrete path around a swimming pool. The degree or level of accuracy required usually depends on the intended purpose of the measurement.

All measurements are approximate. Errors can happen as a result of the equipment being used or the person using the measuring device.

Accuracy is a measure of how close a recorded measurement is to the exact measurement. Precision is the ability to obtain the same result over and over again.

Major track events are electronically timed to the millisecond and rounded to hundredths. An electronic beep has replaced the pistol sound that took 0.15 s to reach the farthest athlete. A camera scans the finish line 2000 times/second and signals the timer as athletes finish.

Lesson starter: Rounding a decimal

- A piece of timber is measured to be 86 cm, correct to the nearest centimetre.
  - **a** What is the smallest measurement possible that rounds to 86 cm when rounded to the nearest cm?
  - **b** What is the largest measurement possible that rounds to 86 when rounded to the nearest cm?
- A measurement is recorded as 6.0 cm, correct to the nearest millimetre.
  - **a** What units were used when measuring?
  - **b** What is the smallest decimal that could be rounded to this value?
  - **c** What is the largest decimal that would have resulted in 6.0 cm?
- Consider a square with side length 7.8941 cm.
  - **a** What is the perimeter of the square if the side length is:
    - **i** used with the four decimal places?
    - **ii** rounded to one decimal place?
    - **iii** truncated at one decimal place (i.e. 7.8)?
  - **b** What is the difference between the perimeters if the decimal is rounded to two decimal places or truncated at two decimal places or written with two significant figures?

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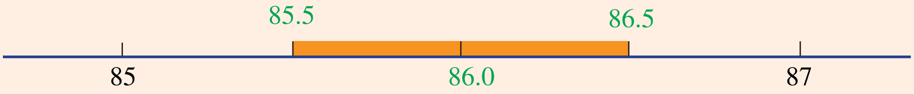
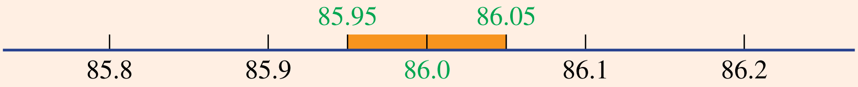
KEY IDEAS

- The **limits of accuracy** tell you what the upper and lower boundaries are for the true measurement.
  - Usually, it is ± 0.5 × the smallest unit of measurement.
    For example, when measuring to the nearest centimetre, 86 cm has limits from 85.5 cm up to (but not including) 86.5 cm.



  - When measuring to the nearest millimetre, the limits of accuracy for 86.0 cm are 85.95 cm to 86.05 cm.



- Errors can also occur in measurement calculations that involve a number of steps.
  - It is important to use exact values or a large number of decimal places throughout calculations to avoid an accumulated error.

BUILDING UNDERSTANDING

1 State a decimal that gives 3.4 when rounded from two decimal places.

2 State a measurement of 3467 mm, correct to the nearest:

a centimetre **b** metre

3 What is the smallest decimal that could result in an answer of 6.7 when rounded to one decimal place?

4 Complete these calculations.

a **i** 8.7 × 3.56 rounded to one decimal place

ii Take your rounded answer from part **a i**, multiply it by 1.8 and round to one decimal place.

b **i** 8.7 × 3.56 answering with three decimal places

ii Take your exact answer from part **b i**, multiply it by 1.8 and round to one decimal place.

c Compare your answers from parts **a ii** and **b ii**. What do you notice? Which answer is more accurate?

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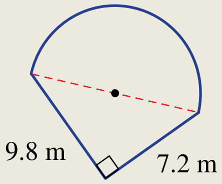
Example 19 Avoiding accumulated errors

Consider the shape shown.



a Use Pythagoras' theorem to find the length of the diameter of the semicircle, rounding to one decimal place.

b Using your rounded answer from part **a**, find the area of the semicircle and round to one decimal place.

c Find the area of the triangle rounding to one decimal place.

d Hence, find the total area using your answers to parts **b** and **c**.

e Now recalculate the total area by retaining more precise answers for the calculations to parts **a–c** above. Round your final answer correct to one decimal place.

f Compare your answers to parts **d** and **e** above. How can you explain the difference?

SOLUTION

a
$$\begin{aligned} d^2 &= 9.8^2 + 7.2^2 \\ &= 147.88 \\ d &= \sqrt{147.88} \\ &= 12.16059\ldots \end{aligned}$$

Diameter is 12.2 m (to 1 d.p.)

b
$$\begin{aligned} \text{Area}_{\text{semicircle}} &= \frac{1}{2} \times \pi \times \left(\frac{12.2}{2}\right)^2 \\ &= 58.449\ldots \\ &= 58.4 \text{ m}^2 \text{ (to 1 d.p.)} \end{aligned}$$

c
$$\begin{aligned} \text{Area}_{\text{triangle}} &= \frac{1}{2} \times 7.2 \times 9.8 \\ &= 35.3 \text{ m}^2 \text{ (to 1 d.p.)} \end{aligned}$$

d
$$\begin{aligned} \text{Total area} &= 58.4 + 35.3 \\ &= 93.7 \text{ m}^2 \end{aligned}$$

e
$$\begin{aligned} \text{Area}_{\text{semicircle}} &= \frac{1}{2} \times \pi \times \left(\frac{\sqrt{147.88}}{2}\right)^2 \\ &= 58.0723\ldots \text{ m}^2 \end{aligned}$$

$$\begin{aligned} \text{Area}_{\text{triangle}} &= \frac{1}{2} \times 7.2 \times 9.8 \\ &= 35.28 \text{ m}^2 \end{aligned}$$

$$\begin{aligned} \text{Total area} &= 58.0723\ldots + 35.28 \\ &= 93.3523\ldots \text{ m}^2 \end{aligned}$$

Total area is 93.4 m^2 (to 1 d.p.)

f The answers differ by 0.3 m^2 when rounded to one decimal place.
The error results in part **d** from the rounding in intermediate steps in parts **a–c**.

EXPLANATION

a Apply Pythagoras' theorem to calculate the diameter (hypotenuse).
Take the square root and round to one decimal place.

b Area of a semicircle $= \frac{1}{2}\pi r^2$ where r is the diameter ÷ 2.
Round to one decimal place.

c Triangle area $= \frac{1}{2}bh$.

d Combine rounded areas of semicircle and triangle.

e Use the exact diameter length to calculate the area of the semicircle.
Retain a number of decimal places for the semicircle area.

Combine the areas to calculate the total area.

Round final answer to one decimal place.

f Compare 93.7 m^2 and 93.4 m^2.
Rounding errors have accumulated to give a difference of 0.3 m^2.

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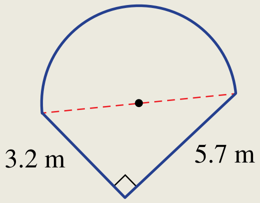
Now you try

Consider the shape shown.



a Use Pythagoras' theorem to find the length of the diameter of the semicircle, rounding to one decimal place.
b Using your rounded answer from part **a**, find the area of the semicircle and round to one decimal place.
c Find the area of the triangle rounding to one decimal place.
d Hence, find the total area using your answers to parts **b** and **c**.
e Now recalculate the total area by retaining more precise answers for the calculations to parts **a–c** above. Round your final answer correct to one decimal place.
f Compare your answers to parts **d** and **e** above. How can you explain the difference?

Example 20 Finding limits of accuracy

Give the limits of accuracy for these measurements.

a 72 cm **b** 86.6 mm

SOLUTION	EXPLANATION
a $72 \pm 0.5 \times 1$ cm $= 72 - 0.5$ cm to $72 + 0.5$ cm $= 71.5$ cm to 72.5 cm	Smallest unit of measurement is one whole cm. Error $= 0.5 \times 1$ cm This error is subtracted and added to the given measurement to find the limits of accuracy.
b $86.6 \pm 0.5 \times 0.1$ mm $= 86.6 \pm 0.05$ mm $= 86.6 - 0.05$ mm to $86.6 + 0.05$ mm $= 86.55$ mm to 86.65 mm	Smallest unit of measurement is 0.1 mm. Error $= 0.5 \times 0.1$ mm $= 0.05$ mm This error is subtracted and added to the given measurement to find the limits of accuracy.

Now you try

Give the limits of accuracy for these measurements.

a 45 cm **b** 15.7 mm

Example 21 Applying the limits of accuracy

Janis measures each side of a square as 6 cm. Find:

a the upper and lower limits for the sides of the square
b the upper and lower limits for the perimeter of the square
c the upper and lower limits for the square's area.

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SOLUTION

a $6 \pm 0.5 \times 1$ cm
$= 6 - 0.5$ cm to $6 + 0.5$ cm
$= 5.5$ cm to 6.5 cm

b $$\begin{aligned}\text{Lower limit } P &= 4 \times 5.5\\ &= 22 \text{ cm}\\ \text{Upper limit } P &= 4 \times 6.5\\ &= 26 \text{ cm}\end{aligned}$$

c $$\begin{aligned}\text{Lower limit } A &= 5.5^2\\ &= 30.25 \text{ cm}^2\\ \text{Upper limit } A &= 6.5^2\\ &= 42.25 \text{ cm}^2\end{aligned}$$

EXPLANATION

Smallest unit of measurement is one whole cm.
Error $= 0.5 \times 1$ cm

The lower limit for the perimeter uses the lower limit for the measurement taken and the upper limit for the perimeter uses the upper limit of 6.5 cm.

The lower limit for the area is 5.5^2, whereas the upper limit will be 6.5^2.

Now you try

Janis measures each side of a square as 9 cm. Find:

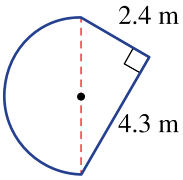
a the upper and lower limits for the sides of the square
b the upper and lower limits for the perimeter of the square
c the upper and lower limits for the square's area.

Exercise 4H

FLUENCY	1–4	1, 3(½), 4–6	1, 3(⅓), 4, 5(½), 6

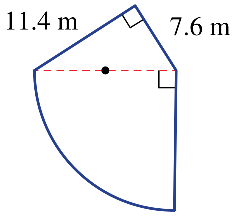
Example 19

1 Consider the shape shown.



a Use Pythagoras' theorem to find the length of the diameter of the semicircle, rounding to one decimal place.
b Using your rounded answer from part **a**, find the area of the semicircle and round to one decimal place.
c Find the area of the triangle rounding to one decimal place.
d Hence, find the total area using your answers to parts **b** and **c**.
e Now recalculate the total area by retaining more precise answers for the calculations to parts **a–c** above. Round your final answer correct to one decimal place.
f Compare your answers to parts **d** and **e** above. How can you explain the difference?

2 Consider the shape shown.



- **a** Use Pythagoras' theorem to find the length of the radius of the quarter circle, rounding to one decimal place.
- **b** Using your rounded answer from part **a**, find the area of the quarter circle and round to one decimal place.
- **c** Find the area of the triangle rounding to one decimal place.
- **d** Hence, find the total area using your answers to parts **b** and **c**.
- **e** Now recalculate the total area by retaining more precise answers for the calculations to parts **a–c** above. Round your final answer correct to one decimal place.
- **f** Compare your answers to parts **d** and **e** above. How can you explain the difference?

Example 20

3 For each of the following:

- **i** Give the smallest unit of measurement (e.g. 0.1 cm is the smallest unit in 43.4 cm).
- **ii** Give the limits of accuracy.

a 45 cm	**b** 6.8 mm	**c** 12 m	**d** 15.6 kg
e 56.8 g	**f** 10 m	**g** 673 h	**h** 9.84 m
i 12.34 km	**j** 0.987 km	**k** 1.65 L	**l** 9.03 mL

4 What are the limits of accuracy for the amount $4500 when it is written:

- **a** to two significant figures?
- **b** to three significant figures?
- **c** to four significant figures?

5 Write the following as a measurement, given that the lower and upper limits of these measurements are as follows.

a 29.5 m to 30.5 m	**b** 14.5 g to 15.5 g
c 4.55 km to 4.65 km	**d** 8.95 km to 9.05 km
e 985 g to 995 g	**f** 989.5 g to 990.5 g

6 Martha writes down the length of her fabric as 150 cm. As Martha does not give her level of accuracy, give the limits of accuracy of her fabric if it was measured correct to the nearest:

a centimetre **b** 10 centimetres **c** millimetre.

PROBLEM–SOLVING	7, 8	7, 8	8, 9

7 A length of copper pipe is given as 25 cm, correct to the nearest centimetre.

- **a** What are the limits of accuracy for this measurement?
- **b** If 10 pieces of copper, each with a given length of 25 cm, are joined end to end, what is the minimum length that it could be?
- **c** What is the maximum length for the 10 pieces of pipe in part **b**?

Example 21

8 The side of a square is recorded as 9.2 cm, correct to two significant figures.

- **a** What is the minimum length that the side of this square could be?
- **b** What is the maximum length that the side of this square could be?
- **c** Find the upper and lower boundaries for this square's perimeter.
- **d** Find the upper and lower limits for the area of this square.

9 The side of a square is recorded as 9.20 cm, correct to three significant figures.

- **a** What is the minimum length that the side of this square could be?
- **b** What is the maximum length that the side of this square could be?
- **c** Find the upper and lower boundaries for this square's perimeter.
- **d** Find the upper and lower limits for the area of this square.
- **e** How has changing the level of accuracy from 9.2 cm (see Question **8**) to 9.20 cm affected the calculation of the square's perimeter and area?

REASONING	10	10	10, 11

10 Cody measures the mass of a baby to be 6 kg. Jacinta says the same baby is 5.8 kg and Luke gives his answer as 5.85 kg.

- **a** Explain how all three people could have different answers for the same measurement.
- **b** Write down the level of accuracy being used by each person.
- **c** Are all their answers correct? Discuss.

11 Write down a sentence explaining the need to accurately measure items in our everyday lives and the accuracy required for each of your examples. Give three examples of items that need to be measured correct to the nearest:

a kilometre **b** millimetre **c** millilitre **d** litre.

ENRICHMENT: Percentage error	–	–	12(½)

12 To calculate the percentage error of any measurement, the error (i.e. ± the smallest unit of measurement) is compared to the given or recorded measurement and then converted to a percentage.

For example: 5.6 cm

Error $= \pm 0.5 \times 0.1 = \pm 0.05$

$$\begin{aligned}\text{Percentage error} &= \frac{\pm 0.05}{5.6} \times 100\% \\ &= \pm 0.89\% \text{ (to two significant figures)}\end{aligned}$$

Find the percentage error for each of the following. Round to two significant figures.

a 28 m **b** 9 km **c** 8.9 km **d** 8.90 km

e 178 mm **f** \$8.96 **g** \$4.25 **h** 701 mL

4I Surface area of prisms and cylinders

LEARNING INTENTIONS

- To know what defines a prism and a cylinder
- To know the meaning of the term surface area
- To know how to use a net to identify the surfaces of prisms and cylinders
- To be able to find the surface area of prisms
- To know how the formula for the surface area of a cylinder is developed and be able to apply it
- To be able to identify visible surfaces of a composite solid to include in surface area calculations

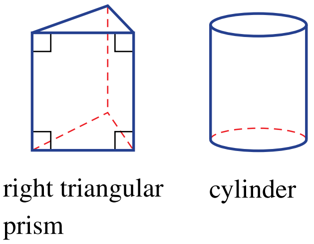
Knowing how to find the area of simple shapes combined with some knowledge about three-dimensional objects helps us to find the surface area of a range of solids.

A cylindrical can, for example, has two circular ends and a curved surface that could be rolled out to form a rectangle. Finding the sum of the two circles and the rectangle will give the surface area of the cylinder.

You will recall the following information about prisms and cylinders.

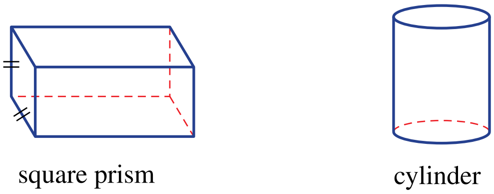
- A **prism** is a polyhedron with a uniform cross-section and two congruent ends.
  - A prism is named by the shape of the cross-section.
  - The remaining sides are parallelograms.
- A **cylinder** has a circular cross-section.
  - A cylinder is similar to a prism in that it has a uniform cross-section and two congruent ends.

Steel cans used for food are coated with tin-plate (2% tin), as tin doesn't corrode. Cans are manufactured by cutting a rectangle, forming a tube, attaching the base, sterilising, filling with food and then joining the circular top.



Lesson starter: Drawing nets

Drawing or visualising a net can help when finding the surface area of a solid. Try drawing a net for these solids.



By labelling the dimensions, can you come up with a formula for the surface area of these solids?

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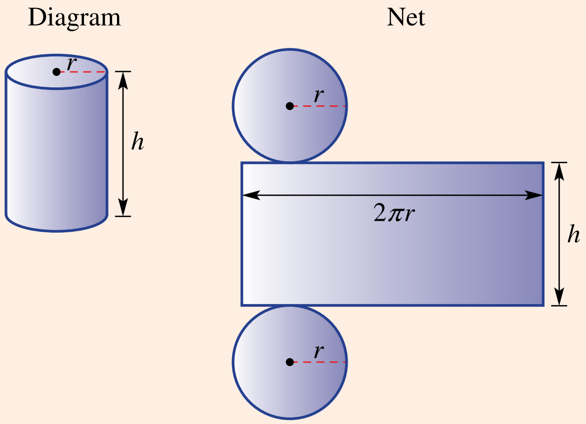
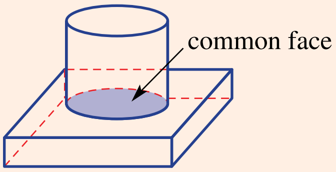
KEY IDEAS

- The **surface area** of a three-dimensional object can be found by finding the sum of the areas of each of the shapes that make up the surface of the object.
- A **net** is a two-dimensional illustration of all the surfaces of a solid object.
- Given below are the net and surface area of a **cylinder**.



$$\begin{aligned} A &= \text{2 circles} + \text{1 rectangle} \\ &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r(r + h) \end{aligned}$$

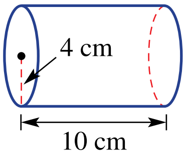
- **Composite solids** are solids made up of two or more basic solids.
  - To find a surface area do not include any common faces.
    - In this example, the top circular face area of the cylinder is equal to the common face area, so the Surface area = surface area of prism + curved surface area of cylinder.



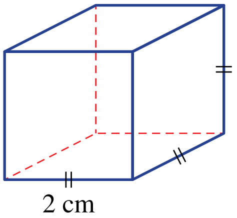
BUILDING UNDERSTANDING

1. Draw an example of these solids.
   a cylinder **b** rectangular prism **c** triangular prism
2. Draw a net for each of these solids.

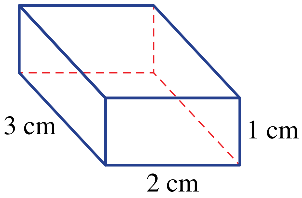
a



b



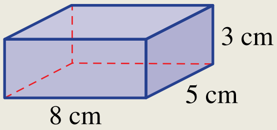
c



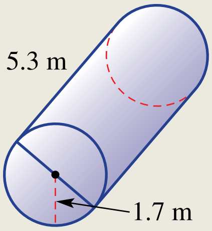
Example 22 Finding the surface area of prisms and cylinders

Find the surface area of this rectangular prism and cylinder. Round your answer to two decimal places where necessary.

a



b

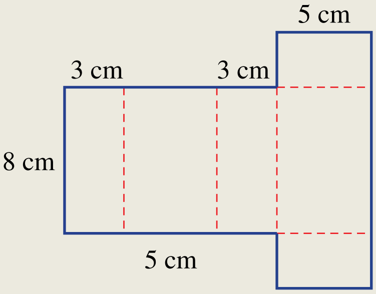


SOLUTION

a $A = 2 \times (8 \times 3) + 2 \times (5 \times 3) + 2 \times (8 \times 5)$
$= 48 + 30 + 80$
$= 158 \text{ cm}^2$

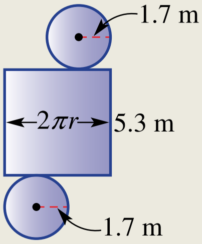
EXPLANATION

Draw the net of the solid if needed to help you. Sum the areas of the rectangular surfaces.



b $A = 2\pi r^2 + 2\pi rh$
$= 2\pi(1.7)^2 + 2\pi(1.7) \times 5.3$
$= 74.77 \text{ m}^2$ (to 2 d.p.)

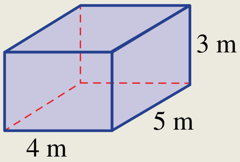
Write the formula and substitute the radius and height.



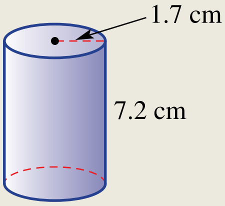
Now you try

Find the surface area of this rectangular prism and cylinder. Round your answer to two decimal places where necessary.

a



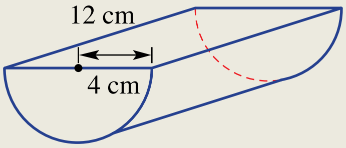
b



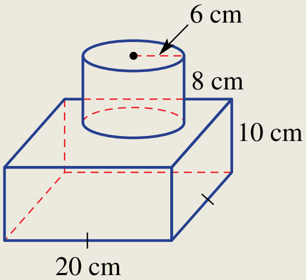
Example 23 Finding the surface area of solids involving sectors and composite solids

Find the surface area of the following, correct to one decimal place.

a A solid with a semicircular cross-section



b A composite object consisting of a square-based prism and a cylinder



SOLUTION

a $A = 2 \times \left(\frac{1}{2} \times \pi \times 4^2\right) + \frac{1}{2} \times 2\pi \times 4 \times 12 + 12 \times 8$

$= 64\pi + 96$

$= 297.1\ \text{cm}^2$ (to 1 d.p.)

b $A = 4 \times (20 \times 10) + 2 \times (20 \times 20) + 2 \times \pi \times 6 \times 8$

$+ \pi(6)^2 - \pi(6)^2$

$= 1600 + 96\pi$

$= 1901.6\ \text{cm}^2$ (to 1 d.p.)

EXPLANATION

The half-cylinder is made up of two semi-circular ends, half the curved surface of a cylinder (i.e. $\frac{1}{2} \times 2\pi rh$) plus a rectangular surface on top. Sum the areas to get the exact answer and round to one decimal place.

The common circular area (base of cylinder), which should not be included, is added back on with the top of the cylinder. So the surface area of the prism is added to only the curved area of the cylinder.

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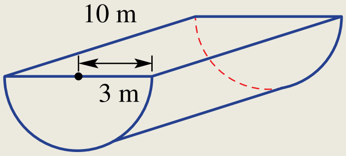
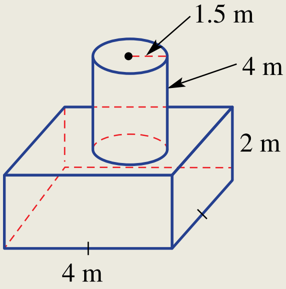
Now you try

Find the surface area of the following, correct to one decimal place.

a A solid with a semicircular cross-section



b A composite object consisting of a square-based prism and a cylinder



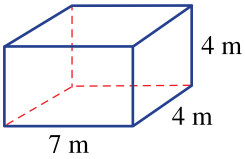
Exercise 4I

FLUENCY	1, 2, 3(½)	1, 2, 3(½), 4	1–2(⅓), 3(½), 4, 5

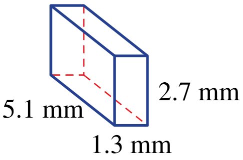
Example 22a

1 Find the surface area of these rectangular prisms.

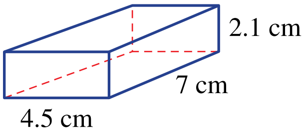
a



b



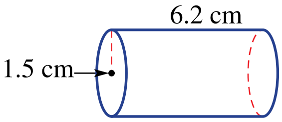
c



Example 22b

2 Find the surface area of these cylinders. Round your answers to two decimal places.

a



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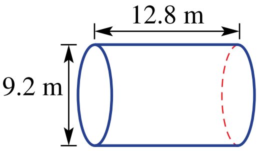

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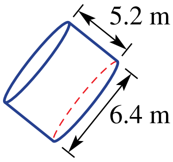

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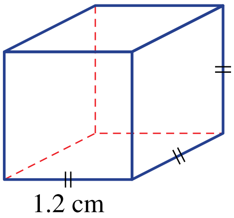
3 Find the surface area of these solids.

a



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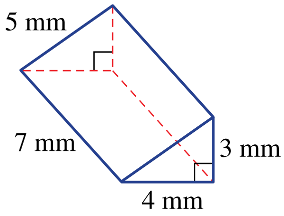

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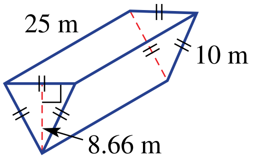

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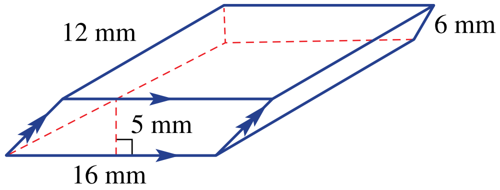

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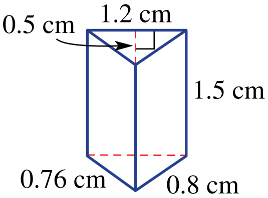

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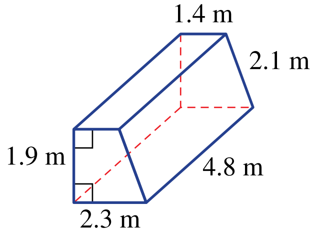


4 Find the surface area, in square metres, of the outer surface of an open pipe with radius 85 cm and length 4.5 m, correct to two decimal places.

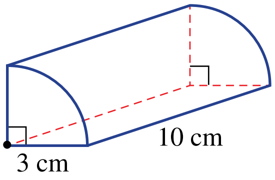
5 What is the minimum area of paper required to wrap a box with dimensions 25 cm wide, 32 cm long and 20 cm high?

PROBLEM–SOLVING	6–7(½)	6–8(½)	6–8(½), 9

Example 23a

6 The cross-sections of these solids are sectors. Find the surface area, rounding to one decimal place. Remember to include any rectangular surfaces also.

a



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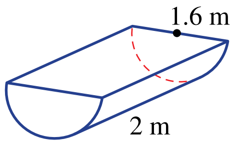

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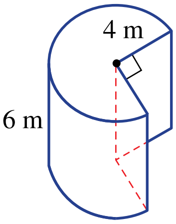

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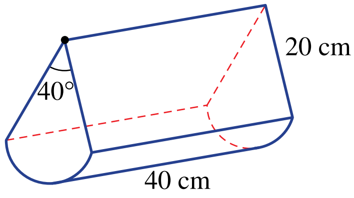
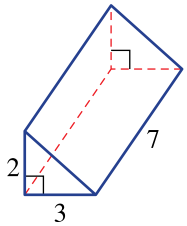


7 Use Pythagoras' theorem to determine any unknown side lengths and find the surface area of these solids, correct to one decimal place.

a



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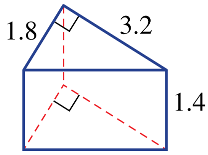

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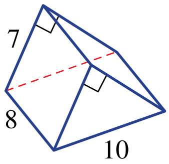

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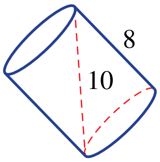

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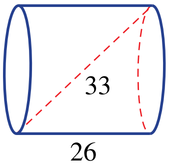

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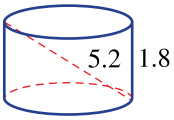
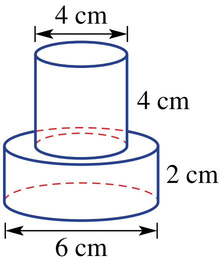


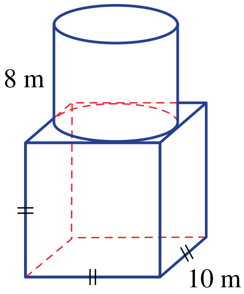
Example 23b

8 Find the surface area of these composite solids. Answer correct to one decimal place.

a



b



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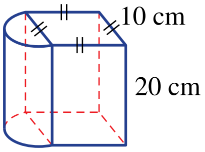

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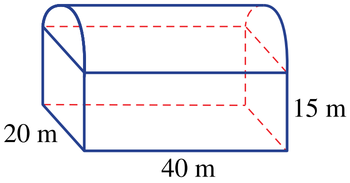
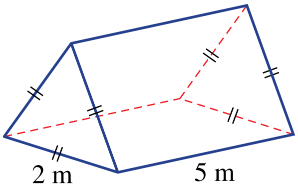
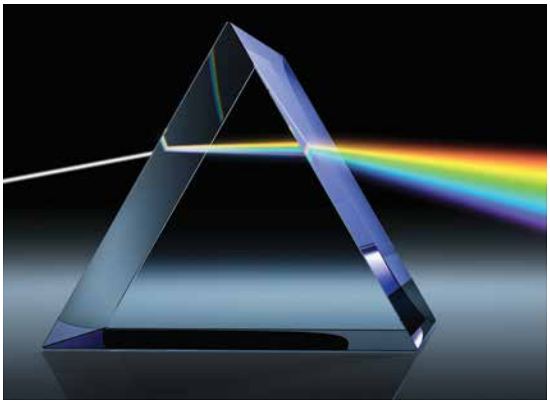


9 Find the surface area of this triangular prism, correct to one decimal place.

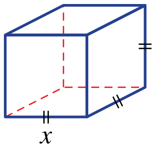




REASONING	10	10, 11	11–13

10 Find a formula for the surface area of these solids, using the given pronumerals.

a



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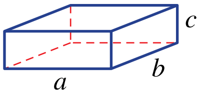

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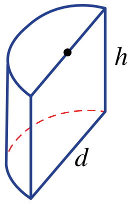

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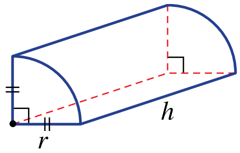


11 Find the exact surface area for a cylinder with the given dimensions. Your exact answer will be in terms of π.

a $r = 1$ and $h = 2$

b $r = \frac{1}{2}$ and $h = 5$

12 The surface area of a cylinder is given by the rule:

$$\text{Surface area} = 2\pi r(r + h)$$

Find the height, to two decimal places, of a cylinder that has a radius of 2 m and a surface area of:

a 35 m^2

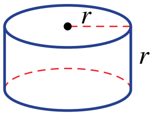
b 122 m^2

13 Can you find the exact radius of the base of a cylinder if its surface area is 8π cm^2 and its height is 3 cm?

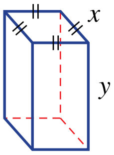
ENRICHMENT: Deriving formulas for special solids	–	–	14

14 Derive the formulas for the surface area of the following solids.

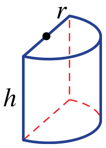
a a cylinder with its height equal to its radius r



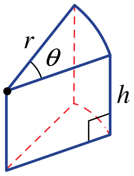
b a square-based prism with square side length x and height y



c a half cylinder with radius r and height h



d a solid with a sector cross-section, radius r, sector angle θ and height h



4J Surface area of pyramids and cones EXTENDING

LEARNING INTENTIONS

- To know the shape of pyramids and cones and their associated nets
- To know the formula for the surface area of a cone
- To be able to find the surface area of a pyramid and a cone
- To be able to use Pythagoras' theorem to find the vertical height or slant height of a cone

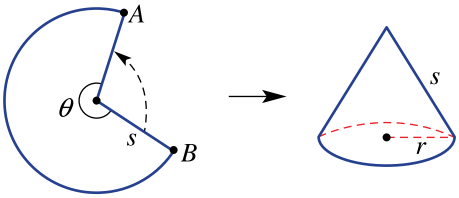
Pyramids and cones are solids for which we can also calculate the surface area by finding the sum of the areas of all the outside surfaces.

The surface area of a pyramid involves finding the sum of the areas of the base and its triangular faces. The rule for the surface area of a cone can be developed after drawing a net that includes a circle (base) and sector (curved surface).

Mechanical engineers and sheet metal workers apply surface area and volume formulas when designing and constructing stainless steel equipment for the food and beverage industries. Cylinder and cone formulas are used when designing brewery vats.

Lesson starter: The cone formula

Use a pair of compasses to construct a large sector. Use any sector angle θ that you like. Cut out the sector and join the points A and B to form a cone of radius r.



- Give the rule for the area of the base of the cone.
- Give the rule for the circumference of the base of the cone.
- Give the rule for the circumference of a circle with radius s.
- Use the above to find an expression for the area of the base of the cone as a fraction of the area πs^2.
- Hence, explain why the rule for the surface area of a cone is given by:
  Surface area $= \pi r^2 + \pi rs$.

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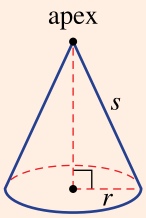
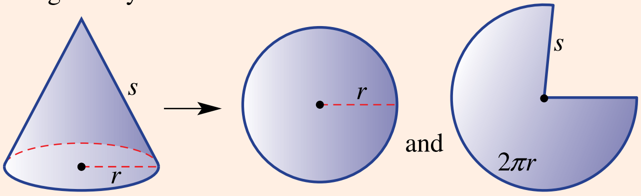
KEY IDEAS

- A **cone** is a solid with a circular base and a curved surface that reaches from the base to a point called the **apex**.
  - A right cone has its apex directly above the centre of the base.
  - The pronumeral s is used for the slant height and r is the radius of the base.
  - Cone surface area is given by:



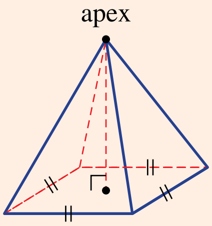
right cone



$$\text{Area (base)} = \pi r^2 \qquad \text{Area} = \frac{2\pi r}{2\pi s} \times \pi s^2 = \pi rs$$

$$\therefore A(\text{cone}) = \pi r^2 + \pi rs = \pi r(r + s)$$

- A **pyramid** has a base that is a polygon and its remaining faces are triangles that meet at the apex.
  - A pyramid is named by the shape of its base.
  - A right pyramid has its apex directly above the centre of the base.



right square-
based pyramid

BUILDING UNDERSTANDING

① State the rule for the following.

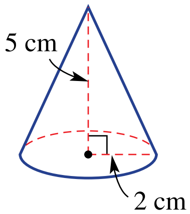
a area of a triangle

b surface area of the base of a cone with radius r

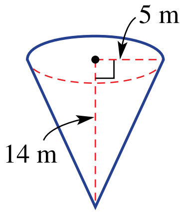
c surface area of the curved part of a cone with slant height s and radius r

② Find the exact slant height for these cones, using Pythagoras' theorem. Express exactly, using a square root sign.

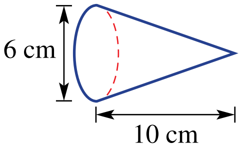
a



b

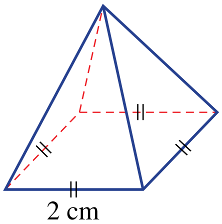


c



③ Draw a net for each of these solids.

a



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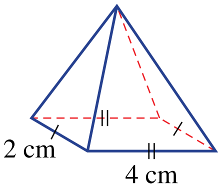

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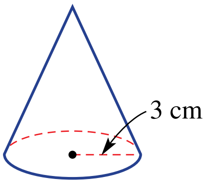

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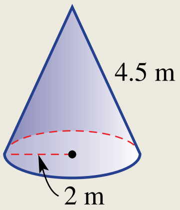
Example 24 Finding the surface area of a cone and a pyramid

Find the surface area of these solids, using two decimal places for part **a**.

a a cone with radius 2 m and slant height 4.5 m

b a square-based pyramid with square-base length 25 mm and triangular face height 22 mm



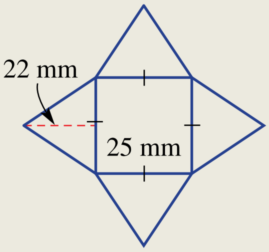
SOLUTION

a
$$\begin{aligned} A &= \pi r^2 + \pi r s \\ &= \pi(2)^2 + \pi(2) \times (4.5) \\ &= 40.84 \text{ m}^2 \text{ (to 2 d.p.)} \end{aligned}$$

b
$$\begin{aligned} A &= l^2 + 4 \times \frac{1}{2}bh \\ &= 25^2 + 4 \times \frac{1}{2} \times 25 \times 22 \\ &= 1725 \text{ mm}^2 \end{aligned}$$

EXPLANATION

The cone includes the circular base plus the curved part. Substitute $r = 2$ and $s = 4.5$.



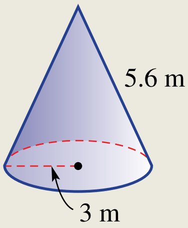
Base area plus four triangular faces.

Now you try

Find the surface area of these solids, using two decimal places for part **a**.

a a cone with radius 3 m and slant height 5.6 m

b a square-based pyramid with square-base length 20 mm and triangular face height 19 mm



Example 25 Finding the slant height and the vertical height of a cone

A cone with radius 3 cm has a curved surface area of 100 cm^2.

a Find the slant height of the cone, correct to one decimal place.

b Find the height of the cone, correct to one decimal place.

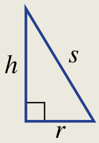
SOLUTION

a
$$\begin{aligned}\text{Surface area} &= \pi r s\\ 100 &= \pi \times 3 \times s\\ s &= \frac{100}{3\pi}\\ &= 10.6\text{ cm (to 1 d.p.)}\end{aligned}$$

EXPLANATION

Substitute the given information into the rule for the curved surface area of a cone and solve for s.

b
$$\begin{aligned}h^2 + r^2 &= s^2\\ h^2 + 3^2 &= \left(\frac{100}{3\pi}\right)^2\\ h^2 &= \left(\frac{100}{3\pi}\right)^2 - 9\\ h &= \sqrt{\left(\frac{100}{3\pi}\right)^2 - 9}\\ &= 10.2\text{ cm (to 1 d.p.)}\end{aligned}$$



Identify the right-angled triangle within the cone and use Pythagoras' theorem to find the height h. Use the exact value of s from part **a** to avoid accumulating errors.

Now you try

A cone with radius 2 cm has a curved surface area of 80 cm^2.

a Find the slant height of the cone, correct to one decimal place.

b Find the height of the cone, correct to one decimal place.

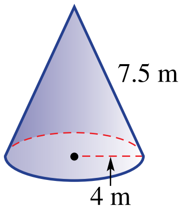
Exercise 4J

FLUENCY	1–3	1–4	1–4

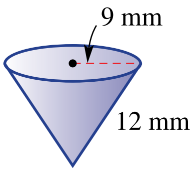
Example 24a

1 Find the surface area of these cones, correct to two decimal places, with radius and slant height as shown.

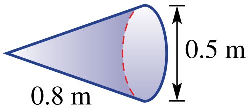
a



b



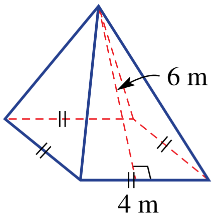
c



Example 24b

2 Find the surface area of these square-based pyramids.

a



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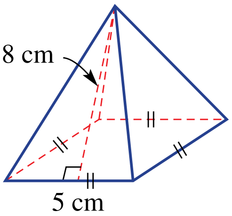

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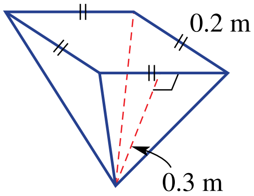
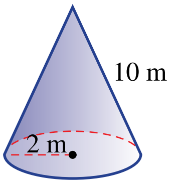


3 For each cone, find the area of the *curved surface* only, correct to two decimal places.

a



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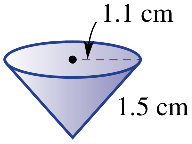

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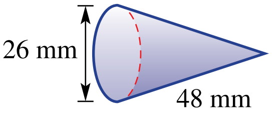


4 A cone has height 10 cm and radius 3 cm.

a Use Pythagoras' theorem to find the slant height of the cone, rounding your answer to two decimal places.

b Find the surface area of the cone, correct to one decimal place.

PROBLEM–SOLVING 5, 6 6–8 6–8, 9($\frac{1}{2}$)

Example 25

5 A cone with radius 5 cm has a curved surface area of 400 cm^2.

a Find the slant height of the cone, correct to one decimal place.

b Find the height of the cone, correct to one decimal place.

6 A cone with radius 6.4 cm has a curved surface area of 380 cm^2.

a Find the slant height of the cone, correct to one decimal place.

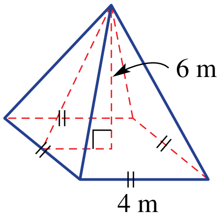
b Find the height of the cone, correct to one decimal place.

7 Party hats A and B are in the shape of open cones with no base. Hat A has radius 7 cm and slant height 25 cm, and hat B has radius 9 cm and slant height 22 cm. Which hat has the greater surface area?

8 This right square-based pyramid has base side length 4 m and vertical height 6 m.

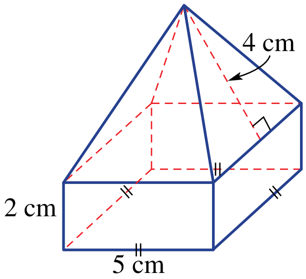
a Find the height of the triangular faces, correct to one decimal place.

b Find the surface area, correct to one decimal place.

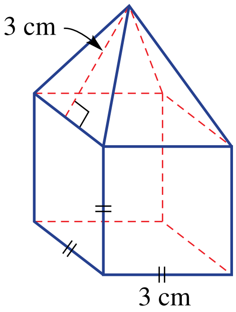


9 Find the surface area of these composite solids, correct to one decimal place as necessary.

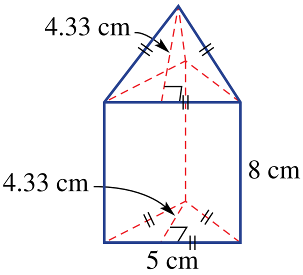
a



b



c



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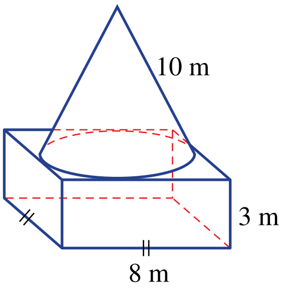
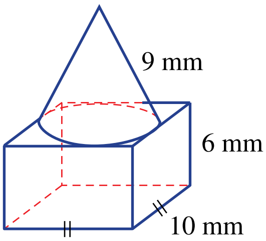
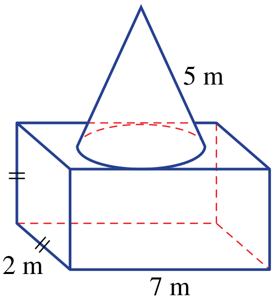


e



f

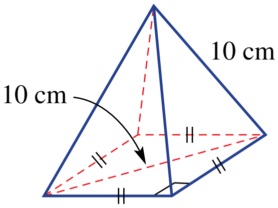


REASONING	10	10, 11	11, 12

10 Explain why the surface area of a cone with radius r and height h is given by the expression $\pi r(r + \sqrt{r^2 + h^2})$.

11 A cone has a height equal to its radius (i.e. $h = r$). Show that its surface area is given by the expression $\pi r^2(1 + \sqrt{2})$.

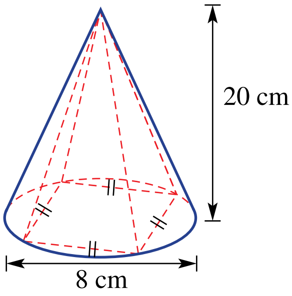
12 There is enough information in this diagram to find the surface area, although the side length of the base and the height of the triangular faces are not given. Find the surface area, correct to one decimal place.



ENRICHMENT: Carving pyramids from cones	–	–	13

13 A woodworker uses a rotating lathe to produce a cone with radius 4 cm and height 20 cm. From that cone the woodworker then cut slices off the sides of the cone to produce a square-based pyramid of the same height.

- **a** Find the exact slant height of the cone.
- **b** Find the surface area of the cone, correct to two decimal places.
- **c** Find the exact side length of the base of the square-based pyramid.
- **d** Find the height of the triangular faces of the pyramid, correct to three decimal places.
- **e** Find the surface area of the pyramid, correct to two decimal places.
- **f** Express the surface area of the pyramid as a percentage of the surface area of the cone. Give the answer correct to the nearest whole percentage.



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4K Volume of prisms and cylinders

LEARNING INTENTIONS

- To understand the concept of volume and capacity of an object
- To know how to use the cross-section of a prism or cylinder to find its volume
- To be able to convert between units of volume and capacity
- To be able to find the volume of right prisms and cylinders
- To be able to identify the regular 3D shapes that comprise a composite solid and find its volume

Volume is the amount of space contained within the outer surfaces of a three-dimensional object and is measured in cubic units.

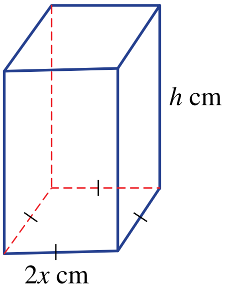
The common groups of objects considered in this section are the prisms and the cylinders.

The volume of grain or cereal that a silo can store is calculated using the volume formulas for cylinders and cones.

Lesson starter: Percentage of volume occupied

Consider the volume of a square-based rectangular prism with side length $2x$ cm and height h cm.

- What is the volume of this prism in terms of x and h?

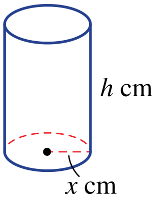


Consider the largest cylinder that can fit inside this prism.

- What is the radius of this cylinder?
- Find the volume of the cylinder using Volume = area of base × height.
- What fraction of the prism's volume does the cylinder occupy? Give this value also to the nearest per cent.

Alternatively, a square-based rectangular prism is fitted in a cylinder of radius x cm and height h cm.

- What is the volume of the cylinder in terms of x and h?
- What is the area of the square base of the rectangular prism?
- Express the volume of the prism as a fraction of the volume of the cylinder. What is this as a percentage?



KEY IDEAS

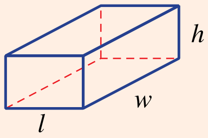
- Metric units for **volume** include cubic kilometres (km^3), cubic metres (m^3), cubic centimetres (cm^3) and cubic millimetres (mm^3).

$$\begin{tikzcd} km^3 \arrow[r, bend left, "\times 1000^3"] & m^3 \arrow[l, bend left, "\div 1000^3"] \arrow[r, bend left, "\times 100^3"] & cm^3 \arrow[l, bend left, "\div 100^3"] \arrow[r, bend left, "\times 10^3"] & mm^3 \arrow[l, bend left, "\div 10^3"] \end{tikzcd}$$

- Units for **capacity** include megalitres (ML), kilolitres (kL), litres (L) and millilitres (mL).
  - $1\ cm^3 = 1\ mL$

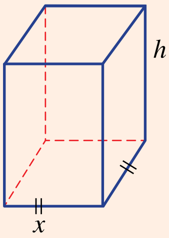
$$\begin{tikzcd} ML \arrow[r, bend left, "\times 1000"] & kL \arrow[l, bend left, "\div 1000"] \arrow[r, bend left, "\times 1000"] & L \arrow[l, bend left, "\div 1000"] \arrow[r, bend left, "\times 1000"] & mL \arrow[l, bend left, "\div 1000"] \end{tikzcd}$$

- For right prisms and cylinders, the volume is given by $V = Ah$, where:
  - A is the area of the base
  - h is the perpendicular height.



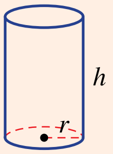
right rectangular prism

$$\begin{aligned} V &= Ah \\ &= lwh \end{aligned}$$



right square prism

$$\begin{aligned} V &= Ah \\ &= x^2h \end{aligned}$$



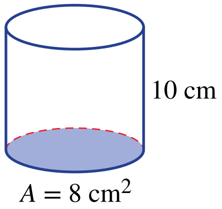
right cylinder

$$\begin{aligned} V &= Ah \\ &= \pi r^2h \end{aligned}$$

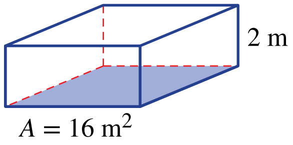
BUILDING UNDERSTANDING

1 Find the volume of these solids with the given base areas.

a



b



2 State the rule for the volume of the following.

a right rectangular prism with length a, width b and height c

b right square prism with base dimensions y by y and perpendicular height h

c cylinder with radius r and height h

Example 26 Converting between units of volume and capacity

Convert these volume measurements to the units given in brackets.

a 0.024 m^3 (cm^3)

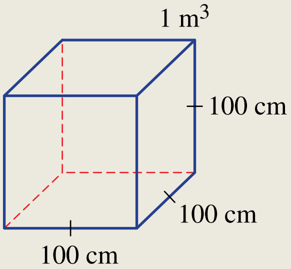
b 12 500 mL (kL)

SOLUTION

a $0.024 \text{ m}^3 = 0.024 \times 100^3 \text{ cm}^3$
$= 0.024 \times 1\,000\,000 \text{ cm}^3$
$= 24\,000 \text{ cm}^3$

EXPLANATION

$1 \text{ m}^3 = 100 \times 100 \times 100 \text{ cm}^3$



b $12\,500 \text{ mL} = 12\,500 \div 1000 \div 1000 \text{ kL}$
$= 0.0125 \text{ kL}$

Divide by 1000 to convert to litres and divide by 1000 again to convert to kilolitres.

Now you try

Convert these volume measurements to the units given in brackets.

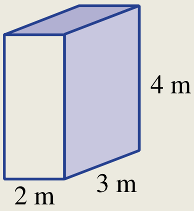
a 42.5 cm^3 (mm^3)

b 124 000 L (ML)

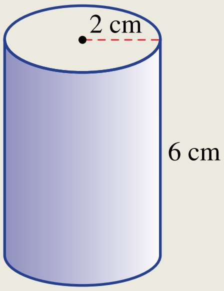
Example 27 Finding the volume of right prisms and cylinders

Find the volume of these solids, rounding to two decimal places for part **b**.

a



b



SOLUTION

a $V = lwh$
$= 2 \times 3 \times 4$
$= 24 \text{ m}^3$

EXPLANATION

Write the volume formula for a rectangular prism.

Substitute $l = 2$, $w = 3$ and $h = 4$.

b $V = \pi r^2 h$
$= \pi(2)^2 \times 6$
$= 75.40 \text{ cm}^3$ (to 2 d.p.)

The prism is a cylinder with base area πr^2.

Substitute $r = 2$ and $h = 6$.

Evaluate and round your answer as required.

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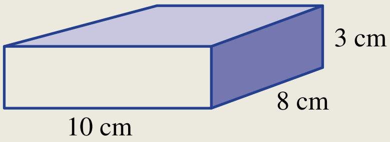
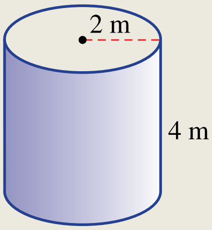
Now you try

Find the volume of these solids, rounding to two decimal places for part **b**.

a

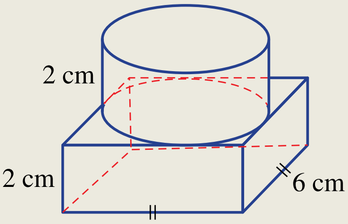


b



Example 28 Finding the volume of a composite solid

Find the volume of this composite solid, correct to the nearest millilitre.



SOLUTION

Radius of cylinder $= \frac{6}{2} = 3$ cm

$$\begin{aligned} V &= lwh + \pi r^2 h \\ &= 6 \times 6 \times 2 + \pi \times 3^2 \times 2 \\ &= 72 + 18\pi \\ &= 128.548\ldots\ \text{cm}^3 \\ &= 129\ \text{mL (to nearest mL)} \end{aligned}$$

EXPLANATION

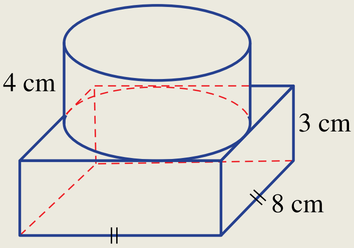
First, find the radius length, which is half the side length of the square base.

Add the volume of the square-based prism and the volume of the cylinder.

1 cm^3 = 1 mL, so 128.548... cm^3 = 128.548... mL and round to the nearest millilitre.

Now you try

Find the volume of this composite solid, correct to the nearest millilitre.



Exercise 4K

FLUENCY	1(½), 2, 3	1(½), 2, 3, 4(½)	1(⅓), 2, 3, 4(½)

Example 26

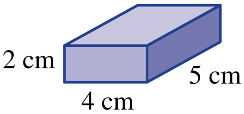
1 Convert these volume measurements to the units given in brackets. (Refer to the Key ideas for help.)

a 2 cm^3 (mm^3)
b 0.2 m^3 (cm^3)
c 0.015 km^3 (m^3)
d 5700 mm^3 (cm^3)
e $28\,300\,000\text{ m}^3$ (km^3)
f $762\,000\text{ cm}^3$ (m^3)
g 0.13 m^3 (cm^3)
h 0.000001 km^3 (m^3)
i 2.094 cm^3 (mm^3)
j 2.7 L (mL)
k 342 kL (ML)
l 35 L (kL)
m 5.72 ML (kL)
n 74 250 mL (L)
o 18.44 kL (L)

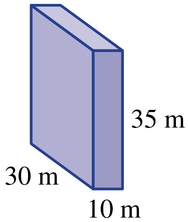
Example 27a

2 Find the volume of each rectangular prism.

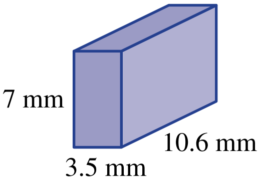
a



b



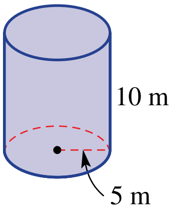
c



Example 27b

3 Find the volume of each cylinder, correct to two decimal places.

a



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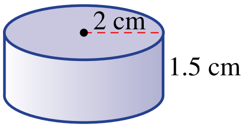

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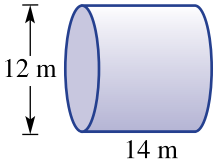
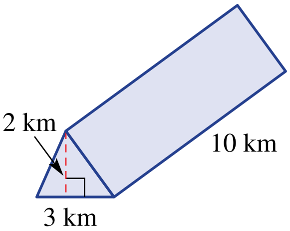
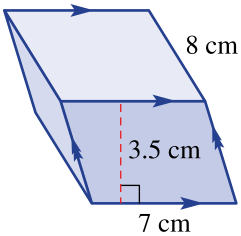


4 Find the volume of these solids, rounding your answers to three decimal places where necessary.

a



b



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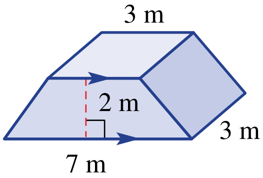

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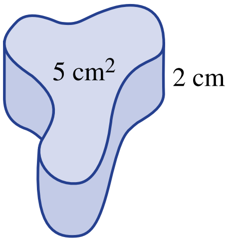

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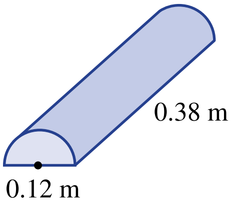
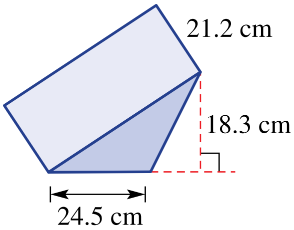


f



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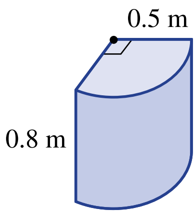

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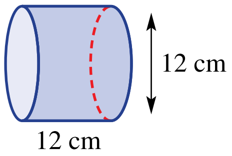
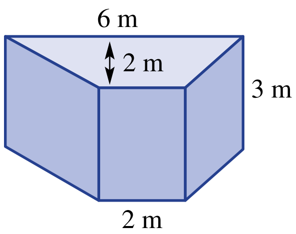


i



PROBLEM–SOLVING

5, 6	6, 7(½)	7(½), 8

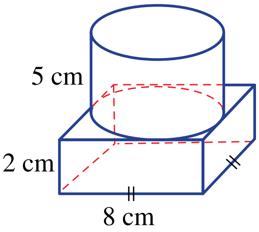
5 How many containers holding 1000 cm^3 (1 L) of water are needed to fill 1 m^3?

6 How many litres of water are required to fill a rectangular fish tank that is 1.2 m long, 80 cm wide and 50 cm high?

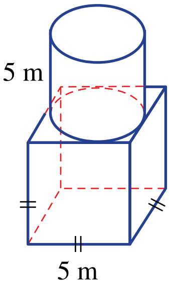
Example 28

7 Find the volume of these composite objects, rounding to two decimal places where necessary in parts **a–c** and to the nearest millilitre in parts **d–f**.

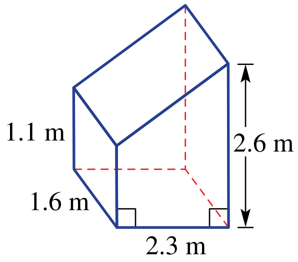
a



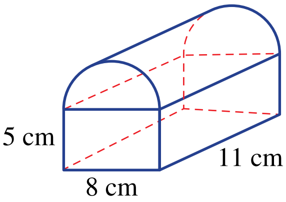
b



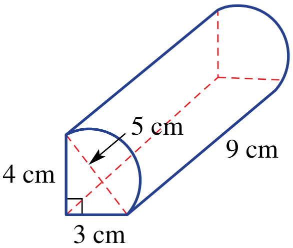
c



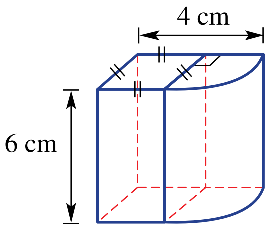
d



e



f



8 Find the exact volume of a cube if its surface area is:

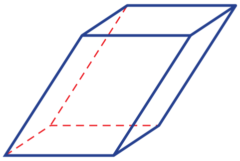
a 54 cm^2

b 18 m^2

REASONING	9	9, 10	10–12

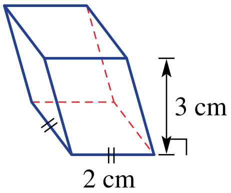
9 Use the rule $V = \pi r^2 h$ to find the height of a cylinder, to one decimal place, with radius 6 cm and volume 62 cm^3.

10 An oblique prism is where the edges and faces are not perpendicular to the base as shown.

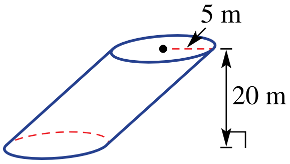


The volume of an oblique prism is calculated as Volume = area of base × perpendicular height. Find the volume of these oblique solids. Round to one decimal place for part **b**.

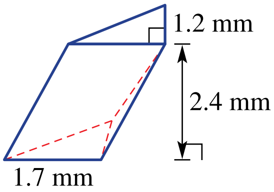
a



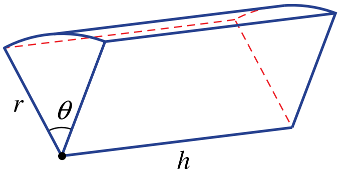
b



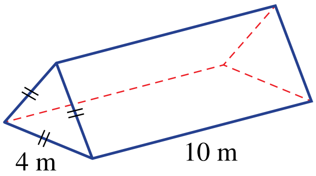
c



11 Find a formula for the volume of a cylindrical portion with angle θ, radius r and height h, as shown.



12 Decide whether there is enough information in this diagram of a triangular prism to find its volume. If so, find the volume, correct to one decimal place.

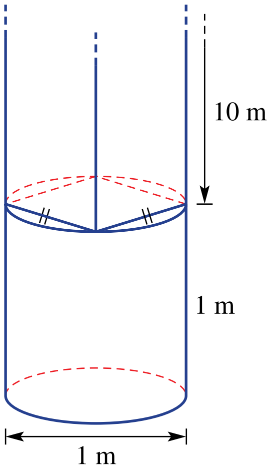


ENRICHMENT: Concrete poles	–	–	13

13 A concrete support structure for a building is made up of a cylindrical base and a square-based prism as the main column. The cylindrical base is 1 m in diameter and 1 m high, and the square prism is 10 m long and sits on the cylindrical base as shown.

a Find the exact side length of the square base of the prism.

b Find the volume of the entire support structure, correct to one decimal place.



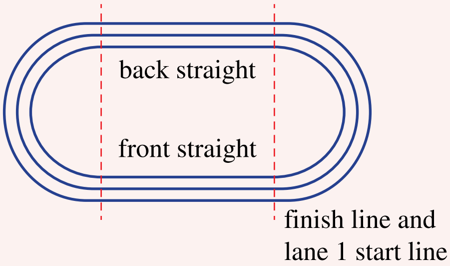
The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Athletics stagger

1 An athletics 400 m track is made up of two straight sections of equal length and two semicircular bends. The first two lanes of the eight lanes of the track are shown.

International regulations state that the radius of the semicircle to the inside edge of the track is 36.500 m.

A coach is interested in the length of each lane of the running track and how a staggered start is used to ensure that each runner has the same distance to cover.



a The 400 m distance for lane 1 is measured for the *running line* of that lane. The *running line* is taken at 300 mm in from the lane's inside edge. By first finding the radius for the lane 1 *running line*, calculate the length of the straight sections, correct to three decimal places.

Each lane is 1.22 m wide, with the running line for each lane after lane 1 considered to be 200 mm in from the lane's inside edge.

b If the competitor from lane 2 started from the lane 1 start line, how far would they be required to run, based on the *running lines*, to complete one lap? Round to two decimal places.

To ensure everyone runs 400 m, competitors need a staggered start.

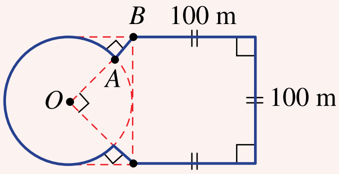
c From your answer to part **b**, what should be the stagger for the lane 2 competitor on their *running line*, correct to two decimal places?

d Calculate the stagger for each of the competitors in lanes 3 to 8, correct to two decimal places.

e Determine a rule for the stagger, s m, of lane number l on the *running line*.

Glass skyscraper

2 A modern city building encased with glass is 120 metres high and has a floor cross-section which combines a portion of a cylinder, triangle and square as shown.



The construction company needs to consider the surface area of the building and decide on a budget for the purchase of glass panels which will be used to clad the building.

a Use the information given in the diagram to find the following distances giving exacts answers.

i OA **ii** OB **iii** AB

b Find the perimeter of the cross-section giving your answer as an exact value.

c Find the exact area of the cross-section.

d Given the height of the building, find the outside glass surface area of the building. Do not include the top or base and round your answer to the nearest square metre.

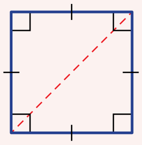
The glass used for the building costs $180 per square metre and the budget provided for the construction company for the purchase of the glass is $10 million.

e Decide if the budget provided for the purchase of the glass is sufficient. Give reasons.

Square diagonals

3 Square sand boxes produced by a company for playgrounds are labelled on the packaging with their diagonal length.

A landscaper is interested in the relationship between this diagonal length and other properties of the sand box including perimeter and area.



a A square sand box has a diagonal length of $\sqrt{3}$ m. Give the area and perimeter of this sand box in simplified form.

b A second square sand box has diagonal length $(2 + 2\sqrt{2})$ m.

- **i** Find the exact area occupied by this sand box in m^2, using $(a + b)(c + d) = ac + ad + bc + bd$ to expand.
- **ii** Express the side length of the sand box in metres in the form $\sqrt{a + b\sqrt{c}}$ where a, b and c are integers.

c To determine the side length of the sand box in part **b** in simplified form, consider the following.

- **i** Use expansion to show that $(\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{xy}$ where x and y are positive integers.
- **ii** Make use of the result in part **i** to simplify $\sqrt{7 + 2\sqrt{10}}$ and $\sqrt{7 + 4\sqrt{3}}$.
- **iii** Hence, simplify your answer to part **b ii** and give the perimeter of the sand box.

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4L Volume of pyramids and cones EXTENDING

LEARNING INTENTIONS

- To understand that the volume of a pyramid or cone is a fraction of the volume of the prism or cylinder with the same base area
- To know the formulas for the volume of pyramids and cones
- To be able to find the volume of pyramids and cones

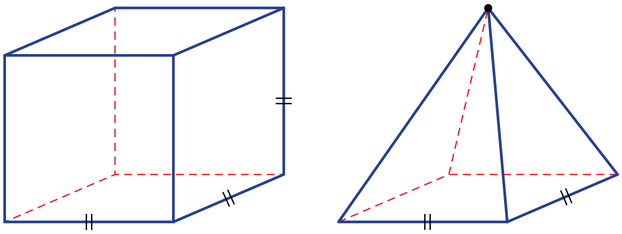
The volume of a cone or pyramid is a certain fraction of the volume of a prism with the same base area.

This particular fraction is the same for both cones and pyramids and will be explored in this section.

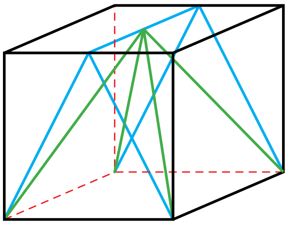
When applying fertiliser, farmers use a container called a spreader, made in a pyramid or cone shape. Agricultural equipment engineers calculate a spreader's volume using pyramid or cone formulas.

Lesson starter: Is a pyramid half the volume of a prism?

Here is a cube and a square pyramid with equal base side lengths and equal heights.



- Discuss whether or not you think the pyramid is half the volume of the cube.
- Now consider this diagram of the cube with the pyramid inside.



The cube is black.
The pyramid is green.
The triangular prism is blue.

- Compared to the cube, what is the volume of the triangular prism (blue)? Give reasons.
- Is the volume of the pyramid (green) more or less than the volume of the triangular prism (blue)?
- Do you know what the volume of the pyramid is as a fraction of the volume of the cube?

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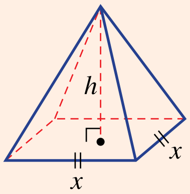
KEY IDEAS

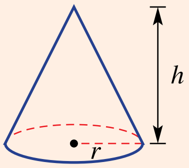
- For pyramids and cones the volume is given by $V = \frac{1}{3}Ah$, where A is the area of the base and h is the perpendicular height.

right square pyramid



$$V = \frac{1}{3}Ah$$
$$= \frac{1}{3}x^2h$$

right cone



$$V = \frac{1}{3}Ah$$
$$= \frac{1}{3}\pi r^2h$$

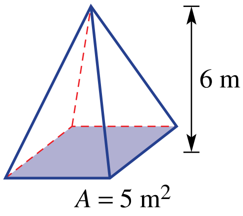
BUILDING UNDERSTANDING

① A cylinder has volume 12 cm^3. What will be the volume of a cone with the same base area and perpendicular height?

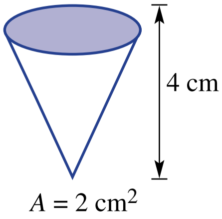
② A pyramid has volume 5 m^3. What will be the volume of a prism with the same base area and perpendicular height?

③ State the volume of these solids with the given base areas.

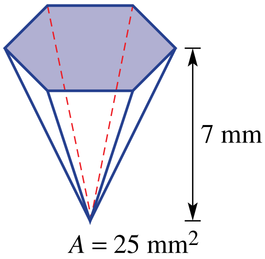
a



b



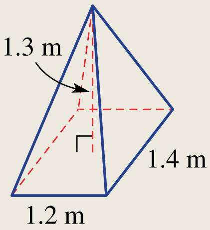
c



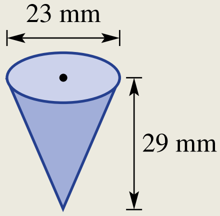
Example 29 Finding the volume of pyramids and cones

Find the volume of this rectangular-based pyramid and cone. Give the answer for part **b**, correct to two decimal places.

a



b

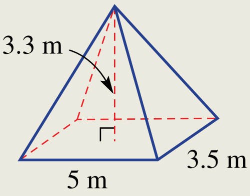


SOLUTION	EXPLANATION
a $V = \frac{1}{3}Ah$	The pyramid has a rectangular base with area $l \times w$.
$= \frac{1}{3}(l \times w) \times h$	
$= \frac{1}{3}(1.4 \times 1.2) \times 1.3$	Substitute $l = 1.4$, $w = 1.2$ and $h = 1.3$.
$= 0.728\text{ m}^3$	
b $V = \frac{1}{3}Ah$	The cone has a circular base of area πr^2.
$= \frac{1}{3}\pi r^2 h$	
$= \frac{1}{3}\pi(11.5)^2 \times 29$	Substitute $r = \frac{23}{2} = 11.5$ and $h = 29$.
$= 4016.26\text{ mm}^3$ (to 2 d.p.)	

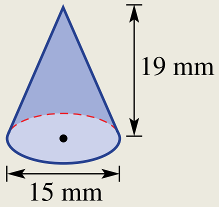
Now you try

Find the volume of this rectangular-based pyramid and cone. Give the answer for part **b**, correct to two decimal places.

a



b



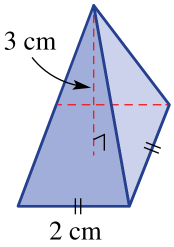
Exercise 4L

FLUENCY	1–2(½)	1–2(½)	1–2(⅓)

Example 29a

1 Find the volume of the following pyramids.

a



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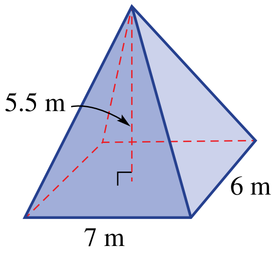

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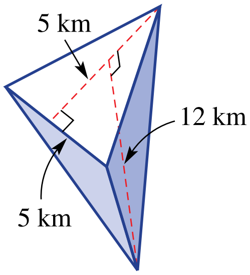
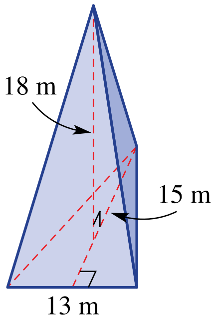


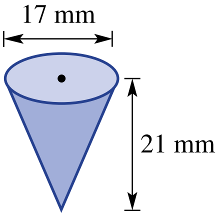
d



Example 29b

2 Find the volume of the following cones, correct to two decimal places.

a



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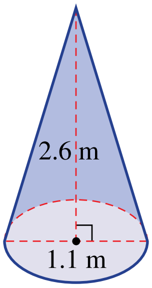

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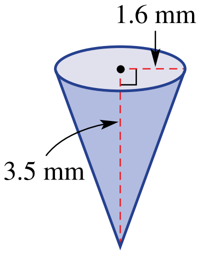

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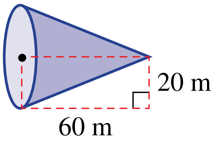
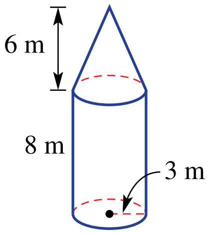


PROBLEM–SOLVING	3, 4(½)	3, 4(½)	4(½), 5

3 A medicine cup is in the shape of a cone with base radius 3 cm and height 5 cm. Find its capacity in millilitres, correct to the nearest millilitre.

4 Find the volume of these composite objects, rounding to two decimal places where necessary.

a



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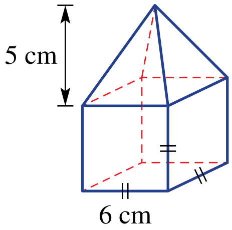

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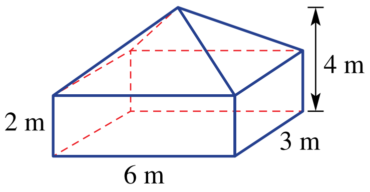

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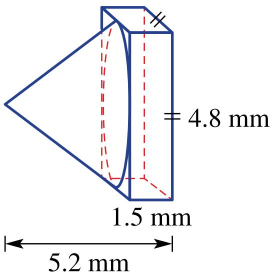

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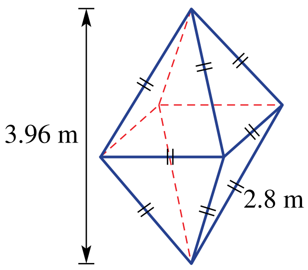

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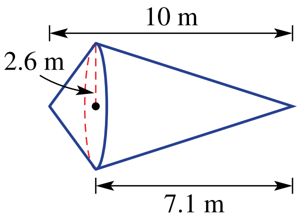
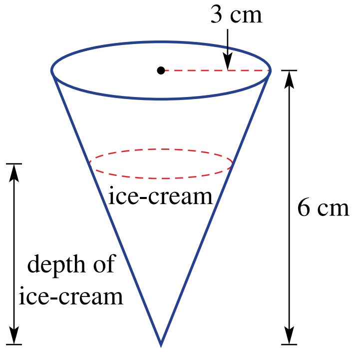


5 The volume of ice-cream in the cone is half the volume of the cone. The cone has a 3 cm radius and 6 cm height. What is the depth of the ice-cream, correct to two decimal places?



REASONING	6	6, 7	7, 8

6 A wooden cylinder is carved to form a cone that has the same base area and the same height as the original cylinder. What fraction of the wooden cylinder is wasted? Give a reason.

7 A square-based pyramid and a cone are such that the diameter of the cone is equal to the length of the side of the square base of the pyramid. They also have the same height.

a Using x as the side length of the pyramid and h as its height, write a rule for:

  i the volume of the pyramid in terms of x and h

  ii the volume of the cone in terms of x and h.

b Express the volume of the cone as a fraction of the volume of the pyramid. Give an exact answer.

8 **a** Use the rule $V = \frac{1}{3}\pi r^2 h$ to find the base radius of a cone, to one decimal place, with height 23 cm and volume 336 cm^3.

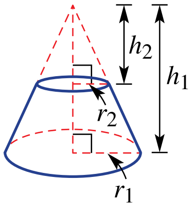
b Rearrange the rule $V = \frac{1}{3}\pi r^2 h$ to write:

  i h in terms of V and r

  ii r in terms of V and h.

ENRICHMENT: Truncated cones	–	–	9

9 A truncated cone is a cone that has its apex cut off by an intersecting plane. In this example, the top has radius r_2, the base has radius r_1 and the two circular ends are parallel.



a Give reasons why $\frac{r_1}{r_2} = \frac{h_1}{h_2}$.

b Find a rule for the volume of a truncated cone.

c Find the volume, to one decimal place, of a truncated cone when $r_1 = 2$ cm, $h_1 = 5$ cm and h_2 equals:

  i $\frac{1}{2}h_1$

  ii $\frac{2}{3}h_1$

4M Surface area and volume of spheres EXTENDING

LEARNING INTENTIONS
- To know the shape of a sphere and a hemisphere
- To know the formulas for the surface area and volume of a sphere and be able to use them
- To be able to use the formulas to find the volume and surface area of composite solids and spherical portions

Planets are spherical in shape due to the effects of gravity. This means that we can describe a planet's size using only one measurement – its diameter or radius. Mars, for example, has a diameter of about half that of the Earth, which is about 12 756 km. The Earth's volume is about 9 times that of Mars and this is because the volume of a sphere varies with the cube of the radius. The surface area of the Earth is about 3.5 times that of Mars because the surface area of a sphere varies with the square of the radius.


Spherical tanks store compressed or liquid gases for the petroleum and chemical industries. The spherical shape uses the smallest land area for the storage volume and distributes pressure evenly over the sphere's surface area.

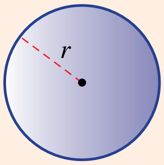
Lesson starter: What percentage of a cube is a sphere?

A sphere of radius 1 unit just fits inside a cube.

- First, guess the percentage of space occupied by the sphere.
- Draw a diagram showing the sphere inside the cube.
- Calculate the volume of the cube and the sphere. For the sphere, use the formula $V = \frac{4}{3}\pi r^3$.
- Now calculate the percentage of space occupied by the sphere. How close was your guess?

KEY IDEAS

- The surface area of a **sphere** depends on its radius, r, and is given by:
  Surface area $= 4\pi r^2$
- The volume of a sphere depends on its radius, r, and is given by:
  Volume $= \frac{4}{3}\pi r^3$



BUILDING UNDERSTANDING

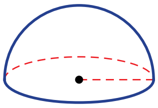
1 Evaluate and round your answer to two decimal places.

a $4 \times \pi \times 5^2$ **b** $4 \times \pi \times \left(\frac{1}{2}\right)^2$ **c** $\frac{4}{3} \times \pi \times 2.8^3$ **d** $\frac{4\pi(7)^3}{3}$

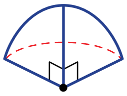
2 Rearrange $12 = 4\pi r^2$ to write r as the subject.

3 What fraction of a sphere is shown in these diagrams?

a



b



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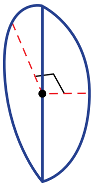


Example 30 Finding the surface area and volume of a sphere

Find the surface area and volume of a sphere of radius 7 cm, correct to two decimal places.

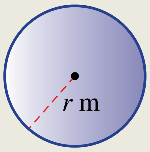
SOLUTION	EXPLANATION
$A = 4\pi r^2$ $= 4\pi(7)^2$ $= 615.75 \text{ cm}^2$ (to 2 d.p.)	Write the rule for the surface area of a sphere and substitute $r = 7$. Evaluate and round the answer.
$V = \frac{4}{3}\pi r^3$ $= \frac{4}{3}\pi(7)^3$ $= 1436.76 \text{ cm}^3$ (to 2 d.p.)	Write the rule for the volume of a sphere and substitute $r = 7$. Evaluate and round the answer.

Now you try

Find the surface area and volume of a sphere of radius 5 cm, correct to two decimal places.

Example 31 Finding the radius of a sphere

Find the radius of a sphere with volume 10 m^3, correct to two decimal places.



SOLUTION

$$\begin{aligned} V &= \tfrac{4}{3}\pi r^3 \\ 10 &= \tfrac{4}{3}\pi r^3 \\ 30 &= 4\pi r^3 \\ \tfrac{15}{2\pi} &= r^3 \\ \therefore r &= \sqrt[3]{\tfrac{15}{2\pi}} \\ &= 1.34 \text{ (to 2 d.p.)} \end{aligned}$$

$\therefore$ The radius is 1.34 m.

EXPLANATION

Substitute $V = 10$ into the formula for the volume of a sphere.

Solve for r^3 by multiplying both sides by 3 and then dividing both sides by 4π. Simplify $\frac{30}{4\pi} = \frac{15}{2\pi}$.

Take the cube root of both sides to make r the subject and evaluate.

Now you try

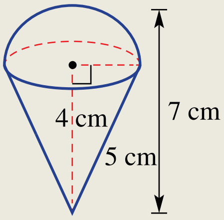
Find the radius of a sphere with volume 6 m^3, correct to two decimal places.

Example 32 Finding the surface area and volume of composite solids with spherical portions

This composite object includes a hemisphere and cone, as shown.

a Find the surface area, rounding to two decimal places.

b Find the volume, rounding to two decimal places.



SOLUTION

a Radius $r = 7 - 4 = 3$

$$\begin{aligned} A &= \tfrac{1}{2} \times 4\pi r^2 + \pi r s \\ &= \tfrac{1}{2} \times 4\pi(3)^2 + \pi(3)(5) \\ &= 33\pi \\ &= 103.67 \text{ cm}^2 \text{ (to 2 d.p.)} \end{aligned}$$

EXPLANATION

First find the radius, r cm, of the hemisphere.
Write the rules for the surface area of each component and note that the top shape is a hemisphere (i.e. half sphere).
Only the curved surface of the cone is required.
Substitute $r = 3$ and $s = 5$.
Simplify and then evaluate, rounding as required.

b

$$\begin{aligned} V &= \tfrac{1}{2} \times \tfrac{4}{3}\pi r^3 + \tfrac{1}{3}\pi r^2 h \\ &= \tfrac{1}{2} \times \tfrac{4}{3}\pi(3)^3 + \tfrac{1}{3}\pi(3)^2(4) \\ &= 18\pi + 12\pi \\ &= 30\pi \\ &= 94.25 \text{ cm}^3 \text{ (to 2 d.p.)} \end{aligned}$$

Volume (object) $= \frac{1}{2}$Volume (sphere) + Volume (cone)

Substitute $r = 3$ and $h = 4$.

Simplify and then evaluate, rounding as required.

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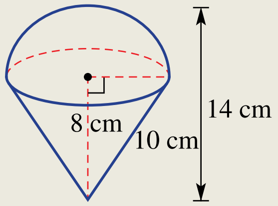
Now you try

This composite object includes a hemisphere and cone, as shown.

a Find the surface area, rounding to two decimal places.

b Find the volume, rounding to two decimal places.

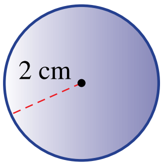


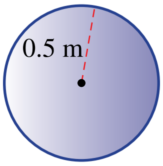
Exercise 4M

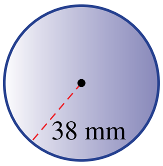
FLUENCY 1–2(½) 1–3(½) 1–3(⅓)

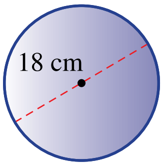
Example 30

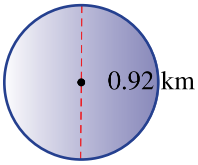
1 Find the surface area and volume of the following spheres, correct to two decimal places.

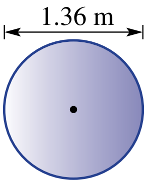
a


b


c


d


e


f


2 Find the surface area and volume of a sphere with the given dimensions. Give the answer correct to two decimal places.

a radius 3 cm

b radius 4 m

c radius 7.4 m

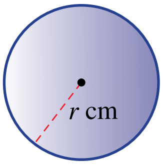
d diameter $\sqrt{5}$ mm

e diameter $\sqrt{7}$ m

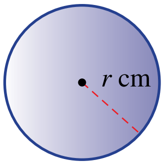
f diameter 2.2 km

Example 31

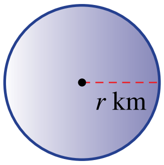
3 **a** Find the radius of these spheres with the given volumes, correct to two decimal places.

i


$V = 15\text{ cm}^3$

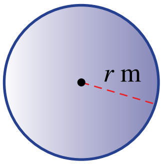
ii


$V = 180\text{ cm}^3$

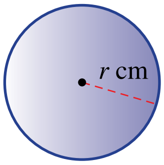
iii


$V = 0.52\text{ km}^3$

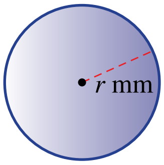
b Find the radius of these spheres with the given surface area, correct to two decimal places.

i


$A = 10\text{ m}^2$

ii


$A = 120\text{ cm}^2$

iii


$A = 0.43\text{ mm}^2$

PROBLEM–SOLVING	4–6	6, 7, 8–9($\frac{1}{2}$)	8–9($\frac{1}{2}$), 10–12

4 A box with dimensions 30 cm long, 30 cm wide and 30 cm high holds 50 tennis balls of radius 3 cm. Find:

a the volume of one tennis ball, correct to two decimal places

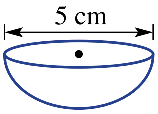
b the volume of 50 tennis balls, correct to one decimal place

c the volume of the box not taken up by the tennis balls, correct to one decimal place.

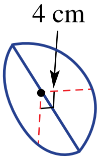
5 An expanding spherical storage bag has 800 cm^3 of water pumped into it. Find the diameter of the bag, correct to one decimal place, after all the water has been pumped in.

6 Find the volume of these portions of a sphere, correct to two decimal places.

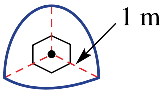
a



b



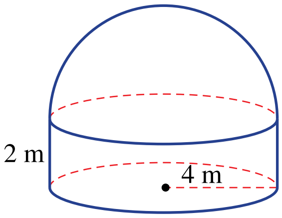
c



7 A monolithic structure has a cylindrical base of radius 4 m and height 2 m and a hemispherical top.

a What is the radius of the hemispherical top?

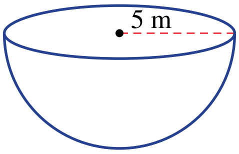
b Find the total volume of the entire monolithic structure, correct to one decimal place.



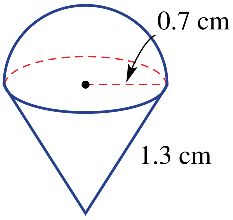
Example 32a

8 Find the surface area for these solids, correct to two decimal places.

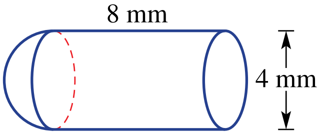
a



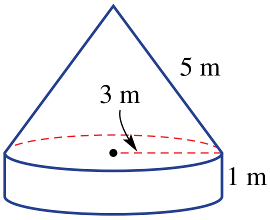
b



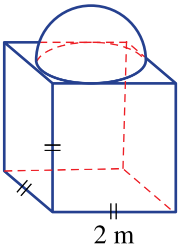
c



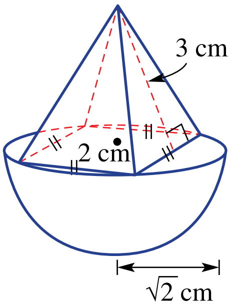
d



e



f



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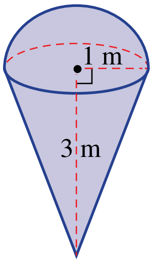
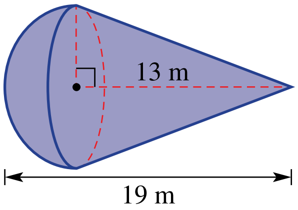
Example 32b

9 Find the volume of the following composite objects, correct to two decimal places.

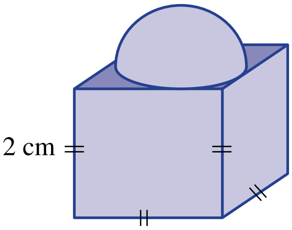
a



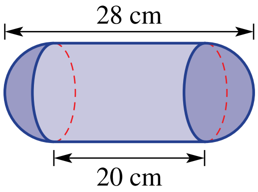
b



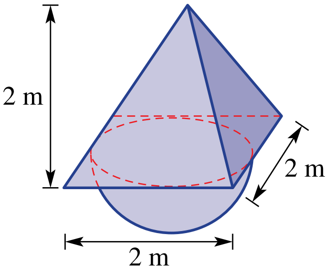
c



d



e



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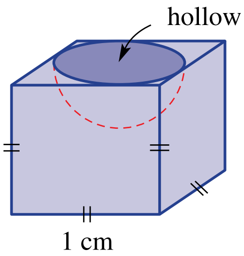


10 A sphere just fits inside a cube. What is the surface area of the sphere as a percentage of the surface area of the cube? Round your answer to the nearest whole percentage.

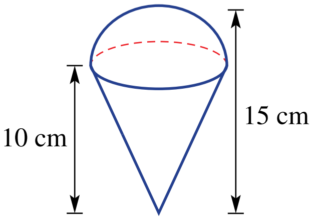
11 A spherical party balloon is blown up to help decorate a room.
If the balloon pops when the volume of air reaches 120 000 cm^3, find the diameter of the balloon at that point, correct to one decimal place.

12 A hemisphere sits on a cone and two height measurements are given as shown. Find:

a the radius of the hemisphere

b the exact slant height of the cone in surd form

c the surface area of the solid, correct to one decimal place.



REASONING	13	13, 14	14–16

13 **a** Find a rule for the radius of a sphere with surface area A.
b Find a rule for the radius of a sphere with volume V.

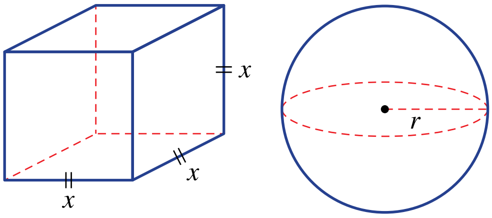
14 A ball's radius is doubled.
a By how much does its surface area change?
b By how much does its volume change?

15 Show that the volume of a sphere is given by $V = \frac{1}{6}\pi d^3$, where d is the diameter.

16 A cylinder and a sphere have the same radius, r, and volume, V. Find a rule for the height of the cylinder in terms of r.

ENRICHMENT: Comparing surface areas	–	–	17

17 Imagine a cube and a sphere that have the same volume.



a If the sphere has volume 1 unit^3, find:
  i the exact radius of the sphere
  ii the exact surface area of the sphere
  iii the value of x (i.e. the side length of the cube)
  iv the surface area of the cube
  v the surface area of the sphere as a percentage of the surface area of the cube, correct to one decimal place.

b Now take the radius of the sphere to be r units. Write:
  i the rule for the surface area of the sphere
  ii the rule for x in terms of r, given the volumes are equal
  iii the surface area of the cube in terms of r.

c Now write the surface area of the sphere as a fraction of the surface area of the cube, using your results from part **b** and simplify to show that the result is $\sqrt[3]{\frac{\pi}{6}}$.

d Compare your answers from part **a v** with that of part **c** (i.e. as a percentage).

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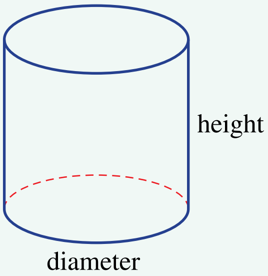
Cylindrical park seats

A local council is designing a simple but artistic set of cylindrical park seats, each with a volume of $0.75\,\text{m}^3$ and made from poured concrete. The top and curved surface areas are then to be painted. The council wants each cylinder to be least 30 cm high and is keen to also minimise the amount of paint needed.



Present a report for the following tasks and ensure that you show clear mathematical workings and explanations where appropriate.

Preliminary task

a If the diameter of the cylinder is 90 cm, find the height given that the volume must be $0.75\,\text{m}^3$. Round to the nearest cm.

b Find the surface area to be painted for the cylinder found in part **a.** Round to the nearest cm^2.

c If the height of the cylinder is 70 cm, find the diameter given that the volume must be $0.75\,\text{m}^3$. Round to the nearest cm. Does this cylinder meet the council's conditions?

d Find the surface area to be painted for the cylinder found in part **c**. Round to the nearest cm^2.

Modelling task

Formulate

a The problem is to find suitable dimensions for the cylindrical seats which meet the council's conditions. Write down all the relevant information that will help solve this problem with the aid of a diagram.

b Use the rule for the volume of a cylinder ($V = \pi r^2 h$) with ($V = 0.75\,\text{m}^3$) and determine a rule for:
  - **i** h in terms of r
  - **ii** r in terms of h.

Solve

c For two chosen values of r determine:
  - **i** the height of the cylindrical seats
  - **ii** the surface area to be painted.

d For two chosen values of h determine:
  - **i** the diameter of the cylindrical seats
  - **ii** the surface area to be painted.

Evaluate and verify

e Compare your choices of r and h and decide which choice gives the least surface area.

f Investigate other possible choices of r and h and try to minimise the surface area.

Communicate

g Summarise your results and describe any key findings.

Extension question

a Derive a rule for the surface area of the cylinder in terms of r only. Use a graph to find the dimensions of the cylinder which gives the minimum surface area. (Use technology where appropriate.)

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Maximising and minimising with solids

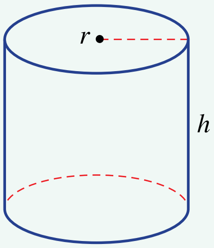
Key technology: Spreadsheets

When working with solids like prisms and cylinders, you might be interested in either of the following:

- Minimising the surface area for a fixed volume
- Maximising the volume for a fixed surface area.

You will recall these rules for the surface area and volume of cylinders.

$$A = 2\pi r^2 + 2\pi rh$$
$$V = \pi r^2 h$$



1 Getting started

A company is making drums to hold chemicals and requires each cylindrical drum to be 50 litres in volume which is 50 000 cm^3.

a Use the volume formula for a cylinder to show that $h = \dfrac{50\,000}{\pi r^2}$.

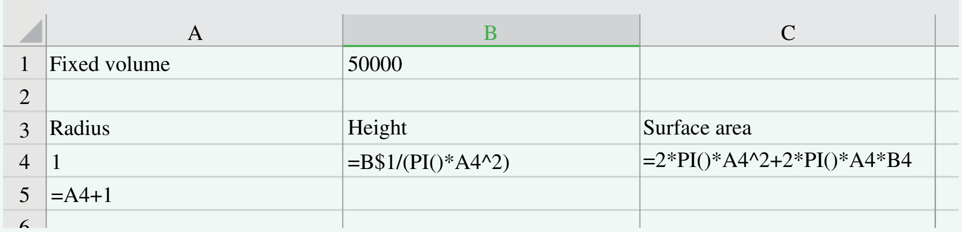
b Find the height of the cylinder correct to two decimal places if the radius is:
  i 20 cm **ii** 10 cm

c Find the surface area of the cylinder correct to two decimal places if the radius is:
  i 20 cm **ii** 10 cm

d Find a radius that gives a smaller surface area compared to the examples in part **c** above.

2 Using technology

a Construct a spreadsheet to find the height and surface area for a cylinder with fixed volume 50 000 cm^3. Use a radius of 1 cm to start and increase by 1 cm each time as shown.



	A	B	C
1	Fixed volume	50000	
2			
3	Radius	Height	Surface area
4	1	=B$1/(PI()*A4^2)	=2*PI()*A4^2+2*PI()*A4*B4
5	=A4+1		
6			

b Fill down from cells A5, B4 and C4 to find the heights and surface areas for cylindrical drums of volume 50 000 cm^3. Locate the integer radius value which provides the minimum surface area.

c Do you think that the integer value of the radius gives the true minimum value of the surface area? Give reasons.

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3 Applying an algorithm

We will now systematically alter the increment made to the radius value in our spreadsheet to find a more accurate solution.

a Apply this algorithm to your spreadsheet and continue until you are satisfied that you have found the radius value that minimises the surface area correct to two decimal places.

- Step 1: Alter the formula in cell A5 so that the increment is smaller. e.g. 0.1 rather than 1.
- Step 2: Fill down until you have located the radius value that minimises the surface area.
- Step 3: Adjust cell A4 to a higher value so you don’t need to scroll through so many cells.
- Step 4: Repeat from Step 1 but use smaller and smaller increments (0.01 and 0.001) until you have found the radius value which minimises the surface area correct to two decimal places.

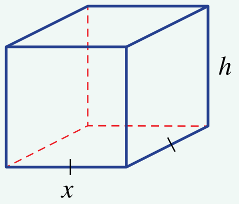
b Write down the value for r, h and A correct to two decimal places which gives the minimum surface area of a cylindrical drum.

c Now alter the fixed volume of the cylinder and repeat the above algorithm.

d What do you notice about the relationship between r and h at the point where there is a minimum surface area? Experiment with different volumes to confirm your conjecture.

4 Extension

Now imagine you are trying to maximise a given volume given a fixed surface area of 1000 cm^2 for a square-based prism as shown.



a Find a rule for h in terms of x using a surface area formula.

b Set up a spreadsheet like the one for the cylinder with columns for x, h and the volume V.

c Use your spreadsheet to find the values of x and h which maximises the volume for the given surface area. Describe the shape of the prism that results from these values of x and h. Confirm your ideas by experimenting with different fixed surface areas.

Density

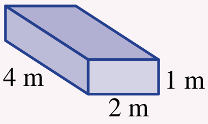
Density is defined as the mass or weight of a substance per cubic unit of volume.

$$\text{Density} = \frac{\text{mass}}{\text{volume}} \quad \text{or} \quad \text{Mass} = \text{density} \times \text{volume}$$

Finding mass

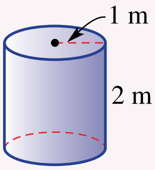
Find the total mass of these objects with the given densities, correct to one decimal place where necessary.

a



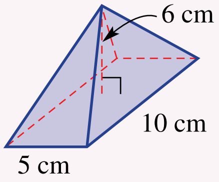
Density = 50 kg per m^3

b



Density = 100 kg per m^3

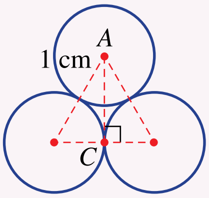
c

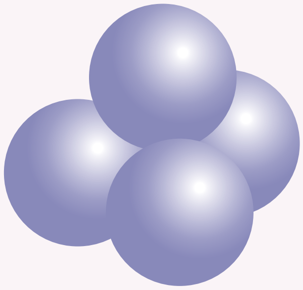


Density = 0.05 kg per cm^3

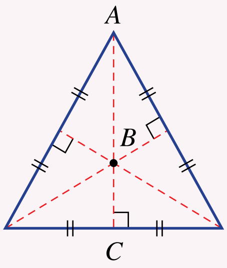
Finding density

- **a** Find the density of a compound with the given mass and volume measurements, rounding to two decimal places where necessary.
  - **i** mass 30 kg, volume 0.4 m^3
  - **ii** mass 10 g, volume 2 cm^3
  - **iii** mass 550 kg, volume 1.8 m^3
- **b** The density of a solid depends somewhat on how its molecules are packed together. Molecules represented as spheres are tightly packed if they are arranged in a triangular form. The following relates to this packing arrangement.



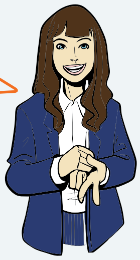


  - **i** Find, the length AC for three circles, each of radius 1 cm, as shown. Use exact values.
  - **ii** Find the total height of four spheres, each of radius 1 cm, if they are packed to form a triangular-based pyramid. Use exact values. First, note that $AB = 2BC$ for an equilateral triangle (shown at right). Pythagoras' theorem can be used to prove this, but this is trickier.



Problems and challenges

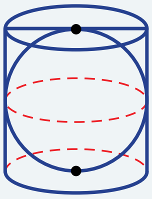
Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



1 A cube has a surface area that has the same value as its volume. What is the side length of this cube?

2 The wheels of a truck travelling at 60 km/h make 4 revolutions per second. What is the diameter of each wheel in metres, correct to one decimal place?

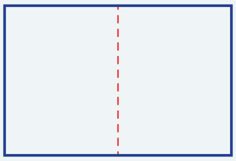
3 A sphere fits exactly inside a cylinder, and just touches the top, bottom and curved surface.



- **a** Show that the surface area of the sphere equals the curved surface area of the cylinder.
- **b** What percentage of the volume of the cylinder is taken up by the sphere? Round your answer to the nearest whole percentage.

4 A sphere and cone with the same radius, r, have the same volume. Find the height of the cone in terms of r.

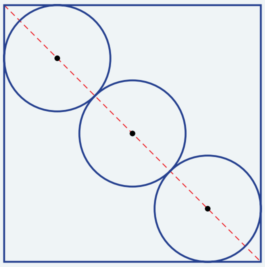
5 A rectangular piece of paper has an area of $100\sqrt{2}\ \text{cm}^2$. The piece of paper is such that, when it is folded in half along the dashed line as shown, the new rectangle is similar (i.e. of the same shape) to the original rectangle. What are the dimensions of the piece of paper?



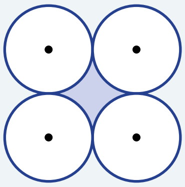
6 Simplify the following, leaving your answer with a rational denominator.

$$\frac{\sqrt{2}}{2\sqrt{2}+1}+\frac{2}{\sqrt{3}+1}$$

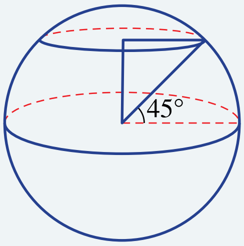
7 Three circles, each of radius 1 unit, fit inside a square such that the two outer circles touch the middle circle and the sides of the square, as shown. Given the centres of the circle lie on the diagonal of the square, find the exact area of the square.



8 Four of the same circular coins of radius r are placed such that they are just touching, as shown. What is the area of the shaded region enclosed by the coins in terms of r?



9 Find the exact ratio of the equator to the distance around Earth at latitude 45° north. (Assume that Earth is a perfect sphere.)



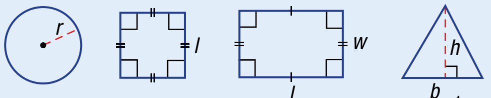
Measurement and surds

Area

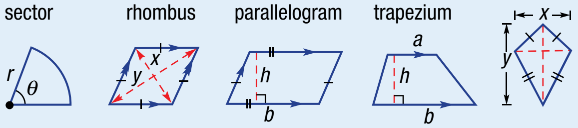
Units of area

$$\text{km}^2 \underset{\div 1000^2}{\overset{\times 1000^2}{\rightleftarrows}} \text{m}^2 \underset{\div 100^2}{\overset{\times 100^2}{\rightleftarrows}} \text{cm}^2 \underset{\div 10^2}{\overset{\times 10^2}{\rightleftarrows}} \text{mm}^2$$

Formulas

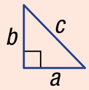


circle	square	rectangle	triangle
$A = \pi r^2$	$A = l^2$	$A = lw$	$A = \frac{1}{2}bh$



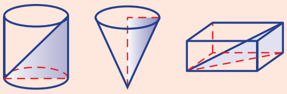
sector	rhombus	parallelogram	trapezium	kite
$A = \frac{\theta}{360} \times \pi r^2$	$A = \frac{1}{2}xy$	$A = bh$	$A = \frac{1}{2}(a+b)h$	$A = \frac{1}{2}xy$

Pythagoras' theorem



$c^2 = a^2 + b^2$

Can occur in 3D shapes



Length

Units of length

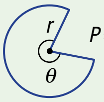
$$\text{km} \underset{\div 1000}{\overset{\times 1000}{\rightleftarrows}} \text{m} \underset{\div 100}{\overset{\times 100}{\rightleftarrows}} \text{cm} \underset{\div 10}{\overset{\times 10}{\rightleftarrows}} \text{mm}$$

Perimeter is the distance around the outside of a closed shape.

Circumference of a circle

$C = 2\pi r = \pi d$

Perimeter of a sector



$P = 2r + \frac{\theta}{360} \times 2\pi r$

Simplifying surds (Opt)

A surd uses symbol $\sqrt{}$ and as a decimal is infinite and non-recurring

Simplify surds: use the highest square factor.

e.g. **1** $\sqrt{20} = \sqrt{4 \times 5}$
$= \sqrt{4} \times \sqrt{5}$
$= 2\sqrt{5}$

2 $4\sqrt{27} = 4 \times \sqrt{9} \times \sqrt{3}$
$= 12\sqrt{3}$

Multiply/Divide (Opt)

$\sqrt{x} \times \sqrt{y} = \sqrt{xy}$

$a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$

$\sqrt{x} \div \sqrt{y} = \sqrt{\frac{x}{y}}$

$a\sqrt{x} \div (b\sqrt{y}) = \frac{a}{b}\sqrt{\frac{x}{y}}$

Rationalise the denominator (Opt)

Express with a whole number in the denominator

$\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{2\sqrt{5}}{5}$

Add/Subtract (Opt)

Like surds only

e.g. **1** $3\sqrt{2} + 4\sqrt{2} = 7\sqrt{2}$

$4\sqrt{2} + \sqrt{5} - \sqrt{2} = 3\sqrt{2} + \sqrt{5}$

Simplify first

e.g. **2** $3\sqrt{8} - 4\sqrt{2}$
$= 3 \times \sqrt{4} \times \sqrt{2} - 4\sqrt{2}$
$= 6\sqrt{2} - 4\sqrt{2}$
$= 2\sqrt{2}$

Errors and accuracy

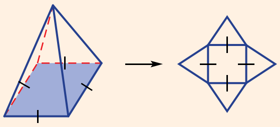
Limits of accuracy are usually ± 0.5 × the smallest unit of measurement e.g. 72 cm is 71.5 cm to 72.5 cm

Errors will accumulate if answers are rounded in intermediate calculations.

Surface area

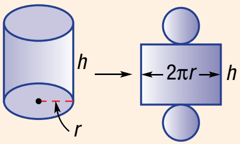
For prisms and pyramids (Ext) draw the net and add the areas of all the faces.

e.g.



$A = 4 \times$ triangles + square base

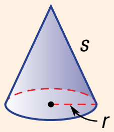
Cylinder



$A = 2\pi rh + 2\pi r^2$

curved surface + base ends

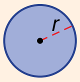
Cone (Ext)



$A = \pi rs + \pi r^2$

curved + base

Sphere (Ext)



$A = 4\pi r^2$

For composite solids consider which surfaces are exposed.

Volume

Units of volume

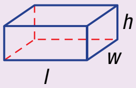
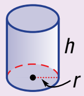
$$\text{km}^3 \underset{\div 1000^3}{\overset{\times 1000^3}{\rightleftarrows}} \text{m}^3 \underset{\div 100^3}{\overset{\times 100^3}{\rightleftarrows}} \text{cm}^3 \underset{\div 10^3}{\overset{\times 10^3}{\rightleftarrows}} \text{mm}^3$$

Units of capacity

$$\text{megalitres (ML)} \underset{\div 1000}{\overset{\times 1000}{\rightleftarrows}} \text{kilolitres (kL)} \underset{\div 1000}{\overset{\times 1000}{\rightleftarrows}} \text{litres (L)} \underset{\div 1000}{\overset{\times 1000}{\rightleftarrows}} \text{millilitres (mL)}$$

$1 \text{ cm}^3 = 1 \text{ mL}$

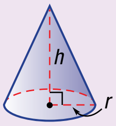
For right prisms and cylinders $V = Ah$, where A is the area of the base and h is the perpendicular height.

rectangular prism	cylinder
	
$V = lwh$	$V = \pi r^2 h$

For pyramids and cones: (Ext)

$V = \frac{1}{3}Ah$, where A is the area of the base.

Cone: $V = \frac{1}{3}\pi r^2 h$



Sphere: $V = \frac{4}{3}\pi r^3$ (Ext)

Chapter checklist with success criteria

A printable version of this checklist is available in the Interactive Textbook ✔

4A **1. I can decide if a number is rational or irrational.** Opt ☐
e.g. Express $\sqrt{6}$ as a decimal and decide if it is rational or irrational.

4A **2. I can simplify a surd using the highest square number factor.** Opt ☐
e.g. Simplify $\sqrt{75}$.

4A **3. I can express a surd as a square root of an integer.** Opt ☐
e.g. Express $3\sqrt{7}$ in the form $\sqrt{a}$ where a is an integer.

4B **4. I can add and subtract expressions involving like surds.** Opt ☐
e.g. Simplify $\sqrt{3} + 5\sqrt{2} + 3\sqrt{3} - 2\sqrt{2}$.

4B **5. I can simplify surds to add or subtract.** Opt ☐
e.g. Simplify $3\sqrt{12} + 2\sqrt{27}$.

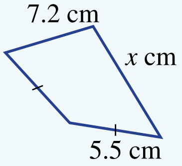
4C **6. I can multiply surds.** Opt ☐
e.g. Simplify $3\sqrt{5} \times 2\sqrt{10}$.

4C **7. I can simplify surds using division.** Opt ☐
e.g. Simplify $4\sqrt{15} \div \sqrt{3}$.

4C **8. I can apply the distributive law to expressions involving surds.** Opt ☐
e.g. Expand and simplify $2\sqrt{5}(3\sqrt{3} - \sqrt{5})$.

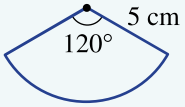
4D **9. I can rationalise a denominator.** Opt ☐
e.g. Rationalise $\frac{4\sqrt{2}}{\sqrt{7}}$.

4E **10. I can work with the perimeter of a polygon.** ☐
e.g. Find the value of x when the perimeter is 26.6 cm.

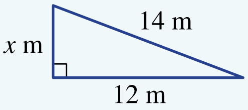


4E **11. I can use the formula for the circumference of a circle.** ☐
e.g. A circle has a radius of r cm, if its circumference is 12 cm, find r correct to two decimal places.

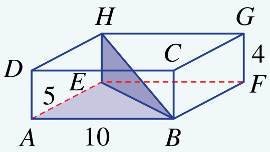
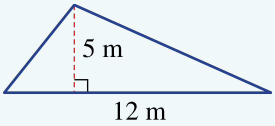
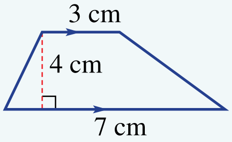
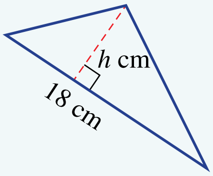
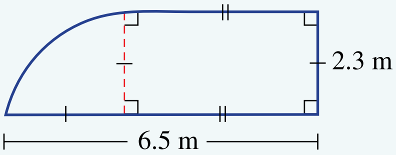
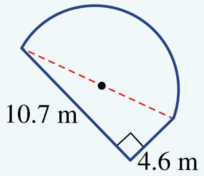
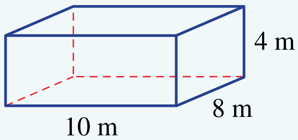
4E **12. I can find the perimeter of a sector.** ☐
e.g. Find the exact perimeter of this sector.



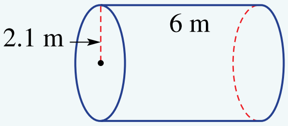
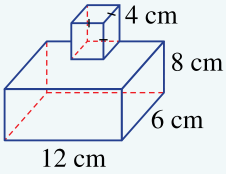
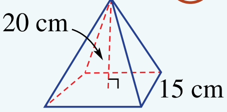
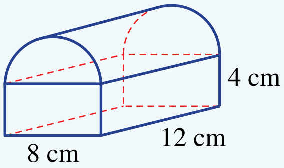
4F **13. I can find a side length using Pythagoras' theorem.** ☐
e.g. Find the length of the unknown side correct to two decimal places.



Chapter checklist with success criteria

		✔
4F	**14. I can use Pythagoras' theorem in 3D.** e.g. For the rectangular prism shown find *BE* in exact form and find *BH* correct to two decimal places. 	☐
4G	**15. I can convert between units of area.** e.g. Convert 1200 cm^2 to m^2.	☐
4G	**16. I can work with area formulas.** e.g. Find the area of these shapes.  	☐
4G	**17. I can use a given area to find an unknown length.** e.g. Find the value of the pronumeral for this shape with area 54 cm^2. 	☐
4G	**18. I can find the area of composite shapes involving sectors.** e.g. Find the area of this composite shape. Give your answer as an exact value and as a decimal correct to two decimal places. 	☐
4H	**19. I can avoid accumulating errors from rounding.** e.g. a Calculate the area of this composite shape: i. by rounding each value calculated to one decimal place ii. by working with exact answers throughout. b Compare your answers. Explain why they are different. 	☐
4H	**20. I can find limits of accuracy.** e.g. Give the limits of accuracy for 85 cm.	☐
4H	**21. I can work with limits of accuracy.** e.g. Each side of a square is measured as 10 cm. Find the upper and lower limits for the perimeter of the square.	☐
4I	**22. I can find the surface area of a prism.** e.g. Find the surface area of this rectangular prism. 	☐

Chapter checklist with success criteria

		✔
4I	**23. I can find the surface area of a cylinder.** e.g. Find the surface area of the cylinder shown correct to two decimal places.  	☐
4I	**24. I can find the surface area of composite solids.** e.g. A composite solid consists of a rectangular prism and a cube as shown. Find the surface area of the object.  	☐
4J	**25. I can find the surface area of a cone.** Ext e.g. Find the surface area of a cone with radius 4 cm and slant height 10 cm. Round to two decimal places.	☐
4J	**26. I can find the surface area of a pyramid.** Ext e.g. Find the surface area of the square-based pyramid shown.  	☐
4J	**27. I can find the slant height or vertical height of a cone.** Ext e.g. A cone has radius 5 cm and a curved surface area of 120 cm^2. Find the vertical height of the cone correct to one decimal place.	☐
4K	**28. I can convert between units of volume.** e.g. Convert these volume measurements to the units shown in brackets. i 1.2 m^3 (cm^3) ii 1 490 000 mL (kL)	☐
4K	**29. I can find the volume of a cylinder or prism.** e.g. Find the volume of a cylinder with height 10 cm and radius 2 cm. Round to two decimal places.	☐
4K	**30. I can find the volume of a composite solid.** e.g. Find the volume of the composite solid correct to one decimal place.  	☐

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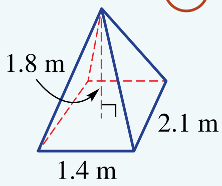
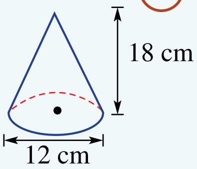
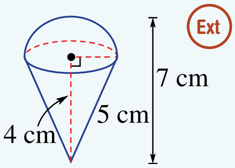
Chapter checklist with success criteria

		✔
4L	**31. I can find the volume of a pyramid.** Ext e.g. Find the volume of this rectangular-based pyramid.  	☐
4L	**32. I can find the volume of a cone.** Ext e.g. Find the volume of this cone correct to two decimal places.  	☐
4M	**33. I can find the surface area and volume of a sphere.** Ext e.g. Find the volume and surface area of a sphere of radius 3 cm, correct to two decimal places.	☐
4M	**34. I can find the radius of a sphere.** Ext e.g. Find the radius of a sphere with volume 12 m^3, correct to two decimal places.	☐
4M	**35. I can work with composite solids with spherical portions.** Ext e.g. Find the surface area and volume of this cone and hemisphere correct to two decimal places.  	☐

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Short-answer questions

4A Opt

1 Simplify the following surds.

a $\sqrt{24}$ **b** $3\sqrt{200}$ **c** $\sqrt{\frac{8}{9}}$ **d** $\frac{2\sqrt{45}}{15}$

4B/C Opt

2 Simplify the following.

a $2\sqrt{3}+4+5\sqrt{3}$ **b** $6\sqrt{5}-\sqrt{7}-4\sqrt{5}+3\sqrt{7}$ **c** $\sqrt{8}+3\sqrt{2}$

d $4\sqrt{3}+2\sqrt{18}-4\sqrt{2}$ **e** $2\sqrt{5}\times\sqrt{6}$ **f** $-3\sqrt{2}\times 2\sqrt{10}$

g $\frac{2\sqrt{15}}{\sqrt{3}}$ **h** $\frac{5\sqrt{14}}{15\sqrt{2}}$ **i** $\frac{\sqrt{27}}{3}-\sqrt{3}$

4C Opt

3 Expand and simplify.

a $\sqrt{2}(2\sqrt{3}+4)$ **b** $2\sqrt{3}(2\sqrt{15}-\sqrt{3})$ **c** $(\sqrt{11})^2$ **d** $(4\sqrt{3})^2$

4D Opt

4 Rationalise the denominator.

a $\frac{1}{\sqrt{6}}$ **b** $\frac{6\sqrt{3}}{\sqrt{2}}$ **c** $\frac{3\sqrt{3}}{2\sqrt{6}}$ **d** $\frac{4\sqrt{2}-\sqrt{3}}{\sqrt{3}}$

4E/G/K

5 Convert the given measurements to the units in the brackets.

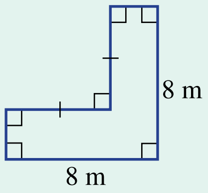
a 0.23 m (cm) **b** 270 mm^2 (cm^2) **c** 2.6 m^3 (cm^3)

d 8.372 L (mL) **e** 638 250 mm^2 (m^2) **f** 0.0003 km^2 (cm^2)

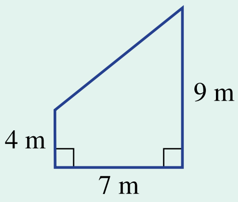
4E/F

6 Find the perimeter of these shapes, correct to one decimal place where necessary. (Note: Pythagoras' theorem may be required in part **b**.)

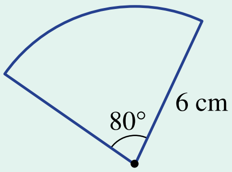
a



b



c



4E/G

7 A floral clock at the Botanic Gardens is in the shape of a circle and has a circumference of 14 m.

a Find the radius of the clock, in exact form.

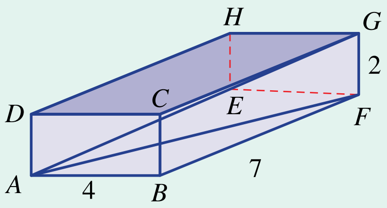
b Hence, find the area occupied by the clock. Answer to two decimal places.

4F

8 For the rectangular prism with dimensions as shown, use Pythagoras' theorem to find:

a AF, leaving your answer in exact form

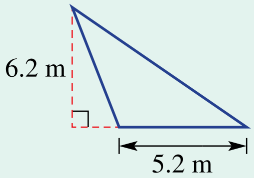
b AG, to two decimal places.



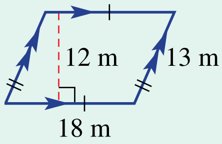
4G

9 Find the area of these shapes. Round to two decimal places where necessary.

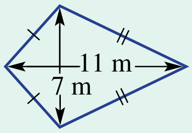
a



b



c



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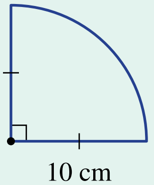

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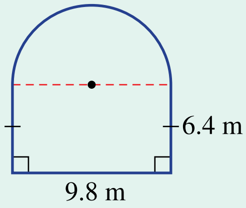

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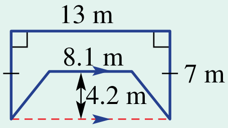
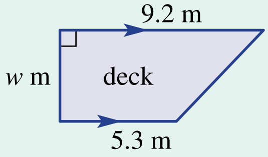


4G

10 A backyard deck, as shown, has an area of 34.8 m^2.

a Find the width, w metres, of the deck.

b Calculate the perimeter of the deck, correct to two decimal places. Pythagoras' theorem will be required to calculate the missing length.



4H/K

11 Consider the composite solid shown.

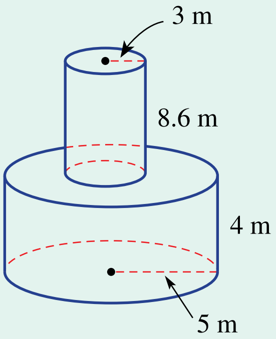
a Find the volume of the base cylinder, rounding to one decimal place.

b Find the volume of the top cylinder, rounding to one decimal place.

c Use your answers from parts **a** and **b** to find the total volume.

d Recalculate the total volume by retaining more precise answers for the calculations in parts **a** and **b**. Round your final answer correct to one decimal place.

e Compare your answers to parts **c** and **d**. How can you explain the difference?



4H

12 Give the limits of accuracy for these measurements.

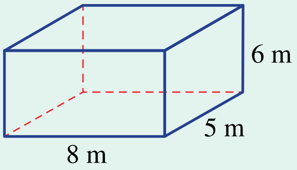
a 8 m **b** 10.3 kg **c** 4.75 L

4I–M

13 For each of these solids find, correct to two decimal places where necessary:

i the surface area **ii** the volume.

a



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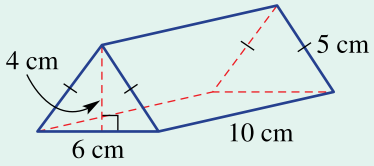

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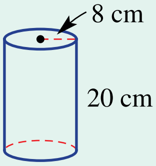

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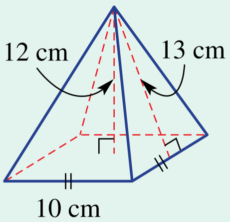

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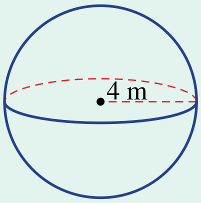
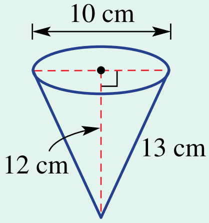
Ext **e**



Ext **f**



4J Ext

14 A cone has a radius of 6 cm and a curved surface area of 350 cm^2.

a Find the slant height of the cone, in exact form.

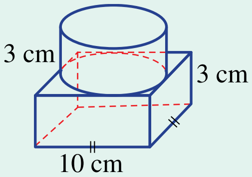
b Find the height of the cone, correct to one decimal place.

4I/K

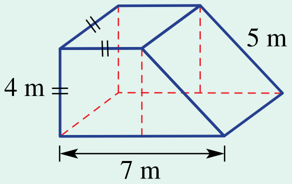
15 For each of the following composite solids find, correct to two decimal places where necessary:

i the surface area **ii** the volume.

a



b



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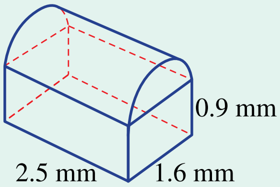
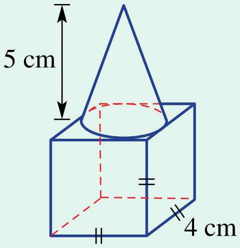


4I–M Ext

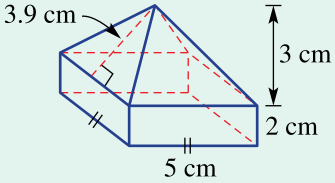
16 For each of the following composite solids find, correct to two decimal places where necessary:

i the surface area **ii** the volume.

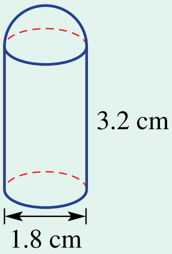
a



b

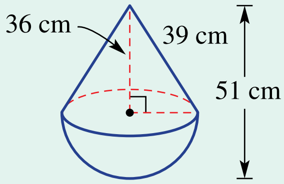


c



4J/L/M Ext

17 A water buoy is in the shape shown. Find:



a the volume of air inside the buoy, in exact form

b the surface area of the buoy, in exact form.

Multiple-choice questions

4A Opt

1 Which of the following is a surd?

A $\sqrt{36}$ **B** π **C** $\sqrt{7}$ **D** $\sqrt[3]{8}$ **E** 1.6

4A Opt

2 $4\sqrt{5}$ is equivalent to:

A $\sqrt{100}$ **B** $\sqrt{80}$ **C** $2\sqrt{10}$ **D** $\sqrt{20}$ **E** $\sqrt{40}$

4C Opt

3 The expanded form of $2\sqrt{5}(5 - 3\sqrt{3})$ is:

A $10\sqrt{5} - 6\sqrt{15}$ **B** $7\sqrt{5} - 5\sqrt{15}$ **C** $10\sqrt{5} - 12\sqrt{2}$

D $10 - 5\sqrt{15}$ **E** $7\sqrt{5} - 5\sqrt{3}$

4D Opt

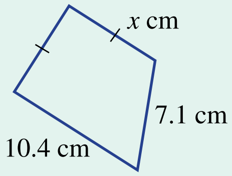
4 $\dfrac{2\sqrt{5}}{\sqrt{6}}$ is equivalent to:

A $\dfrac{2\sqrt{30}}{\sqrt{6}}$ **B** $\dfrac{5\sqrt{6}}{3}$ **C** $2\sqrt{5}$ **D** $\dfrac{\sqrt{30}}{3}$ **E** $\dfrac{\sqrt{30}}{10}$

4E

5 If the perimeter of this shape is 30.3 cm, then the value of x is:

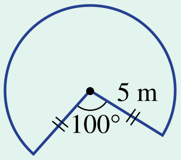
A 12.8
B 6.5
C 5.7
D 6.4
E 3.6



4E

6 The perimeter of the sector shown, rounded to one decimal place, is:

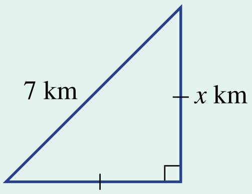
A 8.7 m
B 22.7 m
C 18.7 m
D 56.7 m
E 32.7 m



4F

7 The value of x in this triangle is closest to:

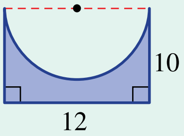
A 4.9
B 3.5
C 5.0
D 4.2
E 3.9



4G

8 The exact shaded (purple) area in square units is:

A $32 - 72\pi$
B $120 - 36\pi$
C $32 + 6\pi$
D $120 - 18\pi$
E $48 + 18\pi$



4G

9 $0.128\,\text{m}^2$ is equivalent to:

A $12.8\,\text{cm}^2$ **B** $128\,\text{mm}^2$ **C** $1280\,\text{cm}^2$

D $0.00128\,\text{cm}^2$ **E** $1280\,\text{mm}^2$

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4I

10 A cube has a surface area of $1350\,\text{cm}^2$. The side length of the cube is:

A 15 cm **B** 11 cm **C** 18 cm **D** 12 cm **E** 21 cm

4I

11 A cylindrical tin of canned food has a paper label glued around its curved surface. If the can is 14 cm high and has a radius of 4 cm, the area of the label is closest to:

A 452 cm^2 **B** 352 cm^2 **C** 126 cm^2 **D** 704 cm^2 **E** 235 cm^2

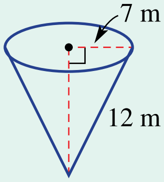
4J Ext

12 The exact surface area of a cone of diameter 24 cm and slant height 16 cm is:

A $216\pi\text{ cm}^2$ **B** $960\pi\text{ cm}^2$ **C** $528\pi\text{ cm}^2$ **D** $336\pi\text{ cm}^2$ **E** $384\pi\text{ cm}^2$

4F

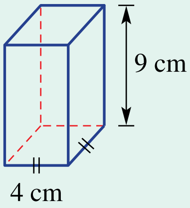
13 A cone has a radius of 7 m and a slant height of 12 m.
The cone's exact height, in metres, is:



A $\sqrt{52}$
B $\sqrt{193}$
C $\sqrt{85}$
D $\sqrt{137}$
E $\sqrt{95}$

4K

14 The volume of liquid that this square-based prism can hold is:



A 72 mL
B 48 mL
C 176 mL
D 144 mL
E 120 mL

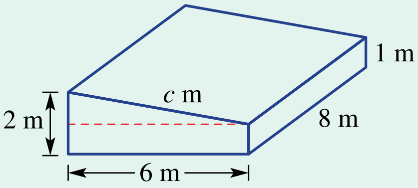
4M Ext

15 The volume of air in a sphere is 100 cm^3. The radius of the sphere, correct to two decimal places, is:

A 1.67 cm **B** 10.00 cm **C** 2.82 cm **D** 23.87 cm **E** 2.88 cm

Extended-response questions

1 A waterski ramp consists of a rectangular flotation container and a triangular angled section, as shown.



a What volume of air is contained within the entire ramp structure?

b Find the length of the angled ramp (c metres), in exact surd form.

The entire structure is to be painted with a waterproof paint costing \$20 per litre. One litre of paint covers 25 square metres.

c Find the surface area of the ramp, correct to one decimal place.

d Find the number of litres and the cost of paint required for the job. Assume you can purchase only one litre tins of paint.

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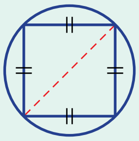
2 A circular school oval of radius 50 metres is marked with spray paint to form a square pitch, as shown.



a State the diagonal length of the square.

b Use Pythagoras' theorem to find the side length of the square, in exact surd form.

c Find the area of the square pitch.

d Find the percentage area of the oval that is not part of the square pitch. Round your answer to the nearest whole percentage.

Two athletes challenge each other to a one-lap race around the oval. Athlete A runs around the outside of the oval at an average rate of 10 metres per second. Athlete B runs around the outside of the square at an average rate of 9 metres per second. Athlete B's average running speed is less because of the need to slow down at each corner.

e Find who comes first and the difference in times, correct to the nearest hundredth of a second.

Opt **3** A small rectangular jewellery box has a base with dimensions $3\sqrt{15}$ cm by $(12 + \sqrt{3})$ cm and a height of $(2\sqrt{5} + 4)$ cm.

a Determine the exact area of the base of the box, in expanded and simplified form.

b Julie's earring boxes occupy an area of $9\sqrt{5}\,\text{cm}^2$. What is the exact simplified number of earring boxes that would fit across the base of the jewellery box?

c The surface of Julie's rectangular dressing table has dimensions $(\sqrt{2} - 1)$ m by $(\sqrt{2} + 1)$ m.

- **i** Find the area of the dressing table, in square centimetres. Recall $(a - b)(a + b) = a^2 - b^2$.
- **ii** What percentage of the area of the dressing table does the jewellery box occupy? Give your answer to one decimal place.

5

Quadratic expressions and equations

Maths in context: Quadratic equations and stopping distances

When the Black Death pandemic struck in 1665, Cambridge University closed, and Isaac Newton moved to his grandparent's farm. At age 24, in isolation for 2 years, Newton's brilliant mind built on Galileo's discoveries and altered our world forever by inventing the maths of continuously changing quantities, Calculus, and also the laws of motion.

Consider the motion equation: $v^2 = u^2 + 2as$ with initial speed u m/s, final speed v m/s, constant acceleration a m/s^2, over a distance of s m. For cars braking heavily to a stop on a dry road $a \approx -10$ m/s^2 and the final speed $v = 0$. Hence stopping distance $s = \frac{-u^2}{2a} = \frac{u^2}{20}$.

What is the stopping distance at 30 km/h? (Use $u = \frac{30}{3.6} = 8.3$ m/s) By what factor does the stopping distance increase if the speed doubles? triples? quadruples? To add a driver's perception and reaction time of $t = 1.5$ *seconds*, use the motion equation $s = ut + 0.5\,at^2$ *where* $a \approx -10$ m/s^2.

Civil engineers solve quadratic equations for assigning speed limits, the design of road lanes, intersections, and train station lengths. For a braking train, deceleration $a \approx -1.5$ m/s^2.

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Chapter contents

Australian Curriculum 9.0

ALGEBRA

Expand, factorise and simplify expressions and solve equations algebraically, applying exponent laws involving products, quotients and powers of variables, and the distributive property (AC9M10A01)

Use mathematical modelling to solve applied problems involving growth and decay, including financial contexts; formulate problems, choosing to apply linear, quadratic or exponential models; interpret solutions in terms of the situation; evaluate and modify models as necessary and report assumptions, methods and findings (AC9M10A04)

Algebraic representations of quadratic functions of the form $f(x) = ax^2 + bx + c$ where a, b and c are non-zero integers, and their transformation to the form $f(x) = a(x + h)^2 + k$ where h and k are non-zero rational numbers, and the solution of related equations (Year 10 optional content)

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

5A Expanding expressions CONSOLIDATING

LEARNING INTENTIONS

- To review how to apply the distributive law to expand brackets
- To be able to expand binomial products including perfect squares
- To be able to form a difference of two squares by expansion

You will recall that expressions that include numerals and pronumerals are central to the topic of algebra. Sound skills in algebra are essential for solving most mathematical problems and this includes the ability to expand expressions involving brackets. This includes binomial products, perfect squares and the difference of two squares. Exploring how projectiles fly subject to Earth's gravity, for example, can be modelled with expressions with and without brackets.

Business analysts develop profit equations, which are quadratics, when sales and profit/item are linear relations of the selling price, e.g. \$*p*/ice-cream:

$$\begin{aligned}\text{Profit/week} &= \text{weekly sales} \times \text{profit/item}\\ &= 150(10 - p) \times (p - 2)\\ &= -150(p^2 - 12p + 20)\end{aligned}$$

Lesson starter: Five key errors

Here are five expansion problems with incorrect answers. Discuss what error has been made and then give the correct expansion.

- $-2(x - 3) = -2x - 6$
- $(x + 3)^2 = x^2 + 9$
- $(x - 2)(x + 2) = x^2 + 4x - 4$
- $5 - 3(x - 1) = 2 - 3x$
- $(x + 3)(x + 5) = x^2 + 8x + 8$

KEY IDEAS

- **Like terms** have the same pronumeral part.
    - They can be collected (i.e. added and subtracted) to form a single term. For example: $7x - 11x = -4x$ and $4a^2b - 7ba^2 = -3a^2b$
- The **distributive law** is used to expand brackets.
    - $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$
    - $(a + b)(c + d) = ac + ad + bc + bd$
    - $(a + b)(c + d)$ is called a binomial product because each expression in the brackets has two terms.

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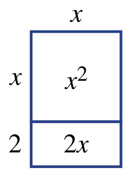
- **Perfect squares**
  - $(a+b)^2 = (a+b)(a+b)$
    $= a^2 + 2ab + b^2$
  - $(a-b)^2 = (a-b)(a-b)$
    $= a^2 - 2ab + b^2$
- **Difference of two squares**
  - $(a+b)(a-b) = a^2 - b^2$
- By definition, a perfect square is an integer that is the square of an integer; however, the rules above also apply for a wide range of values for a and b, including all real numbers.

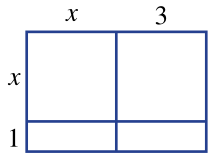
BUILDING UNDERSTANDING

1 Use each diagram to help expand the expressions.

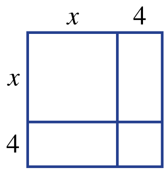
a $x(x+2)$



b $(x+3)(x+1)$



c $(x+4)^2$



2 Simplify these expressions.

a $2 \times 3x$
b $-4 \times 5x$
c $-x \times 4x$
d $5x \div 10$
e $-6x^2 \div (2x)$
f $3x - 21x$
g $-3x + 8x$
h $-5x - 8x$

Example 1 Expanding simple expressions

Expand and simplify where possible.

a $-3(x-5)$
b $2x(1-x)$
c $x(2x-1) - x(3-x)$

SOLUTION	EXPLANATION
a $-3(x-5) = -3x + 15$	Use the distributive law: $a(b-c) = ab - ac$. $-3 \times x = -3x$ and $-3 \times (-5) = 15$
b $2x(1-x) = 2x - 2x^2$	Recall that $2x \times (-x) = -2x^2$.
c $x(2x-1) - x(3-x) = 2x^2 - x - 3x + x^2$ $= 3x^2 - 4x$	Apply the distributive law to each set of brackets first, then simplify by collecting like terms. Recall that $-x \times (-x) = x^2$.

Now you try

Expand and simplify where possible.

a $-2(x-4)$
b $5x(4-x)$
c $x(5x-1) - x(2-3x)$

Example 2 Expanding binomial products, perfect squares and difference of two squares

Expand the following.

a $(x+5)(x+4)$ **b** $(x-4)^2$ **c** $(2x+1)(2x-1)$

SOLUTION

a $$\begin{aligned}(x+5)(x+4) &= x^2+4x+5x+20\\ &= x^2+9x+20\end{aligned}$$

b $$\begin{aligned}(x-4)^2 &= (x-4)(x-4)\\ &= x^2-4x-4x+16\\ &= x^2-8x+16\end{aligned}$$

Alternatively:

$$\begin{aligned}(x-4)^2 &= x^2-2(x)(4)+4^2\\ &= x^2-8x+16\end{aligned}$$

c $$\begin{aligned}(2x+1)(2x-1) &= 4x^2-2x+2x-1\\ &= 4x^2-1\end{aligned}$$

Alternatively:

$$\begin{aligned}(2x+1)(2x-1) &= (2x)^2-(1)^2\\ &= 4x^2-1\end{aligned}$$

EXPLANATION

For binomial products use $(a+b)(c+d)=ac+ad+bc+bd$. Simplify by collecting like terms.

Rewrite and expand using the distributive law.

Alternatively for perfect squares $(a-b)^2=a^2-2ab+b^2$. Here $a=x$ and $b=4$.

Expand, recalling that $2x \times 2x = 4x^2$. Cancel the $-2x$ and $+2x$ terms.

Alternatively for difference of two squares $(a-b)(a+b)=a^2-b^2$. Here $a=2x$ and $b=1$.

Now you try

Expand the following.

a $(x+2)(x+5)$ **b** $(x-2)^2$ **c** $(3x+2)(3x-2)$

Example 3 Expanding more binomial products

Expand and simplify.

a $(2x-1)(3x+5)$ **b** $2(x-3)(x-2)$ **c** $(x+2)(x+4)-(x-2)(x-5)$

SOLUTION

a $$\begin{aligned}(2x-1)(3x+5) &= 6x^2+10x-3x-5\\ &= 6x^2+7x-5\end{aligned}$$

b $$\begin{aligned}2(x-3)(x-2) &= 2(x^2-2x-3x+6)\\ &= 2(x^2-5x+6)\\ &= 2x^2-10x+12\end{aligned}$$

EXPLANATION

Expand using the distributive law and simplify.

Note: $2x \times 3x = 2 \times 3 \times x \times x = 6x^2$.

First expand the brackets using the distributive law, simplify and then multiply each term by 2.

c $(x+2)(x+4)-(x-2)(x-5)$

$$\begin{aligned}&= \left(x^2+4x+2x+8\right)-\left(x^2-5x-2x+10\right)\\ &= \left(x^2+6x+8\right)-\left(x^2-7x+10\right)\\ &= x^2+6x+8-x^2+7x-10\\ &= 13x-2\end{aligned}$$

Expand each binomial product.

Remove brackets in the last step before simplifying.

$$\begin{aligned}-\left(x^2-7x+10\right) &= -1\times x^2+(-1)\times(-7x)\\ &\quad +(-1)\times 10\\ &= -x^2+7x-10\end{aligned}$$

Now you try

Expand and simplify.

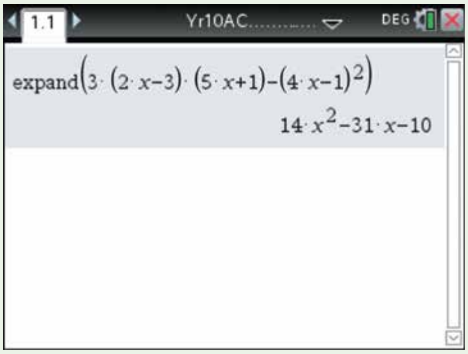
a $(3x-1)(2x+7)$ **b** $3(x-1)(x-4)$ **c** $(x+3)(x+1)-(x-3)(x-4)$

Using calculators to expand and simplify

Expand and simplify $3(2x-3)(5x+1)-(4x-1)^2$.

Using the TI-Nspire:

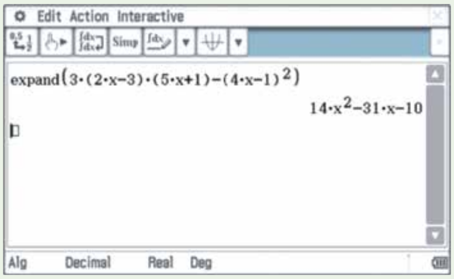
In a **calculator** page use menu **>Algebra>Expand** and type as shown.



```
expand(3·(2·x−3)·(5·x+1)−(4·x−1)^2)
                        14·x^2−31·x−10
```

Using the ClassPad:

In the **Main** application, type and highlight the expression, then tap **Interactive**, **Transformation**, **expand** and type in as shown below.



```
expand(3·(2·x−3)·(5·x+1)−(4·x−1)^2)
                        14·x^2−31·x−10
```

Exercise 5A

FLUENCY	1–5(1/3)	1–5(1/3)	1–5(1/4)

Example 1a, b

1 Expand and simplify where possible.

a $2(x+5)$ **b** $3(x-4)$ **c** $-5(x+3)$ **d** $-4(x-2)$

e $3(2x-1)$ **f** $4(3x+1)$ **g** $-2(5x-3)$ **h** $-5(4x+3)$

i $x(2x+5)$ **j** $x(3x-1)$ **k** $2x(1-x)$ **l** $3x(2-x)$

m $-2x(3x+2)$ **n** $-3x(6x-2)$ **o** $-5x(2-2x)$ **p** $-4x(1-4x)$

q $\frac{2}{5}(10x+4)$ **r** $\frac{3}{4}(8x-5)$ **s** $-\frac{1}{3}(6x+1)$ **t** $-\frac{1}{2}(4x-3)$

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Example 1c

2 Expand and simplify.

a $x(3x - 1) + x(4 - x)$ **b** $x(5x + 2) + x(x - 5)$ **c** $x(4x - 3) - 2x(x - 5)$
d $3x(2x + 4) - x(5 - 2x)$ **e** $4x(2x - 1) + 2x(1 - 3x)$ **f** $2x(2 - 3x) - 3x(2x - 7)$

Example 2a

3 Expand the following.

a $(x + 2)(x + 8)$ **b** $(x + 3)(x + 4)$ **c** $(x + 7)(x + 5)$
d $(x + 8)(x - 3)$ **e** $(x + 6)(x - 5)$ **f** $(x - 2)(x + 3)$
g $(x - 7)(x + 3)$ **h** $(x - 4)(x - 6)$ **i** $(x - 8)(x - 5)$

Example 2b, c

4 Expand the following.

a $(x + 5)^2$ **b** $(x + 7)^2$ **c** $(x + 6)^2$
d $(x - 3)^2$ **e** $(x - 8)^2$ **f** $(x - 10)^2$
g $(x + 4)(x - 4)$ **h** $(x + 9)(x - 9)$ **i** $(2x - 3)(2x + 3)$
j $(3x + 4)(3x - 4)$ **k** $(4x - 5)(4x + 5)$ **l** $(8x - 7)(8x + 7)$

Example 3a

5 Expand the following using the distributive law.

a $(2x + 1)(3x + 5)$ **b** $(4x + 5)(3x + 2)$ **c** $(5x + 3)(2x + 7)$
d $(3x + 2)(3x - 5)$ **e** $(5x + 3)(4x - 2)$ **f** $(2x + 5)(3x - 5)$
g $(4x - 5)(4x + 5)$ **h** $(2x - 9)(2x + 9)$ **i** $(5x - 7)(5x + 7)$
j $(7x - 3)(2x - 4)$ **k** $(5x - 3)(5x - 6)$ **l** $(7x - 2)(8x - 2)$
m $(2x + 5)^2$ **n** $(5x + 6)^2$ **o** $(7x - 1)^2$

PROBLEM–SOLVING	6–7(½)	6–8(⅓)	6–8(⅓), 9

6 Write the missing number.

a $(x + ?)(x + 2) = x^2 + 5x + 6$ **b** $(x + ?)(x + 5) = x^2 + 8x + 15$
c $(x + 7)(x - ?) = x^2 + 4x - 21$ **d** $(x + 4)(x - ?) = x^2 - 4x - 32$
e $(x - 6)(x - ?) = x^2 - 7x + 6$ **f** $(x - ?)(x - 8) = x^2 - 10x + 16$

Example 3b

7 Expand the following.

a $2(x + 3)(x + 4)$ **b** $3(x + 2)(x + 7)$ **c** $-2(x + 8)(x + 2)$
d $-4(x + 9)(x + 2)$ **e** $5(x - 3)(x + 4)$ **f** $3(x + 5)(x - 3)$
g $-3(a + 2)(a - 7)$ **h** $-5(a + 2)(a - 8)$ **i** $4(a - 3)(a - 6)$
j $3(y - 4)(y - 5)$ **k** $-2(y - 3)(y - 8)$ **l** $-6(y - 4)(y - 3)$
m $3(2x + 3)(2x + 5)$ **n** $6(3x - 4)(x + 2)$ **o** $-2(x + 4)(3x - 7)$
p $2(x + 3)^2$ **q** $4(m + 5)^2$ **r** $2(a - 7)^2$
s $-3(y - 5)^2$ **t** $3(2b - 1)^2$ **u** $-3(2y - 6)^2$

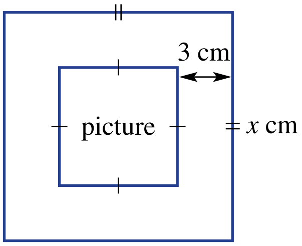
Example 3c

8 Expand and simplify the following.

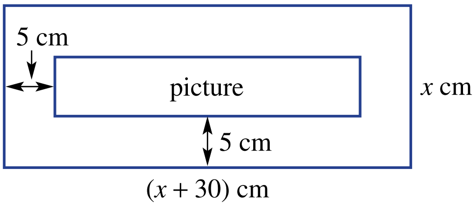
a $(x + 1)(x + 3) + (x + 2)(x + 4)$ **b** $(x + 8)(x + 3) + (x + 4)(x + 5)$
c $(y + 3)(y - 1) + (y - 2)(y - 4)$ **d** $(y - 7)(y + 4) + (y + 5)(y - 3)$
e $(2a + 3)(a - 5) - (a + 6)(2a + 5)$ **f** $(4b + 8)(b + 5) - (3b - 5)(b - 7)$
g $(x + 5)^2 - 7$ **h** $(x - 7)^2 - 9$
i $3 - (2x - 9)^2$ **j** $14 - (5x + 3)^2$

9 Find an expanded expression for the area of the rectangular pictures centred in these rectangular frames.

a



b



REASONING	10–11(½)	10–12(½)	10(½), 12–13(½)

10 Use the distributive law to evaluate the following without the use of a calculator.
For example: $4 \times 102 = 4 \times 100 + 4 \times 2 = 408$.

a 6×103
b 4×55
c 9×63
d 8×208
e 7×198
f 3×297
g 8×495
h 5×696

11 Each problem below has an incorrect answer. Find the error and give the correct answer.

a $-x(x - 7) = -x^2 - 7x$
b $3a - 7(4 - a) = -4a - 28$
c $(2x + 3)^2 = 4x^2 + 9$
d $(x + 2)^2 - (x + 2)(x - 2) = 0$

12 Prove the following by expanding the left-hand side.

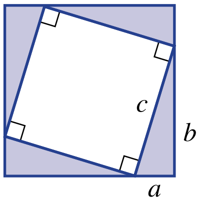
a $(a + b)(a - b) = a^2 - b^2$
b $(a + b)^2 = a^2 + 2ab + b^2$
c $(a - b)^2 = a^2 - 2ab + b^2$
d $(a + b)^2 - (a - b)^2 = 4ab$

13 Expand these cubic expressions.

a $(x + 2)(x + 3)(x + 1)$
b $(x + 4)(x + 2)(x + 5)$
c $(x + 3)(x - 4)(x + 3)$
d $(x - 4)(2x + 1)(x - 3)$
e $(x + 6)(2x - 3)(x - 5)$
f $(2x - 3)(x - 4)(3x - 1)$

ENRICHMENT: Expanding to prove	–	–	14

14 One of the ways to prove Pythagoras' theorem is to arrange four congruent right-angled triangles around a square to form a larger square, as shown.



a Find an expression for the total area of the four shaded triangles by multiplying the area of one triangle by 4.
b Find an expression for the area of the four shaded triangles by subtracting the area of the inner square from the area of the outer square.
c By combining your results from parts **a** and **b**, expand and simplify to prove Pythagoras' theorem: $a^2 + b^2 = c^2$.

5B Factorising expressions

LEARNING INTENTIONS

- To understand what it means to write an expression in factorised form
- To know to always look for a common factor before trying other factorising techniques
- To be able to recognise a difference of two squares including ones involving surds
- To be able to factorise using a common factor or a difference of two squares
- To be able to use the grouping technique to factorise

A common and key step in the simplification and solution of equations involves factorisation. Factorisation is the process of writing a number or expression as a product of its factors.

In this section we look at expressions in which all terms have a common factor, expressions that are a difference of two squares and four-term expressions, which can be factorised by grouping.

After a car accident, crash investigators use the length of tyre skid marks to determine a vehicle's speed before braking. The quadratic equation $u^2 + 2as = 0$ relates to speed, u, to a known braking distance, s, and deceleration $a = -10$ m/s^2 on a dry, flat bitumen road.

Lesson starter: But there are no common factors!

An expression such as $xy + 4x + 3y + 12$ has no common factors across all four terms, but it can still be factorised. The method of grouping can be used.

- Complete this working to show how to factorise the expression.
  $$\begin{aligned} xy + 4x + 3y + 12 &= x(\_\_\_\_\_\_) + 3(\_\_\_\_\_\_) \\ &= (\_\_\_\_\_\_)(x + 3) \end{aligned}$$
- Now repeat with the expression rearranged.
  $$\begin{aligned} xy + 3y + 4x + 12 &= y(\_\_\_\_\_\_) + 4(\_\_\_\_\_\_) \\ &= (\_\_\_\_\_\_)(\_\_\_\_\_\_) \end{aligned}$$
- Are the two results equivalent?

KEY IDEAS

- **Factorise** expressions with **common factors** by 'taking out' the common factors.
  For example: $-5x - 20 = -5(x + 4)$ and $4x^2 - 8x = 4x(x - 2)$.
- Factorise a **difference of two squares** using $a^2 - b^2 = (a + b)(a - b)$.
  - We use surds when a^2 or b^2 is not a perfect square, such as 1, 4, 9, …
    For example: $x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$ using $(\sqrt{5})^2 = 5$.
- Factorise four-term expressions if possible by **grouping** terms and factorising each pair.
  For example: $$\begin{aligned} x^2 + 5x - 2x - 10 &= x(x + 5) - 2(x + 5) \\ &= (x + 5)(x - 2) \end{aligned}$$

BUILDING UNDERSTANDING

1 Determine the highest common factor of these pairs of terms.

a $7x$ and 14 b $-5y$ and -25 c $12a^2$ and $9a$ d $-3x^2y$ and $-6xy$

2 State the missing parts.

a If $x(x - 1) = x^2 - x$, then $x^2 - x = x($______$)$.

b If $2(1 - x) = 2 - 2x$, then $2 - 2x =$ ________________.

c If $(x + 2)(x - 2) =$ _______, then $x^2 - 4 =$ ________________________.

d If $(3x - 7)($_______$) = 9x^2 - 49$, then $9x^2 - 49 =$ ________________________.

Example 4 Taking out common factors

Factorise by taking out common factors.

a $-3x - 12$ **b** $20a^2 + 30a$ **c** $2(x + 1) - a(x + 1)$

SOLUTION	EXPLANATION
a $-3x - 12 = -3(x + 4)$	-3 is common to both $-3x$ and -12. Divide each term by -3 to determine the terms in the brackets. Expand to check.
b $20a^2 + 30a = 10a(2a + 3)$	The HCF of $20a^2$ and $30a$ is $10a$.
c $2(x + 1) - a(x + 1) = (x + 1)(2 - a)$	$(x + 1)$ is a common factor to both parts of the expression.

Now you try

Factorise by taking out common factors.

a $-2x - 8$ **b** $15a^2 + 20a$ **c** $3(x + 2) - a(x + 2)$

Example 5 Factorising a difference of two squares

Factorise the following difference of two squares. You may need to look for a common factor first.

a $x^2 - 16$ **b** $9a^2 - 4b^2$ **c** $12y^2 - 1200$ **d** $(x+3)^2 - 4$

SOLUTION

a $$\begin{aligned} x^2 - 16 &= (x)^2 - (4)^2 \\ &= (x+4)(x-4) \end{aligned}$$

b $$\begin{aligned} 9a^2 - 4b^2 &= (3a)^2 - (2b)^2 \\ &= (3a+2b)(3a-2b) \end{aligned}$$

c $$\begin{aligned} 12y^2 - 1200 &= 12(y^2 - 100) \\ &= 12(y+10)(y-10) \end{aligned}$$

d $$\begin{aligned} (x+3)^2 - 4 &= (x+3)^2 - (2)^2 \\ &= (x+3+2)(x+3-2) \\ &= (x+5)(x+1) \end{aligned}$$

EXPLANATION

Use $a^2 - b^2 = (a+b)(a-b)$, where $a = x$ and $b = 4$.

$9a^2 = (3a)^2$ and $4b^2 = (2b)^2$.

First, take out the common factor of 12. $100 = 10^2$, use $a^2 - b^2 = (a+b)(a-b)$.

Use $a^2 - b^2 = (a+b)(a-b)$, where $a = x+3$ and $b = 2$. Simplify each bracket.

Now you try

Factorise the following difference of two squares. You may need to look for a common factor first.

a $x^2 - 25$ **b** $16a^2 - 9b^2$ **c** $2y^2 - 98$ **d** $(x+2)^2 - 36$

Example 6 Factorising a difference of two squares using surds

Factorise these difference of two squares using surds.

a $x^2 - 10$ **b** $(x-1)^2 - 5$

SOLUTION

a $$\begin{aligned} x^2 - 10 &= x^2 - (\sqrt{10})^2 \\ &= (x+\sqrt{10})(x-\sqrt{10}) \end{aligned}$$

b $$\begin{aligned} (x-1)^2 - 5 &= (x-1)^2 - (\sqrt{5})^2 \\ &= (x-1+\sqrt{5})(x-1-\sqrt{5}) \end{aligned}$$

EXPLANATION

Recall that $(\sqrt{10})^2 = 10$.

Use $a^2 - b^2 = (a+b)(a-b)$, where $a = x-1$ and $b = \sqrt{5}$.

Now you try

Factorise these difference of two squares using surds.

a $x^2 - 7$ **b** $(x-5)^2 - 2$

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Example 7 Factorising by grouping

Factorise by grouping $x^2 - x + ax - a$.

SOLUTION

$$\begin{aligned} x^2 - x + ax - a &= x(x-1) + a(x-1) \\ &= (x-1)(x+a) \end{aligned}$$

EXPLANATION

Factorise two pairs of terms, then take out the common binomial factor $(x - 1)$.

Now you try

Factorise by grouping $x^2 - 2x + ax - 2a$.

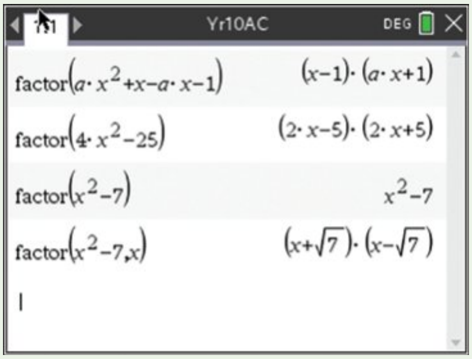
Using calculators to factorise

Factorise:

a $ax^2 + x - ax - 1$ **b** $4x^2 - 25$ **c** $x^2 - 7$

Using the TI-Nspire:

In a **Calculator** page use menu **>Algebra>Factor** and type as shown.

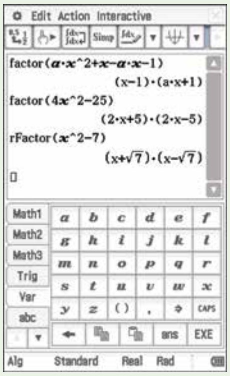


```
factor(a·x^2+x−a·x−1)        (x−1)·(a·x+1)
factor(4·x^2−25)             (2·x−5)·(2·x+5)
factor(x^2−7)                x^2−7
factor(x^2−7,x)              (x+√7)·(x−√7)
```

Note: Use a multiplication sign between the *a* and *x*.
Note: factor only factorises using rational numbers; factor with ,*x* will factorise with surds.

Using the ClassPad:

Use the **VAR** keyboard to type the expression as shown. Highlight the expression and tap **Interactive**, **Transformation**, **factor**.



```
factor(a·x^2+x−a·x−1)
                 (x−1)·(a·x+1)
factor(4x^2−25)
               (2·x+5)·(2·x−5)
rFactor(x^2−7)
                 (x+√7)·(x−√7)
```

Note: factor only factorises using rational numbers; rfactor will factorise with surds.

Exercise 5B

FLUENCY	1–4(1/3), 6(1/2)	1–6(1/3)	1–6(1/4)

Example 4a, b

1 Factorise by taking out the common factors.

a $3x - 18$ **b** $4x + 20$ **c** $7a + 7b$ **d** $9a - 15$
e $-5x - 30$ **f** $-4y - 2$ **g** $-12a - 3$ **h** $-2ab - bc$
i $4x^2 + x$ **j** $5x^2 - 2x$ **k** $6b^2 - 18b$ **l** $14a^2 - 21a$
m $10a - 5a^2$ **n** $12x - 30x^2$ **o** $-2x - x^2$ **p** $-4y - 8y^2$
q $ab^2 - a^2b$ **r** $2x^2yz - 4xy$ **s** $-12m^2n - 12mn^2$ **t** $6xyz^2 - 3z^2$

Example 4c

2 Factorise, noting the common binomial factor. (*Hint*: For parts **g**–**i**, insert a 1 where appropriate.)

a $5(x-1)-a(x-1)$ **b** $b(x+2)+3(x+2)$ **c** $a(x+5)-4(x+5)$
d $x(x+2)+5(x+2)$ **e** $x(x-4)-2(x-4)$ **f** $3(x+1)-x(x+1)$
g $a(x+3)+(x+3)$ **h** $x(x-2)-(x-2)$ **i** $(x-6)-x(x-6)$

Example 5a, b

3 Factorise the following difference of two squares.

a x^2-9 **b** x^2-25 **c** y^2-49 **d** y^2-1
e $4x^2-9$ **f** $36a^2-25$ **g** $1-81y^2$ **h** $100-9x^2$
i $25x^2-4y^2$ **j** $64x^2-25y^2$ **k** $9a^2-49b^2$ **l** $144a^2-49b^2$

Example 5c, d

4 Factorise the following.

a $2x^2-32$ **b** $5x^2-45$ **c** $6y^2-24$ **d** $3y^2-48$
e $3x^2-75y^2$ **f** $3a^2-300b^2$ **g** $12x^2-27y^2$ **h** $63a^2-112b^2$
i $(x+5)^2-16$ **j** $(x-4)^2-9$ **k** $(a-3)^2-64$ **l** $(a-7)^2-1$
m $(3x+5)^2-x^2$ **n** $(2y+7)^2-y^2$ **o** $(5x+11)^2-4x^2$ **p** $(3x-5y)^2-25y^2$

Example 6

5 Factorise using surds.

a x^2-7 **b** x^2-5 **c** x^2-19
d x^2-21 **e** x^2-14 **f** x^2-30
g $(x+4)^2-3$ **h** $(x+2)^2-6$ **i** $(x+5)^2-10$
j $(x-3)^2-11$ **k** $(x-1)^2-7$ **l** $(x-6)^2-15$

Example 7

6 Factorise by grouping.

a $x^2+4x+ax+4a$ **b** $x^2+7x+bx+7b$ **c** $x^2-3x+ax-3a$
d $x^2+2x-ax-2a$ **e** $x^2+5x-bx-5b$ **f** $x^2+3x-4ax-12a$
g $x^2-ax-4x+4a$ **h** $x^2-2bx-5x+10b$ **i** $3x^2-6ax-7x+14a$

PROBLEM–SOLVING	7(½)	7–9(½)	7–10(½)

7 Factorise by first rearranging, so that each pair has a common factor.

a $xy-6-3x+2y$ **b** $ax-12+3a-4x$ **c** $ax-10+5x-2a$
d $xy+12-3y-4x$ **e** $2ax+3-a-6x$ **f** $2ax-20+8a-5x$

8 Factorise fully. Use surds in parts **e**–**h**.

a $x^2-\frac{1}{9}$ **b** $x^2-\frac{1}{4}$ **c** $x^2-\frac{9}{16}$ **d** $x^2-\frac{25}{36}$
e $3x^2-4$ **f** $5x^2-9$ **g** $-9+2x^2$ **h** $-16+5x^2$

Opt **9** Factorise using surds, simplifying surds where possible.

a x^2-18 **b** x^2-45 **c** x^2-20
d x^2-32 **e** x^2-48 **f** x^2-50

Opt **10** Factorise fully by first removing any common factor and simplifying any surds.

a $5x^2-120$ **b** $3x^2-162$ **c** $7x^2-126$ **d** $2x^2-96$
e $2(x+3)^2-10$ **f** $3(x-1)^2-21$ **g** $4(x-4)^2-48$ **h** $5(x+6)^2-90$

REASONING	11(½)	11(½), 12	11(¼), 12–14

11 Evaluate the following, without the use of a calculator, by first factorising.

a $16^2 - 14^2$ **b** $18^2 - 17^2$ **c** $13^2 - 10^2$ **d** $15^2 - 11^2$

e $17^2 - 15^2$ **f** $11^2 - 9^2$ **g** $27^2 - 24^2$ **h** $52^2 - 38^2$

12 **a** Show that $4 - (x + 2)^2 = -x(x + 4)$ by factorising the left-hand side.

b Now factorise the following.

i $9 - (x + 3)^2$ **ii** $16 - (x + 4)^2$ **iii** $25 - (x - 5)^2$

iv $25 - (x + 2)^2$ **v** $49 - (x - 1)^2$ **vi** $100 - (x + 4)^2$

13 **a** Prove that, in general, $(x + a)^2 \neq x^2 + a^2$.

b Are there any values of x for which $(x + a)^2 = x^2 + a^2$? If so, what are they?

14 Show that $x^2 - \frac{4}{9} = \frac{1}{9}(3x + 2)(3x - 2)$ using two different methods.

ENRICHMENT: Hidden difference of two squares	–	–	15 (½), 16

15 Factorise and simplify the following without initially expanding the brackets.

a $(x + 2)^2 - (x + 3)^2$ **b** $(y - 7)^2 - (y + 4)^2$

c $(a + 3)^2 - (a - 5)^2$ **d** $(b + 5)^2 - (b - 5)^2$

e $(s - 3)^2 - (s + 3)^2$ **f** $(y - 7)^2 - (y + 7)^2$

g $(2w + 3x)^2 - (3w + 4x)^2$ **h** $(d + 5e)^2 - (3d - 2e)^2$

i $(4f + 3j)^2 - (2f - 3j)^2$ **j** $(3r - 2p)^2 - (2p - 3r)^2$

16 **a** Is it possible to factorise $x^2 + 5y - y^2 + 5x$? Can you show how?

b Also try factorising:

i $x^2 + 7x + 7y - y^2$ **ii** $x^2 - 2x - 2y - y^2$

iii $4x^2 + 4x + 6y - 9y^2$ **iv** $25y^2 + 15y - 4x^2 + 6x$

Factorising is a key component of the proof of Fermat's last theorem, which states that there are no solutions to $x^n + y^n = z^n$ for $n \geqslant 3$. Although it looks simple, it took the best mathematicians on Earth 358 years to find a proof of this theorem. It was finally proved in 1994 by Andrew Wiles, and his proof is almost 130 pages long!

5C Multiplying and dividing algebraic fractions

LEARNING INTENTIONS

- To understand that expressions need to be in factorised form in order to cancel common factors in algebraic fractions
- To know that it is helpful to cancel common factors in fractions before multiplying or dividing
- To be able to multiply and divide fractions involving algebraic expressions

Since pronumerals represent numbers, the rules for algebraic fractions are the same as those for simple numerical fractions. This includes processes such as cancelling common factors, adding or subtracting with a lowest common denominator (LCD) and multiplying by the reciprocal of the fraction that follows the division sign. In this section we focus on multiplying and dividing algebraic fractions.

The study of air-conditioning uses algebraic fractions to model airflow, air temperatures and humidity. The mechanical engineers who design ventilation systems, and the electricians who install and repair them, all require algebraic skills.

Lesson starter: Describe the error

Here are three problems involving algebraic fractions. Each simplification contains one critical error. Find and describe the errors, then give the correct answer.

a $$\begin{aligned}\frac{6x - \cancel{8}^{2}}{\cancel{4}_{1}} &= \frac{6x - 2}{1} \\ &= 6x - 2\end{aligned}$$

b $$\begin{aligned}\frac{2a}{9} \div \frac{2}{3} &= \frac{2a}{9} \times \frac{2}{3} \\ &= \frac{4a}{27}\end{aligned}$$

c $$\begin{aligned}\frac{3b}{7} \div \frac{2b}{3} &= \frac{3b}{7} \times \frac{3b}{2} \\ &= \frac{9b^2}{14}\end{aligned}$$

KEY IDEAS

- Simplify **algebraic fractions** by factorising expressions where possible and cancelling common factors.
- For multiplication, cancel common factors and then multiply the numerators together and the denominators together.
- For division, multiply by the **reciprocal** of the fraction that follows the division sign. The reciprocal of a is $\frac{1}{a}$ and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

BUILDING UNDERSTANDING

1 What is the reciprocal of each fraction?

a $\frac{3}{2}$ **b** $\frac{7a}{3}$ **c** $\frac{-4xy}{7t}$ **d** $\frac{-8x^2a}{b^2c}$

2 Simplify to find the answer in simplest form.

a $\frac{2}{3} \times \frac{6}{4}$ **b** $\frac{3}{4} \times \frac{10}{9}$ **c** $\frac{4}{7} \div \frac{2}{7}$ **d** $\frac{3}{6} \div \frac{6}{9}$

3 Simplify by cancelling common factors.

a $\frac{10x}{2}$ **b** $\frac{24x}{6}$

c $\frac{5a}{20}$ **d** $\frac{7}{21a}$

e $\frac{3(x+1)}{9(x+1)}$ **f** $\frac{2(x-2)}{8(x-2)}$

Example 8 Cancelling common factors

Simplify by cancelling common factors.

a $\frac{8a^2b}{2a}$ **b** $\frac{3-9x}{3}$

SOLUTION	EXPLANATION
a $\frac{8a^2b}{2a} = \frac{\cancel{8}^4 \times \cancel{a}^1 \times a \times b}{\cancel{2}_1 \times \cancel{a}_1}$ $= 4ab$	Cancel the common factors 2 and *a*.
b $\frac{3-9x}{3} = \frac{\cancel{3}^1(1-3x)}{\cancel{3}^1}$ $= 1-3x$	Factorise the numerator, then cancel the common factor of 3.

Now you try

Simplify by cancelling common factors.

a $\frac{9ab^2}{3b}$ **b** $\frac{5-10x}{5}$

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Example 9 Multiplying and dividing algebraic fractions

Simplify the following.

a $\frac{3a}{a+2} \times \frac{a+2}{6}$ **b** $\frac{2x-4}{3} \div \frac{x-2}{6}$ **c** $\frac{x^2-4}{x-2} \times \frac{x}{3x+6}$

SOLUTION

a $\frac{\cancel{3}^1 a}{\cancel{a+2}_1} \times \frac{\cancel{a+2}_1}{\cancel{6}_2} = \frac{a}{2}$

b $\frac{2x-4}{3} \div \frac{x-2}{6} = \frac{2x-4}{3} \times \frac{6}{x-2}$

$= \frac{2\cancel{(x-2)}^1}{\cancel{3}^1} \times \frac{\cancel{6}^2}{\cancel{(x-2)}^1}$

$= 4$

c $\frac{x^2-4}{x-2} \times \frac{x}{3x+6} = \frac{\overset{1}{\cancel{(x-2)}}\overset{1}{\cancel{(x+2)}}}{\underset{1}{\cancel{x-2}}} \times \frac{x}{3\underset{1}{\cancel{(x+2)}}}$

$= \frac{x}{3}$

EXPLANATION

Cancel the common factors of 3 and $a + 2$. Then multiply the numerators and the denominators.

Multiply by the reciprocal of the second fraction.

Factorise $2x - 4$ and cancel the common factors.

Factorise each numerator and denominator where possible. $x^2 - 4 = x^2 - 2^2$ is a difference of two squares. Cancel common factors and multiply remaining numerators and remaining denominators.

Now you try

Simplify the following.

a $\frac{6a}{a+1} \times \frac{a+1}{12}$ **b** $\frac{3x-12}{2} \div \frac{x-4}{4}$ **c** $\frac{x^2-9}{x+3} \times \frac{x}{2x-6}$

Exercise 5C

FLUENCY	1–4(½)	1–4(½)	1–4(⅓)

Example 8a

1 Simplify by cancelling common factors.

a $\frac{35x^2}{7x}$ **b** $\frac{-14x^2y}{7xy}$ **c** $\frac{-36ab^2}{4ab}$ **d** $\frac{8xy^3}{-4xy^2}$

e $\frac{-15pq^2}{30p^2q^2}$ **f** $\frac{-20s}{45s^2t}$ **g** $\frac{-48x^2}{16xy}$ **h** $\frac{120ab^2}{140ab}$

Example 8b

2 Simplify by cancelling common factors.

a $\frac{4x+8}{4}$ **b** $\frac{6a-30}{6}$ **c** $\frac{6x-18}{2}$ **d** $\frac{5-15y}{5}$

e $\frac{-2-12b}{-2}$ **f** $\frac{21x-7}{-7}$ **g** $\frac{9t-27}{-9}$ **h** $\frac{44-11x}{-11}$

i $\frac{x^2+2x}{x}$ **j** $\frac{6x-4x^2}{2x}$ **k** $\frac{a^2-a}{a}$ **l** $\frac{7a+14a^2}{21a}$

Example 9a

3 Simplify the following.

a $\frac{x+3}{9} \times \frac{4}{x+3}$ **b** $\frac{y-7}{y} \times \frac{5y}{y-7}$ **c** $\frac{10a^2}{a+6} \times \frac{a+6}{4a}$

d $\frac{3}{x} \times \frac{x-1}{6}$ **e** $\frac{x+4}{10} \times \frac{2}{x}$ **f** $\frac{x-2}{3} \times \frac{6}{x}$

g $\frac{2m+4}{m} \times \frac{m}{m+2}$ **h** $\frac{6-18x}{2} \times \frac{5}{1-3x}$ **i** $\frac{b-1}{10} \times \frac{-5}{b-1}$

Example 9b

4 Simplify the following.

a $\frac{x}{5} \div \frac{x}{15}$ **b** $\frac{x+4}{2} \div \frac{x+4}{6}$ **c** $\frac{6x-12}{5} \div \frac{x-2}{3}$

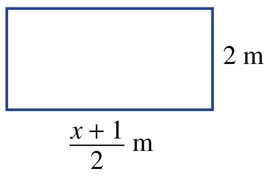
d $\frac{3-6y}{8} \div \frac{1-2y}{2}$ **e** $\frac{2}{a-1} \div \frac{3}{2a-2}$ **f** $\frac{2}{10x-5} \div \frac{10}{2x-1}$

g $\frac{5}{3a+4} \div \frac{15}{-15a-20}$ **h** $\frac{2x-6}{5x-20} \div \frac{x-3}{x-4}$ **i** $\frac{t+1}{9} \div \frac{-t-1}{3}$

PROBLEM–SOLVING	5	5, 6(½)	6–8(½)

5 Find a simplified expression for the area of these rectangles.

a



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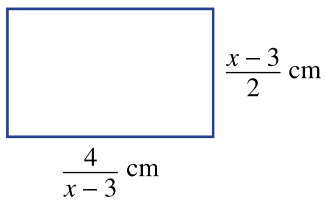
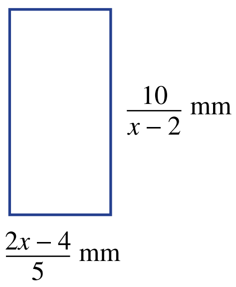


c



Example 9c

6 Simplify the following.

a $\frac{x^2-9}{2x+6}$ **b** $\frac{x+1}{x^2-1}$

c $\frac{x^2-4}{x} \times \frac{6x}{2x+4}$ **d** $\frac{x^2-3x}{x^2-9} \times \frac{4x^2}{2x}$

e $\frac{x^2-16}{x-4} \div \frac{2x+8}{x}$ **f** $\frac{2x^2-8}{x^2-2x} \div \frac{4x+8}{3x}$

7 Simplify these expressions.

a $\frac{x}{3} \times \frac{9x}{5} \times \frac{15}{3x}$ **b** $\frac{2}{a} \times \frac{a}{5} \times \frac{10}{3a}$

c $\frac{x-1}{2} \times \frac{4x}{2x-2} \times \frac{x+3}{5x}$ **d** $\frac{2x-1}{x} \div \frac{2x-1}{2} \div \frac{1}{2}$

e $\frac{2x-3}{5} \div \frac{14x-21}{10} \div \frac{x}{2}$ **f** $\frac{b^2-b}{b} \div \frac{b-1}{b^2} \times \frac{2}{b-1}$

8 Write the missing algebraic fraction.

a $\frac{x+3}{5} \times \square = 2$ **b** $\frac{1-x}{x} \times \square = 3$

c $\square \div \frac{x}{2} = \frac{3(x+2)}{x}$ **d** $\square \div \frac{2x-2}{3} = \frac{5x}{x-1}$

e $\frac{1}{x} \div \square \times \frac{x-1}{2} = 1$ **f** $\frac{2-x}{7} \times \square \div \frac{5x}{x-1} = x$

REASONING	9(½)	9(½), 10	9(½), 10, 11

9 Recall that $(x-1)^2 = (x-1)(x-1)$. Use this idea to simplify the following.

a $\dfrac{(x-1)^2}{x-1}$ **b** $\dfrac{3(x+2)^2}{x+2}$ **c** $\dfrac{4(x-3)^2}{2(x-3)}$

d $\dfrac{4(x+2)}{(x+2)^2}$ **e** $\dfrac{-5(1-x)}{(1-x)^2}$ **f** $\dfrac{(2x-2)^2}{x-1}$

10 Prove that the following all simplify to 1.

a $\dfrac{5x+5}{15} \times \dfrac{3}{x+1}$ **b** $\dfrac{3x-21}{2-x} \times \dfrac{4-2x}{6x-42}$ **c** $\dfrac{10-5x}{2x+6} \div \dfrac{20-10x}{4x+12}$

11 **a** Explain why $\dfrac{x-1}{2} \times \dfrac{4}{1-x} = \dfrac{x-1}{2} \times \dfrac{-4}{x-1}$.

b Use this idea to simplify these expressions.

i $\dfrac{2-a}{3} \times \dfrac{7}{a-2}$ **ii** $\dfrac{6x-3}{x} \div \dfrac{1-2x}{4}$ **iii** $\dfrac{18-x}{3x-1} \div \dfrac{2x-36}{7-21x}$

ENRICHMENT: Same or different?	–	–	12

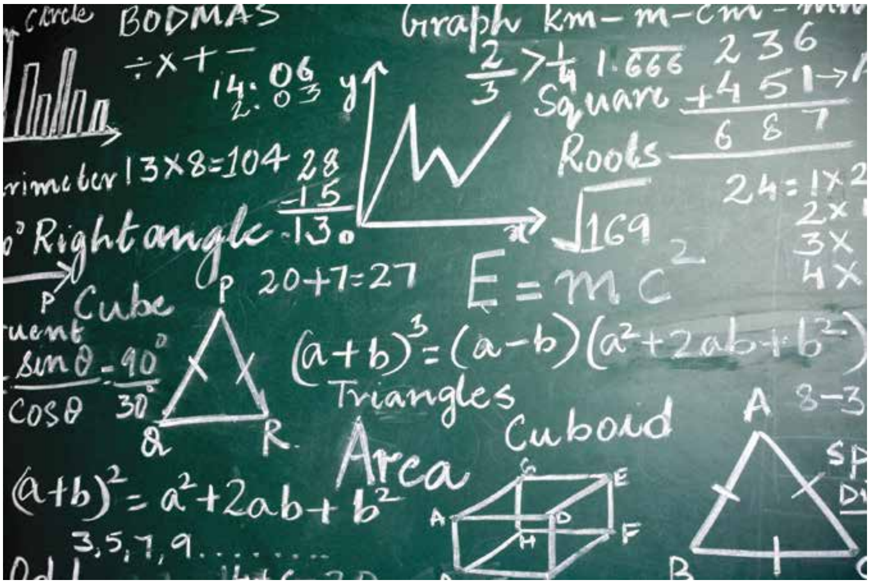
12 **a** Expand $(3-x)^2$ and $(x-3)^2$. What do you notice?

b Prove your result from part **a** for $(a-b)^2$ and $(b-a)^2$.

c Use this result to help simplify these algebraic fractions.

i $\dfrac{3(x-4)^2}{(4-x)^2}$ **ii** $\dfrac{(3-2x)^2}{2x-3}$ **iii** $\dfrac{2x-10}{(5-x)^2}$

iv $\dfrac{6x-36}{4(6-x)^2}$ **v** $\dfrac{2-x}{4x} \times \dfrac{4-2x}{(x-2)^2}$ **vi** $\dfrac{(x-y)^2}{xy} \div \dfrac{y^2-x^2}{x+y}$



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5D Factorising monic quadratic trinomials

LEARNING INTENTIONS

- To be able to identify a monic quadratic trinomial
- To understand the relationship between expanding brackets to form a trinomial and factorising a monic trinomial
- To know how to factorise a monic quadratic trinomial
- To be able to simplify algebraic fractions by first factorising and cancelling common factors

A quadratic trinomial of the form $x^2 + bx + c$ is called a monic quadratic because the coefficient of x^2 is 1.

Now consider:

$$\begin{aligned}(x+m)(x+n) &= x^2 + xn + mx + mn\\ &= x^2 + (m+n)x + mn\end{aligned}$$

We can see from this expansion that mn gives the constant term (c) and $m + n$ is the coefficient of x. This tells us that to factorise a monic quadratic trinomial we should look for factors of the constant term (c) that add to give the coefficient of the middle term (b).

Trinomial quadratics can model the revenue and profits from book publishing. Market research and past sales are used to develop unique quadratic models which find the book's selling price that predicts maximum revenue.

Lesson starter: Factorising $x^2 - 6x - 72$

Discuss what is wrong with each of these statements when trying to factorise $x^2 - 6x - 72$.

- Find factors of 72 that add to 6.
- Find factors of 72 that add to −6.
- Find factors of −72 that add to 6.
- $-18 \times 4 = -72$ so $x^2 - 6x - 72 = (x - 18)(x + 4)$
- $-9 \times 8 = -72$ so $x^2 - 6x - 72 = (x - 9)(x + 8)$

Can you write a correct statement that correctly factorises $x^2 - 6x - 72$?

KEY IDEAS

- **Monic quadratics** have a coefficient of x^2 equal to 1.
- Monic quadratics of the form $x^2 + bx + c$ can be factorised by finding the two numbers that multiply to give the constant term (c) and add to give the coefficient of x (i.e. b).

$$x^2 + (\underbrace{m+n}_{b})x + \underbrace{mn}_{c} = (x+m)(x+n)$$

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BUILDING UNDERSTANDING

1 Find two integers that multiply to give the first number and add to give the second number.

a 18, 11 **b** 20, 12 **c** −15, 2 **d** −12, 1

e −24, −5 **f** −30, −7 **g** 10, −7 **h** 36, −15

2 Fill in the missing integers to complete the following.

a $(x + \boxed{})(x + 3) = x^2 + 5x + 6$

b $(x - \boxed{})(x + 4) = x^2 - 2x - 24$

c $(x - \boxed{})(x - \boxed{}) = x^2 - 7x + 10$

Example 10 Factorising trinomials of the form $x^2 + bx + c$

Factorise:

a $x^2 + 8x + 15$ **b** $x^2 - 5x + 6$ **c** $2x^2 - 10x - 28$ **d** $x^2 - 8x + 16$

SOLUTION	EXPLANATION
a $x^2 + 8x + 15 = (x + 3)(x + 5)$	$3 \times 5 = 15$ and $3 + 5 = 8$ Check: $(x + 3)(x + 5) = x^2 + 5x + 3x + 15$ $= x^2 + 8x + 15$
b $x^2 - 5x + 6 = (x - 3)(x - 2)$	Require both numbers to be negative to add to a negative but multiply to a positive. $-3 \times (-2) = 6$ and $-3 + (-2) = -5$ Check: $(x - 3)(x - 2) = x^2 - 2x - 3x + 6$ $= x^2 - 5x + 6$
c $2x^2 - 10x - 28 = 2(x^2 - 5x - 14)$ $= 2(x - 7)(x + 2)$	First, take out the common factor of 2. Require two numbers to × to −14 and + to −5. $-7 \times 2 = -14$ and $-7 + 2 = -5$
d $x^2 - 8x + 16 = (x - 4)(x - 4)$ $= (x - 4)^2$	$-4 \times (-4) = 16$ and $-4 + (-4) = -8$ $(x - 4)(x - 4) = (x - 4)^2$ is a perfect square.

Now you try

Factorise:

a $x^2 + 7x + 12$ **b** $x^2 - 10x + 24$ **c** $2x^2 - 2x - 12$ **d** $x^2 - 6x + 9$

Example 11 Simplifying algebraic fractions

Use factorisation to simplify these algebraic fractions.

a $\dfrac{x^2 - x - 6}{x + 2}$

b $\dfrac{x^2 - 9}{x^2 - 2x - 15} \times \dfrac{x^2 - 4x - 5}{2x - 6}$

SOLUTION

a $\dfrac{x^2 - x - 6}{x + 2} = \dfrac{(x - 3)\cancel{(x + 2)}^1}{\cancel{(x + 2)}^1}$

$= x - 3$

b $\dfrac{x^2 - 9}{x^2 - 2x - 15} \times \dfrac{x^2 - 4x - 5}{2x - 6}$

$= \dfrac{\cancel{(x + 3)}^1\cancel{(x - 3)}^1}{\cancel{(x - 5)}^1\cancel{(x + 3)}^1} \times \dfrac{\cancel{(x - 5)}^1(x + 1)}{2\cancel{(x - 3)}^1}$

$= \dfrac{x + 1}{2}$

EXPLANATION

First, factorise $x^2 - x - 6$ and then cancel the common factor of $(x + 2)$.

First, factorise all expressions in the numerators and denominators. Cancel to simplify where possible.

Now you try

Use factorisation to simplify these algebraic fractions.

a $\dfrac{x^2 - 2x - 8}{x + 2}$

b $\dfrac{x^2 - 4}{x^2 + x - 2} \times \dfrac{x^2 + 3x - 4}{2x - 4}$

Exercise 5D

FLUENCY 1–4(½) 1–4(⅓) 1–4(¼)

Example 10a, b

1 Factorise these quadratic trinomials.

a $x^2 + 7x + 6$
b $x^2 + 5x + 6$
c $x^2 + 6x + 9$
d $x^2 + 7x + 10$
e $x^2 + 7x + 12$
f $x^2 + 11x + 18$
g $x^2 + 5x - 6$
h $x^2 + x - 6$
i $x^2 + 2x - 8$
j $x^2 + 3x - 4$
k $x^2 + 7x - 30$
l $x^2 + 9x - 22$
m $x^2 - 7x + 10$
n $x^2 - 6x + 8$
o $x^2 - 7x + 12$
p $x^2 - 2x + 1$
q $x^2 - 9x + 18$
r $x^2 - 11x + 18$
s $x^2 - 4x - 12$
t $x^2 - x - 20$
u $x^2 - 5x - 14$
v $x^2 - x - 12$
w $x^2 + 4x - 32$
x $x^2 - 3x - 10$

Example 10c

2 Factorise by first taking out the common factor.

a $2x^2 + 14x + 20$
b $3x^2 + 21x + 36$
c $2x^2 + 22x + 36$
d $5x^2 - 5x - 10$
e $4x^2 - 16x - 20$
f $3x^2 - 9x - 30$
g $-2x^2 - 14x - 24$
h $-3x^2 + 9x - 6$
i $-2x^2 + 10x + 28$
j $-4x^2 + 4x + 8$
k $-5x^2 - 20x - 15$
l $-7x^2 + 49x - 42$

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Example 10d

3 Factorise these perfect squares.

a $x^2 - 4x + 4$ **b** $x^2 + 6x + 9$ **c** $x^2 + 12x + 36$ **d** $x^2 - 14x + 49$

e $x^2 - 18x + 81$ **f** $x^2 - 20x + 100$ **g** $2x^2 + 44x + 242$ **h** $3x^2 - 24x + 48$

i $5x^2 - 50x + 125$ **j** $-3x^2 + 36x - 108$ **k** $-2x^2 + 28x - 98$ **l** $-4x^2 - 72x - 324$

Example 11a

4 Use factorisation to simplify these algebraic fractions.

a $\frac{x^2 - 3x - 54}{x - 9}$ **b** $\frac{x^2 + x - 12}{x + 4}$ **c** $\frac{x^2 - 6x + 9}{x - 3}$

d $\frac{x + 2}{x^2 + 9x + 14}$ **e** $\frac{x - 3}{x^2 - 8x + 15}$ **f** $\frac{x + 1}{x^2 - 5x - 6}$

g $\frac{2(x + 12)}{x^2 + 4x - 96}$ **h** $\frac{x^2 - 5x - 36}{3(x - 9)}$ **i** $\frac{x^2 - 15x + 56}{5(x - 8)}$

PROBLEM–SOLVING 5(½) 5–6(½) 5–7(½)

Example 11b

5 Simplify by first factorising.

a $\frac{x^2 - 4}{x^2 - x - 6} \times \frac{5x - 15}{x^2 + 4x - 12}$ **b** $\frac{x^2 + 3x + 2}{x^2 + 4x + 3} \times \frac{x^2 - 9}{3x + 6}$

c $\frac{x^2 + 2x - 3}{x^2 - 25} \times \frac{2x - 10}{x + 3}$ **d** $\frac{x^2 - 9}{x^2 - 5x + 6} \times \frac{4x - 8}{x^2 + 8x + 15}$

e $\frac{x^2 - 4x + 3}{x^2 + 4x - 21} \times \frac{4x + 4}{x^2 - 1}$ **f** $\frac{x^2 + 6x + 8}{x^2 - 4} \times \frac{6x - 24}{x^2 - 16}$

g $\frac{x^2 - x - 6}{x^2 + x - 12} \times \frac{x^2 + 5x + 4}{x^2 - 1}$ **h** $\frac{x^2 - 4x - 12}{x^2 - 4} \times \frac{x^2 - 6x + 8}{x^2 - 36}$

6 Simplify these expressions that involve surds.

a $\frac{x^2 - 7}{x + \sqrt{7}}$ **b** $\frac{x^2 - 10}{x - \sqrt{10}}$ **c** $\frac{\sqrt{5}\,x + 3}{5x^2 - 9}$

d $\frac{\sqrt{3}\,x - 4}{3x^2 - 16}$ **e** $\frac{(x + 1)^2 - 2}{x + 1 + \sqrt{2}}$ **f** $\frac{(x - 3)^2 - 5}{x - 3 - \sqrt{5}}$

7 Simplify using factorisation.

a $\frac{x^2 + 2x - 3}{x^2 - 25} \div \frac{3x - 3}{2x + 10}$ **b** $\frac{x^2 + 3x + 2}{x^2 + 4x + 3} \div \frac{4x + 8}{x^2 - 9}$ **c** $\frac{x^2 - x - 12}{x^2 - 9} \div \frac{x^2 - 16}{3x + 12}$

d $\frac{x^2 - 49}{x^2 - 3x - 28} \div \frac{4x + 28}{6x + 24}$ **e** $\frac{x^2 + 5x - 14}{x^2 + 2x - 3} \div \frac{x^2 + 9x + 14}{x^2 + x - 2}$ **f** $\frac{x^2 + 8x + 15}{x^2 + 5x - 6} \div \frac{x^2 + 6x + 5}{x^2 + 7x + 6}$

REASONING 8 8, 9(½) 9(½), 10, 11

8 A businessman is showing off his new formula to determine the company's profit, in millions of dollars, after t years.

$$\text{Profit} = \frac{t^2 - 49}{5t - 40} \times \frac{t^2 - 5t - 24}{2t^2 - 8t - 42}$$

Show that this is really the same as: Profit $= \frac{t + 7}{10}$.

9 Note that an expression with a perfect square can be simplified as shown.

$$\begin{aligned}\frac{(x+3)^2}{x+3} &= \frac{(x+3)\cancel{(x+3)}^{1}}{\cancel{x+3}^{1}}\\ &= x+3\end{aligned}$$

Use this idea to simplify the following.

a $\dfrac{x^2-6x+9}{x-3}$ **b** $\dfrac{x^2+2x+1}{x+1}$ **c** $\dfrac{x^2-16x+64}{x-8}$

d $\dfrac{6x-12}{x^2-4x+4}$ **e** $\dfrac{4x+20}{x^2+10x+25}$ **f** $\dfrac{x^2-14x+49}{5x-35}$

10 a Prove that $\dfrac{a^2+2ab+b^2}{a^2+ab} \div \dfrac{a^2-b^2}{a^2-ab} = 1$.

b Make up your own expressions, like the one in part **a**, which equal 1. Ask a classmate to check them.

11 Simplify.

a $\dfrac{a^2+2ab+b^2}{a(a+b)} \div \dfrac{a^2-b^2}{a^2-2ab+b^2}$ **b** $\dfrac{a^2-2ab+b^2}{a^2-b^2} \div \dfrac{a^2-b^2}{a^2+2ab+b^2}$

c $\dfrac{a^2-b^2}{a^2-2ab+b^2} \div \dfrac{a^2-b^2}{a^2+2ab+b^2}$ **d** $\dfrac{a^2+2ab+b^2}{a(a+b)} \div \dfrac{a(a-b)}{a^2-2ab+b^2}$

ENRICHMENT: Addition and subtraction with factorisation

– – 12(½)

12 Factorisation can be used to help add and subtract algebraic fractions. Here is an example.

$$\begin{aligned}\frac{3}{x-2}+\frac{x}{x^2-6x+8} &= \frac{3}{x-2}+\frac{x}{(x-2)(x-4)}\\ &= \frac{3(x-4)}{(x-2)(x-4)}+\frac{x}{(x-2)(x-4)}\\ &= \frac{3x-12+x}{(x-2)(x-4)}\\ &= \frac{4x-12}{(x-2)(x-4)}\\ &= \frac{4(x-3)}{(x-2)(x-4)}\end{aligned}$$

Now simplify the following.

a $\dfrac{2}{x+3}+\dfrac{x}{x^2-x-12}$ **b** $\dfrac{4}{x+2}+\dfrac{3x}{x^2-7x-18}$

c $\dfrac{3}{x+4}-\dfrac{2x}{x^2-16}$ **d** $\dfrac{4}{x^2-9}-\dfrac{1}{x^2-8x+15}$

e $\dfrac{x+4}{x^2-x-6}-\dfrac{x-5}{x^2-9x+18}$ **f** $\dfrac{x+3}{x^2-4x-32}-\dfrac{x}{x^2+7x+12}$

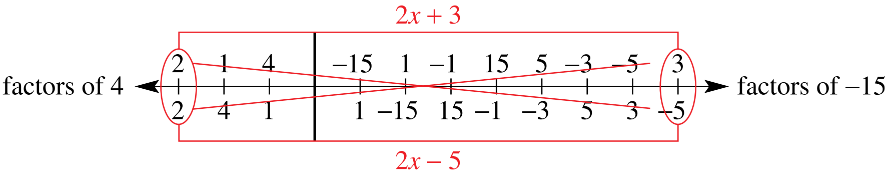
g $\dfrac{x+1}{x^2-25}-\dfrac{x-2}{x^2-6x+5}$ **h** $\dfrac{x+2}{x^2-2x+1}-\dfrac{x+3}{x^2+3x-4}$

5E Factorising non-monic quadratic trinomials OPTIONAL

LEARNING INTENTIONS

- To understand the relationship between expansion and factorisation for binomial products
- To know and be able to apply the process for factorising non-monic quadratic trinomials

There are a number of ways of factorising non-monic quadratic trinomials of the form $ax^2 + bx + c$, where $a \neq 1$. The cross method, for example, uses lists of factors of a and c so that a correct combination can be found. For example, to factorise $4x^2 - 4x - 15$:



$2 \times (-5) + 2 \times 3 = -4$, so choose $(2x + 3)$ and $(2x - 5)$.

$\therefore 4x^2 - 4x - 15 = (2x + 3)(2x - 5)$

The method outlined in this section, however, uses grouping.

Lesson starter: Does the order matter?

To factorise the non-monic quadratic $4x^2 - 4x - 15$ using grouping, we multiply a by c, which is $4 \times (-15) = -60$. Then we look for numbers that multiply to give -60 and add to give -4 (the coefficient of x).

- What are the two numbers that multiply to give -60 and add to give -4?
- Complete the following using grouping.

$$\begin{aligned} 4x^2 - 4x - 15 &= 4x^2 - 10x + 6x - 15 \qquad -10 \times 6 = -60, \qquad -10 + 6 = -4 \\ &= 2x(\underline{\qquad}) + 3(\underline{\qquad}) \\ &= (2x - 5)(\underline{\qquad}) \end{aligned}$$

- If we changed the order of the $-10x$ and $+6x$ do you think the result would change? Copy and complete to find out.

$$\begin{aligned} 4x^2 - 4x - 15 &= 4x^2 - 6x - 10x - 15 \qquad 6 \times (-10) = -60, \qquad 6 + (-10) = -4 \\ &= 2x(\underline{\qquad}) - 5(\underline{\qquad}) \\ &= (\underline{\qquad})(\underline{\qquad}) \end{aligned}$$

KEY IDEAS

- To factorise a **non-monic** trinomial of the form $ax^2 + bx + c$, follow these steps:
  - Find two numbers that multiply to give $a \times c$ and add to give b.
    For $15x^2 - x - 6$, $a \times c = 15 \times (-6) = -90$.
    The factors of -90 that add to $-1\,(b)$ are -10 and 9.

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- Use the two numbers shown in the previous example to split bx, then factorise by grouping.

$$\begin{aligned} 15x^2 - x - 6 &= 15x^2 - 10x + 9x - 6 \\ &= 5x(3x - 2) + 3(3x - 2) = (3x - 2)(5x + 3) \end{aligned}$$

- There are other valid methods that can be used to factorise non-monic trinomials. The cross method is illustrated in the introduction.

BUILDING UNDERSTANDING

1 State the missing numbers in this table.

$ax^2 + bx + c$	$a \times c$	Two numbers that multiply to give $a \times c$ and add to give b
$6x^2 + 13x + 6$	36	9 and ______
$8x^2 + 18x + 4$	32	
$12x^2 + x - 6$		-8 and ______
$10x^2 - 11x - 6$		
$21x^2 - 20x + 4$		-6 and ______
$15x^2 - 13x + 2$		

2 Factorise by grouping pairs.

a $x^2 + 2x + 5x + 10$
b $x^2 - 7x - 2x + 14$
c $6x^2 - 8x + 3x - 4$
d $8x^2 - 4x + 6x - 3$
e $5x^2 + 20x - 2x - 8$
f $12x^2 - 6x - 10x + 5$

Example 12 Factorising non-monic quadratics

Factorise:

a $6x^2 + 23x + 7$
b $9x^2 + 6x - 8$

SOLUTION

a
$$\begin{aligned} 6x^2 + 23x + 7 &= 6x^2 + 2x + 21x + 7 \\ &= 2x(3x + 1) + 7(3x + 1) \\ &= (3x + 1)(2x + 7) \end{aligned}$$

b
$$\begin{aligned} 9x^2 + 6x - 8 &= 9x^2 + 12x - 6x - 8 \\ &= 3x(3x + 4) - 2(3x + 4) \\ &= (3x + 4)(3x - 2) \end{aligned}$$

EXPLANATION

$a \times c = 6 \times 7 = 42$; choose 21 and 2 since $21 \times 2 = 42$ and $21 + 2 = 23$ (b).
Factorise by grouping.

$a \times c = 9 \times (-8) = -72$; choose 12 and -6 since $12 \times (-6) = -72$ and $12 + (-6) = 6$ (b).

Now you try

Factorise:

a $6x^2 + 11x + 3$
b $8x^2 + 10x - 3$

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Example 13 Simplifying algebraic fractions involving quadratic expressions

Simplify $\dfrac{4x^2-9}{10x^2+13x-3}\times\dfrac{25x^2-10x+1}{10x^2-17x+3}$.

SOLUTION

$$\frac{4x^2-9}{10x^2+13x-3}\times\frac{25x^2-10x+1}{10x^2-17x+3}$$

$$=\frac{\cancel{(2x+3)}^1\cancel{(2x-3)}^1}{\cancel{(2x+3)}^1\cancel{(5x-1)}^1}\times\frac{\cancel{(5x-1)}^1\cancel{(5x-1)}^1}{\cancel{(2x-3)}^1\cancel{(5x-1)}^1}$$

$$=1$$

EXPLANATION

First, use the range of factorising techniques to factorise all quadratics.

Cancel to simplify.

Now you try

Simplify $\dfrac{9x^2-4}{12x^2-17x+6}\times\dfrac{16x^2-24x+9}{12x^2-x-6}$.

Exercise 5E

FLUENCY 1, 2(½) 1–3(⅓) 1–3(¼)

Example 12a

1 Factorise the following.

a $3x^2+10x+3$ **b** $2x^2+3x+1$ **c** $3x^2+8x+4$ **d** $5x^2+12x+4$

Example 12b

2 Factorise the following.

a $2x^2-11x+5$ **b** $5x^2+2x-3$ **c** $3x^2-11x-4$ **d** $3x^2-2x-1$
e $7x^2+2x-5$ **f** $2x^2-9x+7$ **g** $3x^2+2x-8$ **h** $2x^2+5x-12$
i $2x^2-9x-5$ **j** $13x^2-7x-6$ **k** $5x^2-22x+8$ **l** $8x^2-14x+5$
m $6x^2+x-12$ **n** $10x^2+11x-6$ **o** $6x^2+13x+6$ **p** $4x^2-5x+1$
q $8x^2-14x+5$ **r** $8x^2-26x+15$ **s** $6x^2-13x+6$ **t** $9x^2+9x-10$

3 Factorise the following.

a $18x^2+27x+10$ **b** $20x^2+39x+18$ **c** $21x^2+22x-8$
d $30x^2+13x-10$ **e** $40x^2-x-6$ **f** $28x^2-13x-6$
g $24x^2-38x+15$ **h** $45x^2-46x+8$ **i** $25x^2-50x+16$

PROBLEM-SOLVING 4(½), 5 4(⅓), 5, 6(⅓) 4(⅓), 5, 6(⅓)

4 Factorise by first taking out the common factor.

a $6x^2+38x+40$ **b** $6x^2-15x-36$ **c** $48x^2-18x-3$
d $32x^2-88x+60$ **e** $16x^2-24x+8$ **f** $90x^2+90x-100$
g $-50x^2-115x-60$ **h** $12x^2-36x+27$ **i** $20x^2-25x+5$

5 A cable is suspended across a farm channel. The height (h), in metres, of the cable above the water surface is modelled by the equation $h = 3x^2 - 19x + 20$, where x metres is the distance from one side of the channel.

- **a** Factorise the right-hand side of the equation.
- **b** Determine the height of the cable when $x = 3$. Interpret this result.
- **c** Determine where the cable is at the level of the water surface.

6 Simplify by first factorising.

- **a** $\dfrac{6x^2 - x - 35}{3x + 7}$
- **b** $\dfrac{8x^2 + 10x - 3}{2x + 3}$
- **c** $\dfrac{9x^2 - 21x + 10}{3x - 5}$
- **d** $\dfrac{10x - 2}{15x^2 + 7x - 2}$
- **e** $\dfrac{4x + 6}{14x^2 + 17x - 6}$
- **f** $\dfrac{20x - 12}{10x^2 - 21x + 9}$
- **g** $\dfrac{2x^2 + 11x + 12}{6x^2 + 11x + 3}$
- **h** $\dfrac{12x^2 - x - 1}{8x^2 + 14x + 3}$

REASONING	7(½)	7–8(½)	7–8(⅓), 9

Example 13

7 Combine all your knowledge of factorising to simplify the following.

- **a** $\dfrac{9x^2 - 16}{x^2 - 6x + 9} \times \dfrac{x^2 + x - 12}{3x^2 + 8x - 16}$
- **b** $\dfrac{4x^2 - 1}{6x^2 - x - 2} \times \dfrac{9x^2 - 4}{8x - 4}$
- **c** $\dfrac{1 - x^2}{15x + 9} \times \dfrac{25x^2 + 30x + 9}{5x^2 + 8x + 3}$
- **d** $\dfrac{20x^2 + 21x - 5}{16x^2 + 8x - 15} \times \dfrac{16x^2 - 24x + 9}{25x^2 - 1}$
- **e** $\dfrac{100x^2 - 25}{2x^2 - 9x - 5} \div \dfrac{2x^2 - 7x + 3}{5x^2 - 40x + 75}$
- **f** $\dfrac{3x^2 - 12}{30x + 15} \div \dfrac{2x^2 - 3x - 2}{4x^2 + 4x + 1}$
- **g** $\dfrac{9x^2 - 6x + 1}{6x^2 - 11x + 3} \div \dfrac{9x^2 - 1}{6x^2 - 7x - 3}$
- **h** $\dfrac{16x^2 - 25}{4x^2 - 7x - 15} \div \dfrac{4x^2 - 17x + 15}{16x^2 - 40x + 25}$

8 Find a method to show how $-12x^2 - 5x + 3$ factorises to $(1 - 3x)(4x + 3)$. Then factorise the following.

- **a** $-8x^2 + 2x + 15$
- **b** $-6x^2 + 11x + 10$
- **c** $-12x^2 + 13x + 4$
- **d** $-8x^2 + 18x - 9$
- **e** $-14x^2 + 39x - 10$
- **f** $-15x^2 - x + 6$

9 Make up your own complex expression like those in Question 7, which simplifies to 1. Check your expression with your teacher or a classmate.

ENRICHMENT: Non-monics with addition and subtraction	–	–	10(½)

10 Factorise the quadratics in the expressions and then simplify using a common denominator.

- **a** $\dfrac{2}{2x - 3} + \dfrac{x}{8x^2 - 10x - 3}$
- **b** $\dfrac{3}{3x - 1} - \dfrac{x}{6x^2 + 13x - 5}$
- **c** $\dfrac{4x}{2x - 5} + \dfrac{x}{8x^2 - 18x - 5}$
- **d** $\dfrac{4x}{12x^2 - 11x + 2} - \dfrac{3x}{3x - 2}$
- **e** $\dfrac{2}{4x^2 - 1} + \dfrac{1}{6x^2 - x - 2}$
- **f** $\dfrac{2}{9x^2 - 25} - \dfrac{3}{9x^2 + 9x - 10}$
- **g** $\dfrac{4}{8x^2 - 18x - 5} - \dfrac{2}{12x^2 - 5x - 2}$
- **h** $\dfrac{1}{10x^2 - 19x + 6} + \dfrac{2}{4x^2 + 8x - 21}$

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Progress quiz

5A

1 Expand brackets and simplify where possible.

a $-2(12x - 5)$ **b** $a(3a - 2) - a(5 - a)$

c $(m + 2)(m + 5)$ **d** $(k - 3)^2$

e $(3m - 2)(3m + 2)$ **f** $(4h + 7)(2h - 5)$

g $5(x - 4)(x - 3)$ **h** $(p + 5)(p + 4) - (p - 2)(p - 8)$

5B

2 Factorise the following.

a $4a - 20$ **b** $-12m^2 + 18m$

c $4(x + 5) - x(x + 5)$ **d** $a^2 - 81$

e $16a^2 - 121b^2$ **f** $5m^2 - 125$

g $(k + 2)^2 - 49$ **h** $(x - 1)^2 - 4$

i $x^2 - 15$ (use surds) **j** $(h + 3)^2 - 7$ (use surds)

k $x^2 + 5x + ax + 5a$ **l** $4x^2 - 8mx - 5x + 10m$

5C

3 Simplify by cancelling common factors.

a $\dfrac{36mk^2}{9mk}$ **b** $\dfrac{3a - 12}{3}$

c $\dfrac{a + 4}{4a} \times \dfrac{18a^2}{a + 4}$ **d** $\dfrac{6h - 15}{6} \div \dfrac{2h - 5}{5}$

5D

4 Factorise:

a $x^2 + x - 20$ **b** $a^2 - 10a + 21$

c $3k^2 - 21k - 54$ **d** $m^2 - 12m + 36$

5D

5 Use factorisation to simplify these algebraic fractions.

a $\dfrac{x^2 + 2x - 15}{x + 5}$

b $\dfrac{x^2 - 25}{x^2 - 9x + 20} \times \dfrac{x^2 + 3x - 28}{2x + 14}$

5E Opt

6 Factorise:

a $6a^2 + 19a + 10$ **b** $8m^2 - 6m - 9$

c $15x^2 - 22x + 8$ **d** $6k^2 - 11k - 35$

5E Opt

7 Simplify $\dfrac{9x^2 - 49}{3x^2 - 4x - 7} \times \dfrac{2x^2 + 7x + 5}{6x^2 + 5x - 21}$.

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5F Factorising by completing the square

LEARNING INTENTIONS

- To know the expanded form of a perfect square
- To be able to carry out the process of completing the square
- To know how to factorise by first completing the square
- To understand that not all quadratic expressions can be factorised and to be able to identify those that can't

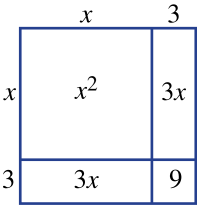
Consider the quadratic expression $x^2 + 6x + 4$. We cannot factorise this using the methods we have established in the previous exercises because there are no factors of 4 that add to 6.

We can, however, use our knowledge of perfect squares and the difference of two squares to help find factors using surds.

The statistical analysis of agricultural research data has found that quadratic equations model harvest yields (kg/ha) versus the quantity of nitrogen fertiliser (kg/ha) used. The CSIRO provides Australian farmers with numerous mathematical models.

Lesson starter: Make a perfect square

This diagram is a square. Its sides are $x + 3$ and its area is given by $x^2 + 6x + 9 = (x + 3)^2$.



Use a similar diagram to help make a perfect square for the following and determine the missing number for each.

- $x^2 + 8x + ?$
- $x^2 + 12x + ?$

Can you describe a method for finding the missing number without drawing a diagram?

KEY IDEAS

- Recall for a perfect square $(x + a)^2 = x^2 + 2ax + a^2$ and $(x - a)^2 = x^2 - 2ax + a^2$.
- To **complete the square** for $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$.
  - $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$
- To factorise by completing the square:
  - Add $\left(\frac{b}{2}\right)^2$ and balance by subtracting $\left(\frac{b}{2}\right)^2$.
  - Factorise the perfect square and simplify.
  - Factorise using difference of two squares: $a^2 - b^2 = (a + b)(a - b)$; surds can be used.

$$\begin{aligned} x^2 + 6x + 4 &= x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 4 \\ &= \left(x + \frac{6}{2}\right)^2 - 5 \\ &= (x + 3)^2 - (\sqrt{5})^2 \\ &= (x + 3 + \sqrt{5})(x + 3 - \sqrt{5}) \end{aligned}$$

- Not all quadratic expressions factorise. This will be seen when you end up with expressions such as $(x + 3)^2 + 6$, which is *not* a difference of two squares.

BUILDING UNDERSTANDING

1 These expressions are of the form $x^2 + bx$. Evaluate $\left(\frac{b}{2}\right)^2$ for each one.

a $x^2 + 6x$ **b** $x^2 + 2x$ **c** $x^2 - 4x$

d $x^2 - 8x$ **e** $x^2 + 5x$ **f** $x^2 - 9x$

2 Factorise these perfect squares.

a $x^2 + 4x + 4$ **b** $x^2 + 8x + 16$ **c** $x^2 + 10x + 25$

d $x^2 - 12x + 36$ **e** $x^2 - 6x + 9$ **f** $x^2 - 18x + 81$

3 Factorise using surds. Recall that $a^2 - b^2 = (a + b)(a - b)$.

a $(x + 1)^2 - 5$ **b** $(x + 4)^2 - 10$ **c** $(x - 3)^2 - 11$

Example 14 Completing the square

Decide what number must be added to these expressions to complete the square. Then factorise the resulting perfect square.

a $x^2 + 10x$ **b** $x^2 - 7x$

SOLUTION	EXPLANATION
a $\left(\frac{10}{2}\right)^2 = 5^2 = 25$	For $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$. Here $b = 10$, and evaluate $\left(\frac{b}{2}\right)^2$.
$x^2 + 10x + 25 = (x + 5)^2$	$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$
b $\left(\frac{-7}{2}\right)^2 = \frac{49}{4}$	In $x^2 - 7x$, $b = -7$ and evaluate $\left(\frac{b}{2}\right)^2$.
$x^2 - 7x + \frac{49}{4} = \left(x - \frac{7}{2}\right)^2$	Factorise the perfect square.

Now you try

Decide what number must be added to these expressions to complete the square. Then factorise the resulting perfect square.

a $x^2 + 12x$ **b** $x^2 - 9x$

Example 15 Factorising by completing the square

Factorise the following by completing the square if possible.

a $x^2 + 8x - 3$

b $x^2 - 2x + 8$

SOLUTION	EXPLANATION
a $x^2 + 8x - 3 = \left(x^2 + 8x + \left(\frac{8}{2}\right)^2\right) - \left(\frac{8}{2}\right)^2 - 3$	Add $\left(\frac{b}{2}\right)^2$ to complete the square and
$= \left(x + \frac{8}{2}\right)^2 - 16 - 3$	balance by subtracting $\left(\frac{b}{2}\right)^2$ also.
$= (x + 4)^2 - 19$	Factorise the resulting perfect square and simplify.
$= (x + 4)^2 - (\sqrt{19})^2$	Express 19 as $(\sqrt{19})^2$ to set up a difference of two squares.
$= (x + 4 - \sqrt{19})(x + 4 + \sqrt{19})$	Apply $a^2 - b^2 = (a + b)(a - b)$ using surds.
b $x^2 - 2x + 8 = \left(x^2 - 2x + \left(\frac{-2}{2}\right)^2\right) - \left(\frac{-2}{2}\right)^2 + 8$	Add $\left(\frac{-2}{2}\right)^2 = (-1)^2$ to complete the square and balance by subtracting $(-1)^2$ also.
$= \left(x - \frac{2}{2}\right)^2 + 7$	Factorise the perfect square and simplify.
$= (x - 1)^2 + 7$	$(x - 1)^2 + 7$ is not a *difference* of two squares.
$\therefore x^2 - 2x + 8$ cannot be factorised.	

Now you try

Factorise the following by completing the square if possible.

a $x^2 + 6x - 1$

b $x^2 - 4x + 7$

Example 16 Factorising with fractions and non-monics

Opt

Factorise the following by completing the square.

a $x^2 + 3x + \frac{1}{2}$

b $2x^2 - 8x + 3$

SOLUTION	EXPLANATION
a $x^2 + 3x + \frac{1}{2} = \left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) - \left(\frac{3}{2}\right)^2 + \frac{1}{2}$	Add $\left(\frac{3}{2}\right)^2$ to complete the square and
$= \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{1}{2}$	balance by subtracting $\left(\frac{3}{2}\right)^2$. Leave in fraction form.
$= \left(x + \frac{3}{2}\right)^2 - \frac{7}{4}$	Factorise the perfect square and simplify.
$= \left(x + \frac{3}{2}\right)^2 - \left(\sqrt{\frac{7}{4}}\right)^2$	$-\frac{9}{4} + \frac{1}{2} = -\frac{9}{4} + \frac{2}{4} = -\frac{7}{4}$
$= \left(x + \frac{3}{2} - \frac{\sqrt{7}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{7}}{2}\right)$	Recall that $\sqrt{\frac{7}{4}} = \frac{\sqrt{7}}{\sqrt{4}} = \frac{\sqrt{7}}{2}$ and use difference of two squares.
$= \left(x + \frac{3 - \sqrt{7}}{2}\right)\left(x + \frac{3 + \sqrt{7}}{2}\right)$	

Continued on next page

b $$\begin{aligned}2x^2 - 8x + 3 &= 2\left(x^2 - 4x + \frac{3}{2}\right)\\ &= 2\left(\left(x - \frac{4}{2}\right)^2 - \left(\frac{-4}{2}\right)^2 + \frac{3}{2}\right)\\ &= 2\left((x-2)^2 - 4 + \frac{3}{2}\right)\\ &= 2\left((x-2)^2 - \frac{8}{2} + \frac{3}{2}\right)\\ &= 2\left((x-2)^2 - \frac{5}{2}\right)\\ &= 2\left(x - 2 - \sqrt{\frac{5}{2}}\right)\left(x - 2 + \sqrt{\frac{5}{2}}\right)\end{aligned}$$

Factor out the coefficient of x^2 i.e. 2.
Complete the square of $x^2 - 4x + \frac{3}{2}$.

Factorise and simplify: $\frac{5}{2} = \left(\sqrt{\frac{5}{2}}\right)^2$ and apply difference of two squares.

Now you try

Factorise the following by completing the square.

a $x^2 + 5x + \frac{1}{2}$ **b** $2x^2 - 4x - 3$

Exercise 5F

FLUENCY 1–3(½) 1–4(½) 1–4(⅓)

Example 14

1 Decide what number must be added to these expressions to complete the square. Then factorise the resulting perfect square.

a $x^2 + 6x$ **b** $x^2 + 12x$ **c** $x^2 + 4x$ **d** $x^2 + 8x$
e $x^2 - 10x$ **f** $x^2 - 2x$ **g** $x^2 - 8x$ **h** $x^2 - 12x$
i $x^2 + 5x$ **j** $x^2 + 9x$ **k** $x^2 + 7x$ **l** $x^2 + 11x$
m $x^2 - 3x$ **n** $x^2 - 7x$ **o** $x^2 - x$ **p** $x^2 - 9x$

Example 15a

2 Factorise by completing the square.

a $x^2 + 4x + 1$ **b** $x^2 + 6x + 2$ **c** $x^2 + 2x - 4$
d $x^2 + 10x - 4$ **e** $x^2 - 8x + 13$ **f** $x^2 - 12x + 10$
g $x^2 - 4x - 3$ **h** $x^2 - 8x - 5$ **i** $x^2 + 14x + 6$

Example 15b

3 Factorise, if possible, by first completing the square.

a $x^2 + 6x + 11$ **b** $x^2 + 4x + 7$ **c** $x^2 + 8x + 1$
d $x^2 + 4x + 2$ **e** $x^2 + 10x + 3$ **f** $x^2 + 4x - 6$
g $x^2 - 10x + 30$ **h** $x^2 - 6x + 6$ **i** $x^2 - 12x + 2$
j $x^2 - 2x + 2$ **k** $x^2 - 8x - 1$ **l** $x^2 - 4x + 6$

Example 16a
Opt

4 Factorise the following.

a $x^2 + 3x + 1$ **b** $x^2 + 7x + 2$ **c** $x^2 + 5x - 2$ **d** $x^2 + 9x - 3$
e $x^2 - 3x + \frac{1}{2}$ **f** $x^2 - 5x + \frac{1}{2}$ **g** $x^2 - 5x - \frac{3}{2}$ **h** $x^2 - 9x - \frac{5}{2}$

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PROBLEM–SOLVING	5(½)	5–7(½)	5–7(⅓)

Example 16b

Opt

5 Factorise by first taking out the common factor.

a $2x^2 + 12x + 8$ **b** $3x^2 + 12x - 3$ **c** $4x^2 - 8x - 16$
d $3x^2 - 24x + 6$ **e** $-2x^2 - 4x + 10$ **f** $-3x^2 - 30x - 3$
g $-4x^2 - 16x + 12$ **h** $-2x^2 + 16x + 4$ **i** $-3x^2 + 24x - 15$

Opt

6 Factorise the following.

a $3x^2 + 9x + 3$ **b** $5x^2 + 15x - 35$ **c** $2x^2 - 10x + 4$
d $4x^2 - 28x + 12$ **e** $-3x^2 - 21x + 6$ **f** $-2x^2 - 14x + 8$
g $-4x^2 + 12x + 20$ **h** $-3x^2 + 9x + 6$ **i** $-2x^2 + 10x + 8$

Opt

7 Factorise the following by completing the square.

a $2x^2 + 4x - 3$ **b** $2x^2 + 8x + 1$ **c** $3x^2 - 6x + 1$
d $3x^2 - 12x - 2$ **e** $2x^2 + 6x + 3$ **f** $3x^2 - 4x + 2$
g $-3x^2 - 7x - 3$ **h** $-2x^2 + 3x + 4$ **i** $-3x^2 - 7x - 4$

REASONING	8	8	8, 9

8 A student factorises $x^2 - 2x - 24$ by completing the square.

a Show the student's working to obtain the factorised form of $x^2 - 2x - 24$.

b Now that you have seen the answer from part **a**, what would you suggest is a better way to factorise $x^2 - 2x - 24$?

9 **a** Explain why $x^2 + 9$ cannot be factorised using real numbers.

b Decide whether the following can or cannot be factorised.

i $x^2 - 25$ **ii** $x^2 - 10$
iii $x^2 + 6$ **iv** $x^2 + 11$
v $(x + 1)^2 + 4$ **vi** $(x - 2)^2 - 8$
vii $(x + 3)^2 - 15$ **viii** $(2x - 1)^2 + 1$

c For what values of m can the following be factorised, using real numbers?

i $x^2 + 4x + m$ **ii** $x^2 - 6x + m$ **iii** $x^2 - 10x + m$

ENRICHMENT: Proof by completing the square	–	–	10

10 Completing the square can be used in a range of proofs.

a Explain why the following statements are true for all values of x in the real number system.

i $x^2 \geqslant 0$ **ii** $(-2x)^2 \geqslant 0$ **iii** $(x - 1)^2 \geqslant 0$
iv $-2(x - 3)^2 \leqslant 0$ **v** $(x - 1)^2 + 1 \geqslant 1$ **vi** $-(2 - x)^2 - 3 \leqslant -3$

b By firstly completing the square, show that the following are true for all x.

i $x^2 + 2x + 1 \geqslant 0$ **ii** $4x^2 - 4x + 1 \geqslant 0$ **iii** $x^2 - 6x + 6 \geqslant -3$
iv $x^2 + 5x + 1 \geqslant -\frac{21}{4}$ **v** $-x^2 - 4x + 3 \leqslant 7$ **vi** $2x^2 - 8x + 3 \geqslant -5$

5G Solving quadratic equations using factorisation

LEARNING INTENTIONS

- To be able to recognise a quadratic equation
- To understand that for the product of two or more numbers to be zero, then one or both of the numbers must be zero
- To know how to rearrange a quadratic equation equal to zero
- To be able to apply the steps required for solving a quadratic equation using the Null Factor Law
- To understand that a quadratic equation can have 0, 1 or 2 solutions

The result of multiplying a number by zero is zero. Consequently, if an expression equals zero then at least one of its factors must be zero. This is called the Null Factor Law and it provides us with an important method that can be utilised to solve a range of mathematical problems involving quadratic equations.

By factorising $2x^2 - 7x - 15$, for example, we can rewrite the equation $2x^2 - 7x - 15 = 0$ as $(2x + 3)(x - 5) = 0$. We can then use the Null Factor Law to solve the equation.

Galileo (17th century) discovered that the path of a thrown or launched object under the influence of gravity follows a precise mathematical rule, the quadratic equation. The flight time, maximum height and range of projectiles could now be calculated.

Lesson starter: Does factorisation beat trial and error?

Set up two teams.

Team A: Trial and error
Team B: Factorisation

Instructions:

- Team A must try to find the two solutions of $x^2 - 2x - 48 = 0$ by guessing and checking values for x that make the equation true.
- Team B must solve the same equation $x^2 - 2x - 48 = 0$ by first factorising the left-hand side.

Which team was the first to find the two solutions for x? Discuss the methods used.

KEY IDEAS

- **Quadratic equations** can be written in the form $ax^2 + bx + c = 0$.
  For example: $2x^2 - 7x - 15 = 0$ is a quadratic equation.
- The **Null Factor Law** states that if the product of two numbers is zero, then either or both of the two numbers is zero.
  - If $p \times q = 0$, then either $p = 0$ or $q = 0$.
  - For example, if $x(x - 3) = 0$, then either $x = 0$ or $x - 3 = 0$ (i.e. $x = 0$ or $x = 3$).

- To solve a quadratic equation, write it in standard form (i.e. $ax^2 + bx + c = 0$) and factorise. Then use the Null Factor Law.
  - If the coefficients of all the terms have a common factor, then first divide by that common factor.
  - A quadratic equation can have 0, 1 or 2 real solutions.

BUILDING UNDERSTANDING

1 State the solutions to these equations, which are already in factorised form.

a $x(x + 1) = 0$ **b** $2x(x - 4) = 0$ **c** $(x - 3)(x + 2) = 0$

d $(x + \sqrt{3})(x - \sqrt{3}) = 0$ **e** $(2x - 1)(3x + 7) = 0$ **f** $(8x + 3)(4x + 3) = 0$

2 Rearrange and state in standard form: $ax^2 + bx + c = 0$ with $a > 0$. Do not solve.

a $x^2 + 2x = 3$ **b** $x^2 - 5x = -6$ **c** $4x^2 = 3 - 4x$

d $2x(x - 3) = 5$ **e** $x^2 = 4(x - 3)$ **f** $-4 = x(3x + 2)$

3 How many different solutions for x will these equations have?

a $(x - 2)(x - 1) = 0$ **b** $(x + 1)(x + 1) = 0$ **c** $(x + \sqrt{2})(x - \sqrt{2}) = 0$

d $(x + 8)(x - \sqrt{5}) = 0$ **e** $(x + 2)^2 = 0$ **f** $3(2x + 1)^2 = 0$

Example 17 Solving quadratic equations using the Null Factor Law

Solve the following quadratic equations.

a $x^2 - 2x = 0$ **b** $x^2 - 15 = 0$ **c** $2x^2 = 50$

SOLUTION	EXPLANATION
a $x^2 - 2x = 0$ $x(x - 2) = 0$ $\therefore x = 0$ or $x - 2 = 0$ $\therefore x = 0$ or $x = 2$	Factorise by taking out the common factor x. Apply the Null Factor Law: if $p \times q = 0$, then $p = 0$ or $q = 0$. Solve for x. Check your solutions by substituting back into the equation.
b $x^2 - 15 = 0$ $(x + \sqrt{15})(x - \sqrt{15}) = 0$ $\therefore x + \sqrt{15} = 0$ or $x - \sqrt{15} = 0$ $\therefore x = -\sqrt{15}$ or $x = \sqrt{15}$	Factorise $a^2 - b^2 = (a - b)(a + b)$ using surds. Alternatively, add 15 to both sides to give $x^2 = 15$, then take the positive and negative square root. So $x = \pm\sqrt{15}$.
c $2x^2 = 50$ $2x^2 - 50 = 0$ $2(x^2 - 25) = 0$ $2(x + 5)(x - 5) = 0$ $\therefore x + 5 = 0$ or $x - 5 = 0$ $\therefore x = -5$ or $x = 5$	First, write in standard form (i.e. $ax^2 + bx + c = 0$). Take out the common factor of 2 and then factorise using $a^2 - b^2 = (a + b)(a - b)$. Apply the Null Factor Law. Alternatively, divide first by 2 to give $x^2 = 25$ and $x = \pm 5$.

Now you try

Solve the following quadratic equations.

a $x^2 - 3x = 0$ **b** $x^2 - 11 = 0$ **c** $3x^2 = 27$

Using calculators to solve quadratic equations

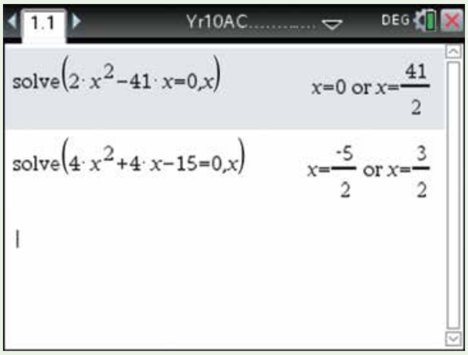
Solve:

a $2x^2 - 41x = 0$

b $4x^2 + 4x - 15 = 0$

Using the TI-Nspire:

In a **Calculator** page use menu **>Algebra>Solve** and type as shown ending with: ,*x*.



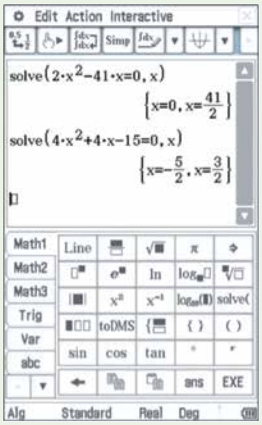
$\text{solve}(2\cdot x^2-41\cdot x=0,x)$ $\quad x=0 \text{ or } x=\frac{41}{2}$

$\text{solve}(4\cdot x^2+4\cdot x-15=0,x)$ $\quad x=\frac{-5}{2} \text{ or } x=\frac{3}{2}$

Note: if your answers are decimals then you can change the **Calculation Mode** to **Auto** in **Settings** on the Home screen.

Using the ClassPad:

In the **Main** application, type and highlight the equation then tap **Interactive**, **Advanced**, **Solve**, **OK**



$\text{solve}(2\cdot x^2-41\cdot x=0, x)$

$\{x=0, x=\frac{41}{2}\}$

$\text{solve}(4\cdot x^2+4\cdot x-15=0, x)$

$\{x=-\frac{5}{2}, x=\frac{3}{2}\}$

Example 18 Solving $ax^2 + bx + c = 0$

Solve the following quadratic equations.

a $x^2 - 5x + 6 = 0$

b $x^2 + 2x + 1 = 0$

Opt **c** $10x^2 - 13x - 3 = 0$

SOLUTION

a

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ (x-3)(x-2) &= 0 \\ \therefore x - 3 = 0 \text{ or } x - 2 &= 0 \\ \therefore x = 3 \text{ or } x = 2 & \end{aligned}$$

EXPLANATION

Factorise by finding two numbers that multiply to 6 and add to -5: $-3 \times (-2) = 6$ and $-3 + (-2) = -5$. Apply the Null Factor Law and solve for x.

Check your solutions by substitution.

b

$$\begin{aligned} x^2 + 2x + 1 &= 0 \\ (x+1)(x+1) &= 0 \\ (x+1)^2 &= 0 \\ \therefore x + 1 &= 0 \\ \therefore x &= -1 \end{aligned}$$

$1 \times 1 = 1$ and $1 + 1 = 2$

$(x+1)(x+1) = (x+1)^2$ is a perfect square.

This gives one solution for x.

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c

$$\begin{aligned} 10x^2 - 13x - 3 &= 0 \\ 10x^2 - 15x + 2x - 3 &= 0 \\ 5x(2x-3) + (2x-3) &= 0 \\ (2x-3)(5x+1) &= 0 \end{aligned}$$

First, factorise using grouping or another method.
$10 \times (-3) = -30$, $-15 \times 2 = -30$ and $-15 + 2 = -13$.

$$\begin{aligned} \therefore 2x - 3 &= 0 \text{ or } 5x + 1 = 0 \\ \therefore 2x &= 3 \text{ or } 5x = -1 \\ \therefore x &= \frac{3}{2} \text{ or } x = -\frac{1}{5} \end{aligned}$$

Solve using the Null Factor Law.

Now you try

Solve the following quadratic equations.

a $x^2 - x - 12 = 0$ **b** $x^2 + 6x + 9 = 0$ **c** $6x^2 + x - 2 = 0$

Example 19 Solving disguised quadratics

Solve the following by first writing in the form $ax^2 + bx + c = 0$ with $a > 0$.

a $x^2 = 4(x + 15)$ **b** $\frac{x+6}{x} = x$

SOLUTION

EXPLANATION

a

$$\begin{aligned} x^2 &= 4(x+15) \\ x^2 &= 4x + 60 \end{aligned}$$

First expand and then write in standard form by subtracting $4x$ and 60 from both sides.

$$\begin{aligned} x^2 - 4x - 60 &= 0 \\ (x-10)(x+6) &= 0 \end{aligned}$$

Factorise and apply the Null Factor Law:
$-10 \times 6 = -60$ and $-10 + 6 = -4$.

$$\begin{aligned} &\therefore x - 10 = 0 \text{ or } x + 6 = 0 \\ &\therefore x = 10 \text{ or } x = -6 \end{aligned}$$

b

$$\begin{aligned} \frac{x+6}{x} &= x \\ x + 6 &= x^2 \\ 0 &= x^2 - x - 6 \end{aligned}$$

First multiply both sides by x and then write in standard form.

$$0 = (x-3)(x+2)$$

Factorise and solve using the Null Factor Law.

$$\begin{aligned} &\therefore x - 3 = 0 \text{ or } x + 2 = 0 \\ &\therefore x = 3 \text{ or } x = -2 \end{aligned}$$

Check your solutions.

Now you try

Solve the following by first writing in the form $ax^2 + bx + c = 0$ with $a > 0$.

a $x^2 = 2(x + 24)$ **b** $\frac{x+20}{x} = x$

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Exercise 5G

FLUENCY 1–3(½) 1–4(½) 1–4(⅓)

Example 17a, b

1 Solve the following quadratic equations.

a $x^2 - 4x = 0$
b $x^2 - 3x = 0$
c $x^2 + 2x = 0$
d $3x^2 - 12x = 0$
e $2x^2 - 10x = 0$
f $4x^2 + 8x = 0$
g $x^2 - 7 = 0$
h $x^2 - 11 = 0$
i $3x^2 - 15 = 0$

Example 17c

2 Solve the following quadratic equations.

a $x^2 = 2x$
b $x^2 = -5x$
c $7x^2 = -x$
d $5x^2 = 20$
e $3x^2 = 27$
f $2x^2 = 72$

Example 18a, b

3 Solve the following quadratic equations.

a $x^2 + 3x + 2 = 0$
b $x^2 + 5x + 6 = 0$
c $x^2 - 6x + 8 = 0$
d $x^2 - 7x + 10 = 0$
e $x^2 + 4x - 12 = 0$
f $x^2 + 2x - 15 = 0$
g $x^2 - x - 20 = 0$
h $x^2 - 5x - 24 = 0$
i $x^2 - 12x + 32 = 0$
j $x^2 + 4x + 4 = 0$
k $x^2 + 10x + 25 = 0$
l $x^2 - 8x + 16 = 0$
m $x^2 - 14x + 49 = 0$
n $x^2 - 24x + 144 = 0$
o $x^2 + 18x + 81 = 0$

Example 18c

Opt

4 Solve the following quadratic equations.

a $2x^2 + 11x + 12 = 0$
b $4x^2 + 16x + 7 = 0$
c $2x^2 - 17x + 35 = 0$
d $2x^2 - 23x + 11 = 0$
e $3x^2 - 4x - 15 = 0$
f $5x^2 - 7x - 6 = 0$
g $6x^2 + 7x - 20 = 0$
h $7x^2 + 25x - 12 = 0$
i $20x^2 - 33x + 10 = 0$

PROBLEM–SOLVING 5–6(½) 5–7(½) 6–7(⅓), 8(½)

5 Solve by first taking out a common factor.

a $2x^2 + 16x + 24 = 0$
b $2x^2 - 20x - 22 = 0$
c $3x^2 - 18x + 27 = 0$
d $5x^2 - 20x + 20 = 0$
Opt **e** $-8x^2 - 4x + 24 = 0$
Opt **f** $18x^2 - 57x + 30 = 0$

Example 19a

6 Solve the following by first writing in the form $ax^2 + bx + c = 0$ with $a > 0$.

a $x^2 = 2(x + 12)$
b $x^2 = 4(x + 8)$
c $x^2 = 3(2x - 3)$
d $x^2 + 7x = -10$
e $x^2 - 8x = -15$
f $x(x + 4) = 4x + 9$
g $2x - 16 = x(2 - x)$
h $x^2 + 12x + 10 = 2x + 1$
i $x^2 + x - 9 = 5x - 4$
j $x^2 - 5x = -15x - 25$
k $x^2 - 14x = 2x - 64$
l $x(x + 4) = 4(x + 16)$
m $2x(x - 2) = 6$
Opt **n** $3x(x + 6) = 4(x - 2)$
Opt **o** $4x(x + 5) = 6x - 4x^2 - 3$

Example 19b

7 Solve the following by first writing in the form $ax^2 + bx + c = 0$ with $a > 0$.

a $\dfrac{5x + 84}{x} = x$
b $\dfrac{9x + 70}{x} = x$
c $\dfrac{18 - 7x}{x} = x$
Opt **d** $\dfrac{20 - 3x}{x} = 2x$
Opt **e** $\dfrac{6x + 8}{5x} = x$
Opt **f** $\dfrac{7x + 10}{2x} = 3x$
g $\dfrac{3}{x} = x + 2$
Opt **h** $\dfrac{1}{x} = 3 - 2x$
i $\dfrac{4}{x - 2} = x + 1$

8 Solving the equation $0.1x^2 + 0.2x - 1.5 = 0$ is equivalent to solving the equation $x^2 + 2x - 15 = 0$, which can be achieved by multiplying both sides of the equation by 10 to remove the decimals. Solve these quadratic equations by first removing the decimals or fractions.

a $0.1x^2 - 0.3x - 2.8 = 0$ **b** $0.01x^2 + 0.12x + 0.2 = 0$ **c** $0.25x^2 + x + 1 = 0$

d $\frac{1}{3}x^2 - 2x + 3 = 0$ Opt **e** $0.4x^2 + x - 2.4 = 0$ Opt **f** $\frac{1}{2}x^2 + \frac{7}{3}x = \frac{5}{6}$

REASONING	9	9, 10	10, 11

9 **a** Write down the solutions to the following equations.
i $2(x - 1)(x + 2) = 0$ **ii** $(x - 1)(x + 2) = 0$

b What difference has the common factor of 2 made to the solutions in the first equation?

c Explain why $x^2 - 5x - 6 = 0$ and $3x^2 - 15x - 18 = 0$ have the same solutions.

10 Explain why $x^2 + 16x + 64 = 0$ has only one solution.

11 When solving $x^2 - 2x - 8 = 7$ a student writes the following.

$x^2 - 2x - 8 = 7$
$(x - 4)(x + 2) = 7$
$x - 4 = 7$ or $x + 2 = 7$
$x = 11$ or $x = 5$

Discuss the problem with this solution and then write a correct solution.

ENRICHMENT: More quadratics in disguise	–	–	12(½)

12 Solve these equations by first multiplying by an appropriate expression.

a $x + 3 = -\frac{2}{x}$ **b** $-\frac{1}{x} = x - 2$ **c** $-\frac{5}{x} = 2x - 11$

d $\frac{x^2 - 48}{x} = 2$ **e** $\frac{x^2 + 12}{x} = -8$ **f** $\frac{2x^2 - 12}{x} = -5$

g $\frac{x - 5}{4} = \frac{6}{x}$ **h** $\frac{x - 2}{3} = \frac{5}{x}$ **i** $\frac{x - 4}{2} = -\frac{2}{x}$

j $\frac{x + 4}{2} - \frac{3}{x - 3} = 1$ **k** $\frac{x}{x - 2} - \frac{x + 1}{x + 4} = 1$ **l** $\frac{1}{x - 1} - \frac{1}{x + 3} = \frac{1}{3}$

5H Applications of quadratic equations

LEARNING INTENTIONS

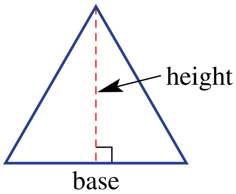
- To be able to set up a quadratic equation from a word problem
- To know how to apply the steps for solving a quadratic equation
- To understand and check the validity of solutions in the context of the given problem

Defining variables, setting up equations, solving equations and interpreting solutions are all important elements of applying quadratic equations to problem solving. The area of a rectangular paddock, for example, that is fenced off using a fixed length of fencing can be found by setting up a quadratic equation, solving it and then interpreting the solutions.

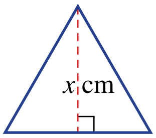
Aerospace engineers model the trajectory of a rocket under the influence of gravity using a quadratic equation of height, h, versus time. The solutions to $h = 0$ are the times when the rocket is at ground level and give its flight time.

Lesson starter: The 10 cm^2 triangle

There are many base and height measurements for a triangle that give an area of 10 cm^2.



- Draw three different triangles that have a 10 cm^2 area. Include the measurements for the base and the height.
- Do any of your triangles have a base length that is 1 cm more than the height? Find the special triangle with area 10 cm^2 that has a base 1 cm more than its height by following these steps.
  - Let x cm be the height of the triangle.
  - Write an expression for the base length.
  - Write an equation if the area is 10 cm^2.
  - Solve the equation to find two solutions for x.
  - Which solution is to be used to describe the special triangle? Why?



KEY IDEAS

- When applying quadratic equations, follow these steps.
  - Define a variable; i.e. 'Let x be …'.
  - Write an equation.
  - Solve the equation.
  - Choose the solution(s) that solves the equation and answers the question in the context in which it was given. Check that the solutions seem reasonable.

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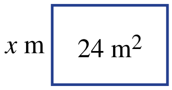
BUILDING UNDERSTANDING

① A rectangle has an area of 24 m^2. Its length is 5 m longer than its width.



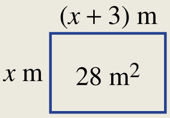
- **a** Complete this sentence: ‘Let x m be the ______________.
- **b** State an expression for the rectangle’s length.
- **c** State an equation using the rectangle’s area.
- **d** Rearrange your equation from part **c** in standard form (i.e. $ax^2 + bx + c = 0$ with $a > 0$) and solve for x.
- **e** Find the dimensions of the rectangle.

② Repeat all the steps in Question ① to find the dimensions of a rectangle with the following properties.

- **a** Its area is 60 m^2 and its length is 4 m more than its width.
- **b** Its area is 63 m^2 and its length is 2 m less than its width.

Example 20 Finding dimensions using a quadratic equation

The area of a rectangle is fixed at 28 m^2 and its length is 3 metres more than its width. Find the dimensions of the rectangle.

SOLUTION	EXPLANATION
Let x m be the width of the rectangle.	Draw a diagram to help.
$\text{Length} = (x + 3)\,\text{m}$ $x(x + 3) = 28$ $x^2 + 3x - 28 = 0$ $(x + 7)(x - 4) = 0$ $x + 7 = 0$ or $x - 4 = 0$ $\therefore x = -7$ or $x = 4$	 Write an equation using the given information. Then write in standard form and solve for x.
$x > 0$ so, choose $x = 4$.	Disregard $x = -7$ because $x > 0$.
Rectangle has width 4 m and length 7 m.	Answer the question in full. Note: Length is $4 + 3 = 7$.

Now you try

The area of a rectangle is fixed at 48 m^2 and its length is 2 metres more than its width. Find the dimensions of the rectangle.

Exercise 5H

FLUENCY	1–5	1, 3–5	3–6

Example 20

1 The area of a rectangle is fixed at 12 m^2 and its length is 1 metre more than its width. Find the dimensions of the rectangle using a quadratic equation.

2 The area of a rectangle is fixed at 54 m^2 and its length is 3 metres more than its width. Find the dimensions of the rectangle using a quadratic equation.

3 Find the height and base lengths of a triangle that has an area of 24 cm^2 and height 2 cm more than its base using a quadratic equation.

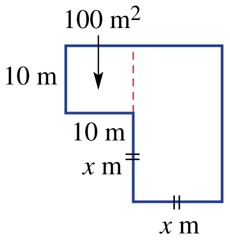
4 Find the height and base lengths of a triangle that has an area of 7 m^2 and height 5 m less than its base using a quadratic equation.

5 The product of two consecutive numbers is 72. Use a quadratic equation to find the two sets of numbers.

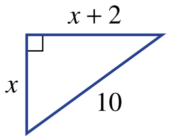
6 The product of two consecutive, even positive numbers is 168. Find the two numbers.

PROBLEM–SOLVING	7, 8	7–10	8–11

7 A 100 m^2 hay shed is to be extended to give 475 m^2 of floor space in total, as shown. All angles are right angles. Find the value of x.



8 Solve for x in these right-angled triangles, using Pythagoras' theorem.

a


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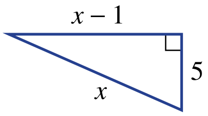
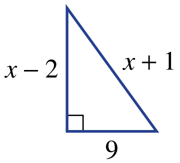


c


9 A square hut of side length 5 m is to be surrounded by a veranda of width x metres. Find the width of the veranda if its area is to be 24 m^2.

10 A father's age is the square of his son's age (x). In 20 years' time the father will be three times as old as his son. What are the ages of the father and son?

11 A rectangular painting is to have a total area (including the frame) of 1200 cm^2. If the painting inside the frame is 30 cm long and 20 cm wide, find the width of the frame.

REASONING	12	12, 13	13, 14

12 The sum of the first n positive integers is given by $\frac{1}{2}n(n+1)$.

a Find the sum of the first 10 positive integers (i.e. use $n = 10$).

b Find the value of n if the sum of the first n positive integers is:

i 28 **ii** 91 **iii** 276

13 A ball is thrust vertically upwards from a machine on the ground. The height (h metres) after t seconds is given by $h = t(4 - t)$.

a Find the height after 1.5 seconds.

b Find when the ball is at a height of 3 metres.

c Why are there two solutions to part **b**?

d Find when the ball is at ground level. Explain.

e Find when the ball is at a height of 4 metres.

f Why is there only one solution for part **e**?

g Is there a time when the ball is at a height of 5 metres? Explain.

14 The height (h metres) of a golf ball is given by $h = -0.01x(x - 100)$, where x metres is the horizontal distance from where the ball was hit.

a Find the values of x when $h = 0$.

b Interpret your answer from part **a**.

c Find how far the ball has travelled horizontally when the height is 1.96 metres.

ENRICHMENT: Fixed perimeter and area	–	–	15, 16

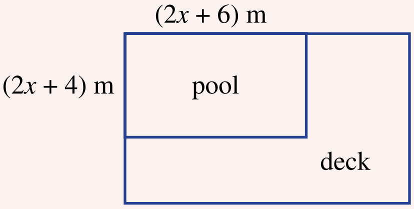
15 A small rectangular block of land has a perimeter of 100 m and an area of 225 m^2. Find the dimensions of the block of land.

16 A rectangular farm has perimeter 700 m and area 30 000 m^2. Find its dimensions.

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

On the pool deck

1 Designs for a rectangular pool are being considered with the dimensions shown such that the length is 2 m more than the width, as shown. The pool will also have a deck built around it as shown. The length and width of the combined rectangular area will be an increase of 50% of the length and width of the pool.



The pool designer wants to explore the areas of possible decks in comparison to the area of the pool.

a Give the length and width of the combined pool and deck area in terms of x.

b Find the area of the deck in m^2 in terms of x.

c If the area of the deck is $100\,m^2$, determine the dimensions of the pool by first finding the value of x.

d Use your answer to part **b** to determine what fraction the pool area is of the deck area.

e Repeat parts **a** and **b** to determine what fraction the pool area is of the deck area, if the deck increases the length and width of the rectangular area by 25%.

Round-robin tournament

2 A round-robin tournament with n teams, where every team plays each other once, requires $\frac{n^2 - n}{2}$ games.

Using this rule, the tournament organisers wish to explore the number of games that need to be scheduled and the number of teams required for a given number of games.

a How many games are played in a round-robin tournament with 6 teams?

b A round-robin tournament has 28 games, solve an appropriate equation to find the number of teams in the competition.

c Investigate if doubling the number of teams, doubles the number of matches required. Prove algebraically.

d Give a simplified expression in terms of n for the number of games required for $n + 1$ teams.

e A tournament has n teams. How many more games are required in the tournament if the number of teams increases by:

i 1 team? **ii** 2 teams? **iii** x teams?

Kayaking along the river

3 A kayaker is paddling up a river which is flowing at a certain speed. He travels 15 km up the river and then back down the river to where he started, kayaking at the same still-water speed, x km/h. The trip takes 4 hours to go up and down the river.

You wish to investigate the effect of the varying river flow speed on the speed of the kayaker who needs to complete the trip of fixed distance in the given time frame.

a If the river is flowing at a rate of 2 km/h and the man is kayaking at a rate of x km/h, find:

i expressions, in terms of x, for the rate the kayaker is moving upstream and the rate the kayaker is moving downstream

ii the value of x for this 4 hour journey.

Consider the same journey, taking the same time, with the river flowing at y km/h.

b Find a rule for the speed of the kayaker in still-water, x km/h, in terms of y.

c Use your rule from part **b** to confirm your answer to part **a** and to find the kayaker's speed if there was no current.

5I Solving quadratic equations by completing the square

LEARNING INTENTIONS

- To understand that completing the square can be used to help factorise a quadratic equation when integers cannot be found
- To be able to solve an equation by using the completing the square method to factorise first
- To recognise a form of a quadratic equation that gives no solutions

In **Section 5F** we saw that some quadratics cannot be factorised using integers but instead could be factorised by completing the square. Surds were also used to complete the factorisation. We can use this method to solve many quadratic equations.

In the 9th century, the great Persian mathematician Al-Khwarizmi first solved quadratic equations by completing the square. His *Al-jabr* book was the principal maths textbook in European universities for 500 years, introducing algebra, algorithms and surds.

Lesson starter: Where does $\sqrt{6}$ come in?

Consider the equation $x^2 - 2x - 5 = 0$ and try to solve it by discussing these points.

- Are there any common factors that can be taken out?
- Are there any integers that multiply to give -5 and add to give -2?
- Try completing the square on the left-hand side. Does this help and how?
- Show that the two solutions contain the surd $\sqrt{6}$.

KEY IDEAS

- To solve quadratic equations of the form $ax^2 + bx + c = 0$ for which you cannot factorise using integers:
  - Complete the square for the quadratic expression and factorise if possible.
  - Solve the quadratic equation using the Null Factor Law or an alternate method.
- Expressions such as $x^2 + 5$ and $(x - 1)^2 + 7$ cannot be factorised further and therefore give no solutions when equal to 0 as they cannot be expressed as a difference of two squares.

BUILDING UNDERSTANDING

1 What number must be added to the following expressions to form a perfect square?

a $x^2 + 2x$ **b** $x^2 + 20x$ **c** $x^2 - 4x$ **d** $x^2 + 5x$

2 Factorise using surds.

a $x^2 - 3 = 0$ **b** $x^2 - 10 = 0$ **c** $(x + 1)^2 - 5 = 0$

3 Solve these equations.

a $(x - \sqrt{2})(x + \sqrt{2}) = 0$ **b** $(x - \sqrt{7})(x + \sqrt{7}) = 0$

c $(x - 3 + \sqrt{5})(x - 3 - \sqrt{5}) = 0$ **d** $(x + 5 + \sqrt{14})(x + 5 - \sqrt{14}) = 0$

Example 21 Solving quadratic equations by completing the square

Solve these quadratic equations by first completing the square.

a $x^2 - 4x + 2 = 0$ Opt **b** $x^2 + 6x - 11 = 0$ Opt **c** $x^2 - 3x + 1 = 0$

SOLUTION

a

$$\begin{aligned} x^2 - 4x + 2 &= 0 \\ x^2 - 4x + 4 - 4 + 2 &= 0 \\ (x - 2)^2 - 2 &= 0 \\ \left(x - 2 + \sqrt{2}\right)\left(x - 2 - \sqrt{2}\right) &= 0 \end{aligned}$$

$$\therefore x - 2 + \sqrt{2} = 0 \text{ or } x - 2 - \sqrt{2} = 0$$

$$\therefore x = 2 - \sqrt{2} \text{ or } x = 2 + \sqrt{2}$$

Alternate method, from

$$\begin{aligned} (x - 2)^2 - 2 &= 0 \\ (x - 2)^2 &= 2 \\ x - 2 &= \pm\sqrt{2} \\ x &= 2 \pm \sqrt{2} \end{aligned}$$

EXPLANATION

Complete the square: $\left(\frac{-4}{2}\right)^2 = 4$.

$x^2 - 4x + 4 = (x - 2)(x - 2) = (x - 2)^2$

Use $a^2 - b^2 = (a + b)(a - b)$.

Apply the Null Factor Law and solve for x.

The solutions can also be written as $2 \pm \sqrt{2}$.

An alternate approach after completing the square is to add 2 to both sides and then take the square root of both sides $\pm\sqrt{2}$ since $(+\sqrt{2})^2 = 2$ and $(-\sqrt{2})^2 = 2$.

SOLUTION

b

$$\begin{aligned} x^2 + 6x - 11 &= 0 \\ x^2 + 6x + 9 - 9 - 11 &= 0 \\ (x + 3)^2 - 20 &= 0 \\ (x + 3 - \sqrt{20})(x + 3 + \sqrt{20}) &= 0 \\ (x + 3 - 2\sqrt{5})(x + 3 + 2\sqrt{5}) &= 0 \end{aligned}$$

$$\therefore x + 3 - 2\sqrt{5} = 0 \quad \text{or} \quad x + 3 + 2\sqrt{5} = 0$$

$$\therefore x = -3 + 2\sqrt{5} \quad \text{or} \quad x = -3 - 2\sqrt{5}$$

Alternatively, $x = -3 \pm 2\sqrt{5}$.

EXPLANATION

Complete the square: $\left(\frac{6}{2}\right)^2 = 9$.

Use difference of two squares with surds. Recall that $\sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$.

Apply the Null Factor Law and solve for x.

$(x + 3)^2 = 20$ can also be solved by taking the square root of both sides.

Alternatively, write solutions using the ± symbol.

Continued on next page

c

$$
\begin{aligned}
x^2 - 3x + 1 &= 0 \\
x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 1 &= 0 \\
\left(x - \frac{3}{2}\right)^2 - \frac{5}{4} &= 0 \\
\left(x - \frac{3}{2} + \sqrt{\frac{5}{4}}\right)\left(x - \frac{3}{2} - \sqrt{\frac{5}{4}}\right) &= 0
\end{aligned}
$$

$\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$

$a^2 - b^2 = (a + b)(a - b)$

$x - \frac{3}{2} + \sqrt{\frac{5}{4}} = 0$ or $x - \frac{3}{2} - \sqrt{\frac{5}{4}} = 0$

Use the Null Factor Law.

$\therefore x = \frac{3}{2} - \frac{\sqrt{5}}{2}$ or $x = \frac{3}{2} + \frac{\sqrt{5}}{2}$

$x = \frac{3 - \sqrt{5}}{2}$ or $x = \frac{3 + \sqrt{5}}{2}$

Recall that $\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2}$

So $x = \frac{3 \pm \sqrt{5}}{2}$

Combine using the ± symbol.

Now you try

Solve these quadratic equations by first completing the square.

a $x^2 - 6x + 2 = 0$ **b** $x^2 + 4x - 14 = 0$ **c** $x^2 - 5x + 2 = 0$

Exercise 5I

FLUENCY	1–3(½)	1–3(⅓)	1–3(⅓)

Example 21a

1 Solve by first completing the square.

a $x^2 + 6x + 3 = 0$ **b** $x^2 + 4x + 2 = 0$ **c** $x^2 + 10x + 15 = 0$

d $x^2 + 4x - 2 = 0$ **e** $x^2 + 8x - 3 = 0$ **f** $x^2 + 6x - 5 = 0$

g $x^2 - 8x - 1 = 0$ **h** $x^2 - 12x - 3 = 0$ **i** $x^2 - 2x - 16 = 0$

j $x^2 - 10x + 18 = 0$ **k** $x^2 - 6x + 4 = 0$ **l** $x^2 - 8x + 9 = 0$

m $x^2 + 6x - 4 = 0$ **n** $x^2 + 20x + 13 = 0$ **o** $x^2 - 14x - 6 = 0$

Example 21b

Opt

2 Solve by first completing the square.

a $x^2 + 8x + 4 = 0$ **b** $x^2 + 6x + 1 = 0$ **c** $x^2 - 10x + 5 = 0$

d $x^2 - 4x - 14 = 0$ **e** $x^2 - 10x - 3 = 0$ **f** $x^2 + 8x - 8 = 0$

g $x^2 - 2x - 31 = 0$ **h** $x^2 + 12x - 18 = 0$ **i** $x^2 + 6x - 41 = 0$

Example 21c

Opt

3 Solve by first completing the square.

a $x^2 + 5x + 2 = 0$ **b** $x^2 + 3x + 1 = 0$ **c** $x^2 + 7x + 5 = 0$

d $x^2 - 3x - 2 = 0$ **e** $x^2 - x - 3 = 0$ **f** $x^2 + 5x - 2 = 0$

g $x^2 - 7x + 2 = 0$ **h** $x^2 - 9x + 5 = 0$ **i** $x^2 + x - 4 = 0$

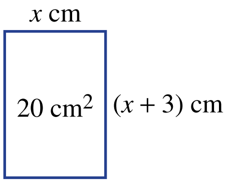
j $x^2 + 9x + 9 = 0$ **k** $x^2 - 3x - \frac{3}{4} = 0$ **l** $x^2 + 5x + \frac{5}{4} = 0$

PROBLEM–SOLVING | 4(½), 5 | 4(⅓), 5, 6–7(½) | 5, 6–7(⅓), 8

4 Decide how many solutions there are to these equations. Try factorising the equations if you are unsure.

a $x^2 - 2 = 0$ **b** $x^2 - 10 = 0$ **c** $x^2 + 3 = 0$
d $x^2 + 7 = 0$ **e** $(x - 1)^2 + 4 = 0$ **f** $(x + 2)^2 - 7 = 0$
g $(x - 7)^2 - 6 = 0$ **h** $x^2 - 2x + 6 = 0$ **i** $x^2 - 3x + 10 = 0$
j $x^2 + 2x - 4 = 0$ **k** $x^2 + 7x + 1 = 0$ **l** $x^2 - 2x + 17 = 0$

5 A rectangle's length is 3 cm more than its width. Find the dimensions of the rectangle if its area is 20 cm^2.



6 Solve the following, if possible, by first factoring out the coefficient of x^2 and then completing the square.

a $2x^2 - 4x + 4 = 0$ **b** $4x^2 + 20x + 8 = 0$ **c** $2x^2 - 10x + 4 = 0$
d $3x^2 + 27x + 9 = 0$ **e** $3x^2 + 15x + 3 = 0$ **f** $2x^2 - 12x + 8 = 0$

7 Solve the following quadratic equations, if possible.

a $x^2 + 3x = 5$ **b** $x^2 + 5x = 9$ **c** $x^2 + 7x = -15$
d $x^2 - 8x = -11$ **e** $x^2 + 12x + 10 = 2x + 5$ **f** $x^2 + x + 9 = 5x - 3$

8 The height, h km, of a ballistic missile launched from a submarine at sea level is given by $h = \frac{x(400 - x)}{20\,000}$, where x km is the horizontal distance travelled.

a Find the height of a missile that has travelled the following horizontal distances.
i 100 km **ii** 300 km

b Find how far the missile has travelled horizontally when the height is:
i 0 km **ii** 2 km

c Find the horizontal distance the missile has travelled when its height is 1 km.
(*Hint:* Complete the square.)

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REASONING	9	9, 10	10, 11

9 Complete the square to show that the following have no (real) solutions.

a $x^2 + 4x + 5 = 0$

b $x^2 - 2x = -3$

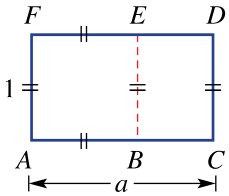
10 A friend starts to solve $x^2 + x - 30 = 0$ by completing the square but you notice there is a much quicker way. What method do you describe to your friend?

11 This rectangle is a golden rectangle.

- *ABEF* is a square.
- Rectangle *BCDE* is similar to rectangle *ACDF*.

a Show that $\frac{a}{1} = \frac{1}{a-1}$.

b Find the exact value of a (which will give you the golden ratio) by completing the square.



ENRICHMENT: Completing the rectangle	–	–	12

12 A rectangular gallery floor is to be partially carpeted around the edge so there is less noise in the gallery as patrons view the paintings on the walls. The hall is 30 metres wide and 40 metres long and the width of the carpeted edge is x metres as shown. Inside the carpeted area is a rectangular timber floor.

a Find an expression in terms of x for the following.

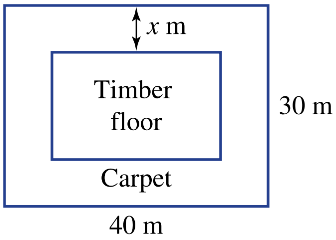
  i The width of the timber floor.

  ii The length of the timber floor.

  iii The area of the timber floor.

b Find the value of x if the timber floor area is to be 600 square metres.

c Find the value of x if the timber floor area is to be 700 square metres. Give an exact answer.



5J Solving quadratic equations using the quadratic formula OPTIONAL

LEARNING INTENTIONS

- To know the quadratic formula and when to apply it
- To be able to use the quadratic formula to solve a quadratic equation
- To know what the discriminant is and what it can be used to determine
- To be able to use the discriminant to determine the number of solutions of a quadratic equation

A general formula for solving quadratic equations can be found by completing the square for the general case.

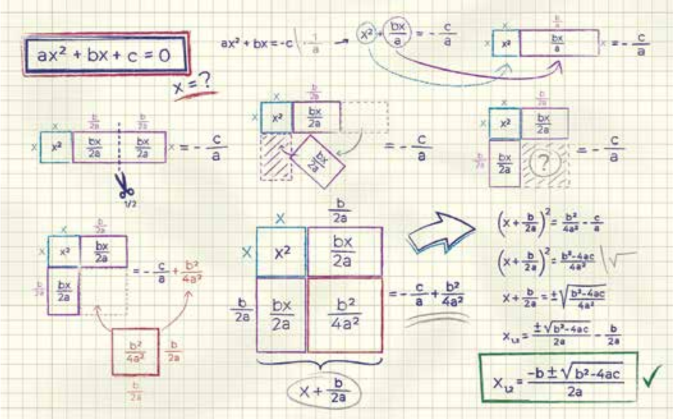
Consider $ax^2 + bx + c = 0$, where a, b, c are constants and $a \neq 0$. Start by dividing both sides by a.

$$\begin{aligned}
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0\\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} &= 0\\
\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} &= 0\\
\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right) &= 0\\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2}\\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}\\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}\\
&= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

Surveyors regularly subdivide land into house blocks. When dimensions are linear expressions of the same variable, an area formula forms a quadratic equation. For a given area, surveyors can solve this equation using the quadratic formula.

This formula now gives us a mechanism to solve quadratic equations and to determine how many solutions the equation has.

The expression under the root sign, $b^2 - 4ac$, is called the discriminant (Δ) and helps us to identify the number of solutions. A quadratic equation can have 0, 1 or 2 real solutions.



Lesson starter: How many solutions?

Complete this table to find the number of solutions for each equation.

$ax^2 + bx + c = 0$	a	b	c	$b^2 - 4ac$	$\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$	$\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$
$2x^2 + 7x + 1 = 0$						
$9x^2 - 6x + 1 = 0$						
$x^2 - 3x + 4 = 0$						

Discuss under what circumstances a quadratic equation has:

- 2 solutions
- 1 solution
- 0 solutions.

KEY IDEAS

- If $ax^2 + bx + c = 0$ (where a, b, c are constants and $a \neq 0$), then
  $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$
  - This is called the **quadratic formula**.
  - The quadratic formula is useful when a quadratic cannot be factorised easily.
- The **discriminant** is $\Delta = b^2 - 4ac$.
  - When $\Delta < 0$, the quadratic equation has 0 real solutions (since $\sqrt{\Delta}$ is undefined when Δ is negative).
  - When $\Delta = 0$, the quadratic equation has 1 real solution $\left(x = -\dfrac{b}{2a}\right)$.
  - When $\Delta > 0$, the quadratic equation has 2 real solutions $\left(x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$.

BUILDING UNDERSTANDING

1 For these quadratic equations in the form $ax^2 + bx + c = 0$, state the values of a, b and c.

a $3x^2 + 2x + 1 = 0$
b $5x^2 + 3x - 2 = 0$
c $2x^2 - x - 5 = 0$
d $-3x^2 + 4x - 5 = 0$

2 Find the value of the discriminant ($b^2 - 4ac$) for each part in Question 1 above.

3 State the number of solutions of a quadratic equation that has:

a $b^2 - 4ac = 0$
b $b^2 - 4ac < 0$
c $b^2 - 4ac > 0$

Example 22 Using the discriminant

Determine the number of solutions to the following quadratic equations using the discriminant.

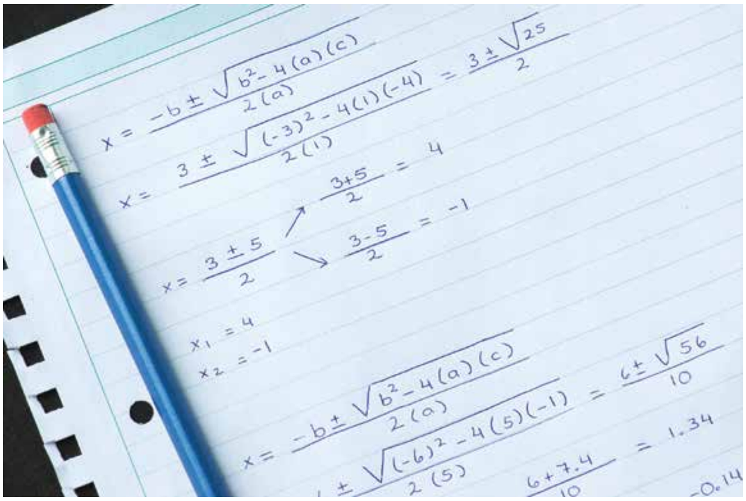
a $x^2 + 5x - 3 = 0$ **b** $2x^2 - 3x + 4 = 0$ **c** $x^2 + 6x + 9 = 0$

SOLUTION	EXPLANATION
a $a = 1, b = 5, c = -3$ $\Delta = b^2 - 4ac$ $= (5)^2 - 4(1)(-3)$ $= 25 + 12$ $= 37$	State the values of a, b and c in $ax^2 + bx + c = 0$. Calculate the value of the discriminant by substituting values.
$\Delta > 0$, so there are 2 solutions.	Interpret the result with regard to the number of solutions.
b $a = 2, b = -3, c = 4$ $\Delta = b^2 - 4ac$ $= (-3)^2 - 4(2)(4)$ $= 9 - 32$ $= -23$	State the values of a, b and c and substitute to evaluate the discriminant. Recall that $(-3)^2 = -3 \times (-3) = 9$.
$\Delta < 0$, so there are no solutions.	Interpret the result.
c $a = 1, b = 6, c = 9$ $\Delta = b^2 - 4ac$ $= (6)^2 - 4(1)(9)$ $= 36 - 36$ $= 0$	Substitute the values of a, b and c to evaluate the discriminant and interpret the result.
$\Delta = 0$, so there is 1 solution.	Note: $x^2 + 6x + 9 = (x + 3)^2$ is a perfect square.

Now you try

Determine the number of solutions to the following quadratic equations using the discriminant.

a $x^2 + 7x - 1 = 0$ **b** $3x^2 - x + 2 = 0$ **c** $x^2 + 8x + 16 = 0$



Example 23 Solving quadratic equations using the quadratic formula

Find the exact solutions to the following using the quadratic formula.

a $x^2 + 5x + 3 = 0$

b $2x^2 - 2x - 1 = 0$

SOLUTION	EXPLANATION
a $a = 1, b = 5, c = 3$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \dfrac{-5 \pm \sqrt{(5)^2 - 4(1)(3)}}{2(1)}$	Determine the values of a, b and c in $ax^2 + bx + c = 0$. Write out the quadratic formula and substitute the values.
$= \dfrac{-5 \pm \sqrt{25 - 12}}{2}$	Simplify.
$= \dfrac{-5 \pm \sqrt{13}}{2}$	Two solutions: $x = \dfrac{-5 - \sqrt{13}}{2}, \dfrac{-5 + \sqrt{13}}{2}$.
b $a = 2, b = -2, c = -1$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$ $= \dfrac{2 \pm \sqrt{4 + 8}}{4}$	Determine the values of a, b and c.
$= \dfrac{2 \pm \sqrt{12}}{4}$	Simplify: $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$.
$= \dfrac{2 \pm 2\sqrt{3}}{4}$ $= \dfrac{1 \pm \sqrt{3}}{2}$	Cancel using the common factor: $\dfrac{2 \pm 2\sqrt{3}}{4} = \dfrac{2(1 \pm \sqrt{3})}{4}$ $= \dfrac{1 \pm \sqrt{3}}{2}$

Now you try

Find the exact solutions to the following using the quadratic formula.

a $x^2 + 3x + 1 = 0$

b $4x^2 - 2x - 3 = 0$

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Exercise 5J

FLUENCY	1–2(½)	1–3(⅓)	1–3(⅓)

Example 22 **1** Using the discriminant, determine the number of solutions for these quadratic equations.

a $x^2 + 5x + 3 = 0$ **b** $x^2 + 3x + 4 = 0$ **c** $x^2 + 6x + 9 = 0$
d $x^2 + 7x - 3 = 0$ **e** $x^2 + 5x - 4 = 0$ **f** $x^2 + 4x - 4 = 0$
g $4x^2 + 5x + 3 = 0$ **h** $4x^2 + 3x + 1 = 0$ **i** $2x^2 + 12x + 9 = 0$
j $-x^2 - 6x - 9 = 0$ **k** $-2x^2 + 3x - 4 = 0$ **l** $-4x^2 - 6x + 3 = 0$

Example 23a **2** Find the exact solutions to the following quadratic equations, using the quadratic formula.

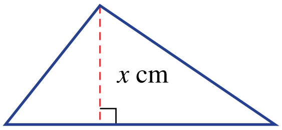
a $x^2 + 3x - 2 = 0$ **b** $x^2 + 7x - 4 = 0$ **c** $x^2 - 7x + 5 = 0$
d $x^2 - 8x + 16 = 0$ **e** $-x^2 - 5x - 4 = 0$ **f** $-x^2 - 8x - 7 = 0$
g $4x^2 + 7x - 1 = 0$ **h** $3x^2 + 5x - 1 = 0$ **i** $3x^2 - 4x - 6 = 0$
j $-2x^2 + 5x + 5 = 0$ **k** $-3x^2 - x + 4 = 0$ **l** $5x^2 + 6x - 2 = 0$

Example 23b **3** Find the exact solutions to the following quadratic equations, using the quadratic formula.

a $x^2 + 4x + 1 = 0$ **b** $x^2 - 6x + 4 = 0$ **c** $x^2 + 6x - 2 = 0$
d $-x^2 - 3x + 9 = 0$ **e** $-x^2 + 4x + 4 = 0$ **f** $-3x^2 + 8x - 2 = 0$
g $2x^2 - 2x - 3 = 0$ **h** $3x^2 - 6x - 1 = 0$ **i** $-5x^2 + 8x + 3 = 0$

PROBLEM–SOLVING	4, 5(½)	4, 5(½), 6	5(½), 7, 8

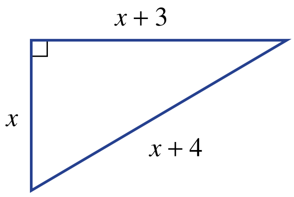
4 A triangle's base is 5 cm more than its height of x cm. Find its height if the triangle's area is 10 cm^2.



5 Solve the following using the quadratic formula.

a $3x^2 = 1 + 6x$ **b** $2x^2 = 3 - 4x$ **c** $5x = 2 - 4x^2$
d $2x - 5 = -\frac{1}{x}$ **e** $\frac{3}{x} = 3x + 4$ **f** $-\frac{5}{x} = 2 - x$
g $5x = \frac{2x + 2}{x}$ **h** $x = \frac{3x + 4}{2x}$ **i** $3x = \frac{10x - 1}{2x}$

6 Find the exact perimeter of this right-angled triangle.



7 Two positive numbers differ by 3 and their product is 11. Find the numbers.

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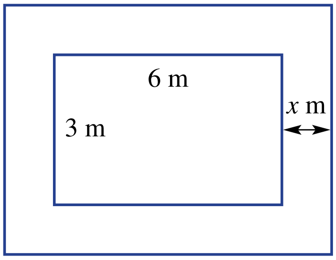
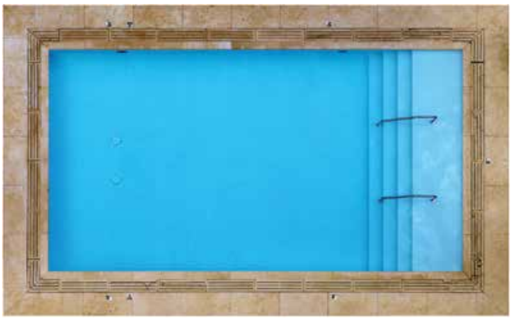
8 A rectangular pool measuring 6 m by 3 m is to have a path surrounding it. If the total area of the pool and path is to be 31 m^2, find the width (x m) of the path, correct to the nearest centimetre.





REASONING	9	9, 10	10, 11

9 Explain why the rule $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ gives only one solution when $b^2 - 4ac = 0$.

10 Make up three quadratic equations that have:

a no solutions **b** 1 solution **c** 2 solutions.

11 For what two values of k does $x^2 + kx + 9 = 0$ have only one solution?

ENRICHMENT: *k* determines the number of solutions	–	–	12(½)

12 The discriminant for $x^2 + 2x + k = 0$ is $4 - 4k$, so there:

- are no solutions for $4 - 4k < 0, \therefore k > 1$
- is 1 solution for $4 - 4k = 0, \therefore k = 1$
- are 2 solutions for $4 - 4k > 0, \therefore k < 1$

a For what values of k does $x^2 + 4x + k = 0$ have:
 i no solutions? **ii** 1 solution? **iii** 2 solutions?

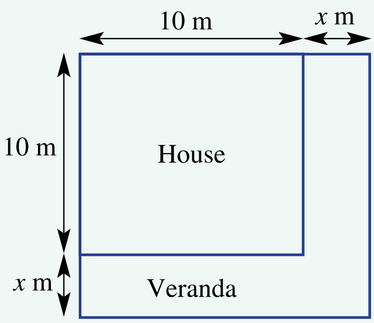
b For what values of k does $kx^2 + 3x + 2 = 0$ have:
 i no solutions? **ii** 1 solution? **iii** 2 solutions?

c For what values of k does $x^2 + kx + 1 = 0$ have:
 i no solutions? **ii** 1 solution? **iii** 2 solutions?

d For what values of k does $3x^2 + kx - 1 = 0$ have:
 i no solutions? **ii** 1 solution? **iii** 2 solutions?

Adding a veranda

Lucas is planning to add a veranda to two adjacent sides of his house. He needs to choose an appropriate veranda width so that the veranda is of a particular total area. Lucas's house is a square of side length 10 m as shown.



Present a report for the following tasks and ensure that you show clear mathematical workings and explanations where appropriate.

Preliminary task

- **a** If $x = 3$, find the total area of the veranda.
- **b** Find an expression for the total veranda area in terms of x.
- **c** Find the width of the veranda (x metres) if the veranda area is to be 44 m^2.
- **d** Find the width of the veranda (x metres) if the veranda area is to be 60 m^2. Round to one decimal place.

Modelling task

Formulate

- **a** The problem is to find integer veranda widths for given veranda areas. Write down all the relevant information that will help solve this problem with the aid of a diagram.
- **b** State the expression for the total veranda area in terms of x.

Solve

- **c** For $x = 2$ and $x = 4.5$:
  - **i** draw a diagram for each
  - **ii** calculate the veranda areas.
- **d** For three veranda areas of your choosing, use your expression for the total veranda area to determine the value of x. Show algebraic working.
- **e** Determine the veranda areas where the resulting equations satisfy both the following conditions:
  - they can be factorised using integers and;
  - the solution is an integer less than 5.

Evaluate and verify

- **f** Explain why when solving for x for a given veranda area, some answers are not integers.
- **g** Deduce the maximum integer width that Lucas can choose if the area of the veranda is to be less than 50 m^2.

Communicate

- **h** Summarise your results and describe any key findings.

Extension question

- **a** Explore the effect on your results if Lucas's house was a rectangle instead of a square.

Technology and computational thinking

Solving equations numerically

Key technology: Graphing software and spreadsheets

We know that we can use algebraic techniques to solve linear, quadratic and even exponential equations; however, in many situations such techniques do not work or are too cumbersome to deal with. In such cases a numerical technique can be used where we repeatedly move closer and closer to the solution until a desired level of accuracy is reached. Technology can help us achieve these numerical steps and find accurate solutions.

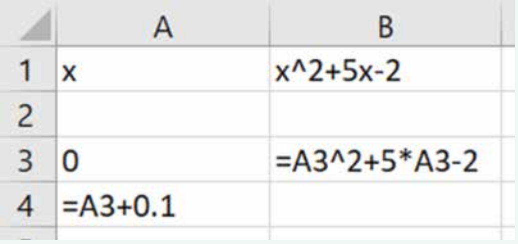
1 Getting started

We will start by looking at the solution to the quadratic equation $x^2 + 5x - 2 = 0$. By hand, we could solve this by completing the square or using the quadratic formula; however, in this case we will use a numerical approach by 'zooming in' to either of the solutions.

- **a** Find the value of $x^2 + 5x - 2$ for the following values of x.
  - **i** $x = 0$ **ii** $x = 0.5$ **iii** $x = 1$
- **b** Which value of x gives a value of $x^2 + 5x - 2$ which is closest to 0?
- **c** Try other values of x between 0 and 1 and try to find a solution to the equation $x^2 + 5x - 2 = 0$ correct to one decimal place.
- **d** Try other values of x between 0 and 1 and try to find a solution to the equation $x^2 + 5x - 2 = 0$ correct to two decimal places.

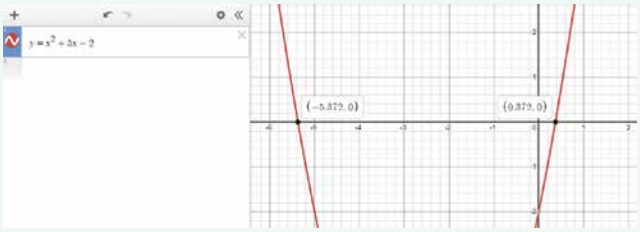
2 Using technology

- **a** Construct a spreadsheet which evaluates $x^2 + 5x - 2$ for various values of x. Use increments of 0.1 as shown.
- **b** Fill down from the cells A4 and B3. For which value of x is $x^2 + 5x - 2$ closest to zero?
- **c** There is another solution to the equation which is negative. Adjust your spreadsheet including the number in cell A3 to find this value of x correct to one decimal place.



	A	B
1	x	x^2+5x-2
2		
3	0	=A3^2+5*A3-2
4	=A3+0.1	

- **d** Another method for zooming in on a solution is to use a graph of $y = x^2 + 5x - 2$ and look at where $y = 0$. Use graphing software like Desmos to set up a graph which focuses on the points where $y = 0$.
- **e** Place points at the place where $y = 0$. By looking at the graph you can see that the solutions are near -5.4 and 0.4.



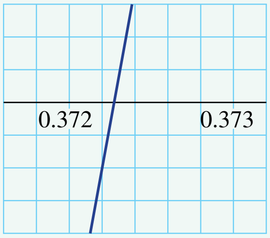
3 Applying an algorithm

To obtain even more accurate solutions to the previous quadratic equation we can zoom in closer using the spreadsheet or graph.

a Use this algorithm with your spreadsheet to find both solutions of the quadratic equation.

- Step 1: Alter the formula in cell A4 so that the increment is smaller. e.g. 0.01 rather than 0.1.
- Step 2: Fill down until you have located the value of x for which $x^2 + 5x - 2$ is closest to zero.
- Step 3: Adjust cell A3 to a different value so you don't need to scroll through so many cells.
- Step 4: Repeat from Step 1 but use smaller and smaller increments (0.001 and 0.0001) until you have found the value of x for which $x^2 + 5x - 2$ is closest to zero, correct to three decimal places.

b Use the functions of your graphing software to zoom into the points where $y = 0$. Use the scale to help find the solution to the equation. Keep zooming in until you are satisfied that your solution is correct to three decimal places.



4 Extension

a All sorts of equations can be solved using the above methods. Now try solving the following equations correct to three decimal places using spreadsheets and/or graphs.

i $2x^2 - 5x - 2 = 0$ **ii** $2^x - 7 = 0$

b An equation like $x^2 = 2^x$ can either be solved by finding an x value where the value of x^2 equals the value of 2^x or by solving $x^2 - 2^x = 0$. Now try to solve the following, correct to three decimal places.

i $x^2 = 2^x$ **ii** $x^2 - 4 = 3^x + 1$

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Investigation

Binomial expansions

Blaise Pascal and expansion

Blaise Pascal (1623–1662) was a French mathematician and philosopher. By the age of 16 he had proved many theorems in geometry and by 17 he had invented and made what is regarded as the first calculator.

One of his mathematical investigations involved exploring the properties and patterns of numbers in a triangular arrangement that is known today as Pascal's triangle. The triangle has many applications in mathematics, including algebraic expansion and probability. The diagram below shows part of this triangle.

Pascal's triangle

row 0	1
row 1	1 1
row 2	1 2 1
row 3	1 3 3 1
row 4	1 4 6 4 1
row 5	1 5 10 10 5 1
row 6	1 1
row 7	
row 8	

Expanding the triangle

a Observe and describe the pattern of numbers shown in rows 0 to 4.

b State a method that might produce the next row in the triangle.

c Complete the triangle to row 8.

Expanding brackets

Consider the expansions of binomial expressions. If you look closely, you can see how the coefficients in each term match the values in the triangle you produced in the triangle above.

$$\begin{aligned}(x+y)^0 &= \mathbf{1}\\ (x+y)^1 &= \mathbf{1}x+\mathbf{1}y\\ (x+y)^2 &= (x+y)(x+y)\\ &= \mathbf{1}x^2+\mathbf{2}xy+\mathbf{1}y^2\end{aligned}$$

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)^2\\ &= (x+y)(x^2+2xy+y^2)\\ &= x^3+2x^2y+xy^2+yx^2+2xy^2+y^3\\ &= \mathbf{1}x^3+\mathbf{3}x^2y+\mathbf{3}xy^2+\mathbf{1}y^3\end{aligned}$$

Expand $(x+y)^4$, $(x+y)^5$ and $(x+y)^6$ by completing the triangle below.

$(x+y)^0$	1
$(x+y)^1$	$1x+1y$
$(x+y)^2$	$1x^2+2xy+1y^2$
$(x+y)^3$	$1x^3+3x^2y+3xy^2+1y^3$
$(x+y)^4$	
$(x+y)^5$	
$(x+y)^6$	

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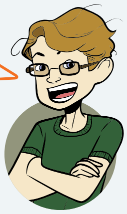
Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



1 Find the monic quadratic in the form $x^2 + bx + c = 0$ with solutions $x = 2 - \sqrt{3}$ and $x = 2 + \sqrt{3}$.

2 If $x + \frac{1}{x} = 7$, what is $x^2 + \frac{1}{x^2}$?

3 Find all the solutions to each equation. (*Hint*: Consider letting $a = x^2$ in each equation.)

a $x^4 - 5x^2 + 4 = 0$

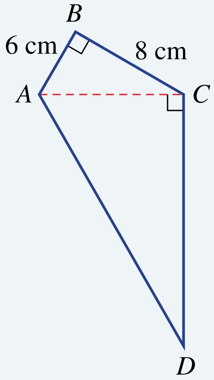
b $x^4 - 7x^2 - 18 = 0$

4 Make a substitution as you did in Question 3 to obtain a quadratic equation to help you solve the following.

a $3^{2x} - 4 \times 3^x + 3 = 0$

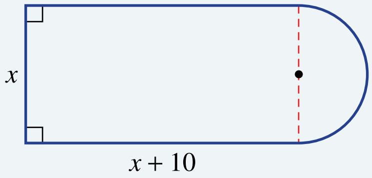
b $4 \times 2^{2x} - 9 \times 2^x + 2 = 0$

5 Quadrilateral *ABCD* has a perimeter of 64 cm with measurements as shown. What is the area of the quadrilateral?



6 A cyclist in a charity ride rides 300 km at a constant average speed. If the average speed had been 5 km/h faster, the ride would have taken 2 hours less. What was the average speed of the cyclist?

7 Find the value of x, correct to one decimal place, in this diagram if the area is to be 20 square units.



8 Prove that $x^2 - 2x + 2 > 0$ for all values of x.

9 A square has the same perimeter as a rectangle of length x cm and width y cm. Determine a simplified expression for the difference in their areas and, hence, show that when the perimeters are equal the square has the greatest area.

10 The equation $x^2 + wx + t = 0$ has solutions α and β, where the equation $x^2 + px + q = 0$ has solutions 3α and 3β. Determine the ratios $w:p$ and $t:q$.

Chapter summary

Quadratic expressions and equations

Factorising non-monic quadratics $ax^2 + bx + c$ (Opt)

Use grouping, split up bx using two numbers that multiply to $a \times c$ and add to give b:

e.g. $6x^2 - 5x - 4$

$a \times c = 6 \times (-4) = -24$

$-8 \times 3 = -24$

$-8 + 3 = -5$

$\therefore \quad 6x^2 + 3x - 8x - 4$

$= 3x(2x + 1) - 4(2x + 1)$

$= (2x + 1)(3x - 4)$

Factorising $x^2 + bx + c$

Two numbers × to give c.
Two numbers + to give b.

e.g. $x^2 - 7x - 18 = (x - 9)(x + 2)$

$-9 \times 2 = -18$

$-9 + 2 = -7$

Factorising and difference of two squares

Always take out common factors first.

Difference of two squares

$a^2 - b^2 = (a - b)(a + b)$

e.g. $4x^2 - 9 = (2x)^2 - (3)^2$

$= (2x - 3)(2x + 3)$

$x^2 - 7 = (x - \sqrt{7})(x + \sqrt{7})$

Factorising by completing the square

e.g. $x^2 + 4x - 3$

$= \left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right) - \left(\frac{4}{2}\right)^2 - 3$

$= \left(x + \frac{4}{2}\right)^2 - 4 - 3$

$= (x + 2)^2 - 7$

$= (x + 2 - \sqrt{7})(x + 2 + \sqrt{7})$

Note, for example, $(x + 2)^2 + 5$ cannot be factorised.

Expanding brackets

$a(b + c) = ab + ac$

$(a + b)(c + d) = ac + ad + bc + bd$

$(a + b)(a - b) = a^2 - b^2$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

Multiplying and dividing algebraic fractions

Factorise and then cancel any common factors first.

e.g. **1** $\frac{\cancel{2}^{1}}{x + 2} \times \frac{3x + 6}{\cancel{8}_{4}}$

$= \frac{1}{\cancel{x + 2}_{1}} \times \frac{3\cancel{(x + 2)}^{1}}{4}$

$= \frac{3}{4}$

2 $\frac{x^2 - 4}{x^2 + 2x}$

$= \frac{(x - 2)\cancel{(x + 2)}}{x\,\cancel{(x + 2)}}$

$= \frac{x - 2}{x}$

To divide, multiply by the reciprocal of the fraction following the ÷.

Solving quadratic equations

Null Factor Law:
If $pq = 0$ then $p = 0$ or $q = 0$.
Write each quadratic in standard form $ax^2 + bx + c = 0$, factorise then apply the Null Factor Law to solve.

e.g. **1** $x^2 - 4x = 0$

$x(x - 4) = 0$

$x = 0$ or $x - 4 = 0$

$x = 0$ or $x = 4$

2 $x^2 = 3x - 10$

$x^2 - 3x + 10 = 0$

$(x - 5)(x + 2) = 0$

$x - 5 = 0$ or $x + 2 = 0$

$x = 5$ or $x = -2$

Applications

1. Define the variable.
2. Set up the equation.
3. Solve by factorising and using the Null Factor Law or quadratic formula.
4. Determine the suitable answer(s).

Quadratic formula (Opt)

If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant $\Delta = b^2 - 4ac$ tells us how many solutions:

$\Delta > 0$ 2 solutions
$\Delta = 0$ 1 solution
$\Delta < 0$ 0 solutions

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Chapter checklist with success criteria

	A printable version of this checklist is available in the Interactive Textbook	✓
5A	**1. I can apply the distributive law to expand and simplify.** e.g. Expand and simplify $2x(3x - 5) - 3(3x - 5)$.	☐
5A	**2. I can expand a binomial product.** e.g. Expand and simplify $(2x - 3)(x + 4)$.	☐
5A	**3. I can expand to form a difference of two squares.** e.g. Expand $(3x + 2)(3x - 2)$.	☐
5A	**4. I can expand a perfect square.** e.g. Expand $(x + 5)^2$.	☐
5B	**5. I can factorise by taking out a common factor.** e.g. Factorise $12x^2 - 18x$.	☐
5B	**6. I can factorise a difference of two squares.** e.g. Factorise $9x^2 - 16$.	☐
5B	**7. I can factorise a difference of two squares involving surds.** e.g. Factorise $x^2 - 7$ using surds.	☐
5B	**8. I can factorise using grouping.** e.g. Factorise $x^2 - ax + 2x - 2a$ by grouping.	☐
5C	**9. I can cancel common factors in algebraic fractions.** e.g. Simplify $\frac{4x - 2}{2}$.	☐
5C	**10. I can multiply and divide simple algebraic fractions.** e.g. Simplify $\frac{3x - 9}{20} \div \frac{2x - 6}{5}$.	☐
5D	**11. I can factorise a monic trinomial.** e.g. Factorise $x^2 - 8x - 20$.	☐
5D	**12. I can factorise a trinomial with a common factor.** e.g. Factorise $3x^2 - 24x + 45$.	☐
5D	**13. I can multiply and divide algebraic fractions by first factorising.** e.g. Simplify by first factorising $\frac{x^2 - 4}{x + 2} \times \frac{3x + 12}{x^2 + 2x - 8}$.	☐
5E	**14. I can factorise a non-monic quadratic.** Opt e.g. Factorise $5x^2 + 13x - 6$.	☐
5F	**15. I can factorise by completing the square.** e.g. Factorise $x^2 + 6x + 2$ by completing the square.	☐
5F	**16. I can factorise non-monic quadratics by completing the square.** Opt e.g. Factorise $2x^2 + 6x + 3$ by completing the square.	☐
5F	**17. I can recognise when a quadratic cannot be factorised.** e.g. Factorise $x^2 - 3x + 4$ by completing the square if possible.	☐

Chapter checklist with success criteria

			✔
5G	**18. I can solve a quadratic equation by factorising and applying the Null Factor Law.** e.g. Solve $3x^2 - 9x = 0$.		☐
5G	**19. I can solve a quadratic equation by first rearranging into standard form.** e.g. Solve $x^2 = 2x + 3$.		☐
5H	**20. I can solve a word problem using a quadratic model.** e.g. The area of a rectangle is 60 m^2 and its length is 4 metres more than its width. Find the dimensions of the rectangle.		☐
5I	**21. I can solve a quadratic equation using completing the square.** e.g. Solve $x^2 + 4x - 22 = 0$ by first completing the square.		☐
5J	**22. I can determine the number of solutions of a quadratic equation using the discriminant.** e.g. Use the discriminant to determine the number of solutions of the equation $2x^2 - 3x - 5 = 0$.	Opt	☐
5J	**23. I can use the quadratic formula to solve a quadratic equation.** e.g. Find the exact solutions of $2x^2 + 3x - 4 = 0$ using the quadratic formula.	Opt	☐

Short-answer questions

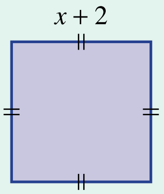
5A
1 Expand the following and simplify where possible.

a $2(x + 3) - 4(x - 5)$ **b** $(x + 5)(3x - 4)$ **c** $(5x - 2)(5x + 2)$

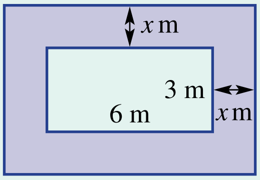
d $(x - 6)^2$ **e** $(x + 4)^2 - (x + 3)(x - 2)$ **f** $(3x - 2)(4x - 5)$

5A
2 Write, in expanded form, an expression for the shaded areas. All angles are right angles.

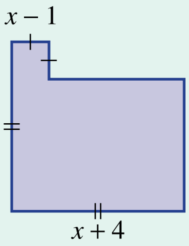
a



b



c



5B
3 Factorise the following.

a $3x^2 + 18x$ **b** $4(x + 1) - b(x + 1)$ **c** $x^2 - ax + 2x - 2a$

5B
4 Factorise the following difference of two squares. Remember to look for a common factor first.

a $x^2 - 49$ **b** $9x^2 - 16$ **c** $3x^2 - 75$

d $x^2 - 11$ **e** $(x + 1)^2 - 16$ **f** $(x - 3)^2 - 10$

5D
5 Factorise these quadratic trinomials.

a $x^2 - 8x + 12$ **b** $x^2 + 10x - 24$ **c** $-3x^2 + 21x - 18$

5E
Opt
6 Factorise these non-monic quadratic trinomials.

a $3x^2 + 17x + 10$ **b** $4x^2 + 4x - 15$ **c** $12x^2 - 16x - 3$ **d** $12x^2 - 23x + 10$

5C/D/E
7 Simplify the following.

a $\dfrac{4x + 8}{x^2 + 2x}$ **b** $\dfrac{4x + 4}{6x} \div \dfrac{x + 1}{3}$

c $\dfrac{12x}{x^2 + 2x - 3} \times \dfrac{x^2 - 1}{6x + 6}$ Opt **d** $\dfrac{4x^2 - 9}{2x^2 + x - 6} \div \dfrac{8x + 12}{x^2 - 2x - 8}$

5F
8 Factorise the following by completing the square.

a $x^2 + 8x + 10$ **b** $x^2 - 6x - 4$

Opt **c** $x^2 + 3x - 2$ Opt **d** $2x^2 - 4x - 3$

5G
9 Solve these quadratic equations by factorising and applying the Null Factor Law.

a $x^2 + 4x = 0$ **b** $3x^2 - 9x = 0$ **c** $x^2 - 25 = 0$

d $x^2 - 10x + 21 = 0$ **e** $x^2 - 8x + 16 = 0$ **f** $x^2 + 5x - 36 = 0$

Opt **g** $2x^2 + 3x - 2 = 0$ Opt **h** $6x^2 + 11x - 10 = 0$ Opt **i** $18x^2 + 25x - 3 = 0$

5G
10 Solve the following quadratic equations by first writing them in standard form.

a $3x^2 = 27$ **b** $x^2 = 4x + 5$

c $2x^2 - 28 = x(x - 3)$ **d** $\dfrac{3x + 18}{x} = x$

5H
11 A rectangular sandpit is 2 m longer than it is wide. If it occupies an area of 48 m^2, determine the dimensions of the sandpit by solving a suitable equation.

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5I

12 Solve these quadratic equations by first completing the square.

a $x^2 + 4x - 3 = 0$ b $x^2 - 6x + 2 = 0$

Opt c $x^2 - 3x - 2 = 0$ Opt d $x^2 + 5x - 5 = 0$

5J Opt

13 For each quadratic equation, determine the number of solutions by finding the value of the discriminant.

a $x^2 + 2x + 1 = 0$ b $x^2 - 3x - 3 = 0$

c $2x^2 - 4x + 3 = 0$ d $-3x^2 + x + 5 = 0$

5J Opt

14 Use the quadratic formula to give exact solutions to these quadratic equations.

a $x^2 + 3x - 6 = 0$ b $x^2 - 2x - 4 = 0$

c $2x^2 - 4x - 5 = 0$ d $-3x^2 + x + 3 = 0$

Multiple-choice questions

5A

1 $(x + 5)^2$ is equivalent to:

A $x^2 + 25$ B $x^2 + 5x$ C $x^2 + 5x + 25$

D $x^2 + 10x + 25$ E $x^2 + 50$

5A

2 $2(2x - 1)(x + 4)$ is equivalent to:

A $4x^2 + 15x - 4$ B $4x^2 + 14x - 8$ C $8x^2 + 28x - 16$

D $8x^2 + 18x - 4$ E $4x^2 + 10x + 8$

5B

3 $4x^2 - 25$ in factorised form is:

A $4(x - 5)(x + 5)$ B $(2x - 5)^2$ C $(2x - 5)(2x + 5)$

D $(4x + 5)(x - 5)$ E $2(2x + 1)(x - 25)$

5D

4 The fully factorised form of $2x^2 - 10x - 28$ is:

A $2(x + 2)(x - 7)$ B $(2x + 7)(x + 4)$ C $2(x - 4)(x - 1)$

D $(2x - 2)(x + 14)$ E $(x - 2)(x + 7)$

5C

5 $\dfrac{x - 4}{8x} \times \dfrac{2x + 8}{x^2 - 16}$ simplifies to:

A $\dfrac{x + 4}{x - 4}$ B $\dfrac{1}{4x}$ C $\dfrac{x + 4}{4(x - 4)}$ D $\dfrac{x}{x - 4}$ E $\dfrac{x + 8}{4x}$

5F

6 The term that needs to be added to make $x^2 - 6x$ a perfect square is:

A 18 B -9 C -3 D 9 E 3

5G

7 The solution(s) to $2x^2 - 8x = 0$ are:

A $x = 0, x = -4$ B $x = 2$ C $x = 0, x = 4$

D $x = 4$ E $x = 0, x = 2$

5G Opt

8 For $8x^2 - 14x + 3 = 0$, the solutions for x are:

A $\frac{1}{8}, -\frac{1}{3}$ B $\frac{3}{4}, -\frac{1}{2}$ C $\frac{1}{4}, \frac{3}{2}$

D $\frac{3}{4}, -\frac{1}{2}$ E $-\frac{1}{2}, -\frac{3}{8}$

5G

9 When written in the standard form $ax^2 + bx + c = 0$, with $a > 0$, $\dfrac{x - 3}{x} = 2x$ is:

A $x^2 + 2x + 3 = 0$ B $x^2 + 3 = 0$ C $2x^2 + x - 3 = 0$

D $2x^2 - x - 3 = 0$ E $2x^2 - x + 3 = 0$

5H **10** The product of two consecutive numbers is 72. If x is the smaller number, an equation to represent this would be:

A $x^2 + x + 72 = 0$ **B** $2x - 71 = 0$ **C** $x^2 + x - 72 = 0$
D $x^2 + 1 = 72$ **E** $x^2 = x + 72$

5I **11** For $(x - 7)^2 - 3 = 0$, the solutions for x are:

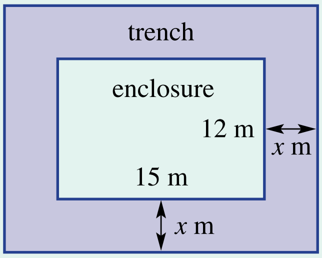
A $7 - \sqrt{3}, 7 + \sqrt{3}$ **B** $-7 - \sqrt{3}, -7 + \sqrt{3}$ **C** $7, -3$
D $-7 - \sqrt{3}, 7 + \sqrt{3}$ **E** $4, 10$

5J Opt **12** If $ax^2 + bx + c = 0$ has exactly two solutions, then:

A $b^2 - 4ac = 0$ **B** $b^2 - 4ac > 0$ **C** $b^2 - 4ac \leqslant 0$
D $b^2 - 4ac \neq 0$ **E** $b^2 - 4ac < 0$

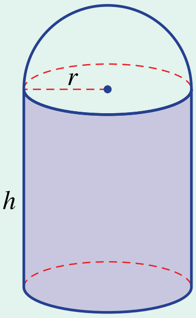
Extended-response questions

1 A zoo enclosure for a rare tiger is rectangular in shape and has a trench of width x m all the way around it to ensure the tiger doesn't get far if it tries to escape. The dimensions are as shown.



- **a** Write an expression in terms of x for:
  - **i** the length of the enclosure and trench combined
  - **ii** the width of the enclosure and trench combined.
- **b** Use your answers from part **a** to find the area of the overall enclosure and trench, in expanded form.
- **c** Hence, find an expression for the area of the trench alone.
- **d** Zoo restrictions state that the trench must have an area of at least 58 m^2. By solving a suitable equation, find the minimum width of the trench.

2 The surface area A of a cylindrical tank with a hemispherical top is given by the equation $A = 3\pi r^2 + 2\pi rh$, where r is the radius and h is the height of the cylinder.



- **a** If the radius of a tank with height 6 m is 3 m, determine its exact surface area.
- **b** If the surface area of a tank with radius 5 m is 250 m^2, determine its height, to two decimal places.
- **c** The surface area of a tank of height 6 m is found to be 420 m^2.
  - **i** Substitute the values and rewrite the equation in terms of r only.
  - **ii** Rearrange the equation and write it in the form $ar^2 + br + c = 0$, with $a > 0$.
  - Opt **iii** Solve for r using the quadratic formula and round your answer to two decimal places.

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Algebra, equations and linear relationships

Short-answer questions

1 **a** Solve these equations for x.

i $2 - 3x = 14$ **ii** $2(2x + 3) = 7x$

iii $\frac{x - 3}{2} = 5$ Opt **iv** $\frac{3x - 2}{4} = \frac{2x + 1}{5}$

b Solve these inequalities for x and graph their solutions on a number line.

i $3x + 2 \leqslant 20$ **ii** $2 - \frac{x}{3} > 1$

Opt **2** Simplify $\frac{x + 2}{3} - \frac{2x + 3}{9}$.

3 **a** Find the gradient and y-intercept for these linear relations and sketch each graph.

i $y = 3x - 2$ **ii** $4x + 3y = 6$

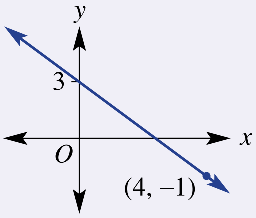
b Sketch by finding the x- and y-intercepts where applicable.

i $y = 2x - 6$ **ii** $3x + 5y = 15$

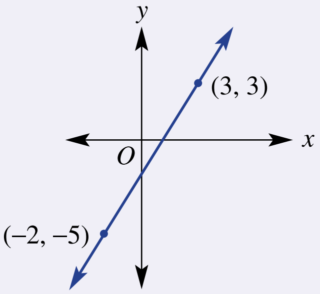
iii $x = 3$ **iv** $y = -2x$

4 Find the equation of the straight lines shown.

a



b



5 Find the value(s) of a in each of the following when:

a the lines $y = ax - 3$ and $y = -3x + 2$ are parallel

b the gradient of the line joining the points $(3, 2)$ and $(5, a)$ is -3

c the distance between $(3, a)$ and $(5, 4)$ is $\sqrt{13}$

d the lines $y = ax + 4$ and $y = \frac{1}{4}x - 3$ are perpendicular.

6 Solve these pairs of simultaneous equations.

a $y = 2x - 1$, $y = 5x + 8$

b $2x - 3y = 8$, $y = x - 2$

c $2x + y = 2$, $5x + 3y = 7$

d $3x - 2y = 19$, $4x + 3y = -3$

7 At a fundraising event, two hot dogs and three cans of soft drink cost \$13, and four hot dogs and two cans of soft drink cost \$18. What are the individual costs of a hot dog and a can of soft drink?

8 Sketch the region for these linear inequalities.

a $y \geqslant 3 - 2x$ **b** $3x - 2y < 9$ **c** $y > -3$

Multiple-choice questions

1 The simplified form of $2x(4 - 3y) - 3(3x - 4xy)$ is:

A $6xy - x$ **B** $2xy$ **C** $x - 18xy$
D $12xy - 7x$ **E** $2x - 3y - 4xy$

2 The point that is not on the line $y = 3x - 2$ is:

A $(-1, -5)$ **B** $(1, 1)$ **C** $(-2, -4)$
D $(4, 10)$ **E** $(0, -2)$

3 The length, d, and midpoint, M, of the line segment joining the points $(-2, 4)$ and $(3, -2)$ are:

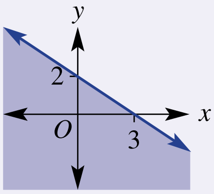
A $d = \sqrt{5}, M = (0.5, 1)$ **B** $d = \sqrt{61}, M = (2.5, 3)$ **C** $d = \sqrt{29}, M = (1, 1)$
D $d = \sqrt{61}, M = (0.5, 1)$ **E** $d = \sqrt{11}, M = (1, 2)$

4 The equation of the line that is perpendicular to the line with equation $y = -2x - 1$ and passes through the point $(1, -2)$ is:

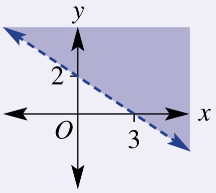
A $y = -\frac{1}{2}x + \frac{3}{2}$ **B** $y = 2x - 2$ **C** $y = -2x - 4$
D $y = x - 2$ **E** $y = \frac{1}{2}x - \frac{5}{2}$

5 The graph of $3x + 2y < 6$ is:

A



B



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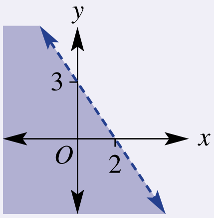
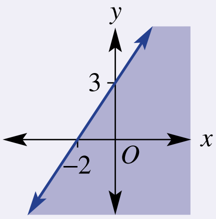
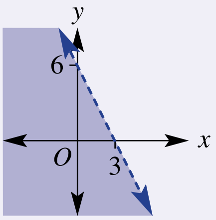


D



E



Extended-response question

A block of land is marked on a map with coordinate axes and with boundaries given by the equations $y = 4x - 8$ and $3x + 2y = 17$.

a Solve the two equations simultaneously to find their point of intersection.

b Sketch each equation on the same set of axes, labelling axis intercepts and the point of intersection.

The block of land is determined by the intersecting region $x \geqslant 0$, $y \geqslant 0$, $y \geqslant 4x - 8$ and $3x + 2y \leqslant 17$.

c Shade the area of the block of land (i.e. the intersecting region on the graph in part **b**).

d Find the area of the block of land if 1 unit represents 100 metres.

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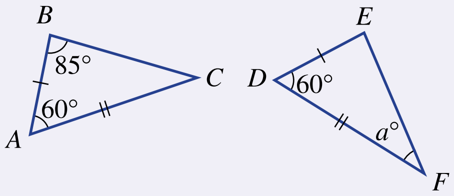
Geometry and networks

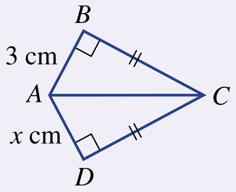
Short-answer questions

1 Prove the following congruence statements, giving reasons, and use this to find the value of the pronumerals.

a $\Delta ABC \equiv \Delta DEF$



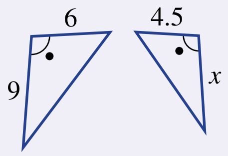
b $\Delta ABC \equiv \Delta ADC$



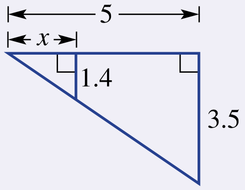
2 Use congruence to prove that a parallelogram (with opposite parallel sides) has equal opposite sides.

3 Find the value of the pronumeral, given these pairs of triangles are similar.

a

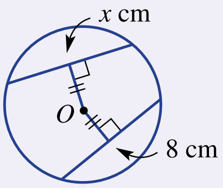


b

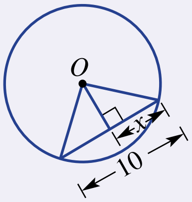


Opt 4 Use the chord and circle theorems to find the value of each pronumeral.

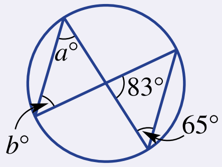
a



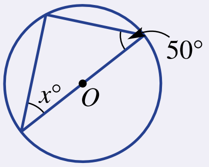
b



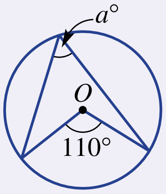
c



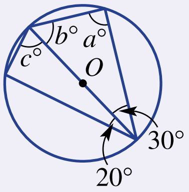
d



e

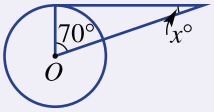


f

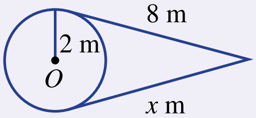


Opt 5 Use tangent properties to find the value of the pronumerals.

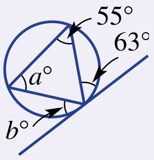
a



b



c



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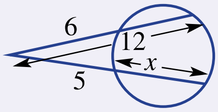
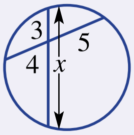
Opt **6** Find the value of x in each figure.

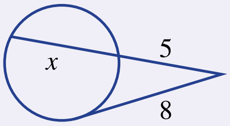
a



b



c



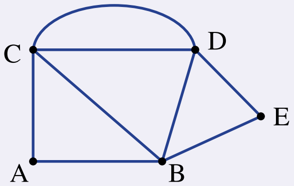
7 Consider the network graph shown.

a State the number of odd vertices (nodes).

b Verify Euler's formula using the information given in the graph.

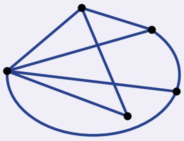
c Decide if the walk $B–A–C–B$ is a path.

d Decide if the graph has an Eulerian trail, with reason and if all Eulerian trails will be circuits.

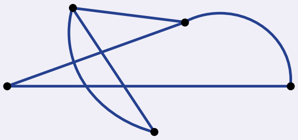


8 Decide if the following graphs are planar or non-planar.

a



b



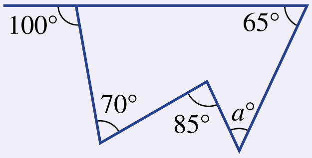
Multiple-choice questions

1 The value of a in the diagram shown is:

A 40 **B** 25

C 30 **D** 50

E 45



2 The values of x and y in these similar figures are:

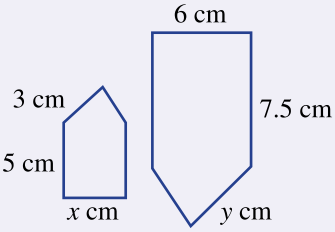
A $x = 2.6, y = 5$

B $x = 4, y = 4.5$

C $x = 4, y = 7.5$

D $x = 3, y = 6$

E $x = 3.5, y = 4.5$

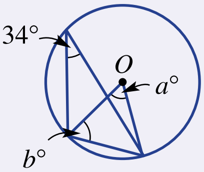


Opt **3** The values of the pronumerals in this diagram are:

A $a = 17, b = 56$ **B** $a = 34, b = 73$

C $a = 68, b = 56$ **D** $a = 34, b = 34$

E $a = 68, b = 34$

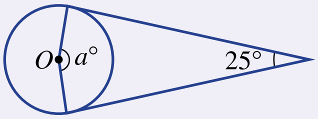


Opt **4** The value of angle a in this diagram involving tangents is:

A 115 **B** 165

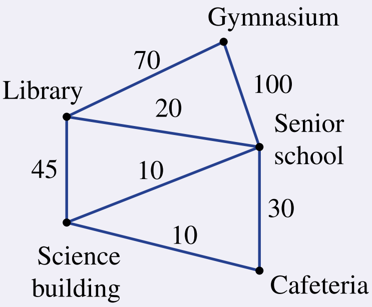
C 140 **D** 130

E 155



5 This weighted graph shows the distances in metres along paths between the main buildings in a school. The shortest distance between the library and the cafeteria is:

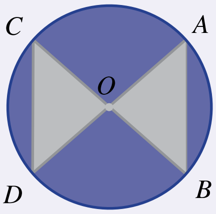
A 50 m   **B** 55 m
C 40 m   **D** 200 m
E 85 m



Extended-response question

A logo for a car manufacturing company is silver and purple and shaped as shown, with O indicating the centre of the circle.

The radius of the logo is 5 cm and chord AB is 6 cm. Given the two chords are equidistant from the centre of the circle, complete the following.



a What is the length of CD? Give a reason.

b Hence, prove that $\Delta OAB \equiv \Delta OCD$.

c By first finding the length of OM, where M is the point such that $OM \perp AB$, find the area of ΔOAB.

d Hence, determine what percentage of the logo is occupied by the silver portion, given the area of a circle is πr^2. Answer correct to one decimal place.

e Given that $\angle OCD = 53.1°$, what is the angle between the two triangles (i.e. $\angle BOD$)?

Indices, exponentials and logarithms

Short-answer questions

1 Use the index laws to simplify the following. Express all answers with positive indices.

a $2x^2 \times 3x^4y^2$ **b** $(3x^3)^4 \div x^7$

c $\left(\frac{3a}{b^4}\right)^2 \times \frac{2b^{10}}{6(2a^5)^0}$ **d** $3a^{-5}b^2$

e $\frac{4}{m^{-3}}$ **f** $\frac{4x^{-2}y^3}{10x^{-4}y^6}$

2 Convert:

a to a basic numeral

i 3.72×10^4 **ii** 4.9×10^{-6}

b to scientific notation, using three significant figures

i 0.000072973   **ii** 4 725 400 000

3 **a** Express in index form.

i $\sqrt{10}$ **ii** $\sqrt{7x^6}$ **iii** $4\sqrt[5]{x^3}$ **iv** $15\sqrt{15}$

b Express in surd form.

i $6^{\frac{1}{2}}$ **ii** $20^{\frac{1}{5}}$ **iii** $7^{\frac{3}{4}}$

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4 Evaluate without using a calculator.

a 5^{-1} **b** 2^{-4} Opt **c** $81^{\frac{1}{4}}$ Opt **d** $8^{-\frac{1}{3}}$

Opt **5** Solve these exponential equations for x.

a $4^x = 64$ **b** $7^{-x} = \frac{1}{49}$ **c** $9^x = 27$ **d** $5^{5x+1} = 125^x$

6 Sketch the graphs:

a $y = 2^x$ **b** $y = 2^{-x}$ **c** $y = -2^x$

7 Determine the final amount after 3 years if:

a \$2000 is compounded annually at 6%

b \$7000 is compounded monthly at 3%.

8 Simplify where necessary and evaluate.

a $\log_4 64$ **b** $\log_5 \frac{1}{25}$ **c** $\log_{10} 1000$ **d** $\log_7 1$

Opt **e** $\log_4 2 + \log_4 8$ **f** $\log_3 54 - \log_3 6$ **g** $\log_8 8$ **h** $\log_a a^3$

9 The following table shows data regarding the share price of some stocks $\$A$ over t years.

t	0	1	2	3	4	5
A	1.5	1.8	2.16	2.592	3.1104	3.7325

a Calculate the values of $\log_{10} A$ correct to one decimal place and plot a graph of $\log_{10} A$ vs t, with t on the horizontal axis.

b What does the shape of the graph of $\log_{10} A$ vs t say about the type of relationship between A and t?

10 Solve for x.

a $\log_6 216 = x$ **b** $\log_x 27 = 3$ **c** $\log_3 x = 4$

Opt **11** **a** Solve for x using the given base.

- **i** $3^x = 30$ **ii** $15 \times 2.4^x = 60$

b Solve for x using base 10 and evaluate, correct to three decimal places.

- **i** $7^x = 120$ **ii** $2000 \times 0.87^x = 500$

Multiple-choice questions

1 $3x^2y^7 \times 2x^4y^{-3}$ simplifies to:

A $\frac{5x^8}{y^{21}}$ **B** $6x^8y^4$ **C** $\frac{6y^4}{x^2}$ **D** $6x^6y^4$ **E** $-6x^6y^{10}$

2 The simplified form of $\frac{12(a^3)^{-2}}{(2ab)^2 \times a^2b^{-1}}$, when written using positive indices, is:

A $\frac{6}{a^2b}$ **B** $3a^2$ **C** $\frac{6a}{b}$ **D** $\frac{3}{a^2b^3}$ **E** $\frac{3}{a^{10}b}$

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3 0.00032379 in scientific notation, using three significant figures, is:

A 3.23×10^{-4} **B** 3.24×10^{4} **C** 3.24×10^{-4}
D 32.4×10^{3} **E** 0.324×10^{-5}

4 A limestone rock loses mass at 6% per year. If it originally weighs 2.5 kg, an equation for its mass M kg after t years is:

A $M = 2.5(1.06)^t$ **B** $M = 2.5 + 0.94^t$ **C** $2.5(0.06)^t$
D $M = 2.5(0.94)^t$ **E** $M = 2.5(0.4)^t$

5 An equivalent statement to $3^x = 20$ is:

A $x = \log_3 20$ **B** $20 = \log_3 x$ **C** $x = \log_{20} 3$
D $3 = \log_x 20$ **E** $x = \log_{10}\left(\frac{20}{3}\right)$

Extended-response question

Lachlan's share portfolio is rising at 8% per year and is currently valued at \$80 000.

a Determine a rule for the value of Lachlan's share portfolio (V dollars) in n years' time.

b What will be the value of the portfolio, to the nearest dollar:

i next year? **ii** in 4 years' time?

c Use trial and error to find when, to two decimal places, the share portfolio will be worth \$200 000. Alternatively, solve using logarithms.

d After 4 years, however, the market takes a downwards turn and the share portfolio begins losing value. Two years after the downturn, Lachlan sells his shares for \$96 170. If the market was declining in value at a constant percentage per year, what was this rate of decline, to the nearest percentage?

Measurement and surds

Short-answer questions

Opt **1** Simplify:

a $\sqrt{54}$ **b** $4\sqrt{75}$ **c** $\frac{3\sqrt{24}}{2}$
d $\sqrt{5} \times \sqrt{2}$ **e** $3\sqrt{7} \times \sqrt{7}$ **f** $3\sqrt{6} \times 4\sqrt{8}$
g $\sqrt{15} \div \sqrt{5}$ **h** $\frac{3\sqrt{30}}{9\sqrt{6}}$ **i** $\sqrt{\frac{200}{49}}$

Opt **2** Simplify fully.

a $2\sqrt{5} + 3\sqrt{7} + 5\sqrt{5} - 4\sqrt{7}$ **b** $\sqrt{20} - 2\sqrt{5}$ **c** $\sqrt{18} - 4 + 6\sqrt{2} - 2\sqrt{50}$

Opt **3** Expand any simplify these expressions.

a $2\sqrt{3}(\sqrt{5} - 2)$ **b** $(3\sqrt{5})^2$

Opt **4** Rationalise the denominator.

a $\frac{3}{\sqrt{2}}$ **b** $\frac{2\sqrt{3}}{5\sqrt{6}}$ **c** $\frac{2 - \sqrt{5}}{\sqrt{5}}$

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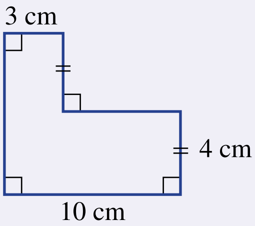
5 Find the perimeter and area of these shapes. Give answers correct to one decimal place where necessary. You will need to use Pythagoras' theorem for part **c**.

a



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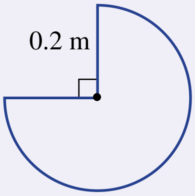
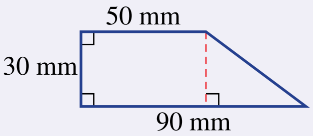
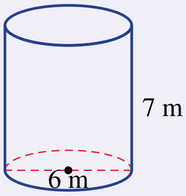


c



6 Find the surface area and volume for these solids. Give your answers to one decimal place.

a



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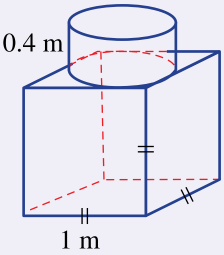
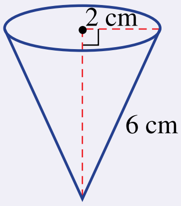


Ext **c**



7 Give the limits of accuracy for these measurements.

a 64 cm

b 14.2 kg

8 A rectangular prism has length 5 cm, width 3 cm and volume 27 cm^3.

a Find the height of the prism.

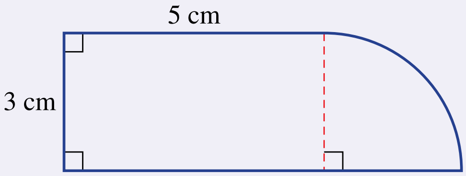
b Find the surface area of the prism.

Ext **9** A cone has volume 90 cm^3 and height 10 cm. Find the exact radius of the cone.

Multiple-choice questions

1 Correct to two decimal places, the perimeter and area, respectively, for this shape are:

A 12.71 cm, 16.77 cm^2
B 20.71 cm, 22.07 cm^2
C 25.42 cm, 29.14 cm^2
D 18.36 cm, 43.27 cm^2
E 17.71 cm, 17.25 cm^2



2 0.04 m^2 is equivalent to:

A 4 cm^2
B 40 mm^2
C 0.0004 cm^2
D 400 cm^2
E 0.00004 km^2

3 A prism has a base area of 30 m^2 and a height of 7 m. Its volume is:

A 105 m^3
B 70 m^3
C 210π m^3
D 210 m^3
E 140 m^3

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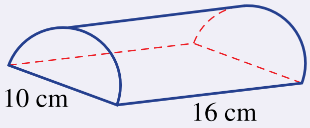
4 The curved surface area of this half cylinder, in exact form, is:



A 80π cm^2
B 105π cm^2
C 92.5π cm^2
D 120π cm^2
E 160π cm^2

5 The volume of a sphere of diameter 30 cm is closest to:

A 113 097 cm^3 **B** 2827 cm^3 **C** 11 310 cm^3 **D** 14 137 cm^3 **E** 7069 cm^3

Extended-response question

A cylindrical glass vase is packaged inside a box that is a rectangular prism, so that the vase touches the box on all four sides and is the same height as the box. The vase has a diameter of 8 cm and height 15 cm. Round your answers to two decimal places where necessary.

a Find the volume of the vase.
b Find the volume of space inside the box but outside the vase.
c A glass stirring rod is included in the vase. Find the length of the longest rod that can be packaged inside the vase.
d Find the difference in the length of rod in part **c** and the longest rod that can fit inside the empty box. Round your answer to two decimal places.

Quadratic expressions and equations

Short-answer questions

1 Expand and simplify.
a $(3x + 1)(3x - 1)$
b $(2x - 5)^2$
c $(2x + 3)(x + 5) - (3x - 5)(x - 4)$

2 Factorise fully these quadratics. Remember to take out any common factors first.
a $4x^2 - y^2$
b $(x + 2)^2 - 7$
c $3x^2 - 48$
d $x^2 + 5x - 14$
e $x^2 - 10x + 25$
f $2x^2 - 16x + 24$

3 Simplify these algebraic fractions.
a $\dfrac{12 - 8x}{4}$
b $\dfrac{5x - 10}{3} \times \dfrac{12}{x - 2}$
c $\dfrac{x^2 - 4}{x^2 + 3x - 10} \div \dfrac{2x + 4}{6}$

4 Factorise these non-monic quadratics.
a $3x^2 - 2x - 8$
b $6x^2 + 7x - 3$
c $10x^2 - 23x + 12$

5 Solve these quadratic equations using the Null Factor Law.
a $2x(x - 3) = 0$
b $(x + 4)(2x - 1) = 0$
c $x^2 + 5x = 0$
d $x^2 - 16 = 0$
e $x^2 - 7 = 0$
f $x^2 - 4x + 4 = 0$
g $x^2 - 5x - 24 = 0$
h $3x^2 + 5x - 2 = 0$

6 Solve these quadratic equations by first writing them in standard form.

a $x^2 = 40 - 3x$ **b** $x(x - 6) = 4x - 21$ **c** $\frac{x + 20}{x} = x$

7 **a** Factorise by completing the square.

i $x^2 - 6x + 4$ **ii** $x^2 + 4x + 7$ Opt **iii** $x^2 + 3x + 1$

b Use your answers to part **a** to solve these equations, if possible.

i $x^2 - 6x + 4 = 0$ **ii** $x^2 + 4x + 7 = 0$ Opt **iii** $x^2 + 3x + 1 = 0$

Opt **8** Solve these quadratic equations using the quadratic formula. Leave your answers in exact surd form.

a $2x^2 + 3x - 6 = 0$ **b** $x^2 - 4x - 6 = 0$

Multiple-choice questions

1 The expanded form of $2(2x - 3)(3x + 2)$ is:

A $12x^2 - 5x - 6$ **B** $12x^2 - 12$ **C** $12x^2 - 10x - 12$
D $24x^2 - 20x - 24$ **E** $12x^2 - x - 6$

2 The factorised form of $25y^2 - 9$ is:

A $(5y - 3)^2$ **B** $(5y - 3)(5y + 3)$ **C** $(25y - 3)(y + 3)$
D $(5y - 9)(5y + 1)$ **E** $5(y + 1)(y - 9)$

3 The factorised form of $2ax - 6x + 5a - 15$ is:

A $(x + a)(2a - 3)$ **B** $(a - 3)(2x + 5)$ **C** $(2x - 5)(a + 3)$
D $(2x - 3)(a + 5)$ **E** $2a(x - 6) + 5(a - 15)$

4 The solution(s) to the quadratic equation $x^2 - 4x + 4 = 0$ is/are:

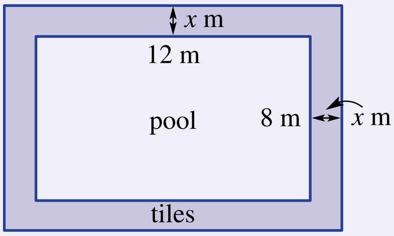
A $x = 0, 4$ **B** $x = 2$ **C** $x = 1, 4$
D $x = 2, -2$ **E** $x = -1, 4$

Opt **5** A quadratic equation $ax^2 + bx + c = 0$ has a discriminant equal to 17. This tells us that:

A the equation has a solution $x = 17$. **B** the equation has no solutions.
C $a + b + c = 17$. **D** the equation has two solutions.
E the equation has one solution.

Extended-response question

A rectangular backyard swimming pool, measuring 12 metres by 8 metres, is surrounded by a tiled path of width x metres, as shown.



a Find a simplified expression for the area of the tiled path.

b If $x = 1$, what is the tiled area?

🖩 **c** Solve an appropriate equation to determine the width, x metres, if the tiled area is 156 m^2.

Opt **d** Find the width, x metres, if the tiled area is 107.36 m^2. Use the quadratic formula.

6

Trigonometry

Maths in context: The Surveying profession

A surveyor writes up legally binding documents, so lengths need to be accurate, and calculations need to be correct. Technological advances make measuring procedures easier, however the knowledge and application of algebra, geometry, trigonometry, and calculus is essential.

Surveyors take outdoor measurements using specialised equipment, then analyse and compile the data, and finally present their completed project to other professionals on the team.

For example:

- Land surveyors work with town planners and focus on new subdivisions. They locate exact boundaries for roads and house blocks and the exact locations for power, sewage, and mains water supply.
- Engineering surveyors plan the details of civil engineering projects: complex constructions such as high-rise buildings, roads, bridges, railways, and tunnels.
- Hydrographic surveyors develop 3D topographic maps of riverbeds, harbour, and ocean seafloors. These maps are used to review routes for shipping and underwater cables; supply location data for offshore exploration, fish farming and wind farms; and, for military operations. To be able to identify which trigonometric ratio to apply

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based on the information in a given right-angled triangle
- Geodetic surveyors and engineers use spherical trigonometry working with curved lines and spherical angles. They update the curvature of the earth, the movement of continents, sea level rise, earthquake zones, and satellite tracking. Australia has moved approximately 2 m NE from its 1994 location on the globe.

Chapter contents

Australian Curriculum 9.0

MEASUREMENT

Solve practical problems applying Pythagoras' theorem and trigonometry of right-angled triangles, including problems involving direction and angles of elevation and depression (AC9M10M03)

Use mathematical modelling to solve practical problems involving proportion and scaling of objects; formulate problems and interpret solutions in terms of the situation; evaluate and modify models as necessary, and report assumptions, methods and findings (AC9M10M05)

ALGEBRA

The graphs of $y = \sin(x)$ and $y = \cos(x)$ as functions of a real variable and the solution of related equations (Year 10 optional content)

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

6A Trigonometric ratios

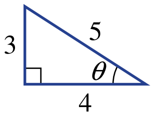
LEARNING INTENTIONS

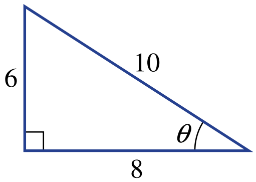
- To understand how the trigonometric ratios relate the angles and side lengths of right-angled triangles
- To know the trigonometric ratios involving sine, cosine and tangent
- To be able to identify which trigonometric ratio to apply based on the information in a given right-angled triangle
- To be able to use trigonometry to find an unknown side length in a right-angled triangle

The study of trigonometry explores the relationship between the angles and side lengths of triangles. Trigonometry can be applied to simple problems, such as finding the angle of elevation of a kite, to solving complex problems in surveying and design.

Trigonometry is built upon the three ratios sine, cosine and tangent. These ratios do not change for triangles that are similar in shape

Engineers use trigonometry to determine the horizontal and vertical components of the forces acting on bridge trusses and cables. Equations are formed by equating forces in opposite directions at joints; solving simultaneously calculates each load.





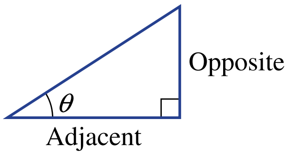
$$\sin\theta = \frac{3}{5} \qquad \sin\theta = \frac{6}{10} = \frac{3}{5}$$

$$\cos\theta = \frac{4}{5} \qquad \cos\theta = \frac{8}{10} = \frac{4}{5}$$

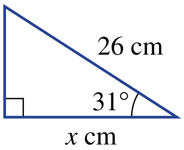
$$\tan\theta = \frac{3}{4} \qquad \tan\theta = \frac{6}{8} = \frac{3}{4}$$

Lesson starter: Which ratio?

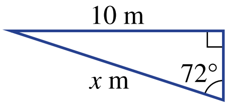
In a group or with a partner, see if you can recall some facts from Year 9 trigonometry to answer the following questions.



- What is the name given to the longest side of a right-angled triangle?
- $\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$ is one trigonometric ratio. What are the other two?
- Which ratio would be used to find the value of x in this triangle? Can you also find the answer?



- Which ratio would be used to find the value of x in this triangle? Can you also find the answer?



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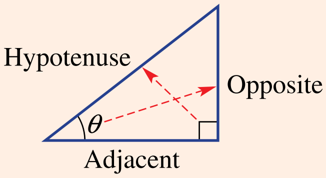
KEY IDEAS

- The **hypotenuse** is the longest side of a right-angled triangle. It is opposite the right angle.
- Given a right-angled triangle and another angle θ, the three trigonometric ratios are:
  - The sine ratio: $\sin\theta = \dfrac{\text{length of the opposite side}}{\text{length of the hypotenuse}}$
  - The cosine ratio: $\cos\theta = \dfrac{\text{length of the adjacent side}}{\text{length of the hypotenuse}}$
  - The tangent ratio: $\tan\theta = \dfrac{\text{length of the opposite side}}{\text{length of the adjacent side}}$



- Many people like to use SOHCAHTOA to help remember the three ratios.
  - $\sin\theta = \dfrac{\text{O}}{\text{H}}$ $\cos\theta = \dfrac{\text{A}}{\text{H}}$ $\tan\theta = \dfrac{\text{O}}{\text{A}}$
- To find an unknown length on a right-angled triangle:
  - choose a trigonometric ratio that links one known angle and a known side length with the unknown side length
  - solve for the unknown side length.

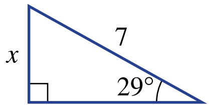
BUILDING UNDERSTANDING

① Use a calculator to evaluate the following, correct to three decimal places.

a $\cos 37^\circ$ **b** $\sin 72^\circ$ **c** $\tan 50^\circ$ **d** $\cos 21.4^\circ$

② Decide which ratio $\left(\text{i.e. } \sin\theta = \dfrac{\text{O}}{\text{H}}, \cos\theta = \dfrac{\text{A}}{\text{H}} \text{ or } \tan\theta = \dfrac{\text{O}}{\text{A}}\right)$ would be best to help find the value of x in these triangles. Do not find the value of x.

a



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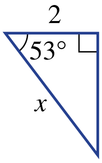
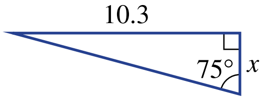


c



③ Solve for x in these equations, correct to two decimal places.

a $\tan 31^\circ = \dfrac{x}{3}$

b $\cos 54^\circ = \dfrac{x}{5}$

c $\sin 15.6^\circ = \dfrac{x}{12.7}$

d $\sin 57^\circ = \dfrac{2}{x}$

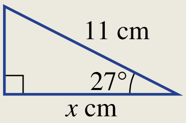
e $\cos 63.4^\circ = \dfrac{10}{x}$

f $\tan 71.6^\circ = \dfrac{37.5}{x}$

Example 1 Solving for an unknown in the numerator

Find the value of x in these right-angled triangles, correct to two decimal places.

a



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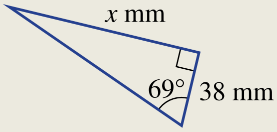
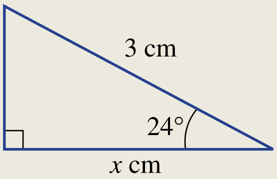


SOLUTION	EXPLANATION
a $\cos\theta = \frac{A}{H}$ $\cos 27° = \frac{x}{11}$ $\therefore x = 11 \times \cos 27°$ $= 9.80$ (to 2 d.p.)	Choose the ratio $\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$. Multiply both sides by 11, then use a calculator. Round your answer as required.
b $\tan\theta = \frac{O}{A}$ $\tan 69° = \frac{x}{38}$ $\therefore x = 38 \times \tan 69°$ $= 98.99$ (to 2 d.p.)	The tangent ratio uses the opposite and the adjacent sides. Multiply both sides by 38.

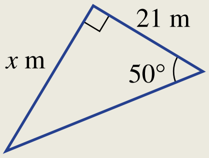
Now you try

Find the value of x in these right-angled triangles, correct to two decimal places.

a



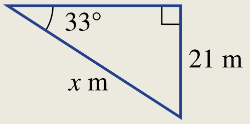
b



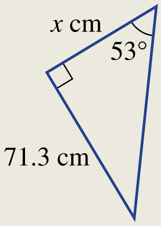
Example 2 Solving for an unknown in the denominator

Find the value of x in these right-angled triangles, rounding your answer to two decimal places.

a



b



SOLUTION

a

$$\begin{aligned} \sin\theta &= \frac{O}{H} \\ \sin 33^\circ &= \frac{21}{x} \\ x \times \sin 33^\circ &= 21 \\ x &= \frac{21}{\sin 33^\circ} \\ &= 38.56 \text{ (to 2 d.p.)} \end{aligned}$$

b

$$\begin{aligned} \tan\theta &= \frac{O}{A} \\ \tan 53^\circ &= \frac{71.3}{x} \\ x \times \tan 53^\circ &= 71.3 \\ x &= \frac{71.3}{\tan 53^\circ} \\ &= 53.73 \text{ (to 2 d.p.)} \end{aligned}$$

EXPLANATION

Choose the sine ratio since the adjacent side is not marked.

Multiply both sides by x to remove the fraction, then divide both sides by sin 33°.

Evaluate using a calculator and round your answer as required.

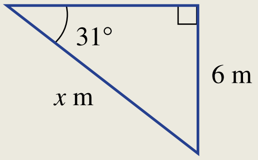
The hypotenuse is unmarked, so use the tangent ratio.

Multiply both sides by x, then solve by dividing both sides by tan 53°.

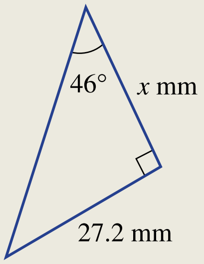
Now you try

Find the value of x in these right-angled triangles, rounding your answer to two decimal places.

a



b



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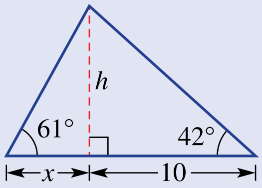
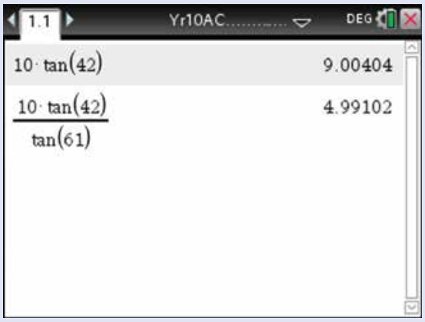
Using calculators in trigonometry

Find the value of the unknowns in this triangle, correct to two decimal places.



Using the TI-Nspire:

First, find the height *h* of the triangle using tan. Do not round this value before using it for the next step. Then use this result to find the value of *x* also using tan. Ensure your General Settings include Degree and Approximate (decimal) modes.



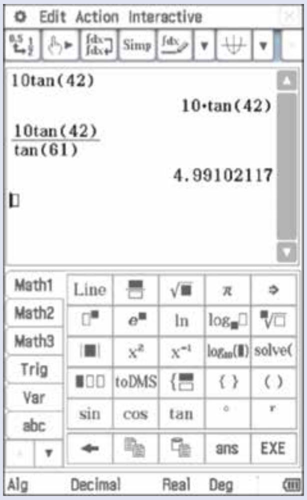
```
10·tan(42)            9.00404
10·tan(42)            4.99102
 tan(61)
```

Hint: use the [trig] key to access tan.

Hint: you can also include a degree symbol ([ctrl] [📖] and select ° or use the [π] key to access °) in your entries if desired.

Using the ClassPad:

In **Standard Degree** mode, first find the height of the triangle using tan. Use this result to find the value of *x* also using tan. Do this calculation in **Decimal Degree** mode.



```
10tan(42)
                 10·tan(42)
10tan(42)
tan(61)
                 4.99102117
```

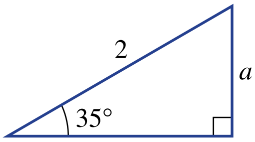
Exercise 6A

FLUENCY	1–2(½)	1–2(½)	1–2(⅓), 3

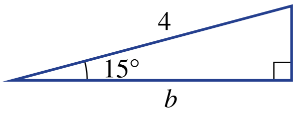
Example 1

1 Use trigonometric ratios to find the values of the pronumerals, to two decimal places.

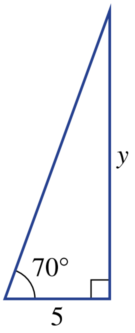
a



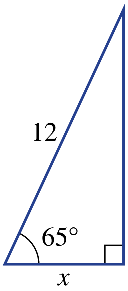
b



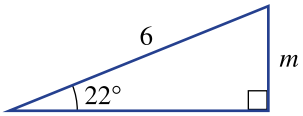
c



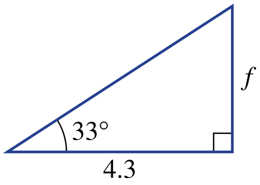
d



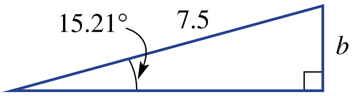
e



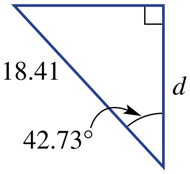
f



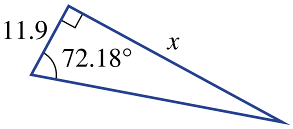
g



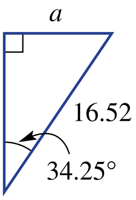
h



i



j



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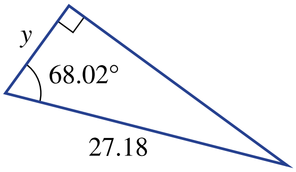
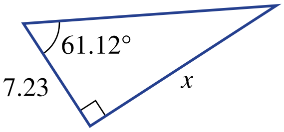


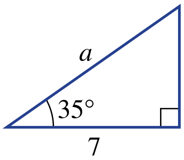
l



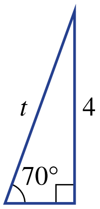
Example 2

2 Use trigonometric ratios to find the values of the pronumerals, to two decimal places, for these right-angled triangles.

a



b



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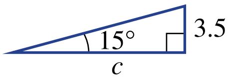
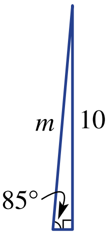


d



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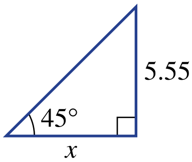
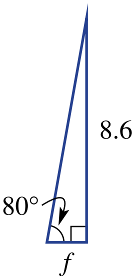
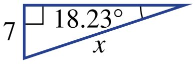


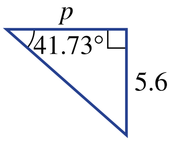
f



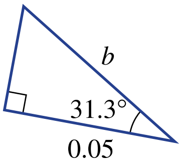
g



h



i



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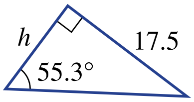

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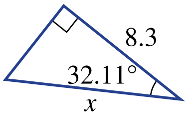
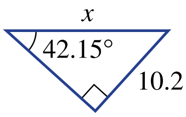
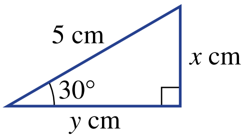


l

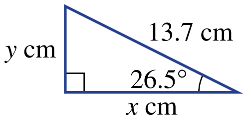


3 Find the unknown side lengths for these right-angled triangles, correct to two decimal places where necessary.

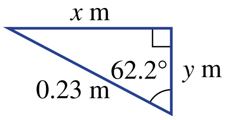
a



b



c

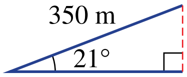


PROBLEM–SOLVING	4, 5	5–7	6–9

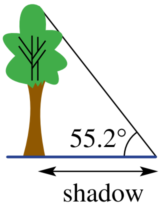
4 A 4WD climbs a 350 m straight slope at an angle of 21° to the horizontal.

a Find the vertical distance travelled, correct to the nearest metre.

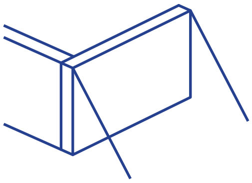
b Find the horizontal distance travelled, correct to the nearest metre.



5 The angle from the horizontal of the line of sight from the end of a tree's shadow to the top of the tree is 55.2°. The length of the shadow is 15.5 m. Find the height of the tree, correct to one decimal place.



6 On a construction site, large concrete slabs of height 5.6 metres are supported at the top by steel beams positioned at an angle of 42° from the vertical. Find the length of the steel beams, to two decimal places.



7 By measuring the diagonals, a surveyor checks the dimensions of a rectangular revegetation area of length 25 metres. If the angle of the diagonal to the side length is 28.6°, find the length of the diagonals, correct to one decimal place.

8 A right-angled triangular flag is made for the premiers of a school competition. The second-longest edge of the flag is 25 cm and the largest non-right angle on the flag is 71°. Find the length of the longest edge of the flag, to the nearest millimetre.

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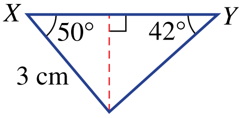
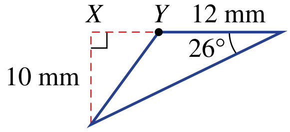
9 Find the length XY in these diagrams, correct to one decimal place.

a



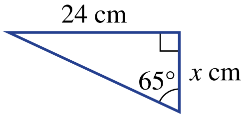
b



REASONING	10	10	10, 11

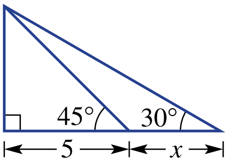
10 A student solves for x, to two decimal places, in the given triangle and gets 11.21, as shown. But the answer is 11.19. Explain the student's error.

$$\begin{aligned} \tan 65^\circ &= \frac{24}{x} \\ x\tan 65^\circ &= 24 \\ x &= \frac{24}{\tan 65^\circ} \\ &= \frac{24}{2.14} \\ &= 11.21 \end{aligned}$$

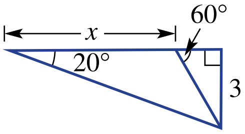


11 Find the value of x, correct to one decimal place, in these triangles.

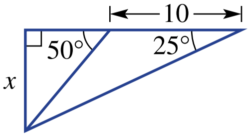
a



b

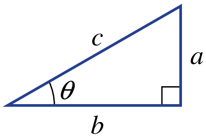


c



ENRICHMENT: Exploring identities	–	–	12

12 For the following proofs, consider the right-angled triangle shown.



a Show that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ by completing these steps.

- **i** Write a in terms of c and θ.
- **ii** Write b in terms of c and θ.
- **iii** Write $\tan\theta$ in terms of a and b.
- **iv** Substitute your expressions from parts **i** and **ii** into your expression for $\tan\theta$ in part **iii**. Simplify to prove $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.
- **v** Can you find a different way of proving the rule described above?

b Show that $(\sin\theta)^2 + (\cos\theta)^2 = 1$ by completing these steps.

- **i** Write a in terms of c and θ.
- **ii** Write b in terms of c and θ.
- **iii** State Pythagoras' theorem using a, b and c.
- **iv** Use your results from parts **i**, **ii** and **iii** to show that $(\sin\theta)^2 + (\cos\theta)^2 = 1$.

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6B Finding unknown angles

LEARNING INTENTIONS

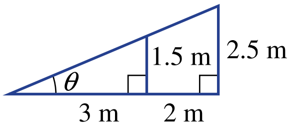
- To know that the inverse trigonometric functions are used to find angles in right-angled triangles
- To be able to use the inverse trigonometric functions to find an angle in a right-angled triangle given two side lengths

The three trigonometric ratios discussed earlier can also be used to find unknown angles in right-angled triangles if at least two side lengths are known. If, for example, $\cos\theta = \frac{1}{2}$ then we use the inverse trigonometric function for cosine, $\cos^{-1}\left(\frac{1}{2}\right)$, to find θ. Calculators are used to obtain these values.

The Eleanor Schonell Bridge in Brisbane is a cable-stayed bridge in which each cable forms a right-angled triangle with the pylons and the bridge deck. Trigonometry and geometry are essential tools for engineers.

Lesson starter: The ramp

A ski ramp is 2.5 m high and 5 m long (horizontally) with a vertical strut of 1.5 m placed as shown.



- Discuss which triangle could be used to find the angle of incline, θ. Does it matter which triangle is used?
- Which trigonometric ratio is to be used and why?
- How does $\tan^{-1}$ on a calculator help to calculate the value of θ?
- Discuss how you can check if your calculator is in degree mode.

KEY IDEAS

- **Inverse trigonometric functions** are used to find angles in right-angled triangles.

If $\sin\theta = k$ then $\theta = \sin^{-1}(k)$	If $\cos\theta = k$ then $\theta = \cos^{-1}(k)$	If $\tan\theta = k$ then $\theta = \tan^{-1}(k)$

where $-1 \leqslant k \leqslant 1$ for $\sin\theta$ and $\cos\theta$.

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BUILDING UNDERSTANDING

1 State the missing part in each sentence.

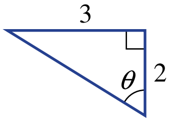
- **a** If $\cos 60^\circ = 0.5$, then $\cos^{-1}(0.5) = \_\_\_\_\_\_$.
- **b** If $\sin 30^\circ = \frac{1}{2}$, then $\sin^{-1}(\_\_\_\_\_\_) = 30^\circ$.
- **c** If $\tan 37^\circ \approx 0.75$, then $\tan^{-1}(\_\_\_\_\_\_) \approx 37^\circ$.

2 Find θ in the following, rounding your answer to two decimal places where necessary.

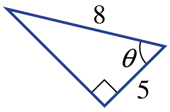
a $\sin\theta = 0.4$ **b** $\cos\theta = 0.5$ **c** $\tan\theta = 0.2$ **d** $\sin\theta = 0.1$

3 Decide which trigonometric ratio (i.e. sine, cosine or tangent) would be used to find θ in these triangles.

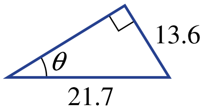
a



b



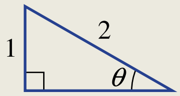
c



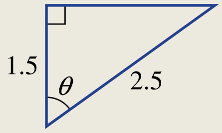
Example 3 Finding angles

Find the value of θ in the following right-angled triangles, rounding to two decimal places in part **b**.

a



b

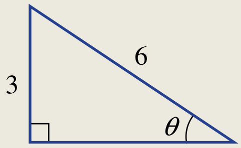


SOLUTION	EXPLANATION
a $\sin\theta = \frac{1}{2}$ $\therefore \theta = \sin^{-1}\left(\frac{1}{2}\right)$ $\theta = 30^\circ$	Use $\sin\theta$, as the opposite side and the hypotenuse are given. Use inverse sine on a calculator to find the angle.
b $\cos\theta = \frac{1.5}{2.5}$ $\therefore \theta = \cos^{-1}\left(\frac{1.5}{2.5}\right)$ $\theta = 53.13^\circ$ (to 2 d.p.)	The adjacent side and the hypotenuse are given, so use $\cos\theta$. Use inverse cosine on a calculator to find the angle and round your answer to two decimal places.

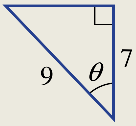
Now you try

Find the value of θ in the following right-angled triangles, rounding to two decimal places in part **b**.

a



b



Example 4 Working with simple trigonometric applications

A long, straight mine tunnel is sunk into the ground. Its final depth is 120 m and the end of the tunnel is 100 m horizontally from the ground entrance. Find the angle the tunnel makes with the horizontal (θ), correct to one decimal place.

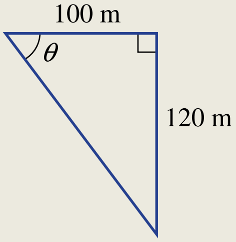
SOLUTION

$$\begin{aligned}\tan\theta &= \frac{120}{100}\\ \theta &= \tan^{-1}\left(\frac{120}{100}\right)\\ &= 50.2^\circ \text{ (to 1 d.p.)}\end{aligned}$$

$\therefore$ 50.2° is the angle the tunnel makes with the horizontal.

EXPLANATION

Start with a labelled diagram, using the given information.



Use $\tan\theta$ since the opposite and adjacent are known sides.

Use inverse tan on a calculator to find the required angle.

Now you try

A straight rabbit burrow is dug into the ground. Its final depth is 4 m and the end of the burrow is 5 m horizontally from the ground entrance. Find the angle the burrow makes with the horizontal (θ), correct to one decimal place.

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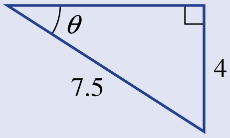
Using calculators in trigonometry

Find the value of the unknown in this triangle, correct to two decimal places.



Using the TI-Nspire:

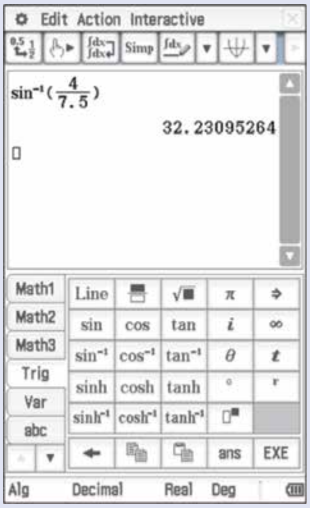
Use the inverse sine function in Degree mode.

$\sin^{-1}\left(\frac{4}{7.5}\right)$ 32.231

Hint: use the trig key to access $\sin^{-1}$.

Using the ClassPad:

Use the inverse sine function in **Decimal Degree** mode.



$\sin^{-1}\left(\frac{4}{7.5}\right)$

32.23095264

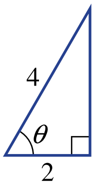
Exercise 6B

FLUENCY	1–2(½)	1–3(½)	1–3(⅓)

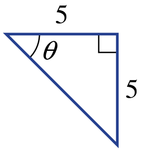
Example 3

1 Find the value of θ in the following right-angled triangles, rounding your answer to two decimal places where necessary.

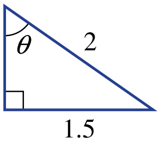
a



b



c



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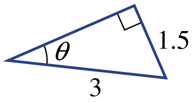

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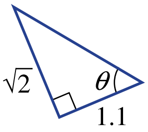

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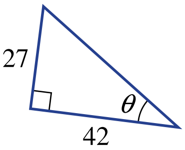
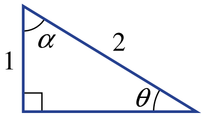


2 Find the value of α and θ, to one decimal place where necessary, for these special triangles.

a



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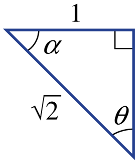

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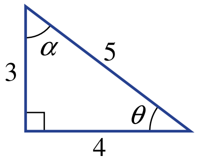

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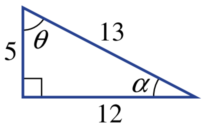

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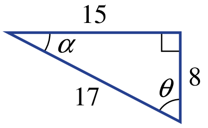

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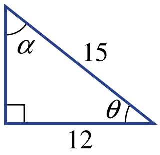


3 The lengths of two sides of a right-angled triangle are provided. Use this information to find the size of the two interior acute angles, and round each answer to one decimal place.

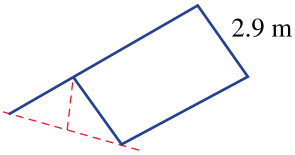
a Hypotenuse 5 cm, opposite 3.5 cm
b Hypotenuse 7.2 m, adjacent 1.9 m
c Hypotenuse 0.4 mm, adjacent 0.21 mm
d Opposite 2.3 km, adjacent 5.2 km
e Opposite 0.32 cm, adjacent 0.04 cm
f Opposite $\sqrt{5}$ cm, hypotenuse $\sqrt{11}$ cm

PROBLEM–SOLVING	4, 5	5, 6	6, 7

Example 4

4 A ladder reaches 5.5 m up a wall and sits 2 m from the base of the wall. Find the angle the ladder makes with the horizontal, correct to two decimal places.

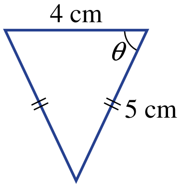
5 A tarpaulin with a simple A-frame design is set up as a shelter. The width of half of the tarpaulin is 2.9 metres, as shown. Find the angle to the ground that the sides of the tarpaulin make if the height at the middle of the shelter is 1.5 metres. Round your answer to the nearest 0.1 of a degree.



6 A diagonal cut of length 2.85 metres is to be made on a rectangular wooden slab from one corner to the other. The front of the slab measures 1.94 metres. Calculate the angle with the front edge at which the carpenter needs to begin the cut. Round your answer to one decimal place.

7 Find the value of θ in these diagrams, correct to one decimal place.

a



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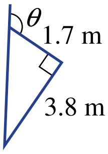

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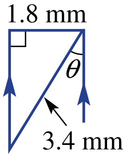


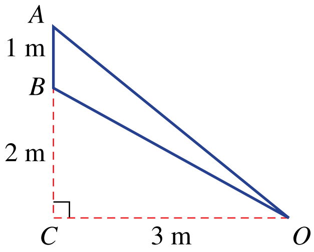
REASONING	8	8, 9	9, 10

8 Consider ΔOAC and ΔOBC.

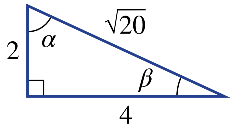
a Find, correct to one decimal place where necessary:

i $\angle AOC$
ii $\angle BOC$

b Hence, find the angle $\angle AOB$.



9 This triangle includes the unknown angles α and β.

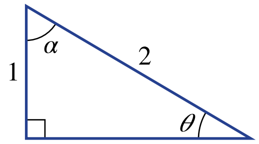


- **a** Explain why only one inverse trigonometric ratio needs to be used to find the values of both α and β.
- **b** Find α and β, correct to one decimal place, using your method from part **a**.

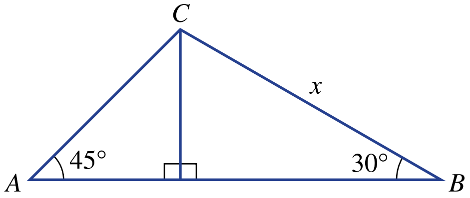
10
- **a** Draw a right-angled isosceles triangle and show all the internal angles.
- **b** If one of the shorter sides is of length x, show that $\tan 45° = 1$.
- **c** Find the exact length of the hypotenuse in terms of x.
- **d** Show that $\sin 45° = \cos 45°$.

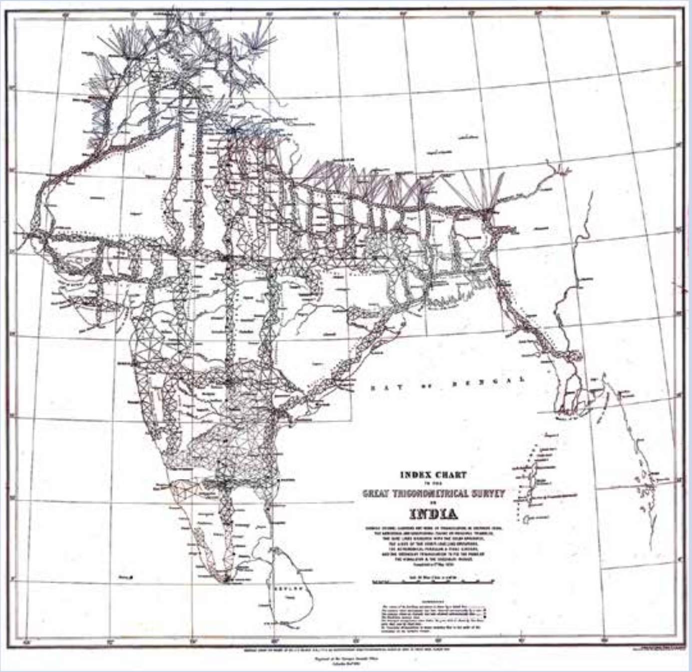
ENRICHMENT: A special triangle	–	–	11

11 Consider this special triangle.



- **a** Find the value of θ.
- **b** Find the value of α.
- **c** Use Pythagoras' theorem to find the exact length of the unknown side, in surd form.
- **d** Hence, write down the exact value for the following, in surd form.
  - **i** $\sin 30°$ **ii** $\cos 60°$ **iii** $\sin 60°$ **iv** $\cos 30°$ **v** $\tan 30°$ **vi** $\tan 60°$
- **e** For the diagram on the right, show that $AB = \left(\dfrac{\sqrt{3}+1}{2}\right)x$.





A map showing the triangles and transects used in the Great Trigonometric Survey of India, produced in 1870

6C Applications in two dimensions

LEARNING INTENTIONS
- To know how angles of elevation and depression are measured
- To be able to draw and label an appropriate diagram from a word problem description and identify a right-angled triangle
- To know how to apply the correct trigonometric relationship to solve a problem

There are many situations where a two-dimensional right-angled triangle can be drawn so that trigonometry can be used to solve a problem. An angle of elevation or depression is commonly used in such triangles.

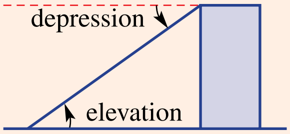
Pilots are trained in trigonometry. Starting the final descent, a pilot will check that the plane's altitude and its horizontal distance from the runway allow for the required angle of descent (i.e. depression) of 3° below the horizontal.

Lesson starter: Mountain peaks

Two mountain peaks in Victoria are Mt Stirling (1749 m) and Mt Buller (1805 m). A map shows a horizontal distance between them of 6.8 km.

- Discuss if you think there is enough information to find the angle of elevation of Mt Buller from Mt Stirling.
- What diagram can be used to summarise the information?
- Show how trigonometry can be used to find this angle of elevation.
- Discuss what is meant by the words *elevation* and *depression* in this context.

KEY IDEAS



- The **angle of elevation** is measured *up* from the horizontal.
- The **angle of depression** is measured *down* from the horizontal.
  - On the same diagram, the angle of elevation and the angle of depression are equal. They are alternate angles in parallel lines.
- To solve more complex problems involving trigonometry:
  - Visualise and draw a right-angled triangle and add any given information.
  - Use a trigonometric ratio to find the unknown.
  - Answer the question in words.

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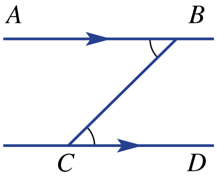
BUILDING UNDERSTANDING

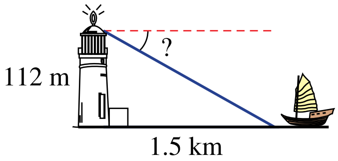
1 Name the two marked angles which are equal in this diagram.



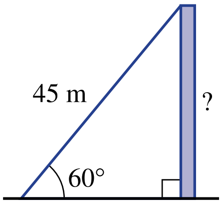
2 Choose the diagram (A, B or C) which matches the description (a, b or c). (Do not try to find the answer.)

- a A cable of length 45 metres is anchored from the ground to the top of a communications mast. The angle of elevation of the cable to the top of the mast is 60°. Find the height of the communications mast.
- b The angle of depression from the top of a 25 metre tall viewing tower to a crocodile on the ground is 62°. Find the direct distance from the top of the tower to the crocodile.
- c Find the angle of depression from a lighthouse beacon that is 112 metres above sea level to a boat that is at a horizontal distance of 1.5 kilometres from the lighthouse.

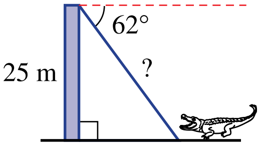
A



B

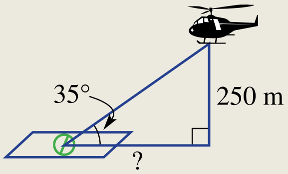


C



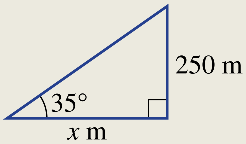
Example 5 Working with an angle of elevation

A helicopter is hovering at an altitude of 250 metres. The angle of elevation from the helipad to the helicopter is 35°. Find the horizontal distance of the helicopter from the helipad, to the nearest centimetre.



SOLUTION

Let x metres be the horizontal distance from the helicopter to the helipad.



$$\begin{aligned} \tan 35^\circ &= \frac{250}{x} \\ \therefore x \times \tan 35^\circ &= 250 \\ x &= \frac{250}{\tan 35^\circ} \\ &= 357.04 \text{ (to nearest cm)} \end{aligned}$$

The horizontal distance from the helicopter to the helipad is 357.04 m.

EXPLANATION

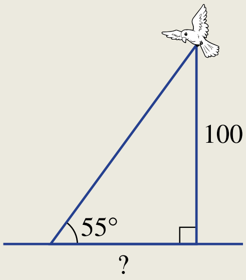
Use $\tan\theta = \frac{\text{O}}{\text{A}}$ since the opposite and adjacent sides are being used. Solve for x.

There are 100 cm in 1 m, so round to two decimal places for the nearest centimetre.

Answer the question in words.

Now you try

A bird is hovering at an altitude of 100 m. The angle of elevation from the observation point to the bird is 55°. Find the horizontal distance of the bird from the observation point, to the nearest centimetre.



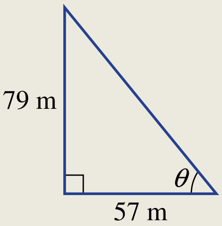
Example 6 Working with an angle of depression

Two vertical buildings 57 metres apart are 158 metres and 237 metres high. Find the angle of depression from the top of the taller building to the top of the shorter building, correct to two decimal places.

SOLUTION

Let θ be the angle of depression from the top of the taller building to the top of the shorter building.

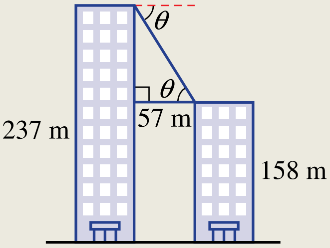
$$\begin{aligned}\text{Height difference} &= 237 - 158\\ &= 79\text{ m}\end{aligned}$$



$$\begin{aligned}\tan\theta &= \frac{79}{57}\\ \theta &= \tan^{-1}\left(\frac{79}{57}\right)\\ &= 54.19^\circ \text{ (to 2 d.p.)}\end{aligned}$$

The angle of depression from the top of the taller building to the top of the shorter building is 54.19°.

EXPLANATION



The angle of depression is below the horizontal, alternate angles can be used to mark the angle inside the triangle formed.

Draw the relevant right-angled triangle separately. We are given the opposite (O) and the adjacent (A) sides; hence, use tan.

Use the inverse tan function to find θ, correct to two decimal places.

Answer the question in words.

Now you try

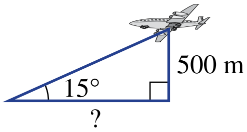
Two vertical poles 32 metres apart are 62 metres and 79 metres high. Find the angle of depression from the top of the taller pole to the top of the shorter pole, correct to two decimal places.

Exercise 6C

FLUENCY	1–5	1–5	2, 4–6

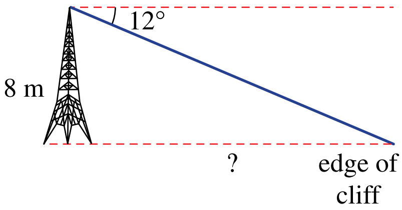
Example 5

1 The altitude of an aeroplane is 500 metres, and the angle of elevation from the runway to the aeroplane is 15°. Find the horizontal distance from the aeroplane to the runway, to the nearest centimetre.



2 The height of a building is 46 m. The angle of elevation from the street corner to the top of the building is 62°. What is the direct distance from the street corner to the top of the building, to the nearest metre?

3 The angle of depression from the top of a tower to the edge of a cliff is 12°. If the tower is 8 m high, how far is the edge of the cliff from the base of the tower correct to one decimal place?



4 The angle of depression from one mountain summit to another is 15.9°. If the two mountains differ in height by 430 metres, find the horizontal distance between the two summits, to the nearest centimetre.

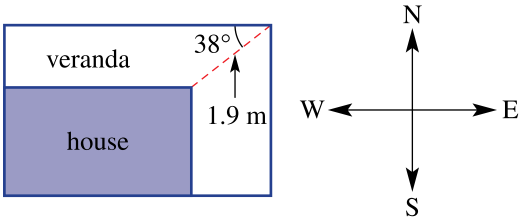
Example 6

5 Two vertical buildings positioned 91 metres apart are 136 metres and 192 metres tall, respectively. Find the angle of depression from the top of the taller building to the top of the shorter building, to the nearest degree.

6 An L-shaped veranda has dimensions as shown. Find the width, to the nearest centimetre, of the veranda for the following sides of the house:

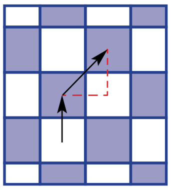
a north side

b east side.



PROBLEM–SOLVING	7	7, 8	8, 9

7 A knight on a chessboard is moved forward 3.6 cm from the centre of one square to another, then diagonally across at 45° to the centre of the destination square. How far did the knight move in total? Give your answer to two decimal places.



8 Two unidentified flying discs are detected by a receiver. The angle of elevation from the receiver to each disc is 39.48°. The discs are hovering at a direct distance of 826 m and 1.296 km from the receiver. Find the difference in height between the two unidentified flying discs, to the nearest metre.

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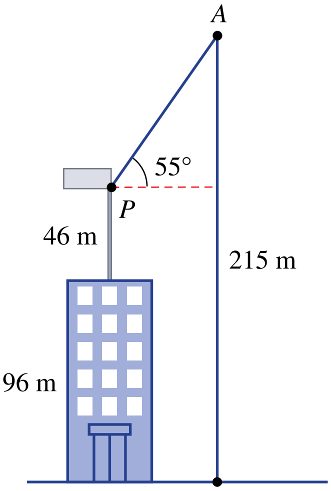
9 Initially a ship and a submarine are stationary at sea level, positioned 1.78 kilometres apart. The submarine then manoeuvres to position A, 45 metres directly below its starting point. In a second manoeuvre, the submarine dives a further 62 metres to position B. Give all answers to two decimal places.

- **a** Find the angle of elevation of the ship from the submarine when the submarine is at position A.
- **b** Find the angle of elevation of the ship from the submarine when the submarine is at position B.
- **c** Find the difference in the angles of elevation from the submarine to the ship when the submarine is at positions A and B.

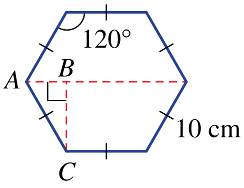
REASONING	10	10, 11	11–13

10 A communications technician claims that when the horizontal distance between two television antennas is less than 12 metres, then an interference problem will occur. The heights of two antennas above ground level are 7.5 metres and 13.9 metres, respectively, and the angle of elevation from the top of the shorter antenna to the top of the taller antenna is 29.5°. According to the technician's claim, will there be an interference problem for these two antennas?

11 The pivot point (P) of the main supporting arm (AP) of a construction crane is 46 metres above the top of a 96 metre tall office building. When the supporting arm is at an angle of 55° to the horizontal, the length of cable dropping from the point A to the ground is 215 metres. Find the length of the main supporting arm (AP), to the nearest centimetre.



12 Consider a regular hexagon with internal angles of 120° and side lengths of 10 cm.



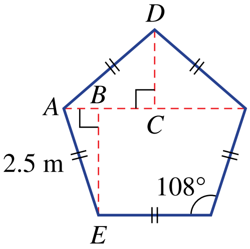
- **a** For the given diagram find, to the nearest millimetre, the lengths:
  - **i** BC
  - **ii** AB
- **b** Find the distance, to the nearest millimetre, between:
  - **i** two parallel sides
  - **ii** two opposite vertices.
- **c** Explore and describe how changing the side lengths of the hexagon changes the answers to part **b**.

13 An aeroplane is flying horizontally, directly towards the city of Melbourne at an altitude of 400 metres. At a given time the pilot views the city lights of Melbourne at an angle of depression of 1.5°. Two minutes later the angle of depression of the city lights is 5°. Find the speed of the aeroplane in km/h, correct to one decimal place.

ENRICHMENT: Vegetable garden design

–	–	14

14 A vegetable garden is to be built in the shape of a regular pentagon using redgum sleepers of length 2.5 metres, as shown. It is known that the internal angles of a regular pentagon are 108°.



a Find the size of the following angles.
  - **i** $\angle AEB$
  - **ii** $\angle EAB$
  - **iii** $\angle CAD$
  - **iv** $\angle ADC$

b Find these lengths, to two decimal places.
  - **i** AB
  - **ii** BE
  - **iii** AC
  - **iv** CD

c Find the distance between a vertex on the border of the vegetable garden and the centre of its opposite side, to two decimal places.

d Find the distance between any two non-adjacent vertices on the border of the vegetable garden, to two decimal places.

e Show that when the length of the redgum sleepers is x metres, the distance between a vertex and the centre of its opposite side of the vegetable garden will be $1.54x$ metres, using two decimal places.

6D Directions and bearings

LEARNING INTENTIONS
- To understand how true bearings are measured and written
- To be able to state a true bearing and its opposite direction from a diagram
- To be able to apply bearings in word problems using a diagram and trigonometry

True bearings are used to communicate direction and therefore are important in navigation. Ship and aeroplane pilots, bushwalkers and military personnel all use bearings to navigate and communicate direction.

Accurate navigation is vital to military personnel, ship and plane pilots, geologists and bushwalkers, who all use bearings and maps to navigate and communicate direction. GPS signals are weak, unreliable and not accurate enough for precise navigation.

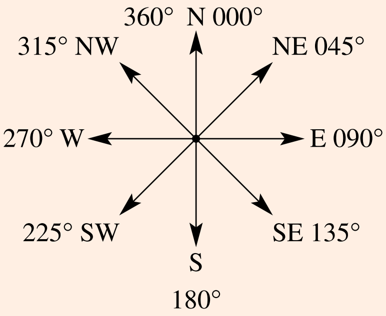
Lesson starter: Navigating a square

A mining surveyor starts walking from base camp to map out an area for soil testing. She starts by walking 2 km on a true bearing of 020° and wants to map out an area that is approximately square.

- Draw a diagram showing the first leg of the walk and the direction of north.
- If the surveyor turns right for the next leg, what will be the true bearing for this section?
- List the direction (as a true bearing) and the distance for all four legs of the walk. Remember that the mapped area must be a square.

KEY IDEAS

- **True bearings (°T)** are measured clockwise from due north. Some angles and directions are shown in this diagram; for example, NE means north-east.
  - True bearings are usually written using three digits.
  - Opposite directions differ by 180°.



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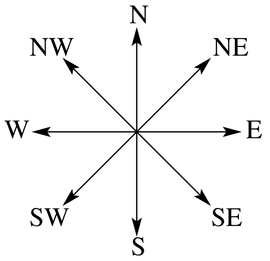
BUILDING UNDERSTANDING

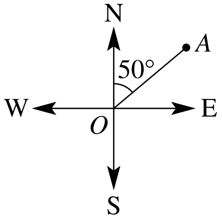
1 Give the true bearing for each of these directions.

a N **b** NE
c E **d** SE
e S **f** SW
g W **h** NW

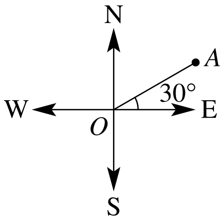


2 For each diagram, give the true bearing from O to A.

a



b



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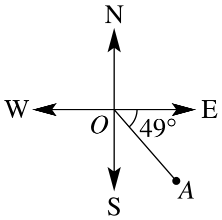


3 State the bearing that is the opposite direction to the following.

a 020°T **b** 262°T **c** 155°T **d** 344°T

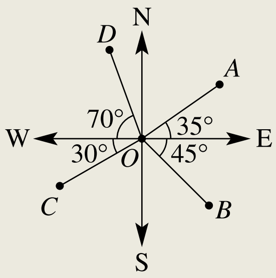
Example 7 Stating a direction

A, B, C and D are four points, as shown.

a Give the true bearing of each point from the origin, O, in this diagram.

b Give the true bearing of:

i O from A **ii** O from D.



SOLUTION

a The bearing of A is $90° - 35° = 055°$T.

The bearing of B is $90° + 45° = 135°$T.

The bearing of C is $270° - 30° = 240°$T.

The bearing of D is $270° + 70° = 340°$T.

b **i** The bearing of O from A is $180° + 55° = 235°$T.

ii The bearing of O from D is $340° - 180° = 160°$T.

EXPLANATION

East is 090° so subtract 35° from 90°.

B is 90° plus the additional 45° in a clockwise direction.

West is 270° so subtract 30° from 270°.

Alternatively for D, subtract 20° from 360°.

The bearing of A from O is 055°T and an opposite direction differs by 180°.

Subtract 180° from the opposite direction (340°T), so that the bearing is between 000°T and 360°T.

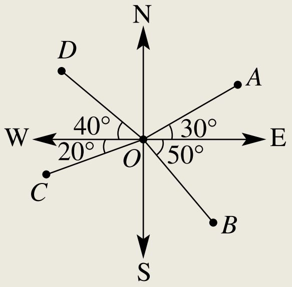
Now you try

A, B, C and D are four points, as shown.

a Give the true bearing of each point from the origin, O, in this diagram.

b Give the true bearing of:

i O from A **ii** O from D.



Example 8 Using bearings with trigonometry

A ship travels due south for 5 km, then on a true bearing of 120° for 11 km.

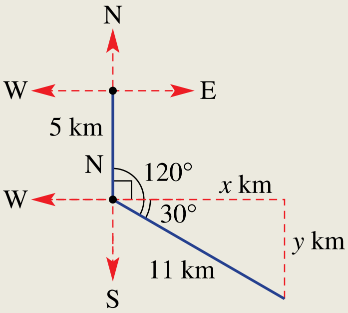
a Find how far east the ship is from its starting point, correct to two decimal places.

b Find how far south the ship is from its starting point.

SOLUTION

EXPLANATION

a



Draw a clear diagram, labelling all relevant angles and lengths. Draw a compass at each change of direction. Clearly show a right-angled triangle, which will help to solve the problem.

$$\cos 30^\circ = \frac{x}{11}$$

$$x = 11 \times \cos 30^\circ$$
$$= 9.53 \text{ (to 2 d.p.)}$$

As x is adjacent to 30° and the hypotenuse has length 11 km, use cosine.

The ship is 9.53 km east of its initial position.

Answer in words.

b $$\sin 30^\circ = \frac{y}{11}$$

$$y = 11 \times \sin 30^\circ$$
$$= 5.5$$

Use sine for opposite and hypotenuse. Use the value provided (11) rather than your answer from part **a**.

Multiply both sides by 11.

Distance south $= 5.5 + 5 = 10.5$ km

The ship is 10.5 km south of its initial position.

Find total distance south by adding the initial 5 km. Answer in words.

Now you try

A ship travels due south for 8 km, then on a bearing of 160° for 12 km.

a Find how far east the ship is from its starting point, correct to two decimal places.

b Find how far south the ship is from its starting point, correct to two decimal places.

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Exercise 6D

FLUENCY	1–5	1, 2, 4–6	2, 4, 5, 7

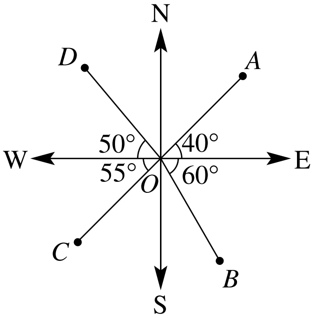
Example 7

1 A, B, C and D are four points, as shown.

a Give the true bearing of each point from the origin, O, in this diagram.

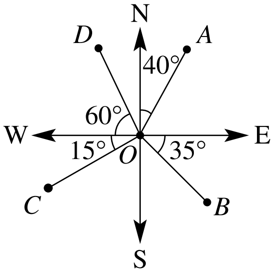
b Give the true bearing of:

i O from A **ii** O from D.



2 Find the true bearing of O from each of the following points, shown in this simple map. (*Hint*: First, find the bearing of each point from O.)

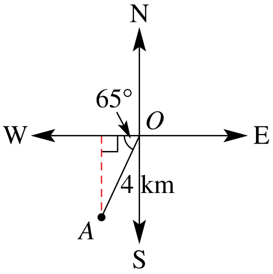
a A **b** B **c** C **d** D



3 For this simple map, find the following, correct to one decimal place.

a How far west is point A from O?

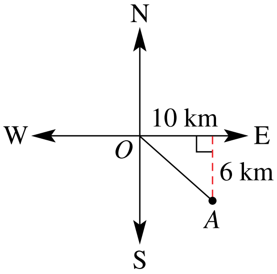
b How far south is point A from O?



4 Find the true bearing, correct to the nearest degree, of:

a point A from O

b point O from A.



Example 8

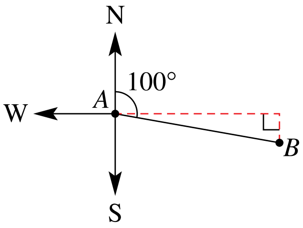
5 A ship travels due south for 3 km, then on a true bearing of 130° for 5 km.

a Find how far east the ship is from its starting point, correct to two decimal places.

b Find how far south the ship is from its starting point, correct to two decimal places.

6 Two points, A and B, positioned 15 cm apart, are such that B is on a true bearing of 100° from A.

- **a** Find how far east point B is from A, correct to two decimal places.
- **b** Find how far south point B is from A, correct to the nearest millimetre.



7 An aeroplane flies 138 km in a southerly direction from a military air base to a drop-off point. The drop-off point is 83 km west of the air base. Find the true bearing, correct to the nearest degree, of:

- **a** the drop-off point from the air base
- **b** the air base from the drop-off point.

PROBLEM–SOLVING	8, 9	8–10	10, 11

8 A bushwalker hikes due north from a resting place for 1.5 km to a waterhole and then on a true bearing of 315° for 2 km to base camp.

- **a** Find how far west the base camp is from the waterhole, to the nearest metre.
- **b** Find how far north the base camp is from the waterhole, to the nearest metre.
- **c** Find how far north the base camp is from the initial resting place, to the nearest metre.

9 On a map, point C is 4.3 km due east of point B, whereas point B is 2.7 km on a true bearing of 143° from point A. Give your answer to two decimal places for the following.

- **a** Find how far east point B is from A.
- **b** Find how far east point C is from A.
- **c** Find how far south point C is from A.

10 A military desert tank manoeuvres 13.5 km from point A on a true bearing of 042° to point B. From point B, how far due south must the tank travel to be at a point due east of point A? Give the answer correct to the nearest metre.

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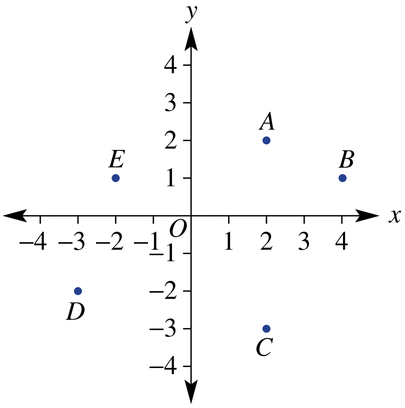
11 Consider the points O, A, B, C, D and E on this Cartesian plane. Round the answers to one decimal place.

a Find the true bearing of:

- **i** A from O
- **ii** D from O
- **iii** B from C
- **iv** E from C.

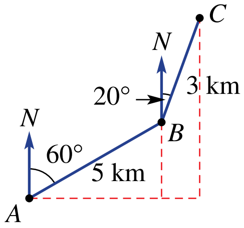
b Find the true bearing from:

- **i** O to E
- **ii** A to B
- **iii** D to C
- **iv** B to D.



REASONING	12	12, 13	12–14

12 An overall direction and distance of a journey can be calculated by considering two (or more) smaller parts (or legs). Find the bearing of C from A and the length AC in this journey by answering these parts.



a Find, correct to two decimal places where necessary, how far north:

- **i** point B is from A
- **ii** point C is from B
- **iii** point C is from A.

b Find, correct to two decimal places, how far east:

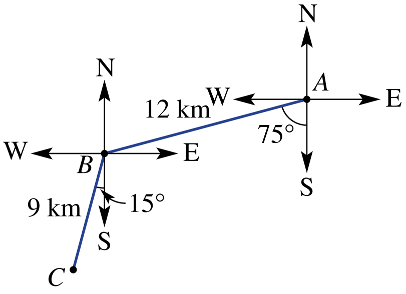
- **i** point B is from A
- **ii** point C is from B
- **iii** point C is from A.

c Now use your answers above to find the following, correct to one decimal place.

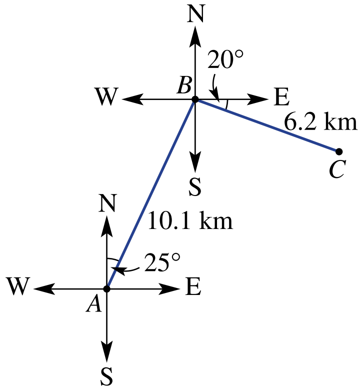
- **i** the true bearing of C from A
- **ii** the distance from A to C. (*Hint*: Use Pythagoras' theorem.)

13 Use the technique outlined in Question **12** to find the distance AC and the bearing of C from A in these diagrams. Give your answers correct to one decimal place.

a



b



14 Tour groups A and B view a rock feature from different positions on a road heading east–west.

Group A views the rock at a distance of 235 m on a bearing of 155° and group B views the rock feature on a bearing of 162° at a different point on the road. Round all answers to two decimal places in the following.

- **a** Find how far south the rock feature is from the road.
- **b** Find how far east the rock feature is from:
  - **i** group A
  - **ii** group B.
- **c** Find the distance between group A and group B.

ENRICHMENT: Navigation challenges	–	–	15, 16

15 A light aeroplane is flown from a farm airstrip to a city runway that is 135 km away. The city runway is due north from the farm airstrip. To avoid a storm, the pilot flies the aeroplane on a bearing of 310° for 50 km, and then due north for 45 km. The pilot then heads directly to the city runway. Round your answers to two decimal places in the following.

- **a** Find how far west the aeroplane diverged from the direct line between the farm airstrip and the city runway.
- **b** Find how far south the aeroplane was from the city runway before heading directly to the city runway on the final leg of the flight.
- **c** Find the bearing the aeroplane was flying on when it flew on the final leg of the flight.

16 A racing yacht sails from the start position to a floating marker on a bearing of 205.2° for 2.82 km, then to a finish line on a bearing of 205.9° for 1.99 km. Round each of the following to two decimal places.

- **a** Find how far south the finish line is from the start position.
- **b** Find how far west the finish line is from the start position.
- **c** Use Pythagoras’ theorem to find the distance between the finish line and the start position.

6E Applications in three dimensions

LEARNING INTENTIONS

- To be able to visualise right-angled triangles in 3D objects
- To be able to draw and label right-angled triangles formed in 3D objects
- To know how to apply the trigonometric ratios to find an unknown and relate this to the original 3D object

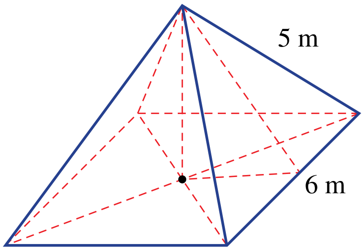
Although a right-angled triangle is a two-dimensional shape, it can also be used to solve problems in three dimensions. Being able to visualise right-angled triangles included in three-dimensional diagrams is an important part of the process of finding angles and lengths associated with three-dimensional objects.

Surveyors use trigonometry to calculate distances and angles between points in three dimensions. Surveyors accurately locate corners for new buildings, boundaries of property for legal ownership, and the placement of roads, bridges, dams, water pipes, power pylons, etc.

Lesson starter: How many right-angled triangles?

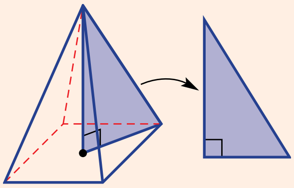
A right square-based pyramid has the apex above the centre of the base. In this example, the base length is 6 m and the slant height is 5 m. Other important lines are dashed.

- Using the given dashed lines and the edges of the pyramid, how many different right-angled triangles can you draw?
- Is it possible to determine the exact side lengths of all your right-angled triangles?
- Is it possible to determine all the angles inside all your right-angled triangles?



KEY IDEAS

- Using trigonometry to solve problems in three dimensions involves:
  - visualising and drawing any relevant two-dimensional triangles
  - using trigonometric ratios to find unknowns
  - relating answers from two-dimensional diagrams to the original three-dimensional object.



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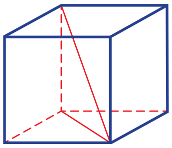
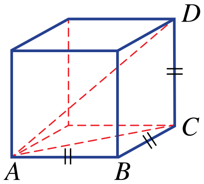
BUILDING UNDERSTANDING

1. By considering only the lines drawn inside this rectangular prism, how many right-angled triangles are formed?



2. The cube shown here has side length 2 m.
   - **a** Draw the right-angled triangle *ABC* and find and label all the side lengths. Pythagoras' theorem can be used. Answer using exact values (e.g. $\sqrt{5}$).
   - **b** Draw the right-angled triangle *ACD* and find and label all the side lengths. Pythagoras' theorem can be used. Answer using exact values.
   - **c** Use trigonometry to find ∠*DAC*, correct to one decimal place.
   - **d** Find the size of ∠*CAB*.

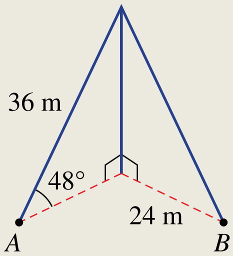


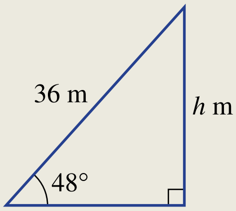
Example 9 Applying trigonometry in 3D

A vertical mast is supported at the top by two cables reaching from two points, *A* and *B*. The cable reaching from point *A* is 36 metres long and is at an angle of 48° to the horizontal. Point *B* is 24 metres from the base of the mast.

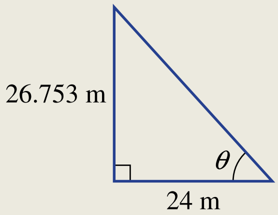
a Find the height of the mast, correct to three decimal places.

b Find the angle to the horizontal of the cable reaching from point *B*, to two decimal places.



SOLUTION	EXPLANATION
a Let h be the height of the mast, in metres.	First, draw the right-angled triangle, showing the information given.
	
$\sin 48^\circ = \frac{h}{36}$	The opposite (O) and hypotenuse (H) are given, so use sine.
$h = 36 \times \sin 48^\circ$ $= 26.753$ (to 3 d.p.)	Multiply both sides by 36 and round to three decimal places.
The height of the mast is 26.753 m.	Answer the question in words.

b



Draw the second triangle, including the answer from part **a**.

$$\tan\theta = \frac{26.753\ldots}{24}$$
$$\theta = \tan^{-1}\left(\frac{26.753\ldots}{24}\right)$$
$$= 48.11^\circ \text{ (to 2 d.p.)}$$

More precisely, use the height of the mast as $36 \times \sin 48^\circ$.
So $\theta = \tan^{-1}\left(\frac{36 \times \sin 48^\circ}{24}\right)$

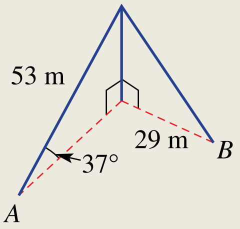
The cable reaching from point *B* is at an angle of 48.11° to the horizontal.

Answer the question in words, rounding your answer appropriately.

Now you try

A vertical mast is supported at the top by two cables reaching from two points, *A* and *B*. The cable reaching from point *A* is 53 metres long and is at an angle of 37° to the horizontal. Point *B* is 29 metres from the base of the mast.

a Find the height of the mast, correct to three decimal places.
b Find the angle to the horizontal of the cable reaching from point *B*, to two decimal places.

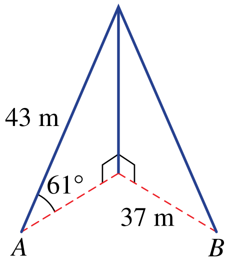


Exercise 6E

FLUENCY 1–4 | 1, 2, 4, 5 | 3–5

Example 9

1 A vertical mast is supported at the top by two cables reaching from two points, *A* and *B*. The cable reaching from point *A* is 43 metres long and is at an angle of 61° to the horizontal. Point *B* is 37 metres from the base of the mast.
a Find the height of the mast, correct to three decimal places.
b Find the angle to the horizontal of the cable reaching from point *B*, to two decimal places.

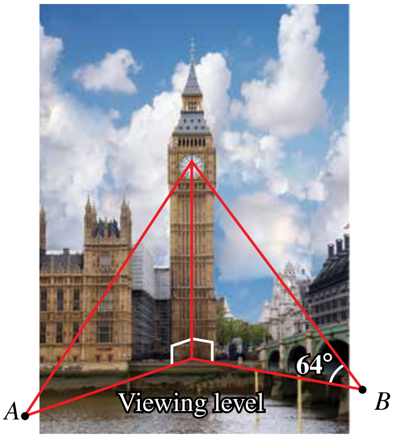


2 A vertical tent pole is supported at the top by two ropes reaching from two pegs, *A* and *B*. The rope reaching from peg *A* is 3 m long and is at an angle of 39° to the horizontal. Peg *B* is 2 m from the base of the pole.
a Find the height of the pole correct to three decimal places.
b Find the angle to the horizontal of the cable reaching from peg *B*, to two decimal places.

3 Viewing points A and B are at a horizontal distance from a clock tower of 36 metres and 28 metres, respectively. The viewing angle to the clockface at point B is 64°.

a Find the height of the clockface above the viewing level, to three decimal places.

b Find the viewing angle to the clockface at point A, to two decimal places.

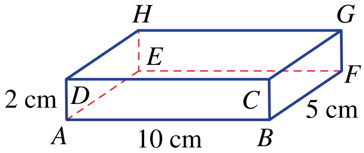


4 A rectangular prism, $ABCDEFGH$, is 5 cm wide, 10 cm long and 2 cm high.

a By drawing the triangle ABF find, to two decimal places:

- **i** $\angle BAF$
- **ii** AF

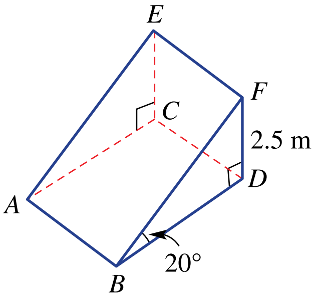
b By drawing the triangle AGF, find $\angle GAF$, to two decimal places.



5 A ramp, $ABCDEF$, rests at an angle of 20° to the horizontal and the highest point on the ramp is 2.5 metres above the ground, as shown. Give your answers to two decimal places in the following questions.

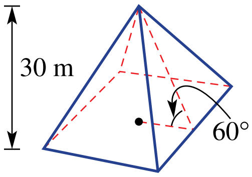
a Find the length of the ramp BF.

b Find the length of the horizontal BD.



PROBLEM–SOLVING 6, 7 6–8 7–9

6 The triangular faces of a right square-based pyramid are at an angle of 60° to the base. The height of the pyramid is 30 m. Find the perimeter of the base of the pyramid, correct to one decimal place.



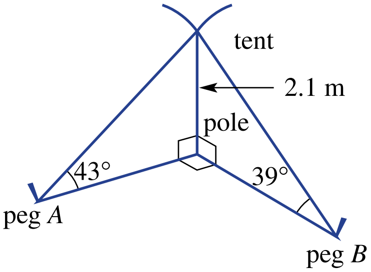
7 A tent pole 2.1 metres tall is secured by ropes in two directions. The ropes are held by pegs A and B at angles of 43° and 39°, respectively, from the horizontal. The line from the base of the pole to peg A is at right angles to the line from the base of the pole to peg B. Round your answers to two decimal places in these questions.

a Find the distance from the base of the tent pole to:

- **i** peg A
- **ii** peg B.

b Find the angle at peg A formed by peg A, peg B and the base of the pole.

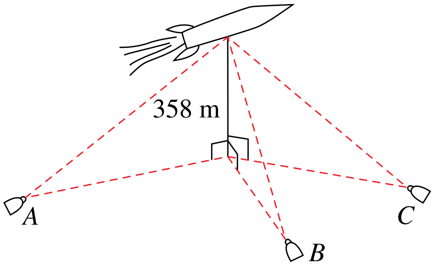
c Find the distance between peg A and peg B.



8 The communities of Wood Town and Green Village live in a valley. Communication between the two communities is enhanced by a repeater station on the summit of a nearby mountain. It is known that the angles of depression from the repeater station to Wood Town and Green Village are 44.6° and 58.2°, respectively. Also, the horizontal distances from the repeater to Wood Town and Green Village are 1.35 km and 1.04 km, respectively.

- **a** Find the vertical height, to the nearest metre, between the repeater station and:
  - **i** Wood Town
  - **ii** Green Village.
- **b** Find the difference in height between the two communities, to the nearest metre.

9 Three cameras operated at ground level view a rocket being launched into space.



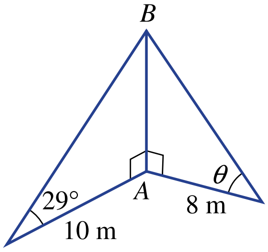
At 5 seconds immediately after launch, the rocket is 358 m above ground level and the three cameras, A, B and C, are positioned at an angle of 28°, 32° and 36°, respectively, to the horizontal.

At the 5 second mark, find:

- **a** which camera is closest to the rocket
- **b** the distance between the rocket and the closest camera, to the nearest centimetre.

REASONING	10	10	10, 11

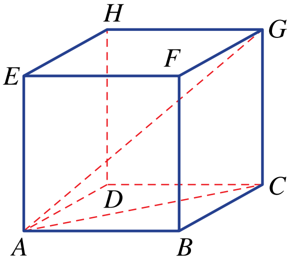
10 It is important to use a high degree of accuracy for calculations that involve multiple parts.



For this 3D diagram complete these steps.

- **a** Find AB, correct to one decimal place.
- **b** Use your answer from part **a** to find θ, correct to one decimal place.
- **c** Now recalculate θ using a more accurate value for AB. Round θ to one decimal place.
- **d** What is the difference between the answers for parts **b** and **c**?

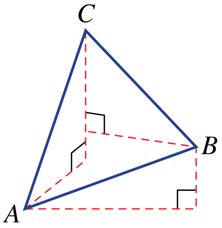
11 For a cube, $ABCDEFGH$, of side length 1 unit, as shown, use trigonometry to find the following, correct to two decimal places where necessary. Be careful that errors do not accumulate.



- **a** $\angle BAC$
- **b** AC
- **c** $\angle CAG$
- **d** AG

ENRICHMENT: Three points in 3D

–	–	12, 13



12 Three points, A, B and C, in three-dimensional space are such that $AB = 6$, $BC = 3$ and $AC = 5$.

The angles of elevation from A to B and from B to C are 15° and 25°, respectively. Round your answer to two decimal places in the following.

- **a** Find the vertical difference in height between:
  - **i** A and B
  - **ii** B and C
  - **iii** A and C.
- **b** Find the angle of elevation from A to C.

13 The points A, B and C in 3D space are such that:

- $AB = 10$ mm, $AC = 17$ mm and $BC = 28$ mm
- the angle of elevation from A to B is 20°
- the angle of elevation from A to C is 55°.

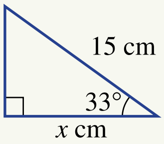
Find the angle of elevation from B to C, to the nearest degree.

Triangulation points or 'trig stations' such as this are used in geodetic surveying to mark points at which measurements are made to calculate local altitude. The calculations involved are similar to those in the Enrichment questions above.

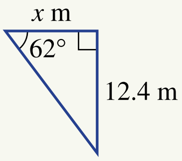
6A

1 Find the value of x in these right-angled triangles, rounding your answer to two decimal places.

a



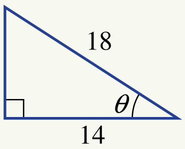
b



6B

2 Find the value of θ in the following right-angled triangles, correct to the nearest degree.

a



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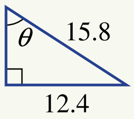


6B

3 At what angle to the horizontal must a 4.5 m ladder be placed against a wall if it must reach up to just below a window that is 4 m above the level ground? Round your answer to the nearest degree.

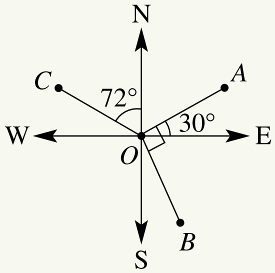
6C

4 The angle of depression from the top of a 20 m building to a worker standing on the ground below is 40°. Find the distance of the worker from the base of the building, correct to two decimal places.

6D

5 **a** Give the true bearing of A, B and C from the origin, O, in the given diagram.

b Give the true bearing of O from A.



6D

6 A man leaves camp C at 11 a.m. and walks 12 km on a true bearing of 200°. He then stops. A woman also leaves camp C at 11 a.m. However, she walks on a true bearing of 110° for 6.5 km before stopping.

a How far apart are the man and the woman once they stop? Give your answer correct to two decimal places.

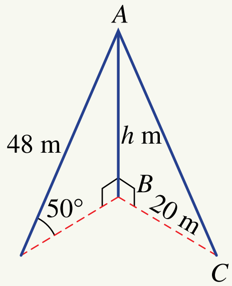
b If the man changes direction and walks to where the woman is waiting, on what bearing should he walk? Round your answer to one decimal place.

6E

7 Consider the given 3D diagram on the right.

a Find the value of h, correct to two decimal places.

b Find $\angle ACB$, to the nearest degree.

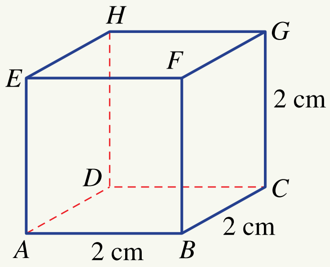


6E

8 A cube has vertices A, B, C, D, E, F, G and H and has side length 2 cm.

a Use Pythagoras' theorem to find the length AC as an exact value.

b Find the angle of elevation of the diagonal AG, i.e. find $\angle CAG$. Round to the nearest degree.



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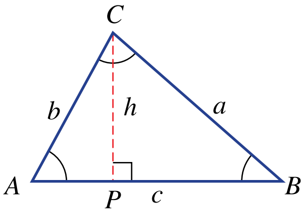
6F The sine rule EXTENDING

LEARNING INTENTIONS
- To know how the sine rule relates the ratio of sides and angles in non right-angled triangles
- To know the criteria of a triangle that must be known to apply the sine rule
- To know how to apply the sine rule to find an angle or a side length in non right-angled triangles
- To understand that some information of triangles can lead to two possible triangles involving either an acute angle or an obtuse angle
- To be able to use the sine rule to find acute or obtuse angles

The use of sine, cosine and tangent functions can be extended to non right-angled triangles.

First consider the triangle below with sides a, b and c and with opposite angles $\angle A$, $\angle B$ and $\angle C$. Height h is also shown.



$$\text{From } \Delta CPB,\ \sin B = \frac{h}{a}$$
$$\text{so} \quad h = a\sin B$$
$$\text{From } \Delta CPA,\ \sin A = \frac{h}{b}$$
$$\text{so} \quad h = b\sin A$$
$$\therefore a\sin B = b\sin A \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B}$$

Pilots need to compensate for cross-winds. In a triangle *ABC*, if *AB* shows a plane's speed and direction and *BC* the wind's speed and direction, then side *AC* gives the plane's resultant speed and direction, calculated using the sine and cosine rules.

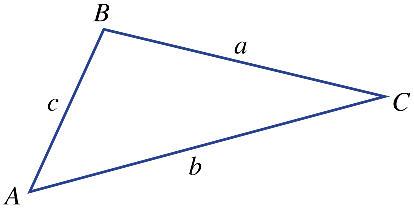
Similarly, it can be shown that $\frac{a}{\sin A} = \frac{c}{\sin C}$ and $\frac{b}{\sin B} = \frac{c}{\sin C}$.

In this section we will consider the sine of angles larger than 90°. This will be discussed in more detail in **Section 6I** but for the moment we will accept such angles with our trigonometric functions.

Lesson starter: Explore the sine rule

Use a ruler and a protractor to measure the side lengths (a, b and c) in centimetres, correct to one decimal place, and the angles (A, B and C), correct to the nearest degree, for this triangle. Dynamic geometry software could also be used for this activity.

- Calculate the following.
  a $\frac{a}{\sin A}$ **b** $\frac{b}{\sin B}$ **c** $\frac{c}{\sin C}$
- What do you notice about the three answers above?
- Draw your own triangle and check to see if your observations are consistent for any triangle.



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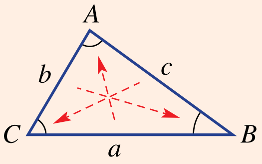
KEY IDEAS

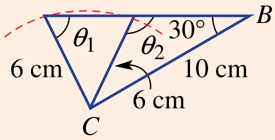
- When using the sine rule, label triangles with capital letters for vertices and the corresponding lower-case letter for the side opposite the angle.



- The **sine rule** states that the ratios of each side of a triangle to the sine of the opposite angle are equal.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{or} \qquad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

  - The sine rule holds true for both acute- and obtuse-angled triangles.
  - Use the sine rule when you know:
    - one side length and
    - the angle opposite that side length and
    - another side length or angle.

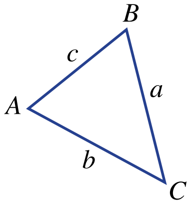


- The **ambiguous case** can arise when we are given two sides and an angle that is not the included angle.
  - This example shows a diagram with two given side lengths and one angle. Two triangles are possible.
  - Using $\frac{6}{\sin 30°} = \frac{10}{\sin \theta}$ could give two results for θ (i.e. θ_1 or θ_2). You will need to choose the correct angle (i.e. acute or obtuse) to suit your triangle (if known).
  - θ_1 and θ_2 are supplementary so to find the obtuse angle θ_2 use $\theta_2 = 180° - \theta_1$.

BUILDING UNDERSTANDING

1 State the missing parts of the sine rule for this triangle.

$$\frac{a}{} = \frac{}{\sin B} = \frac{c}{}$$



2 Solve each equation for a or b, correct to one decimal place.

a $\frac{a}{\sin 47°} = \frac{2}{\sin 51°}$

b $\frac{5}{\sin 63°} = \frac{b}{\sin 27°}$

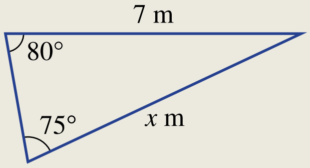
3 Find θ, correct to one decimal place, if θ is acute.

a $\frac{4}{\sin 38°} = \frac{5}{\sin \theta}$

b $\frac{1.2}{\sin \theta} = \frac{1.8}{\sin 47°}$

Example 10 Finding a side length using the sine rule

Find the value of x in this triangle, correct to one decimal place.



SOLUTION

$$\frac{x}{\sin 80^\circ} = \frac{7}{\sin 75^\circ}$$
$$x = \frac{7}{\sin 75^\circ} \times \sin 80^\circ$$
$$= 7.1 \text{ (to 1 d.p.)}$$

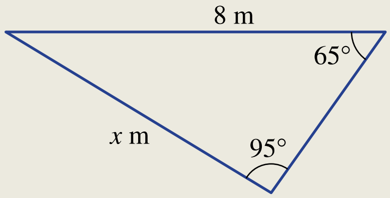
EXPLANATION

Use the sine rule $\frac{a}{\sin A} = \frac{b}{\sin B}$.

Multiply both sides by sin 80°.

Now you try

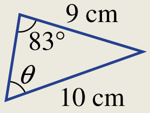
Find the value of x in this triangle, correct to one decimal place.



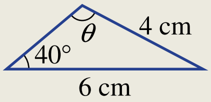
Example 11 Finding an angle using the sine rule

Find the value of θ in these triangles, correct to one decimal place.

a θ is acute.



b θ is obtuse.



SOLUTION

a
$$\frac{10}{\sin 83^\circ} = \frac{9}{\sin \theta}$$
$$10 \times \sin \theta = 9 \times \sin 83^\circ$$
$$\sin \theta = \frac{9 \times \sin 83^\circ}{10}$$
$$\theta = \sin^{-1}\left(\frac{9 \times \sin 83^\circ}{10}\right)$$
$$= 63.3^\circ \text{ (to 1 d.p.)}$$

EXPLANATION

Cross multiply and solve for $\sin \theta$ or alternatively, use

$\frac{\sin A}{a} = \frac{\sin B}{b}$: so, $\frac{\sin \theta}{9} = \frac{\sin 83^\circ}{10}$.

$\therefore \sin \theta = \frac{9 \times \sin 83^\circ}{10}$

Use $\sin^{-1}$ on your calculator to find the value of θ.

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b
$$\begin{aligned}\frac{4}{\sin 40^\circ} &= \frac{6}{\sin\theta}\\ 4\times\sin\theta &= 6\times\sin 40^\circ\\ \sin\theta &= \frac{6\times\sin 40^\circ}{4}\\ \theta &= \sin^{-1}\left(\frac{6\times\sin 40^\circ}{4}\right)\end{aligned}$$

$\theta = 74.6^\circ$ or $180^\circ - 74.6^\circ = 105.4^\circ$

θ is obtuse, so $\theta = 105.4^\circ$ (to 1 d.p.).

Cross multiply or alternatively, use $\frac{\sin\theta}{6} = \frac{\sin 40^\circ}{4}$.

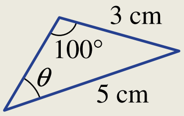
So, $\sin\theta = \frac{6\times\sin 40^\circ}{4}$.

This is an example of the ambiguous case of the sine rule but as θ is obtuse, you will need to choose the supplement of 74.6°.

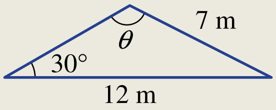
Now you try

Find the value of θ in this triangle, correct to one decimal place.

a θ is acute.



b θ is obtuse.



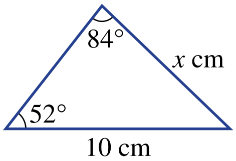
Exercise 6F

FLUENCY	1–2(½)	1–2(½)	1–2(⅓)

Example 10

1 Find the value of x in these triangles, correct to one decimal place.

a



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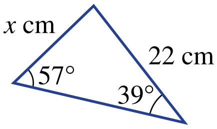

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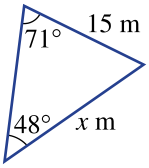

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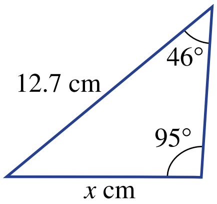

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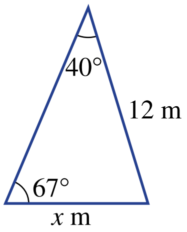

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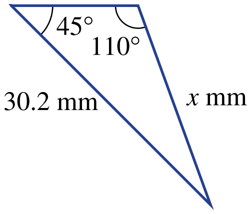
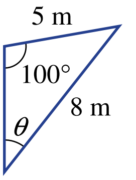


Example 11a

2 Find the value of θ, correct to one decimal place, if θ is acute.

a



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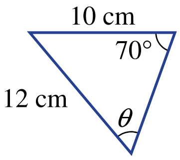

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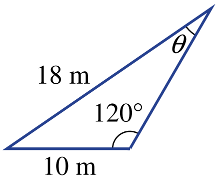

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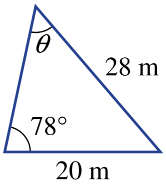

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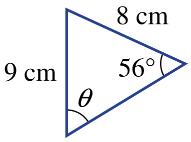

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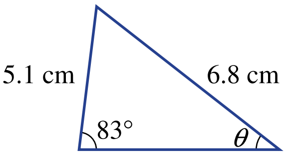
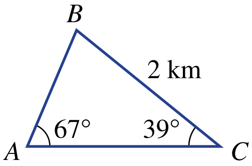


PROBLEM–SOLVING

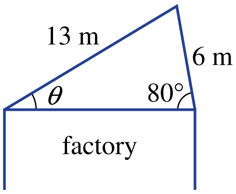
3–5 4–6 5–7

3 Three markers, A, B and C, map out the course for a cross-country race. The angles at A and C are 67° and 39°, respectively, and BC is 2 km.

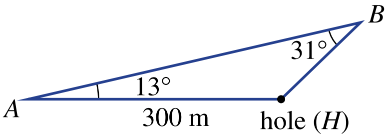
- **a** Find the length AB, correct to three decimal places.
- **b** Find the angle at B.
- **c** Find the length AC, correct to three decimal places.



4 A factory roof has a steep 6 m section at 80° to the horizontal and another 13 m section. What is the angle of elevation of the 13 m section of roof? Give your answer to one decimal place.

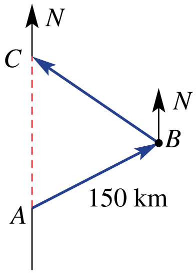


5 A golf ball is hit off-course by 13° to point B. The shortest distance to the hole is 300 m and the angle formed by the new ball position is 31°, as shown. Find the new distance to the hole (BH), correct to one decimal place.



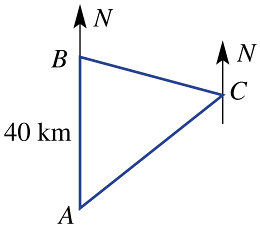
6 An aeroplane is flying due north but, to avoid a storm, it flies 150 km on a bearing of 060°T and then on a bearing of 320°T until it reaches its original course.

- **a** Find the angles $\angle ABC$ and $\angle ACB$.
- **b** As a result of the diversion, how much farther did the aeroplane have to fly? Round your answer to the nearest kilometre.



7 A ship heads due north from point A for 40 km to point B, and then heads on a true bearing of 100° to point C. The bearing from C to A is 240°.

- **a** Find $\angle ABC$.
- **b** Find the distance from A to C, correct to one decimal place.
- **c** Find the distance from B to C, correct to one decimal place.



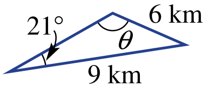
REASONING

REASONING	8(½)	8(½), 9	8(⅓), 9, 10

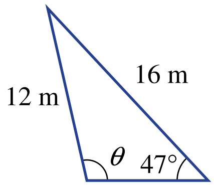
Example 11b

8 Find the value of θ, correct to one decimal place, if θ is obtuse.

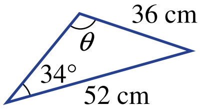
a



b



c



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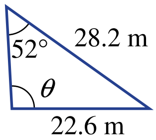

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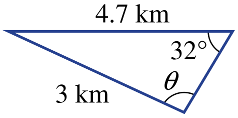

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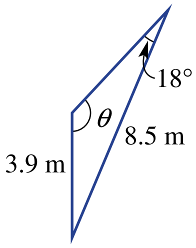
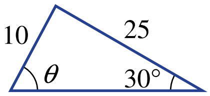


9 Try to find the angle θ in this triangle. What do you notice? Can you explain this result?



10 A triangle ABC has $\angle C = 25°$, $AC = 13$ cm and $AB = 9$ cm. Find all possible values of $\angle B$, correct to one decimal place.

ENRICHMENT: More on the ambiguous case

ENRICHMENT: More on the ambiguous case	–	–	11

11 When finding a missing angle θ in a triangle, the number of possible solutions for θ can be one or two, depending on the given information.

Two solutions: A triangle ABC has $AB = 3$ cm, $AC = 2$ cm and $\angle B = 35°$.

a Find the possible values of $\angle C$, correct to one decimal place.
b Draw a triangle for each angle for $\angle C$ in part **a**.

One solution: A triangle ABC has $AB = 6$ m, $AC = 10$ m and $\angle B = 120°$.

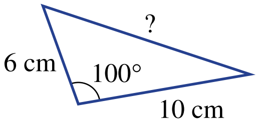
c Find the possible values of $\angle C$, correct to one decimal place.
d Explain why there is only one solution for $\angle C$ and not the extra supplementary angle, as in parts **a** and **b** above.
e Draw a triangle for your solution to part **c**. See the poster on the inside back cover of your book for more information about solving unfamiliar problems.

6G The cosine rule EXTENDING

LEARNING INTENTIONS

- To know that the cosine rule relates one angle and three sides of any triangle
- To be able to use the cosine rule to find any angle (given all three sides) or a third side of a triangle (given two sides and the included angle)

When a triangle is defined by two sides and the included angle, the sine rule is unhelpful in finding the length of the third side because at least one of the other two angles is needed.



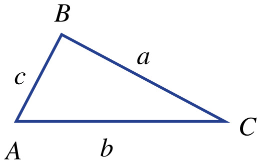
In such situations a new rule called the cosine rule can be used. It relates all three side lengths and the cosine of one angle. This means that the cosine rule can also be used to find an angle inside a triangle when given all three sides.

The proof of the cosine rule will be considered in the Enrichment question of this section.

Lesson starter: Cosine rule in three ways

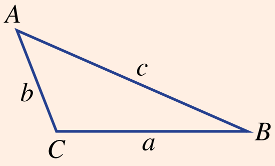
One way to write the cosine rule is like this:

$c^2 = a^2 + b^2 - 2ab\cos C$, where c^2 is the subject of the formula.



- Rewrite the cosine rule by replacing c with a, a with c and C with A.
- Rewrite the cosine rule by replacing c with b, b with c and C with B.

KEY IDEAS



- The **cosine rule** relates one angle and three sides of any triangle.
- The cosine rule is used to find:
  - the third side of a triangle when given two sides and the included angle
  - an angle when given three sides.

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \text{or} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

- If θ is obtuse, then note that $\cos\theta$ is negative. This will be discussed in more detail in **Section 6I**.

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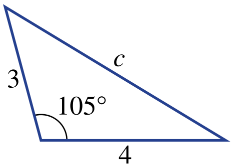
BUILDING UNDERSTANDING

1 State the missing parts to the cosine rule for each triangle.

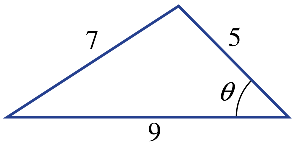
a $c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 3^2 + \_\_\_ - 2 \times \_\_\_ \times \_\_\_ \times \cos \_\_\_$



b $c^2 = a^2 + b^2 - 2ab\cos C$

$\_\_\_\_\_ = 5^2 + 9^2 - 2 \times \_\_\_\_\_ \times \_\_\_\_\_ \times \cos\theta$



2 Simplify and solve for the unknown (i.e. c or θ) in these equations, correct to one decimal place.

a $c^2 = 4^2 + 7^2 - 2 \times 4 \times 7 \times \cos 120°$

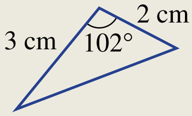
b $c^2 = 1.5^2 + 1.1^2 - 2 \times 1.5 \times 1.1 \times \cos 70°$

c $10^2 = 7^2 + 6^2 - 2 \times 7 \times 6 \times \cos\ \theta$

d $18^2 = 21^2 + 30^2 - 2 \times 21 \times 30 \times \cos\ \theta$

Example 12 Finding a side length using the cosine rule

Find the length of the third side in this triangle, correct to two decimal places.

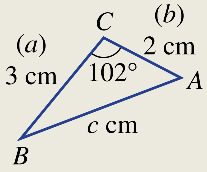


SOLUTION

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= 3^2 + 2^2 - 2(3)(2)\cos 10° \\ &= 13 - 12\cos 102° \\ &= 15.49494\ldots \\ \therefore c &= 3.94 \text{ (to 2 d.p.)} \end{aligned}$$

The length of the third side is 3.94 cm.

EXPLANATION

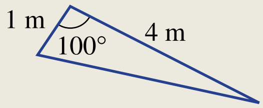


Let c be the length of the unknown side, so $a = 3$ and $b = 2$. Alternatively, let $b = 3$ and $a = 2$.

$c = \sqrt{15.494904\ldots}$

Now you try

Find the length of the third side in this triangle, correct to two decimal places.



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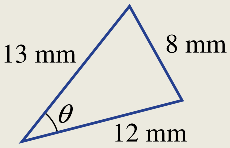
Example 13 Finding an angle using the cosine rule

Find the angle θ in this triangle, correct to two decimal places.



SOLUTION

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ 8^2 &= 13^2 + 12^2 - 2(13)(12)\cos\theta \\ 64 &= 313 - 312\cos\theta \\ 312\cos\theta &= 249 \\ \cos\theta &= \frac{249}{312} \\ \theta &= \cos^{-1}\left(\frac{249}{312}\right) \\ &= 37.05^\circ \text{ (to 2 d.p.)} \end{aligned}$$

EXPLANATION

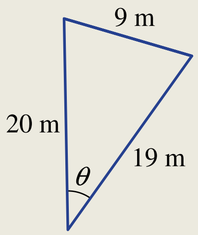
Choose θ to represent $\angle C$, so this makes $c = 8$.

Alternatively, use $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$ to give the same result.

$313 - 64 = 249$

Now you try

Find the angle θ in this triangle, correct to two decimal places.



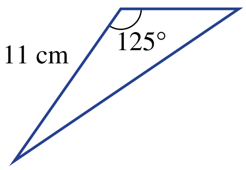
Exercise 6G

FLUENCY	1–2(½)	1–2(½)	1–2(⅓)

Example 12

1 Find the length of the third side, correct to two decimal places.

a



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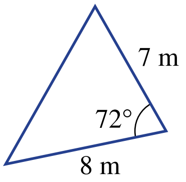

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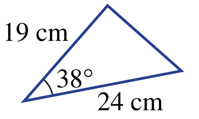

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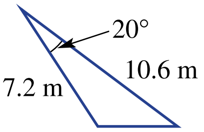

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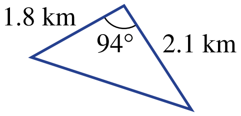

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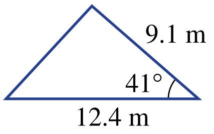
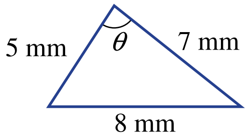


Example 13

2 Find the angle θ, correct to two decimal places.

a



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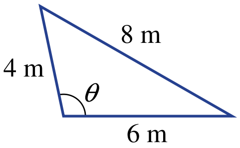

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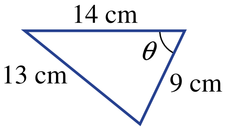

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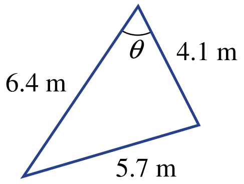

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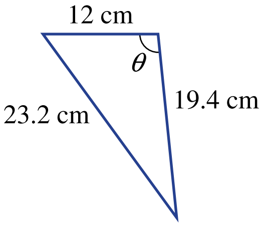

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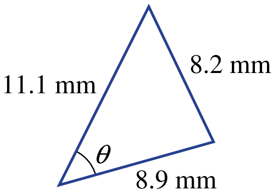


PROBLEM-SOLVING	3–5	4–6	5–7

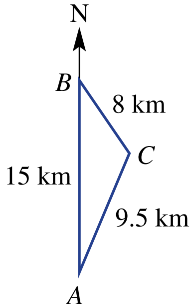
3 A triangular goat paddock has two sides of lengths 320 m and 170 m, and a 71° angle between them. Find the length of the third side, correct to the nearest metre.

4 Find the size of all three angles in a triangle that has side lengths 10 m, 7 m and 13 m. Round each angle to one decimal place.

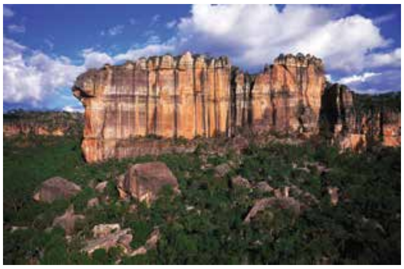
5 Three camp sites, A, B and C, are planned for a hike and the distances between the camp sites are 8 km, 15 km and 9.5 km, as shown. If camp site B is due north of camp site A, find the following, correct to one decimal place.

a The bearing from camp site B to camp site C.

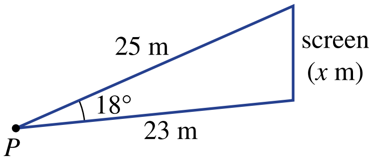
b The bearing from camp site C to camp site A.



6 A helicopter on a joy flight over Kakadu National Park travels due east for 125 km, then on a bearing of 215°T for 137 km before returning to its starting point. Find the total length of the journey, correct to the nearest kilometre.



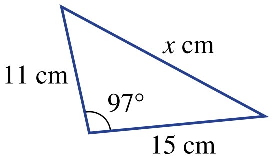
7 The viewing angle to a vertical screen is 18° and the distances between the viewing point, P, and the top and bottom of the screen are 25 m and 23 m, respectively. Find the height of the screen (x m), correct to the nearest centimetre.



REASONING	8(½)	8(½), 9	8(⅓), 9, 10

8 Decide whether the cosine rule or sine rule would be used to calculate the value of x in these triangles.

a



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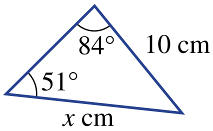

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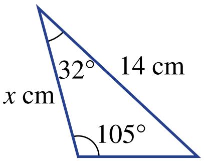

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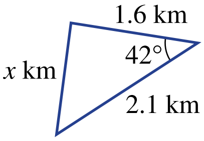

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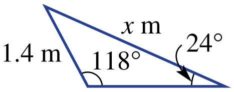

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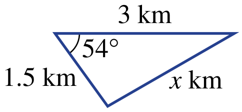
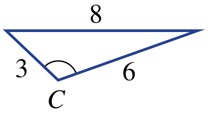


9 **a** Rearrange $c^2 = a^2 + b^2 - 2ab\cos C$ to make $\cos C$ the subject.

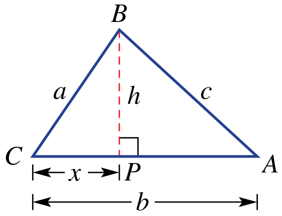
b Use your rule to find angle C in this triangle, correct to one decimal place.



10 A student uses the cosine rule to find an angle in a triangle and simplifies the equation to $\cos\theta = -0.17$. Is the triangle acute or obtuse? Give a reason.

ENRICHMENT: Proof of the cosine rule	–	–	11

11 Triangle ABC shown here includes point P such that $PB \perp CA$, $BP = h$ and $CP = x$.



a Write an expression for length AP.

b Use Pythagoras' theorem and ΔCBP to write an equation in a, x and h.

c Use Pythagoras' theorem and ΔAPB to write an equation in b, c, x and h.

d Combine your equations from parts **b** and **c** to eliminate h. Simplify your result.

e Use $\cos\theta = \dfrac{\text{Adjacent}}{\text{Hypotenuse}}$ to write an expression for $\cos C$.

f Combine your equations from parts **d** and **e** to prove $c^2 = a^2 + b^2 - 2ab\cos C$.

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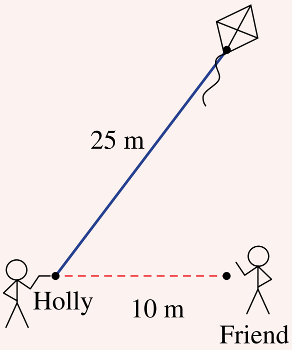
The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Flying a kite

1 Holly and her friend are flying a kite on a 25 m long string (pulled tight).

Holly and her friend are interested in the relationship between the angle of elevation and the height of the kite.



- **a** Holly's friend is standing 10 m away from her and notices the kite is directly overhead. Determine the angle of elevation of the kite from Holly's hand. Round to one decimal place.
- **b** The kite rises with a gush of wind and the angle of elevation becomes 75°. How high is the kite (vertically) above Holly's hand level correct to one decimal place?
- **c** The kite can range from being 0 m to 25 m above Holly's hand level. At what height is the kite when the angle of elevation is 45°? Round to one decimal place.
- **d** Determine the angle of elevation when the kite is 12.5 m above Holly's hand level.
- **e** If the string is x m long, find in terms of x:
  - **i** the height of the kite at an angle of elevation of 30°
  - **ii** the angle of elevation when the kite's vertical height is $\frac{3}{4}$ the length of the string. Round to one decimal place.

Off the beaten track

2 A walker travels off a straight-line track that runs east–west and then later returns to a point further west along the track.

The walker is interested in the relationship between the chosen bearing heading off the track and the return bearing as well as the distances that need to be walked in order to complete the journey.

- **a** Initially the walker takes a true bearing off the track at 200° for 5 km.
  - **i** To return to the track they travel on a true bearing of 340°. What distance do they need to walk to meet up with the track? Give reasons.
  - **ii** Another walker walks for x km from the track on a true bearing of 230°. They head back to the track on a true bearing of 310°. What distance do they need to walk to meet up with the track?
  - **iii** If a walker was to walk in a south-westerly direction from the track for x km on a true bearing of $q°$, where $180 < q < 270$, on what true bearing do they need to travel for x km to arrive back at the track?

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b A walker heads from the track on a true bearing of 210° for 4 km.

- **i** Determine three different bearings and distances that would get them back on the track. Include diagrams to represent these walks.

Three kilometres west along the track from where they start begins a 500 m section of the track that is not accessible. To avoid this section the walker must rejoin the track either before or after this section.

- **ii** On what possible bearings does the walker need to travel to get back to the track but avoid the inaccessible part? Assume that the walker does not want to backtrack, and round bearings to the nearest whole number.

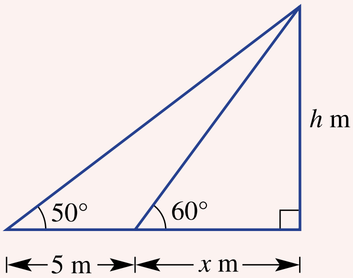
How high is the building?

3 To determine the height of a building two angles of elevation are recorded on level ground a set distance apart.

By applying basic right-angled trigonometry we can investigate the height of the building using angles of elevation and the distance between the points at ground level.

a For the case shown, with two angles of elevation 50° and 60°, 5 m apart:

- **i** Write down two equations involving x and h (one from each right-angled triangle).
- **ii** Use the expressions from part **i** to solve for h correct to the nearest centimetre.



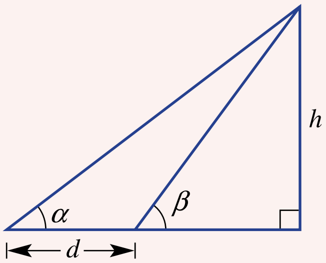
b **i** Repeat part **a**, with the two angles of elevation 50° and 60°, now taken d m apart, to find h in terms of d.

- **ii** Use your answer from part **i**, to confirm your answer to part **a** and to find the height of the building if the measurements were taken 8 m apart. Round to the nearest centimetre.

c Using the diagram from part **a**:

- **i** make use of the sine rule to find the value of h correct to the nearest centimetre.
- **ii** compare your working from part **i** with part **a**. Is either method preferable?

d For the general case shown below, use the method from part **a** to find an expression for h in terms of α, β and d. Check by using your values from part **a**.



6H Area of a triangle EXTENDING

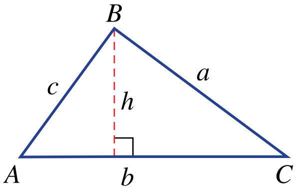
LEARNING INTENTIONS

- To understand how the area of a triangle can be found when two sides and the included angle are known
- To be able to use the area of a triangle formula, $A = \frac{1}{2}ab\sin C$

We can use trigonometry to establish a rule for the area of a triangle using two sides and the included angle.

We can see in this triangle that $\sin C = \frac{h}{a}$, so $h = a\sin C$.

$\therefore A = \frac{1}{2}bh$ becomes $A = \frac{1}{2}ba\sin C$.

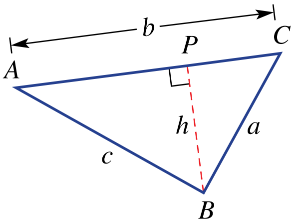


A polygon's area can be found by dividing it into oblique triangles and measuring relevant angles. This method is useful for finding polygon-shaped areas such as irregular farm paddocks, blocks of land or a space that is to be landscaped or paved.

Lesson starter: Calculating area in two ways

Draw any triangle *ABC* and construct the height *PB*. Measure the following as accurately as possible.

a *AC* **b** *BC* **c** *BP* **d** $\angle C$



Now calculate the area using:

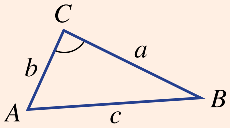
- Area $= \frac{1}{2}bh$
- Area $= \frac{1}{2}ab\sin C$

How close are your answers? They should be equal!

KEY IDEAS

- The area of a triangle is equal to half the product of two sides and the sine of the included angle.

  Area $= \frac{1}{2}ab\sin C$



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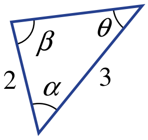
BUILDING UNDERSTANDING

① Evaluate $\frac{1}{2}ab\sin C$, correct to one decimal place, for the given values of a, b and C.

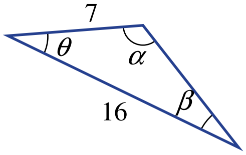
a $a = 3, b = 4, C = 38°$
b $a = 15, b = 7, C = 114°$

② Which angle pronumeral (i.e. α, β or θ) represents the included angle between the two given sides in these triangles?

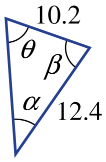
a



b



c



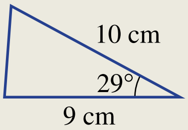
③ Solve these equations for C. Round your answer to two decimal places.

a $10 = \frac{1}{2} \times 4 \times 6 \times \sin C$
b $25 = \frac{1}{2} \times 7 \times 10 \times \sin C$

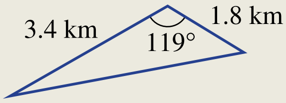
Example 14 Finding the area of a triangle

Find the area of these triangles, correct to one decimal place.

a



b



SOLUTION

a Area $= \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 9 \times 10 \times \sin 29°$

$= 21.8\text{ cm}^2$ (to 1 d.p.)

b Area $= \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 1.8 \times 3.4 \times \sin 119°$

$= 2.7\text{ km}^2$ (to 1 d.p.)

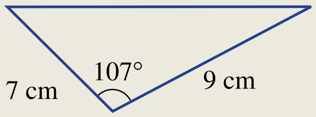
EXPLANATION

Substitute the two sides (a and b) and the included angle (C) into the rule.

Substitute the two sides (a and b) and the included angle (C) into the rule.
The rule works in the same way for an obtuse-angled triangle.

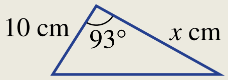
Now you try

Find the area of this triangle, correct to one decimal place.



Example 15 Finding a side length given the area

Find the value of x, correct to two decimal places, given that the area of this triangle is 70 cm^2.



SOLUTION

$$\begin{aligned} \text{Area} &= \frac{1}{2}ab\sin C \\ 70 &= \frac{1}{2} \times 10 \times x \times \sin 93^\circ \\ 14 &= x \sin 93^\circ \\ x &= \frac{14}{\sin 93^\circ} \\ &= 14.02 \text{ (to 2 d.p.)} \end{aligned}$$

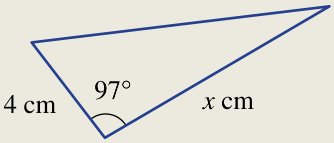
EXPLANATION

Substitute all the given information into the rule, letting $a = 10$ and $b = x$. Use $\angle C = 93^\circ$ as the included angle.

$\frac{1}{2} \times 10 = 5$, so divide both sides by 5 $(70 \div 5 = 14)$ and then solve for x.

Now you try

Find the value of x, correct to two decimal places, given that the area of this triangle is 37 cm^2.



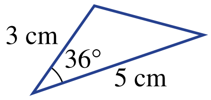
Exercise 6H

FLUENCY	1(½), 3(½)	1(½), 2, 3(½)	1–3(⅓)

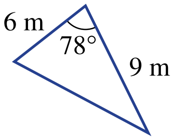
Example 14

1 Find the area of these triangles, correct to one decimal place.

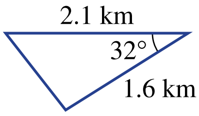
a



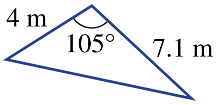
b



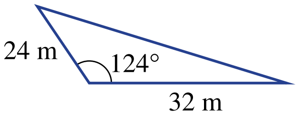
c



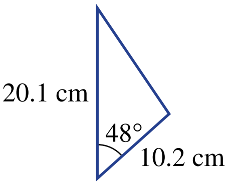
d



e



f



2 Find the area of these triangles, correct to one decimal place.

a ΔXYZ if $XY = 5$ cm, $XZ = 7$ cm and $\angle X = 43^\circ$

b ΔSTU if $ST = 12$ m, $SU = 18$ m and $\angle S = 78^\circ$

c ΔEFG if $EF = 1.6$ km, $FG = 2.1$ km and $\angle F = 112^\circ$

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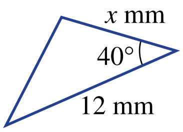
Example 15

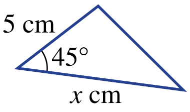
3 Find the value of x, correct to one decimal place, for these triangles with given areas.

a



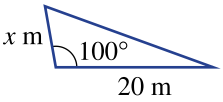
$\text{Area} = 22\,\text{mm}^2$

b



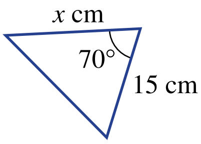
$\text{Area} = 14\,\text{cm}^2$

c



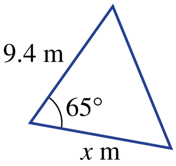
$\text{Area} = 90\,\text{m}^2$

d



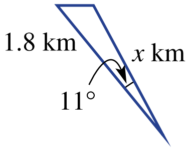
$\text{Area} = 128\,\text{cm}^2$

e



$\text{Area} = 45\,\text{m}^2$

f

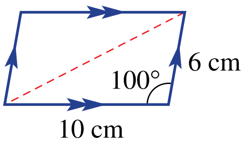


$\text{Area} = 0.23\,\text{km}^2$

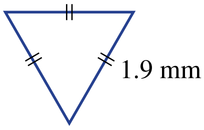
PROBLEM–SOLVING	4, 5	4–6	4–7(1/3)

4 Find the area of these shapes, correct to two decimal places.

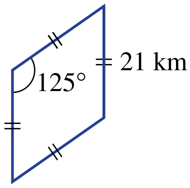
a



b

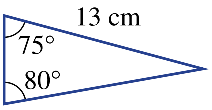


c

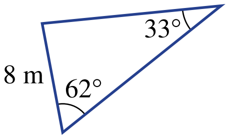


5 First use the sine rule to find another side length, and then find the area of these triangles, correct to two decimal places.

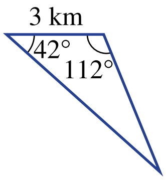
a



b

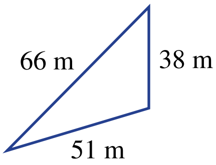


c

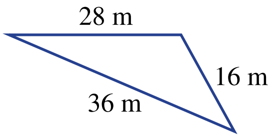


6 First use the cosine rule to find an angle, and then calculate the area of these triangles, correct to two decimal places.

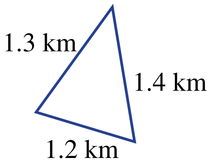
a



b

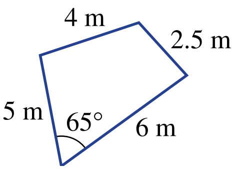


c

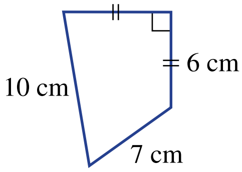


7 Find the area of these quadrilaterals, correct to one decimal place.

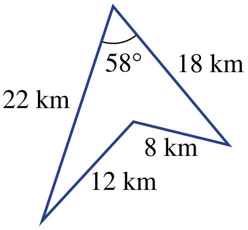
a



b



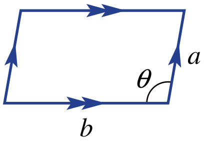
c



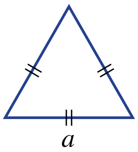
REASONING	8	8, 9	8–10

8 Write a rule for the area of these shapes, using the given pronumerals.

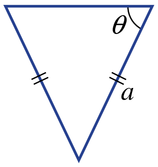
a



b

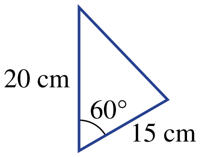


c

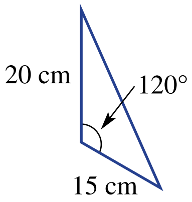


9 **a** Find the area of these two triangles, correct to one decimal place.

i

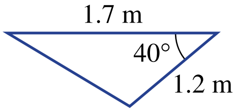


ii



b What do you notice about your answers in part **a**? How can you explain this?

c Draw another triangle that has the same two given lengths and area as the triangle on the right.



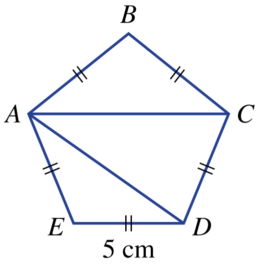
10 **a** Use the rule Area $= \frac{1}{2}ab\sin C$ to find the two possible values of θ (one acute and one obtuse) in the triangle detailed below. Round your answer to one decimal place.

$$\Delta ABC \text{ with } AB = 11\text{ m},\ AC = 8\text{ m, included angle } \theta \text{ and Area} = 40\text{ m}^2.$$

b Draw the two triangles for the two sets of results found in part **a**.

ENRICHMENT: Polygon areas	–	–	11

11 You will recall that the sum (S) of the interior angles of a polygon with n sides is given by $S = (n - 2) \times 180°$.



a This regular pentagon has each side measuring 5 cm.

- **i** Calculate the angle sum of a pentagon.
- **ii** Calculate the size of one interior angle of a regular pentagon.
- **iii** Find the area of ΔAED, correct to two decimal places.
- **iv** Find the length AD, correct to two decimal places.
- **v** Find $\angle ADC$ and $\angle DAC$.
- **vi** Find the area of ΔADC, correct to two decimal places.
- **vii** Find the total area of the pentagon, correct to one decimal place.

b Use a similar approach to find the area of a regular hexagon of side length 5 cm, correct to one decimal place.

c Can this method be used for other regular polygons? Explore and give examples.

6I The unit circle OPTIONAL

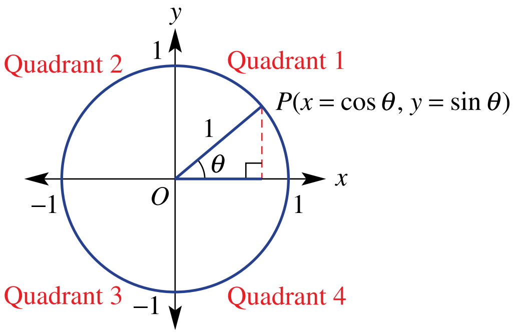
LEARNING INTENTIONS

- To know what the unit circle represents
- To understand how a point on a unit circle can be defined by coordinates related to the cosine of the angle in its triangle and the sine of the angle
- To know the four quadrants of the unit circle and the sign and symmetry properties in these quadrants for the trigonometric ratios
- To be able to identify in which quadrant an angle lies and determine whether its different trigonometric ratios will be positive or negative
- To be able to write an angle in terms of its reference angle in the first quadrant
- To know how tan can be expressed in terms of sin and cos

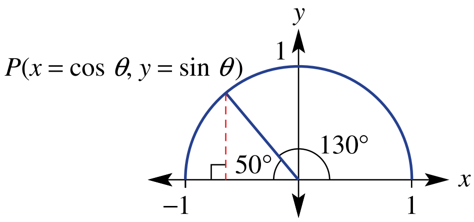
From early trigonometry we calculated $\sin\theta$, $\cos\theta$ and $\tan\theta$ using acute angles. We will now extend this to include the four quadrants of the unit circle, using $0° \leqslant \theta \leqslant 360°$.

Note the following.

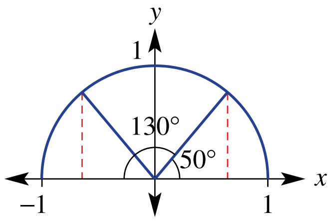
- The unit circle has radius one unit and has centre (0, 0) on a number plane.
- θ is defined anticlockwise from the positive x-axis.
- There are four quadrants, as shown.
- Using a point $P(x, y)$ on the unit circle we define the three trigonometric ratios
  - $\sin\theta = \dfrac{\text{Opposite}}{\text{Hypotenuse}} = \dfrac{y}{1} = y$
  - $\cos\theta = \dfrac{\text{Adjacent}}{\text{Hypotenuse}} = \dfrac{x}{1} = x$
  - $\tan\theta = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta}$



- The coordinates of P, a point on the unit circle, are $(x, y) = (\cos\theta, \sin\theta)$.
- In the second quadrant $\sin\theta$ is positive, $\cos\theta$ is negative and $\tan\theta$ is negative. In this diagram we can see $P(\cos 130°, \sin 130°)$, where $\cos 130°$ is negative, $\sin 130°$ is positive and so $\tan 130°$ will be negative.



In the diagram at right showing 130°, a 50° angle (180° − 130°) drawn in the first quadrant can help relate trigonometric values from the second quadrant to the first quadrant. By symmetry we can see that $\sin 130° = \sin 50°$ and $\cos 130° = -\cos 50°$. This 50° angle is called the **reference angle** (or related angle).



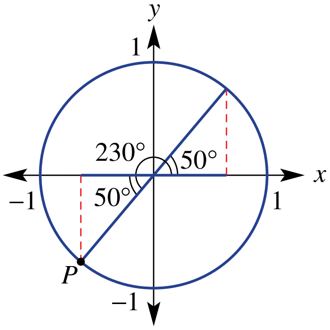
In this section we explore these symmetries and reference angles in the second, third and fourth quadrants.

Lesson starter: Positive or negative

For the angle 230°, the reference angle is 50° and $P = (\cos 230°, \sin 230°)$.

Since P is in the third quadrant, we can see that:

- $\cos 230° = -\cos 50°$, which is negative.
- $\sin 230° = -\sin 50°$, which is negative.



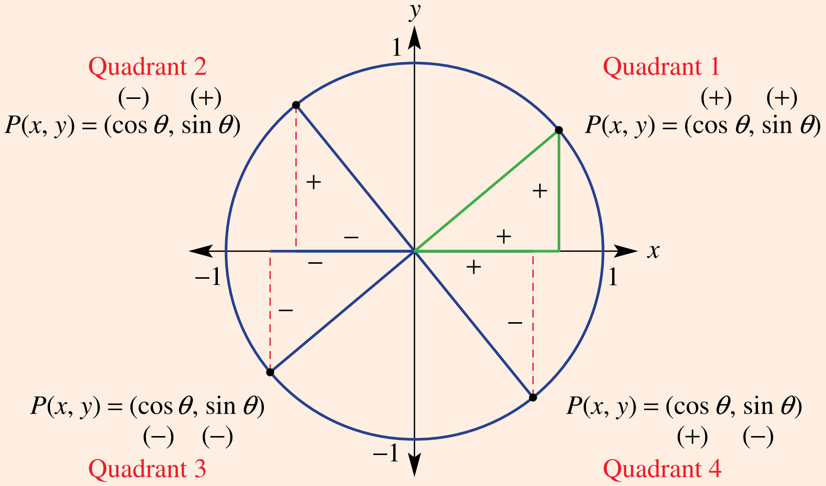
Now determine the following for each value of θ given below.
You should draw a unit circle for each.

- What is the reference angle?
- Is $\cos\theta$ positive or negative?
- Is $\sin\theta$ positive or negative?
- Is $\tan\theta$ positive or negative?

a $\theta = 240°$ **b** $\theta = 210°$ **c** $\theta = 335°$ **d** $\theta = 290°$ **e** $\theta = 162°$

KEY IDEAS

- Every point $P(x, y)$ on the unit circle can be described in terms of the angle θ such that: $x = \cos\theta$ and $y = \sin\theta$, where $-1 \leqslant \sin\theta \leqslant 1$ and $-1 \leqslant \cos\theta \leqslant 1$.



  - θ is measured anticlockwise from the positive x-axis.
  - Negative angles are measured clockwise from the positive x-axis.
- For different quadrants, $\cos\theta$ and $\sin\theta$ can be positive or negative.
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- ASTC means:
  - Quadrant 1: All $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive.
  - Quadrant 2: Only Sin θ is positive.
  - Quadrant 3: Only Tan θ is positive.
  - Quadrant 4: Only Cos θ is positive.

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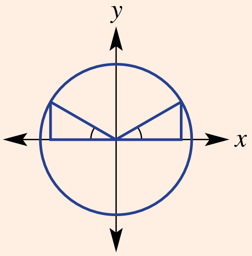
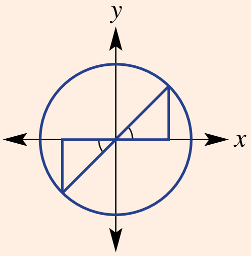
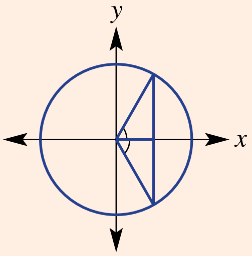
- A **reference angle** (sometimes called a related angle) is an acute angle that helps to relate $\cos\theta$ and $\sin\theta$ to the first quadrant.

Angle θ	90° to 180°	180° to 270°	270° to 360°
Reference angle	$180° - \theta$  	$\theta - 180°$  	$360° - \theta$ 

- Multiples of 90°.

θ	0°	90°	180°	270°	360°
$\sin\theta$	0	1	0	-1	0
$\cos\theta$	1	0	-1	0	1
$\tan\theta$	0	undefined	0	undefined	0

BUILDING UNDERSTANDING

1 Which quadrant in the unit circle corresponds to these values of θ?

a $0° < \theta < 90°$

b $180° < \theta < 270°$

c $270° < \theta < 360°$

d $90° < \theta < 180°$

2 Decide which quadrants make the following true.

a $\sin\theta$ is positive

b $\tan\theta$ is negative

c $\cos\theta$ is negative

d $\cos\theta$ is positive

e $\tan\theta$ is positive

f $\sin\theta$ is negative

3 State the missing values in this table.

θ	0°	90°	180°	270°	360°
$\sin\theta$					0
$\cos\theta$			-1		
$\tan\theta$		undefined			

4 Use a calculator to evaluate the following, correct to three decimal places.

a sin 172°

b sin 212°

c cos 143°

d cos 255°

e tan 222°

f tan 134°

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Example 16 Choosing supplementary angles

Choose an obtuse angle to complete each statement.

a $\sin 30^\circ = \sin$ ___ **b** $\cos 57^\circ = -\cos$ ___ **c** $\tan 81^\circ = -\tan$ ___

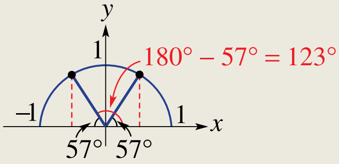
SOLUTION

a $\sin 30^\circ = \sin 150^\circ$

EXPLANATION

Choose the supplement of 30°, which is $180^\circ - 30^\circ = 150^\circ$.

b $\cos 57^\circ = -\cos 123^\circ$



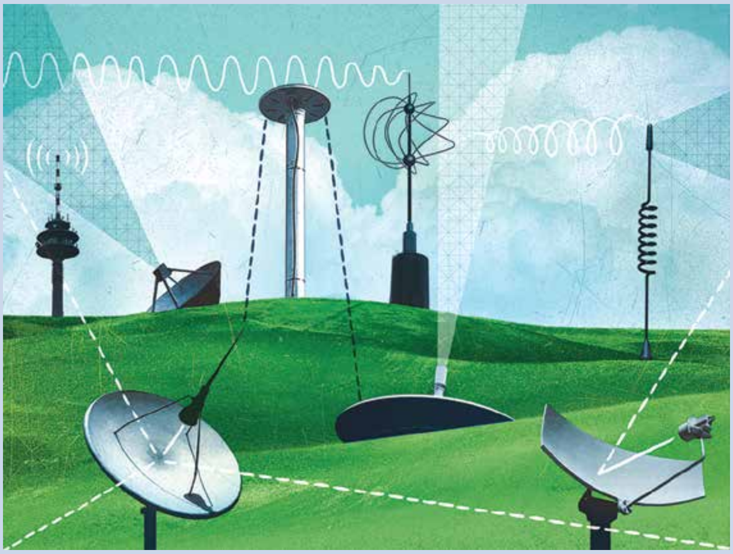
c $\tan 81^\circ = -\tan 99^\circ$

The supplement of 81° is 99°.

Now you try

Choose an obtuse angle to complete each statement.

a $\sin 40^\circ = \sin$ ___ **b** $\cos 74^\circ = -\cos$ ___ **c** $\tan 47^\circ = -\tan$ ___



Extending trigonometry to any sized angle led to the discovery that their values regularly repeat, like the periodic change in the height of a wave. A world-changing application of trigonometry is the modelling of electromagnetic waves.

Example 17 Positioning a point on the unit circle

Decide in which quadrant θ lies and state whether $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive or negative.

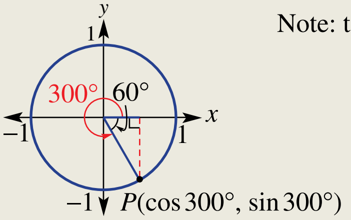
a $\theta = 300°$ **b** $\theta = 237°$ **c** $\theta = -212°$

SOLUTION

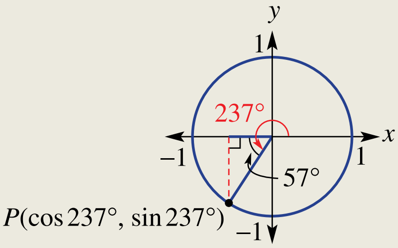
a $\theta = 300°$ is in quadrant 4.
$\sin\theta$ is negative
$\cos\theta$ is positive
$\tan\theta$ is negative

EXPLANATION

Note: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

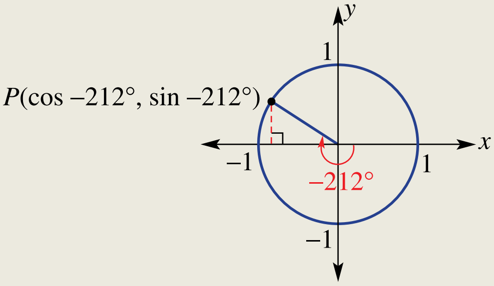


b $\theta = 237°$ is in quadrant 3.
$\sin\theta$ is negative
$\cos\theta$ is negative
$\tan\theta$ is positive



c $\theta = -212°$ is in quadrant 2.
$\sin\theta$ is positive
$\cos\theta$ is negative
$\tan\theta$ is negative

Negative angles are measured clockwise from the positive x-axis.



Now you try

Decide in which quadrant θ lies and state whether $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive or negative.

a $\theta = 320°$ **b** $\theta = 215°$ **c** $\theta = -236°$

Example 18 Using a reference angle

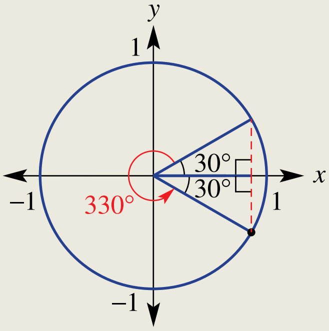
Write the following using their reference angle.

a $\sin 330°$ **b** $\cos 162°$ **c** $\tan 230°$

SOLUTION **EXPLANATION**

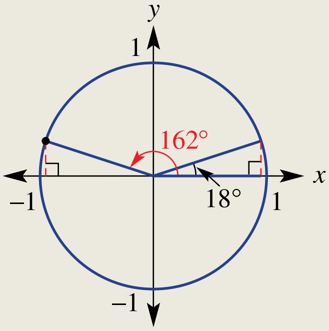
a $\sin 330° = -\sin 30°$

$\sin 330°$ is negative (quadrant 4) and the reference angle is $360° - 330° = 30°$.



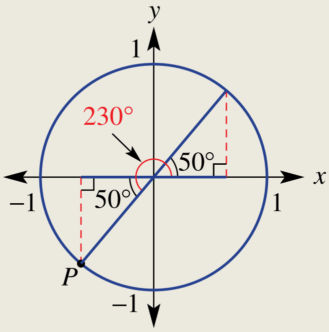
b $\cos 162° = -\cos 18°$

$\cos 162°$ is negative (quadrant 2) and the reference angle is $180° - 162° = 18°$.



c $\tan 230° = \tan 50°$

$\tan 230°$ is positive (quadrant 3 and negative ÷ negative) and the reference angle is $230° - 180° = 50°$.



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Now you try

Write the following using their reference angle.

a $\sin 310^\circ$ **b** $\cos 126^\circ$ **c** $\tan 260^\circ$

Exercise 6I

FLUENCY	1–4(½)	1–5(½)	1–5(⅓)

Example 16

1 Choose an obtuse angle to complete each statement.

a $\sin 40^\circ = \sin$ ___ **b** $\sin 65^\circ = \sin$ ___
c $\cos 25^\circ = -\cos$ ___ **d** $\cos 81^\circ = -\cos$ ___
e $\tan 37^\circ = -\tan$ ___ **f** $\tan 8^\circ = -\tan$ ___

2 Choose an acute angle to complete each statement.

a $\sin 150^\circ = \sin$ ___ **b** $\sin 94^\circ = \sin$ ___
c $-\cos 110^\circ = \cos$ ___ **d** $-\cos 171^\circ = \cos$ ___
e $-\tan 159^\circ = \tan$ ___ **f** $-\tan 143^\circ = \tan$ ___

Example 17

3 Decide in which quadrant θ lies and state whether $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive or negative.

a $\theta = 172^\circ$ **b** $\theta = 295^\circ$ **c** $\theta = 252^\circ$ **d** $\theta = 73^\circ$
e $\theta = 318^\circ$ **f** $\theta = 154^\circ$ **g** $\theta = 197^\circ$ **h** $\theta = 221^\circ$
i $\theta = 210^\circ$ **j** $\theta = 53^\circ$ **k** $\theta = 346^\circ$ **l** $\theta = 147^\circ$
m $\theta = -35^\circ$ **n** $\theta = -324^\circ$ **o** $\theta = -105^\circ$ **p** $\theta = -192^\circ$

Example 18

4 Write each of the following using its reference angle.

a $\sin 280^\circ$ **b** $\cos 300^\circ$ **c** $\tan 220^\circ$ **d** $\sin 140^\circ$
e $\cos 125^\circ$ **f** $\tan 315^\circ$ **g** $\sin 345^\circ$ **h** $\cos 238^\circ$
i $\tan 227^\circ$ **j** $\sin 112^\circ$ **k** $\cos 294^\circ$ **l** $\tan 123^\circ$

5 If θ is acute, find the value of θ.

a $\sin 150^\circ = \sin\theta$ **b** $\sin 240^\circ = -\sin\theta$ **c** $\sin 336^\circ = -\sin\theta$
d $\cos 220^\circ = -\cos\theta$ **e** $\cos 109^\circ = -\cos\theta$ **f** $\cos 284^\circ = \cos\theta$
g $\tan 310^\circ = -\tan\theta$ **h** $\tan 155^\circ = -\tan\theta$ **i** $\tan 278^\circ = -\tan\theta$

PROBLEM–SOLVING	6, 7	6, 7(½)	7(½), 8

6 For what values of θ, in degrees and positive, are the following true?

a All of $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive.
b Only $\sin\theta$ is positive.
c Only $\cos\theta$ is positive.
d Only $\tan\theta$ is positive.

7 Write the reference angle (i.e. related angle) in the first quadrant for these angles.

a 138° **b** 227° **c** 326° **d** 189°
e 213° **f** 298° **g** 194° **h** 302°
i $-40°$ **j** $-207°$ **k** $-114°$ **l** $-316°$

8 Complete the table by finding a second angle, θ_2, that gives the same value for the trigonometric function as θ_1. Use the unit circle to help in each case and assume $0 \leqslant \theta_2 \leqslant 360°$.

Trigonometric function	$\sin\theta$	$\cos\theta$	$\sin\theta$	$\tan\theta$	$\cos\theta$	$\tan\theta$	$\cos\theta$	$\sin\theta$	$\tan\theta$
θ_1	30°	45°	190°	15°	125°	320°	260°	145°	235°
θ_2									

REASONING	9	9, 10, 12(½)	9(½), 10, 11, 12(½)

9 Decide which quadrant suits the given information.

a $\sin\theta < 0$ and $\cos\theta > 0$ **b** $\tan\theta > 0$ and $\cos\theta > 0$
c $\tan\theta < 0$ and $\cos\theta < 0$ **d** $\sin\theta > 0$ and $\tan\theta < 0$
e $\sin\theta > 0$ and $\tan\theta > 0$ **f** $\sin\theta < 0$ and $\cos\theta < 0$

10 Explain why $\tan\theta > 0$ when $180° < \theta < 270°$.

11 Explain why tan 90° and tan 270° are undefined.

12 By considering a unit circle, state whether the following are true or false.

a $\sin 10° < \cos 10°$ **b** $\sin 50° < \tan 50°$
c $\cos 80° > \sin 80°$ **d** $\cos 90° = \sin 0°$
e $\tan 180° = \sin 180°$ **f** $\cos 170° > \sin 170°$
g $\sin 120° > \tan 120°$ **h** $\sin 90° = \cos 180°$
i $\tan 230° < \cos 230°$ **j** $\cos 350° < \sin 85°$
k $\sin 260° < \cos 110°$ **l** $\tan 270° = \cos 180°$

ENRICHMENT: Trigonometric identities	–	–	13, 14

13 You will recall that complementary angles sum to 90°. Answer these questions to explore the relationship between sine and cosine ratios of complementary angles.

a Evaluate the following, correct to two decimal places.

i $\sin 10°$ **ii** $\cos 80°$ **iii** $\sin 36°$ **iv** $\cos 54°$
v $\cos 7°$ **vi** $\sin 83°$ **vii** $\cos 68°$ **viii** $\sin 22°$

b Describe what you notice from part **a**.

c Complete the following.

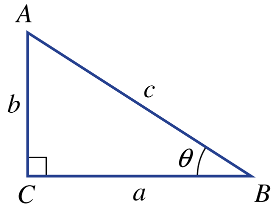
i $\cos\theta = \sin(\_\_\_\_)$ **ii** $\sin\theta = \cos(\_\_\_\_)$

d State the value of θ if θ is acute.

i $\sin 20° = \cos\theta$ **ii** $\sin 85° = \cos\theta$
iii $\cos 71° = \sin\theta$ **iv** $\cos 52° = \sin\theta$

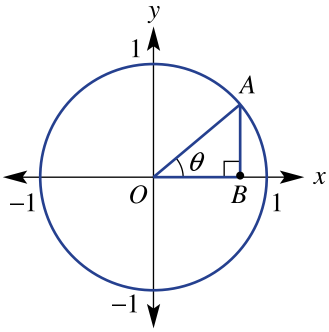
e For this triangle $\angle B = \theta$.

i Write $\angle A$ in terms of θ.
ii Write a ratio for $\sin\theta$ in terms of b and c.
iii Write a ratio for $\cos(90° - \theta)$ in terms of b and c.



f If $\cos(90° - \theta) = \frac{2}{3}$, find $\tan\theta$.

14 Trigonometric identities are mathematical statements that may involve $\sin\theta$ and/or $\cos\theta$ and/or $\tan\theta$ and hold true for all values of θ.
In previous exercises you will have already considered the trigonometric identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$.



a Consider the triangle OAB in the unit circle shown.

i Given that $OA = 1$, $OB = \cos\theta$ and $AB = \sin\theta$, use Pythagoras' theorem to prove the trigonometric identity: $\sin^2\theta + \cos^2\theta = 1$. (Note: $\sin^2\theta = (\sin\theta)^2$.)

ii Check your identity using a calculator to see if it holds true for $\theta = 30°, 145°, 262°$ and $313°$.

b **i** Evaluate the given pairs of numbers using a calculator.
$(\sin 60°, \cos 30°)$, $(\sin 80°, \cos 10°)$, $(\sin 110°, \cos(-20°))$, $(\sin 195°, \cos(-105°))$

ii What do you notice about the value of each number in the pairs above? Drawing a unit circle illustrating each pair of values may help.

iii What is the relationship between θ in $\sin\theta$ and θ in $\cos\theta$ that is true for all pairs in part **b i**?

iv In terms of θ, complete this trigonometric identity: $\sin\theta = \cos(\_\_\_\_)$.

v Check this identity for $\theta = 40°, 155°, 210°$ and $236°$.

c Explore other trigonometric identities by drawing diagrams and checking different angles, such as:

i $\sin\theta = \sin(180° - \theta)$
ii $\cos\theta = \cos(360° - \theta)$
iii $\tan\theta = \tan(180° + \theta)$
iv $\sin 2\theta = 2\sin\theta\cos\theta$
v $\cos 2\theta = \cos^2\theta - \sin^2\theta$
vi $\cos 2\theta = 2\cos^2\theta - 1$
vii $\cos 2\theta = 1 - 2\sin^2\theta$

6J Graphs of trigonometric functions OPTIONAL

LEARNING INTENTIONS

- To know the meaning of the terms amplitude and period and be able to relate them to sine and cosine graphs
- To understand the shape of the sine and cosine graphs and their periodic nature
- To be able to use a sine or cosine graph to find the approximate solution of an equation
- To be able to use symmetry of the unit circle and graphs to compare the trigonometric ratios of angles

As the angle θ increases from 0° to 360°, the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ increase or decrease depending on the value of θ. Graphing the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ against θ gives a clear picture of this.

These wave-like graphs based on trigonometric functions are used to model many variables from the height of the tide on a beach to the width of a soundwave giving a high or low pitch sound.

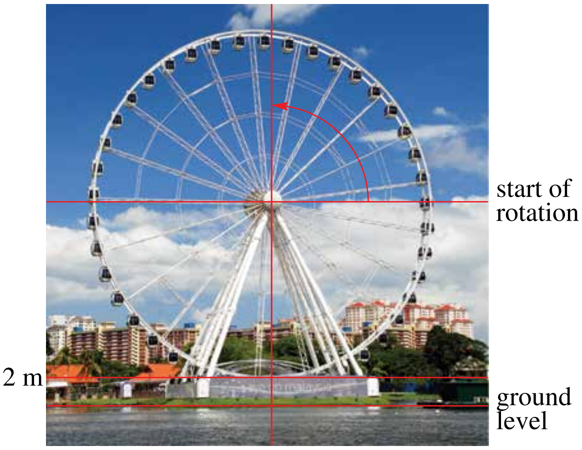
Incoming solar energy is essential for agriculture and solar power production. Local light intensity is determined by the sun's angle of elevation, which has a periodic variation. Trigonometric graphs can model light intensity versus day of the year, time of the day and latitude.

Lesson starter: Ferris wheel ride

Have you ever had a ride on a Ferris wheel? Imagine yourself riding a Ferris wheel again. The wheel rotates at a constant rate, but on which part of the ride will your vertical upwards movement be fastest? On which part of the ride will your vertical movement be slowest?

Work in groups to discuss these questions and help each other to complete the table and graph.

For this example, assume that the bottom of the Ferris wheel is 2 m above the ground and the diameter of the wheel is 18 m. Count the start of a rotation from halfway up on the right, as shown. The wheel rotates in an anticlockwise direction.



Position	Angle of rotation, θ, from halfway up	Vertical height, h, above ground level (m)
Halfway up	0°	
Top	90°	
Halfway down		
Bottom		2
Halfway up		
Top		
Halfway down		

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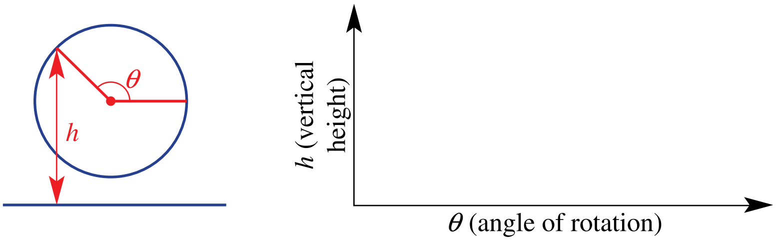
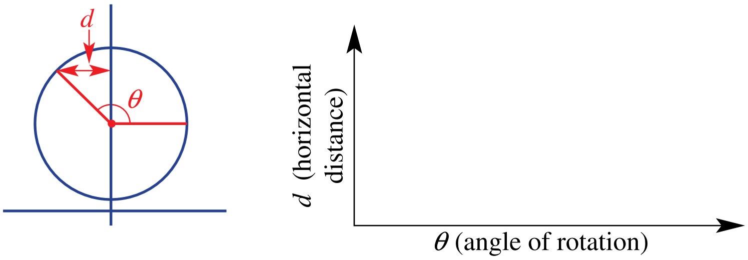
Now draw a graph of the **vertical height** (h) above the ground (vertical axis) versus the **angle** (θ) of anticlockwise rotation for two complete turns of the Ferris wheel.



As a group, discuss some of the key features of the graph.

- What are the maximum and minimum values for the height?
- Discuss any symmetry you see in your graph. How many values of θ (rotation angle) have the same value for height? Give some examples.

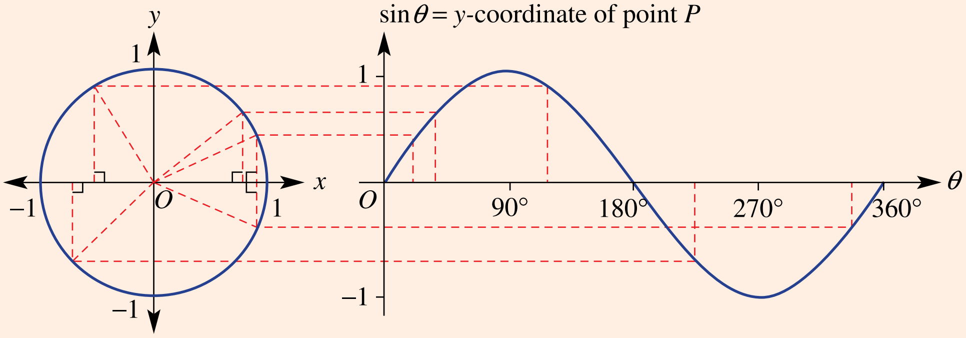
Discuss how the shape would change for a graph of the **horizontal distance** (d) from the circumference (where you sit) to the central **vertical** axis of the Ferris wheel versus the angle (θ) of rotation. Sketch this graph.



The shapes of the Ferris wheel graphs you have drawn are examples of **periodic functions** because the graph shape continuously repeats one cycle (for each period of 360°) as the wheel rotates. The graph of height above the ground illustrates a sine function ($\sin\theta$). The graph of the distance from a point on the circumference to the central vertical axis of the Ferris wheel illustrates a cosine function ($\cos\theta$).

KEY IDEAS

- By plotting θ on the x-axis and $\sin\theta$ on the y-axis we form the graph of $\sin\theta$. $\sin\theta = y$-coordinate of point P on the unit circle. For $0 \leqslant \theta \leqslant 360°$, one full cycle is shown.
  - $y = \sin\theta$



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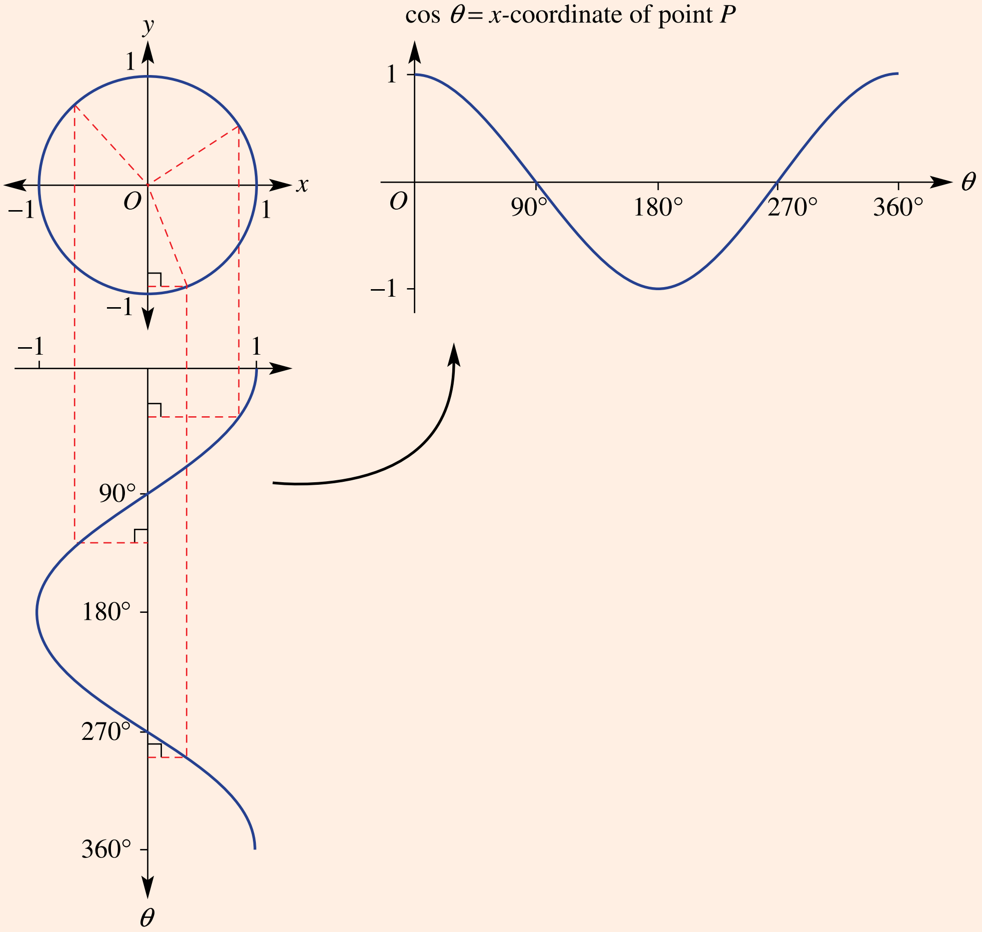
- By plotting θ on the x-axis and $\cos\theta$ on the y-axis we form the graph of $\cos\theta$.
  $\cos\theta = x$-coordinate of point P on the unit circle. For $0 \leqslant \theta \leqslant 360°$, one full cycle is shown.
  - When we write $y = \cos\theta$, the y variable is not to be confused with the y-coordinate of the point P on the unit circle.
  - $y = \cos\theta$



- **Amplitude** is the maximum displacement of the graph from a reference level (here it is the x-axis).
- The **period** of a graph is the time taken (or number of degrees) to make one complete cycle.
- Both $y = \sin\theta$ and $y = \cos\theta$ have Amplitude = 1 and Period = 360°.

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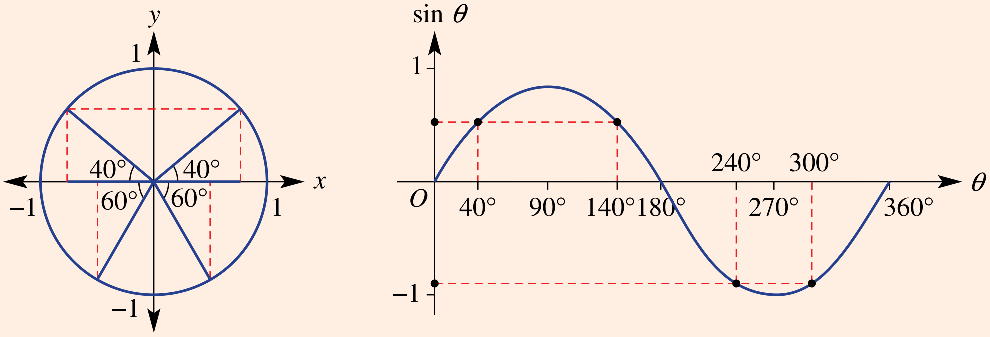
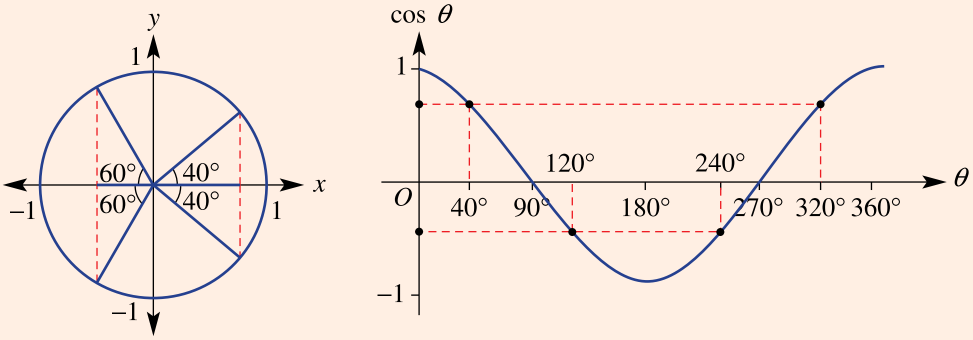
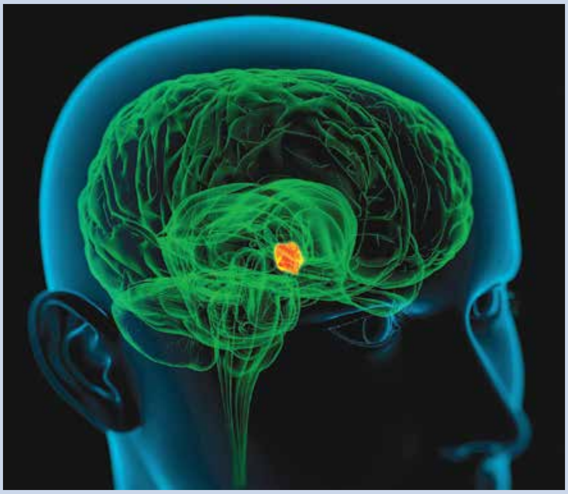
- **Symmetry** within the unit circle using reference angles can be illustrated using graphs of trigonometric functions.
  - This shows sin 40° = sin 140° (reference angle 40°) and sin 240° = sin 300° (reference angle 60°).



  - This shows cos 40° = cos 320° (reference angle 40°) and cos 120° = cos 240° (reference angle 60°).



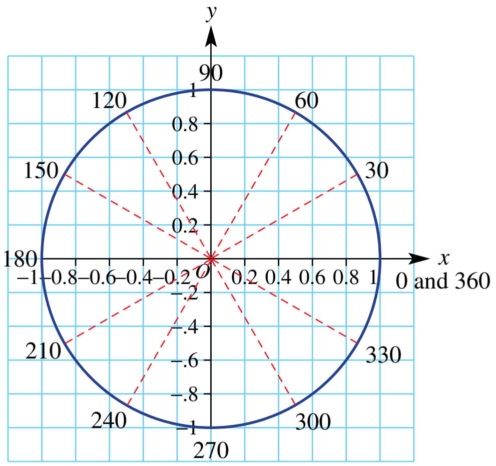


Circadian rhythms, such as the brain's 24-hour sleep-wake cycle, can be modelled with trigonometric graphs. Research shows that digital devices' blue light causes the hypothalamus to suppress the sleep hormone, delaying the brain's sleep waves.

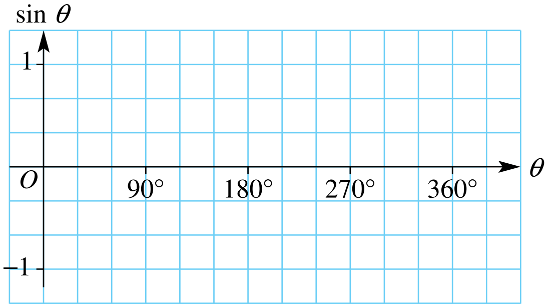
BUILDING UNDERSTANDING

① **a** Find the missing values in the table below for $\sin\theta$, stating the y-coordinate of each point at which the angle intersects the unit circle (shown below).

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin\theta$	0	0.5			0.87			−0.5					



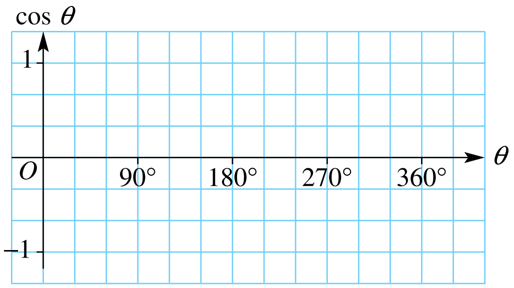
b Graph the points above and join them to make a smooth curve for $\sin\theta$.



② **a** Using the unit circle diagram in Question ①, find the missing values in the table below for $\cos\theta$, stating the x-coordinate of each point at which the angle intersects the unit circle.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos\theta$	1	0.87			−0.5			−0.87					

b Graph the points above and join them to make a smooth curve for $\cos\theta$.



③ **a** For the graph of $\sin\theta$ and using $0° \leqslant \theta \leqslant 360°$, state:

- i the maximum and minimum values of $\sin\theta$
- ii the values of θ for which $\sin\theta = 0$.

b For the graph of $\cos\theta$ and using $0° \leqslant \theta \leqslant 360°$, state:

- i the maximum and minimum values of $\cos\theta$
- ii the values of θ for which $\cos\theta = 0$.

c State the values of θ for $0° \leqslant \theta \leqslant 360°$ such that:

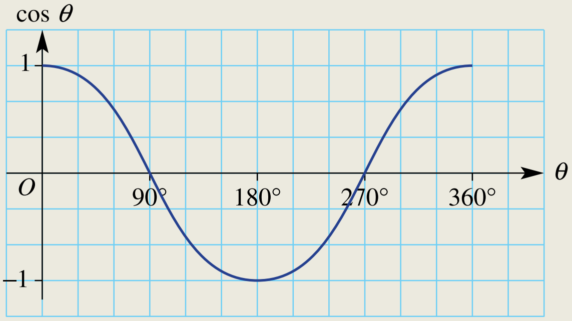
- i $\cos\theta < 0$
- ii $\sin\theta < 0$

Example 19 Reading off a trigonometric graph

Use this graph of $\cos\theta$ to estimate:

a the value of $\cos\theta$ for $\theta = 200°$

b the two values of θ for which $\cos\theta = 0.7$



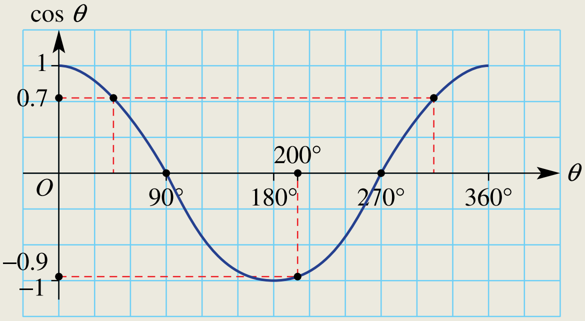
SOLUTION

a $\cos 200° \approx -0.9$

b $\cos\theta = 0.7$

$\theta \approx 46°$ or $314°$

EXPLANATION



Now you try

Use the above graph of $\cos\theta$ (from above) to estimate:

a the value of $\cos\theta$ for $\theta = 100°$

b the two values of θ for which $\cos\theta = -0.6$

Example 20 Comparing the size of the sine of angles

Use the graph of $y = \sin\theta$ to state whether the following are true or false.

a $\sin 70° < \sin 130°$

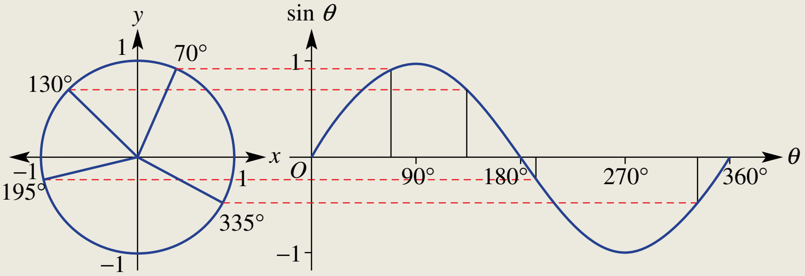
b $\sin 195° > \sin 335°$

SOLUTION

a False

b True

EXPLANATION



Now you try

Use the graph of $y = \sin\theta$ to state whether the following are true or false.

a $\sin 30^\circ > \sin 140^\circ$

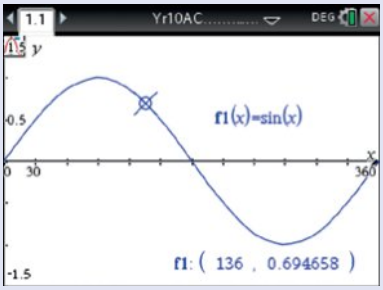
b $\sin 240^\circ > \sin 290^\circ$

Using calculators to graph trigonometric functions

1. Sketch the graph of $y = \sin(x)$ for $0^\circ \leqslant x \leqslant 360^\circ$ and trace to explore the behaviour of y.
2. Sketch the graph of $y = \cos(x)$ for $0^\circ \leqslant x \leqslant 360^\circ$ and $y = \cos(x)$ for $-180^\circ \leqslant x \leqslant 180^\circ$ on separate axes.

Using the TI-Nspire:

1. In a **Graphs** page, define $f1(x) = \sin(x)$ and press enter. Use menu>**Window/Zoom> Window Settings** and set x from 0 to 360 and y from -1.5 to 1.5. Use menu>**Trace>Graph Trace** and scroll along the graph.

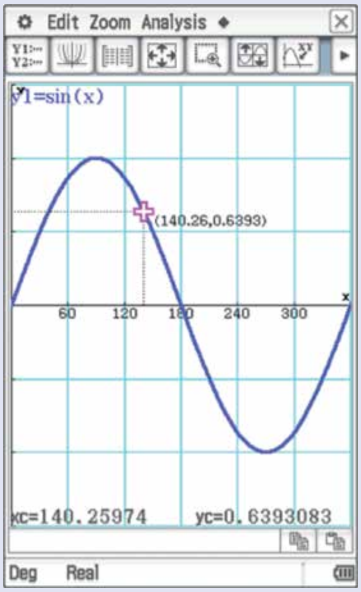


Note: Ensure you are in Degree mode. This setting can be accessed using menu>**Settings** whilst in the **Graphs** page.

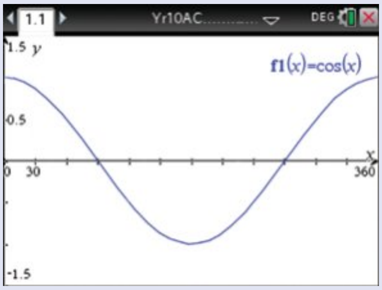
Hint: you can double click on the end axes values and edit if preferred.

Using the ClassPad:

1. With the calculator in **Degree** mode, go to the **Graph&Table** application. Enter the rule $y1 = \sin(x)$ followed by **EXE**. Tap to see the graph. Tap and set x from 0 to 360 with a scale of 60 and y from about -1.5 to 1.5 with a scale of 0.5. Tap **Analysis**, **Trace** and then scroll along the graph.

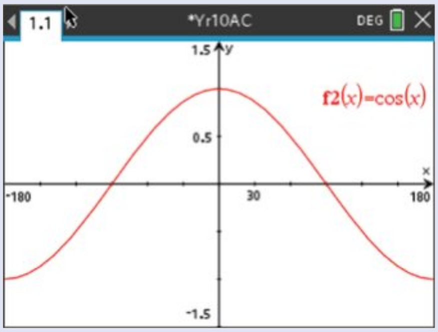


2 In a **Graphs** page, define $f1(x) = \cos(x)$ and use the same settings as before.

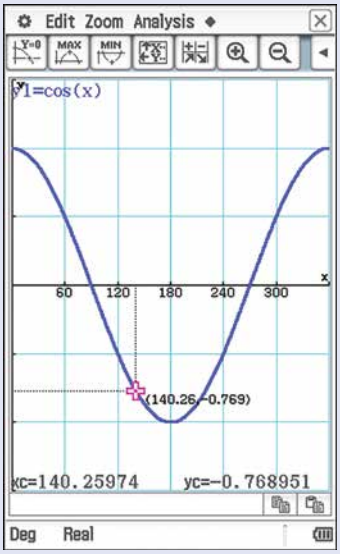


Hint: if the graph entry line is not showing, press tab or double click in an open area.

On another page, define $f2(x) = \cos(x)$ and set x from –180 to 180.

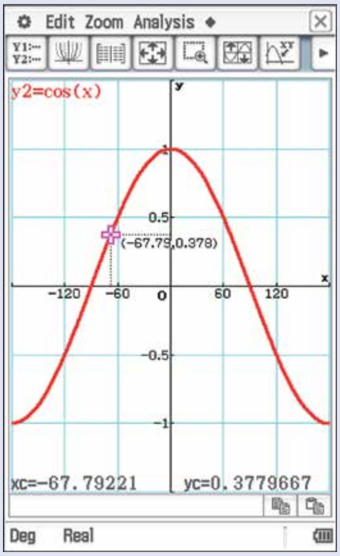


2 In the **Graph&Table** application, enter the rule $y1 = \cos(x)$ followed by **EXE**. Tap . Use settings as before.



xc=140.25974 yc=−0.768951

Now define and select $y2 = \cos(x)$ and set x from –180 to 180.



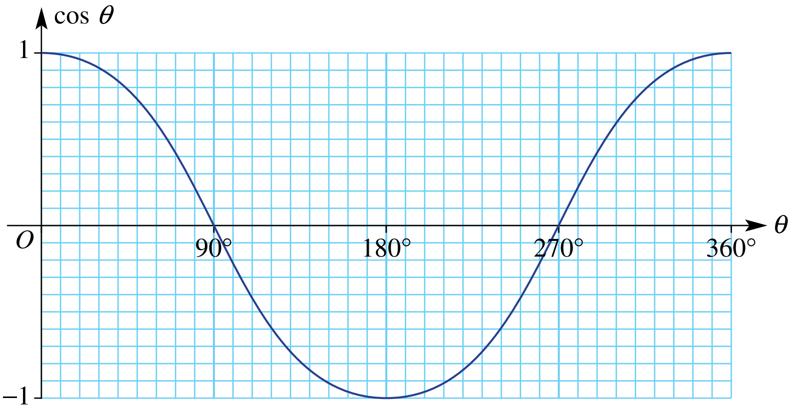
xc=−67.79221 yc=0.3779667

Exercise 6J

FLUENCY 1–2 1–2 1–2

Example 19

1 This graph shows $\cos\theta$ for $0^\circ \leqslant \theta \leqslant 360^\circ$.



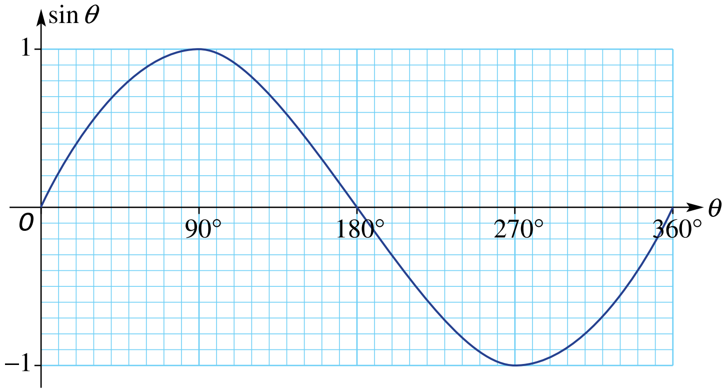
a Use this graph to estimate the value of $\cos\theta$ for the following.

- **i** $\theta = 35^\circ$ **ii** $\theta = 190^\circ$ **iii** $\theta = 330^\circ$ **iv** $\theta = 140^\circ$
- **v** $\theta = 260^\circ$ **vi** $\theta = 75^\circ$ **vii** $\theta = 115^\circ$ **viii** $\theta = 305^\circ$

b Use the same graph to estimate the two values of θ for each of the following.

- **i** $\cos\theta = 0.8$ **ii** $\cos\theta = 0.6$ **iii** $\cos\theta = 0.3$ **iv** $\cos\theta = 0.1$
- **v** $\cos\theta = -0.4$ **vi** $\cos\theta = -0.2$ **vii** $\cos\theta = -0.8$ **viii** $\cos\theta = -0.6$

2 This graph shows $\sin\theta$ for $0^\circ \leqslant \theta \leqslant 360^\circ$.



a Use this graph to estimate the value of $\sin\theta$ for the following.

- **i** $\theta = 25^\circ$ **ii** $\theta = 115^\circ$ **iii** $\theta = 220^\circ$ **iv** $\theta = 310^\circ$
- **v** $\theta = 160^\circ$ **vi** $\theta = 235^\circ$ **vii** $\theta = 320^\circ$ **viii** $\theta = 70^\circ$

b Use the same graph to estimate the two values of θ for each of the following.

- **i** $\sin\theta = 0.6$ **ii** $\sin\theta = 0.2$ **iii** $\sin\theta = 0.3$ **iv** $\sin\theta = 0.9$
- **v** $\sin\theta = -0.4$ **vi** $\sin\theta = -0.8$ **vii** $\sin\theta = -0.7$ **viii** $\sin\theta = -0.1$

PROBLEM–SOLVING

3–5(½) 3–6(½) 3–6(¼)

Example 20

3 By considering the graphs of $y = \sin\theta$ and $y = \cos\theta$, state whether the following are true or false.

a $\sin 60° > \sin 200°$ **b** $\sin 100° < \sin 300°$ **c** $\sin 135° < \sin 10°$
d $\sin 200° = \sin 340°$ **e** $\cos 70° < \cos 125°$ **f** $\cos 315° > \cos 135°$
g $\cos 310° = \cos 50°$ **h** $\cos 95° > \cos 260°$ **i** $\sin 90° = \cos 360°$
j $\cos 180° = \sin 180°$ **k** $\sin 210° > \sin 285°$ **l** $\cos 15° > \cos 115°$

4 For each of the following angles, state the second angle between 0° and 360° that gives the same value for $\sin\theta$.

a 70° **b** 120° **c** 190° **d** 280°
e 153° **f** 214° **g** 307° **h** 183°

5 For each of the following angles, state the second angle between 0° and 360° that gives the same value for $\cos\theta$.

a 80° **b** 10° **c** 165° **d** 285°
e 224° **f** 147° **g** 336° **h** 199°

6 For each of the following angles, state the other angle between $-180°$ and 180° that gives the same value for:

i $\sin\theta$ **ii** $\cos\theta$

a 70° **b** 40° **c** 110° **d** 155°
e $-15°$ **f** $-68°$ **g** $-100°$ **h** $-145°$

REASONING

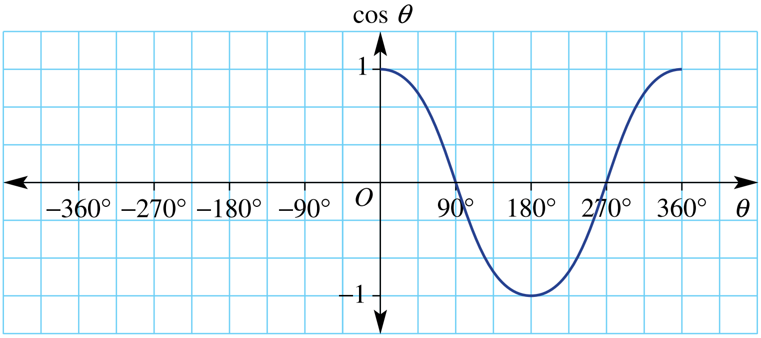
7 7, 8(½) 8(½), 9

7 a How many values of θ satisfy $\sin\theta = 2$? Give a reason.
b How many values of θ satisfy $\cos\theta = -4$? Give a reason.

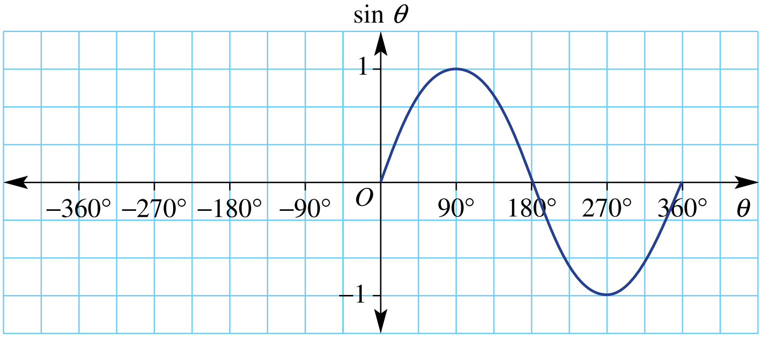
8 Use a calculator to find the two values of θ for $0° \leqslant \theta \leqslant 360°$, correct to one decimal place, for these simple equations.

a $\sin\theta = 0.3$ **b** $\sin\theta = 0.7$ **c** $\cos\theta = 0.6$
d $\cos\theta = 0.8$ **e** $\sin\theta = -0.2$ **f** $\sin\theta = -0.8$
g $\cos\theta = -0.4$ **h** $\cos\theta = 0.65$ **i** $\sin\theta = 0.48$

9 a Given that $\cos(-\theta) = \cos\theta$, complete the graph of $y = \cos\theta$ for $-360° \leqslant \theta \leqslant 360°$.



b Given that $\sin(-\theta) = -\sin\theta$, complete the graph of $y = \sin\theta$ for $-360° \leqslant \theta \leqslant 360°$.



ENRICHMENT: Trigonometric functions with technology

–	–	10

10 Use technology to sketch the graph of the following families of curves on the same axes, and then write a sentence describing the effect of the changing constant.

a	**i** $y = \sin x$	**ii** $y = -\sin x$	
b	**i** $y = \cos x$	**ii** $y = -\cos x$	
c	**i** $y = \sin x$	**ii** $y = 3\sin x$	**iii** $y = \frac{1}{2}\sin x$
d	**i** $y = \cos x$	**ii** $y = \cos(2x)$	**iii** $y = \cos\left(\frac{x}{3}\right)$
e	**i** $y = \sin x$	**ii** $y = \sin(x) + 2$	**iii** $y = \sin(x) - 1$
f	**i** $y = \cos x$	**ii** $y = \cos(x - 45°)$	**iii** $y = \cos(x + 60°)$

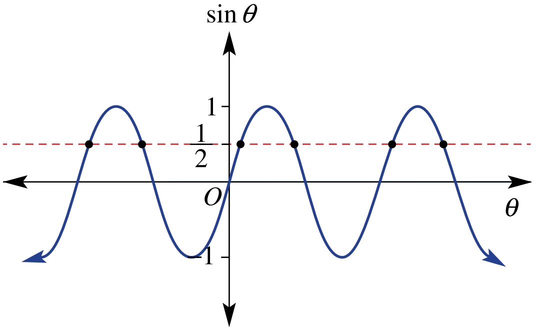
6K Exact values and solving trigonometric equations OPTIONAL

LEARNING INTENTIONS

- To know the exact values of 30°, 45° and 60° for the three trigonometric ratios
- To be able to write certain angles in terms of a reference angle of 30°, 45° or 60° and determine their exact value
- To understand that trigonometric equations can have multiple solutions
- To be able to solve simple trigonometric equations involving exact values

The one cycle shown in the graphs for sine and cosine in **Section 6J** continues forever. When solving a trigonometric equation there are an infinite number of solutions (points of intersection) if the values of θ are not restricted.

$$\sin \theta = \frac{1}{2}$$



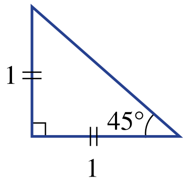
The trigonometric ratio sin 30° has an exact value of $\frac{1}{2}$. Some other common exact values will be established in this section.

Trigonometric equations can be used in areas such as oceanography to calculate the times when tides are at a particular height.

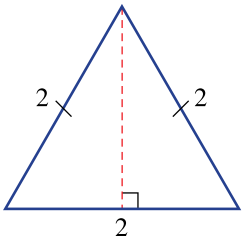
Lesson starter: Special triangles

The angles 30°, 45° and 60° when used with the sine, cosine and tangent ratios produce exact values.

These values can be established from two special triangles.



- A right-angled isosceles triangle with two side lengths of 1 unit is shown.
  - **a** Use Pythagoras' theorem to find the length of the hypotenuse.
  - **b** Hence, use SOHCAHTOA to find:
    - **i** sin 45°
    - **ii** cos 45°
    - **iii** tan 45°
- An equilateral triangle of side length 2 units.
  - **a** In the right-angled triangle formed, label each angle and side length.
  - **b** Hence, use SOHCAHTOA to find:
    - **i** sin 30°
    - **ii** cos 30°
    - **iii** tan 30°
    - **iv** sin 60°
    - **v** cos 60°
    - **vi** tan 60°



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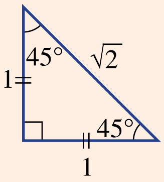
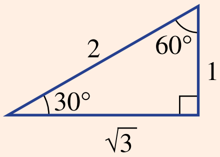
KEY IDEAS

- Exact values for sin θ, cos θ and tan θ can be obtained using two special triangles. Pythagoras' theorem can be used to confirm the length of each side.





$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad\sin 30° = \frac{1}{2}$ $\quad\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad\cos 30° = \frac{\sqrt{3}}{2}$ $\quad\cos 60° = \frac{1}{2}$

$\tan 45° = \frac{1}{1} = 1$ $\quad\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ $\quad\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$

- Exact values for sin θ, cos θ and tan θ for angles of 0°, 30°, 45°, 60° and 90° are given in this table.

$\theta°$	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	undefined

- The unit circle and the graphs of sine and cosine show that trigonometric equations have multiple solutions.
  - Exact values and the reference angle can be used to find one solution.
  - Symmetry can be used to find further solutions.

BUILDING UNDERSTANDING

1 Explain why $\cos 0 = 1$ and $\sin 0 = 0$.

2 Use the unit circle to find:

a cos 90° **b** sin 90°

3 State the missing values to complete this table.

$\theta°$	0	30	45	60	90	180	270	360
$\sin\theta$								
$\cos\theta$								
$\tan\theta$								

4 Fill in the blanks where the missing value is between 0° and 90°.

a $\sin \_\_\_ = \frac{1}{2}$ **b** $\cos \_\_\_ = \frac{\sqrt{3}}{2}$ **c** $\tan \_\_\_ = 1$

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Example 21 Using exact values

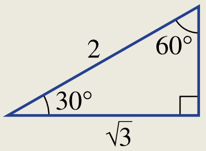
Find the exact value of each of the following.

a tan 30° **b** sin 150° **c** cos 240°

SOLUTION

a $\tan 30° = \frac{1}{\sqrt{3}}$

EXPLANATION

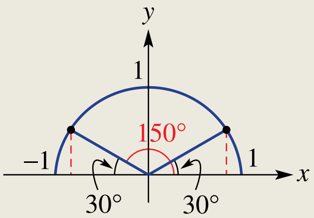


$\tan 30° = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{1}{\sqrt{3}}$

$= \frac{\sqrt{3}}{3}$ also

b $\sin 150° = \sin 30°$

$= \frac{1}{2}$



The sine of supplementary angles are equal and the exact value of sin 30° is $\frac{1}{2}$.

c $\cos 240° = -\cos 60°$

$= -\frac{1}{2}$

cos 240° is negative and 60° is the quadrant 1 reference angle for 240° since $240° - 180° = 60°$.

Now you try

Find the exact value of each of the following.

a cos 30° **b** sin 135° **c** tan 300°

Example 22 Solving simple trigonometric equations

Solve the following trigonometric equations for $0° \leqslant \theta \leqslant 360°$.

a $\sin\theta = \frac{1}{\sqrt{2}}$ **b** $\cos\theta = -\frac{\sqrt{3}}{2}$

SOLUTION

a $\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = 45°, 180° - 45°$

$\theta = 45°, 135°$

EXPLANATION

Use the table of exact values to find the quadrant 1 angle (45°). Sine is also positive in quadrant 2. Use symmetry to obtain the quadrant 2 angle: $180° - \theta$.

SOLUTION

b $\cos\theta = -\frac{\sqrt{3}}{2}$

$\theta = 180° - 60°,\ 180° + 60°$

$\theta = 120°, 240°$

EXPLANATION

Use the table of values to get the quadrant 1 reference angle: $\cos 60° = \frac{\sqrt{3}}{2}$. cos is negative in quadrants 2 and 3. Use symmetry to obtain the angles $180° - \theta$, $180° + \theta$.

Now you try

Solve the following trigonometric equations for $0° \leqslant \theta \leqslant 360°$.

a $\cos\theta = \frac{1}{2}$ **b** $\sin\theta = -\frac{1}{\sqrt{2}}$

Exercise 6K

FLUENCY	1(½), 2, 3–4(½)	1(½), 2, 3(½), 4	1(¼), 2, 3–4(½)

Example 21a,b

1 Find an exact value for each of the following.

a cos 30° **b** sin 45° **c** tan 60° **d** cos 45°
e cos 150° **f** tan 120° **g** sin 135° **h** cos 135°
i sin 120° **j** tan 150° **k** cos 120° **l** sin 150°
m tan 135° **n** sin 90° **o** cos 90° **p** tan 90°

2 Recall the exact sine, cosine and tangent values for 30°, 45° and 60°.

a State the reference angle for 225°.
b Hence, give the exact value of the following.
 i sin 225° **ii** cos 225° **iii** tan 225°
c State the reference angle for 330°.
d Hence, give the exact value of the following.
 i sin 330° **ii** cos 330° **iii** tan 330°

θ	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Example 21c

3 Give the exact value of the following.

a sin 210° **b** cos 240° **c** tan 300° **d** sin 240°
e tan 315° **f** cos 210° **g** sin 330° **h** tan 180°
i cos 225° **j** sin 270° **k** tan 330° **l** cos 300°
m sin 180° **n** tan 270° **o** tan 225° **p** cos 180°

Example 22a

4 Solve the following trigonometric equations for $0° \leqslant \theta \leqslant 360°$.

a $\sin\theta = \frac{1}{2}$ **b** $\sin\theta = \frac{\sqrt{3}}{2}$ **c** $\cos\theta = \frac{1}{2}$
d $\cos\theta = \frac{1}{\sqrt{2}}$ **e** $\tan\theta = \sqrt{3}$ **f** $\tan\theta = 1$

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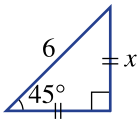
PROBLEM–SOLVING

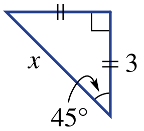
5–6(½) 5–7(½) 5–7(⅓)

5 Use trigonometric ratios to find the exact value of x. Calculators are not required.

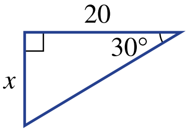
a



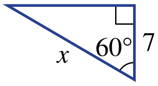
b



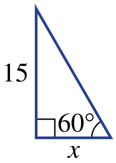
c



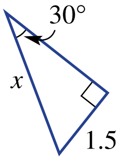
d



e

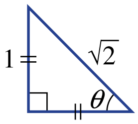


f

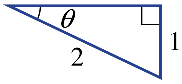


6 Find the exact value of θ without the use of a calculator.

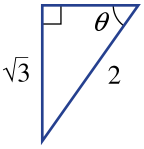
a



b



c



Example 22b

7 Solve the following trigonometric equations for $0° \leqslant \theta \leqslant 360°$.

a $\cos\theta = -\frac{1}{2}$ **b** $\sin\theta = -\frac{1}{\sqrt{2}}$ **c** $\tan\theta = -1$

d $\cos\theta = -\frac{1}{\sqrt{2}}$ **e** $\sin\theta = -\frac{\sqrt{3}}{2}$ **f** $\tan\theta = -\frac{1}{\sqrt{3}}$

REASONING

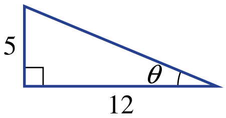
8 8, 9 8, 9

8 This right-angled triangle has its two shorter sides of length 5 and 12.

a Use Pythagoras' theorem to find the length of the hypotenuse.

b Find:

i $\sin\theta$ **ii** $\cos\theta$ **iii** $\tan\theta$



c Use $\tan\theta = \frac{\sin\theta}{\cos\theta}$ to verify your result from part **b iii**.

9 After 360° the cycle of sine and cosine repeats. For example, $\sin 30° = \sin(360° + 30°) = \sin 390°$. That is, $\sin\theta = \sin(360° + \theta)$ and $\cos\theta = \cos(360° + \theta)$. Use this result to solve the following equations for $0° \leqslant \theta \leqslant 720°$.

a $\sin\theta = \frac{1}{2}$ **b** $\cos\theta = \frac{1}{\sqrt{2}}$ **c** $\sin\theta = -\frac{\sqrt{3}}{2}$

ENRICHMENT: Rearranging trigonometric equations

– – 10–11(½)

10 Solve the following equations for $0° \leqslant \theta \leqslant 360°$ by first rearranging to make the trigonometric function the subject.

a $2\sin x - 1 = 0$ **b** $\sqrt{2}\cos x - 1 = 0$ **c** $\sqrt{3}\tan x - 2 = -1$

d $2\cos x + \sqrt{3} = 0$ **e** $6\sin x + 3 = 0$ **f** $1 + \tan x = 0$

11 Just as $\sin\theta = \sin(360° + \theta)$, $\sin\theta = \sin(\theta - 360°)$. The same applies for cosine. Use these results to solve the following equations.

a $2\sin x - \sqrt{3} = 0$ for $-360° \leqslant \theta \leqslant 360°$

b $\sqrt{2}\sin x + 1 = 0$ for $-360° \leqslant \theta \leqslant 360°$

c $2\cos x - 4 = -3$ for $-180° \leqslant \theta \leqslant 720°$

d $\sqrt{2}\cos x + 6 = 5$ for $-180° \leqslant \theta \leqslant 180°$

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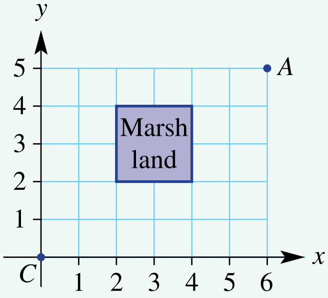
Around the marshland

On a bush hike, Andrea must walk from her campsite (C) to a navigation point A. Her map indicates that on the direct route between C and A there is a marshland and Andrea intends to avoid it by splitting the walk into two legs. If placed on a set of axes, the points $C(0, 0)$ and $A(6, 5)$ and the marshland can be illustrated as shown. All units are in kilometres.



Present a report for the following tasks and ensure that you show clear mathematical workings and explanations where appropriate.

Preliminary task

Andrea's first leg is on a bearing of 020°T for 5 km to point P.

- **a** Find the coordinates of point P correct to two decimal places.
- **b** Find the bearing and distance from point P to the navigation point A correct to one decimal place.
- **c** How much further did Andrea travel compared to the direct route CA through the marshland? Round to one decimal place.

Modelling task

Formulate

- **a** The problem is to find the shortest route around the marshland. Write down all the relevant information that will help solve this problem, with the aid of a diagram.
- **b** Sketch one possible path around the marshland including the legs CP and PA with the aid of a graph.

Solve

- **c** For the first leg CP, choose a bearing greater than 065°T and a distance for Andrea to walk.
- **d** Explain why your chosen first leg clearly avoids the marshland.
- **e** Determine the second leg bearing and distance from point P to point A.
- **f** Calculate the total extra distance travelled compared to the direct route CA.
- **g** Construct a path which misses the marshland but minimises the distance travelled from point C to point A. Include the following:
  - **i** distance travelled
  - **ii** bearing used
  - **iii** a graph.

Evaluate and verify

- **h** Compare the distances and bearings for the different paths considered earlier. Explain why your path from part **g** minimises the distance travelled.
- **i** Explain why walking to the south of the marshland is preferable to walking to the north of the marshland.

Communicate

- **j** Summarise your results and describe any key findings.

Extension question

- **a** Explore the effect on your results if the point A was shifted to a new position.

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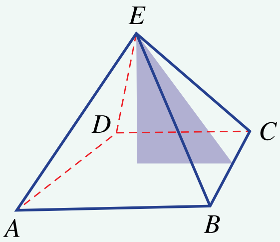
Filming pyramids

Key technology: Dynamic geometry

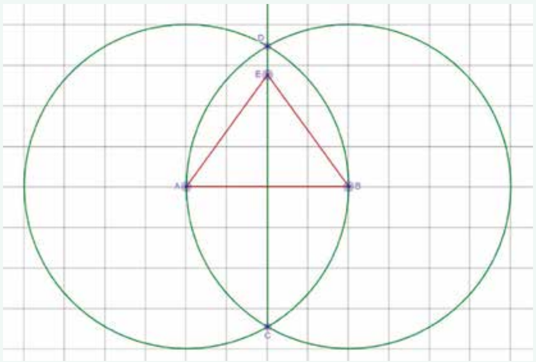
For over 4000 years the great pyramid of Giza was the tallest man-made structure in the world. It's also one of the world's most photographed objects. Today people use drones to photograph or film such objects and are able to capture images from positions that you cannot normally reach on foot. From the air it is much easier to confirm that many of the great Egyptian pyramids are square-based pyramids. The Great Pyramid of Giza has an approximate base length of 230 metres and vertical height 137 metres.

1 Getting started



First we will calculate some lengths and angles associated with the Great Pyramid of Giza, drawn here.

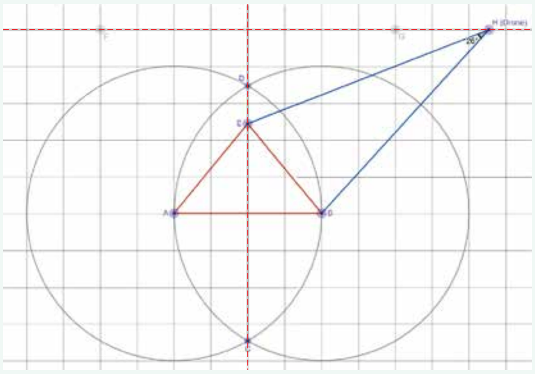
- **a** Draw the two-dimensional triangle shaded in purple and add the base and height measurements taken from the Great Pyramid of Giza information above.
- **b** Use trigonometry to calculate the angle that a face makes with the base rounded to the nearest degree. You will need this later in this investigation.



Dynamic geometry like Desmos geometry can be used to explore three-dimensional objects and two-dimensional representations of them.

- **c** Construct a perpendicular bisector by following this algorithm.
  - Step 1: Construct a line segment *AB*.
  - Step 2: Add the circles with centres *A* and *B* and radii *AB*.
  - Step 3: Add the intersection points of the circles *C* and *D*.
  - Step 4: Construct the perpendicular bisector of *AB* through *CD*.
- **d** Construct an isosceles triangle *ABE* by placing point *E* on *CD* and constructing the segments *AE* and *BE*.

2 Applying an algorithm



- **a** Follow this algorithm and use dynamic geometry software to construct a side view of a pyramid (looking at one face) with a drone camera at a height above the total height of the pyramid. Don't worry about lengths at this stage. Use the diagram shown to help.
  - Step 1: Use a grid and add the pyramid base edge *AB*.
  - Step 2: Construct the perpendicular bisector of *AB* to give *CD*.

- Step 3: Place point E (the apex of the pyramid) on CD.
- Step 4: Construct the horizontal drone path FG above the pyramid.
- Step 5: Place point H (the drone) onto FG.
- Step 6: Measure the viewing angle of the drone $\angle BHE$ to the face of the pyramid BE.

b Drag the point H to show how the viewing angle to the pyramid changes as the drone moves on a horizontal path.

3 Using technology

We will now explore where the drone should be positioned so that the viewing angle is maximised.

a Drag the point H and try to find a position where the viewing angle $\angle BHE$ is maximised.

b Alter the position of the point E to change the height of the pyramid but keep it below the height of the drone. Now re-position the drone to maximise the viewing angle. Repeat with different positions of point E.

c Try experimenting with the length AB and the height of drone path GH above the base AB.

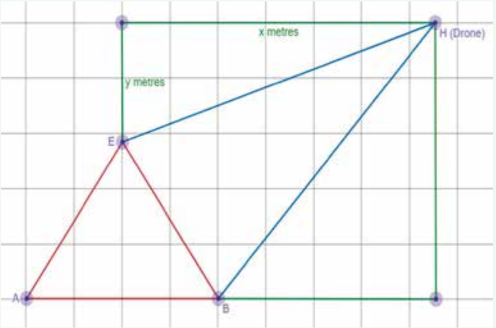
For the Great Pyramid of Giza we know the angle that the face makes with the ground. This was found in section **1** part **b**. Assume that the height of the drone is 1.5 times the distance AB.

d Make adjustments to your dynamic geometry construction so that it is a scale diagram of the Great Pyramid of Giza. Use the above given information. You will need to measure $\angle ABE$ on your construction and ensure it matches the angle found in section **1** part **b**. Also move the height of FG if required.

e Find the maximum viewing angle that the drone can use to capture images of the Great Pyramid of Giza at this height.

4 Extension

The given diagram illustrates a drone travelling at a distance of x metres horizontally from the apex of the pyramid and y metres above the apex.



a Add on the dimensions of the Great Pyramid of Giza.

b Choose a value for x and y and use trigonometry to find the viewing angle $\angle BHE$.

c Now find a formula for the viewing angle in terms of x and y if $x > 115$. Note that 115 metres is half the width of the pyramid.

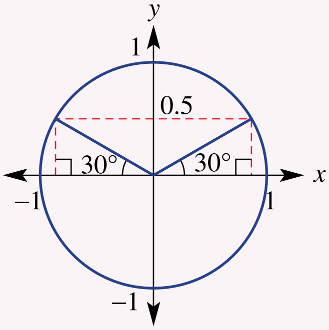
d Use your formula to confirm your answer to part **b** above.

e Now find a formula for the viewing angle in terms of x and y if $x < 115$.

f Use a spreadsheet or graph and your rules from parts **c** and **e** to find the value of x which maximises the viewing angle for a given value of y.

Solving trigonometric equations using a calculator

Trigonometric relations are not necessarily restricted to angles of less than 90°, and this is illustrated by drawing a graph of a trigonometric relation for angles up to 360°. Solving problems using trigonometric relations will therefore result in an equation that can have more than one solution.



For example, consider the equation $\sin\theta = 0.5$ for $0^\circ \leqslant \theta \leqslant 360^\circ$. Since $\sin\theta$ is the y-coordinate on the unit circle, there are two angles that satisfy $\sin\theta = 0.5$.

Solution 1: $\sin\theta = 0.5$

$$\theta = \sin^{-1}(0.5) = 30^\circ$$

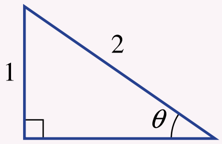
Solution 2:

$$\theta = 180^\circ - 30^\circ = 150^\circ$$

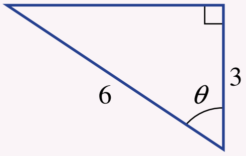
Single solutions ($0^\circ \leqslant \theta \leqslant 90^\circ$)

For these right-angled triangles, write an equation in terms of θ and then solve the equation to find θ.

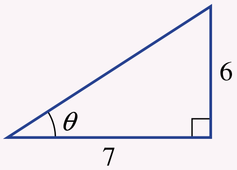
a



b

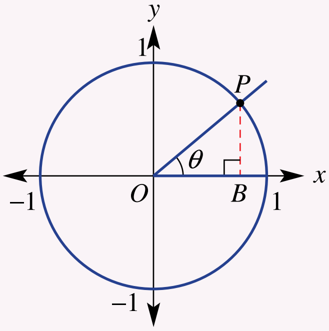


c



Two solutions ($0^\circ \leqslant \theta \leqslant 360^\circ$)

At point P on the unit circle $x = \cos\theta$ and $y = \sin\theta$.



For each of the following:

- **i** use a calculator to find a value for θ between 0° and 360°
- **ii** find a second angle between 0° and 360° that also satisfies the given trigonometric equation.

a $\sin\theta = 0.5$

b $\cos\theta = 0.2$

c $\cos\theta = -0.8$

d $\sin\theta = -0.9$

Harder trigonometric equations

Solve these trigonometric equations for $(0^\circ \leqslant \theta \leqslant 360^\circ)$.

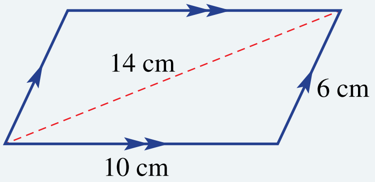
a $5\sin\theta - 1 = 0$

b $2\cos\theta + 3 = 0$

Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.

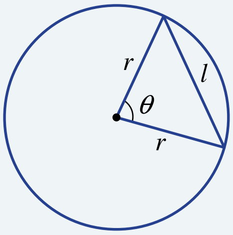
1 Two adjacent sides of a parallelogram have lengths of 6 cm and 10 cm. If the length of the longer diagonal is 14 cm, find:

- **a** the size of the internal angles of the parallelogram
- **b** the length of the other diagonal, to one decimal place.



2 Two cyclists, Stuart and Cadel, start their ride from the same starting point. Stuart travels 30 km on a bearing of 025°T, while Cadel travels the same distance but in a direction of 245°T. What is Cadel's bearing from Stuart after they have travelled the 30 km?

3 Show that for a circle of radius r the length of a chord l that subtends an angle θ at the centre of the circle is given by $l = \sqrt{2r^2(1 - \cos\theta)}$.



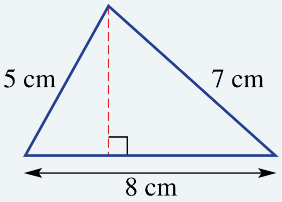
4 Akira measures the angle of elevation to the top of a mountain to be 20°. He walks 800 m horizontally towards the mountain and finds the angle of elevation has doubled. What is the height of the mountain above Akira's position, to the nearest metre?

5 A walking group sets out due east from the town hall at 8 km/h. At the same time, another walking group leaves from the town hall along a different road in a direction of 030°T at 5 km/h.

- **a** How long will it be before the groups are 15 km apart? Give your answer to the nearest minute.
- **b** What is the true bearing of the second group from the first group, to the nearest degree, at any time?

6 Edwina stands due south of a building 40 m tall to take a photograph of it. The angle of elevation to the top of the building is 23°. What is the angle of elevation, correct to two decimal places, after she walks 80 m due east to take another photo?

7 Calculate the height of the given triangle, correct to two decimal places.

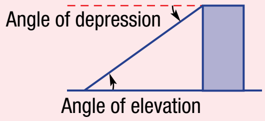


Chapter summary

Trigonometry

Elevation and depression

- Draw right-angled triangle with key information.
- Use trigonometry to find unknown.



Finding angles

If sin $\theta = k$

$\theta = \sin^{-1}(k)$

If cos $\theta = k$

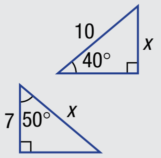
$\theta = \cos^{-1}(k)$

If tan $\theta = k$

$\theta = \tan^{-1}(k)$

$-1 \leqslant k \leqslant 1$ for $\sin\theta$ and $\cos\theta$.

Finding lengths



$\sin 40° = \frac{x}{10}$

$\therefore\ x = 10 \times \sin 40°$

$\cos 50° = \frac{7}{x}$

$x \times \cos 50° = 7$

$x = \frac{7}{\cos 50°}$

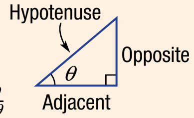
Trigonometric ratios

SOHCAHTOA

$\sin\theta = \frac{\text{O}}{\text{H}}$

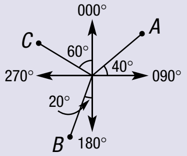
$\cos\theta = \frac{\text{A}}{\text{H}}$

$\tan\theta = \frac{\text{O}}{\text{A}} = \frac{\sin\theta}{\cos\theta}$



Bearings

True bearings are measured clockwise from north.



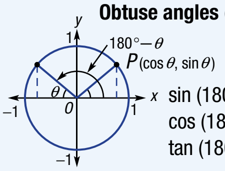
A : 050°T

B : 200°T

C : 300°T

Opposite directions differ by 180°.

Obtuse angles (Opt)



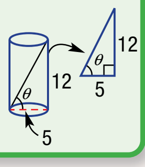
$\sin(180° - \theta) = \sin\theta$

$\cos(180° - \theta) = -\cos\theta$

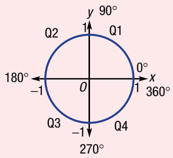
$\tan(180° - \theta) = -\tan\theta$

3D applications

- Identify and redraw right-angled triangle.
- Use trigonometric ratios.
- Answer in words.



4 quadrants of unit circle (Opt)



Q1: All are positive

Q2: only sin positive

Q3: only tan positive

Q4: only cos positive

The reference angle relates an angle to its acute angle in quadrant 1.

Symmetry can then be used to find sin θ or cos θ or tan θ.

e.g. sin 210° in quadrant 3 is negative

210° − 180° = 30°

sin 210° = −sin 30°

$= -\frac{1}{2}$

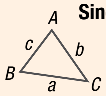
To find reference angle of θ in Q1:

Q2: $180° - \theta$

Q3: $\theta - 180°$

Q4: $360° - \theta$

Sine rule (Ext)

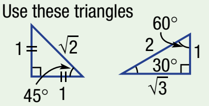


$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

When finding angles, determine whether you require the acute or the obtuse angle.

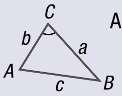
Exact values (Opt)

Use these triangles



θ	0°	30°	45°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

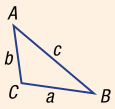
Area of a triangle (Ext)



Area $= \frac{1}{2}ab\sin C$

Cosine rule (Ext)

$c^2 = a^2 + b^2 - 2ab\cos C$



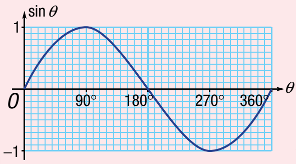
Rule is used to find:

- third side, given two sides and included angle

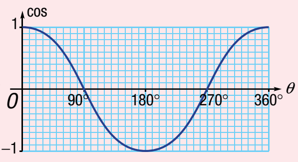
  or
- an angle, given all three sides.

Graphs of sin θ and cos θ (Opt)

$y = \sin\theta,\ 0° \leqslant \theta \leqslant 360°$



$y = \cos\theta,\ 0° \leqslant \theta \leqslant 360°$



Symmetry of the unit circle can be observed in graphs.

Solving trigonometric equations (Opt)

Use the reference angle and symmetry in unit circle quadrants to find the solutions.

e.g. If $\sin\theta = \frac{1}{2}$ $\quad 0° \leqslant \theta \leqslant 360°$

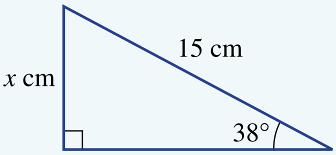
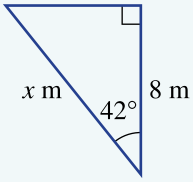
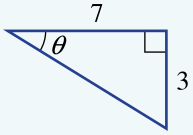
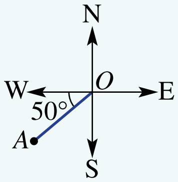
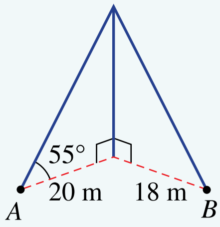
$\theta = 30°$ (Q1 reference angle)

or $\theta = 180° - 30°$ (Q2)

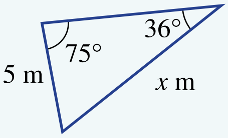
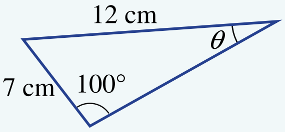
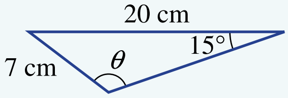
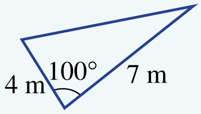
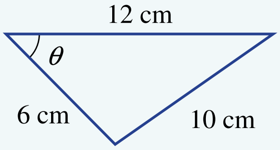
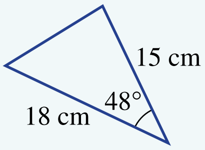
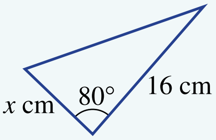
$= 150°$

Chapter checklist with success criteria

A printable version of this checklist is available in the Interactive Textbook

		✔
6A	**1. I can use trigonometry to solve for a side length in the numerator.** e.g. Find the value of x in the triangle shown, correct to two decimal places. 	☐
6A	**2. I can use trigonometry to solve for a side length in the denominator.** e.g. Find the value of x in the triangle shown, correct to two decimal places. 	☐
6B	**3. I can use trigonometry to find the value of an angle in a right-angled triangle.** e.g. For the triangle shown, find the value of θ correct to two decimal places. 	☐
6C	**4. I can apply trigonometry in word problems involving an angle of elevation.** e.g. From a boat at sea level to the top of a lighthouse, the angle of elevation is 25°. If the boat is a horizontal distance of 45 m from the base of the lighthouse, determine the height of the top of the lighthouse above sea level, correct to the nearest metre.	☐
6C	**5. I can apply trigonometry in word problems involving an angle of depression.** e.g. Two buildings 33 metres apart are 82 metres and 56 metres high. Find the angle of depression from the top of the taller building to the top of the shorter building, correct to two decimal places.	☐
6D	**6. I can state a direction and an opposite direction.** e.g. In the diagram, give the true bearing of A from O and O from A. 	☐
6D	**7. I can use bearings with trigonometry.** e.g. A hiker walks due south for 3 km, then on a true bearing of 220° for 2 km. Find how far west the hiker is from his starting point, correct to two decimal places.	☐
6E	**8 I can apply trigonometry in 3D.** e.g. A vertical flagpole is supported by two wires reaching from points A and B on level ground as shown. The wire from point A is attached 20 m from the base of the flag pole and the wire from point B is attached 18 m from the base of the flag pole. If the angle of elevation from point A to the top of the flag pole is 55°, find the angle of elevation from B to the top of the flag pole correct to one decimal place. 	☐

Chapter checklist with success criteria

				✔
6F	**9. I can find a side length using the sine rule.** e.g. Find the value of x in this triangle, correct to one decimal place.		Ext	☐
6F	**10. I can find an acute angle using the sine rule.** e.g. Find the value of θ in this triangle correct to one decimal place.		Ext	☐
6F	**11. I can find an obtuse angle using the sine rule.** e.g. Find the value of θ in this triangle correct to one decimal place if θ is obtuse.		Ext	☐
6G	**12. I can use the cosine rule to find a side length.** e.g. Find the length of the third side in this triangle, correct to two decimal places.		Ext	☐
6G	**13. I can use the cosine rule to find an angle.** e.g. Find the value of θ in this triangle, correct to two decimal places.		Ext	☐
6H	**14. I can find the area of a triangle without knowing the vertical height.** e.g. Find the area of this triangle, correct to one decimal place.		Ext	☐
6H	**15. I can use the area of a triangle to find an unknown side length.** e.g. Find the value of x in this triangle, correct to two decimal places, given that the area of the triangle is $49\,\text{cm}^2$.		Ext	☐
6I	**16. I can position a point on the unit circle.** e.g. Decide in which quadrant $\theta = 250°$ and $\theta = -100°$ each lie and state whether $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive or negative for each.		Opt	☐
6I	**17. I can choose an obtuse angle to relate a trigonometric ratio to an acute angle.** e.g. Choose an obtuse angle to complete the statement: $\cos 33° = -\cos$ ___.		Opt	☐

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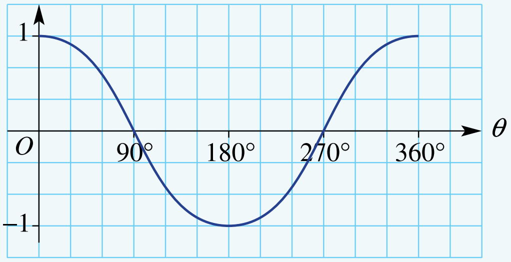
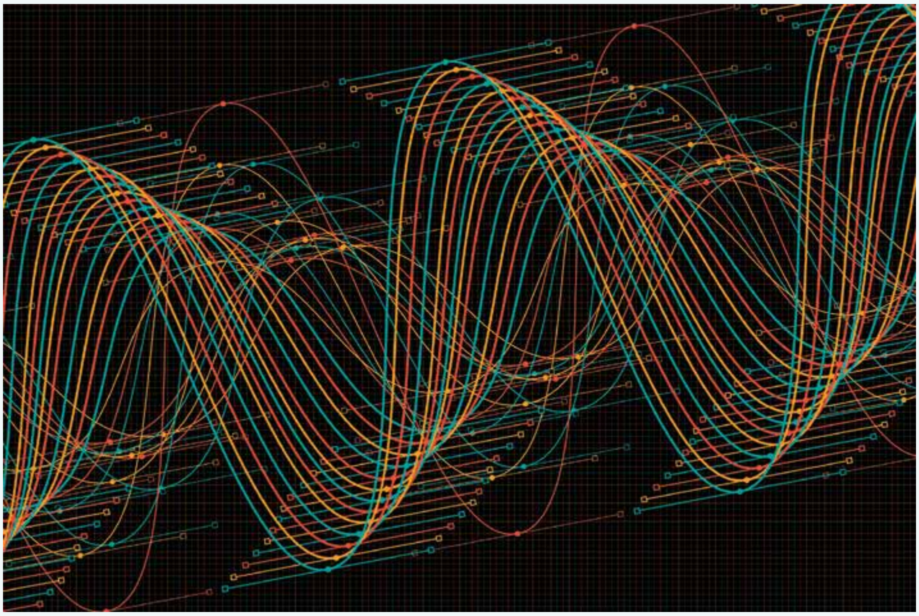
Chapter checklist with success criteria

			✔
6I	**18. I can rewrite a trigonometric value using its reference angle.** e.g. Write cos 300° and sin 195° using their reference angles.	Opt	☐
6J	**19. I can use the graph of sine or cosine.** e.g. Use this graph of $y = \cos\theta$ to estimate the value of cos 130° and the two values of θ for which $\cos\theta = 0.4$.  	Opt	☐
6J	**20. I can use a graph to estimate the size of sine or cosine of an angle.** e.g. Use the graph of $y = \cos\theta$ to state whether or not cos 50° > cos 310°.	Opt	☐
6K	**21. I can use exact values.** e.g. Find the exact value of sin 120° and cos 315°.	Opt	☐
6K	**22. I can solve simple trigonometric equations.** e.g. Solve $\sin\theta = \frac{\sqrt{3}}{2}$ for $0° \leqslant \theta \leqslant 360°$.	Opt	☐

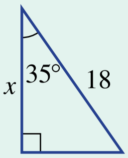


Short-answer questions

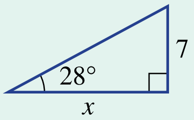
6A

1 Find the value of each pronumeral, rounding your answer to two decimal places.

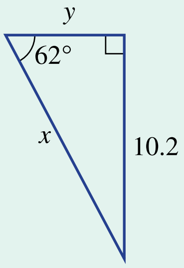
a



b



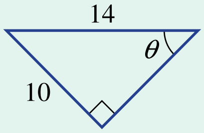
c



6B

2 Find the value of θ, correct to one decimal place.

a



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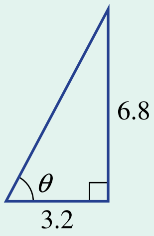
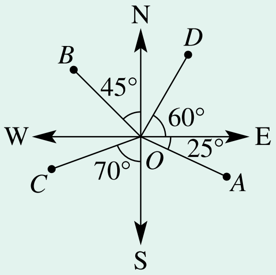


6C

3 An escalator in a shopping centre from level 1 to level 2 is 22 m in length and has an angle of elevation of 16°. Determine how high level 2 is above level 1, to one decimal place.

6D

4 **a** Write each bearing A–D as a true bearing.
  b Give the true bearing of:
    i O from A
    ii O from C.

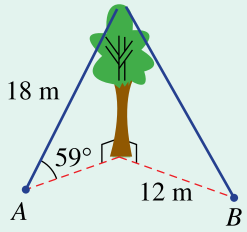


6D

5 A helicopter flies due south for 160 km and then on a bearing of 125°T for 120 km. Answer the following, to one decimal place.
  a How far east is the helicopter from its start location?
  b How far south is the helicopter from its start location?
  c What bearing must it fly on to return directly to the start location?

6E

6 A tree is being supported by two ropes, as shown. The rope to point A is 18 m long and makes an angle of 59° with the ground. Point B is 12 m from the base of the tree.
  a Find the height of the tree, to two decimal places.
  b Find the angle the rope to point B makes with the ground, to the nearest degree.



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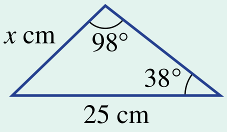
6F/H Ext

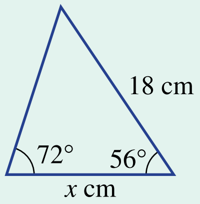
7 For these triangles, find, correct to one decimal place:

- **i** the value of x
- **ii** the area of the triangle.

a

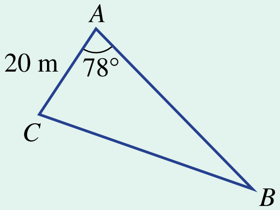


b



6H Ext

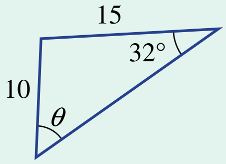
8 Three fences are used to form a triangular pig pen with known dimensions, as shown in the diagram. If the area of the pig pen is 275 m^2, what is the length AB? Round your answer to one decimal place.



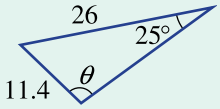
6F Ext

9 Use the sine rule to find the value of θ, correct to one decimal place.

a θ is acute.



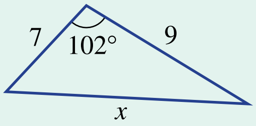
b θ is obtuse.



6G Ext

10 Use the cosine rule to find the value of the pronumeral, to one decimal place.

a



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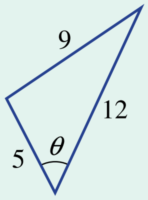


6I/K Opt

11 **a** State whether the following are positive or negative.

- **i** cos 158°
- **ii** tan 231°
- **iii** sin 333°
- **iv** cos 295°

b Rewrite the following using their reference angle.

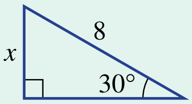
- **i** sin 120°
- **ii** cos 210°
- **iii** tan 315°
- **iv** sin 225°

c Hence, give the exact value of each in part **b**.

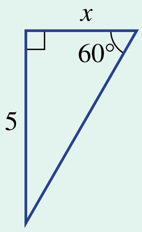
6K Opt

12 Use exact values to find the value of the pronumerals, without using a calculator.

a



b



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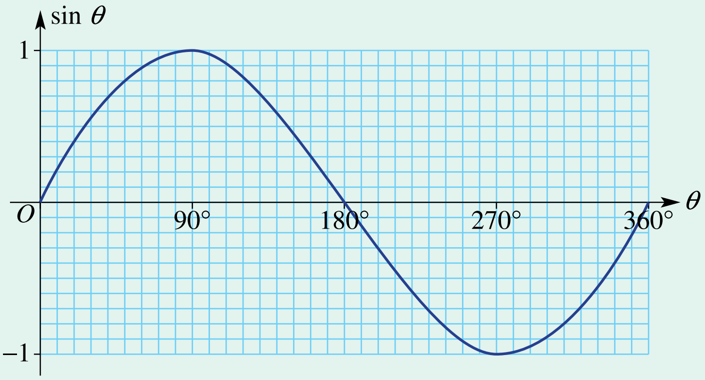
6J Opt

13 Use the graph of $\sin\theta$ shown to complete the following.



a Estimate the value of:
  i $\sin 130°$
  ii $\sin 255°$

b Find the values of θ between 0° and 360° such that:
  i $\sin\theta = 0.8$
  ii $\sin\theta = -0.3$
  iii $\sin\theta = 1.5$

c State if the following are true or false.
  i $\sin 90° = 1$ **ii** $\sin 75° > \sin 140°$ **iii** $\sin 220° < \sin 250°$

6J Opt

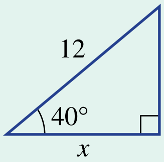
14 Solve the following equations for $0° \leqslant x \leqslant 360°$.

a $\cos x = \frac{1}{\sqrt{2}}$ **b** $\sin x = -\frac{1}{2}$

Multiple-choice questions

6A

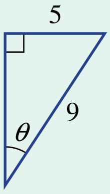
1 The value of x in the diagram shown is equal to:



A $\frac{12}{\cos 40°}$ **B** $12\sin 40°$ **C** $\frac{\sin 40°}{12}$
D $12\cos 40°$ **E** $\frac{12}{\tan 40°}$

6B

2 The angle θ, correct to one decimal place, is:



A 56.3° **B** 33.7° **C** 29.1°
D 60.9° **E** 42.4°

6C

3 The angle of depression from the top of a communications tower measuring 44 m tall to the top of a communications tower measuring 31 m tall is 18°. The horizontal distance between the two towers is closest to:

A 12 m **B** 4 m **C** 14 m **D** 42 m **E** 40 m

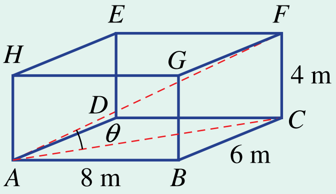
6D

4 A yacht is sailed from A to B on a bearing of 196°T. To sail from B directly back to A the true bearing would be:

A 074°T **B** 096°T **C** 164°T **D** 016°T **E** 286°T

6E

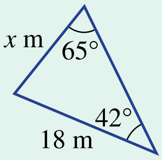
5 The angle θ that AF makes with the base of the rectangular prism is closest to:



A 22° **B** 68° **C** 16°
D 24° **E** 27°

6F Ext

6 The side length x in the triangle shown, correct to one decimal place, is:



A 10.9 **B** 29.7 **C** 13.3
D 12.6 **E** 17.1

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6G Ext

7 The smallest angle in the triangle with side lengths 8 cm, 13 cm and 19 cm, to the nearest degree is:

A 19° **B** 33° **C** 52° **D** 24° **E** 29°

6H Ext

8 The area of the triangle shown can be determined by calculating:

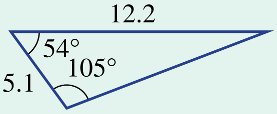
A $\frac{1}{2} \times 5.1 \times 12.2 \times \cos 54°$

B $\frac{1}{2} \times 5.1 \times 12.2 \times \sin 105°$

C $\frac{1}{2} \times 12.2 \times 5.1$

D $\frac{1}{2} \times 12.2 \times 5.1 \times \sin 54°$

E $\frac{1}{2} \times 6.1 \times 5.1 \times \sin 21°$



6I Opt

9 The incorrect statement below is:

A $\cos 110° = -\cos 70°$

B $\cos 246°$ is negative

C $\tan 130°$ is positive

D $\sin 150° = \sin 30°$

E $\sin 300°$ is negative and $\cos 300°$ is positive

6K Opt

10 The exact value of $\sin 120°$ is:

A $\frac{1}{2}$ **B** $-\frac{\sqrt{3}}{2}$ **C** $\frac{\sqrt{3}}{2}$ **D** $\frac{\sqrt{2}}{2}$ **E** $-\frac{1}{2}$

Extended-response questions

1 A group of friends set out on a hike to a waterfall in a national park. They are given the following directions to walk from the park's entrance to the waterfall to avoid having to cross a river.

Walk 5 km on a bearing of 325°T and then 3 km due north.

Round each answer to one decimal place.

a Draw and label a diagram to represent this hike.

b Determine how far east or west the waterfall is from the entrance.

c Find the direct distance from the park's entrance to the waterfall.

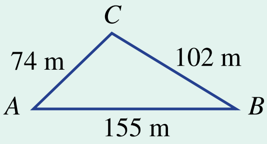
The friends set up tents on level ground at the base of the waterfall at points A (35 m from the base of the waterfall) and B (28 m from the base of the waterfall). The angle of elevation from A to the top of the waterfall is 32°.

d Determine:

- **i** the height of the waterfall
- **ii** the angle of elevation from B to the top of the waterfall.

Ext

2 A paddock, ABC, is fenced off, as shown in the figure.



a Find $\angle A$, to three decimal places.

b Hence, find the area enclosed by the fences. Round your answer to two decimal places.

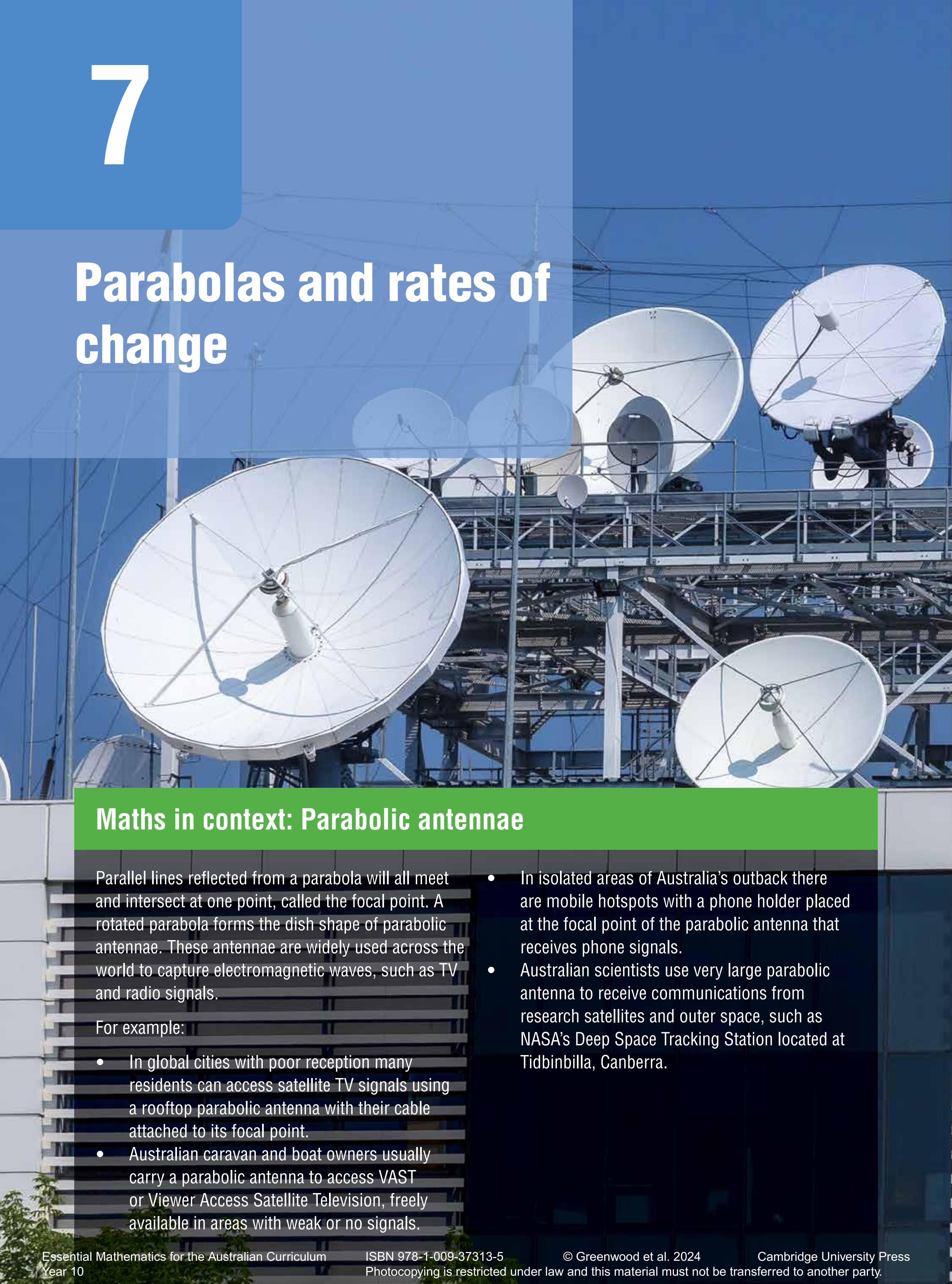
It is planned to divide the paddock into two triangular paddocks by constructing a fence from point C to meet AB at right angles at a point D.

c Determine how many metres of fencing will be required along CD, to the nearest centimetre.

d How far is point D from point A, to the nearest centimetre?

e The person who constructs the fence CD misinterprets the information and builds a fence that does not meet AB at right angles. The fence is 45 metres long.

- **i** Determine, to two decimal places, the two possible angles (i.e. acute and obtuse), this fence line makes with AB.
- **ii** Hence, find the two possible distances of fence post D from A. Round your answer to one decimal place.



7

Parabolas and rates of change

Maths in context: Parabolic antennae

Parallel lines reflected from a parabola will all meet and intersect at one point, called the focal point. A rotated parabola forms the dish shape of parabolic antennae. These antennae are widely used across the world to capture electromagnetic waves, such as TV and radio signals.

For example:

- In global cities with poor reception many residents can access satellite TV signals using a rooftop parabolic antenna with their cable attached to its focal point.
- Australian caravan and boat owners usually carry a parabolic antenna to access VAST or Viewer Access Satellite Television, freely available in areas with weak or no signals.
- In isolated areas of Australia's outback there are mobile hotspots with a phone holder placed at the focal point of the parabolic antenna that receives phone signals.
- Australian scientists use very large parabolic antenna to receive communications from research satellites and outer space, such as NASA's Deep Space Tracking Station located at Tidbinbilla, Canberra.

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Chapter contents

Australian Curriculum 9.0

ALGEBRA

Use mathematical modelling to solve applied problems involving growth and decay, including financial contexts; formulate problems, choosing to apply linear, quadratic or exponential models; interpret solutions in terms of the situation; evaluate and modify models as necessary and report assumptions, methods and findings (AC9M10A04)

Experiment with functions and relations using digital tools, making and testing conjectures and generalising emerging patterns (AC9M10A05)

Algebraic representations of quadratic functions of the form $f(x) = ax^2 + bx + c$ where a, b, and c are non-zero integers, and their transformation to the form $f(x) = a(x + h)^2 + k$ where h and k are non-zero rational numbers, and the solution of related equations (Year 10 optional content)

MEASUREMENT

The effect of increasingly small changes in the value of variables on the average rate of change and in relation to limiting values (Year 10 optional content)

Online resources

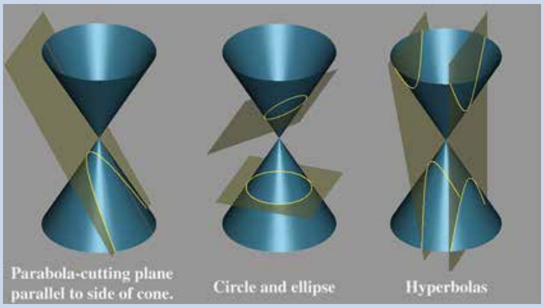
A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

7A Exploring parabolas

LEARNING INTENTIONS

- To know the shape and symmetry of the basic parabola $y = x^2$
- To be able to identify the key features of a parabola from a graph
- To be able to observe the impact of transformations of $y = x^2$

One of the simplest and most important non-linear graphs is the parabola. When a ball is thrown or water streams up and out from a garden hose, the path followed has a parabolic shape. The parabola is the graph of a quadratic relation with the basic rule $y = x^2$. Quadratic rules, such as $y = (x - 1)^2$, $y = 2x^2 - x - 3$ and $y = (x + 4)^2 - 7$, also give graphs that are parabolas and are transformations of the graph of $y = x^2$.

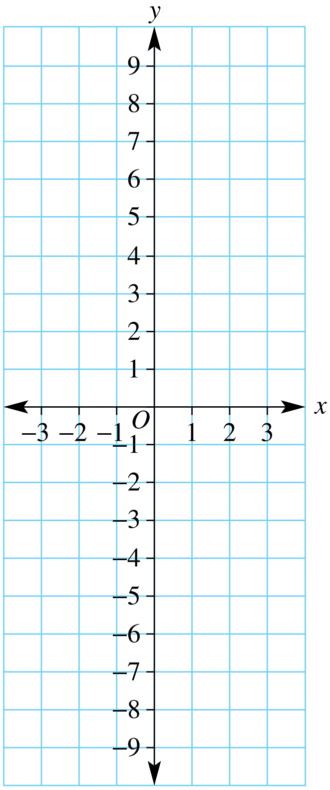


Intersecting a cone with a plane forms curves called the conic sections. Greek scholars analysed the conics using geometry. From the 17th century, Descartes' newly discovered Cartesian geometry enabled a more powerful algebraic analysis.

Lesson starter: To what effect?

To see how different quadratic rules compare to the graph of $y = x^2$, complete this table and plot the graph of each equation on the same set of axes.

x	-3	-2	-1	0	1	2	3
$y_1 = x^2$	9	4					
$y_2 = -x^2$	-9						
$y_3 = (x - 2)^2$							
$y_4 = x^2 - 3$							



- For all the graphs, find such features as the:
  - turning point
  - axis of symmetry
  - y-intercept
  - x-intercepts.
- Discuss how each of the graphs of y_2, y_3 and y_4 compare to the graph of $y = x^2$. Compare the rule with the position of the graph.

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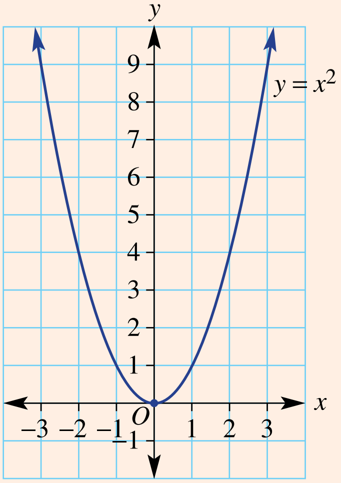
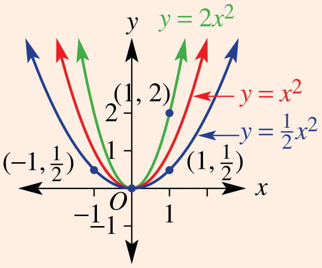
KEY IDEAS

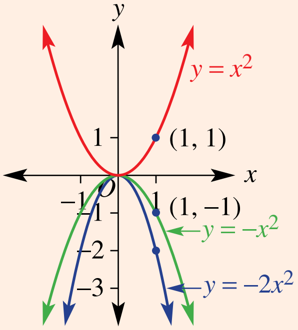
- A **parabola** is the graph of a quadratic relation. The basic parabola has the rule $y = x^2$.
  - The vertex (or turning point) is $(0, 0)$.
  - It is a minimum turning point.
  - Axis of symmetry is $x = 0$.
  - y-intercept has coordinates $(0, 0)$.
  - x-intercept has coordinates $(0, 0)$.



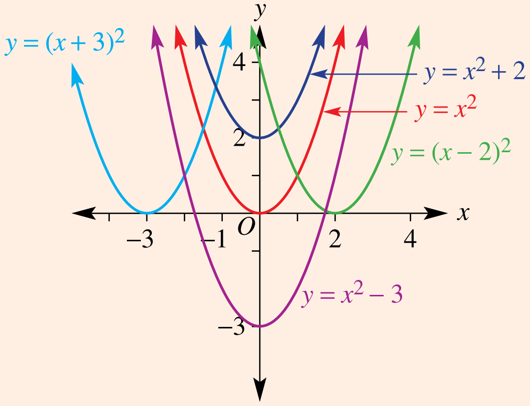
- Simple transformations of the graph of $y = x^2$ include:
  - dilation



  - reflection



  - translation



BUILDING UNDERSTANDING

① Complete the features of this graph.

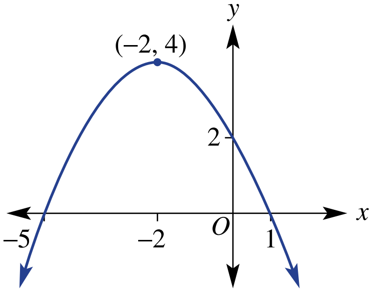
a The parabola has a __________ (*maximum* or *minimum*).

b The coordinates of the turning point are __________.

c The y-intercept coordinates are (0, __).

d The x-intercepts are at (__, 0) and (__, 0).

e The axis of symmetry is __________.



2 Complete the features of this graph.

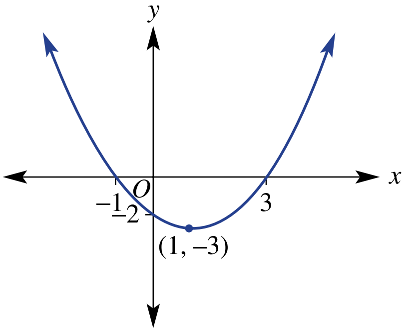
a The parabola has a __________ (*maximum* or *minimum*).

b The coordinates of the turning point are __________.

c The y-intercept coordinates are (0, ___).

d The x-intercepts are (__, 0) and (__, 0).

e The axis of symmetry is __________.



Example 1 Identifying key features of parabolas

Determine the following key features of each of the given graphs.

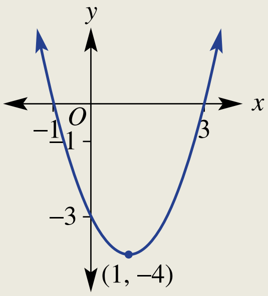
i turning point and whether it is a maximum or minimum

ii axis of symmetry

iii x-intercept coordinates

iv y-intercept coordinates

a



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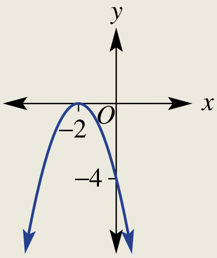

SOLUTION

a **i** Turning point is a minimum at $(1, -4)$.

ii Axis of symmetry is $x = 1$.

iii x-intercepts are at $(-1, 0)$ and $(3, 0)$.

iv y-intercept is at $(0, -3)$.

b **i** Turning point is a maximum at $(-2, 0)$.

ii Axis of symmetry is $x = -2$.

iii x-intercept is at $(-2, 0)$.

iv y-intercept is at $(0, -4)$.

EXPLANATION

Lowest point of graph is at $(1, -4)$.

Line of symmetry is through the x-coordinate of the turning point.

x-intercepts lie on the x-axis ($y = 0$) and the y-intercept on the y-axis ($x = 0$).

Graph has a highest point at $(-2, 0)$.

Line of symmetry is through the x-coordinate of the turning point.

Turning point is also the one x-intercept.

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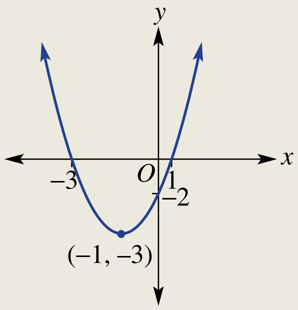
Now you try

Determine the following key features of each of the given graphs.

- **i** turning point and whether it is a maximum or minimum
- **ii** axis of symmetry
- **iii** x-intercept coordinates
- **iv** y-intercept coordinates

a



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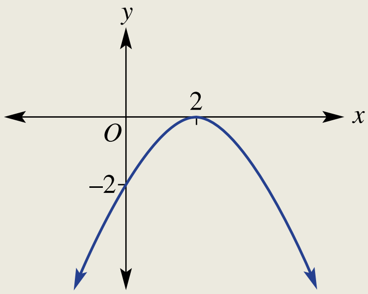
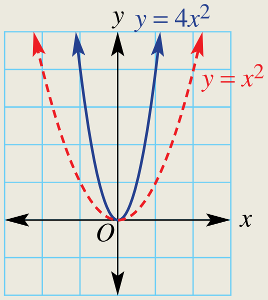


Example 2 Transforming parabolas

Copy and complete the table for the following graphs.

a



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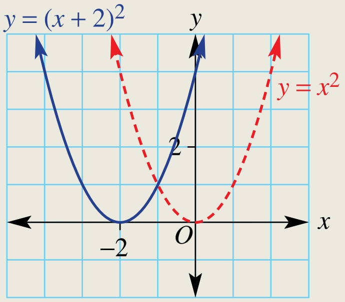
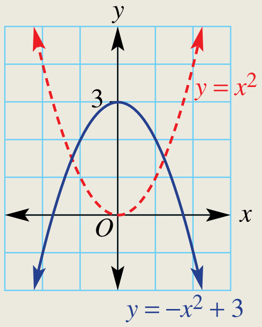


c



	Formula	Maximum or minimum	Reflected in the x-axis (yes/no)	Turning point	y-value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 4x^2$					
b	$y = (x + 2)^2$					
c	$y = -x^2 + 3$					

SOLUTION

	Formula	Maximum or minimum	Reflected in the x-axis (yes/no)	Turning point	y-value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 4x^2$	minimum	no	(0, 0)	4	narrower
b	$y = (x + 2)^2$	minimum	no	(−2, 0)	9	same
c	$y = -x^2 + 3$	maximum	yes	(0, 3)	2	same

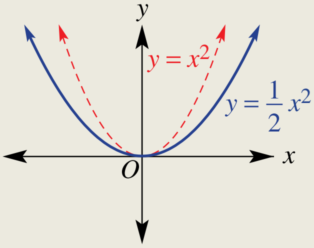
EXPLANATION

Read features from graphs and consider the effect of each change in equation on the graph.

Now you try

Copy and complete the table for the following graphs.

a



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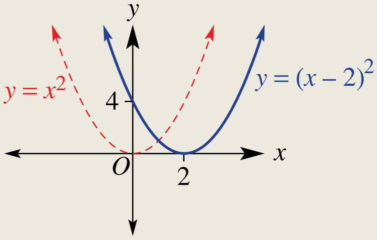
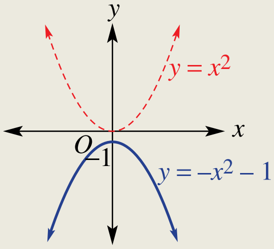


c



	Formula	Maximum or minimum	Reflected in the x-axis (yes/no)	Turning point	y-value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = \frac{1}{2}x^2$					
b	$y = (x - 2)^2$					
c	$y = -x^2 - 1$					

Exercise 7A

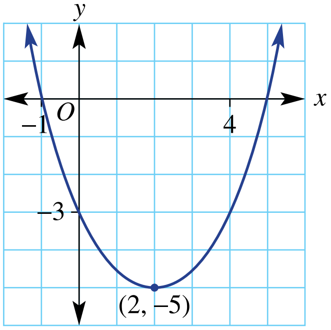
FLUENCY	1–4	1–4(½)	2–4(½)

Example 1

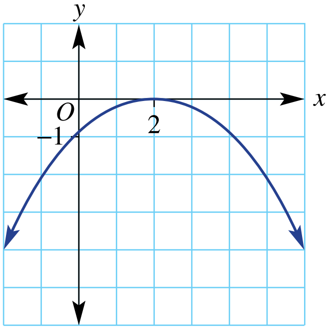
1 Determine these key features of the following graphs.

- **i** turning point and whether it is a maximum or minimum
- **ii** axis of symmetry
- **iii** x-intercept coordinates
- **iv** y-intercept coordinates

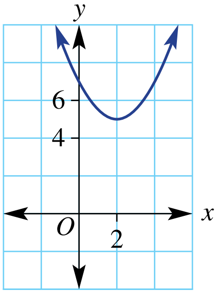
a



b



c



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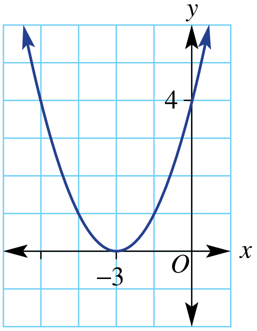

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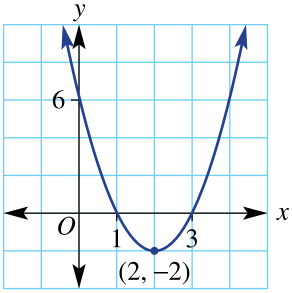

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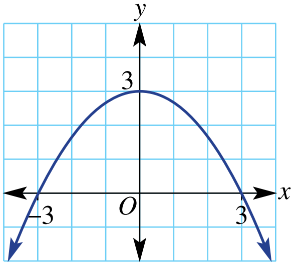
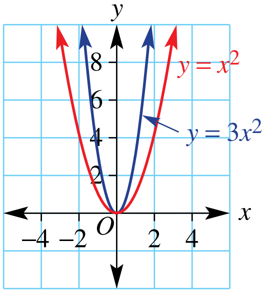


Example 2a

2 Copy and complete the table below for the following graphs.

a



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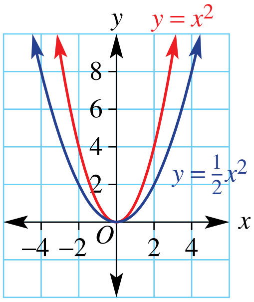

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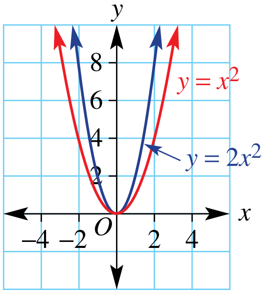

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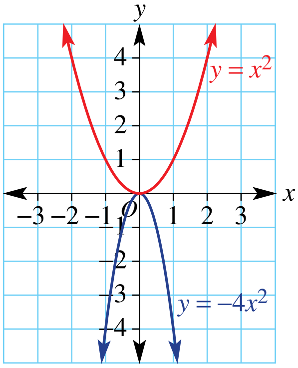

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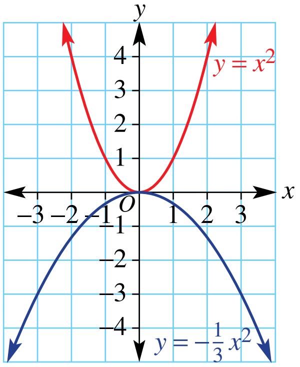

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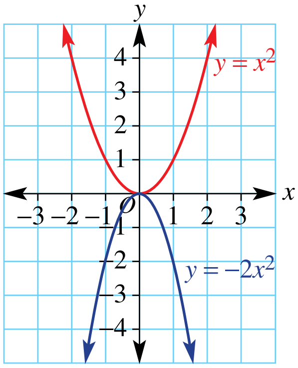

	Formula	**Maximum or minimum**	**Reflected in the x-axis (yes/no)**	**Turning point**	**y-value when $x = 1$**	**Wider or narrower than $y = x^2$**
a	$y = 3x^2$					
b	$y = \frac{1}{2}x^2$					
c	$y = 2x^2$					
d	$y = -4x^2$					
e	$y = -\frac{1}{3}x^2$					
f	$y = -2x^2$					

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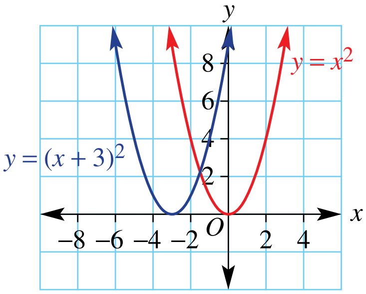
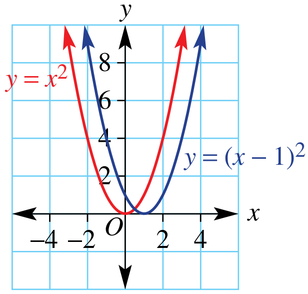
Example 2b

3 Copy and complete the table below for the following graphs.

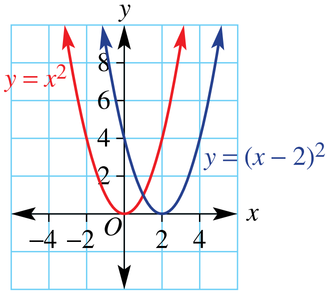
a



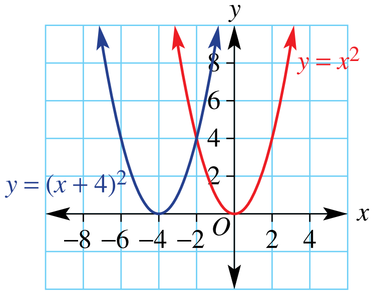
b



c



d

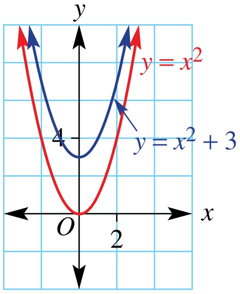


	Formula	Turning point	Axis of symmetry	y-intercept coordinates ($x = 0$)	x-intercept coordinates
a	$y = (x + 3)^2$				
b	$y = (x - 1)^2$				
c	$y = (x - 2)^2$				
d	$y = (x + 4)^2$				

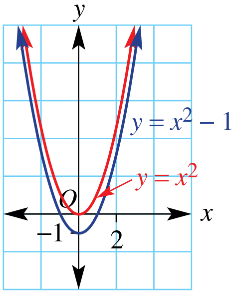
Example 2c

4 Copy and complete the table for the following graphs.

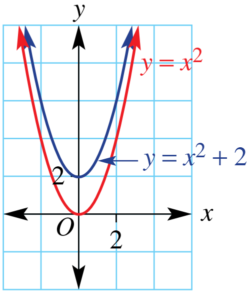
a



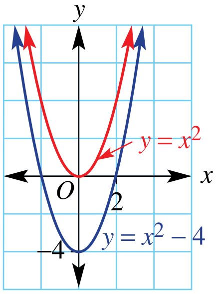
b



c



d



	Formula	Turning point	y-intercept coordinates ($x = 0$)	y-value when $x = 1$
a	$y = x^2 + 3$			
b	$y = x^2 - 1$			
c	$y = x^2 + 2$			
d	$y = x^2 - 4$			

PROBLEM–SOLVING	5–7(½)	5–7(½), 8	5–7(⅓), 8

5 Write down the equation of the axis of symmetry for the graphs of these rules.

a $y = x^2$
b $y = x^2 + 7$
c $y = -2x^2$
d $y = -3x^2$
e $y = x^2 - 4$
f $y = (x - 2)^2$
g $y = (x + 1)^2$
h $y = -(x + 3)^2$
i $y = -x^2 - 3$
j $y = \frac{1}{2}x^2 + 2$
k $y = x^2 - 16$
l $y = -(x + 4)^2$

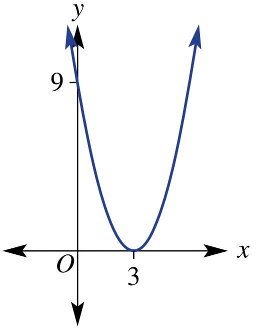
6 Write down the coordinates of the turning point for the graphs of the equations in Question **5**.

7 Find the coordinates of the y-intercept (i.e. when $x = 0$) for the graphs of the equations in Question **5**.

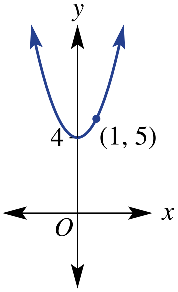
8 Match each of the following equations to one of the graphs below.

a $y = 2x^2$
b $y = x^2 - 6$
c $y = (x + 2)^2$
d $y = -5x^2$
e $y = (x - 3)^2$
f $y = \frac{1}{4}x^2$
g $y = x^2 + 4$
h $y = (x - 2)^2 + 3$

A



B



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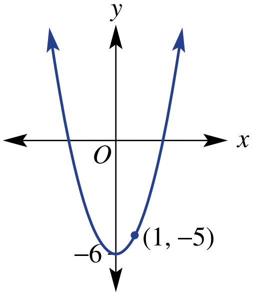

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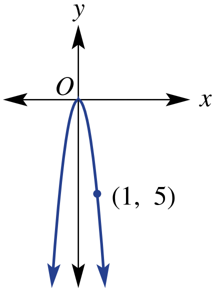

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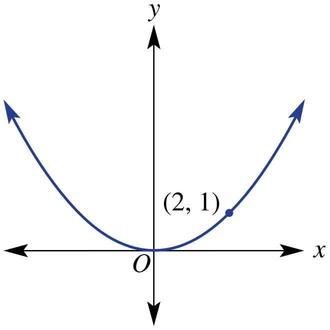

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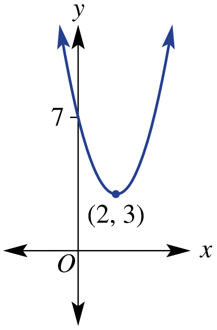
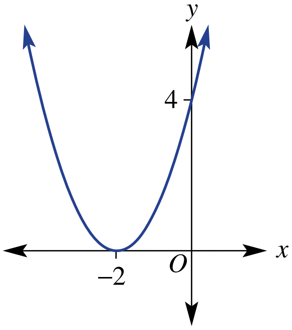
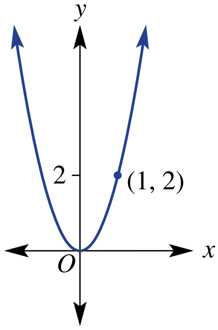


G



H



REASONING	9–11	9–11	9–11, 12

9 **a** Using technology, plot the following pairs of graphs on the same set of axes for $-5 \leqslant x \leqslant 5$ and compare their tables of values.

i $y = x^2$ and $y = 4x^2$
ii $y = x^2$ and $y = \frac{1}{3}x^2$
iii $y = x^2$ and $y = 6x^2$
iv $y = x^2$ and $y = \frac{1}{4}x^2$
v $y = x^2$ and $y = 7x^2$
vi $y = x^2$ and $y = \frac{2}{5}x^2$

b Suggest how the constant a in $y = ax^2$ transforms the graph of $y = x^2$.

10 **a** Using technology, plot the following sets of graphs on the same set of axes for $-5 \leqslant x \leqslant 5$ and compare the turning point of each.

i $y = x^2, y = (x + 1)^2, y = (x + 2)^2, y = (x + 3)^2$
ii $y = x^2, y = (x - 1)^2, y = (x - 2)^2, y = (x - 3)^2$

b Explain how the constant h in $y = (x + h)^2$ transforms the graph of $y = x^2$.

11 **a** Using technology, plot the following sets of graphs on the same set of axes for $-5 \leqslant x \leqslant 5$ and compare the turning point of each.

i $y = x^2, y = x^2 + 1, y = x^2 + 2, y = x^2 + 3$
ii $y = x^2, y = x^2 - 1, y = x^2 - 3, y = x^2 - 5$

b Explain how the constant k in $y = x^2 + k$ transforms the graph of $y = x^2$.

12 Write down an example of a quadratic equation whose graph has:

a two x-intercepts
b one x-intercept
c no x-intercepts.

ENRICHMENT: Finding the rule	–	–	13(½), 14

13 Find a quadratic rule that satisfies the following information.

a turning point (0, 2) and another point (1, 3)
b turning point (0, 2) and another point (1, 1)
c turning point (−1, 0) and y-intercept (0, 1)
d turning point (2, 0) and y-intercept (0, 4)
e turning point (0, 0) and another point (2, 8)
f turning point (0, 0) and another point (−1, −3)
g turning point (−1, 2) and y-intercept (0, 3)
h turning point (4, −2) and y-intercept (0, 0)

14 Plot a graph of the parabola $x = y^2$ for $-3 \leqslant y \leqslant 3$ and describe its features.

This parabolic solar power collecting array is such that a fluid is heated by the sun and its heat is converted into electricity.

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7B Sketching parabolas using transformations

LEARNING INTENTIONS

- To know the types of transformations: dilation, reflection and translation
- To understand the effect of these transformations on the graph of $y = x^2$
- To know how to determine the turning point of a quadratic rule from turning point form
- To be able to sketch a quadratic graph from turning point form
- To be able to find the rule of a quadratic graph given the turning point and another point

Previously we have explored simple transformations of the graph of $y = x^2$ and plotted these on a number plane. We will now formalise these transformations and sketch graphs showing key features without the need to plot every point.

Parabolic flight paths occur in athletic jumping and throwing events. Using photography and parabola transformations, sports scientists can find quadratic equations for specific trajectories. Comparing actual and ideal parabolas may reveal areas for technique improvement.

Lesson starter: So where is the turning point?

Consider the quadratic rule $y = -(x - 3)^2 + 7$.

- Discuss the effect of the negative sign in $y = -x^2$ compared with $y = x^2$.
- Discuss the effect of -3 in $y = (x - 3)^2$ compared with $y = x^2$.
- Discuss the effect of $+7$ in $y = x^2 + 7$ compared with $y = x^2$.
- Now for $y = -(x - 3)^2 + 7$, find:
  - the coordinates of the turning point
  - the axis of symmetry
  - the y-intercept.
- What would be the coordinates of the turning point in these quadratics?
  - $y = (x - h)^2 + k$
  - $y = -(x - h)^2 + k$

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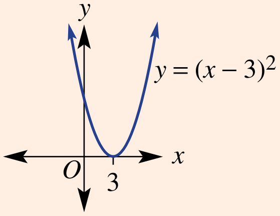
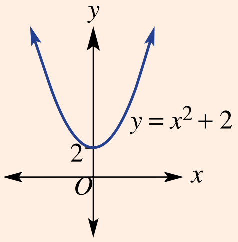
KEY IDEAS

- To sketch a parabola, draw a parabolic curve and label key features including:
  - turning point
  - axis of symmetry
  - y-intercept (substitute $x = 0$).
- For $y = ax^2$, a **dilates** the graph of $y = x^2$.
  - Turning point is $(0, 0)$.
  - y-intercept and x-intercept are both at $(0, 0)$.
  - Axis of symmetry is $x = 0$.
  - When $a > 0$, the parabola is **upright**.
  - When $a < 0$, the parabola is **inverted**.
- For $y = (x - h)^2$, h **translates** the graph of $y = x^2$ horizontally.
  - When $h > 0$, the graph is translated h units to the right.
  - When $h < 0$, the graph is translated h units to the left.



- For $y = x^2 + k$, k translates the graph of $y = x^2$ vertically.
  - When $k > 0$, the graph is translated k units up.
  - When $k < 0$, the graph is translated k units down.
- The **turning point form** of a quadratic is $y = a(x - h)^2 + k$.
  - The turning point is (h, k).
  - The axis of symmetry is $x = h$.



BUILDING UNDERSTANDING

1. Give the coordinates of the turning point for the graphs of these rules.
   - **a** $y = x^2$
   - **b** $y = x^2 + 3$
   - **c** $y = -x^2 - 4$
   - **d** $y = (x - 2)^2$
   - **e** $y = (x + 5)^2$
   - **f** $y = -\frac{1}{3}x^2$
2. Substitute $x = 0$ to find the coordinates of the y-intercept of the graphs with these equations.
   - **a** $y = x^2 + 5$
   - **b** $y = -x^2 - 3$
   - **c** $y = (x + 2)^2$
   - **d** $y = (x + 1)^2 + 1$
3. Choose the word: *left, right, up* or *down* to suit.
   - **a** Compared with the graph of $y = x^2$, the graph of $y = x^2 + 3$ is translated ____________.
   - **b** Compared with the graph of $y = x^2$, the graph of $y = (x - 3)^2$ is translated ____________.
   - **c** Compared with the graph of $y = x^2$, the graph of $y = (x + 1)^2$ is translated ____________.
   - **d** Compared with the graph of $y = x^2$, the graph of $y = x^2 - 6$ is translated ____________.
   - **e** Compared with the graph of $y = -x^2$, the graph of $y = -x^2 - 2$ is translated ____________.
   - **f** Compared with the graph of $y = -x^2$, the graph of $y = -(x + 3)^2$ is translated ____________.
   - **g** Compared with the graph of $y = -x^2$, the graph of $y = -(x - 2)^2$ is translated ____________.
   - **h** Compared with the graph of $y = -x^2$, the graph of $y = -x^2 + 4$ is translated ____________.

Example 3 Sketching with transformations

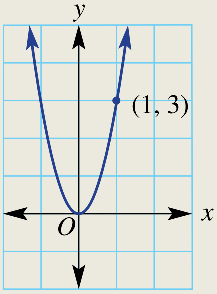
Sketch graphs of the following quadratic relations, labelling the turning point and the y-intercept.

a $y = 3x^2$ **b** $y = -x^2 + 4$ **c** $y = (x - 2)^2$

SOLUTION

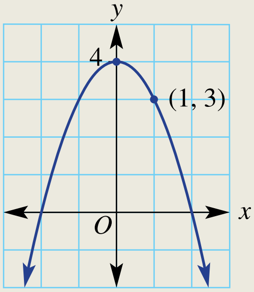
EXPLANATION

a



$y = 3x^2$ is upright and narrower than $y = x^2$. The turning point and y-intercept are at the origin (0, 0). Substitute $x = 1$ to label a second point.

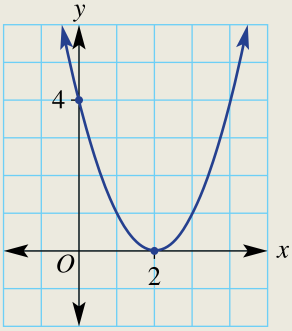
b



$y = -x^2 + 4$ is inverted (i.e. has a maximum) and is translated 4 units up compared with $y = -x^2$. The turning point is at (0, 4) and the y-intercept (i.e. when $x = 0$) is also at (0, 4).

Substitute $x = 1$ to label a second point: $y = -1^2 + 4 = 3$.

c



$y = (x - 2)^2$ is upright (i.e. has a minimum) and is translated 2 units right compared with $y = x^2$. Thus, the turning point is at (2, 0).

Substitute $x = 0$ for the y-intercept: $\begin{aligned} y &= (0-2)^2 \\ &= (-2)^2 \\ &= 4 \end{aligned}$

The y-intercept is at (0, 4).

Now you try

Sketch graphs of the following quadratic relations, labelling the turning point and the y-intercept.

a $y = 2x^2$ **b** $y = -x^2 + 3$ **c** $y = (x + 1)^2$

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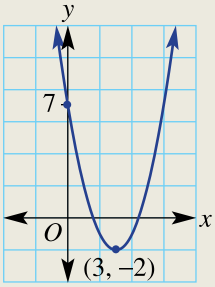
Example 4 Using turning point form

Sketch the graphs of the following, labelling the turning point and the y-intercept.

a $y = (x - 3)^2 - 2$ **b** $y = -(x + 1)^2 + 4$

SOLUTION

a



EXPLANATION

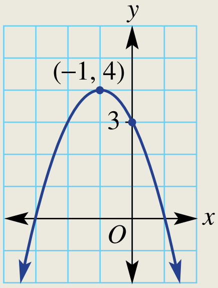
In $y = a(x - h)^2 + k$, $h = 3$ and $k = -2$, so the turning point is $(3, -2)$.

Substitute $x = 0$ to find the y-intercept:

$$\begin{aligned} y &= (0 - 3)^2 - 2 \\ &= 9 - 2 \\ &= 7 \end{aligned}$$

The y-intercept is at $(0, 7)$.

b



The graph is inverted since $a = -1$.

$h = -1$ and $k = 4$, so the turning point is $(-1, 4)$.

$$\text{When } x = 0\text{: } \begin{aligned} y &= -(0 + 1)^2 + 4 \\ &= -1 + 4 \\ &= 3 \end{aligned}$$

The y-intercept is at $(0, 3)$.

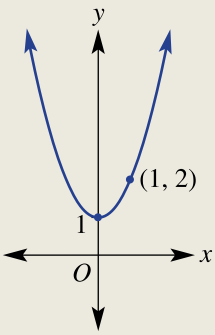
Now you try

Sketch the graphs of the following, labelling the turning point and the y-intercept.

a $y = (x + 1)^2 - 2$ **b** $y = -(x - 2)^2 + 3$

Example 5 Finding a rule from a simple graph

Determine the rule for this parabola with turning point $(0, 1)$ and another point $(1, 2)$.



SOLUTION

$y = ax^2 + 1$

When $x = 1$, $y = 2$ so:

$2 = a(1)^2 + 1$

$\therefore a = 1$

So $y = x^2 + 1$.

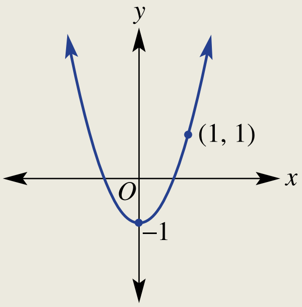
EXPLANATION

In $y = a(x - h)^2 + k$, $h = 0$ and $k = 1$ so the rule is $y = ax^2 + 1$.

We need $y = 2$ when $x = 1$, so $a = 1$.

Now you try

Determine the rule for this parabola with turning point $(0, -1)$ and another point $(1, 1)$.



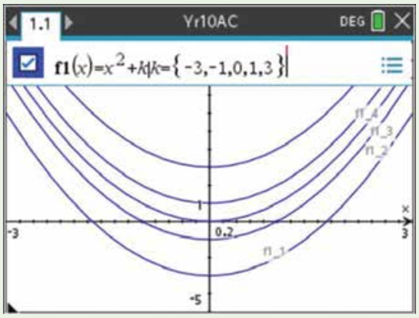
Using calculator to sketch parabolas

Sketch the graph of the family $y = x^2 + k$, using $k = \{-3, -1, 0, 1, 3\}$.

Using the TI-Nspire:

In a **Graphs** page, type the rule in $f1(x)$ using the **given** symbol (|) which is accessed using ctrl =.

$f1(x) = x^2 + k \mid k = \{-3, -1, 0, 1, 3\}$



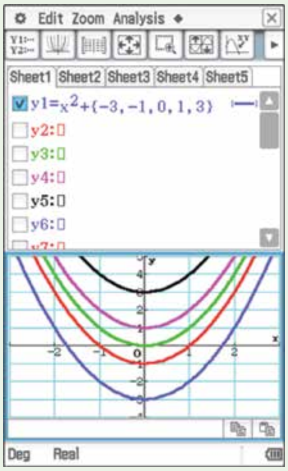
☑ f1(x)=x^2+k|k={-3,-1,0,1,3}

This shows the varying vertical translations.

Using the ClassPad:

In the **Graph&Table** application enter the rule $y1 = x^2 + \{-3, -1, 0, 1, 3\}$ followed by **EXE**.

Tap to see the graph.



☑ y1=x^2+{-3,-1,0,1,3}
☐ y2:□
☐ y3:□
☐ y4:□
☐ y5:□
☐ y6:□

Exercise 7B

FLUENCY	1, 2–4(½)	1–4(½)	1–4(⅓)

Example 3a **1** Sketch graphs of the following quadratics, labelling the turning point and the y-intercept. If the turning point is on the y-axis, also label the point where $x = 1$.

a $y = 2x^2$ **b** $y = -3x^2$ **c** $y = \frac{1}{2}x^2$ **d** $y = -\frac{1}{3}x^2$

Example 3b, c **2** Sketch graphs of the following quadratics, labelling the turning point and the y-intercept. If the turning point is on the y-axis, also label the point where $x = 1$.

a $y = x^2 + 2$ **b** $y = x^2 - 4$ **c** $y = -x^2 + 1$
d $y = -x^2 - 3$ **e** $y = (x + 3)^2$ **f** $y = (x - 1)^2$
g $y = -(x + 2)^2$ **h** $y = -(x - 3)^2$ **i** $y = (x + 4)^2$

3 State the coordinates of the turning point for the graphs of these rules.

a $y = (x + 3)^2 + 1$ **b** $y = (x + 2)^2 - 4$ **c** $y = (x - 1)^2 + 3$
d $y = (x - 4)^2 - 2$ **e** $y = (x - 3)^2 - 5$ **f** $y = (x - 2)^2 + 2$
g $y = -(x - 3)^2 + 3$ **h** $y = -(x - 2)^2 + 6$ **i** $y = -(x + 1)^2 + 4$
j $y = -(x - 2)^2 - 5$ **k** $y = -(x + 1)^2 - 1$ **l** $y = -(x - 4)^2 - 10$

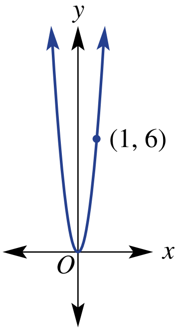
Example 4 **4** Sketch graphs of the following quadratics, labelling the turning point and the y-intercept.

a $y = (x + 1)^2 + 1$ **b** $y = (x + 2)^2 - 1$ **c** $y = (x + 3)^2 + 2$
d $y = (x - 1)^2 + 2$ **e** $y = (x - 4)^2 + 1$ **f** $y = (x - 1)^2 - 4$
g $y = -(x - 1)^2 + 3$ **h** $y = -(x - 2)^2 + 1$ **i** $y = -(x + 3)^2 - 2$
j $y = -(x - 2)^2 + 1$ **k** $y = -(x - 4)^2 - 2$ **l** $y = -(x + 2)^2 + 2$

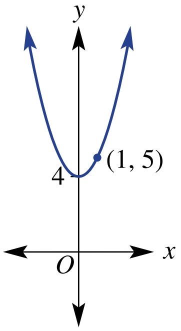
PROBLEM–SOLVING	5(½)	5–6(½)	5–6(⅓), 7

Example 5 **5** Determine the rule for the following parabolas.

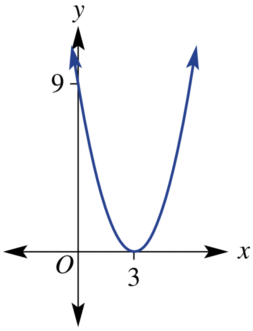
a



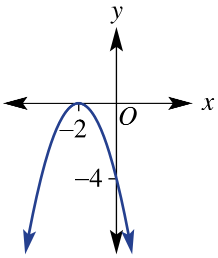
b



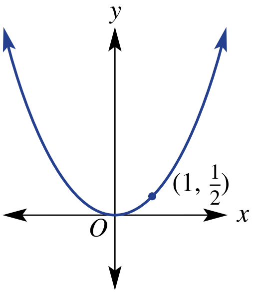
c



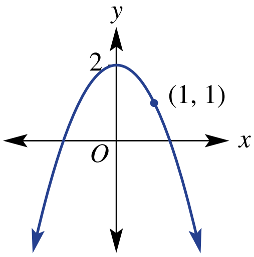
d



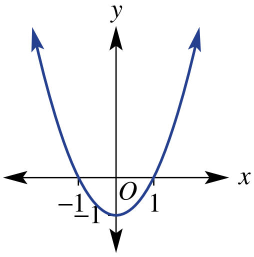
e



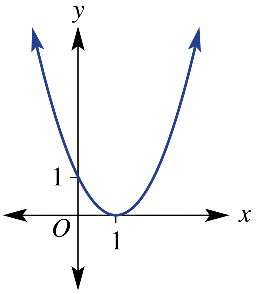
f



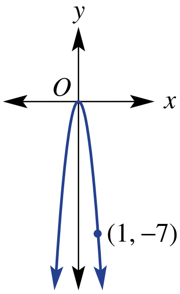
g



h



i



6 Write the rule for each graph when $y = x^2$ is transformed by the following.

- **a** reflected in the x-axis
- **b** translated 2 units to the left
- **c** translated 5 units down
- **d** translated 4 units up
- **e** translated 1 unit to the right
- **f** reflected in the x-axis and translated 2 units up
- **g** reflected in the x-axis and translated 3 units left
- **h** translated 5 units left and 3 units down
- **i** translated 6 units right and 1 unit up

7 The path of a basketball is given by $y = -(x - 5)^2 + 25$, where y metres is the height and x metres is the horizontal distance.

- **a** Is the turning point a maximum or a minimum?
- **b** What are the coordinates of the turning point?
- **c** What are the coordinates of the y-intercept?
- **d** What is the maximum height of the ball?
- **e** What is the height of the ball at these horizontal distances?
  - **i** $x = 3$
  - **ii** $x = 7$
  - **iii** $x = 10$

REASONING

8(½)	8–9(½)	9(½), 10

8 Recall that $y = (x - h)^2 + k$ and $y = a(x - h)^2 + k$ both have the same turning point coordinates. State the coordinates of the turning point for the graphs of these rules.

a $y = 2(x - 1)^2$
b $y = 3(x + 2)^2$
c $y = -4(x + 3)^2$
d $y = 3x^2 - 4$
e $y = 5x^2 - 2$
f $y = -2x^2 + 5$
g $y = 6(x + 4)^2 - 1$
h $y = 2(x + 2)^2 + 3$
i $y = 3(x - 5)^2 + 4$
j $y = -4(x + 2)^2 + 3$
k $y = -2(x + 3)^2 - 5$
l $y = -5(x - 3)^2 - 3$

9 Describe the transformations that take $y = x^2$ to:

a $y = (x - 3)^2$
b $y = (x + 2)^2$
c $y = x^2 - 3$
d $y = x^2 + 7$
e $y = -x^2$
f $y = (x + 2)^2 - 4$
g $y = (x - 5)^2 + 8$
h $y = -(x + 3)^2$
i $y = -x^2 + 6$

10 For $y = a(x - h)^2 + k$, write:

a the coordinates of the turning point
b the coordinates of the y-intercept.

ENRICHMENT: Sketching with many transformations

–	–	11(½)

11 Sketch the graph of the following, showing the turning point and the y-intercept.

a $y = 2(x - 3)^2 + 4$
b $y = 3(x + 2)^2 + 5$
c $y = -2(x - 3)^2 + 4$
d $y = -2(x + 3)^2 - 4$
e $y = \frac{1}{2}(x - 3)^2 + 4$
f $y = -\frac{1}{2}(x - 3)^2 + 4$
g $y = 4 - x^2$
h $y = -3 - x^2$
i $y = 5 - 2x^2$
j $y = 2 + \frac{1}{2}(x - 1)^2$
k $y = 1 - 2(x + 2)^2$
l $y = 3 - 4(x - 2)^2$

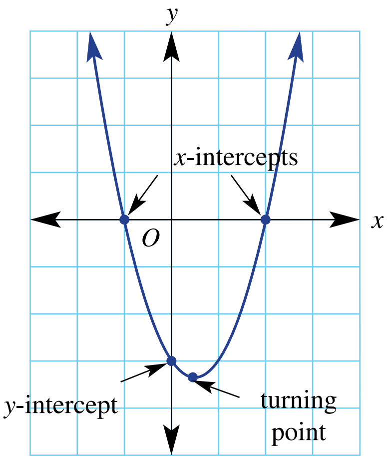
7C Sketching parabolas using factorisation

LEARNING INTENTIONS

- To know the steps for sketching a quadratic graph in the form $y = x^2 + bx + c$
- To understand that quadratic graphs can have 0, 1 or 2 x-intercepts
- To know how to use factorisation to determine the x-intercepts of a quadratic graph
- To know how to use symmetry to locate the turning point once the x-intercepts are known

A quadratic relation written in the form $y = x^2 + bx + c$ differs from that of turning point form, $y = a(x - h)^2 + k$, and so the transformations of the graph of $y = x^2$ to give $y = x^2 + bx + c$ are less obvious. To address this, we have a number of options. We can first try to factorise to find the x-intercepts and then use symmetry to find the turning point or, alternatively, we can complete the square and express the quadratic relation in turning point form. The second of these methods will be studied in **Section 7D**.

Engineers can develop equations of parabolic shapes, such as the St Louis archway, by reversing the factorisation procedure. A sketch is labelled with measurements, the x-intercepts form the factors and expanding the brackets gives the basic quadratic equation.



Lesson starter: Why does the turning point of $y = x^2 - 2x - 3$ have coordinates $(1, -4)$?

- Factorise $y = x^2 - 2x - 3$.
- Hence, find the x-intercepts.
- Discuss how symmetry can be used to locate the turning point.
- Hence, confirm the coordinates of the turning point.

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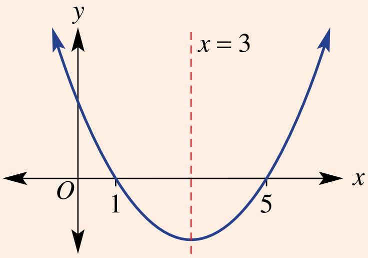
KEY IDEAS

- To sketch a graph of $y = x^2 + bx + c$:
  - find the y-intercept by substituting $x = 0$
  - find the x-intercept(s) by substituting $y = 0$. Factorise where possible and use the Null Factor Law (if $p \times q = 0$, then $p = 0$ or $q = 0$).
- Once the x-intercepts are known, the turning point can be found using symmetry.
  - The axis of symmetry (also the x-coordinate of the turning point) lies halfway between the x-intercepts: $x = \frac{1 + 5}{2} = 3$, for the graph below.



  - Substitute this x-coordinate into the rule to find the y-coordinate of the turning point.

BUILDING UNDERSTANDING

1. Use the Null Factor Law to find the coordinates of the x-intercepts ($y = 0$) for these factorised quadratics.
   a $y = (x + 1)(x - 2)$ **b** $y = x(x - 3)$ **c** $y = -3x(x + 2)$

2. Factorise these quadratics.
   a $y = x^2 - 4x$ **b** $y = x^2 + 2x - 8$
   c $y = x^2 - 8x + 16$ **d** $y = x^2 - 25$

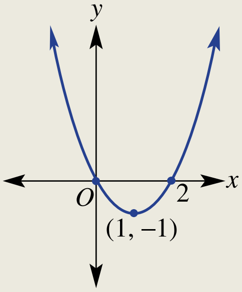
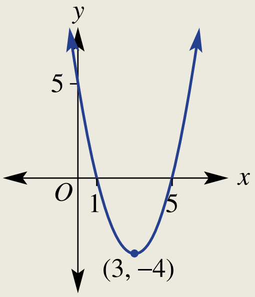
3. Find the coordinates of the y-intercept for the quadratics in Question 2.

4. Use the given rule and x-intercepts to find the coordinates of the turning point.
   a $y = x^2 - 8x + 12$, x-intercepts: at $x = 2$ and $x = 6$
   b $y = -x^2 - 2x + 8$, x-intercepts: at $x = -4$ and $x = 2$

Example 6 Using the x-intercepts to find the turning point

Sketch the graph of the following quadratics by using the x-intercepts to help determine the coordinates of the turning point.

a $y = x^2 - 2x$ **b** $y = x^2 - 6x + 5$

SOLUTION	EXPLANATION
a y-intercept at $x = 0$: $y = 0$ x-intercepts at $y = 0$: $0 = x^2 - 2x$ $0 = x(x - 2)$ $x = 0$ or $x - 2 = 0$ $\therefore x = 0, x = 2$	Identify key features of the graph: y-intercept (when $x = 0$), x-intercepts (when $y = 0$), then factorise by noting the common factor and solve by applying the Null Factor Law. Recall that if $p \times q = 0$, then $p = 0$ or $q = 0$.
Turning point at $x = \frac{0 + 2}{2} = 1$.	Using symmetry the x-coordinate of the turning point is halfway between the x-coordinates of the x-intercepts.
$y = 1^2 - 2(1)$ $= -1$ Turning point is a minimum at $(1, -1)$.	Substitute $x = 1$ into $y = x^2 - 2x$ to find the y-coordinate of the turning point. It is a minimum turning point since the coefficient of x^2 is positive.
	Label key features on the graph and join points in the shape of a parabola.
b y-intercept at $x = 0$: $y = 5$ x-intercepts at $y = 0$: $0 = x^2 - 6x + 5$ $0 = (x - 5)(x - 1)$ $x - 5 = 0$ or $x - 1 = 0$ $\therefore x = 5, x = 1$	Identify key features of the graph: y-intercept ($y = 0^2 - 6(0) + 5$) and x-intercepts by factorising and applying the Null Factor Law.
Turning point at $x = \frac{1 + 5}{2} = 3$. $y = (3)^2 - 6 \times (3) + 5$ $= 9 - 18 + 5$ $= -4$ Turning point is a minimum at $(3, -4)$.	Using symmetry the x-coordinate of the turning point is halfway between the x-coordinates of the x-intercepts. Substitute $x = 3$ into $y = x^2 - 6x + 5$ to find the y-coordinate of the turning point. It is a minimum turning point since the coefficient of x^2 is positive.
	Label key features on the graph and join points in the shape of a parabola.

Now you try

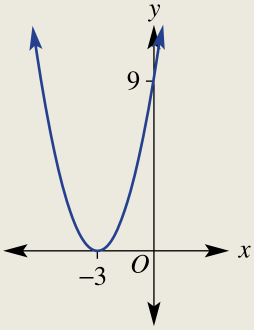
Sketch the graph of the following quadratics by using the x-intercepts to help determine the coordinates of the turning point.

a $y = x^2 - 6x$

b $y = x^2 - 8x + 7$

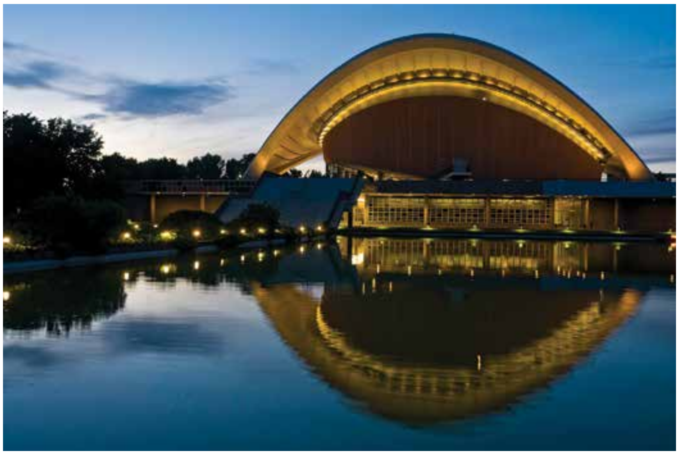
Example 7 Sketching a perfect square

Sketch the graph of the quadratic $y = x^2 + 6x + 9$.

SOLUTION	EXPLANATION
y-intercept at $x = 0$: $y = 9$	For y-intercept substitute $x = 0$.
x-intercepts at $y = 0$: $0 = x^2 + 6x + 9$ $0 = (x + 3)^2$ $x + 3 = 0$ $\therefore x = -3$	For x-intercepts substitute $y = 0$ and factorise: $(x + 3)(x + 3) = (x + 3)^2$. Apply the Null Factor Law to solve for x.
Turning point is at $(-3, 0)$.	As there is only one x-intercept, it is also the turning point.
	Label key features on the graph.

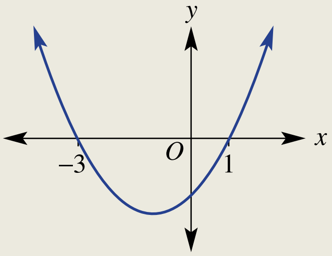
Now you try

Sketch the graph of the quadratic $y = x^2 + 8x + 16$.



Example 8 Finding a turning point from a graph

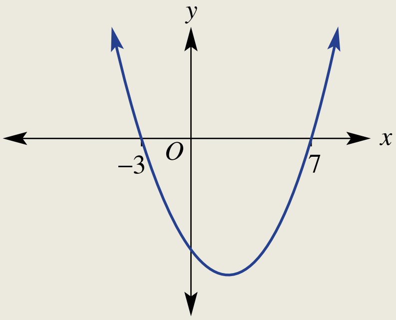
The equation of this graph is of the form $y = (x + a)(x + b)$. Use the x-intercepts to find the values of a and b, then find the coordinates of the turning point.



SOLUTION	EXPLANATION
$a = 3$ and $b = -1$ $y = (x + 3)(x - 1)$	Using the Null Factor Law, $(x + 3)(x - 1) = 0$ gives $x = -3$ and $x = 1$, so $a = 3$ and $b = -1$.
x-coordinate of the turning point is $\frac{-3 + 1}{2} = -1$.	Find the average of the two x-intercepts to find the x-coordinate of the turning point.
y-coordinate is $(-1 + 3)(-1 - 1) = 2 \times (-2) = -4$. Turning point is $(-1, -4)$.	Substitute $x = -1$ into the rule to find the y-value of the turning point.

Now you try

The equation of this graph is of the form $y = (x + a)(x + b)$. Use the x-intercepts to find the values of a and b, then find the coordinates of the turning point.

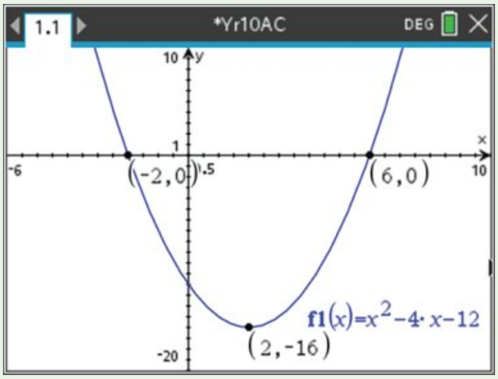


Using calculators to sketch parabolas

Sketch a graph of $y = x^2 - 4x - 12$ and show the x-intercepts and the turning point.

Using the TI-Nspire:

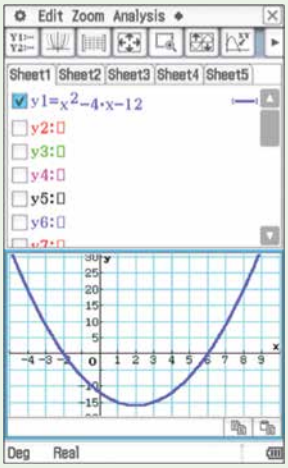
Enter the rule $f1(x) = x^2 - 4x - 12$. Change the scale using the window settings. Use menu **>Analyze Graph** to locate x-intercepts (zeros) and the maximum or minimum.



menu **>Trace>Graph Trace** can also be used. Scroll along the graph to show significant points.

Using the ClassPad:

Enter the rule $y1 = x^2 - 4x - 12$. Tap and set an appropriate scale. Tap **Analysis**, **G-Solve**, **root** to locate the x-intercepts. Tap **Analysis**, **G-Solve**, **Min** to locate the turning point.



Exercise 7C

FLUENCY	1–4(½)	1–5(½)	1–5(⅓)

Example 6a **1** Sketch the graph of the following quadratics by using the x-intercepts to help determine the coordinates of the turning point.

- **a** $y = x^2 + 2x$
- **b** $y = x^2 + 6x$
- **c** $y = x^2 - 4x$
- **d** $y = x^2 - 5x$
- **e** $y = x^2 + 3x$
- **f** $y = x^2 + 7x$

Example 6b **2** Sketch the graph of the following quadratics by using the x-intercepts to help determine the coordinates of the turning point.

- **a** $y = x^2 - 6x + 8$
- **b** $y = x^2 - 8x + 12$
- **c** $y = x^2 + 8x + 15$
- **d** $y = x^2 - 6x - 16$
- **e** $y = x^2 - 2x - 8$
- **f** $y = x^2 - 4x - 21$
- **g** $y = x^2 + 8x + 7$
- **h** $y = x^2 - 12x + 20$

3 Sketch graphs of the following quadratics.

a $y = x^2 - 9x + 20$ **b** $y = x^2 - 5x + 6$ **c** $y = x^2 - 13x + 12$ **d** $y = x^2 + 11x + 30$
e $y = x^2 + 5x + 4$ **f** $y = x^2 + 13x + 12$ **g** $y = x^2 - 4x - 12$ **h** $y = x^2 - x - 2$
i $y = x^2 - 5x - 14$ **j** $y = x^2 + 3x - 4$ **k** $y = x^2 + 7x - 30$ **l** $y = x^2 + 9x - 22$

Example 7

4 Sketch graphs of the following perfect squares.

a $y = x^2 + 4x + 4$ **b** $y = x^2 + 8x + 16$
c $y = x^2 - 10x + 25$ **d** $y = x^2 + 20x + 100$

5 Sketch graphs of the following quadratics that include a difference of two squares.

a $y = x^2 - 9$ **b** $y = x^2 - 16$ **c** $y = x^2 - 4$

PROBLEM–SOLVING	6(½), 7	6(½), 7, 8	7, 9, 10

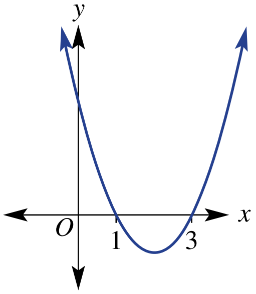
6 Determine the turning points of the following quadratics.

a $y = 2(x^2 - 7x + 10)$ **b** $y = 3(x^2 - 7x + 10)$ **c** $y = 3x^2 + 18x + 24$
d $y = 4x^2 + 24x + 32$ **e** $y = 4(x^2 - 49)$ **f** $y = -4(x^2 - 49)$
g $y = 3x^2 - 6x + 3$ **h** $y = 5x^2 - 10x + 5$ **i** $y = 2x^2 - 4x + 10$

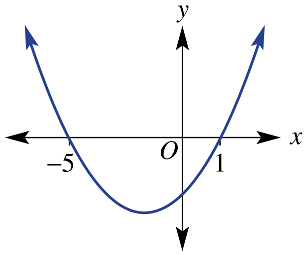
Example 8

7 The equations of these graphs are of the form $y = (x + a)(x + b)$. Use the x-intercepts to find the values of a and b, and then find the coordinates of the turning point.

a



b



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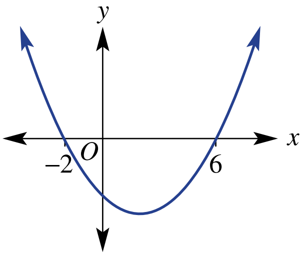


8 State the coordinates of the x-intercepts and turning point for these quadratics.

a $y = x^2 - 2$ **b** $y = x^2 - 11$ **c** $y = 2x^2 - 10$

9 Sketch a graph of these quadratics.

a $y = 9 - x^2$ **b** $y = 1 - x^2$ **c** $y = 4x - x^2$
d $y = 3x - x^2$ **e** $y = -x^2 + 2x + 8$ **f** $y = -x^2 + 8x + 9$

10 If the graph of $y = a(x + 2)(x - 4)$ passes through the point (2, 16), determine the value of a and the coordinates of the turning point for this parabola.

REASONING	11	11, 12	12–14

11 Explain why $y = (x - 3)(x - 5)$ and $y = 2(x - 3)(x - 5)$ both have the same x-intercepts.

12 a Explain why $y = x^2 - 2x + 1$ has only one x-intercept.
b Explain why $y = x^2 + 2$ has zero x-intercepts.

13 Consider the quadratics $y = x^2 - 2x - 8$ and $y = -x^2 + 2x + 8$.
a Show that both quadratics have the same x-intercepts.
b Find the coordinates of the turning points for both quadratics.
c Compare the positions of the turning points.

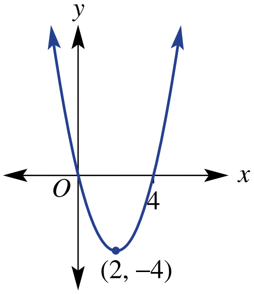
14 A quadratic has the rule $y = x^2 + bx$. Give the coordinates of:

- **a** the y-intercept
- **b** the x-intercepts
- **c** the turning point.

ENRICHMENT: More rules from graphs	–	–	15(½)

15 Determine the equation of each of these graphs in factorised form; for example, $y = 2(x - 3)(x + 2)$.

a



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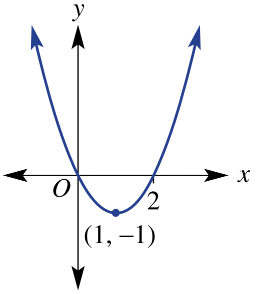

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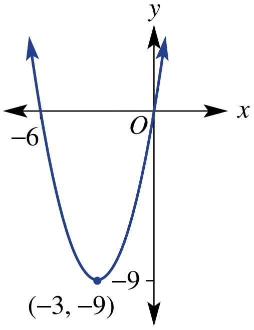

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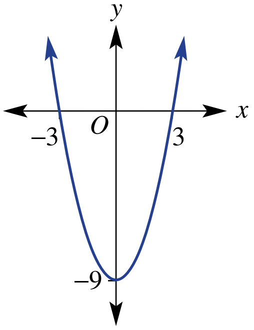

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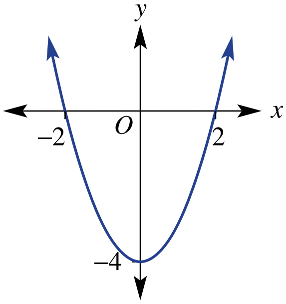

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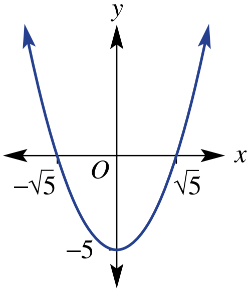

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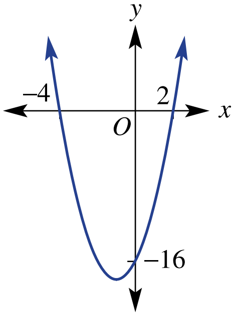

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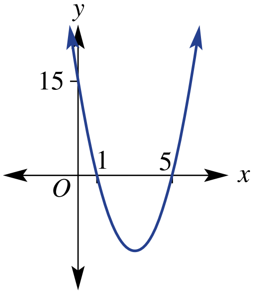

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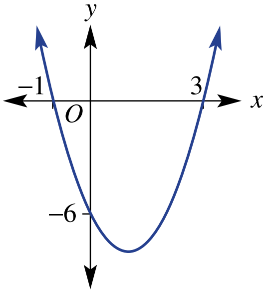

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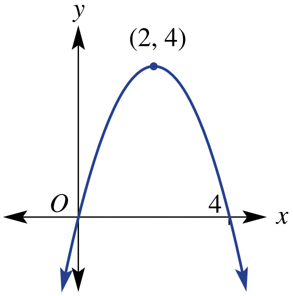

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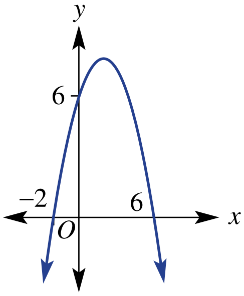

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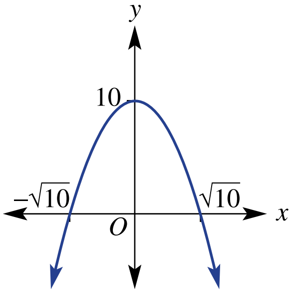

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7D Sketching parabolas by completing the square

LEARNING INTENTIONS

- To know that completing the square can be used to express any quadratic in turning point form
- To be able to find any x-intercepts from the turning point form of a quadratic
- To be able to sketch a quadratic equation in turning point form, labelling key features

We have learnt previously that the turning point of a parabola can be read directly from a rule in the form $y = a(x - h)^2 + k$. This form of quadratic rule can be obtained by completing the square.

Businesses use mathematical modelling to analyse profits. Quadratic equations model revenue vs selling price and its graph is an inverted parabola. With rising prices, revenue grows until the turning point, then it decreases due to declining sales.

Lesson starter: I forgot how to complete the square!

To make $x^2 + 6x$ a perfect square we need to add 9 (from $\left(\frac{6}{2}\right)^2$) since $x^2 + 6x + 9 = (x + 3)^2$.

So to complete the square for $x^2 + 6x + 2$ we write $x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 2 = (x + 3)^2 - 7$.

- Discuss the rules for completing the square and explain how $x^2 + 6x + 2$ becomes $(x + 3)^2 - 7$.
- What does the turning point form of $x^2 + 6x + 2$ tell us about its graph?
- How can you use the turning point form of $x^2 + 6x + 2$ to help find the x-intercepts of $y = x^2 + 6x + 2$?

KEY IDEAS

- By **completing the square**, all quadratics in the form $y = ax^2 + bx + c$ can be expressed in turning point form; i.e. $y = a(x - h)^2 + k$.
- To sketch a quadratic in the form $y = a(x - h)^2 + k$, follow these steps.
  - Determine the coordinates of the turning point (h, k).
    - When a is positive, the parabola has a minimum turning point.
    - When a is negative, the parabola has a maximum turning point.
  - Determine the y-intercept by substituting $x = 0$.
  - Determine the x-intercepts, if any, by substituting $y = 0$ and solving the equation.
- To solve $x^2 = a$, $a > 0$, take the square root of both sides: $x = \pm\sqrt{a}$. i.e. $\sqrt{a}$ and $-\sqrt{a}$.

  For any perfect square, say $(x + 1)^2 = 16$, take the square root of both sides:

$$\begin{aligned} x + 1 &= \pm 4 \\ x &= -1 \pm 4 \\ x &= -1 + 4 \text{ or } x = -1 - 4 \\ x &= 3 \text{ or } x = -5 \end{aligned}$$

BUILDING UNDERSTANDING

1 By completing the square, state the coordinates of the turning point (TP).

a $$\begin{aligned} y &= x^2 + 2x - 5 \\ &= x^2 + 2x + \underline{\qquad} - \underline{\qquad} - \underline{\qquad} \\ &= (\underline{\qquad})^2 - \underline{\qquad} \\ \text{TP} &= (\underline{\qquad}, \underline{\qquad}) \end{aligned}$$

b $$\begin{aligned} y &= x^2 - 6x + 10 \\ &= x^2 - 6x + \underline{\qquad} - \underline{\qquad} + \underline{\qquad} \\ &= \underline{\qquad\qquad} \\ \text{TP} &= (\underline{\qquad}, \underline{\qquad}) \end{aligned}$$

2 Solve these equations for x, giving exact answers.

a $x^2 = 9$ **b** $x^2 = 3$ **c** $(x - 1)^2 = 16$ **d** $(x + 4)^2 = 2$

Example 9 Finding key features of quadratics in turning point form

For $y = -4(x - 1)^2 + 16$:

a determine the coordinates of its turning point and state whether it is a maximum or minimum

b determine the coordinates of the y-intercept

c determine the coordinates of the x-intercepts (if any).

SOLUTION / EXPLANATION

SOLUTION

a Turning point is a maximum at (1, 16).

EXPLANATION

For $y = a(x - h)^2 + k$ the turning point is at (h, k). As $a = -4$ is negative, the parabola has a maximum turning point.

SOLUTION

b y-intercept at $x = 0$:

$$\begin{aligned} y &= -4(0 - 1)^2 + 16 \\ &= -4 + 16 \\ &= 12 \end{aligned}$$

$\therefore$ y-intercept is at (0, 12).

EXPLANATION

Substitute $x = 0$ to find the y-intercept. Recall that $(0 - 1)^2 = (-1)^2 = 1$.

SOLUTION

c x-intercepts at $y = 0$:

$$\begin{aligned} 0 &= -4(x - 1)^2 + 16 \\ 0 &= (x - 1)^2 - 4 \\ (x - 1)^2 &= 4 \\ x - 1 &= \pm 2 \\ x &= 1 \pm 2 \\ x &= -1, 3 \end{aligned}$$

$\therefore$ x-intercepts are at $(-1, 0)$ and $(3, 0)$.

EXPLANATION

Substitute $y = 0$ for x-intercepts.
Divide both sides by -4: $16 \div (-4) = -4$.

Add 4 to both sides, and take the square root of both sides.
Answers are ± 2 since $2^2 = 4$ and $(-2)^2 = 4$.
$1 - 2 = -1$ and $1 + 2 = 3$ are the x-intercepts.
Note: Check that the x-intercepts are evenly spaced either side of the turning point.
Alternatively, use difference of two squares to write in factorised form:
$(x - 1)^2 - 2^2 = 0$
$(x - 1 - 2)(x - 1 + 2) = 0$
and apply the Null Factor Law to solve for x.

Now you try

For $y = -2(x + 1)^2 + 18$:

a determine the coordinates of its turning point and state whether it is a maximum or minimum
b determine the coordinates of the y-intercept
c determine the coordinates of the x-intercepts (if any).

Example 10 Sketching by completing the square

Sketch these graphs by completing the square, giving the x-intercepts in exact form.

a $y = x^2 + 6x + 15$ **b** $y = x^2 - 4x + 2$ Opt **c** $y = x^2 - 3x - 1$

SOLUTION

a Turning point form:

$$\begin{aligned} y &= x^2 + 6x + 15 \\ &= x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 15 \\ &= (x + 3)^2 + 6 \end{aligned}$$

Turning point is a minimum at $(-3, 6)$.

y-intercept at $x = 0$:

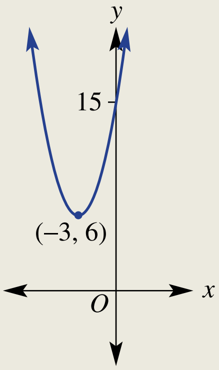
$$\begin{aligned} y &= (0)^2 + 6(0) + 15 \\ &= 15 \end{aligned}$$

$\therefore$ y-intercept is at $(0, 15)$.

x-intercepts at $y = 0$:

$0 = (x + 3)^2 + 6$

There is no solution and there are no x-intercepts.



EXPLANATION

To change the equation into turning point form, complete the square by adding and subtracting $\left(\frac{6}{2}\right)^2 = 9$.

Read off the turning point, which is a minimum, as $a = 1$ is positive.

For the y-intercept, substitute $x = 0$ into the original equation.

For the x-intercepts, substitute $y = 0$ into the turning point form.
This cannot be solved as $(x + 3)^2$ cannot equal -6, hence there are no x-intercepts. Note also that the turning point is a minimum with a lowest y-coordinate of 6, telling us there are no x-intercepts.
Sketch the graph, showing the key points.

Continued on next page

b Turning point form:

$$y = x^2 - 4x + 2$$

$$y = x^2 - 4x + \left(\frac{-4}{2}\right)^2 - \left(\frac{-4}{2}\right)^2 + 2$$

$$= (x - 2)^2 - 2$$

Turning point is a minimum at $(2, -2)$.

y-intercept at $x = 0$:

$$y = 0^2 - 4(0) + 2$$

$$= 2$$

$\therefore y$-intercept is at $(0, 2)$.

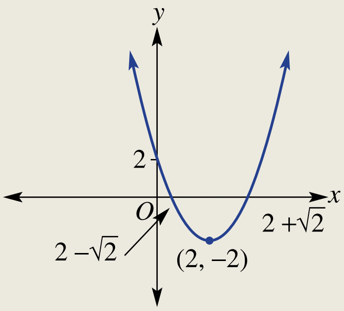
x-intercepts at $y = 0$:

$$0 = (x - 2)^2 - 2$$

$$\therefore (x - 2)^2 = 2$$

$$x - 2 = \pm\sqrt{2}$$

$$x = 2 \pm \sqrt{2}$$



Complete the square to express the rule in turning point form.

Read off the turning point, which is a minimum since the coefficient of x^2 is positive.

Substitute $x = 0$ to find the y-intercept.

Substitute $y = 0$ to find the x-intercepts and solve the resulting equation.

Add 2 to both sides of the equation and take the square root of both sides, remembering to include ±.

Sketch the graph, labelling key points.

Note that the x-intercepts are positioned symmetrically either side of the turning point.

c Turning point form:

$$y = x^2 - 3x - 1$$

$$= x^2 - 3x + \left(-\frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right)^2 - 1$$

$$= \left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$$

Turning point is a minimum at $\left(\frac{3}{2}, -\frac{13}{4}\right)$.

y-intercept at $x = 0$:

$$y = (0)^2 - 3(0) - 1$$

$$= -1$$

$\therefore y$-intercept is at $(0, -1)$.

Complete the square to write in turning point form: $\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$ and $-\frac{9}{4} - 1 = -\frac{9}{4} - \frac{4}{4} = -\frac{13}{4}$.

Substitute $x = 0$ to find the y-intercept.

Substitute $y = 0$ to find the x-intercepts.

x-intercepts at $y = 0$:

$$0 = \left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{13}{4}$$

$$x - \frac{3}{2} = \pm\frac{\sqrt{13}}{2}$$

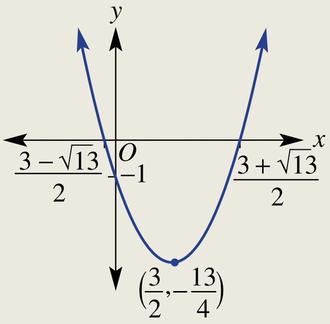
$$x = \frac{3 + \sqrt{13}}{2},\ x = \frac{3 - \sqrt{13}}{2}$$

Add $\frac{13}{4}$ to both sides and take the square root.

$$\sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{\sqrt{4}} = \frac{\sqrt{13}}{2}$$

$x = \frac{3}{2} \pm \frac{\sqrt{13}}{2}$ can also be expressed as

$x = \frac{3 \pm \sqrt{13}}{2}$.



Label key features on the graph, using exact values.

Now you try

Sketch these graphs by completing the square, giving the x-intercepts in exact form.

a $y = x^2 - 2x + 2$ **b** $y = x^2 + 4x + 1$ **c** $y = x^2 - 5x + 1$

Exercise 7D

FLUENCY	1, 2–6(½)	1–6(½)	1–6(⅓)

Example 9a **1** State whether the turning points of the following are a maximum or a minimum and give the coordinates.

a $y = 2(x - 3)^2 + 5$
b $y = -2(x - 1)^2 + 3$
c $y = -4(x + 1)^2 - 2$
d $y = 6(x + 2)^2 - 5$
e $y = 3(x + 5)^2 + 10$
f $y = -4(x - 7)^2 + 2$
g $y = -5(x - 3)^2 + 8$
h $y = 2(x - 3)^2 - 7$

Example 9b **2** Determine the coordinates of the y-intercept of each of the following.

a $y = (x + 1)^2 + 5$
b $y = (x + 2)^2 - 6$
c $y = (x - 3)^2 - 2$
d $y = (x - 4)^2 - 7$
e $y = -(x + 5)^2 + 9$
f $y = -(x - 7)^2 - 6$
g $y = x^2 + 6x + 3$
h $y = x^2 + 5x + 1$
i $y = x^2 + 7x - 5$
j $y = x^2 + x - 8$
k $y = x^2 - 5x + 13$
l $y = x^2 - 12x - 5$

Example 9c

3 Determine the coordinates of the x-intercepts (if any) of the following.

a $y=(x-3)^2-4$ **b** $y=(x+4)^2-9$ **c** $y=(x-3)^2-36$
d $y=2(x+2)^2-10$ **e** $y=-3(x-1)^2+30$ **f** $y=(x-5)^2-3$
g $y=(x-4)^2$ **h** $y=(x+6)^2$ **i** $y=2(x-7)^2+18$
j $y=-2(x-3)^2-4$ **k** $y=-(x-2)^2+5$ **l** $y=-(x-3)^2+10$

4 Determine the coordinates of the x-intercepts (if any) by first completing the square and rewriting the equation in turning point form. Give exact answers.

a $y=x^2+6x+5$ **b** $y=x^2+6x+2$ **c** $y=x^2+8x-5$
d $y=x^2+2x-6$ **e** $y=x^2-4x+14$ **f** $y=x^2-12x-5$

5 Sketch the graphs of the following. Label the turning point and intercepts.

a $y=(x-2)^2-4$ **b** $y=(x+4)^2-9$ **c** $y=(x+4)^2-1$
d $y=(x-3)^2-4$ **e** $y=(x+8)^2+16$ **f** $y=(x+7)^2+2$
g $y=(x-2)^2+1$ **h** $y=(x-3)^2+6$ **i** $y=-(x-5)^2-4$
j $y=-(x+4)^2-9$ **k** $y=-(x+9)^2+25$ **l** $y=-(x-2)^2+4$

Example 10a,b

6 Sketch these graphs by completing the square. Label the turning point and intercepts.

a $y=x^2-2x+6$ **b** $y=x^2+4x+3$ **c** $y=x^2-2x-3$
d $y=x^2+6x+9$ **e** $y=x^2-8x+16$ **f** $y=x^2-8x+20$
g $y=x^2+8x+10$ **h** $y=x^2+6x-5$ **i** $y=x^2+12x$

PROBLEM–SOLVING 7 7–8(½) 7–9(½)

7 Complete the square and decide if the graphs of the following quadratics will have zero, one or two x-intercepts.

a $y=x^2-4x+2$ **b** $y=x^2-4x+4$ **c** $y=x^2+6x+9$
d $y=x^2+2x+6$ **e** $y=x^2-6x+12$ **f** $y=x^2+10x+20$

Example 10c

Opt

8 Sketch these graphs by completing the square. Label the turning point and intercepts with exact values.

a $y=x^2-3x+1$ **b** $y=x^2+5x+2$ **c** $y=x^2-x-2$ **d** $y=x^2+3x+3$

9 Take out a common factor and complete the square to find the y-coordinate of the x-intercepts for these quadratics.

a $y=2x^2+4x-10$ **b** $y=3x^2-12x+9$ **c** $y=2x^2-12x-14$
d $y=4x^2+16x-24$ **e** $y=5x^2+20x-35$ Opt **f** $y=2x^2-6x+2$

REASONING 10(½) 10(½), 11 10(⅓), 11, 12(⅓), 13

10 To sketch a graph of the form $y=-x^2+bx+c$ we can complete the square by taking out a factor of -1. Here is an example.

$$\begin{aligned} y &= -x^2-2x+5 \\ &= -(x^2+2x-5) \\ &= -\left(x^2+2x+\left(\frac{2}{2}\right)^2-\left(\frac{2}{2}\right)^2-5\right) \\ &= -((x+1)^2-6) \\ &= -(x+1)^2+6 \end{aligned}$$

So the turning point is a maximum at $(-1, 6)$.

Sketch the graph of these quadratics using the technique above.

a $y = -x^2 - 4x + 3$ **b** $y = -x^2 + 2x + 2$ **c** $y = -x^2 + 6x - 4$

d $y = -x^2 + 8x - 8$ Opt **e** $y = -x^2 - 3x - 5$ Opt **f** $y = -x^2 - 5x + 2$

11 For what values of k will the graph of $y = (x - h)^2 + k$ have:

a zero x-intercepts? **b** one x-intercept? **c** two x-intercepts?

12 This example recalls how to complete the square with non-monic quadratics of the form $y = ax^2 + bx + c$.

$$\begin{aligned} y &= 3x^2 + 6x + 1 \\ &= 3\left(x^2 + 2x + \frac{1}{3}\right) \\ &= 3\left(x^2 + 2x + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 + \frac{1}{3}\right) \\ &= 3\left((x+1)^2 - \frac{2}{3}\right) \\ &= 3(x+1)^2 - 2 \end{aligned}$$

Key features:

The turning point is $(-1, -2)$.

y-intercept is at $(0, 1)$.

x-intercepts:

$$\begin{aligned} 0 &= 3(x+1)^2 - 2 \\ (x+1)^2 &= \frac{2}{3} \\ x &= \pm\sqrt{\frac{2}{3}} - 1 \end{aligned}$$

Use this technique to sketch the graphs of these non-monic quadratics.

a $y = 4x^2 + 8x + 3$ **b** $y = 3x^2 - 12x + 10$ **c** $y = 2x^2 + 12x + 1$

d $y = 2x^2 + x - 3$ **e** $y = 2x^2 - 7x + 3$ **f** $y = 4x^2 - 8x + 20$

g $y = 6x^2 + 5x + 9$ **h** $y = 5x^2 - 3x + 7$ **i** $y = 5x^2 + 12x$

j $y = 7x^2 + 10x$ **k** $y = -3x^2 - 9x + 2$ **l** $y = -4x^2 + 10x - 1$

13 Show that $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{4}$.

ENRICHMENT: Finding rules using the turning point

–	–	14

14 Find the rule of the quadratic graph with the following features. Express in the form $y = a(x - h)^2 + k$.

- **a** turning point at $(3, 2)$ and passes through $(0, 20)$
- **b** turning point at $(-2, 4)$ and passes through $(0, 6)$
- **c** turning point at $(-1, 2)$ and passes through $(-2, 0)$
- **d** turning point at $(2, 0)$ and passes through $(8, 12)$
- **e** axis of symmetry at $x = 2$ and passes through $(0, 0)$ and $(3, -9)$
- **f** axis of symmetry at $x = -1$ and passes through $(0, 4)$ and $(2, -4)$

7E Sketching parabolas using the quadratic formula and the discriminant OPTIONAL

LEARNING INTENTIONS

- To know the quadratic formula and how it can be used to find the solutions of a quadratic equation
- To be able to use the quadratic formula to determine the x-intercepts of a quadratic graph
- To understand how the discriminant can be used to determine the number of x-intercepts of a quadratic graph
- To be able to use the axis of symmetry rule to locate the turning point of a quadratic graph

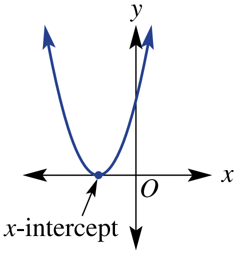
So far we have found x-intercepts for parabolas by factorising (and using the Null Factor Law) and by completing the square. An alternative method is to use the quadratic formula, which states that if $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The discriminant $\Delta = b^2 - 4ac$ determines the number of solutions to the equation.

If $\Delta = 0$, i.e. $b^2 - 4ac = 0$.

The solution to the equation becomes $x = -\dfrac{b}{2a}$.

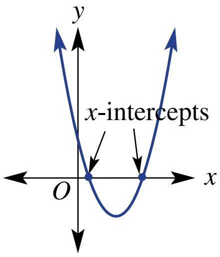
There is one solution and one x-intercept.



If $\Delta > 0$, i.e. $b^2 - 4ac > 0$.

The solution to the equation becomes $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

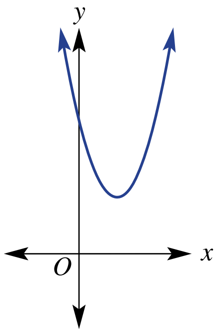
There are two solutions and two x-intercepts.



If $\Delta < 0$, i.e. $b^2 - 4ac < 0$.

Square roots exist for positive numbers only.

There are no solutions nor x-intercepts.



Lesson starter: No working required

Three students set to work to find the x-intercepts for $y = x^2 - 2x + 3$:
Student A finds the intercepts by factorising.
Student B finds the intercepts by completing the square.
Student C uses the discriminant in the quadratic formula.

- Try the method for student A. What do you notice?
- Try the method for student B. What do you notice?
- What is the value of the discriminant for student C? What does this tell them about the number of x-intercepts for the quadratic?
- What advice would student C give students A and B?

KEY IDEAS

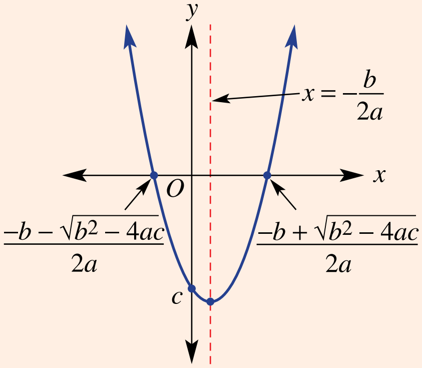
- To sketch the graph of $y = ax^2 + bx + c$, find the following points.
  - y-intercept at $x = 0$: $y = a(0)^2 + b(0) + c = c$
  - x-intercepts when $y = 0$:

    For $0 = ax^2 + bx + c$, use the **quadratic formula**:

    $$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ or } \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

    Alternatively, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
  - Turning point: The x-coordinate lies halfway between the x-coordinates of the x-intercepts, so $x = -\frac{b}{2a}$.

    The y-coordinate is found by substituting the x-coordinate into the original equation.



- $x = -\frac{b}{2a}$ is the **axis of symmetry**.
- To determine if there are zero, one or two x-intercepts, use the **discriminant** $\Delta = b^2 - 4ac$.

  If $\Delta < 0 \rightarrow$ no x-intercepts.

  If $\Delta = 0 \rightarrow$ one x-intercept.

  If $\Delta > 0 \rightarrow$ two x-intercepts.

BUILDING UNDERSTANDING

1 A graph has the rule $y = ax^2 + bx + c$. Determine the number of x-intercepts it will have if:

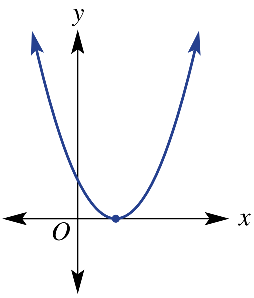
a $b^2 - 4ac > 0$ **b** $b^2 - 4ac < 0$ **c** $b^2 - 4ac = 0$

2 Give the exact value of $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ when:

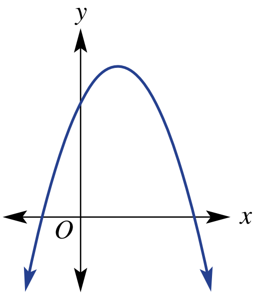
a $a = 1, b = 2, c = -1$ **b** $a = -2, b = 3, c = 5$ **c** $a = -2, b = -1, c = 2$

3 For the following graphs, state whether the discriminant of these quadratics would be zero, positive or negative.

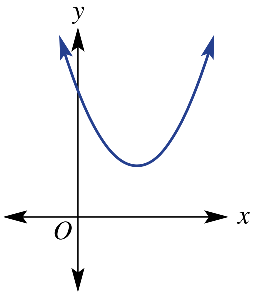
a



b



c



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Example 11 Using the discriminant and using $x = -\frac{b}{2a}$ to find the turning point

Consider the parabola given by the quadratic equation $y = 3x^2 - 6x + 5$.

a Determine the number of x-intercepts.

b Determine the coordinates of the y-intercept.

c Use $x = -\frac{b}{2a}$ to determine the turning point.

SOLUTION

a
$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-6)^2 - 4(3)(5)\\ &= -24\end{aligned}$$
$\Delta < 0$, so there are no x-intercepts.

b y-intercept is at $(0, 5)$.

c
$$\begin{aligned}x &= -\frac{b}{2a}\\ &= -\frac{(-6)}{2(3)}\\ &= 1\\ y &= 3(1)^2 - 6(1) + 5\\ &= 2\end{aligned}$$
$\therefore$ Turning point is at $(1, 2)$.

EXPLANATION

Use the discriminant $\Delta = b^2 - 4ac$ to find the number of x-intercepts. In $3x^2 - 6x + 5$, $a = 3$, $b = -6$ and $c = 5$.
Interpret the result.

Substitute $x = 0$ for the y-intercept.

For the x-coordinate of the turning point use $x = -\frac{b}{2a}$ with $a = 3$ and $b = -6$, as above.

Substitute the x-coordinate into $y = 3x^2 - 6x + 5$ to find the corresponding y-coordinate of the turning point.

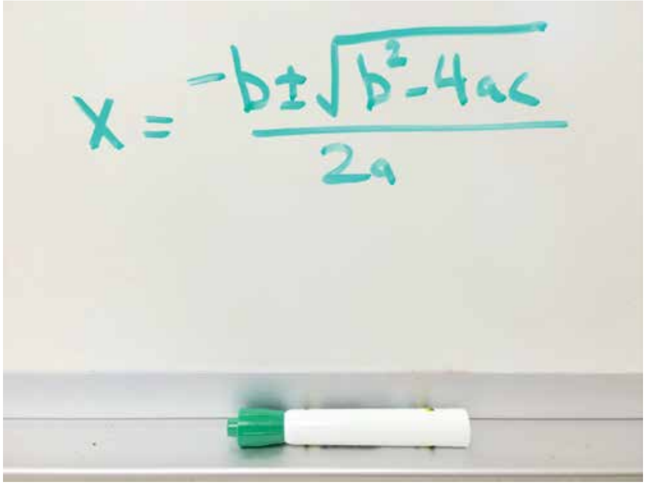
Now you try

Consider the parabola given by the quadratic equation $y = 2x^2 - 4x + 1$.

a Determine the number of x-intercepts.

b Determine the coordinates of the y-intercept.

c Use $x = -\frac{b}{2a}$ to determine the turning point.



Example 12 Sketching graphs using the quadratic formula

Sketch the graph of the quadratic $y = 2x^2 + 4x - 3$, labelling all significant points. Round the x-intercepts to two decimal places.

SOLUTION

y-intercept is at (0, –3).

EXPLANATION

Identify key features; i.e. x- and y-intercepts and the turning point. Substitute $x = 0$ for the y-intercept.

SOLUTION

x-intercepts $(y = 0)$:

$$2x^2 + 4x - 3 = 0$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{4^2 - 4(2)(-3)}}{2(2)} \\ &= \frac{-4 \pm \sqrt{40}}{4} \end{aligned}$$

$x = 0.58, -2.58$ (to 2 d.p.)

$\therefore$ x-intercepts are at $(0.58,\ 0)$, $(-2.58,\ 0)$

EXPLANATION

Use the quadratic formula to find the x-intercepts.

For $y = 2x^2 + 4x - 3$, $a = 2$, $b = 4$ and $c = -3$.

Note: $\dfrac{-4 \pm \sqrt{40}}{4} = \dfrac{-4 \pm 2\sqrt{10}}{4} = \dfrac{-2 \pm \sqrt{10}}{2}$

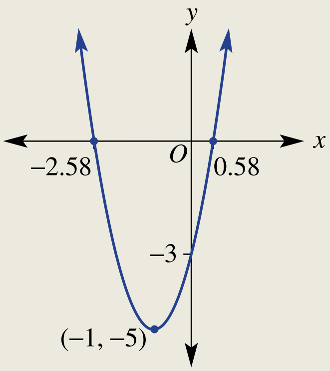
Use a calculator to round to two decimal places.

SOLUTION

$$\begin{aligned} \text{Turning point is at } x &= -\frac{b}{2a} \\ &= -\frac{(4)}{2(2)} \\ &= -1 \end{aligned}$$

and $\therefore y = 2(-1)^2 + 4(-1) - 3 = -5$.

$\therefore$ Turning point is at $(-1,\ -5)$.



EXPLANATION

Substitute $x = -1$ into $y = 2x^2 + 4x - 3$ to find the y-coordinate of the turning point.

Label the key features on the graph and sketch.

Now you try

Sketch the graph of the quadratic $y = 3x^2 - 6x + 1$, labelling all significant points. Round the x-intercepts to two decimal places.

Exercise 7E

FLUENCY 1–4(½) 1–4(½) 1–4(⅓)

Example 11a

1 Use the discriminant to determine the number of x-intercepts for the parabolas given by the following quadratics.

a $y = x^2 + 4x + 4$ **b** $y = x^2 - 3x + 5$ **c** $y = -x^2 + 4x + 2$
d $y = 3x^2 - 4x - 2$ **e** $y = 2x^2 - x + 2$ **f** $y = 2x^2 - 12x + 18$
g $y = 3x^2 - 2x$ **h** $y = 3x^2 + 5x$ **i** $y = -3x^2 - 2x$
j $y = 3x^2 + 5$ **k** $y = 4x^2 - 2$ **l** $y = -5x^2 + x$

Example 11b

2 Determine the coordinates of the y-intercept for the parabolas given by the following quadratics.

a $y = x^2 + 2x + 3$ **b** $y = x^2 - 4x + 5$ **c** $y = 4x^2 + 3x - 2$
d $y = 5x^2 - 2x - 4$ **e** $y = -2x^2 - 5x + 8$ **f** $y = -2x^2 + 7x - 10$
g $y = 3x^2 + 8x$ **h** $y = -4x^2 - 3x$ **i** $y = 5x^2 - 7$

Example 11c

3 Use $x = -\frac{b}{2a}$ to determine the coordinates of the turning point for the parabolas defined by the following quadratics.

a $y = x^2 + 2x + 4$ **b** $y = x^2 + 4x - 1$ **c** $y = x^2 - 4x + 3$
d $y = -x^2 + 2x - 6$ **e** $y = -x^2 - 3x + 4$ **f** $y = -x^2 + 7x - 7$
g $y = 2x^2 + 3x - 4$ **h** $y = 4x^2 - 3x$ **i** $y = -4x^2 - 9$
j $y = -4x^2 + 2x - 3$ **k** $y = -3x^2 - 2x$ **l** $y = -5x^2 + 2$

Example 12

4 Sketch the graph of these quadratics, labelling all significant points. Round the x-intercepts to two decimal places.

a $y = 2x^2 + 8x - 5$ **b** $y = 3x^2 + 6x - 2$ **c** $y = 4x^2 - 2x - 3$
d $y = 2x^2 - 4x - 9$ **e** $y = 2x^2 - 8x - 11$ **f** $y = 3x^2 + 9x - 10$
g $y = -3x^2 + 6x + 8$ **h** $y = -2x^2 - 4x + 7$ **i** $y = -4x^2 + 8x + 3$
j $y = -2x^2 - x + 12$ **k** $y = -3x^2 - 2x$ **l** $y = -5x^2 - 10x - 4$

PROBLEM–SOLVING 5(½) 5–6(½) 5–6(⅓), 7

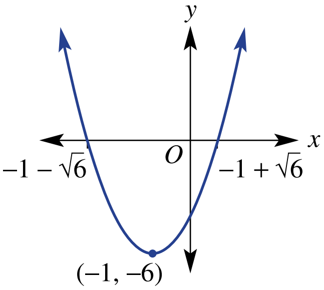
5 Give the exact x-intercepts of the graphs of these parabolas. Simplify any surds.

a $y = 3x^2 - 6x - 1$ **b** $y = -2x^2 - 4x + 3$ **c** $y = -4x^2 + 8x + 6$
d $y = 2x^2 + 6x - 3$ **e** $y = 2x^2 - 8x + 5$ **f** $y = 5x^2 - 10x - 1$

6 Sketch the graphs of these quadratics.

a $y = 4x^2 + 12x + 9$ **b** $y = 9x^2 - 6x + 1$
c $y = -4x^2 - 20x - 25$ **d** $y = -9x^2 + 30x - 25$
e $y = -2x^2 + 8x - 11$ **f** $y = -3x^2 + 12x - 16$
g $y = 4x^2 + 4x + 3$ **h** $y = 3x^2 + 4x + 2$

7 Find a rule in the form $y = ax^2 + bx + c$ that matches this graph.



REASONING	8	8, 9	9, 10

8 Write down two rules in the form $y = ax^2 + bx - c$ that have:

a two x-intercepts **b** one x-intercept **c** no x-intercepts.

9 Explain why the quadratic formula gives only one solution when the discriminant ($b^2 - 4ac$) is equal to 0.

10 Write down the quadratic formula for monic quadratic equations (i.e. where $a = 1$).

ENRICHMENT: Some proof	–	–	11, 12

11 Substitute $x = -\frac{b}{2a}$ into $y = ax^2 + bx + c$ to find the general rule for the y-coordinate of the turning point in terms of a, b and c.

12 Prove the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ by solving $ax^2 + bx + c = 0$.
(*Hint:* Divide both sides by a and complete the square.)

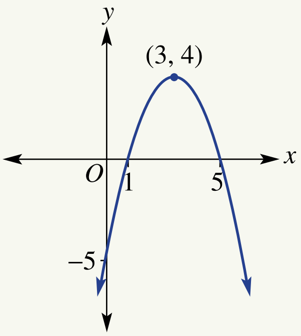
One of the most important applications of quadratic equations is in modelling acceleration, first formulated by Galileo in the early 1600s, and shown in action here in the present day.

Progress quiz

7A

1 For the given graph state:

- **a** the turning point and whether it is a maximum or minimum
- **b** the axis of symmetry
- **c** the coordinates of the intercepts.



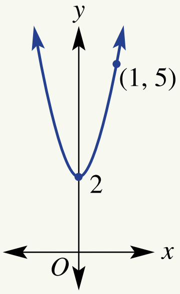
7B

2 Sketch graphs of the following quadratic relations, labelling the turning point and the y-intercept. (The x-intercepts are *not* required.)

a $y = 2x^2$ **b** $y = -x^2 + 3$ **c** $y = (x - 3)^2$ **d** $y = -(x + 2)^2 - 1$

7B

3 Find a rule for this parabola with turning point (0, 2) and another point (1, 5).



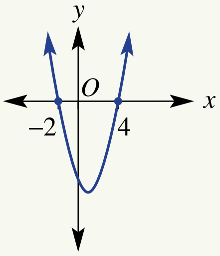
7C

4 Sketch graphs of the following quadratics, label the x- and y-intercepts and determine the coordinates of the turning point, using symmetry.

a $y = x^2 - 2x - 3$ **b** $y = x^2 - 4x + 4$

7C

5 The equation of this graph is of the form $y = (x + a)(x + b)$. Use the x-intercepts to find the values of a and b, then find the coordinates of the turning point.



7D

6 For $y = -2(x - 3)^2 + 8$ determine the:

- **a** coordinates of the turning point and state whether it is a maximum or minimum
- **b** y-intercept coordinates
- **c** x-intercept coordinates (if any).

7D

7 Sketch these graphs by first completing the square to write the equation in turning point form. Label the exact x- and y-intercepts and turning point on the graph.

a $y = x^2 - 4x + 3$ **b** $y = x^2 - 2x - 6$

7E Opt

8 For the parabolas given by the following quadratic equations:

- **i** use the discriminant to determine the number of x-intercepts
- **ii** determine the coordinates of the y-intercept
- **iii** use $x = -\frac{b}{2a}$ to determine the turning point.

a $y = x^2 - 4x + 5$ **b** $y = x^2 + 6x - 7$ **c** $y = -x^2 - 8x - 16$

7E Opt

9 Sketch the graph of the quadratic $y = 2x^2 - 8x + 5$, labelling all significant points. Give the x-intercepts, rounded to two decimal places.

7F Applications of parabolas

LEARNING INTENTIONS

- To be able to set up a quadratic model to solve a word problem
- To know how to apply the processes of quadratics to identify key features of a graph and relate them to real-life contexts
- To be able to identify the possible values of a variable in a given context

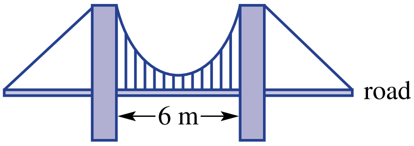
Quadratic equations and their graphs can be used to solve a range of practical problems. These could involve, for example, the path of a projectile or the shape of a bridge's arch. We can relate quantities with quadratic rules and use their graphs to illustrate key features. For example, x-intercepts show where one quantity (y) is equal to zero, and the turning point is where a quantity is a maximum or minimum.

Engineers could design a suspension bridge by placing the road on the *x*-axis and a pylon on the *y*-axis. The support cable forms a parabola and its quadratic equation is used to find the heights of the evenly spaced, vertical supports.

Lesson starter: The civil engineer

Michael, a civil engineer, designs a model for the curved cable of a 6 m suspension bridge using the equation $h = (d - 3)^2 + 2$, where h metres is the height of the hanging cables above the road for a distance d metres from the left pillar.

- What are the possible values for d?
- Sketch the graph of $h = (d - 3)^2 + 2$ for appropriate values of d.
- What is the height of the pillars above the road?
- What is the minimum height of the cable above the road?
- Discuss how key features of the graph have helped to answer the questions above.



KEY IDEAS

- Applying quadratics to solve problems may involve:
  - defining variables
  - forming equations
  - solving equations
  - deciding on a suitable range of values for the variables
  - sketching graphs showing key features
  - finding the maximum or minimum turning point.

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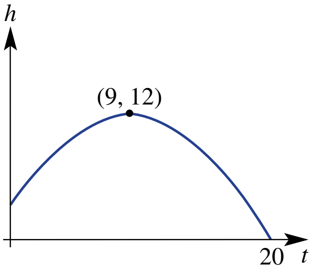
BUILDING UNDERSTANDING



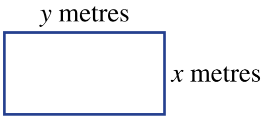
1 This graph shows the height (h metres) of a stone thrown through the air after t seconds.

a How many seconds does it take for the stone to reach its maximum height?

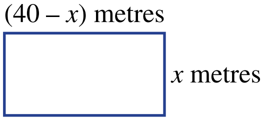
b What is the maximum height of the stone?

c What horizontal distance does the stone travel?

2 **a** If the perimeter of this rectangle is 20 metres, write an equation in terms of x and y.



b Give an expression for the area of this rectangle.



Example 13 Applying parabolas given the rule

The path of a javelin thrown by Jo is given by the formula $h = -\frac{1}{16}(d - 10)^2 + 9$, where h metres is the height of the javelin above the ground and d metres is the horizontal distance travelled.

a Sketch the graph of the rule for $0 \leqslant d \leqslant 22$ by finding the intercepts and the coordinates of the turning point.

b What is the maximum height the javelin reaches?

c What horizontal distance does the javelin travel (i.e. when is $h = 0$)?

SOLUTION	EXPLANATION
a Turning point is $(10, 9)$	If $y = a(x - h)^2 + k$, then the turning point is (h, k).
h-intercept ($d=0$): $h = -\frac{1}{16}(-10)^2 + 9 = 2.75$	Substitute $d = 0$ to find the h-intercept.
d-intercepts ($h=0$): $\begin{aligned} 0 &= -\tfrac{1}{16}(d-10)^2 + 9 \\ 144 &= (d-10)^2 \\ d - 10 &= \pm 12 \\ \therefore d &= -2 \text{ or } 22 \end{aligned}$	Substitute $h = 0$ and solve to find the d-intercepts.

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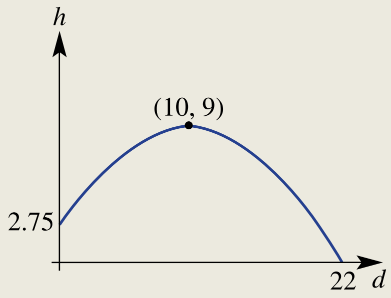


Sketch using $0 \leqslant d \leqslant 22$.

b 9 metres

The maximum value of h is at the turning point.

c 22 metres

$d = 22$ when $h = 0$.

Now you try

A ball is thrown upwards from ground level and reaches a height of h metres after t seconds, given by the formula $h = 20t - 5t^2$.

- **a** Sketch a graph of the rule for $0 \leqslant t \leqslant 4$ by finding the t-intercepts (x-intercepts) and the coordinates of the turning point.
- **b** What maximum height does the ball reach?
- **c** How long does it take the ball to return to ground level ($h = 0$)?

Example 14 Applying parabolas by formulating a rule

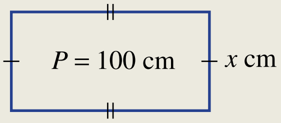
A piece of wire measuring 100 cm in length is bent into the shape of a rectangle. Let x cm be the width of the rectangle.

- **a** Use the perimeter to write an expression for the length of the rectangle in terms of x.
- **b** Write an equation for the area of the rectangle (A cm^2) in terms of x.
- **c** Decide on the suitable values of x.
- **d** Sketch the graph of A versus x for suitable values of x.
- **e** Use the graph to determine the maximum area that can be formed.
- **f** What will be the dimensions of the rectangle to achieve its maximum area?

SOLUTION

a
$$\begin{aligned} 2 \times \text{length} + 2x &= 100 \\ 2 \times \text{length} &= 100 - 2x \\ \therefore \text{Length} &= 50 - x \end{aligned}$$

EXPLANATION



100 cm of wire will form the perimeter.
Length is half of (100 − 2 × width).

b $A = x(50 - x)$

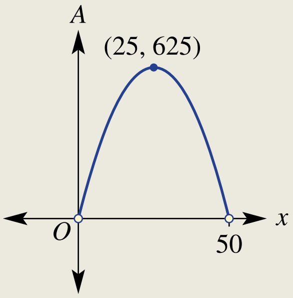
Area of a rectangle = length × width.

Continued on next page

c Length and width must be positive, so we require:
$x > 0$ and $50 - x > 0$
i.e. $x > 0$ and $50 > x$
i.e. $0 < x < 50$

Require each dimension to be positive, solve for x.

d



Sketch the graph, labelling the intercepts and turning point, which has x-coordinate halfway between the x-intercepts; i.e. $x = 25$. Substitute $x = 25$ into the area formula to find the maximum area: $A = 25(50 - 25) = 625$. Note open circles at $x = 0$ and $x = 50$ as these points are not included in the possible x-values.

e The maximum area that can be formed is 625 cm^2.

Read from the graph. The maximum area is the y-coordinate of the turning point.

f Maximum occurs when width $x = 25$ cm, so

$$\begin{aligned}\text{Length} &= 50 - 25\\ &= 25 \text{ cm}\end{aligned}$$

Dimensions that give maximum area are 25 cm by 25 cm, which is, in fact, a square.

From turning point, $x = 25$ gives the maximum area. Substitute to find the corresponding length. Length $= 50 - x$.

Now you try

A piece of wire measuring 80 cm in length is bent into the shape of a rectangle. Let x cm be the width of the rectangle.

a Use the perimeter to write an expression for the length of the rectangle in terms of x.
b Write an equation for the area of the rectangle (A cm^2) in terms of x.
c Decide on the suitable values of x.
d Sketch the graph of A versus x for suitable values of x.
e Use the graph to determine the maximum area that can be formed.
f What will be the dimensions of the rectangle to achieve its maximum area?

Exercise 7F

FLUENCY 1–3 1–4 2–5

Example 13

1 A wood turner carves out a bowl according to the formula $d = \frac{1}{3}x^2 - 27$, where d cm is the depth of the bowl and x cm is the distance from the centre of the bowl.

a Sketch a graph for $-9 \leqslant x \leqslant 9$, showing x-intercepts and the turning point.
b What is the width of the bowl?
c What is the maximum depth of the bowl?

2 The equation for the arch of a particular bridge is given by $h = -\frac{1}{500}(x - 100)^2 + 20$, where h m is the height above the base of the bridge and x m is the distance from the left side.

a Determine the coordinates of the turning point of the graph.

b Determine the x-intercepts of the graph.

c Sketch the graph of the arch for appropriate values of x.

d What is the span of the arch?

e What is the maximum height of the arch?

Richmond Bridge, Tasmania, is the oldest bridge in Australia that is still in use.

Example 14

3 A 20 cm piece of wire is bent to form a rectangle. Let x cm be the width of the rectangle.

a Use the perimeter to write an expression for the length of the rectangle in terms of x.

b Write an equation for the area of the rectangle (A cm^2) in terms of x.

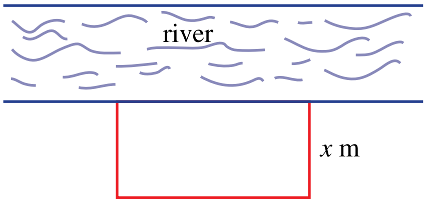
c Decide on suitable values of x.

d Sketch the graph of A versus x for suitable values of x.

e Use the graph to determine the maximum area that can be formed.

f What will be the dimensions of the rectangle to achieve its maximum area?

4 A farmer has 100 m of fencing to form a rectangular paddock with a river on one side (that does not require fencing), as shown.



a Use the perimeter to write an expression for the length of the paddock in terms of the width, x metres.

b Write an equation for the area of the paddock (A m^2) in terms of x.

c Decide on suitable values of x.

d Sketch the graph of A versus x for suitable values of x.

e Use the graph to determine the maximum paddock area that can be formed.

f What will be the dimensions of the paddock to achieve its maximum area?

5 The sum of two positive numbers is 20 and x is the smaller number.

a Write the second number in terms of x.

b Write a rule for the product, P, of the two numbers in terms of x.

c Sketch a graph of P vs x.

d Find the values of x when:

- **i** $P = 0$
- **ii** P is a maximum.

e What is the maximum value of P?

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PROBLEM–SOLVING	6	6, 7	7, 8

6 The equation for a support span is given by $h = -\frac{1}{40}(x - 20)^2$, where h m is the distance below the deck of a bridge and x m is the distance from the left side.

a Determine the coordinates of the turning point of the graph.
b Sketch a graph of the equation using $0 \leqslant x \leqslant 40$.
c What is the width of the support span?
d What is the maximum height of the support span?

7 Jordie throws a rock from the top of a 30 metre high cliff and its height (h metres) above the sea is given by $h = 30 - 5t^2$, where t is in seconds.

a Find the exact time it takes for the rock to hit the water.
b Sketch a graph of h vs t for appropriate values of t.
c What is the exact time it takes for the rock to fall to a height of 20 metres?

8 A bird dives into the water to catch a fish. It follows a path given by $h = t^2 - 8t + 7$, where h is the height in metres above sea level and t is the time in seconds.

a Sketch a graph of h vs t, showing intercepts and the turning point.
b Find the time when the bird:
  i enters the water
  ii exits the water
  iii reaches a maximum depth.
c What is the maximum depth to which the bird dives?
d At what times is the bird at a depth of 8 metres?

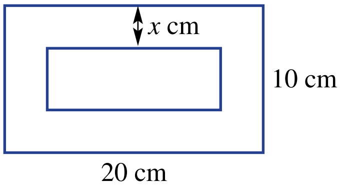
REASONING	9	9–11	11, 12

9 The height, h metres, of a flying kite is given by the rule $h = t^2 - 6t + 10$ for t seconds.

a Find the minimum height of the kite during this time.
b Does the kite ever hit the ground during this time? Give reasons.

10 The sum of two numbers is 64. Show that their product has a maximum of 1024.

11 A rectangular framed picture has a total length and width of 20 cm and 10 cm, respectively. The frame has width x cm.



a Find the rule for the area (A cm^2) of the picture inside.
b What are the minimum and maximum values of x?
c Sketch a graph of A vs x using suitable values of x.
d Explain why there is no turning point for your graph, using suitable values of x.
e Find the width of the frame if the area of the picture is 144 cm^2.

12 A dolphin jumping out of the water follows a path described by $h = -\frac{1}{2}(x^2 - 10x + 16)$, where h is the vertical height, in metres, and x metres is the horizontal distance travelled.

a How far horizontally does the dolphin travel out of the water?
b Does the dolphin ever reach a height of 5 metres above water level? Give reasons.

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ENRICHMENT: The highway and the river and the lobbed ball	–	–	13, 14

13 The path of a river is given by the rule $y = \frac{1}{10}x(x - 100)$ and all units are given in metres. A highway is to be built near or over the river on the line $y = c$.

a Sketch a graph of the path of the river, showing key features.

b For the highway with equation $y = c$, decide how many bridges will need to be built if:

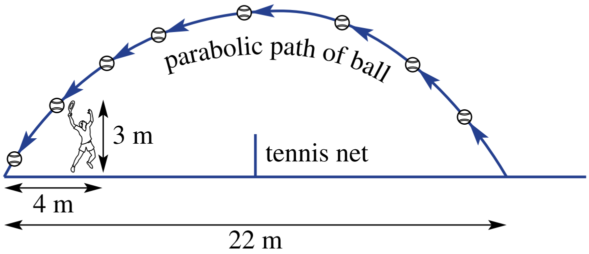
  i $c = 0$ **ii** $c = -300$

c Locate the coordinates of the bridge, correct to one decimal place, if:

  i $c = -200$ **ii** $c = -10$

d Describe the situation when $c = -250$.

14 A tennis ball is lobbed from ground level and must cover a horizontal distance of 22 m if it is to land just inside the opposite end of the court. If the opponent is standing 4 m from the baseline and he can hit any ball less than 3 m high, what is the lowest maximum height the lob must reach to win the point?



7G Intersection of lines and parabolas OPTIONAL

LEARNING INTENTIONS

- To understand how a line can intersect a parabola at 0, 1 or 2 points
- To be able to find the points of intersection of a line and a parabola using substitution
- To know that the discriminant can be used to determine the number of points of intersection of a line and a parabola

We have seen previously when simultaneously solving a pair of linear equations that there is one solution provided that the graphs of these linear equations are not parallel. Graphically, this represents the point of intersection for the two straight lines.

For the intersection of a parabola and a line we can have either zero, one or two points of intersection. As we have done for linear simultaneous equations, we can use the method of substitution to solve a linear equation and a non-linear equation simultaneously.

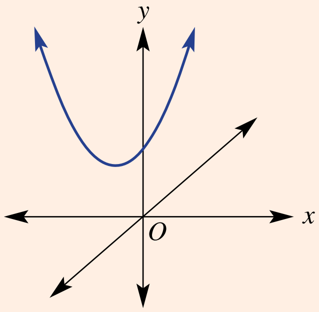

This image shows why the Infinity Bridge, England, is named after the symbol ∞. Architects develop and solve equations in three dimensions to determine the intersection points of lines (support cables) and parabolic curves (the arches).

Lesson starter: How many times does a line cut a parabola?

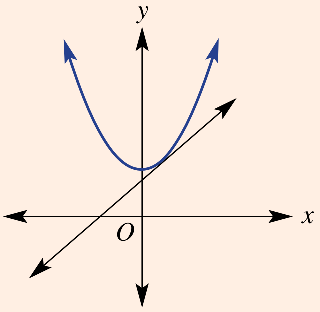
- Use computer graphing software to plot a graph of $y = x^2$.
- By plotting lines of the form $x = h$, determine how many points of intersection a vertical line will have with the parabola.
- By plotting lines of the form $y = c$, determine how many points of intersection a horizontal line could have with the parabola.
- By plotting straight lines of the form $y = 2x + k$ for various values of k, determine the number of possible intersections between a line and a parabola.
- State some values of k for which the line above intersects the parabola:
  - twice
  - never.
- Can you find the value of k for which the line intersects the parabola exactly once?

KEY IDEAS

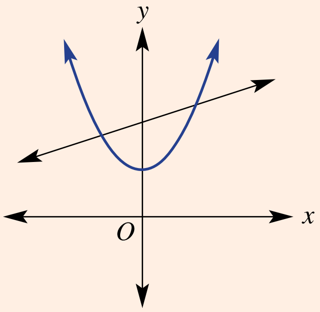
- When solving a pair of simultaneous equations involving a parabola and a line, we may obtain zero, one or two solutions. Graphically, this represents zero, one or two points of intersection between the parabola and the line.
  - A line that intersects a curve twice is called a **secant**.
  - A line that touches a curve at a single point of contact is called a **tangent**.



zero points of intersection

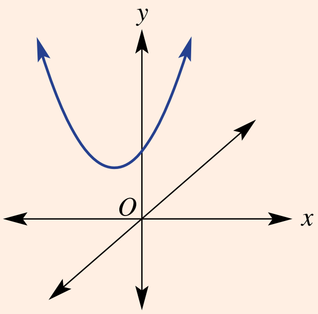


one point of intersection

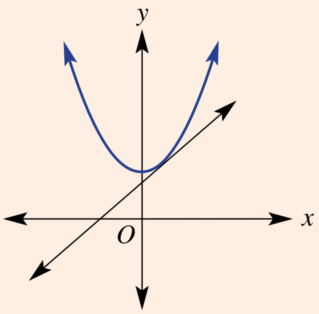


two points of intersection

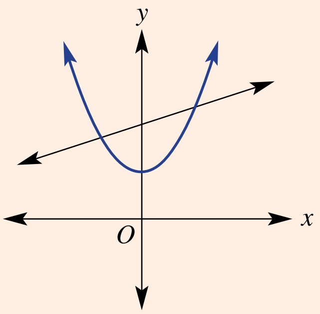
- The method of substitution is used to solve the equations simultaneously.
  - Substitute one equation into the other.
  - Rearrange the resulting equation into the form $ax^2 + bx + c = 0$.
  - Solve for x by factorising and applying the Null Factor Law or use the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
  - Substitute the x-values into one of the original equations to find the corresponding y-value.
- After substituting the equations and rearranging, we arrive at an equation of the form $ax^2 + bx + c = 0$. Hence, the discriminant, $b^2 - 4ac$, can be used to determine the number of solutions (i.e. points of intersection) of the two equations.



zero solutions
$b^2 - 4ac < 0$



one solution
$b^2 - 4ac = 0$



two solutions
$b^2 - 4ac > 0$

BUILDING UNDERSTANDING

① **a** Find the coordinates of the point where the vertical line $x = 2$ intersects the parabola $y = 2x^2 + 5x - 6$.

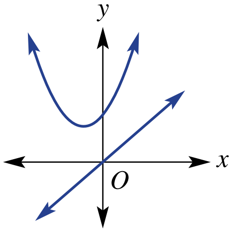
b Find the coordinates of the point where the vertical line $x = -1$ intersects the parabola $y = x^2 + 3x - 1$.

② Rearrange the following into the form $ax^2 + bx + c = 0$, where $a > 0$.

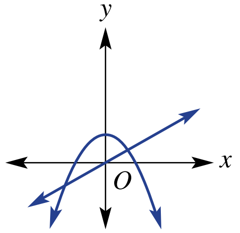
a $x^2 + 5x = 2x - 6$ **b** $x^2 - 3x + 4 = 2x + 1$ **c** $x^2 + x - 7 = -2x + 5$

③ What do we know about the discriminant, $b^2 - 4ac$, of the resulting equation from solving the equations which correspond to these graphs simultaneously?

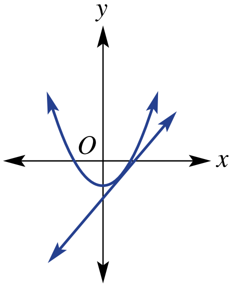
a



b



c



Example 15 Finding points of intersection of a parabola and a horizontal line

Find any points of intersection of these parabolas and horizontal lines.

a $y = x^2 - 3x$
$y = 4$

b $y = x^2 + 2x + 4$
$y = -2$

SOLUTION	EXPLANATION
a By substitution: $x^2 - 3x = 4$ $x^2 - 3x - 4 = 0$ $(x - 4)(x + 1) = 0$	Substitute $y = 4$ from the second equation into the first equation. Write in the form $ax^2 + bx + c = 0$ by subtracting 4 from both sides.
$x - 4 = 0$ or $x + 1 = 0$ $x = 4$ or $x = -1$	Factorise and apply the Null Factor Law to solve for x.
∴ The points of intersection are at $(4,\ 4)$ and $(-1,\ 4)$.	As the points are on the line $y = 4$, the y-coordinate of the points of intersection is 4.
b By substitution:	Substitute $y = -2$ into the first equation.
$x^2 + 2x + 4 = -2$ $x^2 + 2x + 6 = 0$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Apply the quadratic formula to solve $x^2 + 2x + 6 = 0$, where $a = 1$, $b = 2$ and $c = 6$. Alternatively, complete the square: $(x + 1)^2 + 5 = 0$.
$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(6)}}{2(1)}$ $= \frac{-2 \pm \sqrt{-20}}{2}$ ∴ There are no points of intersection.	$\sqrt{-20}$ has no real solutions, or using $(x + 1)^2 + 5 = 0$, there are no solutions. The parabola $y = x^2 + 2x + 4$ and the line $y = -2$ do not intersect.

Now you try

Find any points of intersection of these parabolas and horizontal lines.

a $y = x^2 - x$
$y = 2$

b $y = x^2 + 4x + 1$
$y = -4$

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Example 16 Solving simultaneous equations involving a line and a parabola

Solve the following equations simultaneously.

a $y = x^2$, $y = 2x$

b $y = -4x^2 - x + 6$, $y = 3x + 7$

c $y = x^2 + 1$, $2x - 3y = -4$

SOLUTION	EXPLANATION
a By substitution:	Substitute $y = 2x$ into $y = x^2$.
$\begin{aligned} x^2 &= 2x \\ x^2 - 2x &= 0 \\ x(x-2) &= 0 \\ x = 0 \text{ or } x - 2 &= 0 \\ x = 0 \text{ or } x &= 2 \end{aligned}$	Rearrange the equation so that it is equal to 0. Factorise by removing the common factor x. Apply the Null Factor Law to solve for x.
When $x = 0$, $y = 2 \times (0) = 0$. When $x = 2$, $y = 2 \times (2) = 4$.	Substitute the x-values into $y = 2x$ to obtain the corresponding y-value. Alternatively, the equation $y = x^2$ can be used to find the y-values or it can be used to check the y-values.
∴ The solutions are $x = 0, y = 0$ and $x = 2, y = 4$.	The points $(0, 0)$ and $(2, 4)$ lie on both the line $y = 2x$ and the parabola $y = x^2$.
b By substitution:	Substitute $y = 3x + 7$ into the first equation.
$\begin{aligned} -4x^2 - x + 6 &= 3x + 7 \\ -x + 6 &= 4x^2 + 3x + 7 \\ 6 &= 4x^2 + 4x + 7 \\ 0 &= 4x^2 + 4x + 1 \\ \therefore (2x+1)(2x+1) &= 0 \\ 2x + 1 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$	When rearranging the equation equal to 0, gather the terms on the side that makes the coefficient of x^2 positive, as this will make the factorising easier. Hence, add $4x^2$ to both sides, then add x to both sides and subtract 6 from both sides. Factorise and solve for x.
When $x = -\frac{1}{2}$, $\begin{aligned} y &= 3 \times \left(-\frac{1}{2}\right) + 7 \\ &= \frac{11}{2} \text{ or } 5\frac{1}{2} \end{aligned}$	Substitute the x-value into $y = 3x + 7$ (or $y = -4x^2 - x + 6$ but $y = 3x + 7$ is a simpler equation).
∴ The only solution is $x = -\frac{1}{2}$, $y = \frac{11}{2}$.	Finding only one solution indicates that this line is a tangent to the parabola.
c By substitution:	Replace y in $2x - 3y = -4$ with $x^2 + 1$, making sure you include brackets.
$\begin{aligned} 2x - 3(x^2 + 1) &= -4 \\ 2x - 3x^2 - 3 &= -4 \\ 2x - 3 &= 3x^2 - 4 \\ 2x &= 3x^2 - 1 \end{aligned}$	Expand the brackets and then rearrange into the form $ax^2 + bx + c = 0$.
$\begin{aligned} \therefore 3x^2 - 2x - 1 &= 0 \\ (3x+1)(x-1) &= 0 \\ 3x + 1 &= 0 \text{ or } x - 1 = 0 \\ x &= -\frac{1}{3} \text{ or } x = 1 \end{aligned}$	Factorise and solve for x.

Continued on next page

When $x = -\frac{1}{3}$, $y = \left(-\frac{1}{3}\right)^2 + 1$
$= \frac{1}{9} + 1$
$= \frac{10}{9}$

Substitute the x-values into one of the two original equations to solve for y.

When $x = 1$, $y = (1)^2 + 1$
$= 2$

$\therefore$ The solutions are $x = -\frac{1}{3}$, $y = \frac{10}{9}$ and $x = 1$, $y = 2$.

The line and parabola intersect in two places.

Now you try

Solve the following equations simultaneously.

a $y = x^2$
$y = 4x$

b $y = -x^2 + 6x + 7$
$y = 4x + 8$

c $y = x^2 - 1$
$3x + 2y = 0$

Example 17 Solving simultaneous equations with the quadratic formula

Solve the equations $y = x^2 + 5x - 5$ and $y = 2x$ simultaneously. Round your values to two decimal places.

SOLUTION

By substitution:

$$x^2 + 5x - 5 = 2x$$
$$x^2 + 3x - 5 = 0$$

Using the quadratic formula:

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-5)}}{2(1)}$$
$$= \frac{-3 \pm \sqrt{9 + 20}}{2}$$
$$= \frac{-3 \pm \sqrt{29}}{2}$$
$$= 1.19258\ldots \text{ or } -4.19258\ldots$$

In exact form, $y = 2x$
$$= 2 \times \left(\frac{-3 \pm \sqrt{29}}{2}\right)$$
$$= -3 \pm \sqrt{29}$$

$\therefore$ The solutions are $x = 1.19$, $y = 2.39$ and $x = -4.19$, $y = -8.39$ (to 2 d.p.).

EXPLANATION

Rearrange into standard form.
$x^2 + 3x - 5$ does not factorise with whole numbers.

Quadratic formula: If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Here, $a = 1$, $b = 3$ and $c = -5$.

Use a calculator to evaluate $\frac{-3 - \sqrt{29}}{2}$ and $\frac{-3 + \sqrt{29}}{2}$. Recall that if the number under the square root is negative, then there will be no real solutions.

Substitute exact x-values into $y = 2x$.

Round your values to two decimal places, as required.

Now you try

Solve the equations $y = x^2 + 2x - 1$ and $y = 3x$ simultaneously. Round your values to two decimal places.

Example 18 Determining the number of solutions of simultaneous equations

Determine the number of solutions (points of intersection) of the following pairs of equations.

a $y = x^2 + 3x - 1$
$y = x - 2$

b $y = 2x^2 - 3x + 8$
$y = 5 - 2x$

SOLUTION

a By substitution:

$$\begin{aligned} x^2 + 3x - 1 &= x - 2 \\ x^2 + 2x + 1 &= 0 \end{aligned}$$

Using the discriminant:

$$\begin{aligned} b^2 - 4ac &= (2)^2 - 4(1)(1) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

∴ There is one solution to the pair of equations.

b By substitution:

$$\begin{aligned} 2x^2 - 3x + 8 &= 5 - 2x \\ 2x^2 - x + 3 &= 0 \end{aligned}$$

Using the discriminant:

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4(2)(3) \\ &= 1 - 24 \\ &= -23 < 0 \end{aligned}$$

∴ There is no solution to the pair of equations.

EXPLANATION

Start as if solving the equations simultaneously. Once the equation is in the form $ax^2 + bx + c = 0$, the discriminant $b^2 - 4ac$ can be used to determine the *number* of solutions. Here, $a = 1$, $b = 2$ and $c = 1$.

Recall: $b^2 - 4ac > 0$ means two solutions.
$b^2 - 4ac = 0$ means one solution.
$b^2 - 4ac < 0$ means no solutions.

Substitute and rearrange into the form $ax^2 + bx + c = 0$.

Calculate the discriminant. Here, $a = 2$, $b = -1$ and $c = 3$.

$b^2 - 4ac < 0$ means no solutions.

Now you try

Determine the number of solutions (points of intersection) of the following pairs of equations.

a $y = x^2 + 7x - 4$
$y = x - 1$

b $y = 3x^2 - x + 6$
$y = 5 - 2x$

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Exercise 7G

FLUENCY	1–3(½), 4	1–3(½), 4, 5(½)	1–3(⅓), 4, 5(⅓)

Example 15

1 Find the points of intersection of these parabolas and horizontal lines.

a $y = x^2 + x$
$y = 6$

b $y = x^2 - 4x$
$y = 12$

c $y = x^2 + 3x + 6$
$y = 1$

d $y = 2x^2 + 7x + 1$
$y = -2$

e $y = 4x^2 - 12x + 9$
$y = 0$

f $y = 3x^2 + 2x + 9$
$y = 5$

Example 16a,b

2 Solve these simultaneous equations using substitution.

a $y = x^2$
$y = 3x$

b $y = x^2$
$y = -2x$

c $y = x^2$
$y = 3x + 18$

d $y = x^2 - 2x + 5$
$y = x + 5$

e $y = -x^2 - 11x + 4$
$y = -3x + 16$

f $y = x^2 + 3x - 1$
$y = 4x + 5$

g $y = x^2 - 2x - 4$
$y = -2x - 5$

h $y = -x^2 + 3x - 5$
$y = 3x - 1$

i $y = 2x^2 + 4x + 10$
$y = 1 - 7x$

j $y = 3x^2 - 2x - 20$
$y = 2x - 5$

k $y = -x^2 - 4x + 3$
$y = 2x + 12$

l $y = x^2 + x + 2$
$y = 1 - x$

Example 16c

3 Solve these simultaneous equations by first substituting.

a $y = x^2$
$2x + y = 8$

b $y = x^2$
$x - y = -2$

c $y = x^2$
$2x + 3y = 1$

d $y = x^2 + 3$
$5x + 2y = 4$

e $y = x^2 + 2x$
$2x - 3y = -4$

f $y = -x^2 + 9$
$6x - y = 7$

Example 17

4 Solve the following simultaneous equations, making use of the quadratic formula.

a Give your answers to one decimal place where necessary.

i $y = 2x^2 + 3x + 6$
$y = x + 4$

ii $y = x^2 + 1$
$y = 2x + 3$

iii $y = -2x^2 + x + 3$
$y = 3x + 2$

iv $y = 2x^2 + 4x + 5$
$y = 3 - 2x$

b Give your answers in exact surd form.

i $y = x^2 + 2x - 5$
$y = x$

ii $y = x^2 - x - 1$
$y = 2x$

iii $y = -x^2 - 3x + 3$
$y = -2x$

iv $y = x^2 + 3x - 3$
$y = 2x + 1$

Example 18

5 Determine the number of solutions to the following simultaneous equations.

a $y = x^2 + 2x - 3$
$y = x + 4$

b $y = 2x^2 + x$
$y = 3x - 1$

c $y = 3x^2 - 7x + 3$
$y = 1 - 2x$

d $y = x^2 + 5x + 1$
$y = 2x - 3$

e $y = -x^2$
$y = 2x + 1$

f $y = -x^2 + 2x$
$y = 3x - 1$

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PROBLEM–SOLVING	6, 7(½)	6, 7(½)	7, 8

6 Ben, a member of an indoor cricket team, playing a match in a gymnasium, hits a ball that follows a path given by $y = -0.1x^2 + 2x + 1$, where y is the height above ground, in metres, and x is the horizontal distance travelled by the ball, in metres. The ceiling of the gymnasium is 10.6 metres high. Will this ball hit the roof? Explain.

7 Solve the following equations simultaneously.

a $y = x^2 + 2x - 1$
$y = \dfrac{x-3}{2}$

b $y = x(x-4)$
$y = \dfrac{1}{2}x - 5$

c $y = (x-2)^2 + 7$
$y = 9 - x$

d $y = \dfrac{8 - x^2}{2}$
$y = 2(x-1)$

8 A train track is to be constructed over a section of a lake. On a map, the edge of the lake that the train track will pass over is modelled by the equation $y = 6 - 2x^2$. The segment of train track is modelled by the equation $y = x + 5$.

The section of track to be constructed will start and end at the points at which this track meets the lake.

a Determine the location (i.e. coordinates) of the points on the map where the framework for the track will start and end.

b If 1 unit represents 100 metres, determine the length of track that must be built over the lake, correct to the nearest metre.

REASONING	9	9, 10	10, 11

9 Consider the parabola with equation $y = x^2 - 6x + 5$.

a Use any suitable method to determine the coordinates of the turning point of this parabola.

b Hence, state for which values of c the line $y = c$ will intersect the parabola:

i twice **ii** once **iii** not at all.

10 Consider the parabola with equation $y = x^2$ and the family of lines $y = x + k$.

a Determine the discriminant, in terms of k, obtained when solving these equations simultaneously.

b Hence, determine for which values of k the line will intersect the parabola:

i twice **ii** once **iii** not at all.

11 **a** Use the discriminant to show that the line $y = 2x + 1$ does not intersect the parabola $y = x^2 + 3$.

b Determine for which values of k the line $y = 2x + k$ does intersect the parabola $y = x^2 + 3$.

ENRICHMENT: Multiple tangents?	–	–	12

12 The line $y = mx$ is a tangent to the parabola $y = x^2 - 2x + 4$ (i.e. the line touches the parabola at just one point).

a Find the possible values of m.

b Can you explain why there are two possible values of m? (*Hint*: A diagram may help.)

c If the value of m is changed so that the line now intersects the parabola in two places, what is the set of possible values for m?

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Designer jeans

1 A popular pair of designer jeans sells for \$300. A franchise sells 1200 pairs a month. The company carried out some research and discovered that for every \$10 decrease in price it can sell 100 more pairs a month.

The franchise is interested in maximising profit based on possible changes to the price of the jeans.

- **a** If the jeans are sold for \$290 one month:
  - **i** how many sales are expected?
  - **ii** how much is made in sales (revenue)? How does this compare to the revenue in a month when the jeans are sold for \$300?
- **b** Complete a table like the one shown below, to look at the revenue as the price of jeans is decreased.

Number of \$10 price decreases	Price of jeans (\$)	Number of sales	Revenue (\$)
0	300	1200	$300 \times 1200 = 360\,000$
1	290		
2			
3			

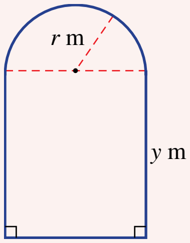
- **c** Use your table to help establish a rule for the revenue, R dollars, based on the number of \$10 price decreases, x.
- **d** Use your rule to determine the price to sell the jeans for to maximise the revenue and state this maximum revenue.

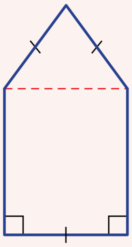
Stained-glass windows

2 Many objects, such as windows, are composite shapes. The stained-glass window shown, for example, is made up of a rectangle and semicircle.

A window company is interested in exploring the relationship between the perimeter of the window and its area. It also wants to look at maximising the area for a fixed perimeter.

- **a** If the perimeter of the window is fixed at 6 m, find:
  - **i** an expression for the height of the rectangular section, y m, in terms of r
  - **ii** a rule for the area, A m^2, of the window in the form $A = ar^2 + br$ where a and b are constants
  - **iii** the dimensions of the window that maximise its area and the maximum area.
- **b** Repeat part **a** for a perimeter of P m. Confirm your result by checking your answer to part **a iii** with $P = 6$.
- **c** Investigate a second stained-glass window, as shown, with an equilateral triangle top section. Compare its maximum area with part **a**, for the same perimeter of 6 m.





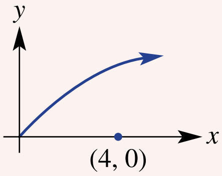
On the lake

3 The parabolic curve can be used to model many shapes in the natural world. A certain lake is represented on a Cartesian plane by the region bound by the curves $y = \frac{1}{2}x^2 - 4x$ and $y = 3x - \frac{1}{2}x^2$.

You will investigate the use of the parabola to model the boundary of the lake and a path of a boat. You will calculate distances on the lake and find minimum distances between objects on or beside the lake.

a **i** Sketch the region represented by the lake including the points of intersection.
  ii At $x = 2$, determine the vertical distance across the lake.
  iii Find a rule in terms of x for the vertical distance (y direction) across the lake.
  iv Hence, find the maximum vertical distance across the lake.

b A speed boat on the lake is following a path as shown on the right. A spectator stands on the sidelines at (4, 0) with 1 unit representing 10 m. The rule for the speedboat's path is given by $y = \sqrt{x}$.



  i Find the direct distance, in metres, from the spectator to the speedboat when the speedboat is at (1, 1).
  ii Find a rule in terms of x for the distance between (4, 0) and a point (x, y) on the speed boat path. (*Hint*: $y = \sqrt{x}$.)
  iii As x increases, explain what happens to the value of $\sqrt{x}$.
  iv Complete the table below by using the minimums of the quadratics in rules **I** and **III** to infer the minimum of their square root graph.

Rule	x-value of minimum	Minimum value
I $y = x^2 + 2$	0	2
II $y = \sqrt{x^2 + 2}$		$\sqrt{2}$
III $y = x^2 - 4x + 7$		
IV $y = \sqrt{x^2 - 4x + 7}$		

  v Hence, if the minimum value of a quadratic rule y is n at $x = m$, give the minimum value of $\sqrt{y}$ and for which x-value it occurs.
  vi Use the ideas above and your rule from part **ii** to determine the coordinates on the speedboat's path, $y = \sqrt{x}$, where it will be closest to the spectator. What is this minimum distance?

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7H Rates of change OPTIONAL

LEARNING INTENTIONS

- To be able to recognise and describe relationships between variables
- To understand the connection between gradient and the rate of change
- To be able to interpret distance–time graphs
- To know that speed is the rate of change of distance over time

We know from previous studies that many pairs of variables are related using linear, exponential or quadratic rules all of which produce different types of graphs. We note that the gradient of a linear graph is constant but for other relations the gradient will vary depending on the point on the curve. It is the gradient of the curve that tells us how quickly the value of y is changing. This is called the rate of change.

The rate of change of distance over time can also be considered as a gradient and is called speed.

Lesson starter: Movement graphs

This is a whole class activity. Two volunteers are needed: the ‘walker’ who completes a journey between the front and back of the classroom and the ‘grapher’ who graphs the journey on the whiteboard.

1. The ‘walker’ will complete a variety of journeys but these are not stated to the class. For example:
    - Walk slowly from the front of the room, stop halfway for a few seconds and then walk steadily to the back of the room. Stop and then return to the front at a fast steady pace.
    - Start quickly from the back of the room and gradually slow down until stopping at the front.
    - Start slowly from the front and gradually increase walking speed all the way to the back. Stop for a few seconds and then return at a steady speed to the front of the room.
2. The ‘grapher’ draws a graph of distance versus time on the whiteboard at the same time as the ‘walker’ moves. The distance is measured from the front of the room. No numbers are needed.
3. The class members also each draw their own distance–time graph as the ‘walker’ moves.
4. After each walk, discuss how well the ‘grapher’ has modelled the ‘walker’s’ movement.
5. This activity can also be done in reverse. The ‘grapher’ draws a distance–time graph on the board and the ‘walker’ moves to match the graph. The class checks that the ‘walker’ is following the graph correctly.

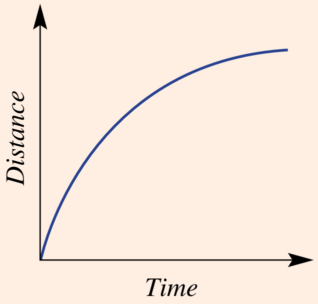
KEY IDEAS

- The shape of a graph shows how both y and the rate of change of y (i.e. the gradient) varies.
  - If y increases as x increases, then the rate of change (the gradient) is positive.
  - If y decreases as x increases, then the rate of change (the gradient) is negative.
  - If y does not change as x increases, then the rate of change (the gradient) is zero.
  - Straight lines have a constant rate of change. There is a fixed change in y for each unit increase in x.
  - Curves have a varying rate of change. The change in y varies for each unit increase in x.
  - Analysing a graph and describing how both y and the rate of change of y varies allows us to check whether a given graph models a situation accurately.

    For example, these distance–time graphs show various journeys from 'home' (distance = 0 at home).

Journey A

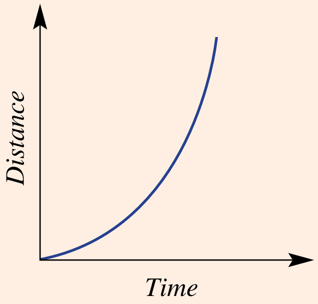
Decelerating away from home.



Distance from home is increasing at a decreasing rate.

Journey B

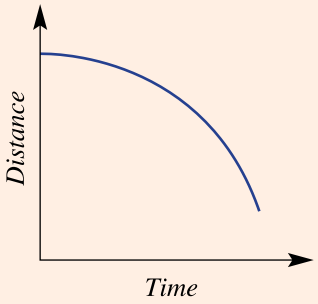
Accelerating away from home.



Distance from home is increasing at an increasing rate.

Journey C

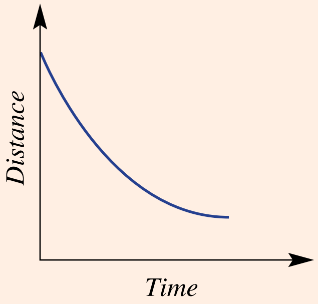
Accelerating towards home.



Distance from home is decreasing at an increasing rate.

Journey D

Decelerating towards home.



Distance from home is decreasing at a decreasing rate.

- Speed is the rate of change of distance over time.

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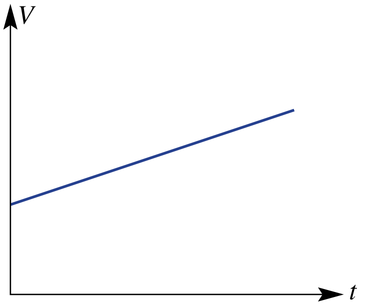
BUILDING UNDERSTANDING

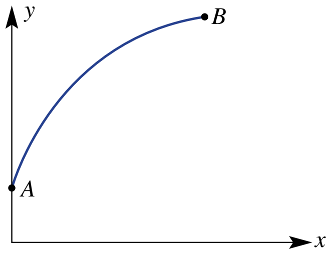


1 This graph shows the relationship between two variables V and t.

a Would you say that the relationship between V and t is exponential, quadratic, linear or other?

b Would you say that the gradient is constant as t varies? Give a reason.

c How would you describe the rate of change of volume (V) over time in this example?



2 This graph shows the relationship between two variables x and y.

a At which point is the y-value the largest?

b At which point is the gradient (rate of change) the greatest?

c Are the y-values increasing or decreasing as x increases?

d Is the gradient increasing or decreasing as x increases?

3 Choose a word to complete these sentences.

a If y increases as x increases, then the rate of change (gradient) is _______________.

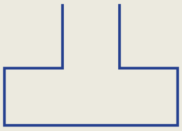
b If y decreases as x increases, then the rate of change (gradient) is _______________.

c If y does not change as x increases, then the rate of change (gradient) is __________.

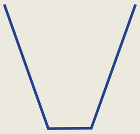
Example 19 Graphing height vs time

Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, h, of water in the container at time, t.

a

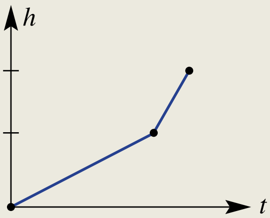


b



SOLUTION

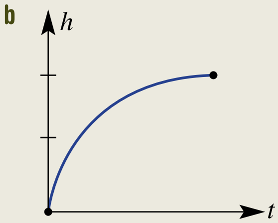
a



EXPLANATION

For the wider part of the container, the rate of change of height (gradient) will be less than the rate of change for the top part of the container. It will take longer to fill the wider part of the container. For each section however the rate of change is constant.

b

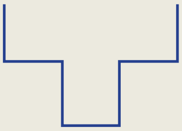


As the water rises the rate of change (gradient) will decrease but still remain positive as the container becomes wider. The rate of change is greatest at the start and least at the end.

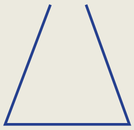
Now you try

Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, *h* of water in the container at time, *t*.

a

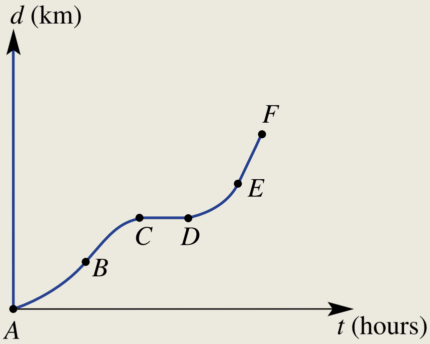


b



Example 20 Working with distance–time graphs

This distance–time graph illustrates a journey by car over a number of hours.



a Between what points does the graph show a speed which is:

i zero?

ii constant?

iii increasing?

iv decreasing?

b Would you say that the car is travelling slower at point *B* compared to point *E*? Give a reason.

c Between the two given points, at what point is the car travelling the fastest?

i Points *A* and *D*

ii Points *C* and *E*

Continued on next page

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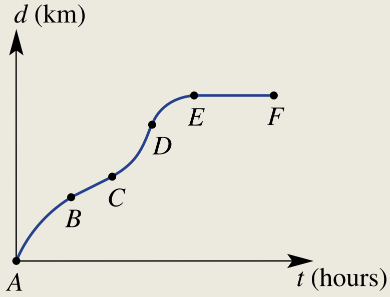
SOLUTION	EXPLANATION
a **i** C to D	Between points C and D the distance has not changed.
ii C to D and E to F	Between these pairs of points the gradient is constant.
iii A to B and D to E	Between these pairs of points the gradient is positive and increasing as the graph becomes steeper.
iv B to C	Between this pair of points the gradient is positive and decreasing as the graphs becomes less steep.
b No	The graph looks steeper at point E compared to the graph at point B.
c **i** B	The graph is steepest at point B within this section.
ii E	The graph is steepest at point E within this section.

Now you try

This distance–time graph illustrates a journey by bicycle over a number of hours.



a Between what points does the graph show a speed which is:

i zero? **ii** constant?

iii increasing? **iv** decreasing?

b Would you say that the bicycle is travelling slower at point B compared to point D? Give a reason.

c Between the two given points, at what point is the bicycle travelling the fastest?

i Points A and C **ii** Points C and F

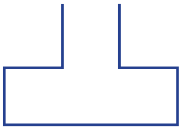
Exercise 7H

FLUENCY 1–5 1–6 2, 3, 5, 6

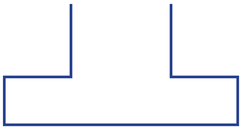
Example 19a

1 Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, *h,* of water in the container at time, *t*.

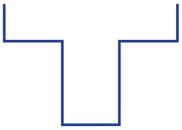
a



b



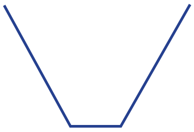
c



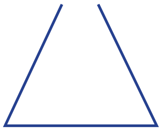
Example 19b

2 Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, *h* of water in the container at time, *t*.

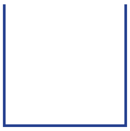
a



b

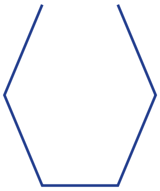


c

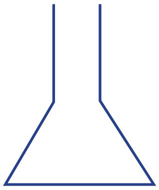


3 Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, *h,* of water in the container at time, *t*.

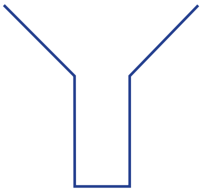
a



b

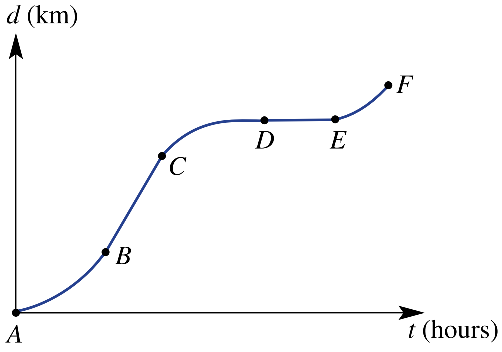


c



Example 20

4 This distance–time graph illustrates a journey by car over a number of hours.



a Between what points does the graph show a speed which is:

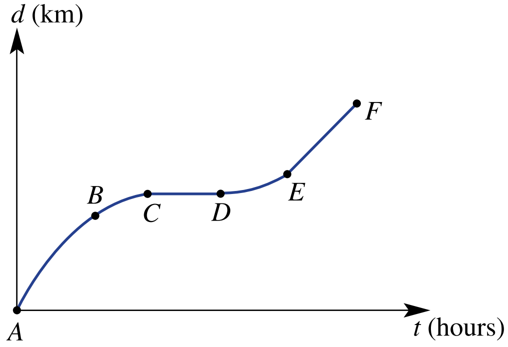
i zero? **ii** constant? **iii** increasing? **iv** decreasing?

b Would you say that the car is travelling slower at point *B* compared to point *F*? Give a reason.

c Between the two given points, at what point is the car travelling the fastest?

i Points *A* and *B* **ii** Points *C* and *E*

5 This distance–time graph illustrates a journey by bicycle over a number of hours.



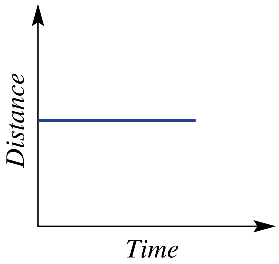
a Between what points does the graph show a speed which is:
- **i** zero?
- **ii** constant?
- **iii** increasing?
- **iv** decreasing?

b Would you say that the bicycle is travelling slower at point B compared to point E? Give a reason.

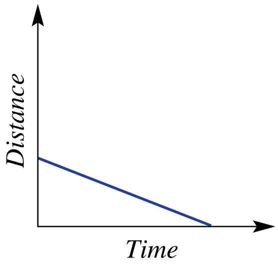
c Between the two given points, at what point is the bicycle travelling the fastest?
- **i** Points B and D
- **ii** Points C and E

6 The distance–time graphs below show various journeys, each with distance measured from ‘home’ (i.e. distance = 0 at home). For each graph, select and copy one correct description from each category below of how the distance from home, the gradient and the speed are varying.

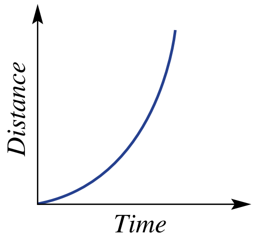
a



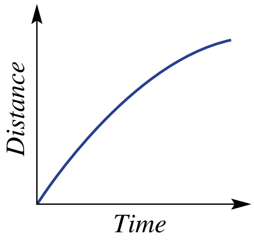
b



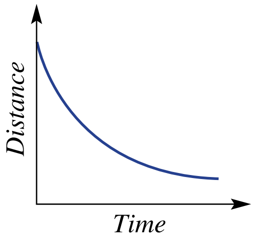
c



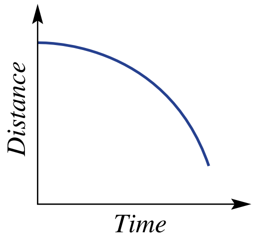
d



e



f



Distance from home

Increasing distance from home
Decreasing distance from home
Fixed distance from home

Gradient of graph

Positive constant gradient
Positive varying gradient
Negative constant gradient
Negative varying gradient
Zero gradient

Speed

Stationary
Lower constant speed
Higher constant speed
Decreasing speed, decelerating
Increasing speed, accelerating

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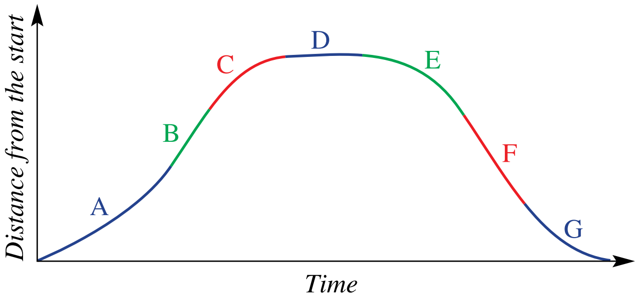
PROBLEM-SOLVING	7, 8	7–9	8–10

7 From the lists below, select and copy the correct description for the rate and speed of each segment of this distance–time graph. The rate of change of distance with respect to time is the gradient.

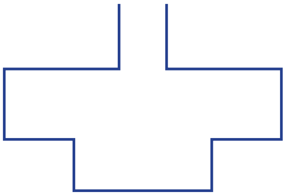


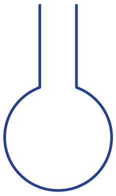
Rate of change of distance with respect to time	Speed
Positive constant rate of change	Stationary
Positive varying rate of change	Constant speed
Negative constant rate of change	Decreasing speed, decelerating
Negative varying rate of change	Increasing speed, accelerating
Zero rate of change	

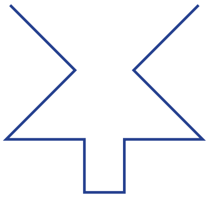
8 Sketch a population–time graph from each of these descriptions

- **a** A population of bilbies is decreasing at a decreasing rate.
- **b** A population of Tasmanian devils is decreasing at an increasing rate.
- **c** A population of camels is increasing at a decreasing rate.
- **d** A population of rabbits is increasing at an increasing rate.

9 Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, h, of water in the container at time, t.

a


b


c


10 Draw a possible graph of speed, s, vs time, t, for an athlete running the following races.

- **a** 100 metres
- **b** 5000 metres

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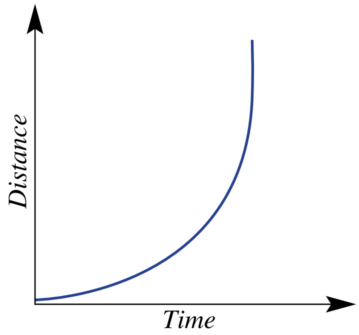
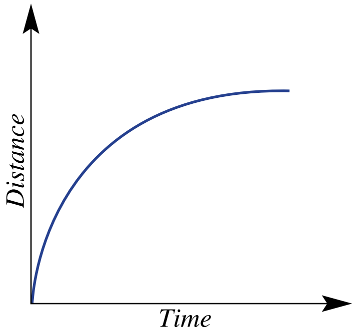
REASONING	11	11, 12	12, 13

11 Which of the following graphs don’t match the journey description correctly or are not physically possible? For each graph, explain the feature that is incorrect and redraw it correctly. Distances are to be interpreted as distance from the original starting point (displacement).

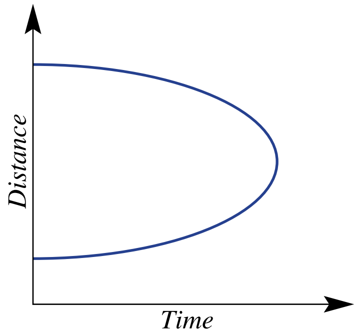
a Stopped, then accelerating to a very high speed.



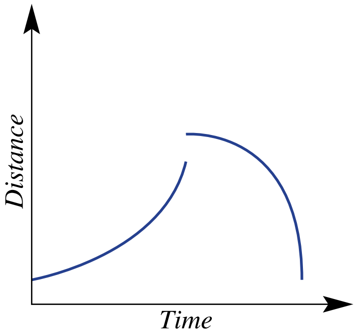
b Moving at very high speed and decelerating to a stop.



c Accelerating, changing direction, then decelerating.

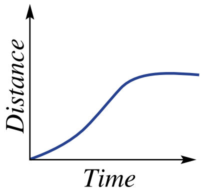


d Accelerating, changing direction, then decelerating.

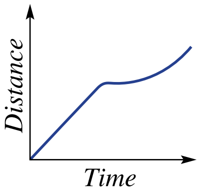


12 Match each distance–time graph (**A–D**) with the correct journey described (**a–d**). Give reasons for each choice, describing how the changing distance and varying rate relates to the movement of the object.

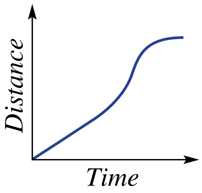
A



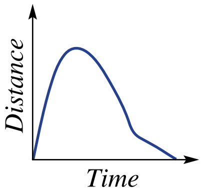
B



C

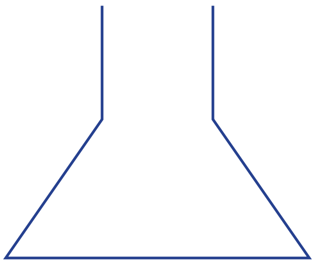


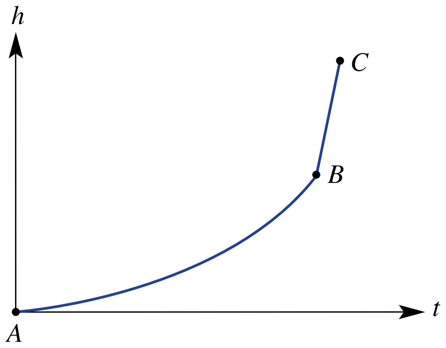
D



- **a** A soccer player runs at a steady speed across the field, stops briefly to avoid a tackle and then accelerates farther away.
- **b** A rocket stage 1 booster accelerates to huge speed, then detaches and quickly decelerates, then accelerates as it falls towards Earth, finally a parachute opens and it slows, falling to Earth at a steady speed.
- **c** A motorbike moves at a steady speed, then accelerates to pass a car, then brakes and decelerates, coming to a stop at traffic lights.
- **d** A school bus accelerates away from the bus stop, then moves at a steady speed and then decelerates and stops as it arrives at the next bus stop.

13 When drawing a graph of height (h) vs time (t) for the level of water in this container as it is being filled, a student draws the following graph.





- **a** Describe the error in the graph at the point B.
- **b** Draw a corrected graph.

Enrichment: Creating distance–time graphs

– – 14

14 Work in small groups to develop distance–time graphs from recorded data. Equipment: 100 m tape measure, stopwatch, recording materials, video camera.

- **a** Determine a suitable method for recording the distance a student has moved after every 5 seconds over a 30 second period.
- **b** Select a variety of activities for the moving student to do in each 30 second period. For example, walking slowly, running fast, starting slowly then speeding up, etc.
- **c** Record and graph distance versus time for each 30 second period.
- **d** For each graph, using sentences with appropriate vocabulary, describe how the distance and rate of change of distance is varying.
- **e** Analyse how accurately each graph has modelled that student's movement.

7I Average and instantaneous rates of change OPTIONAL

LEARNING INTENTIONS

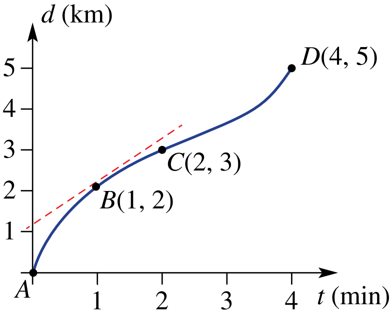
- To understand the difference between average and instantaneous rates of change
- To be able to calculate the average rate of change including average speed
- To be able to approximate an instantaneous rate of change using an average rate of change

When dealing with a linear relation, we know the exact rate of change at all points because the gradient is constant. For other relations however the rate of change will vary across the range of points. The gradient at a particular point is called the instantaneous rate of change and can be illustrated using a tangent passing through that point. The gradient of this tangent cannot be calculated easily using $m = \frac{rise}{run}$ because only one point is known on the tangent. We therefore use the average rate of change between two selected points to approximate the instantaneous rate of change.

The average rate of change can be represented as a straight-line graph joining two points.

Lesson starter: Comparing average and instantaneous rate of change

Consider this distance–time graph.



a Find the gradient between the following pairs of points.

i AB	**ii** AC	**iii** AD
iv BC	**v** BD	**vi** CD

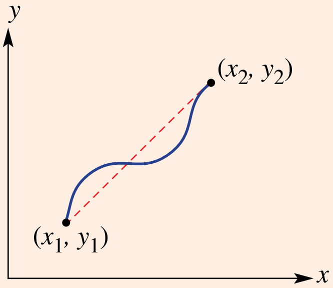
b Knowing that the average rate of change between two points can be found by finding the gradient, find the average rate of change between these pairs of points. Units will be km/min.

i AC **ii** AD

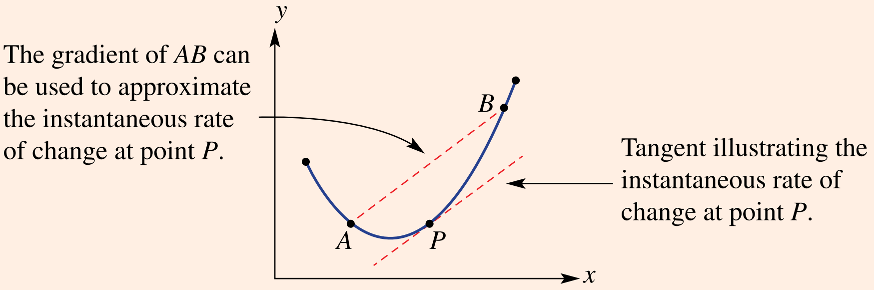
c The instantaneous rate of change at B, shown using a dashed tangent line, can be approximated using the gradients of AB, BC or AC. Which of these gradients would give the best approximation? Give a reason.

KEY IDEAS

- The **average rate of change** of a variable y as x varies between two points is equal to the gradient of the line joining the same two points.
  - Average rate of change $= \dfrac{y_2 - y_1}{x_2 - x_1}$



- The **instantaneous rate of change** is equal to the gradient at a given point.
  - The instantaneous rate of change can be approximated by the average rate of change between two selected points.



- Average speed is equal to the average rate of change of distance over time.

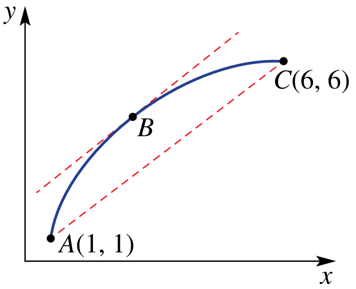
BUILDING UNDERSTANDING

① Consider this curve joining the points $A(1, 1)$ and $C(6, 6)$.

a Find the gradient of the dashed line segment between the points A and C.

b State the average rate of change between the points A and C.

c Which line segment AB, BC or AC gives the best approximation of the instantaneous rate of change at point B? Give a reason.

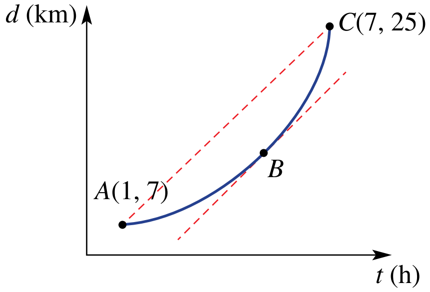


② This graph shows a distance–time graph for part of a journey. Consider this curve joining the points $A(1, 7)$ and $C(7, 25)$.

a Find the gradient of the dashed line segment between the points A and C.

b State the average speed between the points A and C.

c Which line segment AB, BC or AC gives the best approximation of the instantaneous rate of change (speed) at point B? Give a reason.



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Example 21 Finding an average rate of change

Consider the relation with rule $y = x^2 - 3$. Find the average rate of change of y as x changes from:

a 1 to 4

b 1 to 2

SOLUTION

a At $x = 1$, $y = (1)^2 - 3 = -2$

At $x = 4$, $y = (4)^2 - 3 = 13$

$$\begin{aligned}\text{Average rate of change} &= \frac{13-(-2)}{4-1}\\ &= \frac{15}{3}\\ &= 5\end{aligned}$$

EXPLANATION

First find the coordinates of the two points using substitution.

Use $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$ to find the gradient which gives the average rate of change.

SOLUTION

b At $x = 1$, $y = (1)^2 - 3 = -2$

At $x = 2$, $y = (2)^2 - 3 = 1$

$$\begin{aligned}\text{Average rate of change} &= \frac{1-(-2)}{2-1}\\ &= \frac{3}{1}\\ &= 3\end{aligned}$$

EXPLANATION

The y-values at $x = 1$ and $x = 2$ are -2 and 1 respectively.

The gradient gives the average rate of change.

Now you try

Consider the relation with rule $y = -x^2 + 2x$. Find the average rate of change of y as x changes from:

a 0 to 5

b 0 to 1

Example 22 Approximating an instantaneous rate of change

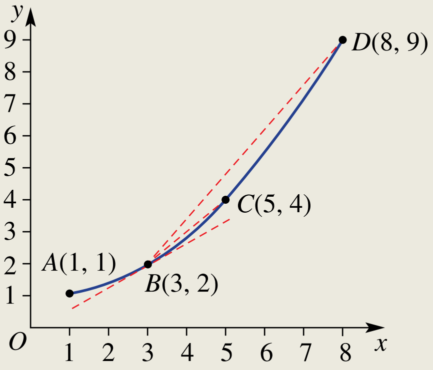
This graph shows part of a curve with a tangent drawn at point B to illustrate the instantaneous rate of change at that point.

a Find the gradient of the following line segments.

i BC

ii BD

b Which of the line segments BC or BD gives a better approximation of the instantaneous rate of change at point B? Give a reason.



SOLUTION

a Gradient $BC = \dfrac{4-2}{5-3} = 1$

Gradient $BD = \dfrac{9-2}{8-3} = \dfrac{7}{5}$

b *BC* gives a better approximation as the gradient of *BC* is closer to the instantaneous rate of change at *B*.

EXPLANATION

Use $m = \dfrac{\text{rise}}{\text{run}}$ to find the gradient of a line segment.

Looking at the given graph, the gradient of *BD* is greater than the gradient of *BC*, which is higher but closer to the gradient of the tangent at *B*.

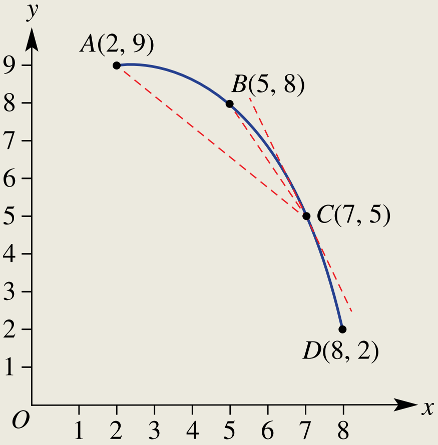
Now you try

This graph shows part of a curve with a tangent drawn at point *B* to illustrate the instantaneous rate of change at that point.

a Find the gradient of the following line segments.

i *AC* **ii** *BC*

b Which of the line segments *AC* or *BC* gives a better approximation of the instantaneous rate of change at point *C*? Give a reason.



Exercise 7I

FLUENCY	1–3, 5, 6	1, 3–6	2, 3, 5, 7

Example 21

1 Consider the relation with rule $y = x^2 + 2$. Find the average rate of change of y as x changes from:

a 1 to 3 **b** 1 to 2

2 Consider the relation with rule $y = -x^2 + 4x$. Find the average rate of change of y as x changes from:

a 0 to 4 **b** 0 to 1

3 Consider the relation with rule $y = x^2 - x + 3$. Find the average rate of change of y as x changes from:

a -1 to 3 **b** -1 to 0

4 Consider the relation with rule $y = -2x^2 + 3x - 1$. Find the average rate of change of y as x changes from:

a -2 to 4 **b** -2 to -1

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Example 22

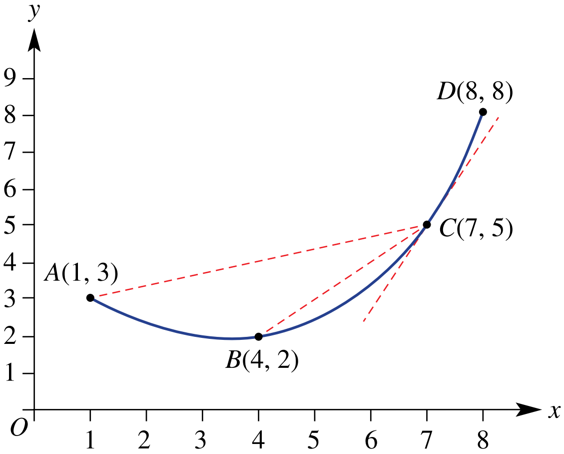
5 This graph shows part of a curve with a tangent drawn at point C to illustrate the instantaneous rate of change at that point.

a Find the gradient of the following line segments.

  i AC

  ii BC

b Which of the line segments AC or BC gives a better approximation of the instantaneous rate of change at point C? Give a reason.



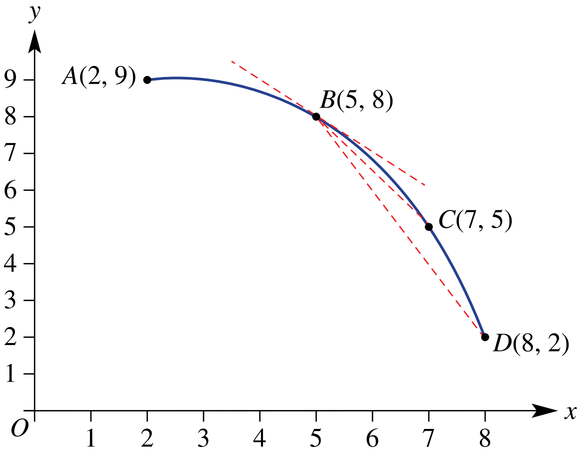
6 This graph shows part of a curve with a tangent drawn at point B to illustrate the instantaneous rate of change at that point.

a Find the gradient of the following line segments.

  i BC

  ii BD

b Which of the line segments BC or BD gives a better approximation of the instantaneous rate of change at point B? Give a reason.



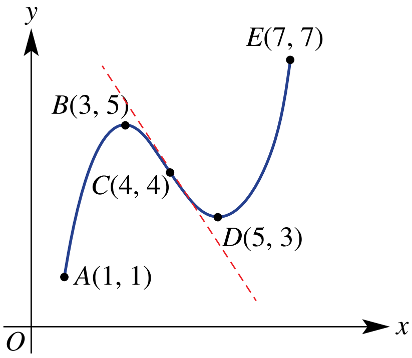
7 This graph shows part of a curve with a tangent drawn at point C to illustrate the instantaneous rate of change at that point.

a Find the gradient of the following line segments.

  i BD

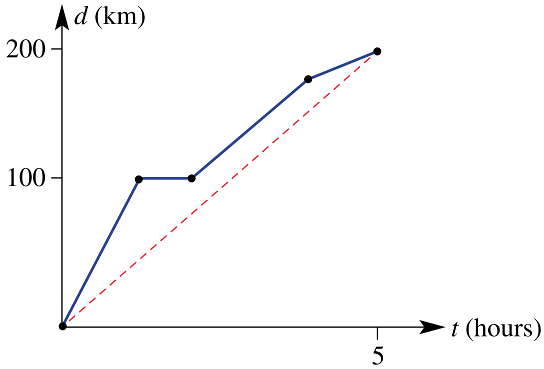
  ii BE

b Which of the line segments BD or BE gives a better approximation of the instantaneous rate of change at point C? Give a reason.

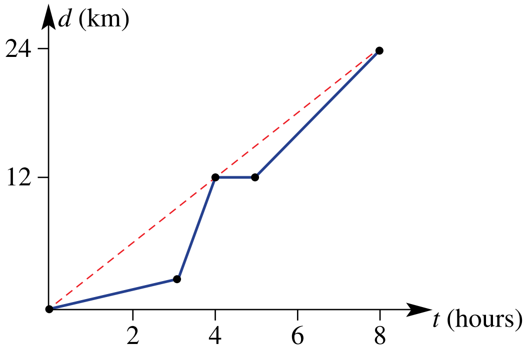


PROBLEM–SOLVING	8, 9	8–10	9–11

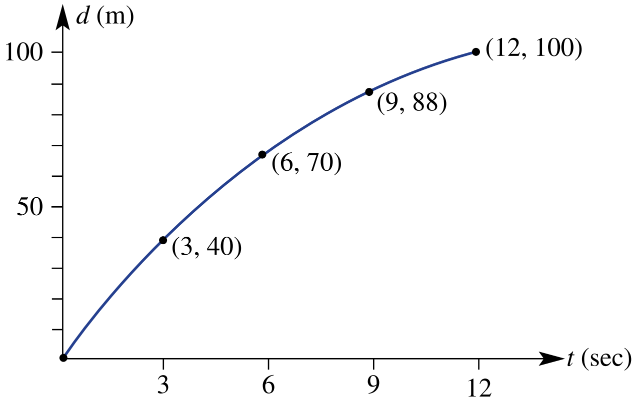
8 A journey by a vehicle covers 200 km in 5 hours as shown. Find the average rate of change of distance (average speed) over the 5 hours.



9 A journey by a hiker covers 24 km in 8 hours as shown. Find the average rate of change of distance (average speed) over the 8 hours.



10 A sprinter's distance-time graph for a 100 m race is shown here.
Use the points (3, 40) and (9, 88) to help estimate the sprinter's speed at the 6 second mark.



11 The volume of liquid in a leaking bucket is modelled by the rule $V = \frac{2}{5}(t - 5)^2$ for $0 \leqslant t \leqslant 5$ where V is in litres and t is in seconds.

a Find the volume of liquid in the bucket at:

i $t = 0$ **ii** $t = 5$.

b Find the average rate of change of volume over the 5 seconds.

c Estimate the instantaneous rate of change of volume after:

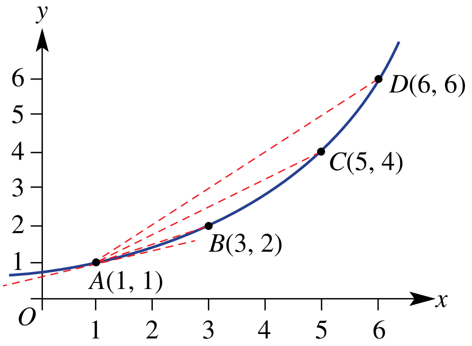
i 3 seconds **ii** 1 second.

REASONING	12	12, 13	13, 14

12 This graph shows a curve passing through four points and a tangent at point A.

a Which segment AB, AC or AD would provide the best approximation to the gradient of the tangent at A? Give a reason.

b Use your answer from part **a** to estimate the instantaneous rate of change at point A.



13 A curve has rule $y = x^2 - 1$.

a Find the average rate of change between the points with the following x-values.

i 1 and 2 **ii** 1 and 1.5 **iii** 1 and 1.1

b Which pair of x-values from part **a** would give the best approximation of the instantaneous rate of change at $x = 1$? Give a reason.

c Explain how you might find an even better approximation of the instantaneous rate of change at $x = 1$.

14 One way to approximate the instantaneous rate of change at a point $x = a$ is to choose two points P and Q, either side of that point. P and Q are chosen so that the average of their x-values is equal to a. We then find the average rate of change between points P and Q.

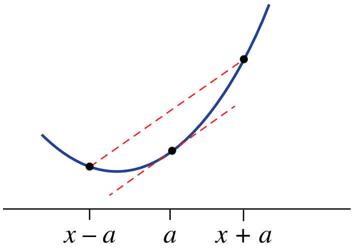
Use this idea to approximate the instantaneous rate of change of the following relations at the given point.

a $y = 2^x$, $(x = 1)$ **b** $y = \frac{1}{x}$, $(x = -2)$

c $y = x^3$, $(x = -2)$ **d** $y = \sqrt{x}$, $(x = 2)$

ENRICHMENT: When an approximate rate of change gives the exact instantaneous rate of change	–	–	15(½)

15 One interesting fact about quadratic relations is that the instantaneous rate of change at $x = a$ is equal to the average rate of change between the points where $x = a - h$ and $x = a + h$ where h is any chosen number.



For example: For $y = x^2$ the instantaneous rate of change at $x = 2$ can be found using $h = 1$ by finding the gradient between the point at $x = 2 - 1 = 1$ and $x = 2 + 1 = 3$.

$$m = \frac{9 - 1}{3 - 1} = 4$$

So the instantaneous rate of change at $x = 2$ is 4.

Use this result to find the instantaneous rate of change of the following quadratics at the given points.

a $y = x^2$, $(x = 3)$ **b** $y = 2x^2 - 1$, $(x = 1)$

c $y = -x^2 + 4$, $(x = 2)$ **d** $y = -3x^2 + 2x - 1$, $(x = -1)$

e $y = -2x^2 + 5x - 3$, $(x = -1)$ **f** $y = 4(x - 2)^2 - 5$, $(x = -2)$

7J Direct variation and inverse variation EXTENDING

LEARNING INTENTIONS

- To understand the relationship between two variables that are directly proportional or inversely proportional
- To understand that the shape of a graph shows how a variable and its rate of change varies
- To be able to find and use rules involving direct and inverse proportion
- To know how to model and interpret a situation using a distance–time graph

Two variables are said to be directly related if they are in a constant (i.e. unchanged) ratio. If two variables are in direct proportion, as one variable increases so does the other. For example, consider the relationship between speed and distance travelled in a given time. In 1 hour, a car can travel 50 km at 50 km/h, 100 km at 100 km/h, etc.

For two variables in inverse or indirect variation, as one variable increases the other decreases. For example, consider a beach house that costs \$2000 per week to rent. As the number of people renting the house increases, then the cost per person decreases.

Renting a holiday house can be expensive. But, when sharing, the rent per person is inversely proportional to the number of people. However, the total rent increases in direct proportion to the length of stay.

Lesson starter: Birthday party relationships

For a birthday party, tickets are purchased at an activity centre for \$22 each.

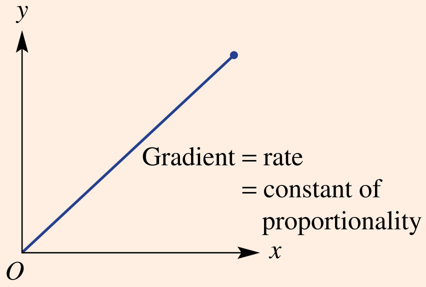
a Describe the relationship between the total cost of the tickets $\$C$ and the number of tickets purchased, n. Express this as a rule.
b Use your rule to find the total cost of purchasing 8 tickets.

At the end of the party, a 2 kg bag of chocolate is distributed evenly among all the friends.

c Describe the relationship between the amount of chocolate received by each person and the number of friends at the party. Express this as a rule.
d Use your rule to find the amount of chocolate received by each person if there there are 8 friends.

KEY IDEAS

- y varies directly with x if their relationship is $y = kx$ or $\frac{y}{x} = k$.
  - k is a constant, and is called the constant of proportionality.
  - The graph of y versus x gives a straight line that passes through the origin, O or $(0, 0)$, where k is the gradient.
  - We write: $y \propto x$ which means that $y = kx$.
  - We say: y varies directly as x, or y is **directly proportional** to x.



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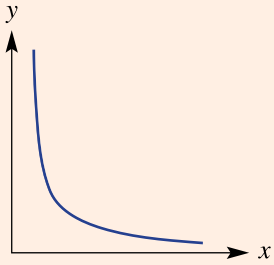
- y varies inversely with x if their relationship is $y = \frac{k}{x}$ or $xy = k$.
  - The graph of y versus x gives a shape called a hyperbola.
  - We write: $y \propto \frac{1}{x}$, which means that $y = \frac{k}{x}$ or $xy = k$.
  - We say: y varies inversely as x, or y is **inversely proportional** to x.



BUILDING UNDERSTANDING

1 For each pair of variables, state whether they are in direct or inverse proportion or neither.

a The number of *hours* worked and *wages* earned at a fixed rate per hour.

b The *volume* of remaining fuel in a car and the *cost* of filling the fuel tank.

c The *speed* and *time* taken to drive a certain distance.

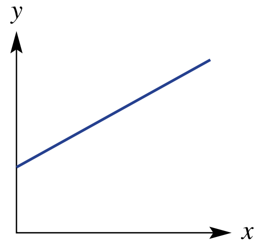
d The *size* of a movie file and the *time* for downloading it to a computer at a constant rate of kB/s.

e The *cost* of a taxi ride and the *distance* travelled. The cost includes flag fall (i.e. a starting charge) and a fixed $/km.

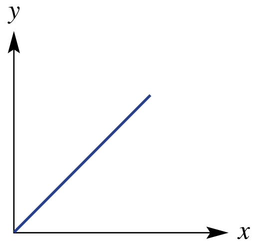
f The *rate* of typing in words per minute and the *time* needed to type a particular assignment.

2 State the main features of each graph and whether it shows direct proportion or inverse (i.e. indirect) proportion or neither.

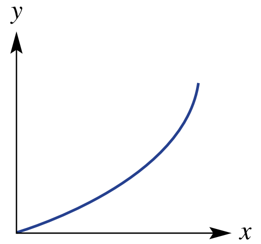
a



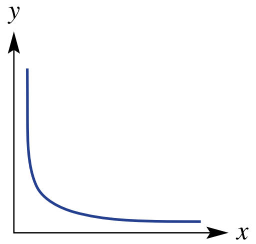
b



c



d



Example 23 Finding and using a direct variation rule

If m is directly proportional to h and $m = 90$ when $h = 20$, determine:

a the relationship between m and h

b m when $h = 16$

c h when $m = 10$.

SOLUTION	EXPLANATION
a $m \propto h$	First, write the variation statement.
$m = kh$	Write the equation, including k.
$90 = k \times 20$	Substitute $m = 90$ and $h = 20$.
$k = \frac{90}{20}$	Divide both sides by 20 to find k.
$k = \frac{9}{2}$	Simplify.
$\therefore m = \frac{9}{2}h$	Write the rule using the value of k found.
b $m = \frac{9}{2} \times 16$	Substitute $h = 16$.
$\therefore m = 72$	Simplify to find m.
c $10 = \frac{9}{2} \times h$	Substitute $m = 10$.
$10 \times \frac{2}{9} = h$ $\therefore h = \frac{20}{9}$	Solve for h.

Now you try

If a is directly proportional to b and $a = 120$ when $b = 80$, determine:

a the relationship between a and b

b a when $b = 40$

c b when $a = 12$.

Example 24 Finding and using an inverse proportion rule

If x and y are inversely proportional and $y = 6$ when $x = 10$, determine:

a the constant of proportionality, k, and write the rule

b y when $x = 15$

c x when $y = 12$.

SOLUTION	EXPLANATION
a $y \propto \frac{1}{x}$	Write the variation statement.
$y = \frac{k}{x}$	Write the equation, including k.
$k = xy$	The constant of proportionality, $k = xy$.
$k = 10 \times 6$ $k = 60$	Use $x = 10$, $y = 6$.
$y = \frac{60}{x}$	Substitute $k = 60$ into the rule $y = \frac{k}{x}$.

Continued on next page

b $y = \frac{60}{15}$
$y = 4$

Substitute $x = 15$ into the rule $y = \frac{60}{x}$.

c $12 = \frac{60}{x}$
$12x = 60$
$x = 5$

Substitute $y = 12$ into the rule $y = \frac{60}{x}$.
Multiply both sides by x.
Divide both sides by 12.

Now you try

If m and n are inversely proportional and $n = 10$ when $m = 6$, determine:

a the constant of proportionality, k, and write the rule

b n when $m = 12$

c m when $n = 30$.

Exercise 7J

FLUENCY	1–3	1–4	1–2(⅓), 4, 5

Example 23

1 **a** If m is directly proportional to h and $m = 50$ when $h = 10$, determine:
   - **i** the relationship between m and h
   - **ii** m when $h = 6$
   - **iii** h when $m = 8$.

b If p is directly proportional to q and $p = 40$ when $q = 10$, determine:
   - **i** the relationship between p and q
   - **ii** p when $q = 15$
   - **iii** q when $p = 100$.

c If p is directly proportional to q and $p = 100$ when $q = 2$, determine:
   - **i** the relationship between p and q
   - **ii** p when $q = 15$
   - **iii** q when $p = 200$.

Example 24

2 **a** If x and y are inversely proportional and $y = 4$ when $x = 6$, determine:
   - **i** the constant of proportionality, k, and write the rule
   - **ii** y when $x = 8$
   - **iii** x when $y = 12$.

b If x and y are inversely proportional and $y = 12$ when $x = 6$, determine:
   - **i** the constant of proportionality, k, and write the rule
   - **ii** y when $x = 36$
   - **iii** x when $y = 3$.

c If y varies inversely with x and $y = 10$ when $x = 5$, determine:
   - **i** the constant of proportionality, k, and write the rule
   - **ii** y when $x = 100$
   - **iii** x when $y = 100$.

3 A vehicle drives at 80 km/h.

- **a** Find a rule linking the distance that the vehicle travels, d km, after t hours.
- **b** Would you say that the distance d is directly proportional or inversely proportional to the time t?
- **c** Find:
  - **i** how far the vehicle travels after 4 hours
  - **ii** how long it takes for the vehicle to travel 200 km.

4 A cyclist completes a 20 km journey at an average speed of s km/h and takes t hours.

- **a** Find a rule linking the average speed that the bike travels, s km/h, after t hours.
- **b** Would you say that the speed s is directly proportional or inversely proportional to the time t?
- **c** Find:
  - **i** the average speed required if the journey is to take 2.5 hours
  - **ii** the time taken if the average speed is 10 km/h.

5 The amount that a farmer earns from selling wheat is in direct proportion to the number of tonnes harvested.

- **a** Find the constant of proportionality, k, given that a farmer receives \$8296 for 34 tonnes of wheat.
- **b** Write the direct proportion equation relating selling price, P, and number of tonnes, n.
- **c** Calculate the selling price of 136 tonnes of wheat.
- **d** Calculate the number of tonnes of harvested wheat that is sold for \$286 700.

PROBLEM–SOLVING	6	6, 7	6, 7

6 A 30-seater school bus costs 20 students \$3.70 each to hire for a day. The overall cost of the bus remains the same regardless of the number of students.

- **a** Write a relationship between the cost per student (c) and the number of students (s).
- **b** If only 15 students use the bus, what would be their individual cost, to the nearest cent?
- **c** What is the minimum a student would be charged, to the nearest cent?

7 For each relationship described below:

  - **i** write a suitable equation
  - **ii** sketch the graph, choosing appropriate values for the initial and final points on the graph.

- **a** The distance that a car travels in 1 hour is directly proportional to the speed of the car. The roads have a 100 km/h speed limit.
- **b** The cost per person of hiring a yacht is inversely proportional to the number of people sharing the total cost. A yacht in the Whitsunday Islands can be hired for \$320 per day for a maximum of eight people on board.
- **c** There is a direct proportional relationship between a measurement given in metric units and in imperial units. A weight measured in pounds is 2.2 times the value of the weight in kilograms.
- **d** The time taken to type 800 words is inversely proportional to the typing speed in words per minute.

REASONING	8	8, 9	9, 10

8 Decide if these pairs of variables are in direct proportion (D) or inverse proportion (I).

- **a** *Amount earned* and *Number of hours worked*
- **b** *Amount of work completed* and *Test grades*
- **c** *Time to complete a job* and *Number of helpers*
- **d** *Speed* and *Time*
- **e** *Distance* and *Time*
- **f** *Distance* and *Speed*
- **g** *Profit* and *Number of sales*
- **h** *Time to paint a house* and *Number of painters*

9 Partial variation occurs when one variable is partly constant and partly varies with another variable. For example: The cost of producing n cakes in a day's work at a bakery involves a fixed cost of \$200 plus a 50 cent cost per cake.

- **a** Find a rule connecting the total cost $\$C$ of producing n cakes.
- **b** Find the total cost of producing 60 cakes.
- **c** Find how many cakes can be produced if the total cost is \$242.50.

10 Joint variation occurs when one variable varies directly with two or more variables. The area of a triangle, for example, varies directly with the base length b and the height h. The rule connecting the variables is $A = \frac{1}{2}bh$.

- **a** What is the constant of proportionality in the above given example?
- **b** Does the value of A increase or decrease as b increases?
- **c** Does the value of A increase or decrease as h decreases?

ENRICHMENT: Combining variables with jigsaw puzzles	–	–	11

11 Combined variation involves a variable y, for example, varying directly with variable x and inversely with variable z. A rule of such variation is of the form $y = \frac{kx}{z}$.

The time it takes to complete a jigsaw puzzle, t hours, for example, depends on the number of pieces, p, and the number of people working on the puzzle, n.

- **a** Write a rule for t in terms of p and n and the constant of variation k.
- **b** Find the value of k if it takes 5 hours for 2 people to complete a 2000-piece puzzle.
- **c** Using your above results find:
  - **i** the time taken to complete a 5000-piece puzzle with 5 people.
  - **ii** the number of people required to complete a 10 000-piece puzzle in 5 hours.

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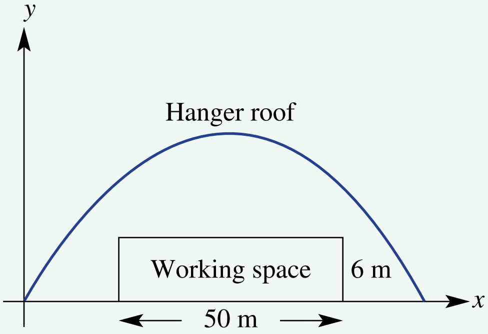
Designing an aircraft hanger

As an engineer you are designing a curved parabolic roof to house a working space for the building of aircraft. The cross-section of the working space is to be 6 m in height and at least 50 m in width. The total width of the hanger cannot be more the 100 m.

Present a report for the following tasks and ensure that you show clear mathematical workings and explanations where appropriate.



Preliminary task

- **a** Sketch a graph of $y = 0.01x(80 - x)$ for $y > 0$, where x and y are in metres.
- **b** Decide if the working space for the planes fits inside the model for the roof defined by the rule in part **a**. Give reasons.
- **c** Repeat parts **a** and **b** above for $y = 0.002x(100 - x)$.

Modelling task

Formulate

- **a** The problem is to find a model for a hanger roof of minimum height to house the working space. Write down all the relevant information that will help solve this problem with the aid of a diagram.

Solve

- **b** Choose at least three realistic values of a using the model $y = ax(80 - x)$ which will house the aircraft working space, 50 m wide and 6 m high. Justify your choices using graphs.
- **c** Determine a value of a for the model $y = ax(80 - x)$ which minimises the height of the hanger.
- **d** Choose your own values of a and b for the model $y = ax(b - x)$ which will:
  - house the aircraft working space, 50 m wide and 6 m high
  - ensure that the total width of the hanger is not more than 100 m.
- **e** Determine the values of a and b for the model $y = ax(b - x)$ which minimises the total height of the hanger.

Evaluate and verify

- **f** Compare the height of the hanger for the different models chosen above.
- **g** Explain why your choices for the values of a and b minimise the total height of the hanger.

Communicate

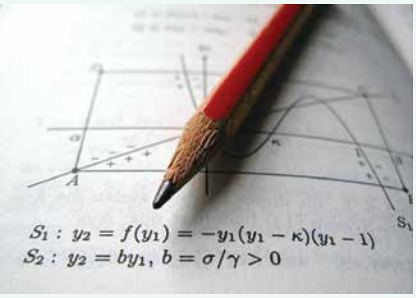
- **h** Summarise your results and describe any key findings.

Extension question

- **a** Choosing the model $y = 0.01x(80 - x)$, determine the maximum possible width of the working space for a given working space height of 6 m.

Exploring constant differences

Key technology: spreadsheets



1 Getting started

We know that for a linear function like $y = 2x - 3$ the difference between the y-values is constant as x increases uniformly. So as x increases by 1, y increases by a constant difference of 2 which is equal to the gradient. Constant differences can also be found for some non-linear functions; however, we need to look deeper to see where they are.

a Complete this table for the linear function $y = 4x - 1$.

x	−2	−1	0	1	2	3
y						
Difference in y						

b What do you notice about the difference between the y-values as x increases by 1.

c Repeat parts **a** and **b** above using the rule $y = -2x + 3$.

d Now let's consider the quadratic function $y = 2x^2 - x + 1$. Complete this table by finding the first difference (the difference in the y values) and the second difference (the difference in the first difference values).

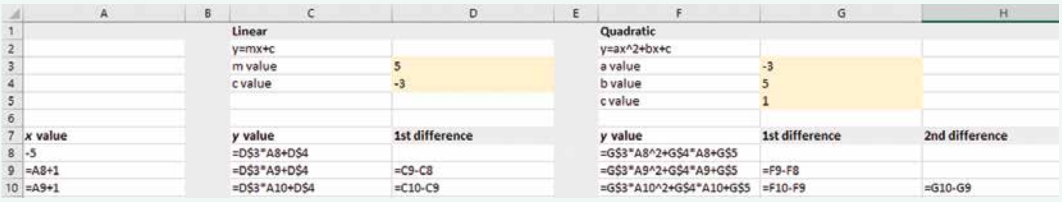
x	−2	−1	0	1	2	3
y						
1st difference						
2nd difference						

e What do you notice about the 2nd difference as x increases by 1.

2 Applying an algorithm

a Set up a spreadsheet to find the constant differences for any linear or quadratic function. Follow these steps:

- Step 1: Start x-values at -5 and increase by 1 using a formula in cell $A9$ as shown.
- Step 2: Enter a formula for the linear function using m in $D3$ and c in $D4$.
- Step 3: Calculate the first difference between the y-values for the linear function.
- Step 4: For the quadratic function, repeat as for the linear function but find a second constant difference by adding a new column.



	A	B	C	D	E	F	G	H
1			**Linear**			**Quadratic**		
2			y=mx+c			y=ax^2+bx+c		
3			m value	5		a value	-3	
4			c value	-3		b value	5	
5						c value	1	
6								
7	**x value**		**y value**	**1st difference**		**y value**	**1st difference**	**2nd difference**
8	-5		=D$3*A8+D$4			=G$3*A8^2+G$4*A8+G$5		
9	=A8+1		=D$3*A9+D$4	=C9-C8		=G$3*A9^2+G$4*A9+G$5	=F9-F8	
10	=A9+1		=D$3*A10+D$4	=C10-C9		=G$3*A10^2+G$4*A10+G$5	=F10-F9	=G10-G9

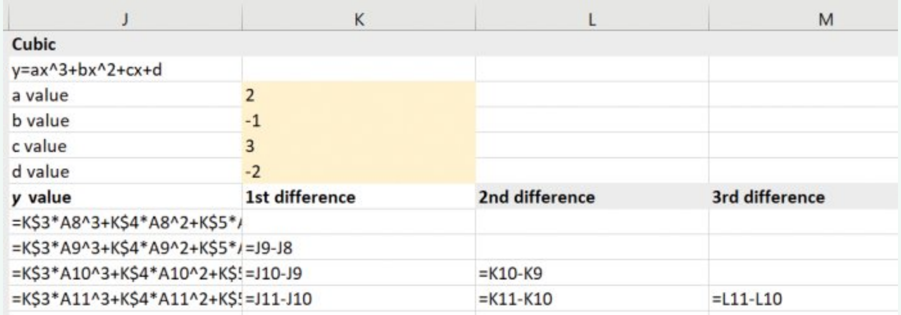
b What do you notice about the second constant difference for the quadratic function?

3 Using technology

a Change the value of m and c for the linear function. What do you notice about the constant difference?

b Change the value of a, b and c for the quadratic function. What do you notice about the second constant difference?

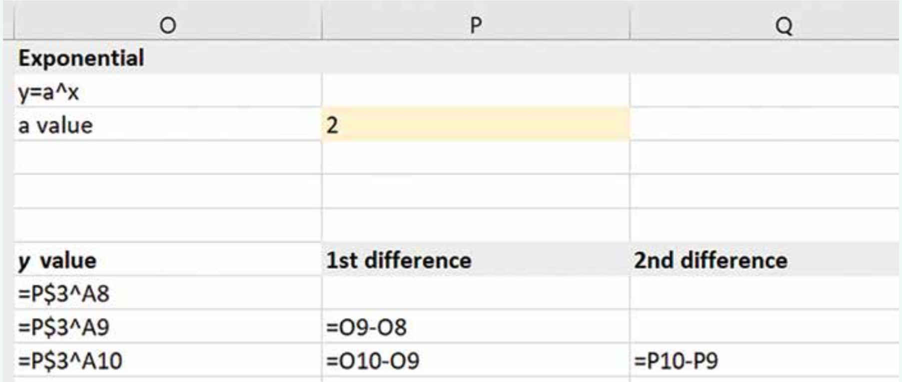
c Expand your spreadsheet to consider a cubic function where the highest power of x is 3.



J	K	L	M
Cubic			
y=ax^3+bx^2+cx+d			
a value	2		
b value	-1		
c value	3		
d value	-2		
y value	**1st difference**	**2nd difference**	**3rd difference**
=K$3*A8^3+K$4*A8^2+K$5*			
=K$3*A9^3+K$4*A9^2+K$5*	=J9-J8		
=K$3*A10^3+K$4*A10^2+K$	=J10-J9	=K10-K9	
=K$3*A11^3+K$4*A11^2+K$	=J11-J10	=K11-K10	=L11-L10

d What do you notice about the third difference for the cubic function?

e Now consider an exponential function like $y = 2^x$. Expand your spreadsheet to find the first and second differences.



O	P	Q
Exponential		
y=a^x		
a value	2	
y value	**1st difference**	**2nd difference**
=P$3^A8		
=P$3^A9	=O9-O8	
=P$3^A10	=O10-O9	=P10-P9

f What do you notice about the first and second differences for an exponential function? Alter the value of a to confirm your ideas.

4 Extension

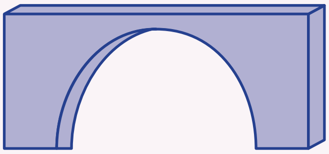
a By experimenting with your spreadsheet, can you find a rule that connects the value of the constant difference for a quadratic function with the value of a. Confirm your rule by experimenting with different values of a, b and c.

b By experimenting with your spreadsheet, can you find a rule that connects the value of the constant difference for a cubic function with the value of a. Confirm your rule by experimenting with different values of a, b, c and d.

c By experimenting with your spreadsheet, describe how the difference for an exponential function relates to the value of a. Confirm your ideas by experimenting with different values of a.

Painting bridges

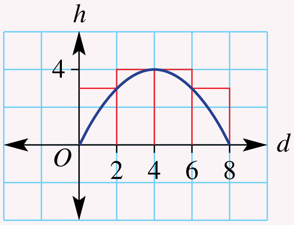
A bridge is 6 m high and has an overall width of 12 m and an archway underneath, as shown.



We need to determine the actual surface area of the face of the arch so that we can order paint to refurbish it. The calculation for this area would be:

$$\begin{aligned}\text{Bridge area} &= 12 \times 6 - \text{area under arch}\\ &= 72 - \text{area under parabola}\end{aligned}$$

Consider an archway modelled by the formula $h = -\frac{1}{4}(d-4)^2 + 4$, where h metres is the height of the arch and d metres is the distance from the left. To estimate the area under the arch, divide the area into rectangular regions. If we draw rectangles above the arch and calculate their areas, we will have an estimate of the area under the arch even though it is slightly too large. Use the rule for h to obtain the height of each rectangle.

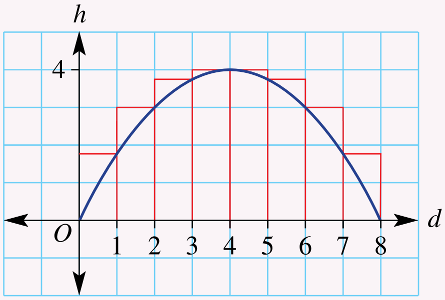


$$\begin{aligned}\text{Area} &= (2 \times 3) + (2 \times 4) + (2 \times 4) + (2 \times 3)\\ &= 6 + 8 + 8 + 6\\ &= 28\ \text{m}^2\end{aligned}$$

$\therefore$ Area is approximately $28\ \text{m}^2$.

We could obtain a more accurate answer by increasing the number of rectangles; i.e. by reducing the width of each rectangle (called the strip width).

Overestimating the area under the arch

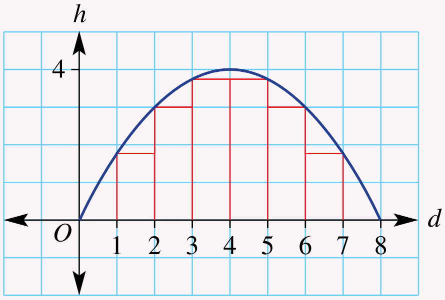


a Construct an accurate graph of the parabola and calculate the area under it using a strip width of 1.

b Repeat your calculations using a strip width of 0.5.

c Calculate the surface area of the face of the arch using your answer from part **b**.

Underestimating the area under the arch



a Estimate the area under the arch by drawing rectangles under the graph with a strip width of 1.

b Repeat the process for a strip width of 0.5.

c Calculate the surface area of the face of the arch using your answer from part **b**.

Improving accuracy

a Suggest how the results from parts 1 and 2 could be combined to achieve a more accurate result.

b Explore how a graphics or CAS calculator can give accurate results for finding areas under curves.

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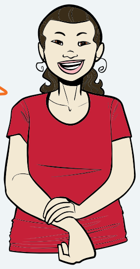
Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



1 Solve these inequalities for x.

a $6x^2 + x - 2 \leqslant 0$

b $12x^2 + 5x - 3 > 0$

c $x^2 - 7x + 2 < 0$

2 Prove the following.

a The graphs of $y = x - 4$ and $y = x^2 - x - 2$ do not intersect.

b The graphs of $y = -3x + 2$ and $y = 4x^2 - 7x + 3$ touch at one point.

c The graphs of $y = 3x + 3$ and $y = x^2 - 2x + 4$ intersect at two points.

3 For what values of k does the graph of $y = kx^2 - 2x + 3$ have:

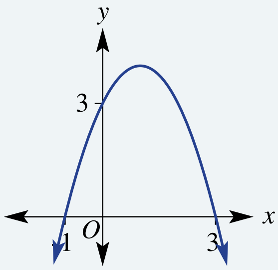
a one x-intercept? **b** two x-intercepts? **c** no x-intercepts?

4 For what values of k does the graph of $y = 5x^2 + kx + 1$ have:

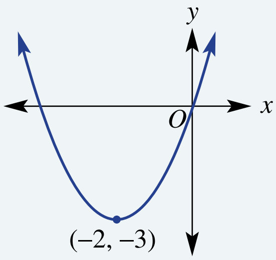
a one x-intercept? **b** two x-intercepts? **c** no x-intercepts?

5 Find the rules for these parabolas.

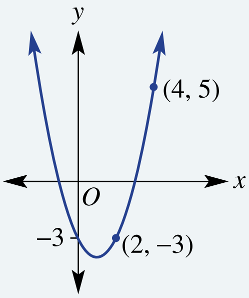
a



b



c



6 A graph of $y = ax^2 + bx + c$ passes through the points $A(0, -8)$, $B(-1, -3)$ and $C(1, -9)$. Use your knowledge of simultaneous equations to find the values of a, b and c and, hence, find the turning point for this parabola, stating the answer using fractions.

7 Determine the maximum vertical distance between these two parabolas at any given x-value between the points where they intersect:

$y = x^2 + 3x - 2$ and

$y = -x^2 - 5x + 10$

8 Two points, P and Q, are on the graph of $y = x^2 + x - 6$. The origin $(0, 0)$ is the midpoint of the line segment PQ. Determine the exact length of PQ.

9 A parabola, $y_1 = (x - 1)^2 + 2$, is reflected in the x-axis to become y_2. Now y_1 and y_2 are each translated 2 units horizontally but in opposite directions, forming y_3 and y_4. If $y_5 = y_3 + y_4$, sketch the graphs of the possible equations for y_5.

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Chapter summary

Parabolas

Sketching $y = x^2 + bx + c$

e.g. $y = x^2 - 4x - 12$

y-intercept: $x = 0$, $y = -12$

x-intercepts: $y = 0$, $x^2 - 4x - 12 = 0$

Factorise $(x - 6)(x + 2) = 0$

Null Factor Law

$x - 6 = 0$ or $x + 2 = 0$

$x = 6$ or $x = -2$

Turning point: halfway between x-intercepts

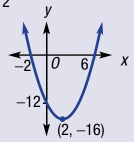
$$x = \frac{6 + (-2)}{2} = 2$$

Substitute $x = 2$:

$y = 2^2 - 4(2) - 12$

$= -16$

$(2, -16)$ minimum



Sketching $y = a(x - h)^2 + k$

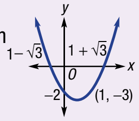
Any quadratic can be written in turning point form by completing the square.

e.g. sketch $y = (x - 1)^2 - 3$

Turning point $(1, -3)$ minimum

y-intercept: $x = 0$, $y = 1 - 3$

$= -2$



x-intercepts: $y = 0$

$0 = (x - 1)^2 - 3$

$3 = (x - 1)^2$

$\pm\sqrt{3} = x - 1$

$\therefore\ x = 1 + \sqrt{3}, 1 - \sqrt{3}$

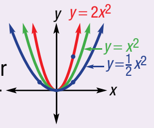
(alternatively use the Null Factor Law)

Transformations and turning point form

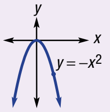
$y = ax^2$ dilates graph compared to $y = x^2$

$0 < a < 1$ graph is wider

$a > 1$ graph is narrower



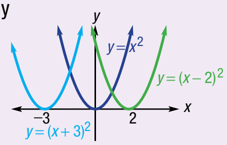
$a < 0$ graph is inverted (reflected in x-axis)



$y = (x - h)^2$ translates $y = x^2$ h units horizontally

$h > 0$ move right

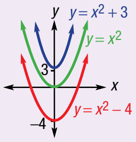
$h < 0$ move left



$y = x^2 + k$ translates $y = x^2$ k units vertically

$k > 0$ move up

$k < 0$ move down



Turning point form $y = a(x - h)^2 + k$ has turning point at (h, k)

Quadratic formula (Opt)

For $y = ax^2 + bx + c$, the quadratic formula can be used to find x-intercepts.

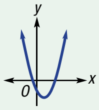
i.e. if $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant $\Delta = b^2 - 4ac$ tells how many x-intercepts:

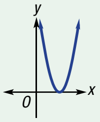
$\Delta > 0$

two x-intercepts



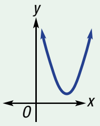
$\Delta = 0$

one x-intercept



$\Delta < 0$

no x-intercepts

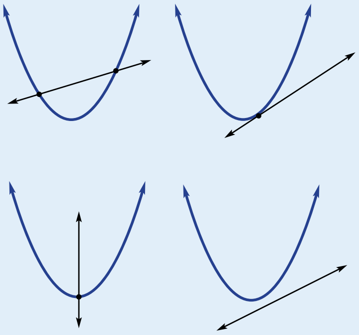


Quadratic applications

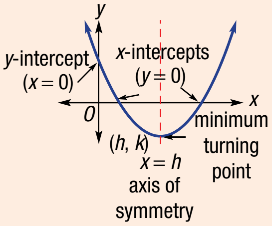
Form a quadratic model based on the formulation in the question. Sketch the graph for suitable values of x, to locate max/min and intercepts.

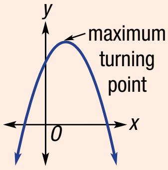
Lines and parabolas (Opt)

Using substitution we can solve simultaneously to find 0, 1 or 2 intersection points.



Key features

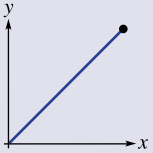




Direct variation and inverse variation (Ext)

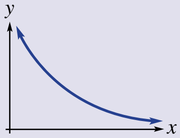
Direct variation: $y = kx$

- Both variables increase or decrease together.
- Graph is a straight line through (0, 0).



Inverse (indirect) variation: $y = \frac{k}{x}$ or $xy = k$

- If one variable increases, the other decreases.
- Graph is a hyperbola shape.

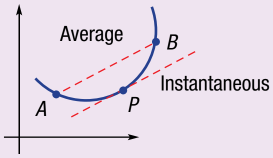


Rates of change

Average and instantaneous rates of change (Opt)

The average rate of change between two points is equal to the gradient of the line joining the two points.

The instantaneous rate of change is equal to the gradient at a particular point and can be approximated by the average rate of change between two selected points.

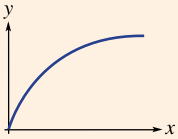


Rates of change (Opt)

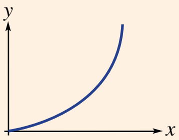
The shape of a graph shows how both y and the rate of change of y (i.e. the gradient) vary.
Straight lines have a constant rate of change; curves have a varying rate of change.

Description of graphs:

- y is increasing or decreasing or constant.
- Rate of change is positive or negative or zero.
- Rate of change is increasing or decreasing or constant.



Positive decreasing gradient



Positive increasing gradient

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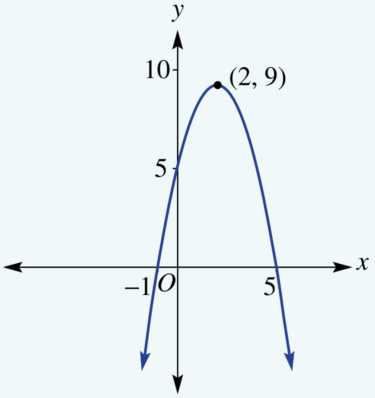
Chapter checklist with success criteria

A printable version of this checklist is available in the Interactive Textbook

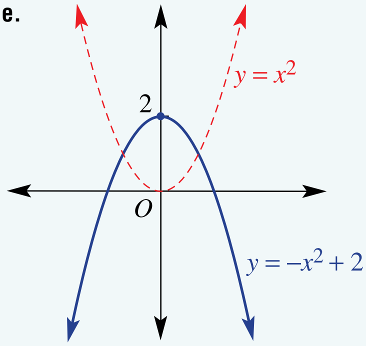
✓

7A **1. I can identify the key features of a parabola.**
e.g. For the graph shown determine the turning point (and whether it's a maximum or a minimum), the axis of symmetry, the x-intercepts and the y-intercept. ☐



7A **2. I can find key features of transformed graphs from a rule.**
e.g. Complete the table for the graph shown ☐

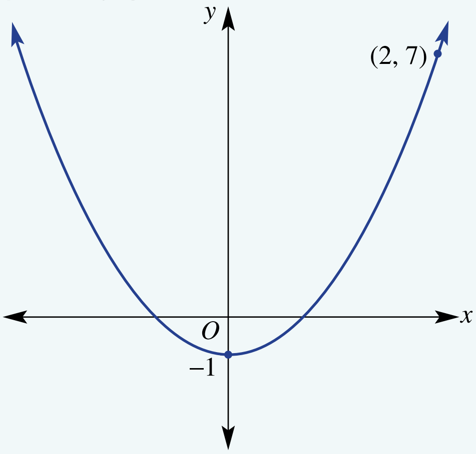
Formula	Max or min	Turning point	y-value when $x = 1$
$y = -x^2 + 2$			



7B **3. I can sketch a quadratic relation involving a dilation or reflection.**
e.g. Sketch the graph of $y = -4x^2$, labelling the turning point and one other point. ☐

7B **4. I can sketch a quadratic relation involving translations.**
e.g. Sketch the graph of $y = (x - 1)^2 + 3$, labelling the turning point and y-intercept. ☐

7B **5. I can find a rule in turning point form from a quadratic graph.**
e.g. Find a rule for this parabola with turning point $(0, -1)$ and another point $(2, 7)$. ☐



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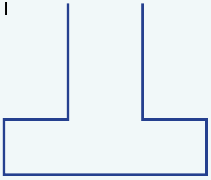
Chapter checklist with success criteria

			✓
7C	**6. I can use the *x*-intercepts to find the turning point and sketch a quadratic graph.** e.g. Sketch the graph of $y = x^2 - 2x - 3$, labelling the intercepts and the turning point.		☐
7C	**7. I can sketch a quadratic graph that is a perfect square.** e.g. Sketch the graph of $y = x^2 - 4x + 4$, labelling the intercepts and turning point.		☐
7D	**8. I can determine the key features of a graph in turning point form.** e.g. For $y = 2(x - 1)^2 - 18$, determine the turning point, *y*-intercept and any *x*-intercepts.		☐
7D	**9. I can sketch a quadratic graph by first completing the square.** e.g. Sketch $y = x^2 + 8x + 20$ by first completing the square.		☐
7E	**10. I can use the discriminant to find the number of *x*-intercepts of a quadratic graph.** e.g. Determine the number of *x*-intercepts of the parabola given by $y = 2x^2 + 3x + 6$.	Opt	☐
7E	**11. I can find the turning point of a quadratic using** $x = -\frac{b}{2a}$. e.g. Determine the turning point coordinates of the parabola given by $y = 2x^2 + 8x - 5$.	Opt	☐
7E	**12. I can sketch a quadratic graph using the quadratic formula.** e.g. Sketch the graph of the quadratic $y = 3x^2 - 6x - 2$, labelling significant points. Round the *x*-intercepts to two decimal places.	Opt	☐
7F	**13. I can apply quadratic models in word problems.** e.g. A piece of wire measuring 80 cm in length is bent into the shape of a rectangle. Let *x* cm be the width of the rectangle. i Use the perimeter to find an expression for the length of the rectangle in terms of *x*. ii Hence, find a rule for the area of the rectangle, $A\text{ cm}^2$, in terms of *x* and sketch its graph for suitable values of *x*. iii Use the graph to determine the maximum area that can be formed and the dimensions of the rectangle that give this area.		☐
7G	**14. I can find the points of intersection of a line and a parabola.** e.g. Find the points of intersection of $y = 2x^2 - 5$ and $y = x + 1$.	Opt	☐
7G	**15. I can determine the number of points of intersection of a line and a parabola.** e.g. Determine the number of solutions (points of intersection) of the equations $4x + y = -4$ and $y = x^2 + 2x + 5$.	Opt	☐
7H	**16. I can graph height vs time to show the rate of change of height.** e.g. Water is poured into these containers at a constant rate. Draw a graph for each showing the relationship between the height, *h*, of water in the container at time *t*. i  ii 	Opt	☐

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Chapter checklist with success criteria

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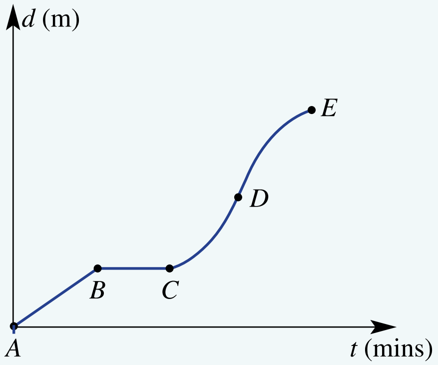
7H **17. I can work with distance-time graphs.** Opt ☐

e.g. The distance-time graph shows a journey by scooter over a number of minutes.



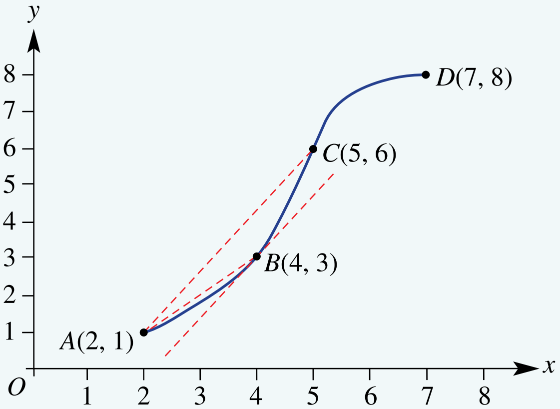
i Between what points does the graph show a speed which is constant?
ii Between what points does the graph show a speed which is decreasing?
iii Between point *C* and *E*, at which point is the scooter traveling the fastest?

7I **18. I can find the average rate of change.** Opt ☐

e.g. Find the average rate of change of *y* in the rule $y = x^2 + 2$ as *x* changes from 1 to 3.

7I **19. I can approximate an instantaneous rate of change.** Opt ☐

e.g. This graph shows part of a curve, with a tangent drawn at point *B* to illustrate the instantaneous rate of change at that point. Find the gradient of line segments *AB* and *AC* and state which one gives a better approximation of the instantaneous rate of change at point *B*.



7J **20. I can work with direct variation.** Ext ☐

e.g. If *y* is directly proportional to *x* and $y = 45$ when $x = 25$, determine the relationship between *y* and *x* and use this to find *y* when $x = 40$.

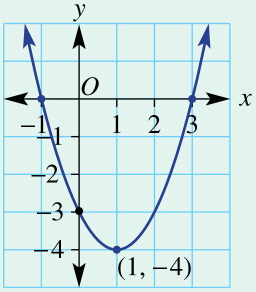
7J **21. I can work with inverse variation.** Ext ☐

e.g. If *x* and *y* are inversely proportional and $y = 5$ when $x = 4$, determine the constant of proportionality *k* and write the rule.

Short-answer questions

7A

1 State the following features of the quadratic graph shown.



a turning point and whether it is a maximum or a minimum

b axis of symmetry

c coordinates of the x-intercepts

d coordinates of the y-intercept

7A/B

2 State whether the graphs of the following quadratics have a maximum or a minimum turning point and give its coordinates.

a $y = (x - 2)^2$ **b** $y = -x^2 + 5$

c $y = -(x + 1)^2 - 2$ **d** $y = 2(x - 3)^2 + 4$

7C

3 Sketch the quadratics below by first finding:

i the y-intercept

ii the x-intercepts, using factorisation

iii the turning point

a $y = x^2 - 4$ **b** $y = x^2 + 8x + 16$ **c** $y = x^2 - 2x - 8$

7D

4 Complete the following for each quadratic below.

i State the coordinates of the turning point and whether it is a maximum or a minimum.

ii Find the coordinates of the y-intercept.

iii Find the coordinates of the x-intercepts (if any).

iv Sketch the graph, labelling the features above.

a $y = -(x - 1)^2 - 3$ **b** $y = 2(x + 3)^2 - 8$

7D

5 Sketch the following quadratics by completing the square. Label all key features with exact coordinates.

a $y = x^2 - 4x + 1$ **b** $y = x^2 + 2x + 6$ Opt **c** $y = x^2 + 3x - 2$

7D/E

6 State the number of x-intercepts of the following quadratics either by using the discriminant or by inspection where applicable.

a $y = (x + 4)^2$ **b** $y = (x - 2)^2 + 5$

Opt **c** $y = x^2 - 2x - 5$ Opt **d** $y = 2x^2 + 3x + 4$

7E

Opt

7 For the following quadratics:

i find the coordinates of the y-intercept.

ii use $x = -\frac{b}{2a}$ to find the coordinates of the turning point.

iii use the quadratic formula to find the x coordinates of the x-intercepts, rounding to one decimal place.

iv sketch the graph.

a $y = 2x^2 - 8x + 5$ **b** $y = -x^2 + 3x + 4$

7G Opt

8 Solve these equations simultaneously.

a $y = x^2 + 4x - 2$
$y = 10$

b $y = 2x^2 + 5x + 9$
$y = -x + 4$

c $y = x^2 + 1$
$2x + 3y = 4$

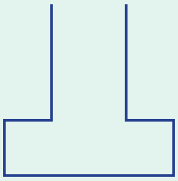
7G Opt

9 Use the discriminant to show that the line $y = x + 4$ intersects the parabola $y = x^2 - x + 5$ in just one place.

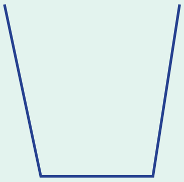
7H Opt

10 Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, h, of water in the container at time t.

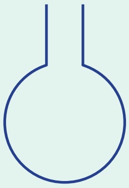
a



b

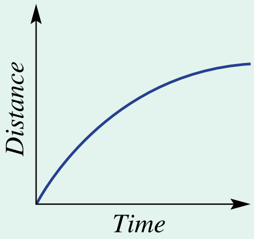


c



7H Opt

11 For the distance–time graph showing a journey from home, describe the journey in relation to distance from home, gradient of graph and speed.



7I Opt

12 For the relations with rules below, find the average rate of change of y as x changes from

i 0 to 2

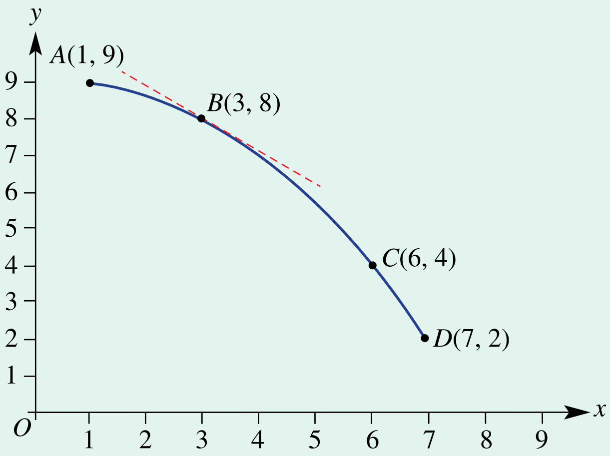
ii −1 to 4

a $y = x^2 + 2x$

b $y = -x^2 - 2x + 3$

7I Opt

13 This graph shows part of a curve with a tangent drawn at point B to illustrate the instantaneous rate of change at the point. Use the points A to D to find the gradient of the line segment that gives the best approximation of the instantaneous rate of change at point B.



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7J Ext

14 **a** If y varies directly with x and $y = 10$ when $x = 2$, find the rule linking y with x.

b If $y = \frac{k}{x}$ and $x = 6$ when $y = 12$, determine:

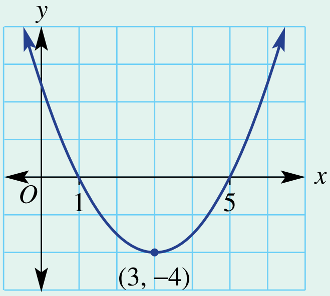
i the value of k **ii** y when $x = 4$ **iii** x when $y = 0.7$

Multiple-choice questions

7A

1 The equation of the axis of symmetry of the graph shown is:

A $y = -4$
B $x = 3$
C $x = -4$
D $y = 3$
E $y = 3x$



7B

2 Compared to the graph of $y = x^2$, the graph of $y = (x - 3)^2$ is:

A translated 3 units down **B** translated 3 units left
C dilated by a factor of 3 **D** translated 3 units right
E translated 3 units up

7B

3 The coordinates and type of turning point of $y = -(x + 2)^2 + 1$ is:

A a minimum at $(-2, -1)$ **B** a maximum at $(2, 1)$
C a minimum at $(2, 1)$ **D** a maximum at $(2, -1)$
E a maximum at $(-2, 1)$

7B

4 The y-intercept of $y = 3(x - 1)^2 + 4$ has coordinates:

A $(0, 1)$ **B** $(0, 4)$ **C** $\left(0, \frac{1}{3}\right)$ **D** $(0, 7)$ **E** $(0, 3)$

7C

5 The x-intercept(s) of the graph of $y = x^2 + 3x - 10$ have x-coordinates at x equals:

A $2, -5$ **B** -10 **C** $5, -2$
D $-5, -2$ **E** $5, 2$

7C

6 A quadratic graph has x-intercepts with x-coordinates at $x = -7$ and $x = 2$. The x-coordinate of the turning point is:

A $x = -\frac{5}{2}$ **B** $x = -\frac{7}{2}$ **C** $x = \frac{9}{3}$
D $x = \frac{5}{2}$ **E** $x = -\frac{9}{2}$

7D

7 The quadratic rule $y = x^2 - 4x - 3$, when written in turning point form, is:

A $y = (x - 2)^2 - 3$ **B** $y = (x - 4)^2 + 1$ **C** $y = (x - 2)^2 - 7$
D $y = (x + 4)^2 - 19$ **E** $y = (x + 2)^2 - 1$

7E Opt

8 A quadratic graph $y = ax^2 + bx + c$ has two x-intercepts. This tells us that:

A The graph has a maximum turning point.
B $-\frac{b}{2a} < 0$
C There is no y-intercept.
D $b^2 - 4ac > 0$
E $b^2 - 4ac = 0$

7F

9 A toy rocket follows the path given by $h = -t^2 + 4t + 6$, where h is the height above ground, in metres, t seconds after launch. The maximum height reached by the rocket is:

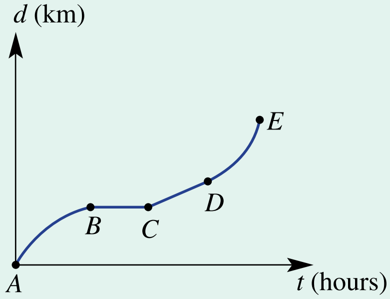
A 10 metres
B 2 metres
C 6 metres
D 8 metres
E 9 metres

7H Opt

10 The distance–time graph shows a journey by bicycle over a number of hours.

Between what points does the graph show a speed which is increasing?

A A to B
B C to D
C D to E
D C to D and D to E
E C to D and A to B



7I Opt

11 The average rate of change of y as x changes from 1 to a $(a > 1)$ in the rule $y = x^2 + 2$ is 4. The value of a is:

A 2 **B** 3 **C** 4 **D** 5 **E** 6

7J

12 If y is inversely proportional to x, the equation is of the form:

A $y = kx$
B $y = kx + c$
C $y = \frac{x}{k}$
D $y = kx^2$
E $y = \frac{k}{x}$

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Extended-response questions

1 The cable for a suspension bridge is modelled by the equation $h = \frac{1}{800}(x - 200)^2 + 30$, where h metres is the distance above the base of the bridge and x metres is the distance from the left side of the bridge.

a Determine the turning point of the graph of the equation.
b If the bridge is symmetrical, determine the suitable values of x.
c Determine the range of values of h.
d Sketch a graph of the equation for the suitable values of x.
e What horizontal distance does the cable span?
f What is the closest distance of the cable from the base of the bridge?
g What is the greatest distance of the cable from the base of the bridge?

2 200 metres of fencing is to be used to form a rectangular paddock. Let x metres be the width of the paddock.

a Write an expression for the length of the paddock in terms of x.
b Write an equation for the area of the paddock ($A\,\text{m}^2$) in terms of x.
c Decide on the suitable values of x.
d Sketch the graph of A versus x for suitable values of x.
e Use the graph to determine the maximum paddock area that can be formed.
f What will be the dimensions of the paddock to achieve its maximum area?

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8

Probability and counting techniques

Maths in context: Predictions and probability

Some outcomes in life can be predicted from a previous or current situation. But other events are not affected by the past. An understanding of probability helps to inform our decisions about the future.

Weather forecasting is informed by an enormous database of statistics. The percentage chance of any future weather is dependent on current weather. A '1 in 100' year flood forecast depends on current weather conditions, not on how many '1 in 100' year floods have occurred that century or even that year.

Insurance cover is priced according to the frequency of past events, based on huge amounts of collected data. Young men pay a higher price for car insurance because data shows that age group has, in the past, had many more car accidents than older adults.

Sports results are predicted using past records of teams and players. What data would you use to choose between two cricket batters? Would the average or mode or median runs/innings be more important? What batting ability is shown by their IQR (Interquartile range)?

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Chapter contents

Australian Curriculum 9.0

PROBABILITY

Use the language of “if .... then”, “given”, “of”, “knowing that” to describe and interpret situations involving conditional probability (AC9M10P01)

Design and conduct repeated chance experiments and simulations using digital tools to model conditional probability and interpret results (AC9M10P02)

Counting principles and factorial notation as a representation that provides efficient counting in multiplicative contexts, including calculations of probabilities (Year 10 optional content)

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

The gamblers fallacy is the mistaken belief that, after consecutive losses when betting on random, independent events, a win becomes more likely. Certain probabilities are totally independent of any historical outcomes, such as tossing a fair coin.

8A Review of probability CONSOLIDATING

LEARNING INTENTIONS

- To review the key terms of probability: trial, sample space, event and outcome
- To understand the possible values of a probability and how they describe the level of chance
- To know how to calculate theoretical probabilities for equally likely outcomes
- To be able to calculate and use an experimental probability

Probability is an area of mathematics concerned with the likelihood of particular random events. In some situations, such as rolling a die, we can determine theoretical probabilities because we know the total number of possible outcomes and the number of favourable outcomes. In other cases, we can use statistics and experimental results to describe the chance that an event will occur. The chance that a particular soccer team will win its next match, for example, could be estimated using various results from preceding games.

A soccer team could win, lose or draw the next match it plays, but these three outcomes do not necessarily have the same probability.

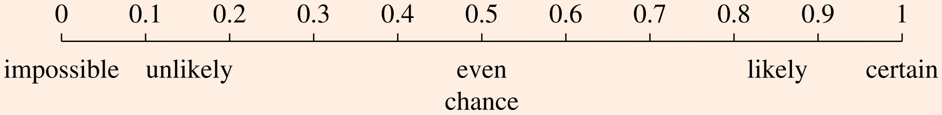
Lesson starter: Name the event

For each number below, describe an event that has that exact or approximate probability. If you think it is exact, then give a reason.

- $\frac{1}{2}$
- 25%
- 0.2
- 0.00001
- $\frac{99}{100}$

KEY IDEAS

- Definitions
  - A **trial** is a single experiment, such as a single roll of a die.
  - The **sample space** is the list of all possible outcomes from an experiment. For example, when rolling a 6-sided die the sample space is $\{1, 2, 3, 4, 5, 6\}$.
  - An **outcome** is a possible result of an experiment.
  - An **event** is the collection of favourable outcomes.
  - **Equally likely outcomes** are outcomes that have the same chance of occurring.
- In the study of probability, a numerical value based on a scale from 0 to 1 is used to describe levels of **chance**.



- The theoretical probability, also known as expected probability, of an event in which outcomes are **equally likely** is calculated as:

$$\text{Pr(event)} = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

- **Experimental probability**, also known as observed probability is calculated in the same way as theoretical probability but uses the results of an experiment:

$$\text{Pr(event)} = \frac{\text{number of favourable outcomes}}{\text{total number of trials}}$$

  - The **long run proportion** is the experimental probability for a sufficiently large number of trials.

BUILDING UNDERSTANDING

1 A coin is flipped once.

a How many different outcomes are possible from a single flip of the coin?

b What is the sample space from a single flip of the coin?

c Are the possible outcomes equally likely?

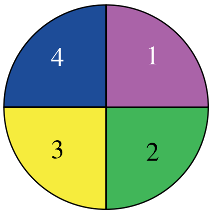
d What is the probability of obtaining a tail?

e What is the probability of not obtaining a tail?

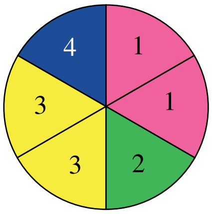
f What is the probability of obtaining a tail or a head?

2 For the following spinners, find the probability that the outcome will be a 4.

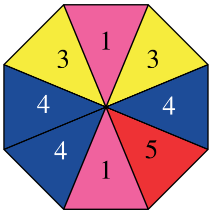
a



b



c



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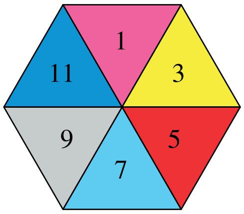

3 Given a spinner has a probability of $\frac{1}{3}$ of spinning a 4, estimate how many 4s you could expect in:

a 12 spins

b 60 spins

c 300 spins.

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Example 1 Calculating simple theoretical probabilities

A letter is chosen from the word TELEVISION. Find the probability that the letter is:

a a V **b** an E **c** not an E **d** an E or a V.

SOLUTION	EXPLANATION
a $\Pr(V) = \frac{1}{10} (= 0.1)$	$\Pr(V) = \frac{\text{number of Vs}}{\text{total number of letters}}$
b $\Pr(E) = \frac{2}{10}$	There are 2 Es in the word TELEVISION.
$= \frac{1}{5} (= 0.2)$	Simplify the fraction.
c $\Pr(\text{not an E}) = \frac{8}{10}$ $= \frac{4}{5} (= 0.8)$	If there are 2 Es in the word TELEVISION, which has 10 letters, then there must be 8 letters that are not E. This is the same as $1 - \Pr(E)$.
d $\Pr(\text{an E or a V}) = \frac{3}{10} (= 0.3)$	The number of letters that are either E or V is 3.

Now you try

A letter is chosen from the word CALCULATION. Find the probability that the letter is:

a a T **b** an A **c** not an A **d** an A or a T.

Example 2 Calculating simple experimental probabilities

An experiment involves tossing three coins and counting the number of heads. Here are the results after running the experiment 100 times.

Number of heads	0	1	2	3
Frequency	11	40	36	13

Find the experimental probability of obtaining:

a zero heads

b two heads

c fewer than two heads

d at least one head.

SOLUTION

a $\Pr(0 \text{ heads}) = \frac{11}{100}$
$= 0.11$

b $\Pr(2 \text{ heads}) = \frac{36}{100}$
$= 0.36$

c $\Pr(\text{fewer than 2 heads}) = \frac{11 + 40}{100}$
$= \frac{51}{100}$
$= 0.51$

d $\Pr(\text{at least 1 head}) = \frac{40 + 36 + 13}{100}$
$= \frac{89}{100}$
$= 0.89$

EXPLANATION

$\Pr(0 \text{ heads}) = \frac{\text{number of times 0 heads are observed}}{\text{total number of trials}}$

$\Pr(2 \text{ heads}) = \frac{\text{number of times 2 heads are observed}}{\text{total number of trials}}$

Fewer than 2 heads means to observe 0 or 1 head.

At least 1 head means that 1, 2 or 3 heads can be observed. This is the same as 1 − Pr(no heads).

Now you try

A experiment involves checking second-hand bicycles for faults. Here are the results after checking 100 bicycles.

Number of faults	0	1	2	3	4
Frequency	10	32	45	9	4

Find the experimental probability that a randomly selected bicycle will have:

a zero faults **b** three faults **c** fewer than three faults **d** at least one fault.

Exercise 8A

FLUENCY 1–4 1, 3–5 2–5($\frac{1}{2}$)

Example 1

1 A letter is chosen from the word TEACHER. Find the probability that the letter is:

a an R **b** an E **c** not an E **d** an R or an E.

2 A letter is chosen from the word EXPERIMENT. Find the probability that the letter is:

a an E **b** a vowel **c** not a vowel **d** an X or a vowel.

3 A 10-sided die numbered 1 to 10 is rolled once. Find these probabilities.

a Pr(8) **b** Pr(odd)
c Pr(even) **d** Pr(less than 6)
e Pr(prime) (Remember that 1 is not prime.) **f** Pr(3 or 8)
g Pr(8, 9 or 10) **h** Pr(greater than 9)

Example 2

4 An experiment involves tossing three coins and counting the number of heads. Here are the results after running the experiment 100 times.

Number of heads	0	1	2	3
Frequency	9	38	43	10

Find the experimental probability of obtaining:

a zero heads **b** two heads **c** fewer than two heads **d** at least one head.

5 An experiment involves rolling two dice and counting the number of sixes. Here are the results after running the experiment 100 times.

Number of sixes	0	1	2
Frequency	62	35	3

Find the experimental probability of obtaining:

a zero sixes **b** two sixes **c** fewer than two sixes **d** at least one six.

PROBLEM–SOLVING

6, 7 7, 8 8, 9

6 Thomas is a prizewinner in a competition and will be randomly awarded a single prize chosen from a collection of 50 prizes. The type and number of prizes to be handed out are listed below.

Prize	car	holiday	iPad	DVD
Number	1	4	15	30

Find the probability that Thomas will be awarded the following.

a a car **b** an iPad **c** a prize that is not a car

7 Find the probability of choosing a red counter if a counter is chosen from a box that contains the following counters.

a 3 red and 3 yellow **b** 3 red and 5 yellow
c 1 red, 1 yellow and 2 blue **d** 5 red, 12 green and 7 orange
e 10 red only **f** 6 blue and 4 green

8 Many of the 50 cars inspected at an assembly plant contained faults. The results of the inspection are as follows.

Number of faults	0	1	2	3	4
Number of cars	30	12	4	3	1

Find the experimental probability that a car selected from the assembly plant will have:

a one fault **b** four faults
c fewer than two faults **d** one or more faults
e three or four faults **f** at least two faults.

9 A quality control inspector examines clothing at a particular factory on a regular basis and records the number of faulty items identified each day. After 20 visits to the factory over the course of the year, the results are summarised in a table.

Number of faulty items	0	1	2	3	4
Frequency	14	4	1	0	1

a Estimate the probability that the inspector will identify the following numbers of faulty items on any particular day.

i 0 **ii** 1 **iii** 2 **iv** 3 **v** 4

b If the factory is fined when two or more faulty items are found, estimate the probability that the factory will be fined on the next inspection.

REASONING	10	10, 11	11, 12

10 A bag contains red and yellow counters. A counter is drawn from the bag and then replaced. This happens 100 times with 41 of the counters drawn being red.

a How many counters drawn were yellow?

b If there were 10 counters in the bag, how many do you expect were red? Give a reason.

c If there were 20 counters in the bag, how many do you expect were red? Give a reason.

11 A card is chosen from a standard deck of 52 playing cards that includes 4 aces, 4 kings, 4 queens and 4 jacks. Find the following probabilities.

a Pr(heart) **b** Pr(king) **c** Pr(king of hearts)

d Pr(heart or club) **e** Pr(king or jack) **f** Pr(heart or king)

g Pr(not a king) **h** Pr(neither a heart nor a king)

12 The probability of selecting a white chocolate from a box is $\frac{1}{5}$ and the probability of selecting a dark chocolate from the same box is $\frac{1}{3}$. The other chocolates are milk chocolates.

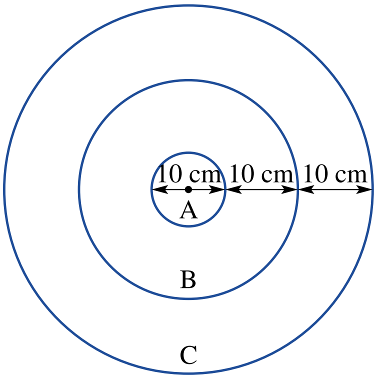
a Find the probability of selecting a milk chocolate.

b How many chocolates in total could be in the box? Give reasons. Is there more than one answer?

ENRICHMENT: Target probability	–	–	13

13 A target board is made up of three rings (A, B and C) that are 10 cm apart, as shown.

An experienced archer shoots an arrow at the board and is guaranteed to hit it, with an equal chance of doing so at any point. Recall that the area of a circle $= \pi r^2$.



a Calculate the total area of the target and express your answer as an exact value (e.g. 10π).

b Calculate, using exact values, the area of the regions labelled:

i A **ii** B **iii** C.

c Calculate the probability that the region in which the archer's arrow will hit will be:

i A **ii** B **iii** C **iv** A or B

v B or C **vi** A or C **vii** A, B or C **viii** not B.

d Investigate whether changing the width of each ring in the target by the same amount changes the answers to part **c**.

8B Set notation in Venn diagrams and two-way tables

LEARNING INTENTIONS

- To know the symbols for set notation for union, intersection and complement and what sets they represent
- To know how to use a Venn diagram or two-way table to display the outcomes of two or more events
- To be able to use Venn diagrams and two-way tables to find associated probabilities

When we consider two or more events it is possible that there are outcomes that are common to both events. A TV network, for example, might be collecting statistics regarding whether or not a person watches cricket and/or tennis or neither over the Christmas holidays. The estimated probability that a person will watch cricket *or* tennis will therefore depend on how many people responded yes to watching both cricket *and* tennis.

TV ratings come from the programs viewed in 3000 randomly selected homes. Statistical analysis gives the proportion of each age group who watch specific programs. This information impacts TV advertising, program development and scheduling.

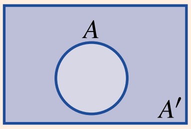
Lesson starter: Duplication in cards

Imagine that you randomly draw one card from a standard deck of 52 playing cards.

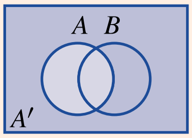
- Discuss what a standard deck includes.
- What is the probability of selecting a heart?
- What is the probability of selecting a king?
- Now find the probability that the card is a king and a heart. Is this possible?
- Find the probability that the card is a king or a heart. Discuss why the probability is not just equal to $\frac{4}{52} + \frac{13}{52} = \frac{17}{52}$.

KEY IDEAS

- Set notation
  - A **set** is a collection or group of elements that can include numbers, letters or other objects.
  - The **sample space**, denoted by S, Ω, ∪ or ξ, is the set of all possible elements or objects considered in a particular situation. This is also called the **universal set**.
  - A **Venn diagram** illustrates how all elements in the sample space are distributed among the events.



or

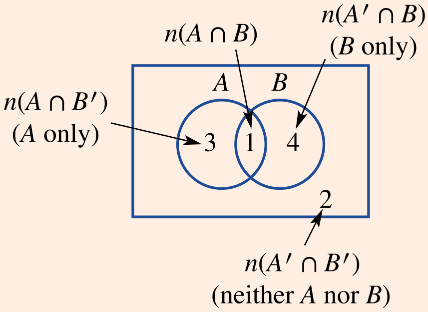


  - A **null** or **empty set** is a set with no elements and is symbolised by { } or ∅.
  - All elements that belong to both A *and* B make up the **intersection**: $A \cap B$.
  - All elements that belong to either events A *or* B make up the **union**: $A \cup B$.
  - Two sets A and B are **mutually exclusive** if they have no elements in common, meaning $A \cap B = \varnothing$.

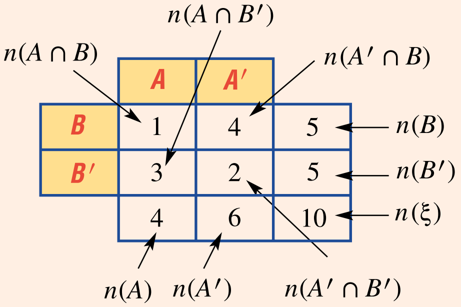
- For an event A, the **complement** of A is A' (or ‘not A’).
- $\Pr(A') = 1 - \Pr(A)$
- A **only** (or $A \cap B'$) is defined as all the elements in A but not in any other set.
- $n(A)$ is the number of elements in set A.

■ Venn diagrams and two-way tables are useful tools when considering two or more events.

Venn diagram



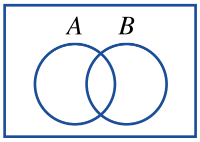
Two-way table



	A	A'	
B	1	4	5
B'	3	2	5
	4	6	10

BUILDING UNDERSTANDING

1 On a Venn diagram like the one shown shade the region represented by each of the following.



a A

b B

c $A \cap B$

d $A \cup B$

e A only

f B only

g A'

h neither A nor B

2 Which symbols $\cup$, $\cap$, or $\varnothing$, would be used to rewrite the following?

a null set

b A and B

c A or B

d A or B or C

3 Decide if the events A and B are mutually exclusive.

a $A = \{1, 3, 5, 7\}$
$B = \{5, 8, 11, 14\}$

b $A = \{-3, -2, \ldots, 4\}$
$B = \{-11, -10, \ldots, -4\}$

c $A = \{\text{prime numbers}\}$
$B = \{\text{even numbers}\}$

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Example 3 Listing sets

Consider the given events A and B that involve numbers taken from the first 10 positive integers.

$A = \{1, 2, 3, 4, 5, 6\}$ $B = \{1, 3, 7, 8\}$

a Represent the two events A and B in a Venn diagram, showing the number of elements belonging to each region.

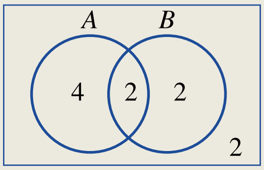
b List the following sets.

i $A \cap B$ **ii** $A \cup B$

c If a number from the first 10 positive integers is randomly selected, find the probability that the following events occur.

i A **ii** $A \cap B$ **iii** $A \cup B$

d Are the events A and B mutually exclusive? Why or why not?

SOLUTION	**EXPLANATION**
a 	The two elements 1 and 3 are common to both sets A and B, $A \cap B$. The two elements 9 and 10 belong to neither set A nor set B, $(A' \cap B')$. A has 6 elements, with 2 in the intersection, so 'A only' has 4 elements.
b **i** $A \cap B = \{1, 3\}$	$A \cap B$ is the intersection of sets A and B.
ii $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$	$A \cup B$ contains elements in either A or B.
c **i** $\Pr(A) = \frac{6}{10} = \frac{3}{5}$	There are 6 elements in A.
ii $\Pr(A \cap B) = \frac{2}{10} = \frac{1}{5}$	$A \cap B$ contains 2 elements.
iii $\Pr(A \cup B) = \frac{8}{10} = \frac{4}{5}$	A ∪ B contains 8 elements.
d The sets A and B are not mutually exclusive since $A \cap B \neq \varnothing$.	The set $A \cap B$ contains at least 1 element.

Now you try

Consider the given events A and B that involve numbers taken from the first 10 positive integers.

$A = \{2, 3, 4, 5, 6, 7, 8\}$ and $B = \{1, 2, 3, 5, 7\}$

a Represent the two events A and B in a Venn diagram, showing the number of elements belonging to each region.

b List the following sets.

i $A \cap B$ **ii** $A \cup B$

c If a number from the first 10 positive integers is randomly selected, find the probability that the following events occur.

i A **ii** $A \cap B$ **iii** $A \cup B$

d Are the events A and B mutually exclusive? Why or why not?

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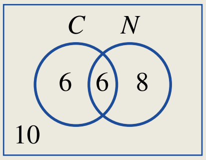
Example 4 Using Venn diagrams

From a class of 30 students, 12 enjoy cricket (*C*), 14 enjoy netball (*N*) and 6 enjoy both cricket and netball.

a Illustrate this information in a Venn diagram.

b State the number of students who enjoy:
  - **i** netball only
  - **ii** neither cricket nor netball.

c Find the probability that a student chosen randomly from the class will enjoy:
  - **i** netball
  - **ii** netball only
  - **iii** both cricket and netball.

SOLUTION	EXPLANATION
a 	First, write 6 in the intersection (i.e. 6 enjoy cricket and netball), then determine the other values according to the given information. Cricket only is $12 - 6 = 6$. The total must be 30.
b **i** $n(N \text{ only}) = 8$	Includes elements in *N* but not in *C*.
ii $n(\text{neither } C \text{ nor } N) = 10$	These are the elements outside both *C* and *N*.
c **i** $\Pr(N) = \frac{14}{30} = \frac{7}{15}$	14 of the 30 students enjoy netball.
ii $\Pr(N \text{ only}) = \frac{8}{30} = \frac{4}{15}$	8 of the 30 students enjoy netball but not cricket.
iii $\Pr(C \cap N) = \frac{6}{30} = \frac{1}{5}$	6 students enjoy both cricket and netball.

Now you try

From a pack of 20 dogs, 9 enjoy fresh meat (*M*), 12 enjoy dry food (*D*) and 7 enjoy both fresh meat and dry food.

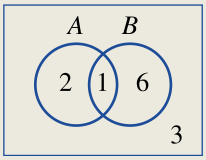
a Illustrate this information in a Venn diagram.

b State the number of dogs who enjoy:
  - **i** fresh meat only
  - **ii** neither fresh meat nor dry food.

c Find the probability that a dog chosen at random from the pack will enjoy:
  - **i** fresh meat
  - **ii** dry food only
  - **iii** both fresh meat and dry food.

Example 5 Using two-way tables

The Venn diagram shows the distribution of elements in two sets, A and B.



a Transfer the information in the Venn diagram to a two-way table.

b Find:

i $n(A \cap B)$ **ii** $n(A' \cap B)$ **iii** $n(A \cap B')$ **iv** $n(A' \cap B')$

v $n(A)$ **vi** $n(B')$ **vii** $n(A \cup B)$

c Find:

i $\Pr(A \cap B)$ **ii** $\Pr(A')$ **iii** $\Pr(A \cap B')$

SOLUTION

a

	A	**A′**	
B	1	6	7
B′	2	3	5
	3	9	12

EXPLANATION

	A	**A′**	
B	$n(A \cap B)$	$n(A' \cap B)$	$n(B)$
B′	$n(A \cap B')$	$n(A' \cap B')$	$n(B')$
	$n(A)$	$n(A')$	$n(\xi)$

b **i** $n(A \cap B) = 1$

$n(A \cap B)$ is the intersection of A and B.

ii $n(A' \cap B) = 6$

$n(A' \cap B)$ is B only.

iii $n(A \cap B') = 2$

$n(A \cap B')$ is A only.

iv $n(A' \cap B') = 3$

$n(A' \cap B')$ is neither A nor B.

v $n(A) = 3$

$n(\mathrm{A}) = n(A \cap B') + n(A \cap B)$

vi $n(B') = 5$

$n(B') = n(A \cap B') + n(A' \cap B')$

vii $n(A \cup B) = 9$

$n(A \cup B) = n(A \cap B) + n(A \cap B') + n(A' \cap B)$

c **i** $\Pr(A \cap B) = \frac{1}{12}$

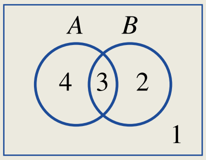
ii $\Pr(A') = \frac{9}{12} = \frac{3}{4}$

iii $\Pr(A \cap B') = \frac{2}{12} = \frac{1}{6}$

When calculating probabilities, you will need to divide the number of elements in each set by the number of elements in the sample space, which is 12.

Now you try

The Venn diagram shows the distribution of elements in two sets, A and B.



a Transfer the information in the Venn diagram to a two-way table.

b Find:

i $n(A \cap B)$ **ii** $n(A' \cap B)$ **iii** $n(A \cap B')$ **iv** $n(A' \cap B')$

v $n(A)$ **vi** $n(B')$ **vii** $n(A \cup B)$

c Find:

i $\Pr(A \cap B)$ **ii** $\Pr(A')$ **iii** $\Pr(A \cap B')$

Exercise 8B

FLUENCY	1, 3, 4	1–5	2, 4–6

Example 3

1 Consider the given events A and B, which involve numbers taken from the first 10 positive integers.

$A = \{1, 2, 4, 5, 7, 8, 10\}$ $B = \{2, 3, 5, 6, 8, 9\}$

a Represent events A and B in a Venn diagram, showing the number of elements belonging to each region.

b List the following sets.

i $A \cap B$ **ii** $A \cup B$

c If a number from the first 10 positive integers is randomly selected, find the probability that the following events occur.

i A **ii** $A \cap B$ **iii** $A \cup B$

d Are the events A and B mutually exclusive? Why/why not?

2 The elements of the events A and B described below are numbers taken from the first 10 prime numbers.

$A = \{2, 5, 7, 11, 13\}$ $B = \{2, 3, 13, 17, 19, 23, 29\}$

a Represent events A and B in a Venn diagram.

b List the elements belonging to the following.

i A and B **ii** A or B

c Find the probability that these events occur.

i A **ii** B **iii** $A \cap B$ **iv** $A \cup B$

Example 4

3 From a group of 50 adults, 35 enjoy reading fiction (F), 20 enjoy reading non-fiction (N) and 10 enjoy reading both fiction and non-fiction.

a Illustrate the information in a Venn diagram.

b State the number of people who enjoy reading:

i fiction only

ii neither fiction nor non-fiction.

c Find the probability that a person chosen at random will enjoy reading:

i non-fiction

ii non-fiction only

iii both fiction and non-fiction.

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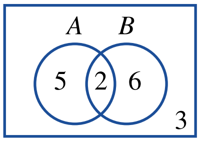
4 At a show, 45 children have the choice of riding on the Ferris wheel (F) and/or the Big Dipper (B). Thirty-five of the children wish to ride on the Ferris wheel, 15 children want to ride on the Big Dipper and 10 children want to ride on both.

- **a** Illustrate the information in a Venn diagram.
- **b** Find:
  - **i** $n(F \text{ only})$
  - **ii** $n(\text{neither } F \text{ nor } B)$
- **c** For a child chosen at random from the group, find the following probabilities.
  - **i** $\Pr(F)$
  - **ii** $\Pr(F \cap B)$
  - **iii** $\Pr(F \cup B)$
  - **iv** $\Pr(F')$
  - **v** $\Pr(\text{neither } F \text{ nor } B)$

Example 5

5 The Venn diagram below shows the distribution of elements in two sets, A and B.



- **a** Transfer the information in the Venn diagram to a two-way table.
- **b** Find:
  - **i** $n(A \cap B)$
  - **ii** $n(A' \cap B)$
  - **iii** $n(A \cap B')$
  - **iv** $n(A' \cap B')$
  - **v** $n(A)$
  - **vi** $n(B')$
  - **vii** $n(A \cup B)$
  - **viii** $n(\xi)$
- **c** Find:
  - **i** $\Pr(A \cap B)$
  - **ii** $\Pr(A')$
  - **iii** $\Pr(A \cap B')$

6 From a total of 10 people, 5 like apples (A), 6 like bananas (B) and 4 like both apples and bananas.

- **a** Draw a Venn diagram for the 10 people.
- **b** Draw a two-way table.
- **c** Find:
  - **i** $n(A' \cap B)$
  - **ii** $n(A' \cap B')$
  - **iii** $\Pr(A \cap B)$
  - **iv** $\Pr(A \cup B)$

PROBLEM–SOLVING	7, 8	7, 9	8–10

7 Decide which of the elements would need to be removed from event A if the two events A and B described below are to become mutually exclusive.

- **a** $A = \{1, 2, 3, 4\}$
  $B = \{4, 5, 6, 7\}$
- **b** $A = \{10, 12, 14, 16, 18\}$
  $B = \{9, 10, 11, 12\}$
- **c** $A = \{a, b, c, d, e\}$
  $B = \{a, c, e, g\}$
- **d** $A = \{1, 3, 5, 8, 10, 15, 20, 22, 23\}$
  $B = \{7, 9, 14, 16, 19, 21, 26\}$

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8 A letter is chosen at random from the word COMPLEMENTARY and two events, C and D, are as follows.

C: choosing a letter belonging to the word COMPLETE
D: choosing a letter belonging to the word CEMENT

a Represent the events C and D in a Venn diagram. Ensure that your Venn diagram includes *all* the letters that make up the word COMPLEMENTARY.

b Find the probability that the randomly chosen letter will:

- **i** belong to C
- **ii** belong to C and D
- **iii** belong to C or D
- **iv** not belong to C
- **v** belong to neither C nor D.

9 Complete the following two-way tables.

a

	A	A'	
B		3	6
B'			
		4	11

b

	A	A'	
B	2	7	
B'			3
	4		

10 In a group of 12 chefs, all enjoy baking cakes and/or tarts. In fact, 7 enjoy baking cakes and 8 enjoy baking tarts. Find out how many chefs enjoy baking both cakes and tarts.

REASONING	11	11, 12	12, 13

11 If events A and B are mutually exclusive and $\Pr(A) = a$ and $\Pr(B) = b$, write expressions for:

- **a** $\Pr(\text{not } A)$
- **b** $\Pr(A \text{ or } B)$
- **c** $\Pr(A \text{ and } B)$

12 Use diagrams to show that $(A \cup B)' = A' \cap B'$.

13 Mario and Erin are choosing a colour to paint the interior walls of their house. They have six colours to choose from: white (w), cream (c), navy (n), sky blue (s), maroon (m) and violet (v).

Mario would be happy with white or cream and Erin would be happy with cream, navy or sky blue. As they can't decide, a colour is chosen at random for them.

Let M be the event that Mario will be happy with the colour and let E be the event that Erin will be happy with the colour.

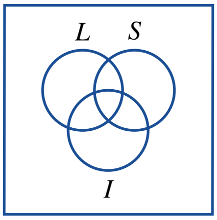
a Represent the events M and E in a Venn diagram.

b Find the probability that the following events occur.

- **i** Mario will be happy with the colour choice; i.e. find $\Pr(M)$.
- **ii** Mario will not be happy with the colour choice.
- **iii** Both Mario and Erin will be happy with the colour choice.
- **iv** Mario or Erin will be happy with the colour choice.
- **v** Neither Mario nor Erin will be happy with the colour choice.

ENRICHMENT: Triple Venn diagrams	–	–	14, 15

14 Of 15 chosen courier companies, 9 offer a local service (L), 7 offer an interstate service (S) and 6 offer an international service (I). Two companies offer all three services, 3 offer both local and interstate services, 5 offer only local services and 1 offers only an international service.



- **a** Draw a Venn diagram displaying the given information.
- **b** Find the number of courier companies that offer neither a local, interstate nor international service.
- **c** If a courier is chosen at random from the 15 examined initially, find the following probabilities.
  - **i** Pr(L)
  - **ii** Pr(L only)
  - **iii** Pr(L or S)
  - **iv** Pr(L and S only)

15 Thirty-eight people were interviewed about their travelling experience in the past 12 months. Although the interviewer did not write down the details of the interviews, she remembers the following information.
In the past 12 months:

- Two people travelled overseas, interstate and within their own state.
- Two people travelled overseas and within their own state only.
- Seven people travelled interstate only.
- 22 people travelled within their own state.
- Three people did not travel at all.
- The number of people who travelled interstate and within their own state only was twice the number of people who travelled overseas and interstate only.
- The number of people who travelled overseas was equal to the number of people who travelled within their own state only.

- **a** Use a Venn diagram to represent the information that the interviewer remembers.
- **b** By writing down equations using the variables x (the number of people who travelled overseas and interstate only) and y (the number of people who travelled overseas only), solve simultaneously and find:
  - **i** the number of people who travelled interstate and overseas only
  - **ii** the number of people who travelled overseas.
- **c** If one person from the 38 is chosen at random, find the probability that the person will have travelled to the following places:
  - **i** within their own state only
  - **ii** overseas only
  - **iii** interstate only
  - **iv** overseas or interstate or within their own state
  - **v** interstate or overseas.

Airlines employ mathematicians to use probability and statistics to predict passenger numbers so they can ensure seats on most flights are filled.

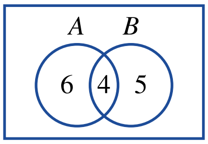
8C The addition rule

LEARNING INTENTIONS

- To understand and know the addition rule for finding the probability of the union of two events
- To be able to apply the addition rule to find unknown probabilities
- To know the meaning of the term mutually exclusive

When two events are mutually exclusive we know that the probability of the union of the events can be found by simply adding the probabilities of each of the individual events. If they are not mutually exclusive then we need to take the intersection into account.

If we take 15 people who like apples (A) or bananas (B), for example, we could illustrate this with the following possible Venn diagram.



$$\Pr(A) = \frac{10}{15}$$

$$\Pr(B) = \frac{9}{15}$$

Restaurants improve customer satisfaction and efficiency by analysing data from orders, loyalty programs, etc. Menus and marketing can use data such as the proportion of brunch customers who order the crab omelette or blueberry pancakes or both.

Clearly, the probability that a person likes apples or bananas is not $\frac{10}{15} + \frac{9}{15} = \frac{19}{15}$ as this is impossible. The intersection needs to be taken into account because, in the example above, this has been counted twice. This consideration leads to the addition rule, which will be explained in this section.

Lesson starter: What's the intersection?

Two events, A and B, are such that $\Pr(A) = 0.5$, $\Pr(B) = 0.4$ and $\Pr(A \cup B) = 0.8$.

- Are the events mutually exclusive? Why?
- Is it possible to find $\Pr(A \cap B)$? If so, find $\Pr(A \cap B)$.
- Can you write a rule connecting $\Pr(A \cup B)$, $\Pr(A)$, $\Pr(B)$ and $\Pr(A \cap B)$?
- Does your rule hold true for mutually exclusive events?

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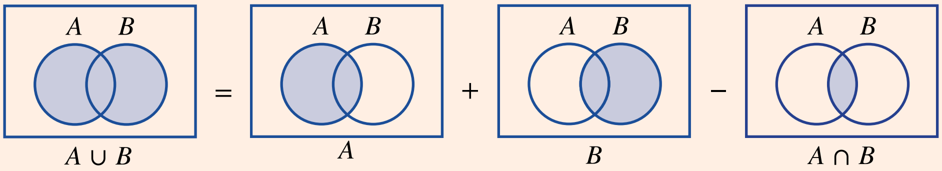
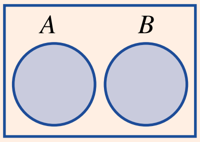
KEY IDEAS

- The **addition rule** for two events, A and B, is:
  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$



- If A and B are mutually exclusive then:
  - $\Pr(A \cap B) = 0$
  - $\Pr(A \cup B) = \Pr(A) + \Pr(B)$



BUILDING UNDERSTANDING

① A fair 6-sided die is rolled.

- Event A is rolling a number greater than 3.
- Event B is rolling an even number.

a State the sets.

i A **ii** B

iii A or B (i.e. $A \cup B$) **iv** A and B (i.e. $A \cap B$)

b Are events A and B mutually exclusive? Give a reason.

c Find $\Pr(A \cup B)$.

② Use the given information and the addition rule to find $\Pr(A \cup B)$.

a $\Pr(A) = 0.7, \Pr(B) = 0.5, \Pr(A \cap B) = 0.4$

b $\Pr(A) = 0.65, \Pr(B) = 0.4, \Pr(A \cap B) = 0.35$

③ Use the addition rule to find $\Pr(A \cap B)$ if $\Pr(A \cup B) = 0.9$, $\Pr(A) = 0.5$ and $\Pr(B) = 0.45$.

Example 6 Applying the addition rule

A card is selected from a standard deck of 52 playing cards (4 suits, no jokers). Let A be the event ‘the card is a diamond’ and B be the event ‘the card is a jack’.

a Find:

i $n(A)$ **ii** $n(B)$ **iii** $n(A \cap B)$

b Find:

i $\Pr(A)$ **ii** $\Pr(A')$ **iii** $\Pr(A \cap B)$

c Use the addition rule to find $\Pr(A \cup B)$.

d Find the probability that the card is a jack or not a diamond.

SOLUTION	EXPLANATION
a **i** $n(A) = 13$	One-quarter of the cards is the diamond suit.
ii $n(B) = 4$	There is one jack in each suit.
iii $n(A \cap B) = 1$	Only one card is both a diamond and a jack.
b **i** $\Pr(A) = \frac{13}{52} = \frac{1}{4}$	13 out of the 52 cards are diamond.
ii $\Pr(A') = 1 - \frac{1}{4}$ $= \frac{3}{4}$	The complement of A is A'.
iii $\Pr(A \cap B) = \frac{1}{52}$	There is one jack of diamonds out of the 52 cards.
c $\Pr(A \cup B)$ $= \Pr(A) + \Pr(B) - \Pr(A \cap B)$ $= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$ $= \frac{16}{52}$ $= \frac{4}{13}$	Substitute $\Pr(A)$, $\Pr(B)$ and $\Pr(A \cap B)$ into the addition rule to find $\Pr(A \cup B)$.
d Pr(jack *or* not a diamond) $= \Pr(B \cup A')$	
$= \Pr(B) + \Pr(A') - \Pr(B \cap A')$	Use the addition rule.
$= \frac{4}{52} + \frac{39}{52} - \frac{3}{52}$	There are 4 jacks and 39 cards that are not diamonds.
$= \frac{40}{52}$ $= \frac{10}{13}$	There are 3 cards that are both jacks and not diamonds.

Now you try

A card is selected from a standard deck of 52 playing cards (4 suits, no jokers). Let A be the event ‘the card is a club’ and B be the event ‘the card is a queen’.

a Find:

i $n(A)$ **ii** $n(B)$ **iii** $n(A \cap B)$

b Find:

i $\Pr(A)$ **ii** $\Pr(A')$ **iii** $\Pr(A \cap B)$

c Use the addition rule to find $\Pr(A \cup B)$.

d Find the probability that the card is a queen or not a club.

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Example 7 Using the addition rule

Two events, A and B, are such that $\Pr(A) = 0.4$, $\Pr(B) = 0.8$ and $\Pr(A \cup B) = 0.85$.
Find:

a $\Pr(A \cap B)$

b $\Pr(A' \cap B')$

SOLUTION

a

$$\begin{aligned} \Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ 0.85 &= 0.4 + 0.8 - \Pr(A \cap B) \\ 0.85 &= 1.2 - \Pr(A \cap B) \\ \therefore \Pr(A \cap B) &= 1.2 - 0.85 \\ &= 0.35 \end{aligned}$$

b

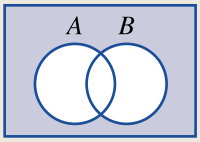
$$\begin{aligned} \Pr(A' \cap B') &= 1 - 0.85 \\ &= 0.15 \end{aligned}$$

EXPLANATION

Write the addition rule and substitute the given information.

Simplify and solve for $\Pr(A \cap B)$.

$A' \cap B' = (A \cup B)'$



Now you try

Two events, A and B, are such that $\Pr(A) = 0.5$, $\Pr(B) = 0.6$ and $\Pr(A \cup B) = 0.75$.
Find:

a $\Pr(A \cap B)$

b $\Pr(A' \cap B')$

Exercise 8C

FLUENCY	1–4	1–5	2–5

Example 6

1 A card is selected from a standard deck of 52 playing cards. Let A be the event ‘the card is a spade’ and B be the event ‘the card is an ace’.

a Find:
- **i** $n(A)$
- **ii** $n(B)$
- **iii** $n(A \cap B)$

b Find:
- **i** $\Pr(A)$
- **ii** $\Pr(A')$
- **iii** $\Pr(A \cap B)$

c Use the addition rule to find $\Pr(A \cup B)$.

d Find the probability that the card is an ace or not a spade.

2 A number is chosen from the set $\{1, 2, 3, \ldots, 20\}$. Let A be the event 'choosing a multiple of 3' and let B be the event 'choosing a prime number'.

a List set:

i A **ii** B

b Find:

i $\Pr(A \cap B)$ **ii** $\Pr(A \cup B)$

c Find the probability that the number is a prime and not a multiple of 3.

3 A 10-sided die numbered 1–10 is rolled. Use the addition rule to find the probability that the number is even or greater than 6.

Example 7

4 Two events, A and B, are such that $\Pr(A) = 0.3$, $\Pr(B) = 0.6$ and $\Pr(A \cup B) = 0.8$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A' \cap B')$

5 Two events, A and B, are such that $\Pr(A) = 0.45$, $\Pr(B) = 0.75$ and $\Pr(A \cup B) = 0.9$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A' \cap B')$

PROBLEM–SOLVING	6, 7	6, 7, 8(½)	8, 9

6 In a race of 24 horses, 18 are brown and 6 are grey. In the race, 8 brown horses and 2 grey horses finished in the top 10.

If a horse was randomly selected, what is the probability the horse is grey or finished in the top 10?

7 Of 32 cars at a show, 18 cars have four-wheel drive, 21 are sports cars and 27 have four-wheel drive or are sports cars.

a Find the probability that a randomly selected car at the show is both four-wheel drive and a sports car.

b Find the probability that a randomly selected car at the show is neither four-wheel drive nor a sports car.

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8 A card is selected from a standard deck of 52 playing cards. Find the probability that the card is:

a a heart or a king
b a club or a queen
c a black card or an ace
d a diamond or not a king
e a king or not a heart
f a 10 or not a spade.

9 **a** Find $\Pr(A \cap B')$ when $\Pr(A \cup B) = 0.8$, $\Pr(A) = 0.5$ and $\Pr(B) = 0.4$.
b Find $\Pr(A' \cap B)$ when $\Pr(A \cup B) = 0.76$, $\Pr(A) = 0.31$ and $\Pr(B) = 0.59$.

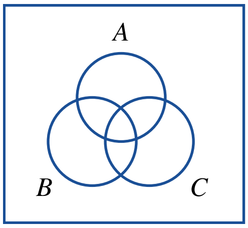
REASONING	10	10, 11	11, 12

10 Why does the addition rule become $\Pr(A \cup B) = \Pr(A) + \Pr(B)$ for mutually exclusive events?

11 Explain why the following represent impossible events.

a $\Pr(A) = 0.3$, $\Pr(B) = 0.5$, $\Pr(A \cap B) = 0.4$
b $\Pr(A \cup B) = 0.75$, $\Pr(A) = 0.32$, $\Pr(B) = 0.39$

12 Write down an addition rule for $\Pr(A \cup B \cup C)$ using sets A, B and C.



ENRICHMENT: Divisibility and the addition rule	–	–	13, 14

13 A number is randomly selected from the first 20 positive integers. Find the probability that it is divisible by:

a 3
b 4
c 2 and 3
d 2 or 3
e 3 or 5
f 2 or 5

14 A number is randomly selected from the first 500 positive integers. Find the probability that it is divisible by:

a 4
b 7
c 3 and 5
d 2 and 7
e 3 and 8
f 3, 7 and 9

8D Conditional probability

LEARNING INTENTIONS

- To understand the notion of conditional probability and that extra information can alter a probability
- To know how to use a Venn diagram or two-way table to determine a conditional probability
- To be able to identify a conditional probability scenario in a word problem

The mathematics associated with the probability that an event occurs given that another event has already occurred is called conditional probability.

Consider, for example, a group of primary school students who have bicycles for a special cycling party. Some of the bicycles have gears, some have suspension and some have both gears and suspension. Consider these two questions.

- What is the probability that a randomly selected bicycle has gears?
- What is the probability that a randomly selected bicycle has gears given that it has suspension?

Loyalty programs that track customers' buying habits assist with targeted advertising. Analysing customer data, a clothing retailer could find the fraction of customers aged under 30 who spend over $100 on one piece of clothing.

The second question is conditional in that we already know that the bicycle has suspension.

Lesson starter: Gears and suspension

Suppose that in a group of 18 bicycles, 9 have gears, 11 have suspension and 5 have both gears and suspension. Discuss the solution to the following question by considering the points below.

What is the probability that a randomly selected bicycle will have gears given that it has suspension?

- Illustrate the information on a Venn diagram.
- How many of the bicycles that have suspension have gears?
- Which areas in the Venn diagram are to be considered when answering the question? Give reasons.
- What would be the answer to the question in reverse; i.e. what is the probability that a bicycle will have suspension given that it has gears?

KEY IDEAS

- The probability of event A occurring given that event B has occurred is denoted by $\Pr(A|B)$, which reads 'the probability of A given B'. This is known as **conditional probability**.

  $$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \text{ and } \Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

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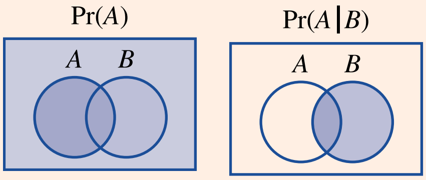
- For problems in this section these rules can be simplified to:

  $\Pr(A|B) = \dfrac{n(A \cap B)}{n(B)}$ and $\Pr(B|A) = \dfrac{n(A \cap B)}{n(A)}$

- $\Pr(A|B)$ differs from $\Pr(A)$ in that the sample space is reduced to the set B, as shown in these Venn diagrams.

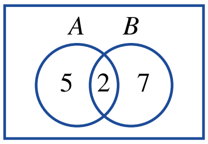


- Phrases that suggest a conditional probability scenario include 'of', 'given', 'knowing that' and 'if … then'.

BUILDING UNDERSTANDING

① In a group of 20 people, 15 are wearing jackets and 10 are wearing hats; 5 are wearing both a jacket and a hat.
  a What fraction of the people who are wearing jackets are wearing hats?
  b What fraction of the people who are wearing hats are wearing jackets?

② Use this Venn diagram to answer these questions.



  a **i** Find $n(A \cap B)$. **ii** Find $n(B)$.
  b Find $\Pr(A|B)$ using $\Pr(A|B) = \dfrac{n(A \cap B)}{n(B)}$.

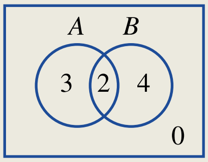
③ Use this two-way table to answer these questions.

	A	**A′**	
B	7	5	12
B′	3	1	4
	10	6	16

  a **i** Find $n(A \cap B)$. **ii** Find $n(A)$.
  b Find $\Pr(B|A)$ using $\Pr(B|A) = \dfrac{n(A \cap B)}{n(A)}$.
  c Find $\Pr(A|B)$.

Example 8 Finding conditional probabilities using a Venn diagram

Consider this Venn diagram displaying the number of elements belonging to the events A and B.



Find the following probabilities.

a $\Pr(A)$ **b** $\Pr(A \cap B)$ **c** $\Pr(A|B)$ **d** $\Pr(B|A)$

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SOLUTION

a $\Pr(A) = \frac{5}{9}$

b $\Pr(A \cap B) = \frac{2}{9}$

c $\Pr(A \mid B) = \frac{2}{6} = \frac{1}{3}$

d $\Pr(B \mid A) = \frac{2}{5}$

EXPLANATION

There are 5 elements in A and 9 in total.

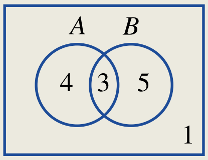
There are 2 elements common to A and B.

2 of the 6 elements in B are in A.

2 of the 5 elements in A are in B.

Now you try

Consider this Venn diagram displaying the number of elements belonging to the events A and B.



Find the following probabilities.

a $\Pr(A)$

b $\Pr(A \cap B)$

c $\Pr(A \mid B)$

d $\Pr(B \mid A)$

Example 9 Finding conditional probabilities using a two-way table

From a group of 15 hockey players at a game of hockey, 13 played on the field, 7 sat on the bench and 5 both played and sat on the bench.

A hockey player is chosen at random from the team.

Let A be the event ‘the person played on the field’ and let B be the event ‘the person sat on the bench’.

a Represent the information in a two-way table.

b Find the probability that the person only sat on the bench.

c Find the probability that the person sat on the bench given that they played on the field.

d Find the probability that the person played on the field given that they sat on the bench.

SOLUTION

a

	A	A'	
B	5	2	7
B'	8	0	8
	13	2	15

b $\Pr(B \cap A') = \frac{2}{15}$

c $\Pr(B \mid A) = \frac{5}{13}$

d $\Pr(A \mid B) = \frac{5}{7}$

EXPLANATION

$n(A \cap B) = 5$, $n(A) = 13$, $n(B) = 7$. The total is 15. Insert these values and then fill in the other places to ensure the rows and columns give the required totals.

Two people sat on the bench and did not play on the field.

$n(B \cap A) = 5$ and $n(A) = 13$.

$n(A \cap B) = 5$ and $n(B) = 7$.

Now you try

In a group of 23 movie goers, 13 bought popcorn, 15 bought a Cola and 9 bought popcorn and a Cola. One of the movie goers is selected at random.

Let A be the event 'the person bought popcorn' and B be the event 'the person bought a Cola'.

a Represent the information in a two-way table.
b Find the probability that the person only bought popcorn.
c Find the probability that the person bought popcorn given that they bought a Cola.
d Find the probability that the person bought a Cola given that they bought popcorn.

Exercise 8D

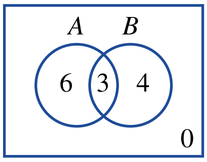
FLUENCY 1–2(½), 3 1–2(½), 3 1–2(¼), 3, 4

Example 8

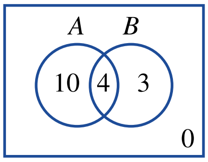
1 The following Venn diagrams display information about the number of elements associated with the events A and B. For each Venn diagram, find:

i $\Pr(A)$ **ii** $\Pr(A \cap B)$ **iii** $\Pr(A \mid B)$ **iv** $\Pr(B \mid A)$

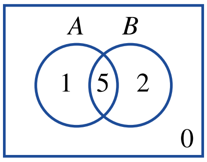
a



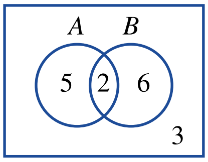
b



c



d



Example 9

2 The following two-way tables show information about the number of elements in the events A and B. For each two-way table, find:

i $\Pr(A)$ **ii** $\Pr(A \cap B)$ **iii** $\Pr(A \mid B)$ **iv** $\Pr(B \mid A)$

a

	A	A'	
B	2	8	10
B'	5	3	8
	7	11	18

b

	A	A'	
B	1	4	5
B'	3	1	4
	4	5	9

c

	A	A'	
B	7	3	10
B'	1	6	7
	8	9	17

d

	A	A'	
B	4	2	6
B'	8	2	10
	12	4	16

3 Of a group of 20 English cricket fans at a match, 13 purchased a pie, 15 drank beer and 9 both purchased a pie and drank beer.
Let A be the event 'the fan purchased a pie'.
Let B be the event 'the fan drank beer'.

a Represent the information in a two-way table.
b Find the probability that a fan in the group only purchased a pie (and did not drink beer).
c Find the probability that a fan in the group purchased a pie given that they drank beer.
d Find the probability that a fan in the group drank beer given that they purchased a pie.

4 Of 15 musicians surveyed to find out whether they play the violin or the piano, 5 play the violin, 8 play the piano and 2 play both instruments.

a Represent the information in a Venn diagram.

b How many of the musicians surveyed do not play either the violin or the piano?

c Find the probability that one of the 15 musicians surveyed plays piano knowing that they play the violin.

d Find the probability that one of the 15 musicians surveyed plays the violin knowing that they play the piano.

PROBLEM–SOLVING	5, 6	5, 6	7, 8

5 On a car production line, 30 cars are due to be completed by the end of the day. Fifteen of the cars have cruise control and 20 have airbags, and 6 have both cruise control and airbags.

a Represent the information provided in a Venn diagram or two-way table.

b Find the probability that a car chosen at random will contain the following.

i cruise control only **ii** airbags only

c If the car chosen has cruise control, then find the probability that the car will have airbags.

d If the car chosen has airbags, then find the probability that the car will have cruise control.

6 For each of the following, complete the given two-way tables and find:

i $n(A' \cap B')$ **ii** $\Pr(B \mid A)$ **iii** $\Pr(A \mid B)$

a

	A	**A′**	
B	2		4
B′			
	5		8

b

	A	**A′**	
B	3		16
B′			
	8		27

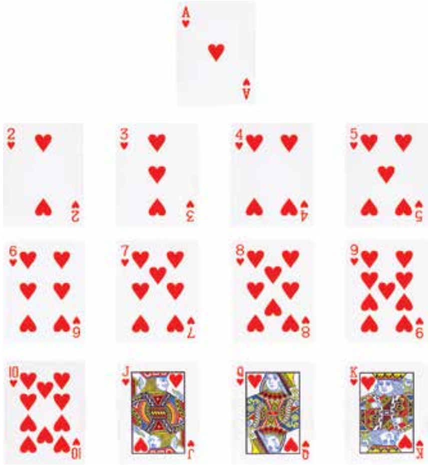
7 A card is drawn from a standard deck of 52 playing cards. Find the probability that:

a the card is a king given that it is a heart

b the card is a jack given that it is a red card

c the card is a diamond given that it is a queen

d the card is a black card given that it is an ace.



8 A number is chosen from the first 24 positive integers. Find the probability that:

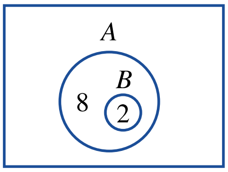
a if the number is divisible by 4 then it is divisible by 3

b if the number is divisible by 3 then it is divisible by 6.

REASONING	9	9, 10	10, 11

9 Two events, A and B, are mutually exclusive. What can be said about the probability of A given B (i.e. $\Pr(A|B)$) or the probability of B given A (i.e. $\Pr(B|A)$)? Give a reason.

10 Two events, A and B, are such that B is a subset of A, as shown in this Venn diagram.



a Find $\Pr(A|B)$.
b Find $\Pr(B|A)$.

11 **a** Rearrange the rule $\Pr(B|A) = \dfrac{\Pr(A \cap B)}{\Pr(A)}$ to make $\Pr(A \cap B)$ the subject.

b Hence, find $\Pr(A \cap B)$ when $\Pr(B|A) = 0.3$ and $\Pr(A) = 0.6$.

ENRICHMENT: Investment workshops	–	–	12

12 People aged between 20 and 50 years attended a workshop on shares, property or cash at an investment conference. The number of people attending each workshop is shown in this table.

Workshop	20–29 years	30–39 years	40–50 years
Shares	40	85	25
Property	18	57	6
Cash	5	32	61

a How many people attended the conference?
b Find the probability that a randomly selected person at the conference is aged between 30 and 39 years.
c Find the probability that a randomly selected person at the conference attends the property workshop.
d Find the probability that a randomly selected person at the conference attends the property workshop given they are not in the 30–39 age group.
e Find the probability that a randomly selected person at the conference is aged between 40 and 50 years given that they do not attend the cash workshop.
f Find the probability that a randomly selected person at the conference does not attend the shares workshop given they are not in the 30–39 age group.

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8E Two-step experiments using arrays

LEARNING INTENTIONS

- To be able to construct a table to systematically display the outcomes of a two-step experiment
- To understand the difference between 'with replacement' and 'without replacement' and their impact on the possible outcomes of a two-step experiment
- To be able to calculate probabilities from a sample space in a table

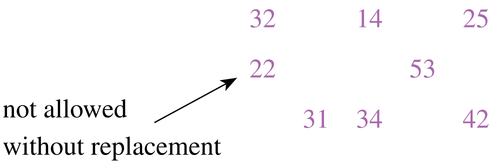
When an experiment involves two or more components, like flipping a coin twice or selecting three chocolates from a box, we are dealing with multi-stage experiments. The outcomes for such an event depend on whether or not they are conducted with or without replacement. For two-step experiments, tables are helpful when listing all the possible outcomes.

Geneticists use 'two-parent' tables to find proportions of inherited traits, such as the maize kernel colours purple (9/16), red (3/16) and white (4/16). Research of inheritance patterns in maize led to revolutionary findings of how chromosomes change during reproduction.

Lesson starter: Does replacement matter?

From the digits $\{1, 2, 3, 4, 5\}$ you select two of these to form a two-digit number.

- How many numbers can be formed if selections are made with replacement?
- How many numbers can be formed if selections are made without replacement?



- Find the probability that the number 35 is formed if selections are made with replacement.
- Find the probability that the number 35 is formed if selections are made without replacement.

KEY IDEAS

- Tables (or arrays) can be used to list the sample space for **two-step experiments**.
- If **replacement** is allowed then outcomes from each selection can be repeated.
- If selections are made **without replacement** then outcomes from each selection cannot be repeated. For example: Two selections are made from the digits $\{1, 2, 3\}$.

With replacement

		1st		
		1	2	3
	1	(1, 1)	(2, 1)	(3, 1)
2nd	2	(1, 2)	(2, 2)	(3, 2)
	3	(1, 3)	(2, 3)	(3, 3)

Without replacement

		1st		
		1	2	3
	1	×	(2, 1)	(3, 1)
2nd	2	(1, 2)	×	(3, 2)
	3	(1, 3)	(2, 3)	×

BUILDING UNDERSTANDING

1 Two letters are chosen from the word DOG.

These tables list the sample space if selections are made:

with replacement

		1st D	O	G
	D	(D, D)	(O, D)	(G, D)
2nd	O	(D, O)	(O, O)	(G, O)
	G	(D, G)	(O, G)	(G, G)

without replacement

		1st D	O	G
	D	×	(O, D)	(G, D)
2nd	O	(D, O)	×	(D, G)
	G	(D, G)	(O, G)	×

a State the total number of outcomes if selection is made:

- **i** with replacement
- **ii** without replacement.

b If selection is made with replacement, find the probability that:

- **i** the two letters are the same
- **ii** there is at least one D
- **iii** there is not an O.

c If selection is made without replacement, find the probability that:

- **i** the two letters are the same
- **ii** there is at least one D
- **iii** there is not an O.

2 Two digits are selected from the set $\{2, 3, 4\}$ to form a two-digit number. Find the number of two-digit numbers that can be formed if the digits are selected:

- **a** with replacement
- **b** without replacement.

Example 10 Constructing a table with replacement

A fair 6-sided die is rolled twice.

a List the sample space, using a table.

b State the total number of outcomes.

c Find the probability of obtaining the outcome (1, 5).

d Find:

- **i** Pr(double)
- **ii** Pr(sum of at least 10)
- **iii** Pr(sum not equal to 7)

e Find the probability of a sum of 12, given that the sum is at least 10.

SOLUTION

a

Roll 1

		1	2	3	4	5	6
	1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
Roll 2	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
	5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
	6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

EXPLANATION

Be sure to place the number from roll 1 in the first position for each outcome.

b 36 outcomes

There is a total of $6 \times 6 = 36$ outcomes.

c $\text{Pr}(1, 5) = \frac{1}{36}$

Only one outcome is (1, 5).

d **i** $\text{Pr}(\text{double}) = \frac{6}{36} = \frac{1}{6}$

Six outcomes have the same number repeated.

ii $\text{Pr}(\text{sum of at least } 10) = \frac{6}{36} = \frac{1}{6}$

Six outcomes have a sum of either 10, 11 or 12.

iii $\text{Pr}(\text{sum not equal to } 7) = 1 - \frac{6}{36} = \frac{5}{6}$

This is the complement of having a sum of 7. Six outcomes have a sum of 7.

e $\text{Pr}(\text{sum of } 12 \mid \text{sum of at least } 10) = \frac{1}{6}$

One of the 6 outcomes with a sum of at least 10 has a sum of 12.

Now you try

A fair 6-sided die is rolled twice.

a Find the probability of obtaining the outcome (6, 4).

b Find:

- **i** Pr((3, 2) or (2, 3))
- **ii** Pr(sum of at least 9)
- **iii** Pr(sum less than 4)

c Find the probability of a sum of 3 given that the sum is at most 4.

Example 11 Constructing a table without replacement

Two letters are chosen from the word KICK without replacement.

a Construct a table to list the sample space.

b Find the probability of:

- **i** obtaining the outcome (K, C)
- **ii** selecting two Ks
- **iii** selecting a K and a C
- **iv** selecting two Ks given that at least one K is selected.

SOLUTION

a

		1st			
		K	I	C	K
	K	×	(I, K)	(C, K)	(K, K)
2nd	I	(K, I)	×	(C, I)	(K, I)
	C	(K, C)	(I, C)	×	(K, C)
	K	(K, K)	(I, K)	(C, K)	×

b **i** $\Pr(K, C) = \frac{2}{12} = \frac{1}{6}$

ii $\Pr(K, K) = \frac{2}{12} = \frac{1}{6}$

iii $\Pr(K \cap C) = \frac{4}{12} = \frac{1}{3}$

iv $\Pr(2 \text{ Ks} \mid \text{at least } 1 \text{ K}) = \frac{2}{10} = \frac{1}{5}$

EXPLANATION

Selection is without replacement, so the same letter (from the same position) cannot be chosen twice.

Two of the 12 outcomes are (K, C).

Two of the outcomes are K and K, which use different Ks from the word KICK.

Four outcomes contain a K and a C.

There are 10 outcomes with at least one K, two of which have two Ks.

Now you try

Two cars are chosen from a group of four without replacement. Of the four cars, two are red (R), one is blue (B) and one is white (W).

a Construct a table to list the sample space.

b Find the probability of:

- **i** obtaining a red car first then a white car
- **ii** selecting two red cars
- **iii** selecting one red car and one blue car
- **iv** selecting two red cars given that at least one of them is a red car.

Exercise 8E

FLUENCY	1–3	1, 3, 4	2–4

Example 10a–d

1 A fair 4-sided die is rolled twice.

- **a** List the sample space, using a table.
- **b** State the total number of possible outcomes.
- **c** Find the probability of obtaining the outcome (2, 4).
- **d** Find the probability of:
  - **i** a double
  - **ii** a sum of at least 5
  - **iii** a sum not equal to 4.

2 Two coins are tossed, each landing with a head (H) or tail (T).

- **a** List the sample space, using a table.
- **b** State the total number of possible outcomes.
- **c** Find the probability of obtaining the outcome (H, T).
- **d** Find the probability of obtaining:
  - **i** one tail
  - **ii** at least one tail.

- **e** If the two coins are tossed 1000 times, how many times would you expect to get two tails?

Example 11

3 Two letters are chosen from the word SET without replacement.

- **a** Show the sample space, using a table.
- **b** Find the probability of:
  - **i** obtaining the outcome (E, T)
  - **ii** selecting one T
  - **iii** selecting at least one T
  - **iv** selecting an S and a T
  - **v** selecting an S or a T.

4 A letter is chosen from the word LEVEL without replacement and then a second letter is chosen from the same word.

- **a** Draw a table displaying the sample space for the pair of letters chosen.
- **b** State the total number of outcomes possible.
- **c** State the number of outcomes that contain exactly one of the following letters.
  - **i** V
  - **ii** L
  - **iii** E
- **d** Find the probability that the outcome will contain exactly one of the following letters.
  - **i** V
  - **ii** L
  - **iii** E
- **e** Find the probability that the two letters chosen will be the same.

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PROBLEM–SOLVING	5, 6	5, 6	6, 7

5 In a quiz, Min guesses that the probability of rolling a sum of 10 or more from two fair 6-sided dice is 10%. Complete the following to decide whether or not this guess is correct.

		Die 1					
		1	2	3	4	5	6
	1	2	3	…			
	2	3	…				
Die 2	3	4					
	4	⋮					
	5	⋮					
	6						

- **a** Copy and complete the table representing all the outcomes for possible totals that can be obtained.
- **b** State the total number of outcomes.
- **c** Find the number of the outcomes that represent a sum of:
  - **i** 3
  - **ii** 7
  - **iii** less than 7.
- **d** Find the probability that the following sums are obtained.
  - **i** 7
  - **ii** less than 5
  - **iii** greater than 2
  - **iv** at least 11
- **e** Find the probability that the sum is at least 10, and decide whether or not Min's guess is correct.

6 A letter is randomly chosen from the word OLD and then a second letter is chosen from the word COLLEGE.

- **a** Draw a table illustrating all possible pairs of letters that can be chosen.
- **b** State the total number of outcomes.
- **c** If a double represents selecting the same letter, find the probability of selecting a double.

7 The 10 students who completed a special flying course are waiting to see if they will be awarded the one Distinction or the one Merit award for their efforts.

- **a** In how many ways can the two awards be given if:
  - **i** the same student can receive both awards?
  - **ii** the same student cannot receive both awards?
- **b** Assuming that a student cannot receive both awards, find the probability that a particular student receives:
  - **i** the Distinction award
  - **ii** the Merit award
  - **iii** neither award.
- **c** Assuming that a student can receive both awards, find the probability that they receive at least one award.

REASONING	8	8, 9	8, 10

Example 10e

8 Two fair 4-sided dice are rolled and the sum is noted.

- **a** Find the probability of:
  - **i** a sum of 5
  - **ii** a sum of less than 6.
- **b**
  - **i** Find the probability of a sum of 5 given that the sum is less than 6.
  - **ii** Find the probability of a sum of 2 given that the sum is less than 6.
  - **iii** Find the probability of a sum of 7 given that the sum is at least 7.

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9 Decide whether the following situations would naturally involve selections with replacement or without replacement.

- **a** selecting two people to play in a team
- **b** tossing a coin twice
- **c** rolling two dice
- **d** choosing two chocolates to eat

10 In a game of chance, six cards numbered 1 to 6 are lying face down on a table. Two cards are selected without replacement and the sum of both numbers is noted.

- **a** State the total number of outcomes.
- **b** Find the probability that the total sum is:
  - **i** equal to 3
  - **ii** equal to 4
  - **iii** at least 10
  - **iv** no more than 5.
- **c** What would have been the answer to part **b i** if the experiment had been conducted with replacement?

ENRICHMENT: Random weights	–	–	11

11 In a gym, Justin considers choosing two weights to fit onto a rowing machine to make the load heavier. There are four different weights to choose from: 2.5 kg, 5 kg, 10 kg and 20 kg, and there are plenty of each weight available. After getting a friend to randomly choose both weights, Justin attempts to operate the machine.

- **a** Complete a table that displays all possible total weights that could be placed on the machine.
- **b** State the total number of outcomes.
- **c** How many of the outcomes deliver a total weight described by the following?
  - **i** equal to 10 kg
  - **ii** less than 20 kg
  - **iii** at least 20 kg
- **d** Find the probability that Justin will be attempting to lift the following total weight.
  - **i** 20 kg
  - **ii** 30 kg
  - **iii** no more than 10 kg
  - **iv** less than 10 kg
- **e** If Justin is unable to lift more than 22 kg, what is the probability that he will not be able to operate the rowing machine?

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Progress quiz

8A **1** A letter is selected at random from the word SUPERCALIFRAGILISTIC. Find the probability that the letter is:

a an S
b an A
c a P
d a vowel
e not a vowel.

8A **2** An experiment involves choosing a card from a standard deck of 52 playing cards and recording its suit. Here are the results of running the experiment 100 times.

Suit	heart	club	spade	diamond
Frequency	17	29	38	16

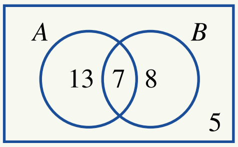
Find the experimental probability of obtaining:

a a heart
b a club
c a red card
d not a spade.

8B **3** From a class of 50 students, 32 enjoy tennis and 20 enjoy squash. If 3 students enjoy both sports:

a display this information in a Venn diagram and a two-way table
b state the number of students who enjoy only tennis
c find the probability that a randomly selected student from the group enjoys neither tennis nor squash.

8B **4** The Venn diagram shows the distribution of elements in two sets, A and B. Find:



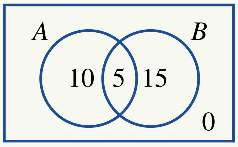
a $n(A \cap B)$
b $n(A)$
c $n(A' \cap B')$
d $\Pr(A \cap B)$
e $\Pr(A')$
f $\Pr(A \cup B)$

8C **5** The probability of a student in a year level owning a scooter is 0.37, the probability of owning a bicycle is 0.78 and the probability of owning both is 0.24. What is the probability that a randomly selected student from the year level owns a scooter or a bicycle?

8C **6** Two events, A and B, are such that $\Pr(A) = 0.7$, $\Pr(B) = 0.45$ and $\Pr(A \cap B) = 0.32$. Find:

a $\Pr(A \cup B)$
b $\Pr(A' \cap B')$

8D **7** Consider the Venn diagram displaying the number of elements belonging to events A and B. Find:



a $\Pr(A)$
b $\Pr(A \cap B)$
c $\Pr(A | B)$
d $\Pr(B | A)$

8D **8** From a survey of 40 students in the playground, 35 like soft drink, 20 like water and 15 like both soft drink and water.

a Display this information in a two-way table.
b Find the probability that a student chosen from the group dislikes soft drink given that they like water.

8E **9** Three cards numbered 1, 2 and 3 are placed in a hat.

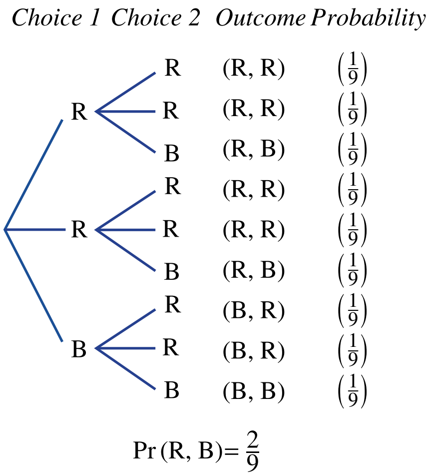
a Two cards are chosen, with replacement, and their product noted. Use a table to find the probability that the product is even.
b Two cards are chosen, without replacement. Find the probability of obtaining the outcome (3, 1).

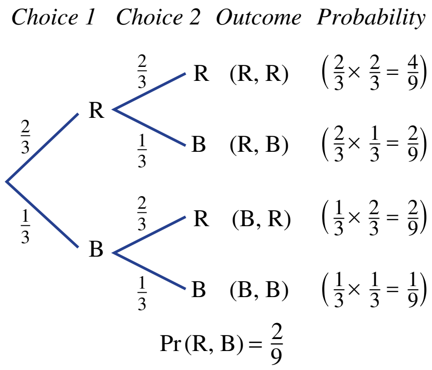
8F Using tree diagrams

LEARNING INTENTIONS

- To be able to draw a tree diagram to show the sample space of two or more stage experiments
- To understand when it is appropriate to use a tree diagram to display outcomes
- To know how to determine probabilities on tree diagram branches using with or without replacement
- To be able to find probabilities of event outcomes using a tree diagram

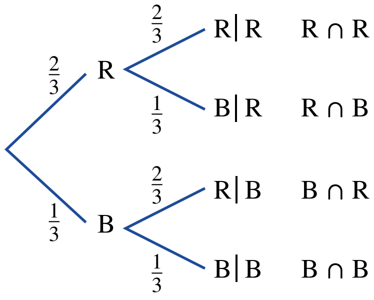
Suppose a bag contains two red counters and one blue counter and that two counters are selected at random with replacement. One way to display the outcomes is with a tree diagram in which all equally likely outcomes are listed in columns, as shown below left. A more efficient way, however, is to group similar outcomes and write their corresponding probabilities on the branches, as shown below right.





You will note that in the tree diagram on the right the probability of each outcome is obtained by multiplying the branch probabilities. The reason for this relates to conditional probabilities.

Using conditional probabilities, the tree diagram above right can be redrawn like this (right).



We know from conditional probability that:

- $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

Using B and R we could write:

- $\Pr(R|B) = \dfrac{\Pr(B \cap R)}{\Pr(B)}$

By rearranging we have:
$$\begin{aligned}\Pr(B \cap R) &= \Pr(B) \times \Pr(R|B)\\ &= \frac{1}{3} \times \frac{2}{3}\\ &= \frac{2}{9}\end{aligned}$$

This explains why we multiply branches on tree diagrams. This also applies when selection is made without replacement.

Lesson starter: Prize probability

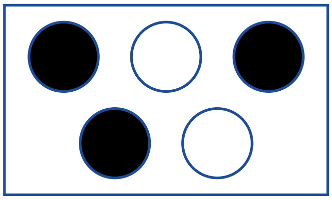
Two lucky door prizes are randomly awarded to a group of 7 adult and 3 child partygoers.

- Use a tree diagram with branch probabilities to show how selection with replacement can be displayed.
- Use a tree diagram with branch probabilities to show how selection without replacement can be displayed.
- Which of these situations has a higher probability?
  - **a** An adult and a child receive one prize each if selection is made with replacement.
  - **b** An adult and a child receive one prize each if selection is made without replacement.

KEY IDEAS

- **Tree diagrams** can be used to list the sample space for experiments involving two or more stages.
  - Branch probabilities are used to describe the chance of each outcome at each step.
  - Each outcome for the experiment is obtained by multiplying the branch probabilities.
  - Branch probabilities will depend on whether selection is made with or without replacement.
  - If more than one outcome meets a criteria, sum the probabilities.

BUILDING UNDERSTANDING



1 A box contains 2 white (W) and 3 black (B) counters.

a A single counter is drawn at random. Find the probability that it is:

i white **ii** black.

b Two counters are now drawn at random. The first one is replaced before the second one is drawn. Find the probability that the second counter is:

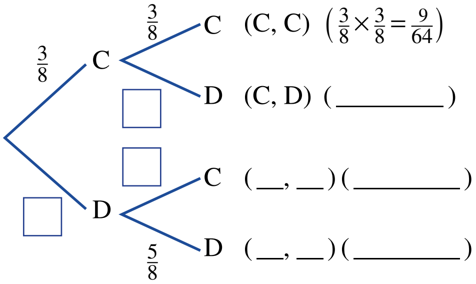
i white **ii** black.

c Two counters are drawn and the first counter is not replaced before the second one is drawn. If the first counter is white, find the probability that the second counter is:

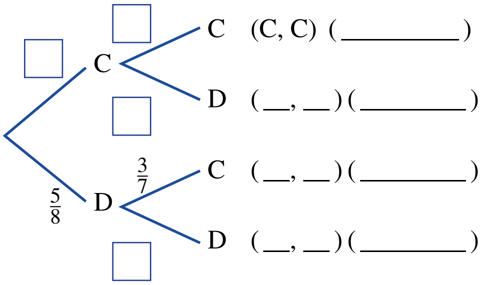
i white **ii** black.

2 Two prizes are awarded to a group of 3 cats (C) and 5 dogs (D) for best in show. State the missing components in these tree diagrams.

a with replacement



b without replacement



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Example 12 Constructing a tree diagram for multi-stage experiments

Boxes A and B contain 4 counters each. Box A contains 2 red and 2 green counters and box B contains 1 red and 3 green counters. A box is chosen at random and then a single counter is selected.

a If box A is chosen, what is the probability that a red counter is chosen from it?
b If box B is chosen, what is the probability that a red counter is chosen from it?
c Represent the options available as a tree diagram that shows all possible outcomes and related probabilities.
d What is the probability of selecting box B and a red counter?
e What is the probability of selecting a red counter?

SOLUTION

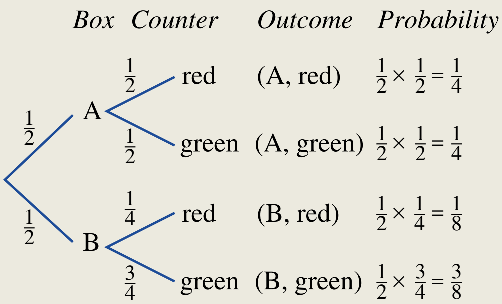
a Pr(red from box A) $= \frac{2}{4} = \frac{1}{2}$

Two of the 4 counters in box A are red.

b Pr(red from box B) $= \frac{1}{4}$

One of the 4 counters in box B is red.

c



First selection is a box followed by a counter. Multiply each of the probabilities along the branch pathways to find the probability of each outcome.

d
$$\begin{aligned}\text{Pr(B, red)} &= \frac{1}{2} \times \frac{1}{4}\\ &= \frac{1}{8}\end{aligned}$$

The probability of choosing box B is $\frac{1}{2}$ and a red counter from box B is $\frac{1}{4}$, so multiply these probabilities for the outcome (B, red).

e
$$\begin{aligned}\text{Pr(1 red)} &= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4}\\ &= \frac{1}{4} + \frac{1}{8}\\ &= \frac{3}{8}\end{aligned}$$

The outcomes (A, red) and (B, red) both contain 1 red counter, so add together the probabilities for these two outcomes.

Now you try

Boxes A and B contain 5 counters each. Box A contains 3 red and 2 green counters and box B contains 1 red and 4 green counters. A box is chosen at random and then a single counter is selected.

a If box A is chosen, what is the probability that a red counter is chosen from it?
b If box B is chosen, what is the probability that a red counter is chosen from it?
c Represent the options available as a tree diagram that shows all possible outcomes and related probabilities.
d What is the probability of selecting box B and a red counter?
e What is the probability of selecting a red counter?

Example 13 Using a tree diagram for experiments without replacement

A bag contains 5 blue (B) and 3 white (W) marbles and 2 marbles are selected without replacement.

a Draw a tree diagram showing all outcomes and probabilities.

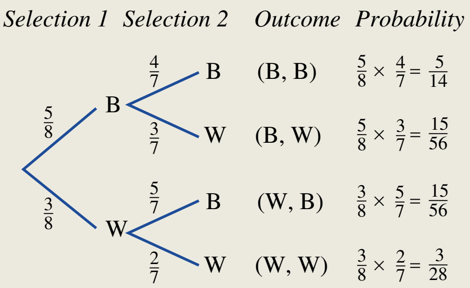
b Find the probability of selecting:

- **i** a blue marble followed by a white marble (B, W)
- **ii** 2 blue marbles
- **iii** exactly 1 blue marble.

c If the experiment is repeated with replacement, find the answers to each question in part **b**.

SOLUTION

a



b **i**
$$\Pr(B, W) = \frac{5}{8}\times\frac{3}{7} = \frac{15}{56}$$

ii
$$\Pr(B, B) = \frac{5}{8}\times\frac{4}{7} = \frac{5}{14}$$

iii
$$\Pr(1 \text{ blue}) = \frac{5}{8}\times\frac{3}{7}+\frac{3}{8}\times\frac{5}{7} = \frac{30}{56} = \frac{15}{28}$$

EXPLANATION

After 1 blue marble is selected there are 7 marbles remaining: 4 blue and 3 white.

After 1 white marble is selected there are 7 marbles remaining: 5 blue and 2 white.

Multiply the probabilities on the (B, W) pathway.

Only 4 blue marbles remain after the first selection. Multiply the probabilities on the (B, B) pathway.

The outcomes (B, W) and (W, B) both have one blue marble. Multiply probabilities to find individual probabilities, then sum for the final result.

c **i** $\Pr(B, W) = \frac{5}{8} \times \frac{3}{8}$
$= \frac{15}{64}$

When selecting objects with replacement, remember that the number of marbles in the bag remains the same for each selection.

That is, $\Pr(B) = \frac{5}{8}$ and $\Pr(W) = \frac{3}{8}$ throughout.

ii $\Pr(B, B) = \frac{5}{8} \times \frac{5}{8}$
$= \frac{25}{64}$

iii $\Pr(1 \text{ blue}) = \frac{5}{8} \times \frac{3}{8} + \frac{3}{8} \times \frac{5}{8}$
$= \frac{30}{64}$
$= \frac{15}{32}$

One blue marble corresponds to the (B, W) or (W, B) outcomes.

Now you try

A bag contains 4 blue (B) and 5 white (W) marbles and 2 marbles are selected without replacement.

a Draw a tree diagram showing all outcomes and probabilities.

b Find the probability of selecting:

- **i** a blue marble followed by a white marble (B, W)
- **ii** 2 blue marbles
- **iii** exactly 1 blue marble.

c If the experiment is repeated with replacement, find the answers to each question in part **b**.

Exercise 8F

FLUENCY	1–3	1, 3, 4	2–4

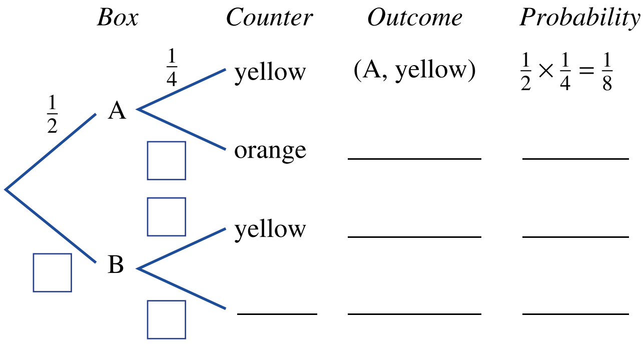
Example 12

1 Boxes A and B contain 4 counters each. Box A contains 1 yellow and 3 orange counters and box B contains 3 yellow and 1 orange counter. A box is chosen at random and then a single counter is selected.

a If box A is chosen, what is the probability of selecting a yellow counter?

b If box B is chosen, what is the probability of selecting a yellow counter?

c Represent the options available by completing this tree diagram.

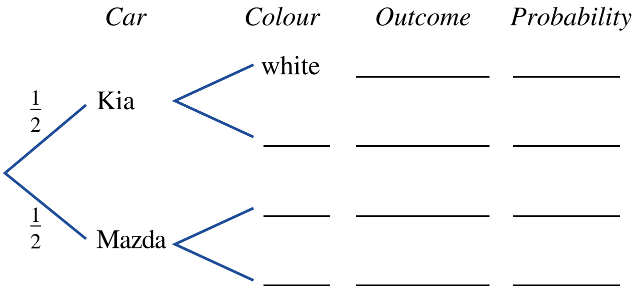


d What is the probability of selecting box B and a yellow counter?

e What is the probability of selecting a yellow counter?

2 As part of a salary package an employee randomly selects a Kia or a Mazda. There are 3 white Kias and 1 silver Kia and 2 white Mazdas and 1 red Mazda to choose from.

a Complete a tree diagram showing a random choice of a car make and then a colour.



b Find the probability that the employee chooses:

- **i** a white Kia
- **ii** a red Mazda
- **iii** a white car
- **iv** a car that is not white
- **v** a silver car or a white car
- **vi** a car that is neither a Kia nor red.

Example 13

3 A bag contains 4 red (R) and 2 white (W) marbles, and 2 marbles are selected without replacement.

a Draw a tree diagram showing all outcomes and probabilities.

b Find the probability of selecting:

- **i** a red marble and then a white marble (R, W)
- **ii** 2 red marbles
- **iii** exactly 1 red marble.

c The experiment is repeated with replacement. Find the answers to each question in part **b**.

4 Two animals are selected from a group of 3 rabbits (R) and 4 guinea pigs (G) without replacement.

a Draw a tree diagram to find the probability of selecting:

- **i** 2 rabbits
- **ii** 2 guinea pigs
- **iii** 1 rabbit and 1 guinea pig
- **iv** 2 animals either both rabbits or both guinea pigs.

b The experiment is repeated with replacement. Find the answers to each question in part **a**.

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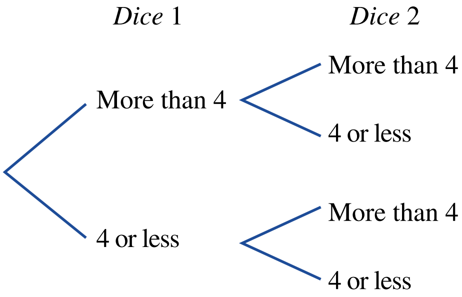
PROBLEM–SOLVING			
	5, 6	5, 7	7, 8

5 Two dice are rolled and it is noted when the dice show a number more than 4 or not more than 4.

a Complete a tree diagram, showing the outcomes of the two dice.



b Find the probability that the two dice are:

i both more than 4
ii both 4 or less
iii not both 4 or less
iv one 4 or less and one more than 4.

6 Two bottles of wine are randomly selected for tasting from a box containing 2 red and 2 white wines. Use a tree diagram to help answer the following.

a If the first bottle is replaced before the second is selected, find:

i Pr(2 red)
ii Pr(1 red)
iii Pr(not 2 white)
iv Pr(at least 1 white)

b If the first bottle is not replaced before the second is selected, find:

i Pr(2 red)
ii Pr(1 red)
iii Pr(not 2 white)
iv Pr(at least 1 white)

7 Cans of sliced peaches produced by 'Just peaches' are sometimes underweight. A box of 10 cans is selected from the factory and then 2 cans from the 10 are tested without replacement. This particular box of 10 cans is known to have 2 cans that are underweight.

a State the probability that the first can chosen will be:

i underweight
ii not underweight.

b Use a tree diagram to find the probability that:

i both cans are underweight
ii one can is underweight
iii at most 1 can is underweight.

c The factory passes the inspection if no cans are found to be underweight. Find the chance that this will occur and express your answer as a percentage, rounded to one decimal place.

8 The probability of rain on any particular day is 0.2. However, the probability of rain on a day after a rainy day is 0.85, whereas the probability of rain on a day after a non-rainy day is 0.1.

a On two consecutive days, find the probability of having:

i two rainy days
ii exactly one rainy day
iii at least one dry day.

b On three consecutive days, find the probability of having:

i three rainy days
ii exactly one dry day
iii at most two rainy days.

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REASONING	9, 10	9–11	10–12

9 Two socks are selected at random from a drawer containing 4 red and 4 yellow socks.

a Find the probability that the two socks will be of the same colour if the socks are drawn without replacement.

b Find the probability that the two socks will not be of the same colour if the socks are drawn without replacement.

10 Two drinks are chosen from a supply of 100 cola cans and 120 lemonade cans.

a Find the probability correct to two decimal places of selecting, without replacement:

- **i** 2 cola cans
- **ii** 2 lemonade cans
- **iii** 1 of each flavour.

b Repeat part **a** with replacement.

c What do you notice about your answers to parts **a** and **b**? Why do you think this is the case?

11 A box contains 2 red (R) and 3 blue (B) counters and three counters are selected without replacement.

a Use a tree diagram to find:

- **i** Pr(R, R, B)
- **ii** Pr(2 red)
- **iii** Pr(3 red)
- **iv** Pr(at least 1 red)
- **v** Pr(at most 2 blue)

b If a fourth selection is made without replacement, find the probability that:

- **i** at least 1 red is selected
- **ii** 3 blue are selected.

12 Containers A, B and C hold 4 marbles each, all of which are the same size. The following table illustrates the marble colours in each container.

	Container A	Container B	Container C
Purple	1	2	3
Green	3	2	1

A container is chosen at random and then a marble is selected from the container.

a Draw a tree diagram to help determine all the possible outcomes and the associated probabilities. *Suggestion*: You will need three branches to start (representing the three different containers that can be chosen), followed by two branches for each of A, B and C (to represent the choice of either a purple or a green marble).

b State the total number of outcomes.

c Find the following probabilities.

- **i** Pr(A, purple)
- **ii** Pr(B, green)
- **iii** Pr(C, purple)

d Find the probability of obtaining a green marble.

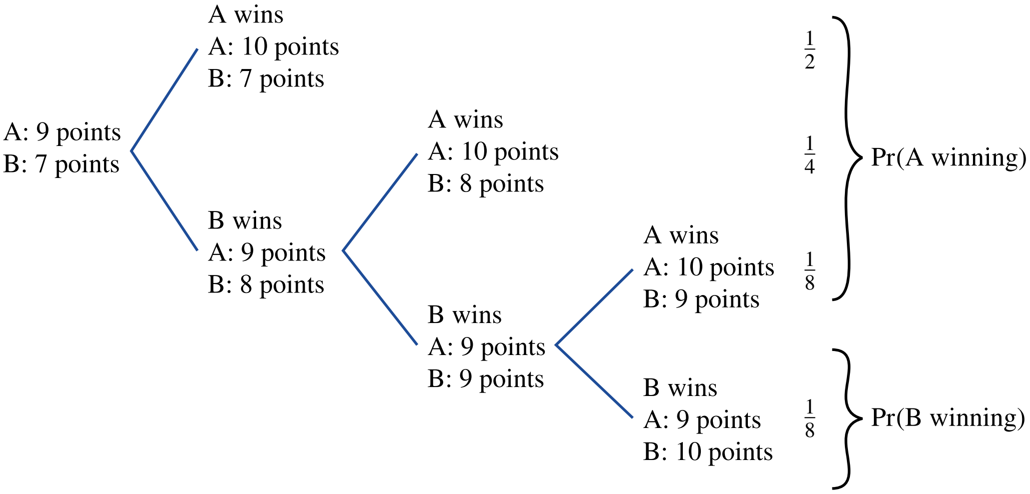
ENRICHMENT: Fermat and Pascal

– – 13

13 The French mathematicians Pierre de Fermat and Blaise Pascal inspired the development of mathematical probability through their consideration of simple games. Here's one of their first problems.

> Two equally skilled people play a game in which the first to earn 10 points wins \$100 and each player has an equal chance of winning a point. At some point in the game, however, one of the players has to leave and the game must be stopped. If the game score is 9 points to 7, how should the \$100 be divided between the two players?

This diagram shows the number of ways the game could have been completed.



- **a** Use this diagram to help calculate the probability that:
  - **i** player A wins the game
  - **ii** player B wins the game.
- **b** Based on your answers from part **a**, describe how the \$100 should be divided between players A and B.
- **c** Investigate how the \$100 should be divided between players A and B if the game is stopped with the following number of points. You will need to draw a new tree diagram each time.
  - **i** player A: 8 points, player B: 7 points
  - **ii** player A: 7 points, player B: 7 points
  - **iii** player A: 8 points, player B: 6 points
  - **iv** player A: 6 points, player B: 7 points
- **d** Choose your own pair of game points and investigate.

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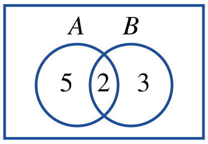
8G Independent events

LEARNING INTENTIONS
- To understand what it means for two events to be independent
- To be able to determine mathematically if two events are independent
- To know that selections made with replacement will be independent



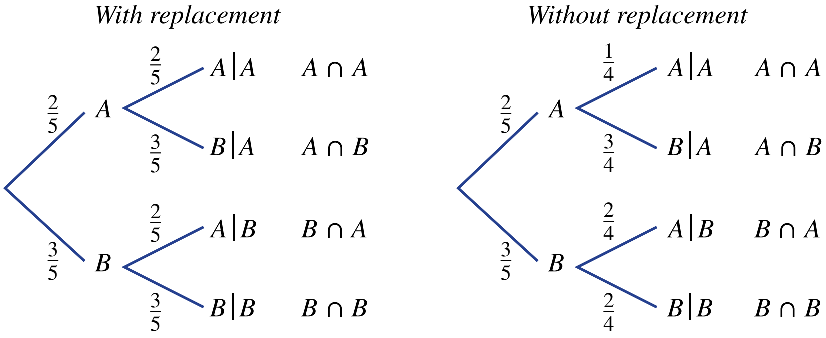
In previous sections we have looked at problems involving conditional probability. This Venn diagram, for example, gives the following results.

$$\Pr(A) = \frac{7}{10} \quad \text{and} \quad \Pr(A\,|\,B) = \frac{2}{5}.$$

Clearly the condition B in $\Pr(A\,|\,B)$ has changed the probability of A. The events A and B are therefore not independent.

For multiple events we can consider events either with or without replacement.

These tree diagrams, for example, show two selections of marbles from a bag of 2 aqua (A) and 3 blue (B) marbles.



In the first tree diagram we can see that $\Pr(A\,|\,B) = \Pr(A)$, so the events are independent. In the second tree diagram we can see that $\Pr(A\,|\,B) \neq \Pr(A)$, so the events are not independent.

So, for independent events we have:

$\Pr(A\,|\,B) = \Pr(A) \qquad (*)$

This implies that $\Pr(A\,|\,B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$ becomes $\Pr(A) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$, using (*).

Then rearranging gives:

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$.

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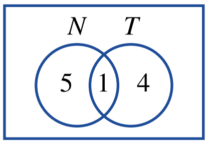
Lesson starter: Are they independent?

Recall that two events are independent if the outcome of one event does not affect the probability of the other event. Discuss whether or not you think the following pairs of events are independent. Give reasons.

- Tossing two coins with the events:
  - getting a tail on the first coin
  - getting a tail on the second coin.
- Selecting two mugs without replacement from a drawer in which there are 3 red and 2 blue mugs and obtaining the events:
  - first is a blue mug
  - second is a red mug.
- Selecting a person from a group of 10 who enjoys playing netball (N) and/or tennis (T), as in the Venn diagram shown.
  - selecting a person from the group who enjoys netball
  - selecting a person from the group who enjoys tennis.



KEY IDEAS

■ Two events are **independent** if the outcome of one event does not change the probability of obtaining the other event.
- $\Pr(A|B) = \Pr(A)$ or $\Pr(B|A) = \Pr(B)$
- $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

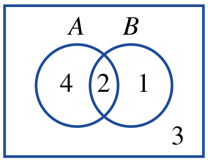
■ For multi-stage experiments with selection made **with replacement**, successive events are independent.

■ For multi-stage experiments with selection made **without replacement**, successive events are not independent.

BUILDING UNDERSTANDING

1. A fair coin is tossed twice. Let A be the event 'the first toss gives a tail' and let B be the event 'the second toss gives a tail'.
   - **a** Find:
     - **i** $\Pr(A)$
     - **ii** $\Pr(B)$
   - **b** Would you say that events A and B are independent?
   - **c** What is $\Pr(B|A)$?

2. This Venn diagram shows the number of elements in events A and B.
   - **a** Find:
     - **i** $\Pr(B)$
     - **ii** $\Pr(B|A)$
   - **b** Is $\Pr(B|A) = \Pr(B)$?
   - **c** Are the events A and B independent?



3. Complete each sentence.
   - **a** For multi-stage experiments, successive events are independent if selections are made ________ replacement.
   - **b** For multi-stage experiments, successive events are not independent if selections are made ________ replacement.

Example 14 Checking for independent events

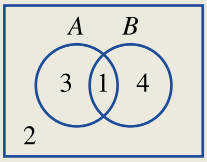
A selection of 10 mobile phone offers includes four with free connection and five with a free second battery, whereas one offer has both free connection and a free second battery.
Let A be the event 'choosing a mobile phone with free connection'.
Let B be the event 'choosing a mobile phone with a free second battery'.

a Summarise the information about the 10 mobile phone offers in a Venn diagram.

b Find:

  i $\Pr(A)$ **ii** $\Pr(A \mid B)$

c State whether or not the events A and B are independent.

SOLUTION	**EXPLANATION**
a 	Start with the 1 element that belongs to both sets A and B and complete the diagram according to the given information.
b **i** $\Pr(A) = \frac{4}{10}$ $= \frac{2}{5}$	4 of the 10 elements belong to set A.
ii $\Pr(A \mid B) = \frac{1}{5}$	1 of the 5 elements in set B belongs to set A.
c The events A and B are not independent.	$\Pr(A \mid B) \neq \Pr(A)$

Now you try

A selection of 14 hotel offers includes 8 with free Wifi and 9 with a free breakfast, whereas 3 offer both free Wifi and a free breakfast.
Let A be the event 'choosing a hotel with free Wifi'.
Let B be the event 'choosing a hotel with a free breakfast'.

a Summarise the information about the 14 hotel offers in a Venn diagram.

b Find:

  i $\Pr(A)$ **ii** $\Pr(A \mid B)$

c State whether or not the events A and B are independent.

Exercise 8G

FLUENCY	1, 2, 3–4(½)	1, 3–4(½)	2, 3–4(½)

Example 14

1 A selection of 8 offers for computer printers includes 3 with a free printer cartridge and 4 with a free box of paper, whereas 2 have both a free printer cartridge and a free box of paper.
Let A be the event 'choosing a printer with a free printer cartridge'.
Let B be the event 'choosing a printer with a free box of paper'.

a Summarise the given information about the 8 computer printer offers in a Venn diagram.

b Find:

i $\Pr(A)$ **ii** $\Pr(A \mid B)$

c State whether or not the events A and B are independent.

2 A selection of 6 different baby strollers includes 3 with a free rain cover, 4 with a free sun shade, and 2 offer both a free rain cover and a free sun shade.
Let A be the event 'choosing a stroller with a free sun shade'.
Let B be the event 'choosing a stroller with a free rain cover'.

a Summarise the given information about the 6 baby strollers in a Venn diagram.

b Find:

i $\Pr(A)$ **ii** $\Pr(A \mid B)$

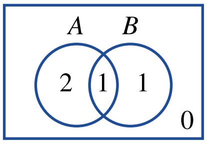
c State whether or not the events A and B are independent.

3 From events A and B in the given Venn diagrams:

i Find $\Pr(A)$ and $\Pr(A \mid B)$.

ii Hence, decide whether or not events A and B are independent.

a



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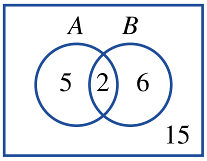
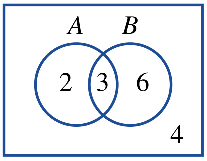


c



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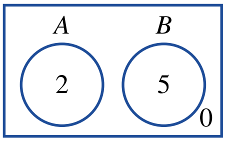

4 For the events A and B, with details provided in the given two-way tables, find $\Pr(A)$ and $\Pr(A \mid B)$ and decide whether or not the events A and B are independent.

a

	A	A'	
B	1	1	2
B'	3	3	6
	4	4	8

b

	A	A'	
B	1	3	4
B'	2	4	6
	3	7	10

c

	A	A'	
B	3	17	20
B'	12	4	16
	15	21	36

d

	A	A'	
B	1		9
B'			
	5		45

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PROBLEM–SOLVING	5	5, 6	6, 7

5 Of 17 leading accountants, 15 offer advice on tax (T), whereas 10 offer advice on business growth (G). Eight of the accountants offer advice on both tax and business growth. One of the 17 accountants is chosen at random.

a Use a Venn diagram or two-way table to help find:

i $\Pr(T)$ **ii** $\Pr(T \text{ only})$ **iii** $\Pr(T|G)$

b Are the events T and G independent?

6 A fair coin is tossed 5 times. Find the probability of obtaining:

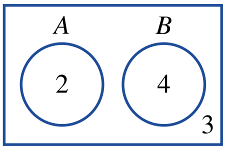
a 5 heads **b** at least 1 tail **c** at least 1 head.

7 A fair 6-sided die is rolled three times. Find the probability that the sum of the three dice is:

a 18 **b** 3 **c** 4 **d** 5

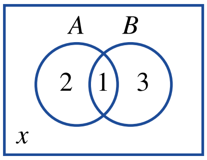
REASONING	8	8, 9	8, 9

8 Use this diagram to help decide if this statement is true or false. If two events, A and B, are mutually exclusive then they are also independent.

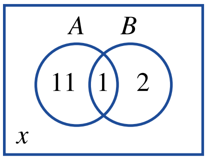


9 Consider the events A and B with the number of elements contained in each event given in the Venn diagrams below. In each case, find the value of x so that the events A and B are independent; i.e. $\Pr(A) = \Pr(A|B)$.

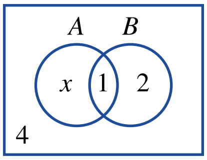
a



b



c



ENRICHMENT: Independence and the addition rule	–	–	10, 11

10 For two independent events A and B, recall that $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$. Two independent events A and B are such that $\Pr(A) = 0.6$ and $\Pr(B) = 0.4$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$

11 For two independent events A and B, we are given $\Pr(A \cup B) = 0.9$ and $\Pr(A) = 0.4$. Find $\Pr(B)$.

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The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

TV news popularity

1 In a country town, viewers have a choice between a 6 p.m. news bulletin on Channel A and another on Channel B. Two independent research companies take a sample of 40 households in the town regarding their news viewing habits over the course of a month.

In both surveys, 3 households didn't watch a 6 p.m. news at all, while 5 households said they watched both channels over the course of the month.

You wish to analyse the given sample data to investigate the news viewing habits of a number of people.

- **a** In the first survey, the number of households that watch only Channel B is 3 times the number of households that watch only Channel A. Complete a diagram or table to determine how many households watched only Channel A.
- **b** In the second survey, twice as many households watched Channel B from time to time compared to Channel A. How many households claimed to watch Channel B?

In the city there are three channels with a 6 p.m. news bulletin, Channels A, B and C. A survey of 400 households showed that over the course of a month, 30 households watched channels A and B but not C, 21 watched channels A and C but not B, 33 watched channels B and C only. 158 households in total watched Channel C, 190 watched Channel B and 55 watched Channel A only. 43 households did not watch a 6 p.m. news bulletin.

- **c** From the survey results, calculate the probability that a randomly selected household watched Channel B only.

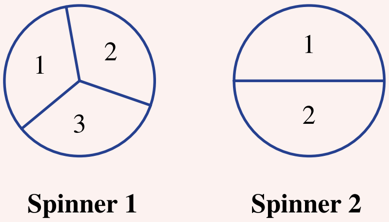
Spin to win

2 In a game at a fair, two spinners are spun and their numbers multiplied together to produce a total. The game costs \$5 to play. An odd total sees you win \$10, while an even total means that you lose your money.

The manager of the game wants to explore its profitability by calculating various probabilities. He also wishes to make adjustments to the game so that the chances of winning can be increased in the hope that more people might play.

- **a** Consider the game which has the two spinners shown.
  - **i** Complete the table below using a tree diagram and use it to determine the probability of winning \$10.

Total	1	2			6
Probability					



Spinner 1 **Spinner 2**

ii Hence, fill in the table for the profit from one game. What outcome does the game favour?

Profit	\$5	–\$5
Probability		

iii In one particular game the player wins. What is the probability they spun an even number on the first spinner?

Word spreads around the fair that not many people are winning at the game. To get customers back, the manager decides to make the game a 50% chance of winning. He leaves the first spinner as is and the second spinner is adjusted to have the odd and even number in unequal portions.

b Let p be the proportion of the second spinner occupied by the odd number.
  i Determine the probability of achieving an odd total and an even total in terms of p.
  ii For the game to be fair (equal chance of win or lose), what should be the value of p?

c Design a pair of spinners such that:
  i the probability of an odd total and an even total is equal
  ii the probability of an odd total is twice the probability of an even total.

d If p is the proportion of the first spinner that has odd numbers and q is the proportion of the second spinner that has odd numbers, what is the requirement for pq for parts **c i** and **ii**?

Rolling 6s

3 A simple dice game involves three rolls of a regular 6-sided die. Points are awarded as follows:

- 10 points for a 6 on the first roll
- 6 points for a 6 on the second roll
- 2 points for a 6 on the third roll
- 5 point bonus if all three rolls are 6s.

You are to investigate the probability of obtaining a certain number of points using a fair die and then reconsider the game if a biased die is used.

a Consider the following game probabilities.
  i What is the probability of obtaining each of 10 points in a round, 6 points in a round and 2 points in a round?
  ii What is probability of obtaining 16 points in a round?
  iii What is the probability of obtaining the maximum 23 points?
  iv Two 6s occur in the three rolls of a dice. What is the probability of this occurring and what are the possible points obtained?
  v A competitor needs to score at least one point in the last round of the game to win. What is the probability that they win?

b A brother challenges his sister to another game but with a biased die. This die has a probability of p of obtaining a 6. Answer in terms of p in parts **i**–**ii**.
  i What is the probability of rolling no 6s in a round with this die?
  ii Hence, what is the probability of scoring points in a round?
  iii If the probability of obtaining maximum points in a round is $\frac{1}{27}$, what is the value of p?
  iv If the probability of obtaining exactly 16 points in a round is 0.032 while the probability of obtaining exactly 10 points is 0.128, determine the value of p.

8H Counting principles and factorial notation OPTIONAL

LEARNING INTENTIONS
- To understand the addition and multiplication principles used in counting
- To be able to apply the addition and multiplication principles in simple counting problems
- To know factorial notation and be able to compute numerical expressions involving factorials

When solving more complex problems in the topic of probability, we may need to count a large number of possible outcomes. For example, if we wanted to know the probability of choosing three particular people from a group of twenty, we would first need to find the total number of ways that any combination of three can be chosen. In this case the number is 1140. Rather than counting each possible outcome one by one, we can use counting techniques, sometimes called combinatorics, to quickly arrive at such a result. Such techniques often involve the product of all integers from some number n down to 1. This is called n factorial, denoted by $n!$

Lesson starter: To add or to multiply?

Hungry Harry chooses a pizza from nine different meat options and seven different vegetarian options. He also considers five dessert options.

- If Harry were to calculate the total number of pizza options (meat or vegetarian), would he use addition or multiplication with the given numbers to find the result?
- If Harry were to calculate the total number of ways he could choose one pizza and one dessert, would he use addition or multiplication to find the number of pizza and dessert options?
- Harry arranges four different cheese topping options in four bowls on his table. How many ways can he arrange the four bowls in a row? How can factorial notation be used to express this answer?

KEY IDEAS

- The **addition principle**
  - If there are A ways of performing one task and B ways of performing a different task, then there is a total of A + B ways of performing any one of the two tasks.
  - Example: If I want to choose one drink from a choice of five fruit juice options or six soft drink options, then there are $5 + 6 = 11$ ways that I could choose a drink.
- The **multiplication principle**
  - If there are A ways of performing one task and B ways of performing a different task, then there are A × B ways of performing both tasks.
  - Example: If I want to choose one drink from a choice of five fruit juice options and one drink from a choice of six soft drink options, then there are $5 \times 6 = 30$ ways of choosing one combination of a fruit juice drink and a soft drink.
- In general, if the problem involves ‘or’ use the addition principle and if it involves ‘and’ use the multiplication principle. Some problems may require both.
- ***n* factorial:** $n! = n \times (n-1) \times (n-2) \times \ldots . \times 3 \times 2 \times 1$, where n is a positive integer
  - Example: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
  - Note that $0! = 1$

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BUILDING UNDERSTANDING

1 Some yellow and green counters are placed on a table, and all are labelled with a different number. There are four yellow counters and seven green counters.

- **a** Imagine one of the counters is to be chosen from either of the yellow or green counters.
  - **i** How many choices are available in total?
  - **ii** Was the addition principle or the multiplication principle used to help find your answer to part **a i**?
- **b** Now imagine one counter being chosen from the yellow group of counters and one also being chosen from the green group of counters.
  - **i** How many different combinations of the two counters are possible?
  - **ii** Was the addition principle or the multiplication principle used to help find your answer to part **b i**?

2 Evaluate the following.

- **a** $3 \times 2 \times 1$
- **b** $4 \times 3 \times 2 \times 1$
- **c** $5 \times 4 \times 3 \times 2 \times 1$
- **d** $\dfrac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$

Example 15 Using the addition and multiplication principles

For the following situations decide if the addition or multiplication principle would be used to find the total number of choices. Then find the total number of choices.

- **a** Andy wishes to choose one book from a small library display. The display includes eight paperback books and five hardcover books.
- **b** Andy wishes to choose one scarf from a group of five different scarfs and one hat from a group of three different hats before he goes for a walk.

SOLUTION	EXPLANATION
a Addition principle $8 + 5 = 13$ choices	Andy is choosing one item from either the set of paperback books or the hardcover books, so the total number of options is obtained using addition.
b Multiplication principle $5 \times 3 = 15$ choices	Andy is choosing one item from each group and for each single choice of scarf there are three choices of hat, so the total number of options is obtained using multiplication.

Now you try

For the following situations decide if the addition or multiplication principle would be used to find the total number of choices. Then find the total number of choices.

- **a** Rebecca wishes to choose one painting from a group of six and one photograph from a group of three to help decorate a room.
- **b** Rebecca wishes to choose one indoor pot plant from a nursery which has nine real plant options and four plastic plant options.

Example 16 Evaluating factorials

Evaluate the following.

a $4!$ **b** $\frac{6!}{3!}$ **c** $\frac{5!}{0!}$

SOLUTION	EXPLANATION
a $4! = 4 \times 3 \times 2 \times 1 = 24$	Use $n! = n \times (n-1) \times (n-2) \times \ldots. \times 3 \times 2 \times 1$
b $\frac{6!}{3!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$ $= 120$	Write both factorials in expanded form, then cancel the common factors to simplify.
c $\frac{5!}{0!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{1}$ $= 120$	Write $5!$ in expanded form and use $0! = 1$.

Now you try

Evaluate the following.

a $3!$ **b** $\frac{5!}{3!}$ **c** $\frac{4!}{0!}$

Exercise 8H

FLUENCY	1–3, 4(½)	1–2(½), 3, 4(½)	2(½), 3, 4(¼)

Example 15a

1 Find how many choices of TV shows I could watch if I choose one show from the following.

a 6 dramas, 3 comedies
b 2 horrors, 4 science fiction
c 5 comedies, 7 murder mysteries, 4 documentaries
d 3 thrillers, 10 children shows, 11 dramas

Example 15b

2 Find how many ways I can choose a shirt and tie combination including one shirt and one tie from the following.

a 3 shirts, 2 ties
b 6 shirts, 3 ties
c 10 shirts, 5 ties
d 20 shirts, 6 ties

3 For the following situations decide if the addition or multiplication principle would be used to find the total number of choices. Then find the total number of choices.

a Sisi wishes to choose one seat on a plane. There are twenty window seats and 14 aisle seats available.

b Sisi can choose one of three main courses and one of two desserts on the plane.

Example 16

4 Evaluate the following.

a $3!$ **b** $5!$ **c** $2!$ **d** $6!$

e $\frac{4!}{3!}$ **f** $\frac{7!}{4!}$ **g** $\frac{6!}{2!}$ **h** $\frac{5!}{1!}$

i $\frac{2!}{0!}$ **j** $\frac{4!}{0!}$ **k** $\frac{0!}{1!}$ **l** $\frac{0!}{3!}$

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PROBLEM–SOLVING

5, 6	5–7	6–8

5 Walt wants to climb a tree in his backyard which contains six oak trees, two gum trees and four peppercorn trees. How many choices of tree does Walt have?

6 Magda is buying a shirt and a jacket. There are ten different shirt options and six different jacket options to choose from. In how many ways can she purchase a shirt and a jacket?

7 Remi needs to buy three lights for a bedroom including one wall light, one down light and one pendant. There are four different wall lights available, three down lights available and five pendant lights available. How many different combinations are possible of the three lights?

8 If symbols can be used more than once, how many different passwords can be created with:

a 4 digits selected from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
b 3 letters selected from A, B, C, ….., Z
c 6 Greek letters selected from $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$

REASONING

9, 10	9, 10, 11(½)	10, 11

9 Mattaes has three rabbits and five guinea pigs in a pen in his backyard and chooses one of each type of animal to show his friends. Mattaes' father says that there are $3 + 5 = 8$ ways of doing this. Is he correct? Explain why or why not.

10 Kevin can travel from city A to city B by using either four different road options, two different flying options and three different train options. He calculates a total of $4 \times 2 \times 3 = 24$ options. Is he correct? Explain why or why not.

11 A expression such as $\frac{7!}{5!}$ can be simplified in the following way.

$$\begin{aligned}\frac{7!}{5!} &= \frac{7 \times 6 \times 5!}{5!} \\ &= 7 \times 6 \qquad \text{(by cancelling the 5!)} \\ &= 42\end{aligned}$$

Use this method to simplify the following without the use of a calculator.

a $\frac{5!}{3!}$ **b** $\frac{10!}{9!}$ **c** $\frac{30!}{29!}$ **d** $\frac{100!}{98!}$

ENRICHMENT: Proofs with factorials

–	–	12–14

12 Prove without a calculator that $\frac{100!}{2!98!} \times \frac{197!}{198!} = 25$.

13 Prove without a calculator that $\frac{8!(x+3)!}{7!(x+2)!} = 8x + 24$ assuming x is a positive integer.

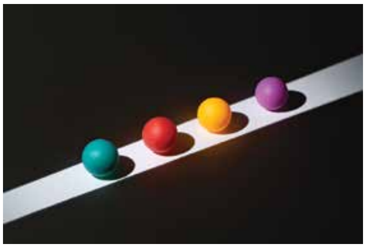
14 Prove that if a is a positive integer greater than 3 then $\frac{a!}{(a-3)!} \times \frac{1}{a^2 - a} = a - 2$.

8I Arrangements OPTIONAL

LEARNING INTENTIONS

- To understand the concept of arrangements (permutations)
- To be able to use factorial notation to find the number of ways objects can be arranged
- To be able to apply counting techniques to find the total number of arrangements given certain conditions

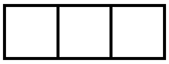
When a set of objects is ordered in a particular way, we say that an arrangement or permutation is formed. If four objects A, B, C and D were to be arranged in a row, then two possible arrangements could be ADBC or BCAD. Clearly each arrangement depends on the order. In this case there are in fact 24 possible arrangements for the four objects. We will see in this section how factorials can be used to count the total number of arrangements in an efficient way.



Lesson starter: Arranging Amanda, Betty and Callum

Imagine arranging three people Amanda (A), Betty (B) and Callum (C) in a row of three seats.

- Using the letters A, B and C, list all the possible seating arrangements.
- Now find the total number of arrangements in a different way by filling in these boxes and using the following instructions.



  - In the first box on the left find the number of ways that the first seat can be filled by the three people.
  - In the second box find the number of ways that it can be filled knowing that one person is no longer available.
  - In the third box find the number of ways that it can be filled knowing that two people are no longer available.
- Using the three numbers in the boxes, decide if the addition or multiplication principle should be used to determine the total number of arrangements of the three people.
- How could factorial notation be used to describe the total number of arrangements in this case?
- Use this technique to find the total number of ways that four people could be arranged in a row.

KEY IDEAS

- An **arrangement** (or permutation) is a particular ordering of a set of objects.
- If n objects are arranged in a row there are $n!$ ways that this can be done.
- In general, if r objects chosen from n are arranged in a row, this is denoted as ${}^nP_r$ where the P stands for permutation and ${}^nP_r = \frac{n!}{(n-r)!}$

  For example: ${}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 60$

- A method using boxes can be used to find the total number of arrangements with or without certain restrictions such as the following:
  - *Arranging all elements of a group*. For example, the total number of ways four objects can be arranged in a row. This is the same as $^4P_4 = 4! = 24$.

    $$\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 4 \times 3 \times 2 \times 1 = 4! = 24$$

  - *Arranging a given number of elements of a group*. For example, the total number of ways three objects chosen from five can be arranged in a row. This is the same as $^5P_3 = 60$.

    $$\boxed{5}\boxed{4}\boxed{3} = 5 \times 4 \times 3 = 60$$

  - *Arranging elements of a group if a given number of them must be together*. For example, the total number of ways that five objects can be arranged in a row if two of them must be together. The two objects that must be together are firstly treated as one, then we multiply by the number of ways that these two can be arranged.

    $$\boxed{4}\boxed{3}\boxed{2}\boxed{1} \times \boxed{2}\boxed{1} = 4! \times 2! = 48$$

  - *Arranging elements of a group if a given number of them are identical.* For example, the number of ways that five objects can be arranged in a row if three of them are identical. We divide by the number of ways of arranging the three identical objects as each of these arrangements do not count as a different option.

    $$\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} \div \boxed{3}\boxed{2}\boxed{1} = \frac{5!}{3!} = 20$$

BUILDING UNDERSTANDING

① Complete the following to find the total number of ways five people can be arranged in a row.

$$\boxed{5}\boxed{4}\boxed{}\boxed{}\boxed{} = 5! = \underline{\qquad}$$

② Complete the following to find the total number of ways two objects chosen from four can be arranged in a row.

$$\boxed{}\boxed{3} = \underline{\qquad}$$

③ Complete the following to find the total number of ways four objects can be arranged in a row if two of them must be together.

$$\boxed{}\boxed{2}\boxed{} \times \boxed{}\boxed{1} = \underline{\qquad} \times \underline{\qquad} = \underline{\qquad}$$

④ Complete the following to find the total number of ways six objects can be arranged in a row if three of them are identical.

$$\frac{\boxed{6}\boxed{}\boxed{4}\boxed{}\boxed{}\boxed{}}{\boxed{3}\boxed{}\boxed{}} = \frac{6!}{3!} = \underline{\qquad}$$

Example 17 Finding the total number of arrangements

Find the total number of ways of arranging the following.

a Five objects in a row.

b Three objects in a row chosen from seven.

SOLUTION

a $\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 5! = 120$

b $\boxed{7}\boxed{6}\boxed{5} = 7 \times 6 \times 5$

$= 210$

EXPLANATION

There are five ways to fill the first position, four ways to fill the second position and so on. Use the multiplication principle to find the total number of arrangements.

There are seven ways to fill the first position, six ways to fill the second position and five ways to fill the third position. The are no positions remaining.

Alternatively, use ${}^7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 210$

Now you try

Find the total number of ways of arranging the following.

a Six objects in a row.

b Two objects in a row chosen from five.

Example 18 Finding arrangements with restrictions

Find the total number of ways of arranging the following.

a Six objects if two of them are to be together.

b Seven objects if three of them are identical.

SOLUTION

a $\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} \times \boxed{2}\boxed{1} = 5! \times 2!$

$= 240$

b $$\frac{\boxed{7}\boxed{6}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1}}{\boxed{3}\boxed{2}\boxed{1}} = \frac{7!}{3!}$$

$$= \frac{5040}{6}$$

$$= 840$$

EXPLANATION

Treat the two objects that need to be together as one object, then arrange five objects rather than six. Then multiply by the number of ways that the two objects, that are together, can be arranged.

First treat every object as unique and find the total number of ways that they can be arranged (7!). Then divide by the total number of ways that the identical objects can be arranged (3!).

Now you try

Find the total number of ways of arranging the following.

a Five objects if three of them are to be together.

b Eight objects if four of them are identical.

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Exercise 8I

FLUENCY	1–2($\frac{1}{2}$), 3, 4	1–2($\frac{1}{2}$), 3–5	1–2($\frac{1}{4}$), 5

Example 17a

1 Find the total number of ways of arranging the following.

a Three objects in a row.

b Seven objects in a row.

c Nine objects in a row.

d Twelve objects in a row.

Example 17b

2 Find the total number of ways of arranging the following.

a Three objects in a row chosen from six.

b Four objects in a row chosen from five.

c Four objects in a row chosen from seven.

d Six objects in a row chosen from ten.

3 In how many ways can:

a eight books be arranged on a shelf?

b three books be arranged on a shelf chosen from eight?

4 In how many ways can:

a the letters of the word PENCIL be arranged in a row?

b four letters chosen from the word PENCIL be arranged in a row?

5 How many ways can the first, second and third places be filled by 10 runners in a sprint race?

PROBLEM–SOLVING	6, 7	6–8	7–9

Example 18a

6 Find the total number of ways of arranging the following.

a four objects if two of them are to be together

b eight objects if three of them are to be together

7 How many ways can eight children be seated in a row if two of them, Lara and Polly, must sit together?

Example 18b

8 Find the total number of ways of arranging the following.

a five objects if three of them are identical

b nine objects if five of them are identical

9 How many ways can I arrange twelve Christmas cards on a table in a row if six of them are identical?

REASONING	10	10, 11	10–12

10 Imagine creating a password of three digits chosen from the integers 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. How many passwords are possible if:

- **a** the digits cannot be reused?
- **b** the digits can be reused?

11 How many ways can the letters of the word STATISTICS be arranged? Note the sets of identical letters.

12 How many ways can I arrange four books and five magazines in a row if:

- **a** the books must be together?
- **b** the books are grouped together and the magazines are grouped together?

ENRICHMENT: How many numbers?	–	–	13–15

13 How many three-digit numbers can be formed if digits are selected from 1, 2, 3, 4 and 5 and the following conditions are satisfied?

- **a** a digit cannot be used more than once
- **b** a digit can be used more than once

14 If no digit can be used more than once, how many numbers can be formed from the digits 2, 3, 4 and 5 which are:

- **a** two-digit numbers?
- **b** four-digit numbers?
- **c** greater than 300 (include both three- and four-digit possibilities)?

15 How many numbers greater than 300 can be formed by choosing any number of digits from 1, 2, 3, 4 and 5, if no digit can be used more than once?

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8J Selections OPTIONAL

LEARNING INTENTIONS

- To understand the concept of selections (combinations)
- To be able to use factorial notation to find the number of ways objects can be selected from a group
- To be able to apply counting techniques to find the total number of selections under given conditions

Previously we learned that when arranging objects in a row, the order is important. For example, when arranging two of the letters A, B or C, we know that AB is a different arrangement compared to BA. However, when selecting two of the letters A, B or C, then AB is the same combination as BA. The order is therefore not important when we are forming combinations of objects selected from a group, such as selecting two lollipops from a choice of four.

Lesson starter: Arranging two from four

Imagine selecting two people chosen from four, Aly (A), Beatrice (B), Cole (C) and Doris (D), to form a pair of two who will go on a hike together.

- How many ways could you arrange two of the people from the four?
- Now list the selections (combinations) of two chosen from the four people.
- Compare the two answers above. What number do you divide by to obtain the number of selections of two people starting with the number of arrangements of two people?
- Repeat the above tasks for the situation involving three people chosen from five.
- Can you come up with a formula for the number of ways you can choose r people from n?

KEY IDEAS

- A **selection** (or combination) is a particular grouping of a set of objects where the order is not important.
- The number of selections of r objects chosen from n is equal to the number of arrangements of r objects chosen from n divided by $r!$ (the number of ways r objects can be arranged in a row).
- The number of ways of choosing r objects from n is denoted ${}^nC_r$ or $\binom{n}{r}$.
- ${}^nC_r = {}^nP_r \div r! = \dfrac{n!}{r!(n-r)!}$

BUILDING UNDERSTANDING

1. Two letters are chosen from the four letters A, B, C and D.
   - **a** List all the arrangements (permutations) of the two letters.
   - **b** List all the selections (combinations) of the two letters.
   - **c** Using boxes or the rule ${}^nP_r = \dfrac{n!}{(n-r)!}$, confirm your answer to part **a**.

d Using ${}^nP_r \div r!$, confirm your answer to part **b**.

e Using the rule ${}^nC_r = \dfrac{n!}{r!(n-r)!}$, confirm your answer to part **b**.

2 Three letters are chosen from the four letters A, B, C and D.

a Using boxes or the rule ${}^nP_r = \dfrac{n!}{(n-r)!}$, find the number of arrangements (permutations) of the three letters.

b List the selections (combinations) of the three letters.

c Using ${}^nP_r \div r!$, confirm your answer to part **b**.

d Using the rule ${}^nC_r = \dfrac{n!}{r!(n-r)!}$, confirm your answer to part **b**.

Example 19 Evaluating ${}^nC_r$ or $\binom{n}{r}$

Evaluate the following.

a ${}^4C_2$ **b** ${}^8C_6$ **c** $\binom{3}{3}$ **d** $\binom{7}{6}$

SOLUTION	EXPLANATION
a $\begin{aligned}{}^4C_2 &= \frac{4!}{2!2!}\\ &= \frac{24}{4}\\ &= 6\end{aligned}$	${}^nC_r = \dfrac{n!}{r!(n-r)!}$ ${}^4C_2 = \dfrac{4!}{2!(4-2)!}$
b $\begin{aligned}{}^8C_6 &= \frac{8!}{6!2!}\\ &= 28\end{aligned}$	${}^nC_r = \dfrac{n!}{r!(n-r)!}$ ${}^8C_6 = \dfrac{8!}{6!(8-6)!}$
c $\begin{aligned}\binom{3}{3} &= {}^3C_3\\ &= \frac{3!}{3!0!}\\ &= 1\end{aligned}$	$\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ $\binom{3}{3} = \dfrac{3!}{3!(3-3)!}$
d $\begin{aligned}\binom{7}{6} &= {}^7C_6\\ &= \frac{7!}{6!1!}\\ &= 7\end{aligned}$	$\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ $\binom{7}{6} = \dfrac{7!}{6!(7-6)!}$

Now you try

Evaluate the following.

a ${}^5C_2$ **b** ${}^7C_4$ **c** $\binom{4}{4}$ **d** $\binom{10}{8}$

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Example 20 Finding the number of possible selections

How many different selections of three books can be chosen from six?

SOLUTION

$$\begin{aligned} {}^6C_3 &= \frac{6!}{3!3!} \\ &= 20 \end{aligned}$$

EXPLANATION

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

$${}^6C_3 = \frac{6!}{3!(6-3)!}$$

Alternatively use

${}^6P_3 \div 3!$ or $\boxed{6}\boxed{5}\boxed{4} \div 3! = 20$

Now you try

How many different selections of four books can be chosen from seven?

Example 21 Making selections from two groups

In a lost animals home, two cats are chosen from five available and three dogs are chosen from four available to distribute to five families. How many ways can the two cats and three dogs be chosen?

SOLUTION

$$\begin{aligned} {}^5C_2 \times {}^4C_3 &= \frac{5!}{2!3!} \times \frac{4!}{3!1!} \\ &= 10 \times 4 \\ &= 40 \end{aligned}$$

EXPLANATION

Using the multiplication principle take the product of the number of selections of cats with the number of selections of dogs.

Now you try

In a shop, Nadia selects three shirts from seven available and two belts from six available. How many ways can the three shirts and two belts be selected?

Exercise 8J

FLUENCY	1–4	1–2(½), 3–5	1–2(¼), 5

Example 19a, b

1 Use ${}^nC_r = \frac{n!}{r!(n-r)!}$ to evaluate the following.

a ${}^5C_3$ **b** ${}^8C_6$ **c** ${}^{12}C_1$ **d** ${}^9C_9$

Example 19c, d

2 Use $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to evaluate the following.

a $\binom{6}{4}$ **b** $\binom{11}{7}$ **c** $\binom{12}{12}$ **d** $\binom{15}{1}$

Example 20

3 How many different selections of two cows can be chosen from seven available?

4 How many different teams of eleven soccer players can be selected from a squad of fifteen?

5 How many different ways can three pizza toppings be selected from nine options?

PROBLEM–SOLVING	6, 7	6, 7	6–8

Example 21

6 A pencil case contains five different pencils and six different pens. How many ways can the following be selected?

- **a** one pencil and one pen
- **b** two pencils and one pen
- **c** two pencils and three pens

7 A hockey squad contains nine defenders and twelve attackers. How many different ways can a team of eleven players be selected including six defenders and five attackers?

8 Use factorials to evaluate the following without a calculator.

a $^{50}C_{49}$ **b** $^{200}C_{198}$ **c** $^{10}C_7$

REASONING	9	9, 10	9–11

9 Explain why:

a $^{n}C_1 = n$ **b** $^{n}C_0 = 1$ **c** $^{n}C_n = 1$

10 A book contains eight chapters including three essays and five stories. How many ways can three different chapters be selected:

- **a** without restriction?
- **b** if one essay and two stories are selected?

11 A television set up offers twelve free-to-air channels and five streaming services. How many different ways can the TV be utilised if:

- **a** one free-to-air channel and one streaming service are selected?
- **b** either two free-to-air or two streaming services are selected?

ENRICHMENT: Choosing any number of options	–	–	12–14

12 To find the number of ways you can choose any number of objects from a group of four, we would calculate $^{4}C_0 + {}^{4}C_1 + {}^{4}C_2 + {}^{4}C_3 + {}^{4}C_4 = 1 + 4 + 6 + 4 + 1 = 16 = 2^4$.

- **a** Find the total number of ways of choosing any number of objects from a group of five.
- **b** Give an expression in terms of n for the total number of ways of choosing any number of objects.

13 How many ways can I choose any number of students from a group of six to represent the school at a competition, assuming that at least one person is chosen?

14 Four students from a group of six Year 11 students and ten Year 12 students are to be selected to attend a leadership conference. How many ways can this be achieved if:

- **a** there is no restriction?
- **b** there must be two Year 11 students and two Year 12 students?
- **c** there must be at least two Year 12 students?

8K Applications of counting in probability OPTIONAL

LEARNING INTENTIONS
- To understand that counting techniques can be used to find the number of elements in a sample space
- To be able to solve probability problems involving arrangements using counting techniques
- To be able to solve probability problems involving selections using counting techniques

To solve a probability problem, we first need to determine the number of elements in the sample space. For example, to find the chance of two particular people being randomly chosen from a group of five, we first find the total number of ways any two people can be chosen from five. We use counting techniques, in this case selections, to achieve this.

Lesson starter: Triple scoop ice-cream

Miriam chooses a stack of three different flavoured scoops chosen from five available for a triple scoop ice-cream.

- How many different ways can the three scoops be arranged on a cone?
- What is the probability that one particular arrangement of the three scoops is provided?
- How many different combinations of the three scoops are available?
- What is the probability that one particular combination of the three scoops is provided?

KEY IDEAS

- Counting techniques are used to find the number of elements in the sample space in a probability problem.
- $\Pr(\text{Event}) = \dfrac{\text{Number of favourable outcomes}}{\text{Number of elements in the sample space}}$

BUILDING UNDERSTANDING

1. Two songs are chosen from a playlist of six songs then played in a particular order.
   - **a** How many different arrangements of the two songs are possible?
   - **b** What is the probability that one particular arrangement of the two songs is played?
2. Four different paintings are to be hung in a row on a wall.
   - **a** How many different arrangements of the four paintings are possible?
   - **b** What is the probability that one particular arrangement of the four paintings is hung?
3. Three movies are to be chosen from a group of seven for a weekend movie binge.
   - **a** How many different combinations of the three movies are possible?
   - **b** What is the probability that one particular selection of the three movies is chosen?

Example 22 Finding probabilities involving arrangements

Six different trees are available for an arrangement of trees in a row along a fence line. Find the probability that:

a one particular arrangement of six trees is planted along the fence

b one particular arrangement of four of the six trees is planted along the fence.

SOLUTION	EXPLANATION
a Number of arrangements $= 6! = 720$ $\Pr(\text{one arrangement}) = \frac{1}{720}$	First find the number of elements in the sample space. $\boxed{6}\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1} = 6! = 720$ The number of favourable outcomes is 1 and the number of elements in the sample space is 720.
b $\boxed{6}\boxed{5}\boxed{4}\boxed{3} = 6 \times 5 \times 4 \times 3 = 360$ Number of arrangements is 360. $\Pr(\text{one arrangement}) = \frac{1}{360}$	First find the total number of ways four trees can be arranged chosen from six. Then find the probability of one of the outcomes occurring.

Now you try

Melba has five rocks in her stone collection arranged in a row on her display shelf. Find the probability that:

a one particular arrangement of the five rocks can be made on her shelf

b one particular arrangement of two of the five rocks is arranged on her shelf.

Example 23 Finding probabilities involving selections

A car dealer has four different cars and three different 4WDs in her showroom. She selects two of them to place in the window for this week's window display. Find the probability that:

a a combination of any two of the seven vehicles is chosen

b a combination of one car and one 4WD is chosen.

SOLUTION	EXPLANATION
a ${}^7C_2 = 21$ $\Pr(\text{Any two vehicles}) = \frac{1}{21}$	First find the total number of ways of choosing two vehicles from seven. ${}^7C_2 = \frac{7!}{2!5!}$ The number of favourable outcomes is 1 and the number of elements in the sample space is 21.
b ${}^4C_1 \times {}^3C_1 = 12$ $\Pr(\text{One car and one 4WD}) = \frac{1}{12}$	First find the total number of ways of choosing one car and one 4WD. Then find the probability knowing that only one outcome is favourable.

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Now you try

A florist chooses three bouquets to display in a shop window and can choose from five fresh flower bouquets and six native bouquets. Find the probability that:

a a combination of any three of the eleven bouquets is chosen

b a combination of one fresh flower and two native bouquets is chosen.

Exercise 8K

FLUENCY	1–4	1, 3–5	2–5

Example 22

1 Seven different posters are available to be pinned on a wall and arranged in a row. Find the probability that:

a one particular arrangement of the seven posters is pinned on the wall

b one particular arrangement of three posters chosen from the seven is pinned on the wall.

2 To form a single row pattern on a newly renovated bathroom wall, five different feature tiles are available. Find the probability that:

a one particular arrangement of five different tiles is arranged on the wall

b one particular arrangement of three of the five different tiles is arranged on the wall.

3 A display of eight different gold coins is available for a three-coin display in an antique shop. Find the probability that one particular arrangement of three of the eight coins is displayed.

Example 23

4 Of twelve basketball players in a squad, eight are experienced and four are beginners. A combination of five is chosen to start a particular game. Find the probability that:

a a combination of any five of the twelve players is chosen

b a combination of three experienced players and two beginners are chosen.

5 Of nine nominated teachers and students for a committee, four are teachers and five are students. A committee of three is chosen. Find the probability that:

a a combination of any three of the nine people is chosen

b a combination of one staff member and two students are chosen.

PROBLEM-SOLVING	6, 7	6–8	7–9

6 Three letter words are formed by choosing letters, without replacement, from the word FISHERMAN. Find the probability that the word HIM is formed.

7 Five young children are seated in a row and two of them must sit together. Find the probability that one particular arrangement of the five children is seated.

8 Of seven computers arranged in a row on a shop desk, three of them are identical. Find the probability that one particular arrangement is chosen for the shop desk.

9 Six fish are randomly selected from a tank which holds five gold fish and eleven black fish. Each fish also has its own distinct markings. Find the probability of selecting:

- **a** any particular combination of six of the sixteen fish in the tank
- **b** two particular gold fish and four particular black fish
- **c** no gold fish and six particular black fish.

REASONING	10	10, 11	10–12

10 Four-digit codes are formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. Find the probability that the code 1234 is formed if:

- **a** the digits cannot be reused
- **b** the digits can be reused.

11 Two digits chosen without replacement from 1, 2, 3, 4, 5 and 6 are to form a two-digit number. Find the probability that:

- **a** the number 62 is formed
- **b** the numbers 24 or 42 are formed.

12 Two marbles are selected without replacement from a bag containing three clear, four mix-coloured and two grey marbles. Find the probability that both marbles are:

- **a** mix-coloured
- **b** grey
- **c** clear or grey
- **d** not grey.

ENRICHMENT: Pet play centre	–	–	13, 14

13 A pet store chooses four pets at a time that are placed into a pen for children to play with. They choose from five different rabbits and eight different cats. Find the probability that on one particular day the children will play with:

- **a** just rabbits
- **b** two rabbits and two cats
- **c** at least three rabbits
- **d** at most one cat.

14 The pet store owner can create three, four or five-digit pins using digits from 1, 2, 3, 4, 5 and 6 for his security door, where digits cannot be used more than once. Each pin can be thought of as a number. For example, the selection 4, 3 and 6 is the number 436. Find the probability that:

- **a** a three-digit pin will be the number 524
- **b** a four-digit pin will be the number 2614
- **c** a four-digit pin will be greater than the number 3000
- **d** a four or five-digit pin will greater than the number 6000.

Modelling

Witnessing the northern lights

Wallace travels to Alaska in the hope of seeing an aurora called the northern lights. An aurora is a light show caused by collisions between electrically charged particles. He takes part in a tour which ventures out for 10 nights. Wallace is hoping to see the lights at least 7 times.

The company that operates the 10-night adventure can show statistically that the probability of witnessing the lights on any given night is 0.5.

Present a report for the following tasks and ensure that you show clear mathematical workings and explanations where appropriate.

Preliminary task

a On average how many times can you expect Wallace to see the lights across the 10 nights?

b Estimate with a guess the probability that Wallace will see the lights at least 7 times.

c Use a spinner (with two halves: Yes and No) or a die (Yes is 1–3 and No is 4–6) or a random number generator to simulate a 10-night adventure and count the number of times the lights are observed.

Modelling task

Formulate

a The problem is to use simulation to help estimate the probability that Wallace will see the northern lights at least 7 times. Write down all the relevant information that will help solve this problem.

b Describe the type of random number tool you will use and how it will be used in the simulation to produce the results.

Solve

c Use your random number tool to simulate the 10-night adventure counting the number of times the lights are observed.

d Repeat to generate results for a total of twenty 10-night adventures. Use a table like the one shown below to record your results.

Number of times lights are seen on each 10-night adventure	0	1	2	3	4	5	6	7	8	9	10
Tally											
Frequency											

e Use your results to determine the experimental probability that Wallace will see the northern lights at least 7 times.

Evaluate and verify

f Try to improve your results from parts **c–e** by increasing the number of times the 10-night adventure is simulated.

g Compare your estimate probability with the theoretical value of 0.172 correct to three decimal places.

Communicate

h Summarise your results and describe any key findings.

Extension question

a Investigate how changing the 0.5 probability of seeing lights on any given night changes the structure of the simulation and the results obtained.

Simulating queues

Key technology: Programming and spreadsheets

The average time waiting in a queue depends on a range of factors. For example, if there is a single queue in a coffee shop, the time that you spend in the queue will depend on the time that you arrive at the queue and how long each person in the queue takes to be served once they reach the front of the queue. While the mathematics of queuing systems can be become quite complex, it is possible to run simulations to collect experimental data and make solid predictions.

1 Getting started

Imagine a queue at a coffee shop where the time taken for the next person to arrive after the previous customer (*a* minutes) is somewhere between 1 and 3 minutes (with all times equally likely) and the time taken to be served once the customer is at the front of the queue (*b* minutes) is somewhere between 1 and 2 minutes (with all times equally likely). Let's now suppose that these times for the next four customers are as follows. Note that the *a* value for customer 1 means that they arrive 3 minutes after the opening of the shop and that there is nobody already in the queue.

Customer	*a* minutes	*b* minutes
1	3	2
2	1	1.5
3	2	1
4	3	2

- **a** How long does it take after the previous customer for the following customers to arrive at the queue?
  - **i** customer 2
  - **ii** customer 4
- **b** How long does it take for the following customers to be served once they are at the start of the queue?
  - **i** customer 1
  - **ii** customer 3
- **c** How long does customer 1 have to wait in the queue?
- **d** How long does customer 2 have to wait in the queue given that they arrive 1 minute after customer 1 arrives? Note that customer 1 takes 2 minutes to be served.
- **e** Use one of the following techniques to create a random number between 1 and 3. Then try to generate a random number between 1 and 2.

Spreadsheet: = RAND()*(3–1)+1   CAS: rand().(3–1)+1

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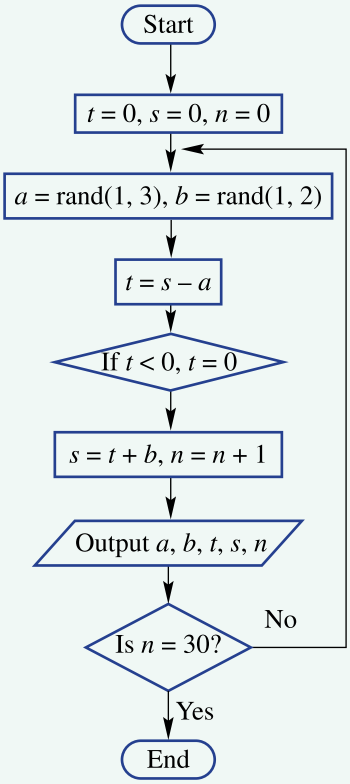
2 Applying an algorithm

First define some variables:

- a = time taken for the next person to arrive after the previous customer
- b = time taken to be served once at the front of the queue
- t = time waiting in queue for each customer
- s = total time waiting and being served for each customer
- n = the number of customers

Here is a flowchart which shows how the algorithm could work for 30 customers after a shop has opened.



a Run through the algorithm shown in the flowchart to find the values of the above variables for each customer. Use this table and complete at least 6 rows. Round all decimals to one decimal place. For each a and b value use a random number generator shown in section 1 and round to one decimal place.

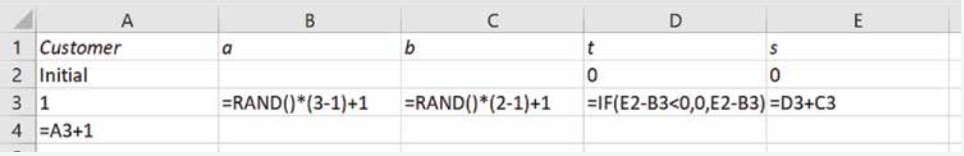
Customer	a min	b min	t min	s min	n
Initial	–	–	0	0	0
1					
2					
3					
4					
5					
6					

b Find the average time the first 6 customers have to wait in the queue; i.e. find the average of t.

c Explain why the line 'If $t < 0, t = 0$' is inserted into the algorithm.

3 Using technology

Here is a spreadsheet that executes the algorithm.



	A	B	C	D	E
1	*Customer*	*a*	*b*	*t*	*s*
2	Initial			0	0
3	1	=RAND()*(3-1)+1	=RAND()*(2-1)+1	=IF(E2-B3<0,0,E2-B3)	=D3+C3
4	=A3+1				

a Set up this spreadsheet and fill down from cells A4, B3, C3, D3 and E3. Fill down for 30 customers.

b Use the syntax = AVERAGE(D3:D32) to find the average wait time (t) in the queue for the 30 customers.

c Use Shift F9 to recalculate and run the simulation again and again. Record the average wait time for each simulation.

d Repeat part **c** for a total of 10 simulations and count how many of the simulations the customer will have to wait in the queue for more than 0.1 minutes (6 seconds). Hence calculate the probability that a customer will have to wait in the queue for more than 0.1 minutes (6 seconds).

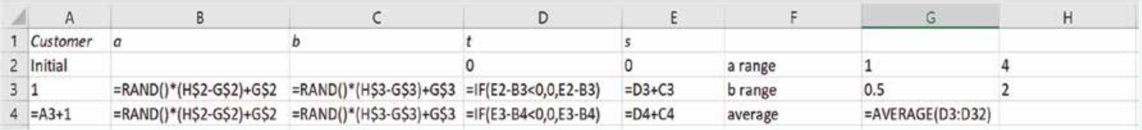
e Use your spreadsheet to experiment with the range of values that are possible for a and b.

  i Describe what happens to the average time waiting in the queue (t) if the range of a values (the time taken for the next person to arrive after the previous customer) is increased or decreased.

  ii Describe what happens to the average time waiting in the queue (t) if the range of b values (the time taken to be served once at the front of the queue) is increased or decreased.

4 Extension

a Modify your spreadsheet so the range limits for a and b can be inputted and modify the formulas referring to a and b. See below then experiment by changing these limits.



	A	B	C	D	E	F	G	H
1	*Customer*	*a*	*b*	*t*	*s*			
2	Initial			0	0	a range	1	4
3	1	=RAND()*(H$2-G$2)+G$2	=RAND()*(H$3-G$3)+G$3	=IF(E2-B3<0,0,E2-B3)	=D3+C3	b range	0.5	2
4	=A3+1	=RAND()*(H$2-G$2)+G$2	=RAND()*(H$3-G$3)+G$3	=IF(E3-B4<0,0,E3-B4)	=D4+C4	average	=AVERAGE(D3:D32)	

b Use your updated spreadsheet to find:

  i a possible range of a and b values which mean that customers rarely needed to wait in a queue at all

  ii a possible range of a and b values which mean that customers need to wait more than 5 minutes on average.

c Write pseudocode for the above algorithm/flowchart but add the following features:

- input range limits for the variables a and b
- the running total of the t values
- the average of the t values after the looping has finished.

d Use a programming language like Python to program the above algorithm. Use your program to calculate the experimental probability that a customer will have to wait for more than a certain amount of time.

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From London to Paris on the Eurostar

On a special work assignment, Helena is to be paid £100 per hour for every hour she spends in Paris after travelling from London on the Eurostar.

Helena is waiting for a train at St Pancras Station in London and is placed on standby; she is not guaranteed a definite seat. If there is no place on a given train, then she waits to see if there is a seat on the next train.

This information is provided for Helena at the station.

Train	Probability of a place	Cost
7 a.m.	$\frac{1}{2}$	£320
8 a.m.	$\frac{2}{3}$	£200
9 a.m.	$\frac{3}{4}$	£150
No further trains	–	£0

Standby

a Illustrate the information given, using a tree diagram and showing all Helena's options and the probabilities for the three trains. (Note: All branch paths might not be the same length.)

b Find the probability that Helena will catch the following trains.

i 7 a.m. **ii** 8 a.m. **iii** 9 a.m.

c What is the probability that Helena will miss all the available trains?

Maximising income

a In terms of pure financial gain, which train is the most desirable for Helena to catch? Remember that she is paid £100 for each extra hour she is in Paris.

b How much money would Helena need to earn per hour on the work assignment if the following trains were the most financially desirable to catch?

i 7 a.m. **ii** 9 a.m.

Expected cost

a Tabulate the cost of travel of each outcome and its corresponding probability using your results from part **b** in the **Standby** section above.

Cost	£320	£200	£150	£0
Probability				

b By finding the sum of the product of each cost and its corresponding probability, find Helena's expected (average) cost for train travel from London to Paris.

c If Helena repeats this journey on 20 occasions, what would be her expected total cost?

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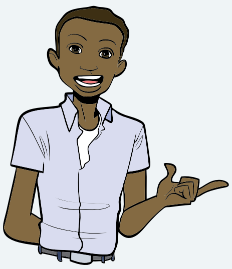
Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



1 A women's tennis match is won by the first player to win two sets. Andrea has a 0.4 chance of winning in each set against Elisa. Find the following probabilities.

- **a** Andrea wins in two sets.
- **b** Andrea wins in three sets.
- **c** Elisa wins after losing the first set.

2 Find $\Pr(A)$ if $\Pr(A \cup B) = 0.74$ and $\Pr(B) = 0.36$, assuming that A and B are independent events.

3 A fair coin is tossed 3 times. Find the probability that:

- **a** at least 1 head is obtained
- **b** at least 1 head is obtained given that the first toss is a head
- **c** at least 2 heads are obtained given that there is at least 1 head.

4 Two digits are chosen without replacement from the set $\{1, 2, 3, 4\}$ to form a two-digit number. Find the probability that the two-digit number is:

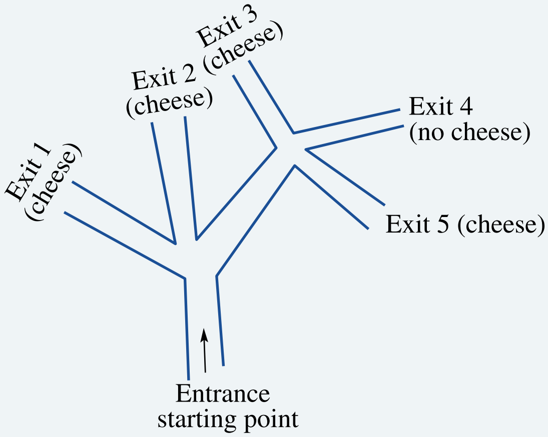
- **a** 32
- **b** even
- **c** less than 40
- **d** at least 22.

5 A fair coin is tossed 6 times. What is the probability that at least one tail is obtained?

6 What is the chance of choosing the correct six numbers in a 49-ball lottery game?

7 The letters of the word DOOR are jumbled randomly. What is the probability that the final arrangement will spell DOOR?

8 In an experiment, a mouse runs into a maze and randomly chooses one of the three paths at each fork.
Cheese is located at four of the five exit points. What is the probability that the mouse finds its way to an exit containing cheese?



9 Abbey has x green party lights and two red party lights for her balcony. If she has 21 possible arrangements of the party lights, how many green party lights does she have?

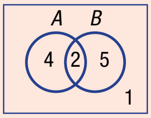
Chapter summary

Probability and counting techniques

Review

- Sample space is the list of all possible outcomes.
- $\Pr(\text{event}) = \dfrac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$

Venn diagram



Two-way table

	A	A'	
B	2	5	7
B'	4	1	5
	6	6	12

Selections (Opt)

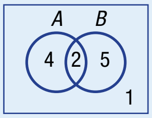
A selection (or combination) is a group of objects where order is not important.
The number of ways of choosing r objects from n is denoted by ${}^nC_r$ or $\binom{n}{r}$

where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

e.g. ${}^7C_3 = \dfrac{7!}{3!\,4!}$
$= 35$

Conditional probability

$\Pr(A\,|\,B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$ or $\dfrac{n(A \cap B)}{n(B)}$



	A	A'	
B	2	5	7
B'	4	1	5
	6	6	12

$\Pr(A\,|\,B) = \dfrac{2}{7}$

Arrangements (Opt)

An arrangement is a particular order of a set of objects. To arrange n objects in a row there are $n!$ arrangements.
If 3 objects are chosen from 7, the number of arrangements is

$$\boxed{7}\boxed{6}\boxed{5} = 7 \times 6 \times 5$$
$$= 210$$

Other restrictions sometimes need to be accounted for also.

Tables

With replacement

	A	B	C
A	(A, A)	(B, A)	(C, A)
B	(A, B)	(B, B)	(C, B)
C	(A, C)	(B, C)	(C, C)

Without replacement

	A	B	C
A	×	(B, A)	(C, A)
B	(A, B)	×	(C, B)
C	(A, C)	(B, C)	×

Factorials (Opt)

$n! = n \times (n-1) \times (n-2) \times \ldots \times 1$
e.g. $4! = 4 \times 3 \times 2 \times 1$
$= 24$
$0! = 1$

Independent events

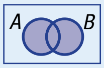
- $\Pr(A|B) = \Pr(A)$
- $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

Addition/multiplication principles (Opt)

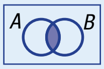
From 5 fiction and 4 non-fiction books, there are $5 + 4 = 9$ choices of books. If choosing one fiction and one non-fiction, there are $5 \times 4 = 20$ choices.

Unions and intersections

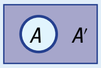
- Union $A \cup B$ (A or B)



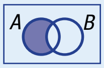
- Intersection $A \cap B$ (A and B)



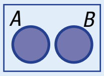
- Complement of A is A' (not A)



- A only is $A \cap B'$



- Mutually exclusive events $A \cap B = \varnothing$



Addition rule

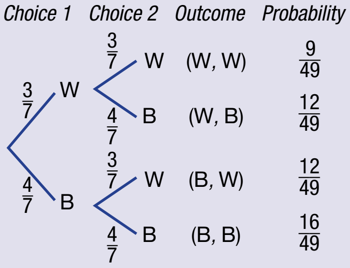
$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

If A and B are mutually exclusive:
$\Pr(A \cap B) = 0$ and
$\Pr(A \cup B) = \Pr(A) + \Pr(B)$

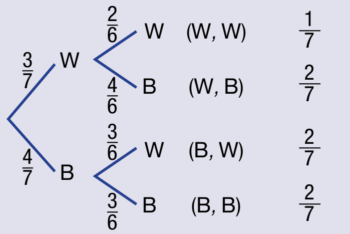
Tree diagrams

3 white
4 black

With replacement



Without replacement



$\Pr(\text{1 white, 1 black}) = \dfrac{2}{7} + \dfrac{2}{7}$
$= \dfrac{4}{7}$

Chapter checklist with success criteria

A printable version of this checklist is available in the Interactive Textbook ✔

8A **1. I can calculate simple theoretical probabilities.** ☐
e.g. A letter is chosen from the word CHRISTMAS. Find the probability that the letter is an S.

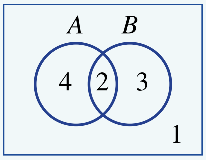
8A **2. I can calculate simple experimental probabilities.** ☐
e.g. An experiment involves rolling a regular 6-sided die 3 times and counting the number of 6s. The results after running the experiment 100 times are:

Number of 6s	0	1	2	3
Frequency	54	32	12	2

Find the experimental probability of obtaining more than 1 six.

8B **3. I can construct and use a Venn diagram.** ☐
e.g. From a class of 20 music students, 14 can play the piano (P), 8 can play the guitar (G) and 5 can play both. Illustrate this information in a Venn diagram and use this to find the probability that a student randomly chosen from the class plays neither the piano nor the guitar.

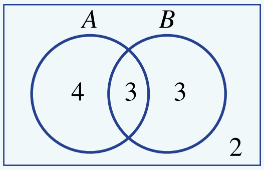
8B **4. I can use a two-way table.** ☐
e.g. The Venn diagram shows the distribution of elements in two sets, A and B.
Transfer the information in the Venn diagram to a two-way table and find $n(A' \cap B)$ and $\Pr(A \cup B)$.



8C **5. I can apply the addition rule.** ☐
e.g. A card is selected from a standard deck of 52 playing cards (4 suits, no jokers). Let A be the event 'the card is red' and B be the event 'the card is an Ace'. Use the addition rule to the find the probability that the card is an Ace or red ($\Pr(A \cup B)$).

8C **6. I can use the addition rule.** ☐
e.g. Two events, A and B, are such that $\Pr(A) = 0.3$, $\Pr(B) = 0.6$ and $\Pr(A \cup B) = 0.8$. Find $\Pr(A \cap B)$.

8D **7. I can find conditional probability from a Venn diagram.** ☐
e.g. Consider the Venn diagram displaying the number of elements belonging to the events A and B. Find $\Pr(A \mid B)$ and $\Pr(B \mid A)$.



8D **8. I can find conditional probabilities using a two-way table.** ☐
e.g. In a team of 11 cricketers in a match, 6 players bowled (A), 5 players batted (B) and 2 both batted and bowled. Represent the information in a two-way table and find the probability that a randomly selected team member bowled given that they batted.

8E **9. I can construct a table for a two-step experiment with replacement.** ☐
e.g. A fair 4-sided die is rolled twice. List the sample space of the sum of the faces of the two die using a table and find Pr(sum of at least 5).

8E **10. I can construct a table for a two-step experiment without replacement.** ☐
e.g. A bag contains 2 blue marbles, 1 red and 1 green. Two marbles are chosen from the bag without replacement. Construct a table to list the sample space and find the probability of selecting a blue marble and a green marble.

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Chapter checklist with success criteria

			✔
8F	**11. I can construct a tree diagram for multi-stage experiments.** e.g. Boxes A and B contain 5 counters each. Box A contains 3 blue and 2 white counters and box B contains 4 blue and 1 white counter. A box is chosen at random followed by a single counter. Draw a tree diagram that shows all possible outcomes and related probabilities. Use this to find the probability of selecting a blue counter.		☐
8F	**12. I can use a tree diagram for experiments without replacement.** e.g. A box contains 5 hard chocolates (H) and 3 soft chocolates (S). Two chocolates are selected at random without replacement. Draw a tree diagram showing all outcomes and find the probability of selecting one of each.		☐
8G	**13. I can check for independent events.** e.g. A selection of 10 gym memberships includes 3 with 24-hour gym access and 5 with personal training sessions, while 2 of the memberships have both of these. Let A be the event 'choosing a membership with 24-hour gym access' and B be the event 'choosing a membership with personal training sessions'. Display the information in a Venn diagram or two-way table and determine whether or not events A and B are independent.		☐
8H	**14. I can use the addition principle.** e.g. How many choices are there if one paint colour is chosen from 3 shades of yellow or 5 shades of blue?	Opt	☐
8H	**15. I can use the multiplication principle.** e.g. How many options are there if you choose one hot drink and one slice of cake from 5 types of hot drink and 6 types of cake?	Opt	☐
8H	**16. I can evaluate factorials.** e.g. Evaluate $5!$ and $\frac{6!}{4!}$.	Opt	☐
8I	**17. I can find the total number of arrangements.** e.g. How many ways can Zoe arrange four of her eight holiday souvenirs in a row on her desk?	Opt	☐
8I	**18. I can find the number of arrangements with restrictions.** e.g. Find the number of ways of arranging six objects: i if three of them are identical ii if three of them are to be together.	Opt	☐
8J	**19. I can use selection notation.** e.g. Evaluate ${}^7C_2$ and $\binom{6}{1}$.	Opt	☐
8J	**20. I can find the number of possible selections.** e.g. How many different selections are there if three chocolate bars are chosen from seven?	Opt	☐
8J	**21. I can find the number of possible selections from multiple groups.** e.g. For a sandwich, two meats are chosen from four available and three salad items are chosen from five available. How many types of sandwiches can be formed?	Opt	☐
8K	**22. I can find probabilities involving arrangements or selections.** e.g. Two different ice-cream scoops are chosen from a selection of ten available flavours. What is the probability that one particular combination of flavours is chosen?	Opt	☐

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Short-answer questions

8A **1** A letter is chosen from the word INTEREST. Find the probability that the letter will be:

a an I
b an E
c a vowel
d not a vowel
e E or T.

8A **2** A letter is chosen from the word POSITIVE. Find the probability that the letter also belongs to these words.

a NEGATIVE
b ADDITION
c DIVISION

8A **3** Belinda, an engineer, inspects 20 houses in a street for cracks. The results are summarised in this table.

Number of cracks	0	1	2	3	4
Frequency	8	5	4	2	1

a From these results estimate the probability that the next house inspected in the street will have the following number of cracks.
  i 0
  ii 1
  iii 2
  iv 3
  v 4

b Estimate the probability that the next house will have:
  i at least 1 crack
  ii no more than 2 cracks.

8B **4** Of 36 people, 18 have an interest in cars, 11 have an interest in homewares and 6 have an interest in both cars and homewares.

a Represent this information using a Venn diagram.
b Represent this information using a two-way table.
c State the number of people surveyed who do not have an interest in either cars or homewares.
d If a person is chosen at random from the group, find the probability that the person will:
  i have an interest in cars and homewares
  ii have an interest in homewares only
  iii not have any interest in cars.

8C **5** All 26 birds in an aviary have clipped wings and/or a tag. In total, 18 birds have tags and 14 have clipped wings.

a Find the number of birds that have both a tag and clipped wings.
b Find the probability that a bird chosen at random will have a tag only.

8B/C **6** A card is selected from a standard deck of 52 playing cards. Let A be the event 'the card is a heart' and let B be the event 'the card is a king'.

a Find:
  i $n(A)$
  ii $n(B)$
  iii $n(A \cap B)$

b Find:
  i $\Pr(A')$
  ii $\Pr(A \cap B)$

c Use the addition rule to find $\Pr(A \cup B)$.
d Find the probability that the card is a king or not a diamond.

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8C

7 Two events, A and B, are such that $\Pr(A) = 0.25$, $\Pr(B) = 0.35$ and $\Pr(A \cup B) = 0.5$. Find:

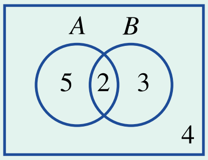
a $\Pr(A \cap B)$

b $\Pr(A' \cap B')$

8D

8 For these probability diagrams, find $\Pr\left(A \mid B\right)$.

a

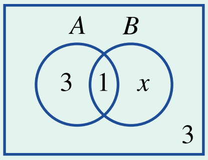


b

	A	***A′***	
B	1		
B′	2	2	
			9

8D/G

9 Two events, A and B, are represented on the following Venn diagram. Also, $n(B \text{ only}) = x$, where x is a positive integer.



a If $x = 4$, find:

i $\Pr(A)$ **ii** $\Pr(B)$ **iii** $\Pr\left(A \mid B\right)$

b If $x = 4$, decide whether the events A and B are independent.

c If $x = 1$, find:

i $\Pr(A)$ **ii** $\Pr(B)$ **iii** $\Pr\left(A \mid B\right)$

d If $x = 1$, decide if the events A and B are independent.

8E

10 A letter is chosen at random from the word HAPPY and a second letter is chosen from the word HEY.

a List the sample space, using a table.

b State the total number of outcomes.

c Find the probability that the two letters chosen will be:

i H then E **ii** the same **iii** not the same.

8F

11 A fair 4-sided die is rolled twice and the total is noted.

a Use a tree diagram to list the sample space, including all possible totals.

b Find these probabilities.

i $\Pr(2)$ **ii** $\Pr(5)$ **iii** $\Pr(1)$ **iv** $\Pr(\text{not } 1)$

8F

12 Two people are selected from a group of two children and three adults, without replacement. Use a tree diagram to find the probability of selecting:

a a child on the first selection

b an adult on the second selection given that a child is chosen on the first selection

c two adults

d one adult

e at least one child.

8G

13 Two independent events, A and B, are such that $\Pr(A) = 0.4$ and $\Pr(B) = 0.3$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$

8H/J

Opt

14 Evaluate the following.

a $6!$ **b** $\frac{5!}{2!}$ **c** ${}^{8}C_{3}$ **d** $\binom{9}{8}$

8I Opt

15 A store owner has a collection of six new dresses in stock. How many arrangements of these dresses are possible in a row in the shop window if:

a all dresses are displayed?

b three of the dresses are chosen to be displayed?

c all dresses are displayed and the two long dresses must be together?

8I/K Opt

16 Jodie has five Christmas decorations to arrange in a row on her shelf. If three of them are identical, what is the probability of one particular arrangement of the decorations on the shelf?

8K Opt

17 A three-digit pin is formed from the digits 0–9, where digits can be repeated. What is the probability that the three-digit pin formed is greater than 799?

8J/K Opt

18 A squad of 12 basketball players is available.

a Five players are chosen for the starting five. How many different starting fives are possible?

b In the squad of 12, 8 players are taller than two metres. What is the probability that a starting five of 3 particular players taller than two metres and 2 particular players not taller than two metres is chosen?

Multiple-choice questions

8A

1 A letter is chosen from the word SUCCESS. The probability that the letter is neither C nor S is:

A $\frac{2}{7}$ **B** $\frac{3}{5}$ **C** $\frac{5}{7}$ **D** $\frac{4}{7}$ **E** $\frac{3}{7}$

8A

2 The number of manufacturing errors spotted in a car plant on 20 randomly selected days is given by this table.

Number of errors	0	1	2	3
Frequency	11	6	2	1

An estimate of the probability that on the next day at least one error will be observed is:

A $\frac{3}{10}$ **B** $\frac{9}{20}$ **C** $\frac{11}{20}$ **D** $\frac{17}{20}$ **E** $\frac{3}{20}$

8B

3 From the list of the first 10 positive integers, $A = \{1, 3, 5, 7, 9\}$ and B is the set of primes less than 10. Therefore, $\Pr(A')$ and $\Pr(A \text{ only})$ are, respectively:

A $\frac{1}{3}, \frac{1}{5}$ **B** $\frac{1}{2}, \frac{1}{2}$ **C** $\frac{1}{2}, \frac{3}{10}$ **D** $\frac{1}{2}, \frac{1}{5}$ **E** $\frac{1}{3}, \frac{2}{5}$

8B

4 For this two-way table, $\Pr(A \cap B')$ is:

A $\frac{2}{3}$ **B** $\frac{1}{4}$ **C** $\frac{1}{7}$

D $\frac{1}{3}$ **E** $\frac{3}{4}$

	A	***A′***	
B	2		3
B′			4
		4	

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8C

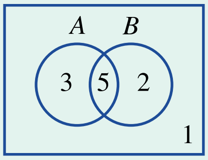
5 The sets A and B are known to be mutually exclusive. Which of the following is therefore true?

A $n(A) = n(B)$
B $n(A \cap B) = 0$
C $A = \varnothing$
D $\Pr(A \cap B) = 1$
E $\Pr(A \cup B) = 0$

8D

6 For this Venn diagram, $\Pr(A \mid B)$ is:

A $\frac{5}{7}$
B $\frac{5}{2}$
C $\frac{5}{8}$
D $\frac{5}{3}$
E $\frac{3}{11}$



8E

7 A letter is chosen from each of the words CAN and TOO. The probability that the pair of letters will not have an O is:

A $\frac{2}{3}$
B $\frac{1}{2}$
C $\frac{1}{3}$
D $\frac{1}{9}$
E $\frac{5}{9}$

8E

8 A box has 3 red and 2 blue counters. If a red counter is selected and not replaced, then the probability that a blue counter will be observed on the second selection is:

A $\frac{1}{2}$
B $\frac{2}{5}$
C $\frac{2}{3}$
D $\frac{1}{4}$
E $\frac{3}{4}$

8F

9 The number of times a coin must be tossed to give 16 possible outcomes is:

A 8
B 2
C 16
D 3
E 4

8G

10 Two events are independent when:

A $\Pr(A) = \Pr(B)$
B $\Pr(A') = \varnothing$
C $\Pr(A \cup B) = 0$
D $\Pr(A \mid B) = \Pr(B)$
E $\Pr(A) = \Pr(A \mid B)$

8H Opt

11 To travel from Milan to Barcelona there are 3 routes from Milan to Lyon and then 4 routes from Lyon to Barcelona. How many possible routes are there from Milan to Barcelona?

A 7
B 25
C 36
D 12
E 19

8J Opt

12 ${}^{7}C_{3}$ gives the same value as:

A ${}^{10}C_{6}$
B ${}^{3}C_{7}$
C ${}^{7}C_{4}$
D ${}^{10}C_{7}$
E ${}^{7}C_{5}$

8J Opt

13 How many combinations of committees of five students are possible if three are chosen from eight Year 12 nominees and two are chosen from six Year 11 nominees?

A 840
B 560
C 366
D 71
E 112

8K Opt

14 Three of seven sports cars are to be displayed in a row in a car showroom window. What is the probability that one particular arrangement of three of the seven cars is in the window?

A $\frac{1}{840}$
B $\frac{1}{210}$
C $\frac{1}{5040}$
D $\frac{1}{35}$
E $\frac{3}{7}$

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Extended-response questions

1 Of 15 people surveyed to find out whether they run or swim for exercise, 6 said they run, 4 said they swim and 8 said they neither run nor swim.

- **a** How many people surveyed run and swim?
- **b** One of the 15 people is selected at random. Find the probability that they:
  - **i** run or swim
  - **ii** only swim.
- **c** Represent the information in a two-way table.
- **d** Find the probability that:
  - **i** a person swims given that they run
  - **ii** a person runs given that they swim.

2 A bakery sells three types of bread: raisin (R) at \$2 each, sourdough (S) at \$3 each, and white (W) at \$1.50 each. Lillian is in a hurry. She randomly selects two loaves and takes them quickly to the counter. Each type of loaf has an equal chance of being selected.

- **a** Draw a table showing the possible combination of loaves that Lillian could have selected.
- **b** Find the probability that Lillian selects:
  - **i** two raisin loaves
  - **ii** two loaves that are the same
  - **iii** at least one white loaf
  - **iv** not a sourdough loaf.

Lillian has only \$4 in her purse.

- **c** How many different combinations of bread will Lillian be able to afford?
- **d** Find the probability that Lillian will not be able to afford her two chosen loaves.

On the next day, there are only two raisin, two sourdough and three white loaves available. Lillian chooses two loaves without replacement from the limited number of loaves.

- **e** Use a tree diagram showing branch probabilities to find:
  - **i** Pr(2 raisin loaves)
  - **ii** Pr(1 sourdough loaf)
  - **iii** Pr(not more than 1 white loaf)
  - **iv** Pr(2 loaves that are not the same)

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9

Statistics

Maths in context: A Biostatistician and an Actuary

Rachel Carson was an American biologist who collected and recorded data showing the environmental effects of synthetic pesticides. Her book Silent Spring (1962) led to the formation of the US Environmental protection agency. DDT is now banned as a pesticide in 15 countries, including US and Australia.

A Biostatistician's university studies include applied maths, statistical computing, coding, and biostatistics. Pharmaceutical companies employ Biostatisticians to assess the health effects of new drug treatments. Also, Biostatisticians can research and record the effects of exposure to harmful chemicals and pollution on people, wildlife, and the environment.

An Actuary is a highly paid statistician whose university Actuarial studies include maths, statistics, finance, economics, and coding. Actuaries collect data and assess financial risk for individuals and businesses. An actuary determines the cost of insurance for travel, houses, and vehicles based on the risk of illness, injury, disability, death, or loss of property. Actuaries also provide financial advice to businesses regarding the risk of various investments.

Chapter contents

Australian Curriculum 9.0

STATISTICS

Analyse claims, inferences and conclusions of statistical reports in the media, including ethical considerations and identification of potential sources of bias (AC9M10ST01)

Compare data distributions for continuous numerical variables using appropriate data displays including boxplots; discuss the shapes of these distributions in terms of centre, spread, shape and outliers in the context of the data (AC9M10ST02)

Construct scatterplots and comment on the association between the 2 numerical variables in terms of strength, direction and linearity (AC9M10ST03)

Construct two-way tables and discuss possible relationship between categorical variables (AC9M10ST04)

Plan and conduct statistical investigations of situations that involve bivariate data; evaluate and report findings with consideration of limitations of any inferences (AC9M10ST05)

Measures of spread, their interpretation and usefulness with respect to different data distributions (Year 10 optional content)

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

9A Collecting representative data

LEARNING INTENTIONS

- To understand how surveys work and the necessary considerations for their construction
- To understand the difference between a population and a sample
- To know how to describe types of data using the key words: categorical (nominal or ordinal) or numerical (discrete or continuous)
- To be able to decide if a survey sample is representative

There are many reports on television and radio that begin with the words 'A recent study has found that …'. These are usually the result of a survey or investigation that a researcher has conducted to collect information about an important issue, such as unemployment, crime or obesity.

Sometimes the results of these surveys are used to persuade people to change their behaviour. Sometimes they are used to pressure the government into changing the laws or to change the way the government spends public money.

Results of surveys and other statistics can sometimes be misused or displayed in a way to present a certain point of view.

Niche marketing is when a product or service is advertised to a specific group, such as people who train for obstacle competitions or dog-owners who use luxury dog groomers. Surveys provide valuable data for niche marketing and sales.

Lesson starter: Improving survey questions

Here is a short survey. It is not very well constructed.

Question 1: How old are you?
Question 2: How much time did you spend sitting in front of the television or a computer yesterday?
Question 3: Some people say that teenagers like you are lazy and spend way too much time sitting around when you should be outside exercising. What do you think of that comment?

Have a class discussion about the following.

- What will the answers to Question 1 look like? How could they be displayed?
- What will the answers to Question 2 look like? How could they be displayed?
- Is Question 2 going to give a realistic picture of your normal daily activity?
- Do you think Question 2 could be improved somehow?
- What will the answers to Question 3 look like? How could they be displayed?
- Do you think Question 3 could be improved somehow?

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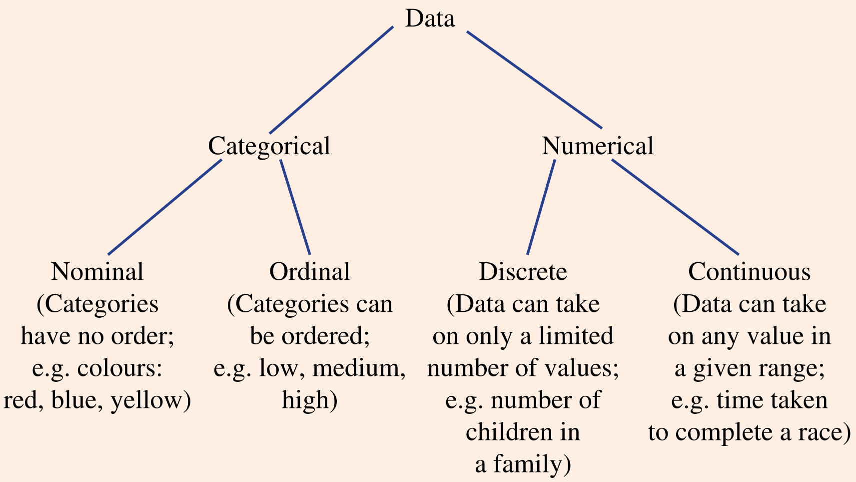
KEY IDEAS

- **Surveys** are used to collect statistical data.
  - Survey questions need to be constructed carefully so that the person knows exactly what sort of answer to give. Survey questions should use simple language and should not be ambiguous.
  - Survey questions should not be worded so that they deliberately try to provoke a certain kind of response.
  - If the question contains an option to be chosen from a list, the number of options should be an odd number, so that there is a 'neutral' choice. For example, the options could be:

strongly agree	agree	unsure	disagree	strongly disagree

- A **population** is a group of people, animals or objects with something in common. Some examples of populations are:
  - all the people in Australia on Census Night
  - all the students in your school
  - all the tigers in the wild in Sumatra
  - all the cars in Brisbane
  - all the wheat farms in NSW.
- A **sample** is a group that has been chosen from a population. Sometimes information from a sample is used to describe the whole population, so it is important to choose the sample carefully.
- **Statistical data** can be divided into subgroups.



BUILDING UNDERSTANDING

1 Match each word (**a**–**e**) with its definition (**A**–**E**).

a population
b census
c sample
d survey
e data

A a group chosen from a population
B a tool used to collect statistical data
C all the people or objects in question
D statistics collected from every member of the population
E the factual information collected from a survey or other source

2 Match each word (**a**–**f**) with its definition (**A**–**F**).

a numerical
b continuous
c discrete
d categorical
e ordinal
f nominal

A categorical data that has no order
B data that are numbers
C numerical data that take on a limited number of values
D data that can be divided into categories
E numerical data that take any value in a given range
F categorical data that can be ordered

3 Classify each set of data as categorical or numerical.

a 4.7, 3.8, 1.6, 9.2, 4.8
b red, blue, yellow, green, blue, red
c low, medium, high, low, low, medium

4 Which one of the following survey questions would generate categorical data?

A How many times do you eat at your favourite fast-food place in a typical week?
B How much do you usually spend buying your favourite fast food?
C How many items did you buy last time you went to your favourite fast-food place?
D Which is your favourite fast-food?

Example 1 Describing types of data

What type of data would the following survey questions generate?

a How many televisions do you have in your home?
b To what type of music do you most like to listen?

SOLUTION	EXPLANATION
a Numerical and discrete	The answer to the question is a number with a limited number of values; in this case, a whole number.
b Categorical and nominal	The answer is a type of music and these categories have no order.

Now you try

What type of data would the following survey questions generate?

a How tall are the students in Year 10?

b What is your level of satisfaction (low, medium and high) with a meal at a restaurant?

Example 2 Choosing a survey sample

A survey is carried out on the internet to determine Australia's favourite musical performer. Why will this sample not necessarily be representative of Australia's views?

SOLUTION	EXPLANATION
An internet survey is restricted to people with a computer and internet access, ruling out some sections of the community from participating in the survey.	The sample may not include some of the older members of the community or those in areas without access to the internet. Also, the survey would need to be set up so that people can do it only once so that 'fake' surveys are not completed.

Now you try

A survey is carried out in a library to determine typical study habits of Year 12 students. Why will this sample not necessarily be representative of all Year 12 students?

Exercise 9A

FLUENCY	1–3	1–3	2, 3

Example 1

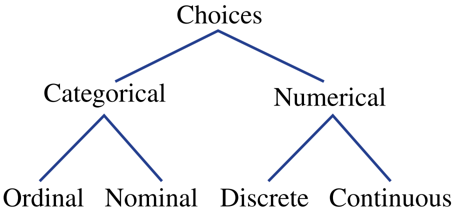
1 What type of data would the following survey questions generate?

a How many people are there in each office room?

b What was the time taken to complete the task?

c What colour are the jackets on a rack?

d How would you rate the movie: good, don't care, bad?



2 Year 10 students were asked the following questions in a survey. Describe what type of data each question generates.

a How many people under the age of 18 years are there in your immediate family?

b How many letters are there in your first name?

c Which company is the carrier of your mobile telephone calls? Optus/Telstra/Vodafone/Other (Please specify.)

d What is your height?

e How would you describe your level of application in Maths? (Choose from very high, high, medium or low.)

Example 2

3 Decide if the following surveys would be representative of the entire Australian population.

- **a** A survey via social media to find out people's favourite news program.
- **b** A survey to find out the average number of pets in a household from people entering a pet store.
- **c** Using census data to determine the average household income.
- **d** Making 10 000 random phone calls to find out who is likely to win the next federal election.

4 A popular Australian 'current affairs' television show recently investigated the issue of spelling. They suspected that people in their twenties are not as good at spelling as people in their fifties, so they decided to conduct a statistical investigation.

They chose a sample of 12 people aged 50–59 years and 12 people aged 20–29 years.

Answer the following questions on paper, then discuss in a small group or as a whole class.

- **a** Do you think that the number of people surveyed is adequate?
- **b** How many people do you think there are in Australia aged 20–29 years?
- **c** How many people do you think there are in Australia aged 50–59 years?
- **d** Use the website of the Australian Bureau of Statistics to look up the answers to parts **b** and **c**.
- **e** Do you think it is fair and reasonable to compare the spelling ability of these two groups of people?
- **f** How would you go about comparing the spelling ability of these two groups of people?
- **g** Would you give the two groups the same set of words to spell?
- **h** How could you give the younger people an unfair advantage?
- **i** What sorts of words would you include in a spelling test for the survey?
- **j** How and where would you choose the people to do the spelling test?

PROBLEM–SOLVING	5, 6	5–7	6–8

5 The principal decides to survey Year 10 students to determine their opinion of Mathematics.

- **a** In order to increase the chance of choosing a representative sample, the principal should:
  - **A** Give a survey form to the first 30 Year 10 students who arrive at school.
  - **B** Give a survey form to all the students studying the most advanced Maths subject.
  - **C** Give a survey form to five students in every Maths class.
  - **D** Give a survey form to 20% of the students in every class.
- **b** Explain your choice of answer in part **a**. Describe what is wrong with the other three options.

6 Discuss some of the problems with the selection of a survey sample for each given topic.

- **a** a survey at the train station of how Australians get to work
- **b** an email survey on people's use of computers
- **c** phoning people on the electoral roll to determine Australia's favourite sport

Is a train station survey of how people get to work representative?

7 Choose a topic in which you are especially interested, such as football, cricket, movies, music, cooking, food, computer games or social media.

Make up a survey about your topic that you could give to the people in your class.

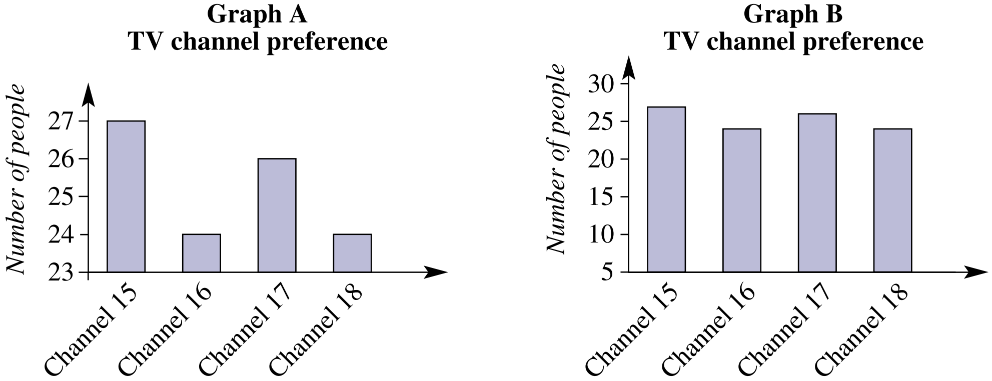
It must have *four* questions.
Question 1 must produce data that are categorical and ordinal.
Question 2 must produce data that are categorical and nominal.
Question 3 must produce data that are numerical and discrete.
Question 4 must produce data that are numerical and continuous.

8 A television news reporter surveyed four companies and found that the profits of three of these companies had reduced over the past year. They report that this means the country is facing an economic downturn and that only one in four companies is making a profit.

- **a** What are some of the problems in this media report?
- **b** How could the news reporter improve their sampling methods?
- **c** Is it correct to say that only one in four companies is making a profit? Explain.

REASONING	9	9, 10	10, 11

9 Here are two column graphs, each showing the same results of a survey that asked people which TV channel they preferred.



- **a** Which graph could be titled ‘Channel 15 is clearly most popular’?
- **b** Which graph could be titled ‘All TV channels have similar popularity’?
- **c** What is the difference between the two graphs?
- **d** Which graph is misleading and why?

10 Describe three ways that graphs or statistics could be used to mislead people and give a false impression about the data.

11 Search the internet or newspaper for ‘misleading graphs’ and ‘how to lie with statistics’. Explain why they are misleading.

ENRICHMENT: The 2021 Australian Census	–	–	12, 13

12 Research the 2021 Australian Census on the website of the Australian Bureau of Statistics. Find out something interesting from the results of the 2021 Australian Census and write a short news report.

13 It is often said that Australia has an ageing population. What does this mean?
Search the internet for evidence showing that the ‘average’ Australian is getting older every year.

9B Review of data displays CONSOLIDATING

LEARNING INTENTIONS

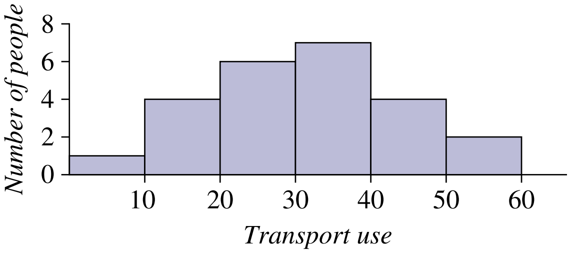
- To review the types of graphs that can be used to display categorical data or numerical data
- To know how to construct a frequency table and histogram from numerical data using class intervals
- To know how to find the measures of centre, mean and median, of a set of data

Statistical graphs are an essential element in the analysis and representation of data. Graphs can help to show the most frequent category, the range of values, the shape of the distribution and the centre of the data. By looking at statistical graphs the reader can quickly draw conclusions about the numbers or categories in the data set and interpret this within the context of the data.

People who specialise in medical biostatistics apply statistical techniques to analyse results from health-related research, such as in genetics, medicine and pharmacy. Data presentation includes using bar charts, line charts, histograms and scatter plots.

Lesson starter: Public transport analysis

A survey was carried out to find out how many times people in the group had used public transport in the past month. The results are shown in this histogram.



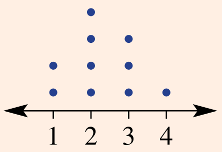
Discuss what the histogram tells you about this group of people and their use of public transport. You may wish to include these points:

- How many people were surveyed?
- Is the data symmetrical or skewed?
- Is it possible to work out the exact mean? Why/why not?
- Do you think these people were selected from a group in your own community? Give reasons.

KEY IDEAS

- The different types of **statistical data** that we saw in the previous section; i.e. categorical (nominal or ordinal) and numerical (discrete or continuous), can be displayed using different types of graphs to represent the different data.

- Graphs for a single set of categorical or discrete data
  - **Dot plot**

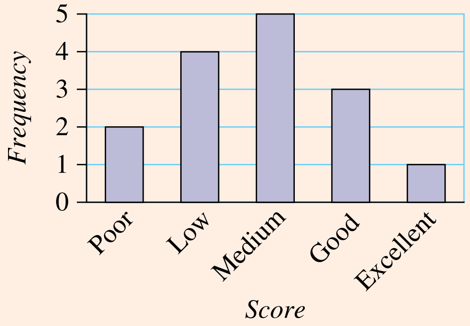


  - **Stem-and-leaf plot**

Stem	Leaf
0	1 3
1	2 5 9
2	1 4 6 7
3	0 4

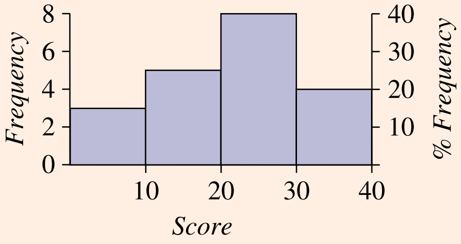
2 | 4 means 24

  - **Column graph**



- **Histograms** can be used for grouped discrete or continuous numerical data.
  The interval 10– includes all numbers from 10 (including 10) to fewer than 20.

Class interval	Frequency	Percentage frequency
0–	3	15
10–	5	25
20–	8	40
30–40	4	20

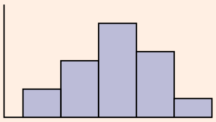


- The two most common measures of centre are:
  - **mean** ($\bar{x}$): $\bar{x} = \dfrac{\text{sum of all data values}}{\text{number of data values}}$
  - **median**: the middle value when data are placed in order

- The **mode** of a data set is the data value that occurs most frequently.

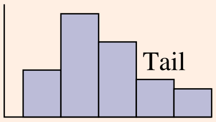
- Data can be **symmetrical** or **skewed**.

Symmetrical



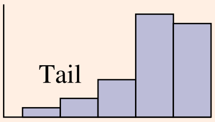
The mean and median will be equal.

Positively skewed



The median will be less than the mean.

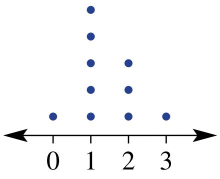
Negatively skewed



The median will be greater than the mean.

BUILDING UNDERSTANDING

① A number of families were surveyed to find the number of children in each. The results are shown in this dot plot.



- **a** How many families were surveyed?
- **b** Find the mean number of children in the families surveyed.
- **c** State the median number of children in the families surveyed.
- **d** State the mode for the number of children in the families surveyed.
- **e** What percentage of the families have, at most, two children?

② State the missing values in this frequency table.

Class interval	Frequency	Percentage frequency
0–	2	
10–	1	
20–	5	
30–40	2	
Total		

Example 3 Presenting and analysing data in frequency histograms

Twenty people were surveyed to find out how many times they use the internet in a week. The raw data are listed.

21, 19, 5, 10, 15, 18, 31, 40, 32, 25
11, 28, 31, 29, 16, 2, 13, 33, 14, 24

- **a** Organise the data into a frequency table using class intervals of 10. Include a percentage frequency column.
- **b** Construct a histogram for the data, showing both the frequency and percentage frequency on the one graph.
- **c** Describe the data in the histogram as symmetrical, positively skewed or negatively skewed.

SOLUTION

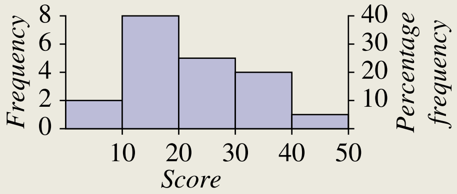
a

Class interval	Frequency	Percentage frequency
0–	2	10
10–	8	40
20–	5	25
30–	4	20
40–50	1	5
Total	20	100

EXPLANATION

Calculate each percentage frequency by dividing the frequency by the total (i.e. 20) and multiplying by 100.

b Number of times the internet is accessed



Transfer the data from the frequency table to the histogram. Axis scales are evenly spaced and the histogram bar is placed across the boundaries of the class interval. There is no space between the bars.

c Positively skewed

The tail of the data is to the right.

Now you try

Sixteen people were surveyed to find out how many phone texts they send in one day. The raw data are as follows.

10, 7, 2, 5, 22, 14, 7, 9, 11, 29, 32, 18, 5, 24, 12, 14

a Organise the data into a frequency table using class intervals of 10. Include a percentage frequency column.

b Construct a histogram for the data, showing both the frequency and percentage frequency on the one graph.

c Describe the data in the histogram as symmetrical, positively skewed or negatively skewed.

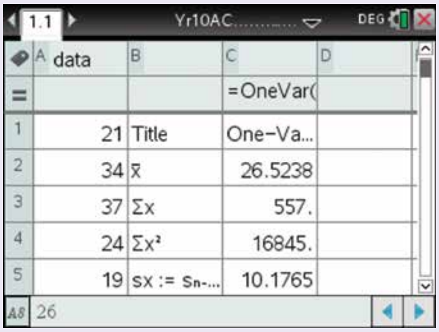
Using calculators to graph grouped data

1 Enter the following data in a list called *data* and find the mean and median.
21, 34, 37, 24, 19, 11, 15, 26, 43, 38, 25, 16, 9, 41, 36, 31, 24, 21, 30, 39, 17

2 Construct a histogram using intervals of 3 and percentage frequency for the data above.

Using the TI-Nspire:

1 In a **Lists and spreadsheets** page type in the list name *data* and enter the values as shown. Use menu **>Statistics>Stat Calculations>One-Variable Statistics** and press enter. Scroll to view the statistics.

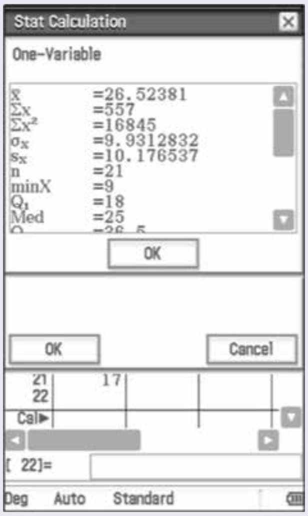


	A data	B	C	D
=			=OneVar(	
1	21	Title	One-Va...	
2	34	x̄	26.5238	
3	37	Σx	557.	
4	24	Σx²	16845.	
5	19	sx := Sn-...	10.1765	

A6 26

Using the ClassPad:

1 In the **Statistics** application enter the data into list1. Tap **Calc, One-Variable** and then **OK**. Scroll to view the statistics.



```
x̄      =26.52381
Σx     =557
Σx²    =16845
σx     =9.9312832
sx     =10.176537
n      =21
minX   =9
Q₁     =18
Med    =25
```

```
21   17
22
[ 22]=
```

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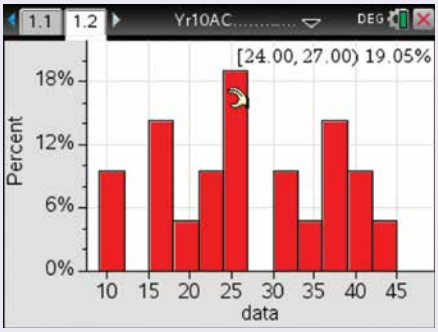
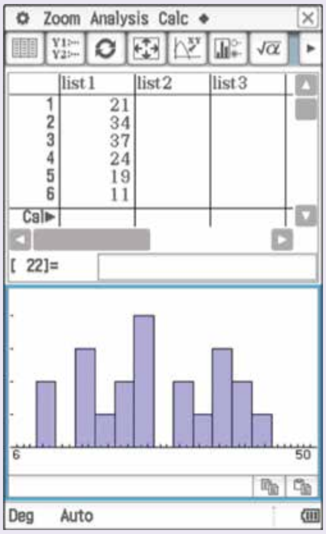
Using the TI-Nspire:

2 Insert a **Data and Statistics** page and select the *data* variable for the horizontal axis. Use menu**>Plot Type>Histogram**. Then use menu**>Plot Properties>Histogram Properties>Bin Settings>Equal Bin Width**. Choose the **Width** to be 3 and **Alignment** to be 0. Use menu**>Window/Zoom>Zoom-Data** to auto rescale. Use menu**>Plot Properties>Histogram Properties>Histogram Scale>Percent** to show the percentage frequency.



Using the ClassPad:

2 Tap **SetGraph**, ensure StatGraph1 is ticked and then tap **Setting**. Change the **Type** to **Histogram**, set **XList** to **list1**, **Freq** to **1** and then tap on **Set**. Tap and set **HStart** to 9 and **HStep** to 3.



	list1	list2	list3
1	21		
2	34		
3	37		
4	24		
5	19		
6	11		

Example 4 Working with stem and leaf plots

Twelve test marks for Rolle in a semester are given as percentages in the raw data below:

66 74 81 68 94 88 74 85 90 72 79 86

a Construct a stem-and-leaf plot for the data.

b Use your stem-and-leaf plot to find:

i the mode **ii** the median **iii** the mean.

SOLUTION

a

Stem	Leaf
6	6 8
7	2 4 4 9
8	1 5 6 8
9	0 4

8|4 means 84%

b **i** Mode is 74.

ii Median $= \dfrac{79+81}{2}$

$= 80$

iii Mean $= \dfrac{66+68+\ldots+94}{12}$

$= 79.75$

EXPLANATION

Use the 10s column for the stem and the units column for the leaf.

Order the data in each leaf and also show a key (e.g. 8|4 means 84).

74 is the most common value, appearing twice.

After counting the scores in order from the lowest value, the two middle values are 79 and 81, so the median is the mean of these two numbers.

Sum all of the data values and divide by the number of the data values.

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Now you try

Payments from Lucy's last twelve babysitting jobs are given below.

$36 $48 $44 $62 $56 $58 $52 $39 $56 $38 $60 $45

a Construct a stem-and-leaf plot for the data.

b Use your stem-and-leaf plot to find:

i the mode **ii** the median **iii** the mean.

Exercise 9B

FLUENCY	1–5	1, 2, 4–6	2, 4–6

Example 3

1 The number of points scored by Milton in a 20-game basketball season is shown:

8, 15, 12, 22, 24, 32, 10, 16, 6, 21, 20, 24, 17, 22, 18, 8, 28, 20, 17, 25

a Organise the data into a frequency table using class intervals of 10 and include a percentage frequency column.

b Construct a histogram for the data, showing both the frequency and percentage frequency on the one graph.

c Describe the data in the histogram as symmetrical, positively skewed or negatively skewed.

2 The number of wins scored this season is given for 20 hockey teams. Here are the raw data.

4, 8, 5, 12, 15, 9, 9, 7, 3, 7, 10, 11, 1, 9, 13, 0, 6, 4, 12, 5

a Organise the data into a frequency table using class intervals of 5 and include a percentage frequency column.

b Construct a histogram for the data, showing both the frequency and percentage frequency on the one graph.

c Describe the data in the histogram as symmetrical, positively skewed or negatively skewed.

3 This frequency table displays the way in which 40 people travel to and from work.

Type of transport	Frequency	Percentage frequency
Car	16	
Train	6	
Tram	8	
Walking	5	
Bicycle	2	
Bus	3	
Total	**40**	

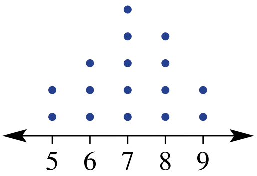
a Copy and complete the table.

b Use the table to find:

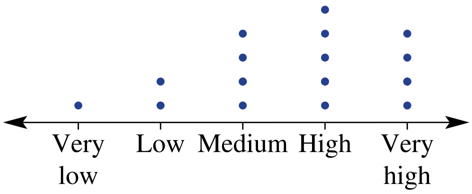
- **i** the frequency of people who travel by train
- **ii** the most popular form of transport
- **iii** the percentage of people who travel by car
- **iv** the percentage of people who walk or cycle to work
- **v** the percentage of people who travel by public transport, including trains, buses and trams.

4 Describe each graph as symmetrical, positively skewed or negatively skewed.

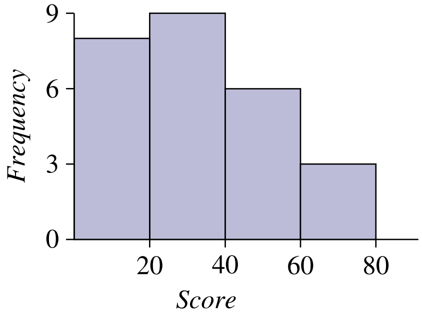
a



b



c



d

Stem	Leaf
4	1 6
5	0 5 4 8
6	1 8 9 9 9
7	2 7 8
8	3 8

4 | 6 means 46

Example 4

5 The maximum temperature for the last 10 days is given below in °C.

18 19 16 18 23
27 26 25 30 18

a Construct a stem-and-leaf plot for the data.

b Use your stem-and-leaf plot to find:

i the mode

ii the median

iii the mean.

6 For the data in these stem-and-leaf plots, find:

i the mean (rounded to one decimal place)

ii the median

iii the mode.

a

Stem	Leaf
2	1 3 7
3	2 8 9 9
4	4 6

3 | 2 means 32

b

Stem	Leaf
0	4
1	0 4 9
2	1 7 8
3	2

2 | 7 means 27

PROBLEM–SOLVING 7, 8 8, 9 9, 10

7 Two football players, Nick and Jack, compare their personal tallies of the number of goals scored for their team over a 12-match season. Their tallies are as follows.

Game	1	2	3	4	5	6	7	8	9	10	11	12
Nick	0	2	2	0	3	1	2	1	2	3	0	1
Jack	0	0	4	1	0	5	0	3	1	0	4	0

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a Draw a dot plot to display Nick's goal-scoring achievement.
b Draw a dot plot to display Jack's goal-scoring achievement.
c How would you describe Nick's scoring habits?
d How would you describe Jack's scoring habits?

8 Three different electric sensors, A, B and C, are used to detect movement in Harvey's backyard over a period of 3 weeks. An in-built device counts the number of times the sensor detects movement each night. The results are as follows.

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Sensor A	0	0	1	0	0	1	1	0	0	2	0	0	0	0	0	1	1	0	0	1	0
Sensor B	0	15	1	2	18	20	2	1	3	25	0	0	1	15	8	9	0	0	2	23	2
Sensor C	4	6	8	3	5	5	5	4	8	2	3	3	1	2	2	1	5	4	0	4	9

a Using class intervals of 3 and starting at 0, draw up a frequency table for each sensor.
b Draw histograms for each sensor.
c Given that it is known that stray cats consistently wander into Harvey's backyard, how would you describe the performance of:
  i sensor A?
  ii sensor B?
  iii sensor C?


Possums could set off the sensors.

9 This tally records the number of mice that were weighed and categorised into particular mass intervals for a scientific experiment.
a Construct a table using these column headings: Mass, Frequency and Percentage frequency.
b Find the total number of mice weighed in the experiment.
c State the percentage of mice that were in the 20– gram interval.
d Which was the most common weight interval?
e What percentage of mice were in the most common mass interval?
f What percentage of mice had a mass of 15 grams or more?

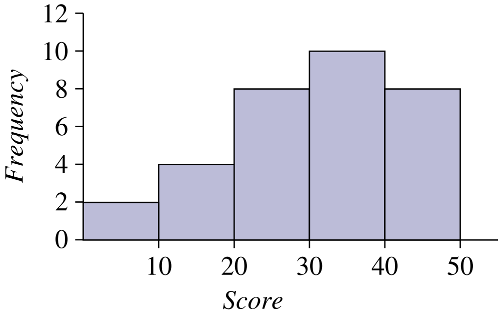
Mass (grams)	Tally
10–	𝍷𝍷𝍷
15–	𝍸 𝍷
20–	𝍸 𝍸 𝍸 𝍷
25–	𝍸 𝍸 𝍸 𝍸 𝍷
30–35	𝍷𝍷𝍷𝍷

10 A school symphony orchestra contains four musical sections: strings, woodwind, brass and percussion. The number of students playing in each section is summarised in this tally.
a Construct and complete a percentage frequency table for the data.
b What is the total number of students in the school orchestra?
c What percentage of students play in the string section?
d What percentage of students do not play in the string section?
e If the number of students in the string section increases by 3, what will be the percentage of students who play in the percussion section? Round your answer to one decimal place.
f What will be the percentage of students in the string section of the orchestra if the entire woodwind section is absent? Round your answer to one decimal place.

Section	Tally
String	𝍸 𝍸 𝍸 𝍸 𝍷
Woodwind	𝍸 𝍷𝍷𝍷
Brass	𝍸 𝍷𝍷
Percussion	𝍷𝍷𝍷𝍷

REASONING	11	11, 12	12, 13

11 This histogram shows the distribution of test scores for a class. Explain why the percentage of scores in the 20–30 range is 25%.



12 Explain why the exact value of the mean, median and mode cannot be determined directly from a histogram that shows grouped data like the one in Question **11**.

13 State the possible values of a, b and c in this ordered stem-and-leaf plot.

Stem	Leaf
3	2 3 a 7
4	b 4 8 9 9
5	0 1 4 9 c
6	2 6

ENRICHMENT: Bimodal weaver ants	–	–	14

14 Weaver ants live in trees and have body lengths which form an interesting distribution. The following data represents the body length of 30 weaver ants measured in millimetres.

4.2, 7.4, 9.1, 8.3, 8.5, 5.3, 5.4, 6.1, 7.9, 7.4
5.2, 7.2, 8.6, 8.0, 7.6, 5.6, 5.0, 4.8, 8.4, 8.1
4.7, 5.0, 7.8, 8.4, 8.6, 4.5, 5.7, 7.9, 8.6, 8.4

a Construct a table including the following columns:
- Body length using 5 mm class intervals: 4.0 –, 4.5 –, 5.0 –, etc
- Frequency
- Percentage frequency rounded to one decimal place.

b Construct a frequency histogram for the data using your table constructed in part **a** above.

c Describe the shape and spread of the data for the weaver ants.

d Find the modal class for the weaver ants with the following body lengths.
- **i** less than 6.0 mm
- **ii** greater than 6.0 mm

e Research other bimodal distributions that occur naturally and give a brief description of one chosen example.

9C Two-way tables

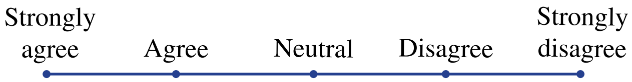
LEARNING INTENTIONS

- To know what a two-way table is as a display of data
- To know that a Likert scale is a list of ordered options for a survey question
- To be able to construct a two-way table
- To be able to interpret a two-way table

In statistics it is common to compare two categorical variables. We might be interested in, for example, the level of salary of university graduates compared to the type of degree they complete. One way to present such paired data in a meaningful way is to use a two-way table which makes it easier to compare the categories and see if there is a connection between the variables. When collecting the data to be entered into a two-way table, many types of surveys could be used. One such survey uses a Likert scale where respondents can answer by giving a degree of opinion on a scale typically using five options.

Lesson starter: Analysing data collected via a 5-point Likert scale

The following 5-point Likert scale is used to gather information about what junior and senior school students think about a particular mathematics competition. The question asked was: Do you think the competition was worthwhile?



The results are shown in this two-way table.

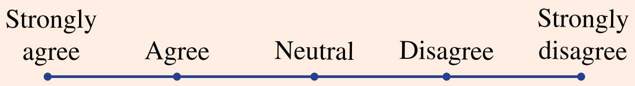
	Strongly agree	Agree	Neutral	Disagree	Strongly disagree	Total
Junior	2	10	15	12	7	46
Senior	20	28	16	4	1	69
Total	22	38	31	16	8	115

- How many students were surveyed in total?
- How many of the junior students responded *Agree* to the given question?
- How many of the senior students responded *Strongly disagree* to the given question?
- How many students in total responded *Neutral* to the given question?
- Approximately what proportion of the students were from the senior school?
- Approximately what proportion of the students were from the senior school and selected *Neutral*?
- What general conclusions can you draw about the opinions of the junior and senior school students regarding the mathematics competition?

KEY IDEAS

- A **two-way table** is a way of displaying data to help compare two related categorical variables.
  - The options for one variable form the rows and the options for the other variable form the columns.
  - The entries in the table, often given as frequencies or proportions, are totalled both vertically and horizontally.

- A **Likert scale** is a list of ordered options which can be responded to as part of a survey question.
  - A 5-point scale is often used as it provides a neutral option and the ability for respondents to offer a degree of opinion.



BUILDING UNDERSTANDING

① This incomplete two-way table shows the results of a survey asking if people are currently working and whether they took a holiday in the last 12 months.

	Holiday	No holiday	Total
Working	10	8	a
Not working	6	12	18
Total	16	b	c

a What are the missing values: a, b and c?

b How many people took the survey?

c How many of the people surveyed were:

- **i** working?
- **ii** not working and did not take a holiday in the last 12 months?

d Based on this data, would you agree with the statement that people who are working take more holidays?

② This table with incomplete totals shows the results of a survey using a Likert scale asking 20 people what they thought of the most recent interest rate increase. The survey also asked if they were pensioners or not.

	Very satisfied	Satisfied	Neutral	Dissatisfied	Very dissatisfied	Total
Pensioner	4	2	1	1	0	
Non-pensioner	0	1	4	4	3	
Total						

Complete the two-way table, and then answer the following questions.

a What fraction of the people surveyed were:

- **i** pensioners?
- **ii** pensioners who are satisfied with the interest rate rise?
- **iii** non-pensioners who are very dissatisfied with the interest rate rise?
- **iv** non-pensioners who are neutral to the interest rate rise?

b What percentage of the people surveyed were:

- **i** pensioners that said that they were dissatisfied?
- **ii** non-pensioners that said that they were satisfied?

c Would you say that this data supports the notion that pensioners are more satisfied with interest rate increases?

Example 5 Interpreting a two-way table

This two-way table summarises data collected from a survey using a 5-point Likert scale. The survey asked 50 people if they are over or under the age of 40 and to what degree they agree with the idea to reduce the driving blood alcohol limit from 0.05 to 0.04.

	Strongly agree	Agree	Neutral	Disagree	Strongly disagree	Total
40 and over	2	7	10	5	5	24
Under 40	0	1	6	12	7	26
Total	2	8	16	17	7	50

a State how many of the people surveyed:
  i were 40 years of age or over
  ii strongly agree with the idea
  iii were 40 years of age or over and responded *Agree* to the question
  iv were under 40 and responded *Disagree* to the question.

b What percentage of the people surveyed:
  i were under 40 years of age?
  ii were neutral to the idea?
  iii were under 40 years of age and responded *Strongly Disagree* with the idea?
  iv were 40 years of age or over and responded *Neutral* to the idea?

c Would you say that this data supports the notion that younger people would prefer a lower driving blood alcohol limit? Give a reason.

SOLUTION	EXPLANATION
a **i** 24 **ii** 2 **iii** 7 **iv** 12	Read directly from the table by choosing the matching cell frequency or column total.
b **i** 52% **ii** 32% **iii** 14% **iv** 20%	Divide the frequency by the total (50) and multiply by 100.
c No. A greater proportion of the people under 40 disagree with the idea.	The majority of the people in the under 40 category *Disagree* or *Strongly disagree* with the idea.

Now you try

This two-way table summarises data collected from a survey using a 5-point Likert scale. The survey asked 20 teachers if they thought there was enough time in the day to prepare for classes. Some teachers were from public schools and some were from private schools.

	Strongly agree	Agree	Neutral	Disagree	Strongly disagree	Total
Public	1	2	4	2	1	10
Private	0	2	6	1	1	10
Total	1	4	10	3	2	20

a State how many of the teachers surveyed:
  i were from public schools
  ii responded *Agree* to the question
  iii were from public schools and responded *Agree* to the question
  iv were from private schools and responded *Disagree* to the question.

b What percentage of the teachers surveyed:
  i were from private schools?
  ii were neutral to the question?
  iii were from private schools and responded *Neutral* to the question?
  iv were from public schools and responded *Disagree* to the question?

c Would you say that this data supports the notion that teachers from public schools feel that there is enough time in the day to prepare for classes compared to private school teachers? Give a reason.

Example 6 Constructing a two-way table

A number of people were surveyed to see if they owned a beach house. They were also asked as to whether they were over 60 years of age. 15 of the 25 people 60 years of age and over said that they owned a beach house and 6 of the 15 people under 60 years of age said that they owned a beach house.

a Represent this information in a two-way table.

b What fraction of the people surveyed:
  i were under 60 years of age?
  ii were under 60 years of age and owned a beach house?

c Would you say that this data supports the notion that older people are more likely to own a beach house? Give a reason.

SOLUTION

a

	Beach house	No beach house	Total
60 and over	15	10	25
Under 60	6	9	15
Total	21	19	40

EXPLANATION

Construct the table with one variable representing the rows and the other representing the columns.

Fill in the given information (in red) then find the missing values by working with the row and column totals.

b **i** $\frac{15}{40} = \frac{3}{8}$	Divide the frequency by the total (40) and simplify.
ii $\frac{6}{40} = \frac{3}{20}$	
c Yes. A much larger proportion of the people over 60 years of age own a beach house.	Look at the proportion of people that own a beach house for each age category.

Now you try

A number of city and country dogs were checked to see if they were of pure or mixed breed. 18 of the 30 country dogs were of pure breed and 13 of the 20 city dogs were of mixed breed.

a Represent this information in a two-way table.

b What fraction of the dogs surveyed:

- **i** were from the city?
- **ii** were from the city and of pure breed?

c Would you say that this data supports the notion that country dogs are more likely to be of mixed breed? Give a reason.

Exercise 9C

FLUENCY	1–3	1, 3, 4	2, 4

Example 5

1 This two-way table summarises data collected from a survey using a 5-point Likert scale. The survey asked 50 people if they are over or under the age of 30 and to what degree they agree with the idea to increase the speed limit on highways from 100 km/h to 120 km/h.

	Strongly agree	Agree	Neutral	Disagree	Strongly disagree	Total
30 and over	2	7	10	5	0	24
Under 30	0	1	6	12	7	26
Total	2	8	16	17	7	50

a State how many of the people surveyed:

- **i** were under 30 years of age
- **ii** strongly disagreed with the idea
- **iii** were under 30 years of age and responded *Agree* to the question
- **iv** were 30 years of age or over and responded *Neutral* to the question.

b What percentage of the people surveyed:

- **i** were over 30 years of age?
- **ii** were neutral to the idea?
- **iii** were under 30 years of age and agreed with the idea?
- **iv** were 30 years of age or over and strongly disagreed with the idea?

c Would you say that this data supports the notion that older people agreed with the idea to increase the speed limit on highways? Give a reason.

2 This two-way table summarises data collected from a survey using a 5-point Likert scale. The survey asked 100 people if they are over or under the age of 50 and to what degree they support a new road tunnel in the area.

	Strongly agree	**Agree**	**Neutral**	**Disagree**	**Strongly disagree**	**Total**
50 and over	4	6	14	19	9	52
Under 50	16	14	11	5	2	48
Total	20	20	25	24	11	100

a State how many of the people surveyed:
- **i** were 50 years of age or over
- **ii** were neutral to the idea
- **iii** were 50 years of age or over and responded *Disagree* to the idea
- **iv** were under 50 and responded *Agree* to the idea.

b What percentage of the people surveyed:
- **i** were under 50 years of age?
- **ii** strongly agreed with the idea?
- **iii** were 50 years of age or over and were neutral to the idea?
- **iv** were under 50 years of age and strongly disagreed with the idea?

c Would you say that this data supports the notion that younger people agree with the idea to build a new road tunnel? Give a reason.

Example 6

3 A number of people were surveyed to see if they smoked. They were also asked as to whether they were over 55 years of age. 5 of the 30 people 55 years of age or over said that they smoked and 8 of the 20 people under 55 years of age said that they smoked.

a Represent this information in a two-way table.

b What fraction of the people surveyed:
- **i** were 55 years of age or older?
- **ii** were under 55 years of age and did not smoke?

c Would you say that this data supports the notion that younger people are more likely to smoke? Give a reason.

4 A number of customers were surveyed to see if they are happy with a particular waiter in the coffee shop. There were also asked as to whether or not they were satisfied with their coffee. 26 of the 32 customers who were satisfied with their coffee said that they were happy with the waiter and 3 of the 8 customers who were not satisfied with their coffee said that they were happy with the waiter.

a Represent this information in a two-way table.

b What fraction of the people surveyed:
- **i** were satisfied with their coffee?
- **ii** were satisfied with their coffee and were not happy with the waiter?

c Would you say that this data supports the notion that if you are not satisfied with your coffee then you will not be happy with the waiter? Give a reason.

PROBLEM–SOLVING	5, 6	5–7	6–8

5 A number of adults and children were surveyed about their favourite film genre and the results are presented in this table.

	Comedy	Horror	Science fiction	Drama	Animated	Total
Children	3	1	6	4	9	23
Adults	3	6	5	11	2	27
Total	6	7	11	15	11	50

- **a** What percentage of the peopled surveyed:
  - **i** were children?
  - **ii** were children and responded *Science fiction* to the question?
  - **iii** were adults and responded *Drama* to the question?
- **b** What fraction of the people who responded *Animated* as their favourite genre were adults?
- **c** What fraction of the people who responded *Horror* as their favourite genre were adults?
- **d** Would you say that this data supports the notion that more adults like *Horror* and *Drama* compared to children? Give a reason.

6 Give the missing values, a, b, c, d and e in this table.

	Walks	No walks	Total
Cats	2	11	b
Dogs	13	a	19
Total	c	d	e

7 Customers at a restaurant were asked if they were satisfied with the quality of meal served either during the week or on weekends. 16 of the 18 weekday customers said that they were satisfied and 14 of the 22 weekend customers said that they were satisfied.

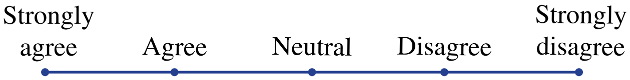
- **a** Use a two-way table to find how many of the customers were both:
  - **i** satisfied and a weekday customer
  - **ii** dissatisfied and a weekend customer.
- **b** Rounded to one decimal place, what percentage of:
  - **i** the weekday customers were satisfied with their meal?
  - **ii** satisfied customers dined on the weekend?
- **c** Would you say that the data supports the notion that it is less likely that customers will be satisfied on the weekend? Give a reason.

8 Of 20 adult and child passports, 3 of the 11 adult passports expire within 12 months and a total of 5 child passports do not expire within 12 months. Find the proportion of passports which are:

- **a** adult passports
- **b** child passports which expire within 12 months.

REASONING	9	9, 10	10, 11

9 Give a reason why a chosen Likert scale like the 5-point scale might include an odd number of options.



10 The table below summarises the results of a survey which asked if people intend on visiting the beach in the next month. It also asked if they were a local resident or a tourist.

	Beach	**No beach**	**Total**
Local	8	24	32
Tourist	2	6	8
Total	10	30	40

a What fraction of the tourists intend to visit the beach in the next month?
b What fraction of the locals intend to visit the beach in the next month?
c Would you say that there is enough evidence to suggest that it is more likely that a local resident intends on visiting the beach in the next month compared to a tourist? Give a reason.

11 Sometimes two-way tables are filled with proportions (numbers between 0 and 1) rather than frequencies. Can you give a reason for doing this?

ENRICHMENT: Hypothesis testing	–	–	12

12 We have seen that two-way tables can be used to help decide if there is any connection between two categorical variables. Formally this is called hypothesis testing.

- The null hypothesis, H_0, is that there is no connection between the two variables.
- The alternative hypothesis, H_1, is that there is a connection between the two variables.

The following tables include two variables A and B. Decide if you would accept H_0 or reject H_0 (accept the alternative hypothesis) in each case and give a reason.

a

	B	**B′**	**Total**
A	4	6	10
A′	21	29	50
Total	25	35	60

b

	B	**B′**	**Total**
A	3	8	11
A′	12	4	16
Total	15	12	27

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9D Summary statistics

LEARNING INTENTIONS

- To understand the concept of quartiles for a set of data
- To be able to find the five-figure summary for a set of data
- To understand how the range and interquartile range describe the spread of a data set
- To know how to determine the outliers of a set of data

In addition to the median of a single set of data, there are two related statistics called the upper and lower quartiles. When data are placed in order, then the lower quartile is central to the lower half of the data and the upper quartile is central to the upper half of the data. These quartiles are used to calculate the interquartile range, which helps to describe the spread of the data, and determine whether or not any data points are outliers.

Australians who rent have a wide spread of ages: roughly 27% are 15–25 years; 31% are 25–35 years; 20% are 35–45 years; 15% are 45–55 years; and 7% are older than 55. A five-figure summary and a box plot would more effectively show this age spread.

Lesson starter: House prices

A real estate agent tells you that the median house price for a town in 2019 was \$753 000 and the mean was \$948 000.

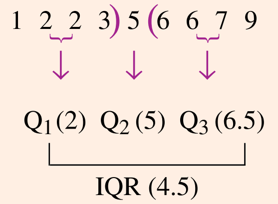
- Is it possible for the median and the mean to differ by so much?
- Under what circumstances could this occur? Discuss.

KEY IDEAS

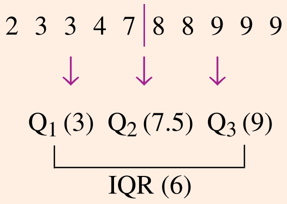
- **Five-figure summary**
  - **Minimum value** (min): the minimum value
  - **Lower quartile** (Q_1): the number above 25% of the ordered data
  - **Median** (Q_2): the middle value above 50% of the ordered data
  - **Upper quartile** (Q_3): the number above 75% of the ordered data
  - **Maximum value** (max): the maximum value

- **Measures of spread**
  - **Range** = max value − min value
  - **Interquartile range** (IQR)
    IQR = upper quartile − lower quartile
    $= Q_3 - Q_1$

    e.g. Odd number of values



Even number of values



  - The **standard deviation** is discussed in **Section 9F**.
- **Outliers** are data elements outside the vicinity of the rest of the data. More formally, a data point is an outlier when it is below the **lower fence** (i.e. lower limit) or above the **upper fence** (i.e. upper limit).
  - Lower fence = $Q_1 - 1.5 \times \text{IQR}$
  - Upper fence = $Q_3 + 1.5 \times \text{IQR}$
  - An outlier does not significantly affect the median of a data set.
  - An outlier does significantly affect the mean of a data set.

BUILDING UNDERSTANDING

1 **a** State the types of values that must be calculated for a five-figure summary.
b Explain the difference between the range and the interquartile range.
c What is an *outlier*?
d How do you determine if a score in a single data set is an outlier?

2 This data set shows the number of cars in 13 families surveyed.
0, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 4, 8
a Find the median (i.e. the middle value).
b By first removing the middle value, determine:
  i the lower quartile Q_1 (middle of lower half)
  ii the upper quartile Q_3 (middle of upper half).
c Determine the interquartile range (IQR).

3 The number of ducks spotted in eight different flocks are given in this data set.
2, 7, 8, 10, 11, 11, 13, 15
a **i** Find the median (i.e. average of the middle two numbers).
  ii Find the lower quartile (i.e. middle of the smallest four numbers).
  iii Find the upper quartile (i.e. middle of the largest four numbers).
b Determine the IQR.
c Calculate $Q_1 - 1.5 \times \text{IQR}$ and $Q_3 + 1.5 \times \text{IQR}$.
d Are there any outliers (i.e. numbers below $Q_1 - 1.5 \times \text{IQR}$ or above $Q_3 + 1.5 \times \text{IQR}$)?

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Example 7 Finding the range and IQR

Determine the range and IQR for these data sets by finding the five-figure summary.

a 2, 2, 4, 5, 6, 8, 10, 13, 16, 20

b 1.6, 1.7, 1.9, 2.0, 2.1, 2.4, 2.4, 2.7, 2.9

SOLUTION

a Range = 20 − 2 = 18

2 2 4 5 6 | 8 10 13 16 20

↑ ↑ ↑

Q_1 Q_2 (7) Q_3

$Q_2 = 7$, so $Q_1 = 4$ and $Q_3 = 13$.

$$\begin{aligned}\text{IQR} &= 13 - 4\\ &= 9\end{aligned}$$

b Range = 2.9 − 1.6 = 1.3

(1.6 1.7 | 1.9 2.0) 2.1 (2.4 2.4 | 2.7 2.9)

↑ ↑ ↑

Q_1 Q_2 Q_3

$$Q_1 = \frac{1.7 + 1.9}{2} = 1.8$$

$$Q_3 = \frac{2.4 + 2.7}{2} = 2.55$$

$$\begin{aligned}\text{IQR} &= 2.55 - 1.8\\ &= 0.75\end{aligned}$$

EXPLANATION

Range = max − min

First, split the ordered data in half to locate the median, which is $\frac{6+8}{2} = 7$.

Q_1 is the median of the lower half and Q_3 is the median of the upper half.

$\text{IQR} = Q_3 - Q_1$

Max = 2.9, min = 1.6

Leave the median out of the upper and lower halves when locating Q_1 and Q_3.

Average the two middle values of the lower and upper halves to find Q_1 and Q_3.

Now you try

Determine the range and IQR for these data sets by finding the five-figure summary.

a 3, 5, 5, 6, 7, 9, 10, 12

b 3.8, 3.9, 4.0, 4.2, 4.5, 4.5, 4.7

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Example 8 Finding the five-figure summary and outliers

The following data set represents the number of flying geese spotted on each day of a 13-day tour of England.

5, 1, 2, 6, 3, 3, 18, 4, 4, 1, 7, 2, 4

a For the data, find:

- **i** the minimum and maximum number of geese spotted
- **ii** the median
- **iii** the upper and lower quartiles
- **iv** the IQR
- **v** any outliers by determining the lower and upper fences.

b Can you give a possible reason for why the outlier occurred?

SOLUTION	EXPLANATION
a **i** Min = 1, max = 18	Look for the largest and smallest numbers and order the data:
ii 1, 1, 2, 2, 3, 3, 4, 4, 4, 5, 6, 7, 18 ∴ Median = 4	$\underset{}{1\ \ 1\ \ 2} \,\Big\vert\, 2\ \ 3\ \ 3) \;4\; (4\ \ 4\ \ 5 \,\Big\vert\, 6\ \ 7\ \ 18$ $\uparrow Q_1 \quad \uparrow Q_2 \quad \uparrow Q_3$
iii $\text{Lower quartile} = \dfrac{2+2}{2} = 2$ $\text{Upper quartile} = \dfrac{5+6}{2} = 5.5$	Since Q_2 falls on a data value, it is not included in the lower or upper halves when Q_1 and Q_3 are calculated.
iv $\text{IQR} = 5.5 - 2 = 3.5$	$\text{IQR} = Q_3 - Q_1$
v $\text{Lower fence} = Q_1 - 1.5 \times \text{IQR} = 2 - 1.5 \times 3.5 = -3.25$ $\text{Upper fence} = Q_3 + 1.5 \times \text{IQR} = 5.5 + 1.5 \times 3.5 = 10.75$ ∴ The outlier is 18.	A data point is an outlier when it is less than $Q_1 - 1.5 \times \text{IQR}$ or greater than $Q_3 + 1.5 \times \text{IQR}$. There are no numbers less than −3.25 but 18 is greater than 10.75.
b Perhaps a flock of geese was spotted that day.	

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Now you try

The following data set represents the number of people on 11 buses in a local area.

36, 24, 15, 23, 26, 0, 19, 24, 26, 33, 19

a For the data, find:
  - **i** the minimum and maximum number of people on the buses.
  - **ii** the median
  - **iii** the upper and lower quartiles
  - **iv** the IQR
  - **v** any outliers by determining the lower and upper fences.

b Can you give a possible reason for why the outlier occurred?

Exercise 9D

FLUENCY	1–3	1(½), 2, 3	2–4

Example 7

1 Determine the range and IQR for these data sets by finding the five-figure summary.

a 3, 4, 6, 8, 8, 10, 13

b 10, 10, 11, 14, 14, 15, 16, 18

c 1.2, 1.8, 1.9, 2.3, 2.4, 2.5, 2.9, 3.2, 3.4

d 41, 49, 53, 58, 59, 62, 62, 65, 66, 68

Example 8

2 The following numbers of cars, travelling on a quiet suburban street, were counted on each day for 15 days.

10, 9, 15, 14, 10, 17, 15, 0, 12, 14, 8, 15, 15, 11, 13

For the given data, find:

a the minimum and maximum number of cars counted

b the median

c the lower and upper quartiles

d the IQR

e any outliers by determining the lower and upper fences

f a possible reason for the outlier.

3 Summarise the data sets below by finding:
  - **i** the minimum and maximum values
  - **ii** the median (Q_2)
  - **iii** the lower and upper quartiles (Q_1 and Q_3)
  - **iv** the IQR
  - **v** any outliers.

a 4, 5, 10, 7, 5, 14, 8, 5, 9, 9

b 24, 21, 23, 18, 25, 29, 31, 16, 26, 25, 27

4 The number 20 is an outlier in this data set 1, 2, 2, 3, 4, 20.

a Calculate the mean if the outlier is:
  - **i** included
  - **ii** excluded.

b Calculate the median if the outlier is:
  - **i** included
  - **ii** excluded.

c By how much does including the outlier increase the:
  - **i** mean?
  - **ii** median?

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PROBLEM–SOLVING	5, 6(½)	6(½), 7	7, 8

5 Twelve different calculators had the following numbers of buttons.

36, 48, 52, 43, 46, 53, 25, 60, 128, 32, 52, 40

- **a** For the given data, find:
  - **i** the minimum and maximum number of buttons on the calculators
  - **ii** the median
  - **iii** the lower and upper quartiles
  - **iv** the IQR
  - **v** any outliers
  - **vi** the mean.
- **b** Which is a better measure of the centre of the data, the mean or the median? Explain.
- **c** Can you give a possible reason why the outlier has occurred?

6 Using the definition of an outlier, decide whether or not any outliers exist in the following sets of data. If so, list them.

- **a** 3, 6, 1, 4, 2, 5, 9, 8, 6, 3, 6, 2, 1
- **b** 8, 13, 12, 16, 17, 14, 12, 2, 13, 19, 18, 12, 13
- **c** 123, 146, 132, 136, 139, 141, 103, 143, 182, 139, 127, 140
- **d** 2, 5, 5, 6, 5, 4, 5, 6, 7, 5, 8, 5, 5, 4

7 For the data in this stem-and-leaf plot, find:

- **a** the IQR
- **b** any outliers
- **c** the median if the number 37 is added to the list
- **d** the median if the number 22 is added to the list instead of 37.

Stem	Leaf
0	1
1	6 8
2	0 4 6
3	2 3

2 | 4 means 24

8 Three different numbers have median 2 and range 2. Find the three numbers.

REASONING	9	9, 10	10–12

9 Explain what happens to the mean of a data set if all the values are:

- **a** increased by 5
- **b** multiplied by 2
- **c** divided by 10.

10 Explain what happens to the IQR of a data set if all values are:

- **a** increased by 5
- **b** multiplied by 2
- **c** divided by 10.

11 Give an example of a small data set that satisfies the following.

- **a** median = mean
- **b** median = upper quartile
- **c** range = IQR

12 Explain why, in many situations, the median is preferred to the mean as a measure of centre.

ENRICHMENT: Some research	–	–	13

13 Use the internet to search for data about a topic that interests you. Try to choose a single set of data that includes between 15 and 50 values.

- **a** Organise the data using:
  - **i** a stem-and-leaf plot
  - **ii** a frequency table and histogram.
- **b** Find the mean and the median.
- **c** Find the range and the interquartile range.
- **d** Write a brief report describing the centre and spread of the data, referring to parts **a** to **c** above.
- **e** Present your findings to your class or a partner.

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9E Box plots

LEARNING INTENTIONS

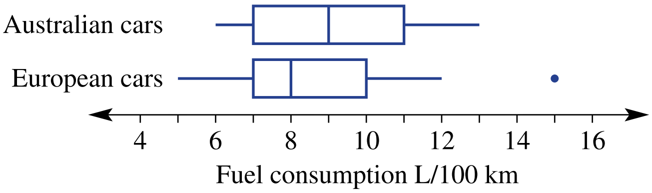
- To understand the features of a box plot in describing the spread of a set of data
- To know how to construct a box plot with outliers
- To be able to compare data sets using parallel box plots

The five-figure summary (min, Q_1, Q_2, Q_3, max) can be represented in graphical form as a box plot. Box plots are graphs that summarise single data sets. They clearly display the minimum and maximum values, the median, the quartiles and any outliers. Box plots also give a clear indication of how data are spread, as the IQR is shown by the width of the central box.

Medical researchers analyse data about the health of babies and mothers. Parallel box plots comparing birth weights of full-term babies born to smoking and non-smoking mothers show significantly lower weights for babies whose mothers smoke.

Lesson starter: Fuel consumption

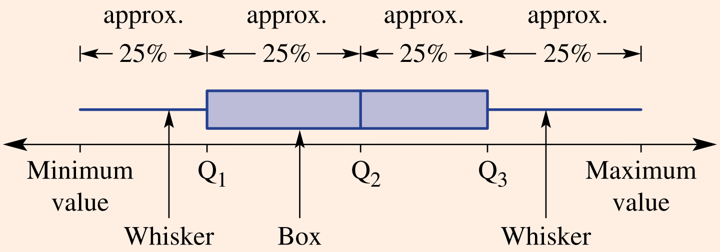
This parallel box plot summarises the average fuel consumption (litres per 100 km) for a group of Australian-made and European-made cars.



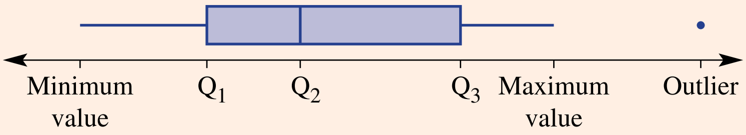
- What do the box plots say about how the fuel consumption compares between Australian-made and European-made cars?
- What does each part of the box plot represent?
- What do you think the dot (●) represents on the European cars box plot?

KEY IDEAS

- A **box plot** (also called a box-and-whisker plot) can be used to summarise a data set.
  - It divides the data set into four groups that are approximately equal in size (25%).



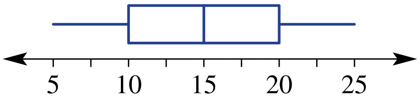
- An **outlier** is marked with a dot (•).
  - An outlier is greater than $Q_3 + 1.5 \times \text{IQR}$ or less than $Q_1 - 1.5 \times \text{IQR}$.
  - The whiskers stretch to the lowest and highest data values that are not outliers.



- **Parallel box plots** are two or more box plots drawn on the same scale. They are used to compare data sets within the same context.

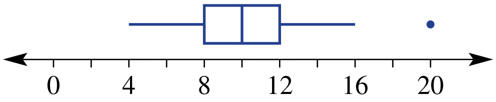
BUILDING UNDERSTANDING

① For this simple box plot, state:



- **a** the median (Q_2)
- **b** the minimum
- **c** the maximum
- **d** the range
- **e** the lower quartile (Q_1)
- **f** the upper quartile (Q_3)
- **g** the interquartile range (IQR).

② Complete the following for this box plot.



- **a** Find the IQR.
- **b** Calculate $Q_1 - 1.5 \times \text{IQR}$.
- **c** Calculate $Q_3 + 1.5 \times \text{IQR}$.
- **d** State the value of the outlier.
- **e** Check that the outlier is greater than $Q_3 + 1.5 \times \text{IQR}$.

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Example 9 Constructing a box plot

Consider the given data set:

5, 9, 4, 3, 5, 6, 6, 5, 7, 12, 2, 3, 5

a Determine whether any outliers exist by first finding Q_1 and Q_3.

b Draw a box plot to summarise the data, marking outliers if they exist.

SOLUTION

a

$$2\ \ 3\ \ \underbrace{3\ \ 4}_{\substack{\uparrow\\ Q_1}}\ \ 5\ \ 5)\underset{\substack{\uparrow\\ Q_2}}{5}(5\ \ 6\ \ \underbrace{6\ \ 7}_{\substack{\uparrow\\ Q_3}}\ \ 9\ \ 12$$

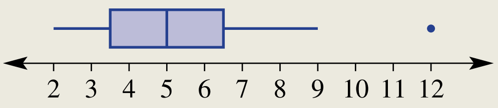
$$Q_1 = \frac{3+4}{2} = 3.5 \qquad Q_3 = \frac{6+7}{2} = 6.5$$

$$\therefore \text{IQR} = 6.5 - 3.5 = 3$$

$$\begin{aligned} Q_1 - 1.5 \times \text{IQR} &= 3.5 - 1.5 \times 3 \\ &= -1 \\ Q_3 + 1.5 \times \text{IQR} &= 6.5 + 1.5 \times 3 \\ &= 11 \end{aligned}$$

$\therefore$ 12 is an outlier.

b



EXPLANATION

Order the data to help find the quartiles.

Locate the median Q_2 then split the data in half above and below this value.

Q_1 is the middle value of the lower half and Q_3 the middle value of the upper half.

Determine IQR = $Q_3 - Q_1$.

Check for any outliers; i.e. values below $Q_1 - 1.5 \times \text{IQR}$ or above $Q_3 + 1.5 \times \text{IQR}$.

There are no data values below -1 but $12 > 11$.

Draw a line and mark in a uniform scale reaching from 2 to 12. Sketch the box plot by marking the minimum 2 and the outlier 12 and Q_1, Q_2 and Q_3. The end of the five-point summary is the nearest value below 11; i.e. 9.

Now you try

Consider the given data set:

12, 8, 19, 13, 22, 15, 1, 17, 24, 19

a Determine whether any outliers exist by first finding Q_1 and Q_3.

b Draw a box plot to summarise the data, marking outliers if they exist.

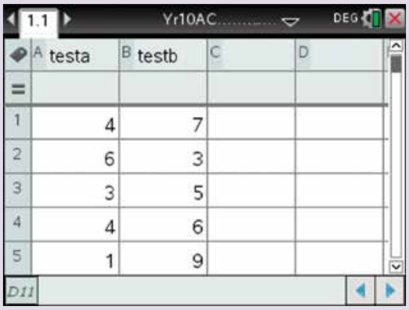
Using calculators to draw box plots

1 Type these data into lists and define them as Test A and Test B.
    Test A: 4, 6, 3, 4, 1, 3, 6, 4, 5, 3, 4, 3
    Test B: 7, 3, 5, 6, 9, 3, 6, 7, 4, 1, 4, 6

2 Draw parallel box plots for the data.

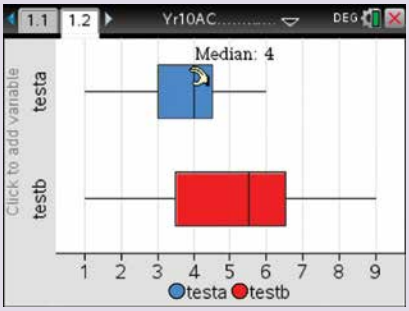
Using the TI-Nspire:

1 In a **Lists and spreadsheets** page type in the list names ***testa*** and ***testb*** and enter the values as shown.



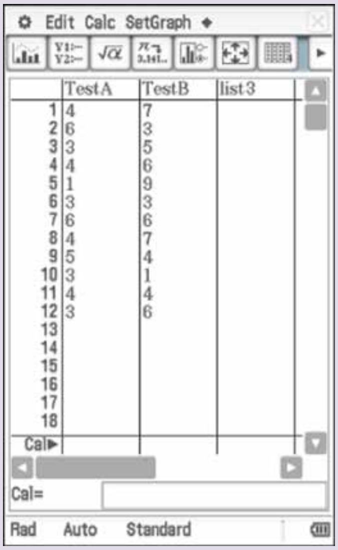
	A testa	B testb	C	D
=				
1	4	7		
2	6	3		
3	3	5		
4	4	6		
5	1	9		

2 Insert a **Data and Statistics** page and select the **testa** variable for the horizontal axis. Change to a box plot using menu**>Plot Type>Box Plot**. Trace (or hover over) to reveal the statistical measures. To show the box plot for ***testb***, use menu**>Plot Properties>Add X Variable** and select ***testb***.



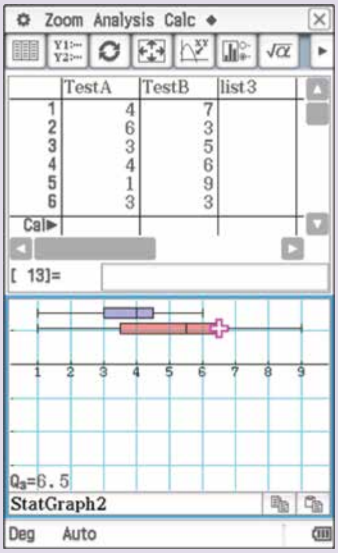
Using the ClassPad:

1 In the **Statistics** application enter the data into the lists. Give each column a title.



	TestA	TestB	list3
1	4	7	
2	6	3	
3	3	5	
4	4	6	
5	1	9	
6	3	3	
7	6	6	
8	4	7	
9	5	4	
10	3	1	
11	4	4	
12	3	6	
13			
14			
15			
16			
17			
18			

2 Tap. For graph **1**, set **Draw** to **On**, **Type** to **MedBox**, **XList** to **mainTestA** and **Freq** to **1**. For graph **2**, set **Draw** to **On**, **Type** to **MedBox**, **XList** to **mainTestB** and **Freq** to **1**. Tap **Set**. Tap



	TestA	TestB	list3
1	4	7	
2	6	3	
3	3	5	
4	4	6	
5	1	9	
6	3	3	

$Q_3=6.5$

StatGraph2

Exercise 9E

FLUENCY	1	1–2(½)	1–2(½)

Example 9

1 Consider the data sets below.

- **i** Determine whether any outliers exist by first finding Q_1 and Q_3.
- **ii** Draw a box plot to summarise the data, marking outliers if they exist.

a 4, 6, 5, 2, 3, 4, 4, 13, 8, 7, 6
b 1.8, 1.7, 1.8, 1.9, 1.6, 1.8, 2.0, 1.1, 1.4, 1.9, 2.2
c 21, 23, 18, 11, 16, 19, 24, 21, 23, 22, 20, 31, 26, 22
d 0.04, 0.04, 0.03, 0.03, 0.05, 0.06, 0.07, 0.03, 0.05, 0.02

2 First, find Q_1, Q_2 and Q_3 and then draw box plots for the given data sets. Remember to find outliers and mark them on your box plot if they exist.

a 11, 15, 18, 17, 1, 2, 8, 12, 19, 15
b 37, 48, 52, 51, 51, 42, 48, 47, 39, 41, 65
c 0, 1, 5, 4, 4, 4, 2, 3, 3, 1, 4, 3
d 124, 118, 73, 119, 117, 120, 120, 121, 118, 122

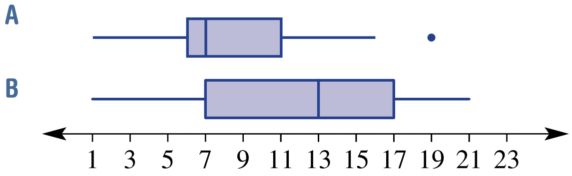
PROBLEM–SOLVING	3, 4	3, 4	3, 5

3 The following masses, in kilograms, of 15 Madagascan lemurs are recorded as part of a conservation project.

14.4, 15.5, 17.3, 14.6, 14.7
15.0, 15.8, 16.2, 19.7, 15.3
13.8, 14.6, 15.4, 15.7, 14.9

a Find Q_1, Q_2 and Q_3.
b Which masses, if any, would be considered outliers?
c Draw a box plot to summarise the lemurs' masses.

4 Two data sets can be compared using parallel box plots on the same scale as shown below.



a What statistical measure do these box plots have in common?
b Which data set (A or B) has a wider range of values?
c Find the IQR for:
- **i** data set A
- **ii** data set B.

d How would you describe the main difference between the two sets of data from which the parallel box plots have been drawn?

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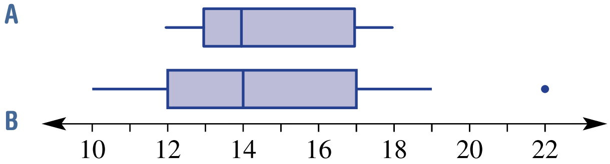
5 Consider these parallel box plots, A and B.



a What statistical measures do these box plots have in common?

b Which data set (A or B) has a wider range of values?

c Find the IQR for:

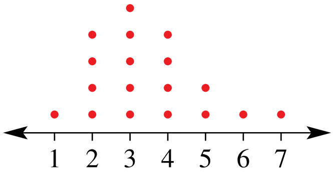
i data set A **ii** data set B.

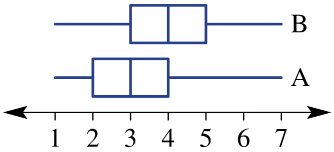
d How would you describe the main difference between the two sets of data from which the parallel box plots have been drawn?

REASONING	6	6, 7	7, 8

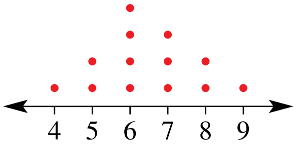
6 Select the box plot (A or B) that best matches the given dot plot or histogram.

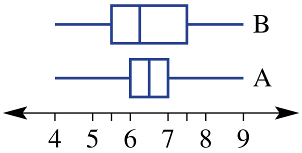
a



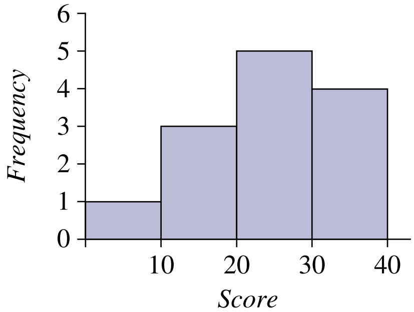


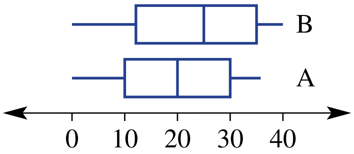
b





c





7 Fifteen essays are marked for spelling errors by a particular examiner and the following numbers of spelling errors are counted.

3, 2, 4, 6, 8, 4, 6, 7, 6, 1, 7, 12, 7, 3, 8

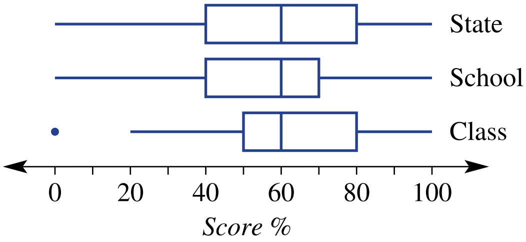
The same 15 essays are marked for spelling errors by a second examiner and the following numbers of spelling errors are counted.

12, 7, 9, 11, 15, 5, 14, 16, 9, 11, 8, 13, 14, 15, 13

a Draw parallel box plots for the data.

b Do you believe there is a major difference in the way the essays were marked by the two examiners? If yes, describe this difference.

8 The results for a Year 12 class are to be compared with the Year 12 results of the school and the State, using the parallel box plots shown.



a Describe the main differences between the performance of:
  i the class against the school
  ii the class against the State
  iii the school against the State.

b Why is an outlier shown on the class box plot but not shown on the school box plot?

ENRICHMENT: Creating your own parallel box plots – – 9

9 **a** Choose an area of study for which you can collect data easily, for example:
- heights or weights of students
- maximum temperatures over a weekly period
- amount of pocket money received each week for a group of students.

b Collect at least two sets of data for your chosen area of study – perhaps from two or three different sources, including the internet.
Examples:
- Measure student heights in your class and from a second class in the same year level.
- Record maximum temperatures for 1 week and repeat for a second week to obtain a second data set.
- Use the internet to obtain the football scores of two teams for each match in the previous season.

c Draw parallel box plots for your data.

d Write a report on the characteristics of each data set and the similarities and differences between the data sets collected.

9A **1** What type of data would these survey questions generate?

a How many pets do you have?

b What is your favourite ice-cream flavour?

9B **2** A Year 10 class records the length of time (in minutes) each student takes to travel from home to school. The results are listed here.

15	32	6	14	44	28	15	9	25	18
8	16	13	20	19	27	23	12	38	15

a Organise the data into a frequency table, using class intervals of 10. Include a percentage frequency column.

b Construct a histogram for the data, showing both the frequency and percentage frequency on the one graph.

c Describe the data in the histogram as symmetrical, positively skewed or negatively skewed.

9B **3** The data below gives the maximum daily wind speed in km/h over 14 days.

28 34 14 24 32 36 18 40 37 34 21 16 19 42

a Construct a stem-and leaf plot to display the data.

b Use the stem-and-leaf plot to find:

i the mode

ii the median

iii the mean (to one decimal place).

9C **4** This two-way table summarises data collected from a survey using a 5-point Likert scale. The survey asked 50 people if they are over or under the age of 50 and to what degree they agree with the idea of Australia becoming a republic.

	Strongly agree	Agree	Neutral	Disagree	Strongly disagree	Total
50 and over	0	4	6	4	8	22
Under 50	8	10	4	4	2	28
Total	6	14	10	10	10	50

a State how many of the people surveyed were under 50 and responded *Agree*.

b What percentage of the people surveyed were 50 and over and neutral to the idea?

c Would you say that this data supports the notion that younger people would prefer a republic? Give a reason.

9D

5 Determine the range and IQR for these data sets by finding the five-figure summary.

a 4, 9, 12, 15, 16, 18, 20, 23, 28, 32

b 4.2, 4.3, 4.7, 5.1, 5.2, 5.6, 5.8, 6.4, 6.6

9D

6 The following numbers of parked cars were counted in the school car park and adjacent street each day at morning recess for 14 school days.

36, 38, 46, 30, 69, 31, 40, 37, 55, 34, 44, 33, 47, 42

a For the data, find:

  i the minimum and maximum number of cars

  ii the median

  iii the upper and lower quartiles

  iv the IQR

  v any outliers.

b Can you give a possible reason for why the outlier occurred?

9E

7 The ages of a team of female gymnasts are given in this data set:

18, 23, 14, 28, 21, 19, 15, 32, 17, 18, 20, 13, 21

a Determine whether any outliers exist by first finding Q_1 and Q_3.

b Draw a box plot to summarise the data, marking outliers if they exist.

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9F Standard deviation OPTIONAL

LEARNING INTENTIONS
- To understand that standard deviation is a number that describes the spread of the data about the mean
- To know that a small standard deviation means data are concentrated about the mean
- To know how to calculate the standard deviation for a small set of data
- To be able to compare two sets of data referring to the mean and standard deviation

For a single data set we have already discussed the range and interquartile range to describe the spread of the data. Another statistic commonly used to describe spread is standard deviation. The standard deviation is a number that describes how far data values are from the mean. A data set with a relatively small standard deviation will have data values concentrated about the mean, and if a data set has a relatively large standard deviation then the data values will be more spread out from the mean.

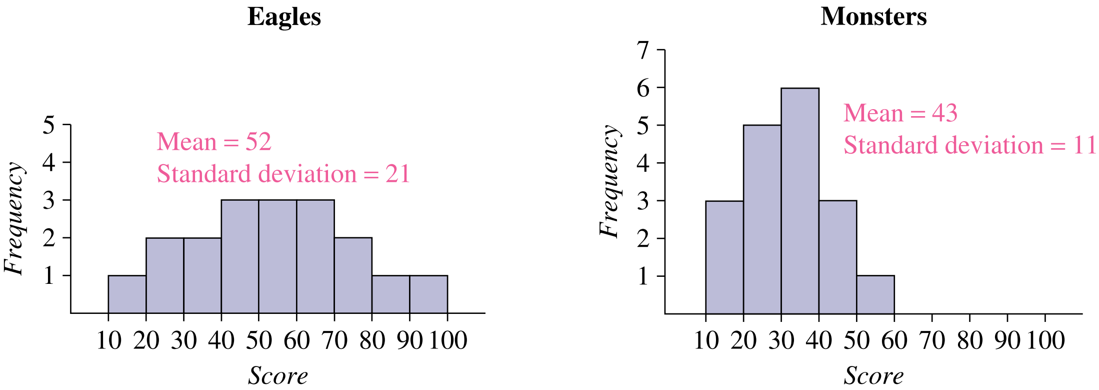
The standard deviation can be calculated by hand but, given the tedious nature of the calculation, technology can be used for more complex data sets. In this section technology is not required but you will be able to find a function on your calculator (often denoted s or σ) that can be used to find the standard deviation.

When selecting a sportsperson for a competition, the average and standard deviation of past results are useful. Two cricketers may have equal average runs per game, but the player with the smaller standard deviation is the more consistent batter.

Lesson starter: Which is the better team?

These histograms show the number of points scored by the Eagles and the Monsters basketball teams in an 18-round competition. The mean and standard deviation are given for each team.

- Which team has the higher mean? What does this say about the team's performance?
- Which team has the smaller standard deviation? What does this say about the team's performance? Discuss.



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KEY IDEAS

- The **standard deviation** is a number that describes how far data values deviate from the mean.
  - If data are concentrated about the mean, then the standard deviation is relatively small.
  - If data are spread out from the mean, then the standard deviation is relatively large.
  - The sample standard deviation is for a sample data set drawn from the population.
  - If every data value from a population is used, then we calculate the population standard deviation.

- To calculate the **sample standard deviation** (s), follow these steps.
  1. Find the mean ($\bar{x}$).
  2. Find the difference between each value and the mean (called the deviation).
  3. Square each deviation.
  4. Sum the squares of each deviation.
  5. Divide by the number of data values less 1 (i.e. $n - 1$).
  6. Take the square root. $s = \sqrt{\dfrac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n - 1}}$

- If the data represent the complete population, then divide by n instead of $(n - 1)$. This would give the **population standard deviation** (σ). Dividing by $(n - 1)$ for the sample standard deviation gives a better estimate of the population standard deviation.

- In many common situations we can expect 95% of the data to be within two standard deviations of the mean.

- The **mean absolute deviation** is calculated using the absolute (positive) difference between each data value and the mean rather than the square of the difference. The median absolute deviation uses the median rather than the mean.
  - The standard deviation will always be equal to or larger than the mean absolute deviation.
  - Extreme outliers will affect the standard deviation more than the mean absolute deviation.

BUILDING UNDERSTANDING

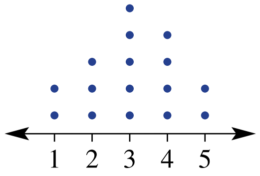
1 Use the word *smaller* or *larger* to complete each sentence.

a If data are more spread out from the mean, then the standard deviation is ___________.

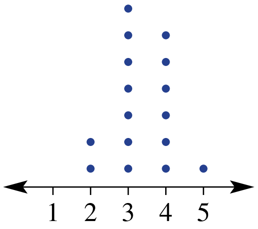
b If data are more concentrated about the mean, then the standard deviation is ____________.

2 Here are two dot plots, A and B.

A



B



a Which data set (A or B) would have the higher mean?

b Which data set (A or B) would have the higher standard deviation?

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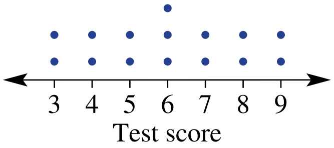
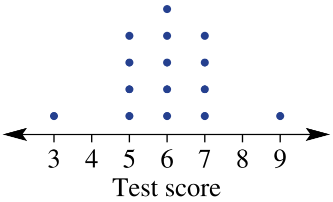
3 These dot plots show the results for a class of 15 students who sat tests A and B. Both sets of results have the same mean and range.

A



B



Which data set (A or B) would have the higher standard deviation? Give a reason.

4 This back-to-back stem-and-leaf plot compares the number of trees or shrubs in the backyards of homes in the suburbs of Gum Heights and Oak Valley.

a Which suburb has the smaller mean number of trees or shrubs? Do not calculate the actual means.

b Without calculating the actual standard deviations, which suburb has the smaller standard deviation?

Gum Heights Leaf	Stem	Oak Valley Leaf
7 3 1	0	
8 6 4 0	1	0
9 8 7 2	2	0 2 3 6 8 8 9
9 6 4	3	4 6 8 9
	4	3 6

2 | 8 means 28

Example 10 Calculating the standard deviation

Calculate the mean and sample standard deviation for this small data set, correct to one decimal place.

2, 4, 5, 8, 9

SOLUTION

$$\bar{x} = \frac{2+4+5+8+9}{5}$$

$$= 5.6$$

$$s = \sqrt{\frac{(2-5.6)^2+(4-5.6)^2+(5-5.6)^2+(8-5.6)^2+(9-5.6)^2}{5-1}}$$

$$= \sqrt{\frac{(-3.6)^2+(-1.6)^2+(-0.6)^2+(2.4)^2+(3.4)^2}{4}}$$

$$= 2.9 \text{ (to 1 d.p.)}$$

EXPLANATION

Sum all the data values and divide by the number of data values (i.e. 5) to find the mean.

Sum the square of all the deviations, divide by $(n-1)$ (i.e. 4) and then take the square root.

Deviation 1 is $2 - 5.6$ (the difference between the data value and the mean).

Now you try

Calculate the mean and sample standard deviation for this small data set, correct to one decimal place.

1, 2, 2, 4, 5

Example 11 Interpreting the standard deviation

This back-to-back stem-and-leaf plot shows the distribution of distances that 17 people in Darwin and Sydney travel to work. The means and standard deviations are given.

Consider the position and spread of the data and then answer the following.

a By looking at the stem-and-leaf plot, suggest why Darwin's mean is less than that of Sydney.

b Why is Sydney's standard deviation larger than that of Darwin?

c Give a practical reason for the difference in centre and spread for the data for Darwin and Sydney.

Darwin Leaf	Stem	Sydney Leaf	Sydney
8 7 4 2	0	1 5	$\bar{x} = 27.9$
9 9 5 5 3	1	2 3 7	$s = 15.1$
8 7 4 3 0	2	0 5 5 6	
5 2 2	3	2 5 9 9	Darwin
	4	4 4 6	$\bar{x} = 19.0$
	5	2	$s = 10.1$

3 | 5 means 35 km

SOLUTION	EXPLANATION
a The maximum score for Darwin is 35. Sydney's mean is affected by several values larger than 35.	The mean depends on every value in the data set.
b The data for Sydney are more spread out from the mean. Darwin's scores are more closely clustered near its mean.	Sydney has more scores with a large distance from its mean. Darwin's scores are closer to the Darwin mean.
c Sydney is a larger city and more spread out, so people have to travel farther to get to work.	Higher populations often lead to larger cities and longer travel distances.

Now you try

This stem-and-leaf plot shows the distribution of hours of television watched by 20 students from each of Year 7 and Year 12 over a one-month period. The means and standard deviations are given.

Consider the position and spread of the data and then answer the following.

a Why is the mean for Year 12 less than that for Year 7?

b Why is Year 7's standard deviation larger than that for Year 12?

c Give a practical reason for the difference in centre and spread for the Year 7 and Year 12 data.

Year 7	Stem	Year 12	Year 7
9	0	4 7	$\bar{x} = 30.1$
9 5 2	1	0 1 3 4 4 6 7 9	$s = 10.7$
9 8 8 4 1	2	1 2 2 4 5 7 8 9	
9 8 7 5 3 2 0	3	3 5	Year 12
6 3 2 2	4		$\bar{x} = 19.6$
			$s = 8.5$

2 | 4 means 24 hours

Exercise 9F

FLUENCY	1(½)	1(½), 2	1(½), 2

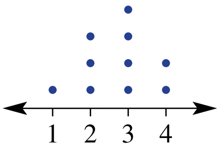
Example 10

1 Calculate the mean and sample standard deviation for these small data sets. Use the formula for the sample standard deviation. Round the standard deviation to one decimal place where necessary.

a 3, 5, 6, 7, 9
b 1, 1, 4, 5, 7
c 2, 5, 6, 9, 10, 11, 13
d 28, 29, 32, 33, 36, 37

2 Calculate the mean and sample standard deviation for the data in these graphs, correct to one decimal place.

a



b

Stem	Leaf
0	4
1	1 3 7
2	0 2

1 | 7 means 17

PROBLEM–SOLVING	3, 4	3, 4	4, 5

Example 11

3 This back-to-back stem-and-leaf plot shows the distribution of distances travelled by students at an inner-city and an outer-suburb school. The means and standard deviations are given.

Inner-city Leaf	Stem	Outer-suburb Leaf
9 6 4 3 1 1	0	3 4 9
9 4 2 0	1	2 8 8 9
7 1	2	1 3 4
	3	4
	4	1

2 | 4 means 24 km

Inner-city
$\bar{x} = 10.6$
$s = 8.0$
Outer-suburb
$\bar{x} = 18.8$
$s = 10.7$

Consider the position and spread of the data and then answer the following.

a Why is the mean for the outer-suburb school larger than that for the inner-city school?
b Why is the standard deviation for the inner-city school smaller than that for the outer-suburb school?
c Give a practical reason for the difference in centre and spread for the two schools.

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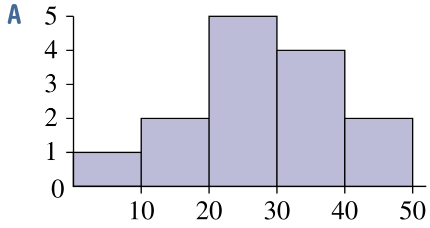
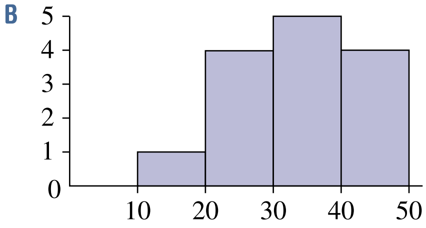
4 Consider these two histograms, and then state whether the following are true or false.





- **a** The mean for set A is greater than the mean for set B.
- **b** The range for set A is greater than the range for set B.
- **c** The standard deviation for set A is greater than the standard deviation for set B.

5 Find the mean and sample standard deviation for the scores in these frequency tables. Round the standard deviations to one decimal place.

a

Score	Frequency
1	3
2	1
3	3

b

Score	Frequency
4	1
5	4
6	3

REASONING	6	6, 7	7–9

6 Two simple data sets, A and B, are identical except for the maximum value, which is an outlier for set B.

A: 4, 5, 7, 9, 10
B: 4, 5, 7, 9, 20

- **a** Is the range for set A equal to the range for set B?
- **b** Is the mean for each data set the same?
- **c** Is the median for each data set the same?
- **d** Would the standard deviation be affected by the outlier? Explain.

7 Data sets 1 and 2 have means $\bar{x}_1$ and $\bar{x}_2$, and standard deviations s_1 and s_2.

- **a** If $\bar{x}_1 > \bar{x}_2$, does this necessarily mean that $s_1 > s_2$? Give a reason.
- **b** If $s_1 < s_2$ does this necessarily mean that $\bar{x}_1 < \bar{x}_2$?

8 Data sets A and B each have 20 data values and are very similar except for an outlier in set A. Explain why the interquartile range might be a better measure of spread than the range or the standard deviation.

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9 The mean absolute deviation takes the positive difference between each data value and the mean. This can be denoted $|x_i - \bar{x}|$ such that $|6 - 3| = 3$ and $|2 - 3| = 1$.

$$\text{mean absolute deviation} = \sqrt{\frac{|x_1 - \bar{x}| + |x_2 - \bar{x}| + \ldots + |x_n - \bar{x}|}{n - 1}}$$

Consider the data set 1, 3, 5, 8, 10.

a Calculate the mean absolute deviation correct to one decimal place.

b Calculate the standard deviation correct to one decimal place.

c Compare your answers from parts **a** and **b**.

Now, consider the data set 2, 4, 10, 18, 80.

d Calculate both the standard deviation and the mean absolute deviation correct to one decimal place. What effect does the extreme value have on the difference in deviation values between the two data sets?

ENRICHMENT: Study scores	–	–	10

10 The Mathematics study scores (out of 100) for 50 students in a school are as listed.

71, 85, 62, 54, 37, 49, 92, 85, 67, 89
96, 44, 67, 62, 75, 84, 71, 63, 69, 81
57, 43, 64, 61, 52, 59, 83, 46, 90, 32
94, 84, 66, 70, 78, 45, 50, 64, 68, 73
79, 89, 80, 62, 57, 83, 86, 94, 81, 65

The mean ($\bar{x}$) is 69.16 and the sample standard deviation (s) is 16.0.

a Calculate:
  - **i** $\bar{x} + s$
  - **ii** $\bar{x} - s$
  - **iii** $\bar{x} + 2s$
  - **iv** $\bar{x} - 2s$
  - **v** $\bar{x} + 3s$
  - **vi** $\bar{x} - 3s$

b Use your answers from part **a** to find the percentage of students with a score within:
  - **i** one standard deviation from the mean
  - **ii** two standard deviations from the mean
  - **iii** three standard deviations from the mean.

c
  - **i** Research what it means when we say that the data are ‘normally distributed’. Give a brief explanation.
  - **ii** For data that are normally distributed, find out what percentage of data are within one, two and three standard deviations from the mean. Compare this with your results for part **b** above.

9G Cumulative frequency and percentiles

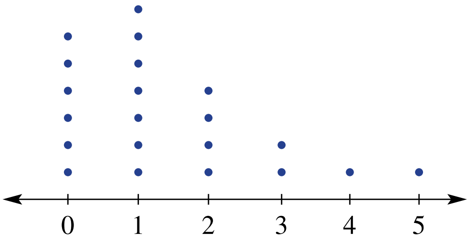
LEARNING INTENTIONS

- To know what is meant by cumulative frequency and percentage cumulative frequency
- To know what is meant by a percentile
- To be able to interpret a cumulative frequency table and curve
- To be able to construct a percentage cumulative frequency curve and use it to find percentiles

We are sometimes interested in the total number or percentage of data elements below or above a particular value. Cumulative frequency tables and curves help us to calculate and visualise this information. Cumulative data and corresponding graphs called cumulative frequency curves can help to find percentiles which are values above a certain percentage of the data. A percentile of 90 in a maths test, for example, is a score which sits above 90% of the other scores obtained by others in the class.

Lesson starter: Exploring percentiles

Here is a dot plot showing the results of a survey which asked 21 people how many pets they had at home.



- Find the median number of pets for the given data. i.e. Find the 50th percentile.
- The 75th percentile is Q_3, the upper quartile. Find Q_3.
- Find the IQR after locating Q_1, the lower quartile.
- What number of pets sits above 40% of the recorded data? i.e. Find the 40th percentile.
- What number of pets sits above 70% of the recorded data? i.e. Find the 70th percentile.

KEY IDEAS

- **Cumulative frequency** is the sum of all the frequencies of the categories (class intervals) up to and including that category.
  - Cumulative frequency is thought of as the ‘running total’.
  - The percentage cumulative frequency is the cumulative frequency expressed as a percentage.
  - Percentage cumulative frequency $= \dfrac{\text{Cumulative frequency}}{\text{Total number of data elements}} \times \dfrac{100}{1}$
- A percentage cumulative frequency curve is a graph of the percentage cumulative frequency values. Such a curve is constructed by:
  - Plotting the point (x, y) where:
    - x is the x value on the right side of each class interval
    - y is the percentage cumulative frequency for that class interval
  - Join the points including (0, 0) with a smooth curve.

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- The pth **percentile** is a data value that sits above $p\%$ of the data.
  - Q_1 is the 25th percentile
  - Q_2 is the 50th percentile
  - Q_3 is the 75th percentile
  - Any percentile can be approximated by reading off the percentage cumulative frequency curve.

BUILDING UNDERSTANDING

① This table shows the average number of hours 20 people spend in front of a screen per day.

Number of hours	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	4	4	20
3–	6	10	50
6–	8	18	90
9–	1	19	95
12–15	1	20	100

a How many people sit in front of a screen for:

i 3 to less than 6 hours?

ii 9 to less than 12 hours?

iii less than 6 hours?

iv less than 9 hours?

b What percentage of the people sit in front of a screen for:

i less than 9 hours?

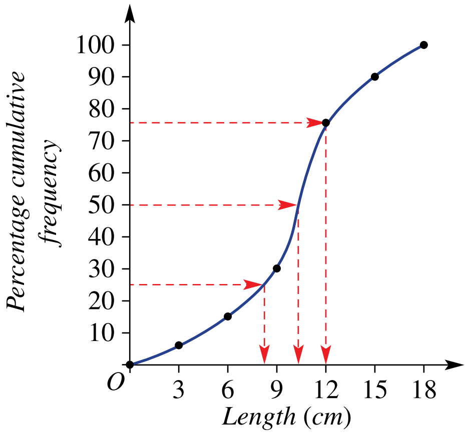
ii less than 15 hours?

iii 6 to less than 9 hours?

iv 12 to less than 15 hours?

② This table and matching percentage cumulative frequency curve show the lengths of various breeds of mice at a zoo recorded in centimetres.

Length (cm)	Percentage cumulative frequency
0–	5
3–	15
6–	30
9–	75
12–	90
15–18	100



a The red dashed lines on the graph show how the 25th (Q_1), 50th (Q_2) and 75th (Q_3) percentiles are found. Use these dashed lines to estimate these percentiles correct to the nearest whole number.

b Use the curve to estimate the following percentiles correct to the nearest whole number.

i 20th

ii 40th

iii 60th

iv 80th

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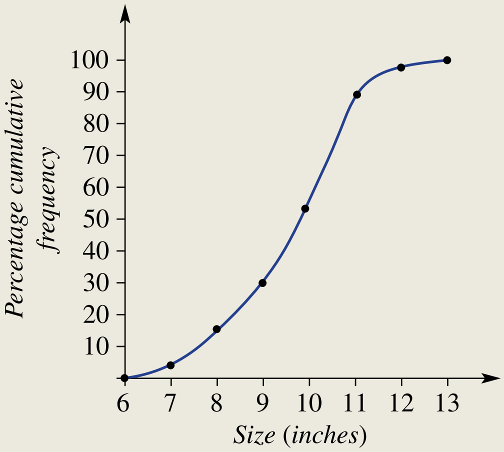
Example 12 Interpreting cumulative frequency tables and curves

40 people who recently purchased tablet devices were surveyed to find out about the screen size of their device. This table and graph summarise the findings with the data being recorded in inches.

Size (inches)	Frequency	Cumulative frequency	Percentage cumulative frequency
6–	1	1	2.5
7–	5	6	15
8–	6	12	30
9–	9	21	52.5
10–	15	36	90
11–	3	39	97.5
12–13	1	40	100



a Using the table, find how many of the tablet devices have the following sizes.
  - **i** from 7 to less than 8 inches
  - **ii** from 10 to less than 11 inches
  - **iii** less than 9 inches
  - **iv** less than 12 inches

b Using the table, find what percentage of the tablet devices have the following sizes.
  - **i** less than 10 inches
  - **ii** more than 11 inches
  - **iii** from 9 to less than 10 inches
  - **iv** from 11 to less than 12 inches

c Using the given percentage cumulative frequency curve, estimate the following.
  - **i** 50th percentile (median)
  - **ii** 20th percentile
  - **iii** 80th percentile

d Use the given graph to estimate the IQR for the given data.

SOLUTION	EXPLANATION
a **i** 5 **ii** 15 **iii** 12 **iv** 39	Use the frequency column in the table to find the total number of tablet devices for a given class interval. Use the cumulative frequency column to find the total number of tablet devices under a certain length.
b **i** 52.5% **ii** 10.0% **iii** 22.5% **iv** 7.5%	Use the percentage cumulative frequency column to find the percentage of a particular category or collection of categories.

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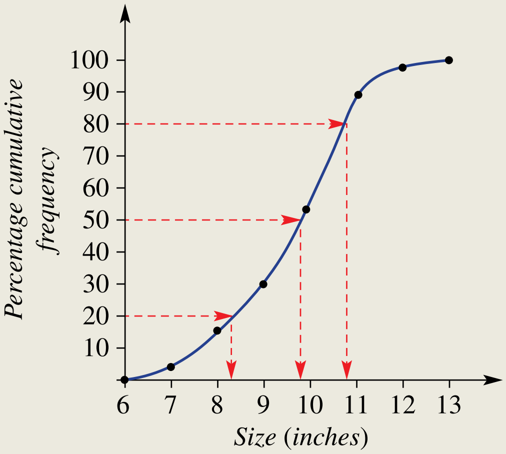
c **i** Approx. 9.9 inches
ii Approx. 8.3 inches
iii Approx. 10.8 inches



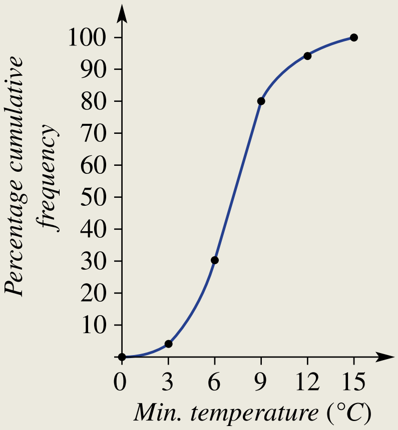
d $Q_1 \approx 8.6$
$Q_3 \approx 10.5$
$\text{IQR} \approx 10.5 - 8.6$
$= 1.9$ inches

Use the percentage cumulative frequency curve to estimate the 25th and 75th percentiles then find their difference.

Now you try

The minimum daily temperature in degrees Celsius was recorded for 30 days in June. This table and graph summarise the findings.

Min temp (°C)	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	1	1	3.3
3–	8	9	30.0
6–	15	24	80.0
9–	4	28	93.3
12–15	2	30	100.0



a Using the table, find how many days had minimum temperatures:
- **i** from 3 to less than 6 degrees Celsius
- **ii** from 9 to less than 12 degrees Celsius
- **iii** less than 3 degrees Celsius
- **iv** less than 9 degrees Celsius.

b Using the table, find what percentage of the days had minimum temperatures:
- **i** less than 6 degrees Celsius
- **ii** more than 9 degrees Celsius
- **iii** from 3 to less than 6 degrees Celsius
- **iv** from 6 to less than 9 degrees Celsius.

c Using the given percentage cumulative frequency curve, estimate the following. Approximate to the nearest integer.
- **i** 50th percentile (median)
- **ii** 20th percentile
- **iii** 80th percentile

d Use the given graph to estimate the IQR for the given data.

Example 13 Constructing a percentage cumulative frequency curve

The number of hours of streaming service shows watched per week by a group of 30 students are summarised in this table.

Number of hours	Frequency
0–	1
4–	4
8–	7
12–	12
16–	3
20–	2
24–28	1

a Construct a cumulative frequency column and percentage cumulative frequency column for the given table.

b Construct a cumulative frequency curve for the data.

c Use the cumulative frequency curve to approximate each of the following percentiles.

- **i** 80th
- **ii** 50th
- **iii** 25th

d Estimate the percentage cumulative frequency corresponding to watching 10 hours of streaming service per week.

e Interpret the 50th percentile.

SOLUTION

a

Number of hours	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	1	1	3.3
4–	4	5	16.7
8–	7	12	40.0
12–	12	24	80.0
16–	3	27	90.0
20–	2	29	96.7
24–28	1	30	100.0

EXPLANATION

Each cumulative frequency value is the sum of all the frequencies up to and including that point.

The percentage cumulative frequency is the cumulative frequency expressed as a percentage. Rounding has been used to one decimal place.

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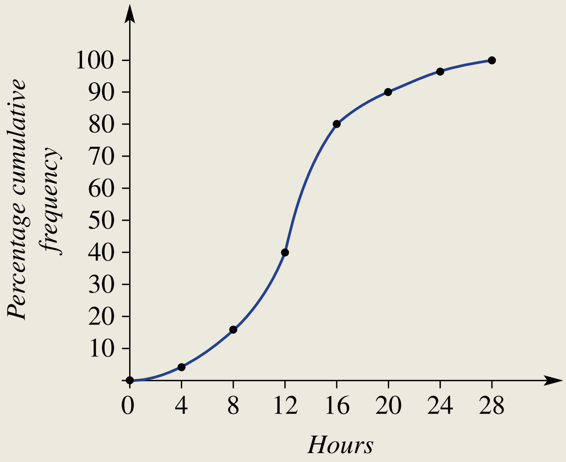
b



Plot the points (x, y) with x being the right side of each class interval and y being the matching percentage cumulative frequency for that interval.
Join with a smooth curve and include the point (0, 0).

c **i** 16 hours
ii Approx. 13 hours
iii Approx. 9.5 hours

Draw in a dashed horizontal line from the chosen percentile to the curve then vertically down to receive the percentile.

d Approximately 30%

Draw in a dashed vertical line rising from 10 hours then across to the y-axis to receive the percentage.

e Approximately 50% of the students watch less than 13 hours of streaming service per week.

The 50th percentile is the number of hours above which 50% of the students watch.

Now you try

Twenty-five marathon runners complete a 42 km run and their finishing times in minutes are summarised below.

Time (min)	Frequency
120–	1
130–	3
140–	4
150 –	8
160–	7
170–180	2

a Construct a cumulative frequency column and percentage cumulative frequency column for the given table.
b Construct a cumulative frequency curve for the data.
c Use the cumulative frequency curve to approximate each of the following percentiles.
i 80th **ii** 50th **iii** 25th
d Estimate the percentage cumulative frequency corresponding to a time of 165 minutes.
e Interpret the 50th percentile.

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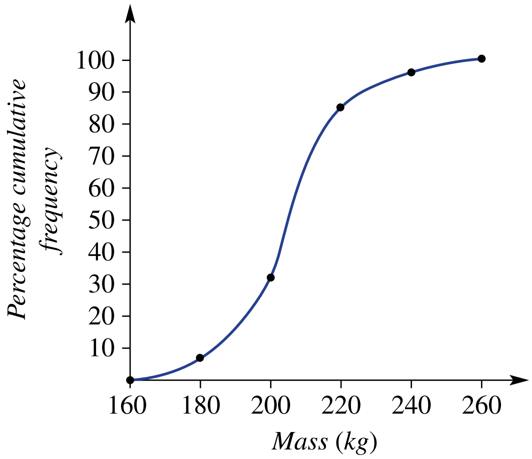
Exercise 9G

FLUENCY	1–3	1, 3, 4	2, 4

Example 12

1 Twenty-eight bales of wool are weighed before leaving the farm shed. This table and graph summarise the findings with the data being recorded in kilograms.

Mass (kg)	Frequency	Cumulative frequency	Percentage cumulative frequency
160–	2	2	7.1
180–	7	9	32.1
200–	15	24	85.7
220–	3	27	96.4
240–260	1	28	100.0



a Using the table, find how many of the wool bales have the following weights:
- **i** from 160 to less than 180 kg
- **ii** from 200 to less than 220 kg
- **iii** less than 220 kg
- **iv** less than 240 kg.

b Using the table, find what percentage of the wool bales have the following weights. Round to one decimal place:
- **i** less than 180 kg
- **ii** more than 200 kg
- **iii** from 180 to less than 200 kg
- **iv** from 240 to less than 260 kg.

c Using the given percentage cumulative frequency curve, estimate the following. Approximate to the nearest integer.
- **i** 50th percentile (median)
- **ii** 30th percentile
- **iii** 85th percentile

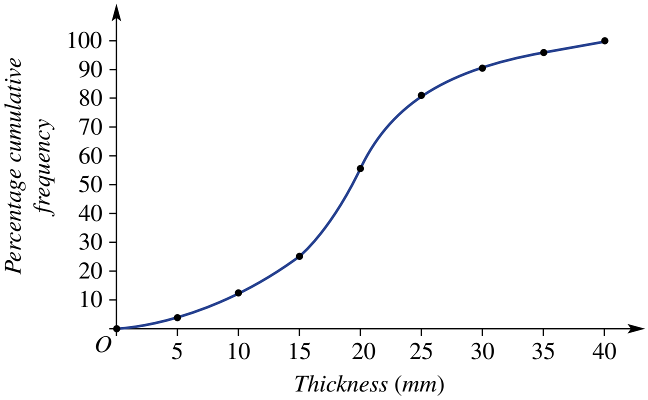
d Use the given graph to estimate the IQR for the given data.

2 Thirty-two library books are randomly selected and their thickness measured in millimetres. The results are summarised in this table and the graph below.

Thickness (mm)	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	1	1	3.1
5–	3	4	12.5
10–	4	8	25.0
15–	10	18	56.3
20–	8	26	81.3
25–	3	29	90.6
30–	2	31	96.9
35–40	1	32	100.0

a Using the table, find how many of the books have the following thickness:
- **i** from 10 to less than 15 mm
- **ii** from 20 to less than 25 mm
- **iii** less than 30 mm
- **iv** less than 10 mm.

b Using the table, find what percentage of the wool bales have the following weights. Round to one decimal place:
- **i** less than 20 mm
- **ii** more than 25 mm
- **iii** from 15 to less than 20 mm
- **iv** from 30 to less than 35 mm.



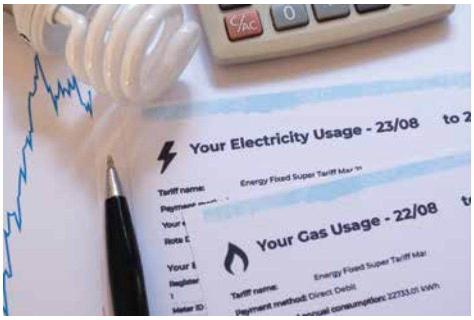
c Using the given percentage cumulative frequency curve estimate the following. Approximate to the nearest integer.
- **i** 50th percentile (median)
- **ii** 20th percentile
- **iii** 70th percentile

d Use the given graph to estimate the IQR for the given data. Approximate to the nearest integer.

Example 13

3 Fifty households were surveyed to find out the size of their most recent gas bill and the results are summarised in this table.

Bill ($)	Frequency
0–	3
100–	6
200–	10
300–	13
400–	10
500–	6
600–700	2



a Construct a cumulative frequency column and percentage cumulative frequency column for the given table.

b Construct a cumulative frequency curve for the data.

c Use the cumulative frequency curve to approximate each of the following percentiles.
- **i** 90th
- **ii** 50th
- **iii** 30th

d Estimate the percentage cumulative frequency corresponding to a gas bill of $450.

e Interpret the 50th percentile.

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4 Fifty-five backyards were measured to find how deep they are and the results are summarised in this table.

Depth (m)	Frequency
0–	0
5–	2
10–	7
15–	18
20–	22
25–	5
30–35	1

a Construct a cumulative frequency column and percentage cumulative frequency column for the given table. Round to one decimal place in the percentage cumulative frequency column.

b Construct a cumulative frequency curve for the data.

c Use the cumulative frequency curve to approximate each of the following percentiles to one decimal place.

i 70th **ii** 50th **iii** 10th

d Estimate the percentage cumulative frequency corresponding to a backyard depth of 24 m.

e Interpret the 50th percentile.

PROBLEM–SOLVING 5, 6 | 5(½), 6 | 5(½), 6, 7

5 State the values of the missing pronumerals in these tables. Round percentages to one decimal place.

a

Score	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	0	0	0.0
5–	4	4	*c*
10–	5	*a*	29.1
15–	8	17	*d*
20–	6	23	71.9
25–	5	*b*	87.5
30–	3	31	96.9
35–40	1	32	100.0

b

Score	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	2	2	6.9
3–	5	7	24,1
6–	7	14	*c*
9–	6	*a*	69.0
12–	5	25	*d*
15–	3	28	96.6
18–21	1	*b*	100.0

c

Score	Frequency	Cumulative frequency	Percentage cumulative frequency
10–	1		
20–	5	*a*	
30–	13		*c*
40–	9		
50–	8		
60–	4		*d*
70–	2	*b*	
80–90	2		

d

Score	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	0		
20–	6		*c*
40–	20	*a*	
60–	12	*b*	
80–	6		*d*
100–	2		

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6 The time in seconds taken for 42 runners at a school athletics carnival to run a 80-metre sprint is summarised in this table.

Score	Frequency
10–	1
11–	5
12–	13
13–	9
14–	8
15–	4
16–17	2

a Find the cumulative frequency for the following times.

i 12 to less than 13 seconds **ii** 14 to less than 15 seconds

b Find the percentage cumulative frequency for the following times. Round to one decimal place.

i 11 to less than 12 seconds **ii** 15 to less than 16 seconds

c What percentages of runners ran the following times? Round to one decimal place.

i less than 13 seconds **ii** more than 13 seconds

7 The heights, in centimetres, of 32 randomly selected people were measured, resulting in the following data.

150, 166, 169, 173, 175, 175, 164, 171, 163, 152, 167, 176, 154, 181, 164, 171
154, 177, 181, 174, 155, 178, 180, 184, 178, 162, 173, 173, 170, 161, 154, 157

Use a frequency table and percentage cumulative frequency curve to estimate the following.

a median **b** IQR

REASONING	8	8, 9	9, 10

8 For these frequency tables, state the class interval which would correspond to the steepest part of a percentage cumulative frequency curve.

a

Class interval	Frequency
0–	2
10–	4
20–	7
30–	5
40–	3
50–60	1

b

Class interval	Frequency
0–	1
2–	8
4–	6
6–	3
8–10	2

9 Without drawing the curve, describe the general shape of the percentage cumulative frequency curve for these frequency tables.

a

Class interval	Frequency
0–	2
5–	10
10–	8
15–	4
20–	3
25–30	1

b

Class interval	Frequency
0–	2
100–	4
200–	7
300–	31
400–	24
500–600	3

10 Explain why using less class intervals would lead to less accurate percentile calculations on a percentage cumulative frequency curve.

ENRICHMENT: Approximating percentiles without a percentage cumulative frequency curve	–	–	11

11 It is possible to make a reasonably good approximation of a percentile by just considering the percentage cumulative frequency values in a table. Consider this given table.

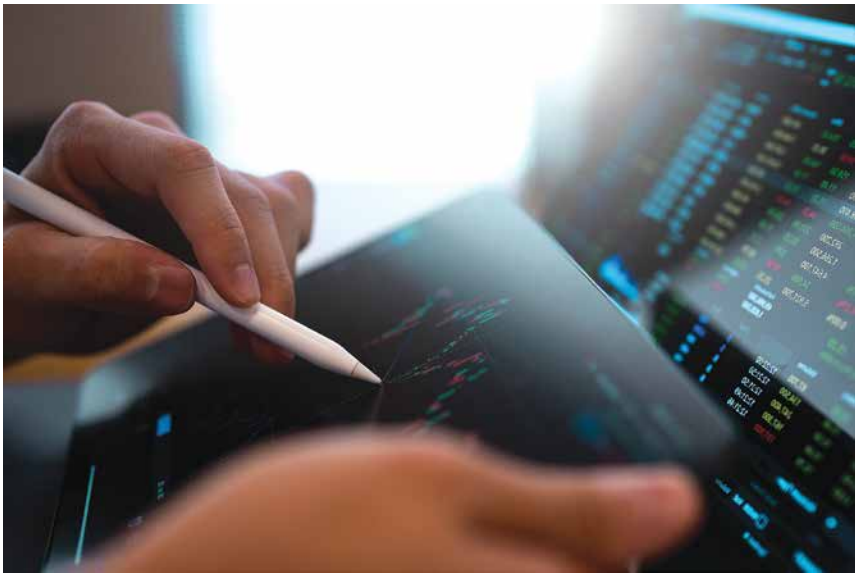
Score	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	1	1	5
10–	3	4	20
20–	6	10	50
30–	5	15	75
40–	3	18	90
50–60	2	20	100

Let's say we are trying to find the 30th percentile. We know that the 30th percentile will sit somewhere in the 20 to less than 30 class interval and we can see that the percentage cumulative frequencies either side are 20% and 50%. Now 30% sits at one third of the way between 20% and 50% and similarly 23.3 sits approximately one third of the way between 20 and 30 in the matching class interval. We can therefore approximate the 30th percentile as 23.3. Note that this would be the result that we would obtain if the percentage cumulative frequency curve was approximated using straight line segments between points.

a Use this technique to approximate the following percentiles using the data in the table above. Round to one decimal place.

i 10th **ii** 40th **iii** 80th **iv** 95th

b Use this technique to estimate the IQR for the given data.



9H Time-series data

LEARNING INTENTIONS

- To understand that time-series data are data recorded at regular time intervals
- To know how to plot a time-series graph with time on the horizontal axis
- To be able to use a time-series plot to describe any trend in the data

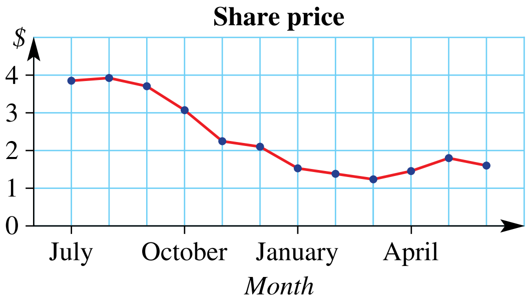
A time series is a sequence of data values that are recorded at regular time intervals. Examples include temperature recorded on the hour, speed recorded every second, population recorded every year and profit recorded every month. A line graph can be used to represent time-series data and these can help to analyse the data, describe trends and make predictions about the future.

The BOM (Bureau of Meteorology) publishes time-series graphs of Australian annual and monthly mean temperature anomalies, i.e. deviations from the overall average. Over recent decades, these graphs show an upward trend of positive and increasing anomalies.

Lesson starter: Share price trends

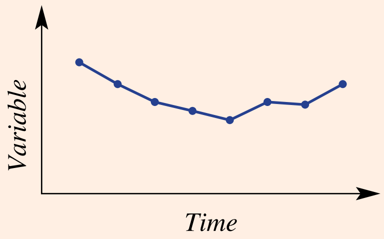
A company's share price is recorded at the end of each month of the financial year, as shown in this time-series graph.

- Describe the trend in the data at different times of the year.
- At what time of year do you think the company starts reporting bad profit results?
- Does it look like the company's share price will return to around $4 in the next year? Why?

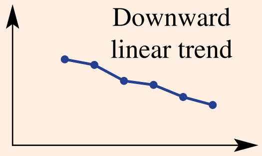


KEY IDEAS

- **Time-series data** are recorded at regular time intervals.
- The graph or plot of a time series uses:
  - time on the horizontal axis as the **independent** variable
  - line segments connecting points on the graph.
  - the variable being considered on the vertical axis as the **dependent** variable.



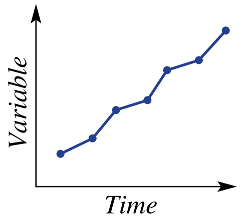
- If the time-series plot results in points being on or near a straight line, then we say that the trend is **linear**.



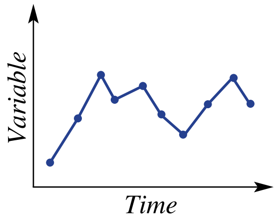
BUILDING UNDERSTANDING

1 Describe the following time-series plots as having a linear (i.e. straight-line trend), non-linear trend (i.e. a curve) or no trend.

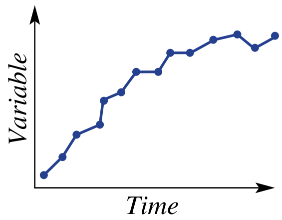
a



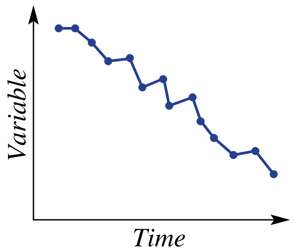
b



c



d



2 This time-series graph shows the temperature over the course of an 8-hour school day.

a State the temperature at:

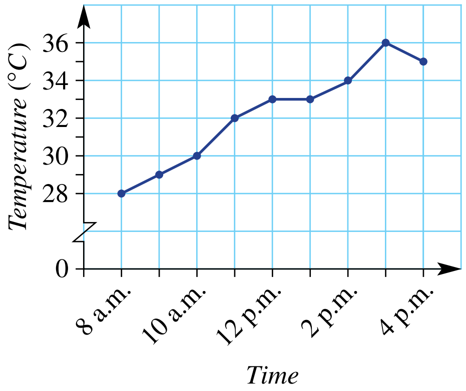
i 8 a.m. **ii** 12 p.m.
iii 1 p.m. **iv** 4 p.m.

b What was the maximum temperature?

c During what times did the temperature:

i stay the same? **ii** decrease?

d Describe the general trend in the temperature for the 8-hour school day.



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Example 14 Plotting and interpreting a time-series plot

The approximate population of an outback town is recorded from 1990 to 2005.

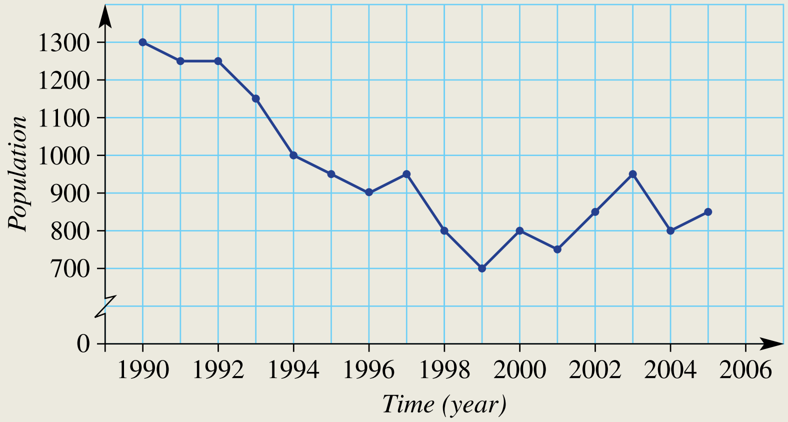
Year	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005
Population	1300	1250	1250	1150	1000	950	900	950	800	700	800	750	850	950	800	850

a Plot the time series.

b Describe the trend in the data over the 16 years.

SOLUTION

a



EXPLANATION

Use time on the horizontal axis. Break the y-axis so as to not include 0 – 700. Join points with line segments.

b The population declines steadily for the first 10 years. The population rises and falls in the last 6 years, resulting in a slight upwards trend.

Interpret the overall rise and fall of the lines on the graph.

Now you try

The average price of lambs at a market over 14 weeks is given in this table.

Week	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Price ($)	82	80	85	89	91	87	93	104	100	111	108	105	112	119

a Plot the time series.

b Describe the trend in the data over the 14 weeks.

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Exercise 9H

FLUENCY	1, 2	1, 3	2, 3

Example 14

1 The approximate population of a small village is recorded from 2010 to 2020.

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Population	550	500	550	600	700	650	750	750	850	950	900

a Plot the time-series graph.

b Describe the general trend in the data over the 11 years.

c For the 11 years, what was the:

i minimum population? **ii** maximum population?

2 A company's share price over 12 months is recorded in this table.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Price ($)	1.30	1.32	1.35	1.34	1.40	1.43	1.40	1.38	1.30	1.25	1.22	1.23

a Plot the time-series graph. Break the y-axis to exclude values from $0 to $1.20.

b Describe the way in which the share price has changed over the 12 months.

c What is the difference between the maximum and minimum share price in the 12 months?

3 The pass rate (%) for a particular examination is given in a table over 10 years.

Year	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024
Pass rate (%)	74	71	73	79	85	84	87	81	84	83

a Plot the time-series graph for the 10 years.

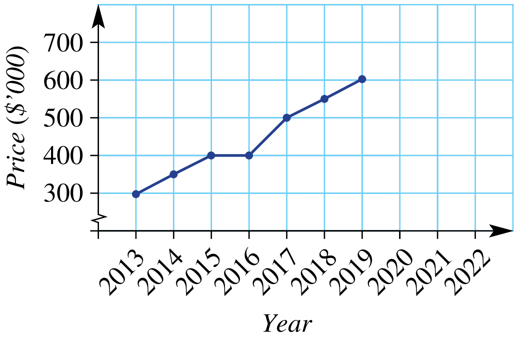
b Describe the way in which the pass rate for the examination has changed in the given time period.

c In what year was the pass rate a maximum?

d By how much had the pass rate improved from 2015 to 2019?

PROBLEM–SOLVING	4, 5	4, 5	5, 6

4 This time-series plot shows the upwards trend of house prices in an Adelaide suburb over 7 years from 2013 to 2019.



a Would you say that the general trend in house prices is linear or non-linear?

b Assuming the trend in house prices continues for this suburb, what would you expect the house price to have been in:

i 2020? **ii** 2022?

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5 The two top-selling book stores for a company list their sales figures for the first 6 months of the year. Sales amounts are in thousands of dollars.

	July	August	September	October	November	December
City Central ($'000)	12	13	12	10	11	13
Southbank ($'000)	17	19	16	12	13	9

a What was the difference in the sales volume for:

i August? **ii** December?

b In how many months did the City Central store sell more books than the Southbank store?

c Construct a time-series plot for both stores on the same set of axes.

d Describe the trend of sales for the 6 months for:

i City Central **ii** Southbank.

e Based on the trend for the sales for the Southbank store, what would you expect the approximate sales volume to be in January?

6 Two pigeons (Green Tail and Blue Crest) each have a beacon that communicates with a recording machine. The distance of each pigeon from the machine is recorded every hour for 8 hours.

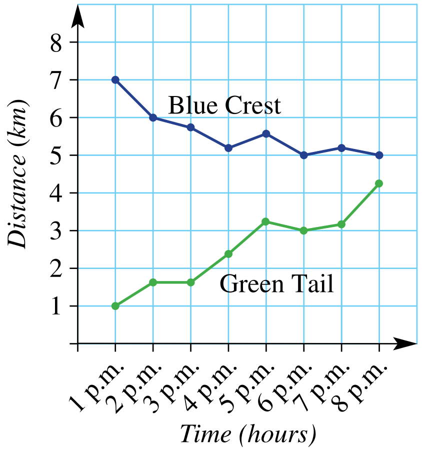
a State the distance from the machine at 3 p.m. for:

i Blue Crest **ii** Green Tail.

b Describe the trend in the distance from the recording machine for:

i Blue Crest **ii** Green Tail.

c Assuming that the given trends continue, predict the time when the pigeons will be the same distance from the recording machine.



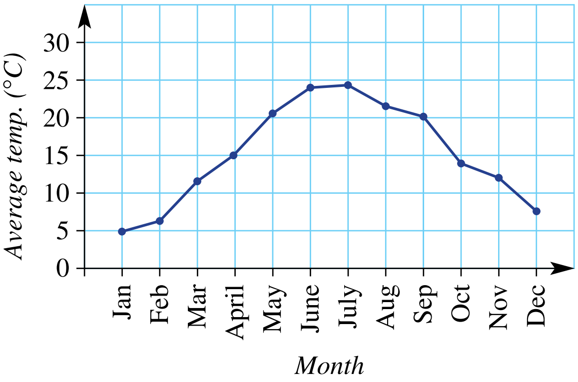
REASONING	7	7, 8	8, 9

7 The average monthly maximum temperature for a city is illustrated in this graph.

a Explain why the average maximum temperature for December is close to the average maximum temperature for January.

b Do you think this graph is for an Australian city?

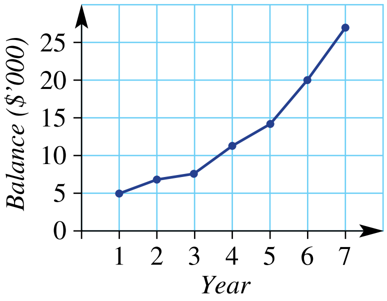
c Do you think the data are for a city in the Northern Hemisphere or the Southern Hemisphere? Give a reason.



8 The balance of an investment account is shown in this time-series plot.

a Describe the trend in the account balance over the 7 years.

b Give a practical reason for the shape of the curve that models the trend in the graph.



9 A drink at room temperature is placed in a fridge that is at 4°C.

a Sketch a time-series plot that might show the temperature of the drink after it has been placed in the fridge.

b Would the temperature of the drink ever get to 3°C? Why?

ENRICHMENT: Moving run average	–	–	10

10 In this particular question, a moving average is determined by calculating the average of all data values up to a particular time or place in the data set.

Consider a batsman in cricket with the following runs scored from 10 completed innings.

Innings	1	2	3	4	5	6	7	8	9	10
Score	26	38	5	10	52	103	75	21	33	0
Moving average	26	32	23							

a Complete the table by calculating the moving average for innings 4–10. Round to the nearest whole number where required.

b Plot the score and moving averages for the batter on the same set of axes.

c Describe the behaviour of the:

i score graph **ii** moving average graph.

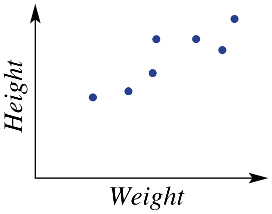
d Describe the main difference in the behaviour of the two graphs. Give reasons.

9I Bivariate data and scatter plots

LEARNING INTENTIONS

- To understand that bivariate data involve data about two variables in a given context
- To know how to draw a scatter plot to compare data from two variables
- To be able to use a scatter plot to describe the correlation between the two variables using key terms

When we collect information about two variables in a given context, we are collecting bivariate data. As there are two variables involved in bivariate data, we use a number plane to graph the data. These graphs are called scatter plots and are used to illustrate a relationship that may exist between the variables. Scatter plots make it very easy to see the strength of the association between the two variables.



Market research analysts find a positive correlation in scatter plots of advertising spending versus product sales. AI (artificial intelligence) algorithms use automated marketing to create highly effective digital advertising, specifically targeted to each person's online presence.

Lesson starter: A relationship or not?

Consider the two variables in each part below.

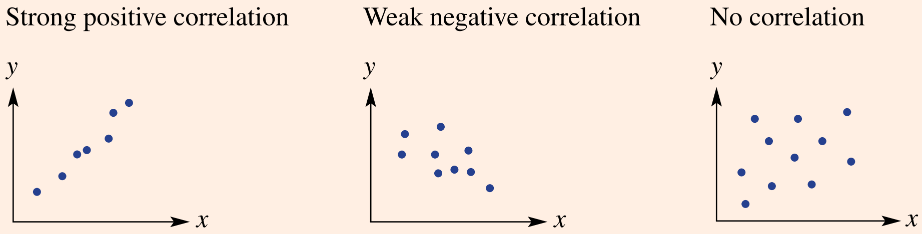
- Would you expect there to be some relationship between the two variables in each of these cases?
- If you think a relationship exists, would you expect the second listed variable to increase or to decrease as the first variable increases?

a Height of person and Weight of person
b Temperature and Life of milk
c Length of hair and IQ
d Depth of topsoil and Brand of motorcycle
e Years of education and Income
f Spring rainfall and Crop yield
g Size of ship and Cargo capacity
h Fuel economy and CD track number
i Amount of traffic and Travel time
j Cost of 2 litres of milk and Ability to swim
k Background noise and Amount of work completed

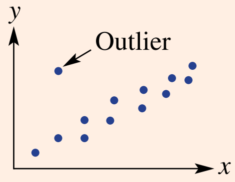
How might the size of a ship and its cargo capacity be related?

KEY IDEAS

- **Bivariate data** include data for two variables.
  - The two variables are usually related; for example, height and weight.
  - The variable that is changed or controlled is the independent variable and is on the x-axis.
  - The variable being tested or measured is the dependent variable and is on the y-axis.
- A **scatter plot** is a graph on a number plane in which the axes variables correspond to the two variables from the bivariate data.
- The words *relationship, correlation* and *association* are used to describe the way in which variables are related.
- Types of correlation:
  For example:



- An **outlier** can clearly be identified as a data point that is isolated from the rest of the data.



BUILDING UNDERSTANDING

① Decide if it is likely for there to be a strong correlation between these pairs of variables.

a Height of door and Thickness of door handle
b Weight of car and Fuel consumption
c Temperature and Length of phone calls
d Size of textbook and Number of textbook chapters
e Diameter of flower and Number of bees
f Amount of rain and Size of vegetables in the vegetable garden

② For each of the following sets of bivariate data with variables x and y, decide whether y generally increases or decreases as x increases.

a

x	1	2	3	4	5	6	7	8	9	10
y	3	2	4	4	5	8	7	9	11	12

b

x	0.1	0.3	0.5	0.9	1.0	1.1	1.2	1.6	1.8	2.0	2.5
y	10	8	8	6	7	7	7	6	4	3	1

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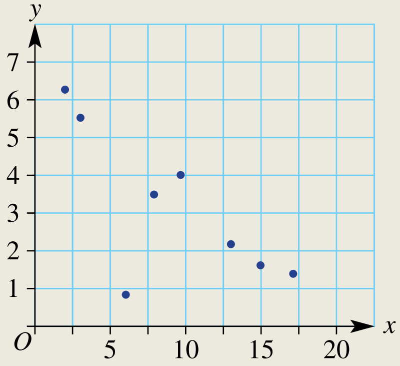
Example 15 Constructing and interpreting scatter plots

Consider this simple bivariate data set.

x	13	9	2	17	3	6	8	15
y	2.1	4.0	6.2	1.3	5.5	0.9	3.5	1.6

a Draw a scatter plot for the data.
b Describe the correlation between x and y as positive or negative.
c Describe the correlation between x and y as strong or weak.
d Identify any outliers.

SOLUTION	EXPLANATION
a 	Plot each point using a • on graph paper.
b Negative correlation	As x increases, y decreases.
c Strong correlation	The downwards trend in the data is clearly defined.
d The outlier is (6, 0.9).	This point defies the trend.

Now you try

Consider this simple bivariate data set.

x	12	2	15	10	4	5	8	13	7
y	4.0	1.3	4.5	3.6	1.8	2.0	2.5	2.0	2.9

a Draw a scatter plot for the data.
b Describe the correlation between x and y as positive or negative.
c Describe the correlation between x and y as strong or weak.
d Identify any outliers.

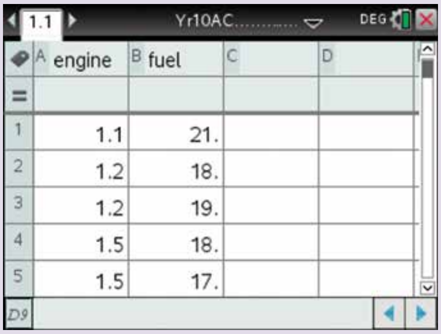
Using calculators to draw scatter plots

Type the following data about car fuel economy into two lists and draw a scatter plot.

Car	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**
Engine size	1.1	1.2	1.2	1.5	1.5	1.8	2.4	3.3	4.2	5.0
Fuel economy	21	18	19	18	17	16	15	20	14	11

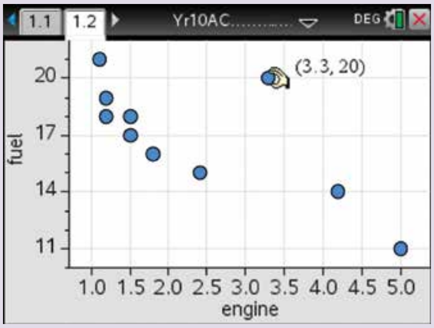
Using the TI-Nspire:

1 In a **Lists and spreadsheets** page type in the list names ***engine*** and ***fuel*** and enter the values as shown.



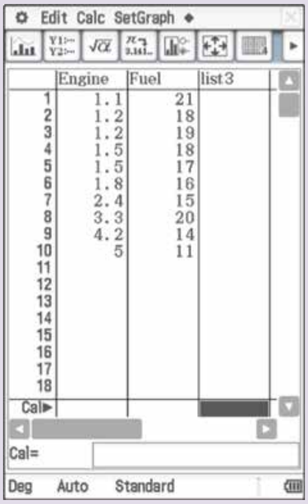
	A engine	B fuel	C	D
=				
1	1.1	21.		
2	1.2	18.		
3	1.2	19.		
4	1.5	18.		
5	1.5	17.		

2 Insert a **Data and Statistics** page and select the ***engine*** variable for the horizontal axis and ***fuel*** for the vertical axis. Hover over points to reveal coordinates.



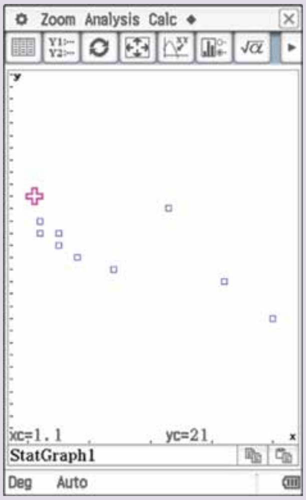
Using the ClassPad:

1 In the **Statistics** application, assign a title to each column then enter the data into the lists.



	Engine	Fuel	list3
1	1.1	21	
2	1.2	18	
3	1.2	19	
4	1.5	18	
5	1.5	17	
6	1.8	16	
7	2.4	15	
8	3.3	20	
9	4.2	14	
10	5	11	
11			
12			
13			
14			
15			
16			
17			
18			

2 Tap . For graph 1 set **Draw** to **On**, **Type** to **Scatter**, **XList** to **mainEngine**, **YList** to **mainFuel**, **Freq** to **1** and **Mark** to **square**. Tap **Set**. Tap . Tap **Analysis**, **Trace** to reveal coordinates.



Exercise 9I

FLUENCY	1–4	1, 3, 4	2–4

Example 15

1 Consider this simple bivariate data set. (Use technology to assist if desired. See page 841.)

x	1	2	3	4	5	6	7	8
y	1.0	1.1	1.3	1.3	1.4	1.6	1.8	1.0

a Draw a scatter plot for the data.
b Describe the correlation between x and y as positive or negative.
c Describe the correlation between x and y as strong or weak.
d Identify any outliers.

2 Consider this simple bivariate data set. (Use technology to assist if desired. See page 841.)

x	14	8	7	10	11	15	6	9	10
y	4	2.5	2.5	1.5	1.5	0.5	3	2	2

a Draw a scatter plot for the data.
b Describe the correlation between x and y as positive or negative.
c Describe the correlation between x and y as strong or weak.
d Identify any outliers.

3 By completing scatter plots (by hand or using technology) for each of the following data sets, describe the correlation between x and y as positive, negative or none.

a

x	1.1	1.8	1.2	1.3	1.7	1.9	1.6	1.6	1.4	1.0	1.5
y	22	12	19	15	10	9	14	13	16	23	16

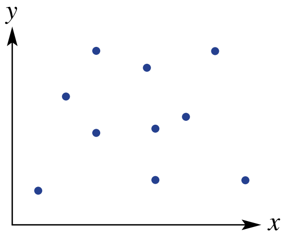
b

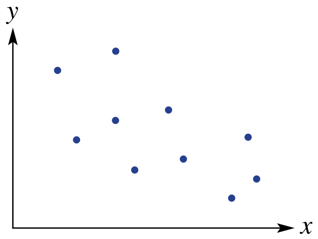
x	4	3	1	7	8	10	6	9	5	5
y	115	105	105	135	145	145	125	140	120	130

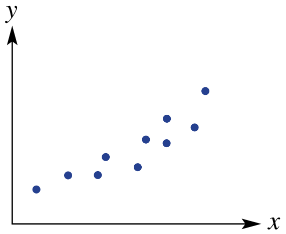
c

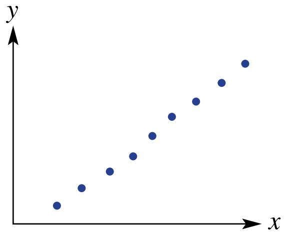
x	28	32	16	19	21	24	27	25	30	18
y	13	25	22	21	16	9	19	25	15	12

4 For the following scatter plots, describe the correlation between x and y.

a


b


c


d


PROBLEM–SOLVING	5, 6	6, 7	6, 8

5 For common motor vehicles, consider the two variables *Engine size* (cylinder volume) and *Fuel economy* (number of kilometres travelled for every litre of petrol).

a Do you expect there to be some relationship between these two variables?

b As the engine size increases, would you expect the fuel economy to increase or decrease?

c The following data were collected for 10 vehicles.

Car	A	B	C	D	E	F	G	H	I	J
Engine size	1.1	1.2	1.2	1.5	1.5	1.8	2.4	3.3	4.2	5.0
Fuel economy	21	18	19	18	17	16	15	20	14	11

i Do the data generally support your answers to parts **a** and **b**?

ii Which car gives a fuel economy reading that does not support the general trend?

6 A tomato grower experiments with a new organic fertiliser and sets up five separate garden beds: A, B, C, D and E. The grower applies different amounts of fertiliser to each bed and records the diameter of each tomato picked.
The average diameter of a tomato from each garden bed and the corresponding amount of fertiliser are recorded below.

Bed	A	B	C	D	E
Fertiliser (grams per week)	20	25	30	35	40
Average diameter (cm)	6.8	7.4	7.6	6.2	8.5

a Draw a scatter plot for the data with ‘Diameter’ on the vertical axis and ‘Fertiliser’ on the horizontal axis. Label the points A, B, C, D and E.

b Which garden bed appears to go against the trend?

c According to the given results, would you be confident in saying that the amount of fertiliser fed to tomato plants does affect the size of the tomato produced?

7 In a newspaper, the number of photos and number of words were counted for 15 different pages. Here are the results.

Page	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of photos	3	2	1	2	6	4	5	7	4	5	2	3	1	0	1
Number of words	852	1432	1897	1621	912	1023	817	436	1132	1201	1936	1628	1403	2174	1829

a Sketch a scatter plot using ‘Number of photos’ on the horizontal axis and ‘Number of words’ on the vertical axis.

b From your scatter plot, describe the general relationship between the number of photos and the number of words per page. Use the words positive, negative, strong correlation or weak correlation.

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8 On 14 consecutive days, a local council measures the volume of sound heard from a freeway at various points in a local suburb. The volume of sound, in decibels, is recorded against the distance (in metres) between the freeway and the point in the suburb.

Distance (m)	200	350	500	150	1000	850	200	450	750	250	300	1500	700	1250
Volume (dB)	4.3	3.7	2.9	4.5	2.1	2.3	4.4	3.3	2.8	4.1	3.6	1.7	3.0	2.2

a Draw a scatter plot of *Volume* against *Distance*, plotting *Volume* on the vertical axis and *Distance* on the horizontal axis.

b Describe the correlation between *Distance* and *Volume* as positive, negative or none.

c Generally, as *Distance* increases does *Volume* increase or decrease?

REASONING

9	9, 10	10, 11

9 A government department is interested in convincing the electorate that a larger number of police on patrol leads to a lower crime rate. Two separate surveys are completed over a one-week period and the results are listed in this table.

	Area	**A**	**B**	**C**	**D**	**E**	**F**	**G**
Survey 1	Number of police	15	21	8	14	19	31	17
	Incidence of crime	28	16	36	24	24	19	21
Survey 2	Number of police	12	18	9	12	14	26	21
	Incidence of crime	26	25	20	24	22	23	19

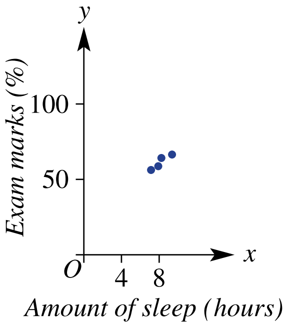
a By using scatter plots, determine whether or not there is a relationship between the number of police on patrol and the incidence of crime, using the data in:

- **i** survey 1
- **ii** survey 2.

b Which survey results do you think the government will use to make its point? Why?

10 A student collects some data and finds that there is a positive correlation between height and the ability to play tennis. Does that mean that if you are tall you will be better at tennis? Explain.

11 A person presents you with this scatter plot and suggests a strong correlation between the amount of sleep and exam marks. What do you suggest is the problem with the person's graph and conclusions?



ENRICHMENT: Does television provide a good general knowledge?	–	–	12

12 A university graduate is conducting a test to see whether a student's general knowledge is in some way linked to the number of hours of television watched.
Twenty Year 10 students sit a written general knowledge test marked out of 50. Each student also provides the graduate with details about the number of hours of television watched per week. The results are given in the table below.

Student	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
Hours of TV	11	15	8	9	9	12	20	6	0	15	9	13	15	17	8	11	10	15	21	3
Test score	30	4	13	35	26	31	48	11	50	33	31	28	27	6	39	40	36	21	45	48

a Which two students performed best on the general knowledge test, having watched TV for the following numbers of hours?
  i fewer than 10
  ii more than 4

b Which two students performed worst on the general knowledge test, having watched TV for the following numbers of hours?
  i fewer than 10
  ii more than 4

c Which four students best support the argument that the more hours of TV watched, the better your general knowledge will be?

d Which four students best support the argument that the more hours of TV watched, the worse your general knowledge will be?

e From the given data, would you say that the graduate should conclude that a student's general knowledge is definitely linked to the number of hours of TV watched per week?

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Twenty20

1 Two teams, the Auckland Aces and the Sunrisers Hyderabad, are part of an international 20/20 cricket tournament. They each play 10 round-robin matches and their batting totals are shown below.

Aces	148	172	186	179	194	132	112	154	142	177
Sunrisers	147	160	166	182	171	163	170	155	152	166

You are to compare the statistics of the two cricket teams using box plots and discuss each team's performance in terms of the number of runs and the consistency of the run scoring across the season.

- **a** Draw parallel box plots for these two data sets.
- **b** Compare the box plots of the two teams, commenting on which team appears capable of getting higher scores and which team appears more consistent.
- **c** The Auckland Aces' lowest two scores were the result of rain delays and the restricted number of overs that they faced. If these two innings were increased by 40 runs each, what changes occur on the box plot?
- **d** In their first final, the Sunrisers Hyderabad's batting total would be an outlier if included in their above set of scores. What possible scores did they get in this innings?

Salaries and payrise

2 A small business has 20 employees with the following monthly salaries.

Salary ($)	Number of employees
4500	5
5400	8
5800	5
6400	2

The small business wishes to calculate measures of centre and spread for its salary data and then investigate the impact on these summary statistics given changes in some specific salaries.

- **a** **i** Calculate the mean, median, range and standard deviation (to the nearest dollar) of these salaries.
  - **ii** The top two earning employees are given an increase of $$x$ per month. Describe the impact on the mean, median and range in terms of x.
  - **iii** Describe the impact on the standard deviation from part **ii**.

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b Employees at another small business think they are paid less given their mean monthly salary is \$4800 with standard deviation \$800.

- **i** In this company 95% of salaries lie within two standard deviations of the mean. What would employees who are in the top or bottom 2.5% of earners be earning?
- **ii** If each employee in this business is given a pay rise of $\$x$, give the new mean and standard deviation of employee salaries in terms of x where appropriate.
- **iii** The employees instead decide to give each person a percentage increase in their salary. If each person's salary is increased by a factor of k, give the new mean and standard deviation of the salaries in terms of k.

Winter getaway

3 A family is planning to escape the winter cold and spend July in Noosa. They wish to be prepared for varying temperatures throughout the day and compare the daily maximum and minimum temperatures of a recent July as shown in the table below.

Min. temp (°C)	19	17	17	16	18	19	18	11	12	14	14	15	15	12	15	15
Max. temp (°C)	23	21	20	20	23	23	24	23	19	17	17	19	20	21	21	23

Min. temp (°C)	14	16	16	16	13	14	15	16	16	16	17	17	17	17	15
Max. temp (°C)	23	23	24	24	22	19	19	23	24	24	24	25	25	26	23

The family is interested in the relationship between the maximum and minimum temperatures for the month of July and use this to make predictions for their upcoming holiday.

a Prepare a scatter plot of these data with the minimum temperature on the horizontal axis.

b To make predictions the family use a straight line to model the data. If the line passes through the points shown in red, find the equation of this line by completing the following:
$\text{max. temp} = \underline{\qquad} \times \text{min. temp} + \underline{\qquad}$.

c Use your equation in part **b** to find the likely:

- **i** max. temp on a day with a min. temp of 13°C, rounding to the nearest degree
- **ii** min. temp on a day with a max temp of 28°C, rounding to the nearest degree.

d Which of your results in part **c** seem the most accurate? Why?

e Select two other points on the graph that a straight line modelling the data could reasonably pass through. Find the equation of this line and repeat part **c**. Comment on the similarities or differences in your results for part **c** using the two different equations.

9J Line of best fit by eye

LEARNING INTENTIONS

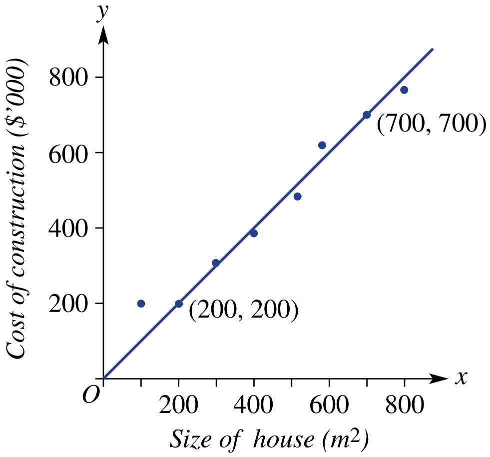
- To understand that a line of best fit can be used as a model for the data when there is a strong linear association
- To know how to fit a line of best fit by eye
- To know how to find the equation of a line of best fit
- To be able to use the line of best fit and its equation to estimate data values within and outside the data range

When bivariate data have a strong linear correlation, we can model the data with a straight line. This line is called a trend line or line of best fit. When we fit the line 'by eye', we try to balance the number of data points above the line with the number of points below the line. This trend line and its equation can then be used to construct other data points within and outside the existing data points.

A scatter plot of product price (y) versus demand (x) shows a negative correlation, with a downward sloping trend line. Businesses use demand equations to forecast sales and make informed decisions about future stock and staffing levels.

Lesson starter: Size versus cost

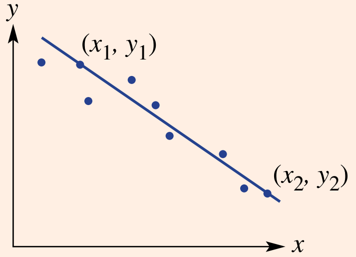
This scatter plot shows the estimated cost of building a house of a given size, as quoted by a building company. The given trend line passes through the points (200, 200) and (700, 700).



- Do you think the trend line is a good fit to the points on the scatter plot? Why?
- How can you find the equation of the trend line?
- How can you predict the cost of a house of $1000\,\text{m}^2$ with this building company?

KEY IDEAS

- A **line of best fit** or **trend line** is positioned by eye by balancing the number of points above the line with the number of points below the line.
  - The distance of each point from the trend line also must be taken into account.
- The equation of the line of best fit can be found using two points that are on the line of best fit.



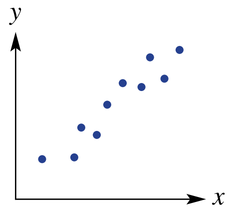
- For $y = mx + c$:

  $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and substitute a point to find the value of c.
  - Alternatively, use $y - y_1 = m(x - x_1)$.
- The line of best fit and its equation can be used for:
  - **interpolation**: constructing points within the given data range
  - **extrapolation**: constructing points outside the given data range.

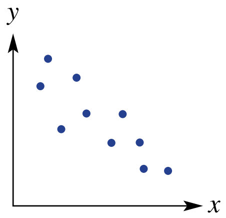
BUILDING UNDERSTANDING

① Practise fitting a line of best fit on these scatter plots by trying to balance the number of points above the line with the numbers of points below the line. (Use the side of a ruler if you don't want to draw a line.)

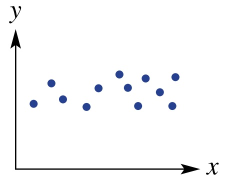
a



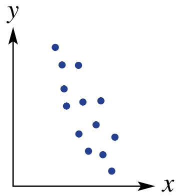
b



c

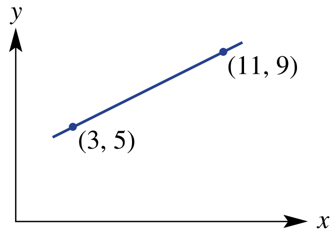


d



② For each graph find the equation of the line in the form $y = mx + c$. First, find the gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and then substitute a point.

a



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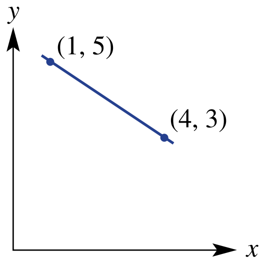

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3 Using $y = \frac{5}{4}x - 3$, find:

a y when:

- **i** $x = 16$
- **ii** $x = 7$

b x when:

- **i** $y = 4$
- **ii** $y = \frac{1}{2}$

Example 16 Fitting a line of best fit

Consider the variables x and y and the corresponding bivariate data.

x	1	2	3	4	5	6	7
y	1	2	2	3	4	4	5

a Draw a scatter plot for the data.

b Is there positive, negative or no correlation between x and y?

c Fit a line of best fit by eye to the data on the scatter plot.

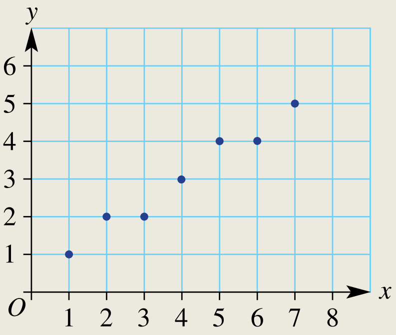
d Use your line of best fit to estimate:

- **i** y when $x = 3.5$
- **ii** y when $x = 0$
- **iii** x when $y = 1.5$
- **iv** x when $y = 5.5$

SOLUTION

EXPLANATION

a

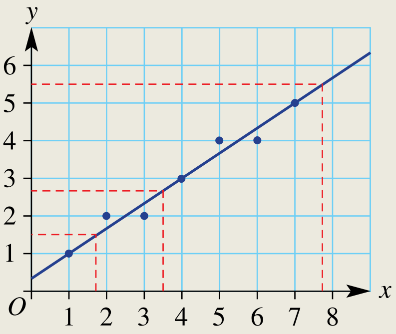


Plot the points on graph paper.

b Positive correlation

As x increases, y increases.

c



Since a relationship exists, draw a line on the plot, keeping as many points above as below the line. (There are no outliers in this case.)

d **i** $y \approx 2.7$
ii $y \approx 0.4$
iii $x \approx 1.7$
iv $x \approx 7.8$

Extend vertical and horizontal lines from the values given and read off your solution. As they are approximations, we use the ≈ sign and not the = sign.

Now you try

Consider the variables x and y and the corresponding bivariate data.

x	1	2	3	4	5	6
y	10	8	8	6	5	3

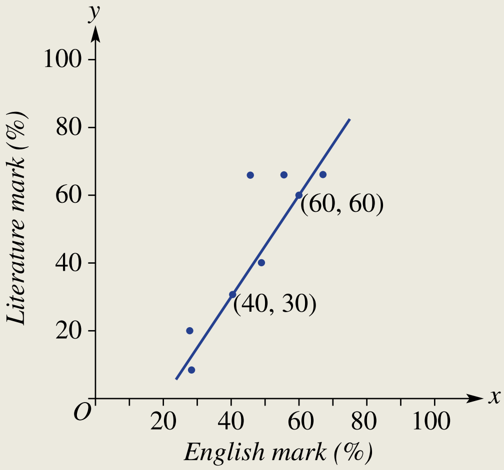
a Draw a scatter plot for the data.
b Is there positive, negative or no correlation between x and y?
c Fit a line of best fit by eye to the data on the scatter plot.
d Use your line of best fit to estimate:
- **i** y when $x = 3.5$
- **ii** y when $x = 0$
- **iii** x when $y = 1.5$
- **iv** x when $y = 5.5$

Example 17 Finding the equation of a line of best fit

This scatter plot shows a linear relationship between English marks and Literature marks in a small class of students. A trend line passes through (40, 30) and (60, 60).

a Find the equation of the trend line.
b Use your equation to estimate a Literature score if the English score is:
- **i** 50
- **ii** 86

c Use your equation to estimate the English score if the Literature score is:
- **i** 42
- **ii** 87



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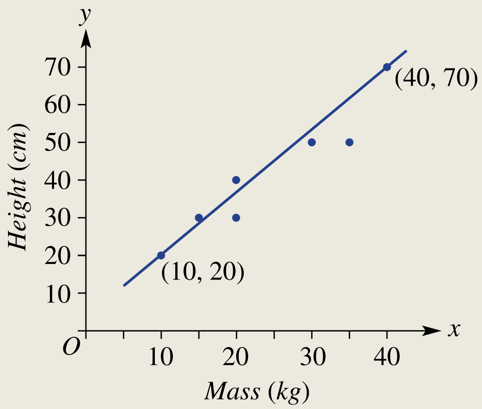
SOLUTION	EXPLANATION
a $y = mx + c$ $m = \frac{60 - 30}{60 - 40}$ $= \frac{30}{20}$ $= \frac{3}{2}$ $\therefore y = \frac{3}{2}x + c$	Use $m = \frac{y_2 - y_1}{x_2 - x_1}$, i.e. $\frac{\text{rise}}{\text{run}}$, for the two given points.
$(40, 30)$: $30 = \frac{3}{2}(40) + c$ $30 = 60 + c$ $c = -30$ $\therefore y = \frac{3}{2}x - 30$	Substitute either (40, 30) or (60, 60) to find c.
b **i** $y = \frac{3}{2}(50) - 30$ $= 45$ $\therefore$ Literature score is 45.	Substitute $x = 50$ and find the value of y.
ii $y = \frac{3}{2}(86) - 30$ $= 99$ $\therefore$ Literature score is 99.	Repeat for $x = 86$.
c **i** $42 = \frac{3}{2}x - 30$ $72 = \frac{3}{2}x$ $x = 48$ $\therefore$ English score is 48.	Substitute $y = 42$ and solve for x.
ii $87 = \frac{3}{2}x - 30$ $117 = \frac{3}{2}x$ $x = 78$ $\therefore$ English score is 78.	Repeat for $y = 87$.

Now you try

This scatter plot shows a linear relationship between the mass and height of a small number of dogs. A trend line passes through (10, 20) and (40, 70).



a Find the equation of the trend line.

b Use your equation to estimate a dog height if its mass is:

i 25 kg **ii** 52 kg

c Use your equation to estimate a dog mass if its height is:

i 60 cm **ii** 80 cm

Using calculators to find equations of regression

Consider the following data and use a graphics or CAS calculator or software to help answer the questions below. Round answers to two decimal places where necessary.

x	1	2	2	4	5	5	6	7	9	11
y	1.8	2	1.5	1.6	1.7	1.3	0.8	1.1	0.8	0.7

a Construct a scatter plot for the data.

b Find the equation of the least squares regression line.

c Sketch the graph of the regression line onto the scatter plot.

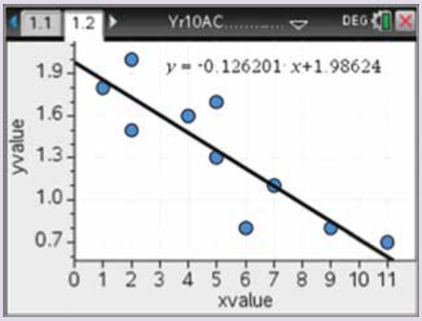
d Use the least squares regression line to estimate the value of y when x is:

- **i** 4.5
- **ii** 15

Using the TI-Nspire:

a, b, c In a **Lists & Spreadsheet** page enter the data in the lists named *xvalue* and *yvalue*. Insert a **Data & Statistics** page and select *xvalue* as the variable on the horizontal axis and *yvalue* as the variable on the vertical axis.

To show the linear regression line and equation use menu **>Analyze>Regression>Show Linear (mx + b)**



Least squares: $y = -0.126201x + 1.986245$

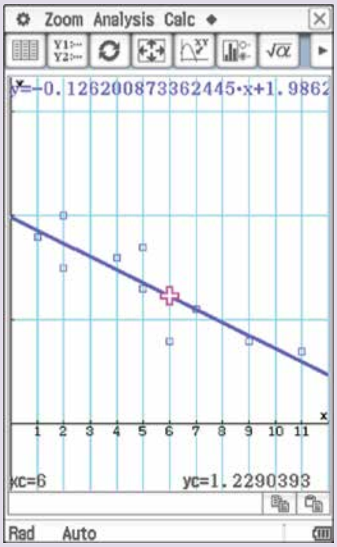
d **i** $y \approx 1.42$

ii $y \approx 0.09$

Using the ClassPad:

a, b, c In the **Statistics** application enter the data into the lists. Tap **Calc**, **Regression**, **Linear Reg** and set **XList** to **list1**, **YList** to **list2**, **Freq** to **1, Copy Formula** to **y1** and **Copy Residual** to **Off**. Tap **OK** to view the regression equation. Tap on OK again to view the regression line.

Tap **Analysis, Trace** and then scroll along the regression line.



d **i** $y \approx 1.42$

ii $y \approx 0.09$

Exercise 9J

FLUENCY	1–3	1, 2(½), 3	1, 3

Example 16

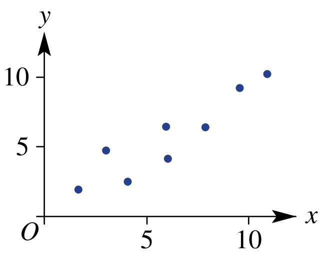
1 Consider the variables x and y and the corresponding bivariate data.

x	1	2	3	4	5	6	7
y	2	2	3	4	4	5	5

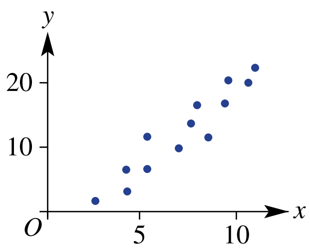
- **a** Draw a scatter plot for the data.
- **b** Is there positive, negative or no correlation between x and y?
- **c** Fit a line of best fit by eye to the data on the scatter plot.
- **d** Use your line of best fit to estimate:
  - **i** y when $x = 3.5$
  - **ii** y when $x = 0$
  - **iii** x when $y = 2$
  - **iv** x when $y = 5.5$

2 For the following scatter plots, pencil in a line of best fit by eye, and then use your line to estimate the value of y when $x = 5$.

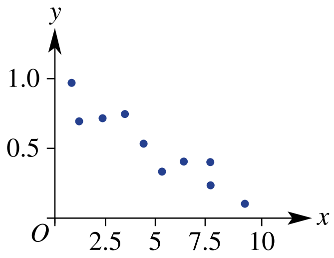
a



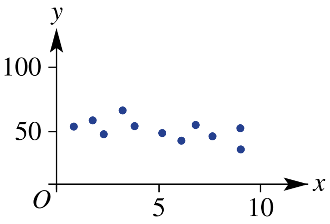
b



c



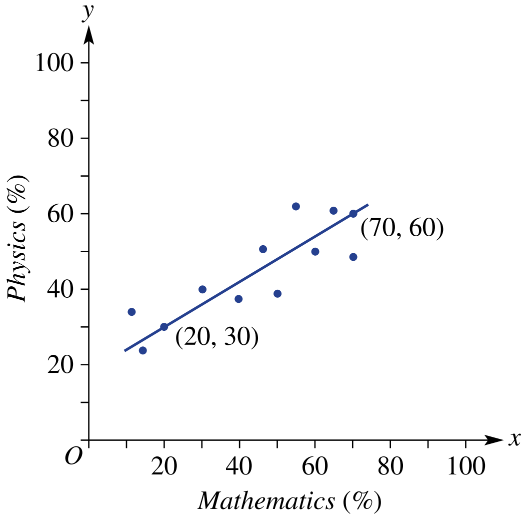
d



Example 17

3 This scatter plot shows a linear relationship between Mathematics marks and Physics marks in a small class of students. A trend line passes through (20, 30) and (70, 60).

- **a** Find the equation of the trend line.
- **b** Use your equation to find the Physics score if the Mathematics score is:
  - **i** 40
  - **ii** 90
- **c** Use your equation to find the Mathematics score if the Physics score is:
  - **i** 36
  - **ii** 78



PROBLEM–SOLVING	4	4, 5	4, 5

4 Over eight consecutive years, a city nursery has measured the growth of an outdoor bamboo species for that year. The annual rainfall in the area where the bamboo is growing was also recorded. The data are listed in the table.

Rainfall (mm)	450	620	560	830	680	650	720	540
Growth (cm)	25	45	25	85	50	55	50	20

- **a** Draw a scatter plot for the data, showing growth on the vertical axis.
- **b** Fit a line of best fit by eye.
- **c** Use your line of best fit to estimate the growth expected for the following rainfall readings. You do not need to find the equation of the line.
  - **i** 500 mm
  - **ii** 900 mm
- **d** Use your line of best fit to estimate the rainfall for a given year if the growth of the bamboo was:
  - **i** 30 cm
  - **ii** 60 cm

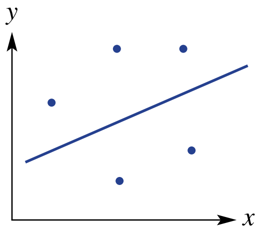
5 A line of best fit for a scatter plot, relating the weight (kg) and length (cm) of a group of dogs, passes through the points (15, 70) and (25, 120). Assume weight is on the x-axis.

- **a** Find the equation of the trend line.
- **b** Use your equation to estimate the length of an 18 kg dog.
- **c** Use your equation to estimate the weight of a dog that has a length of 100 cm.

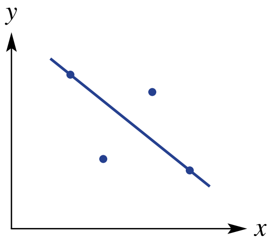
REASONING	6	6, 7	6–8

6 Describe the problem when using each trend line below for interpolation. (Refer to Key ideas)

a

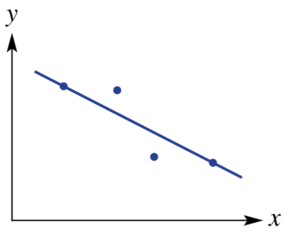


b

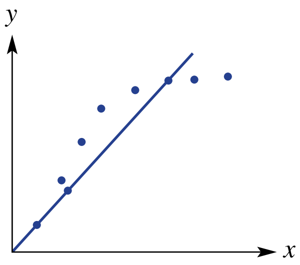


7 Describe the problem when using each trend line below for extrapolation. (Refer to Key ideas)

a



b



8 A trend line relating the percentage scores for Music performance (y) and Music theory (x) is given by $y = \frac{4}{5}x + 10$.

a Find the value of x when:

i $y = 50$ **ii** $y = 98$

b What problem occurs in predicting Music theory scores when using high Music performance scores?

ENRICHMENT: Heart rate and age

– – 9

9 Two independent scientific experiments confirmed a correlation between *Maximum heart rate* (in beats per minute or b.p.m.) and *Age* (in years). The data for the two experiments are as follows.

Experiment 1													
Age (years)	15	18	22	25	30	34	35	40	40	52	60	65	71
Max. heart rate (b.p.m.)	190	200	195	195	180	185	170	165	165	150	125	128	105

Experiment 2													
Age (years)	20	20	21	26	27	32	35	41	43	49	50	58	82
Max. heart rate (b.p.m.)	205	195	180	185	175	160	160	145	150	150	135	140	90

a Sketch separate scatter plots for experiment 1 and experiment 2.

b By fitting a line of best fit by eye to your scatter plots, estimate the maximum heart rate for a person aged 55 years, using the results from:

i experiment 1 **ii** experiment 2.

c Estimate the age of a person who has a maximum heart rate of 190, using the results from:

i experiment 1 **ii** experiment 2.

d For a person aged 25 years, which experiment estimates a lower maximum heart rate?

e Research the average maximum heart rate of people according to age and compare with the results given above.

Different watches are used by people to record heart rate.

Estimating future average maximum daily temperatures

At a weather station near Canberra the following data representing the average monthly maximum temperature was calculated by the Bureau of Meteorology.

Present a report for the following tasks and ensure that you show clear mathematical workings and explanations where appropriate.

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1997	27.2	30.4	24.8	23	16.7	12.8	12.2	13.9	16.1	21.7	26.9	29.9
1998	29.5	30.5	27.8	21.4	16.2	12.8	11.2	13.7	17.7	18.7	21.4	27.8
1999	29.5	27.1	25.1	19	17.8	12.6	13.3	14.9	18.1	20.2	20.9	24.9
2000	25.1	29.2	25.7	20.4	15.1	12.6	11.9	13.5	18.2	19.5	22.8	28
2001	31.2	29	23.7	21.3	16.6	14.3	12.6	13.7	18.8	18.9	22.4	26.5
2002	28.7	24.5	25.5	22.6	16.2	13.4	12.8	14.9	17.9	21.7	27.7	28.4
2003	30.6	29.2	23.9	20.1	17.2	13.2	12.5	13.3	16.8	17.9	24.4	27.6
2004	29.5	30.5	27	22.1	16.5	13.1	12.1	14.6	17.5	20.3	23.7	26.4
2005	28.8	27	24.5	24.3	18.3	13.9	12.7	14.5	16.4	20.4	22.9	28.8
2006	30.6	30.5	27	19.6	16.1	11.8	12.5	15.4	19.3	24	26.4	27.7
2007	30.8	28.3	25.8	22	17.9	11.6	11.4	15.2	17.4	23	24.8	25.1
2008	29.2	24.5	26	19.6	17.2	14.5	11.6	13	18.3	23.1	23.1	24.9
2009	31.2	28.9	26.6	19.6	16.4	12.4	12.6	14.7	17.6	18.8	28.8	29
2010	31.6	27.2	24.9	21.4	16.6	13.3	12.8	12.5	16.7	19.7	22.4	24.9
2011	28.7	26.5	23.3	19.4	15.3	13.4	12.2	15.8	17.9	20.3	25	23.4
2012	26.8	24.7	22.1	20.2	15.5	12.7	12.7	13.5	18	20.9	25.3	27.2
2013	32	26.8	25.1	21.8	16.8	13.4	12.7	14.3	19.7	21.9	23.7	28.5
2014	31.2	28.9	24.1	19.3	17.4	13	12.1	13.9	17.8	22.7	27.3	27.3
2015	26.9	27.1	25.6	18.5	16	13.3	10.6	13.4	17.2	24.8	24.4	28.6
2016	27.7	29.1	26.8	23.4	17.6	12.6	12.6	14.2	15.7	18.5	25	28.1
2017	31.9	29.6	25.5	19.6	16.2	13.4	12.7	13.6	17.8	22.6	23.8	27.4
2018	31	28.3	26.3	24.9	17.3	13	13.3	13.7	17.9	22.4	24.1	29.1

Preliminary task

a For the month of January, plot a graph of the average maximum daily temperature from 1997 to 2018.

b Use technology or otherwise to find a rule for a line of good fit (regression line) to model the January data.

c Use your model to predict the average maximum monthly temperature in Canberra in 2030.

Modelling task

Formulate

a The problem is to use graphs and regression lines to predict the average temperature for a given month in Canberra in 2030. Write down all the relevant information that will help solve this problem.

Solve

b Using the given data, choose a month (not January) to construct a time-series plot for the average maximum monthly temperature from 1997 to 2018.

c Sketch a line of good fit on your plot and find its equation.

d Use your line of good fit to predict the average monthly maximum temperature for Canberra in 2030 for your chosen month.

e Repeat the above for at least one other chosen month.

Evaluate and verify

f Compare the results from part **d** above against the given temperatures in the table.

g Decide if your results imply that future temperatures for Canberra are increasing.

h Explain how the gradient of your line of good fit relates to your answer to part **g** above.

i Use technology if possible to verify the accuracy of your line of best fit.

Communicate

j Summarise your results and describe any key findings.

Extension question

a Use data obtained from the Bureau of Meteorology's website to explore maximum temperature trends in other regions of Australia.

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Australian animal extinctions

Key technology: CAS regression

1 Getting started

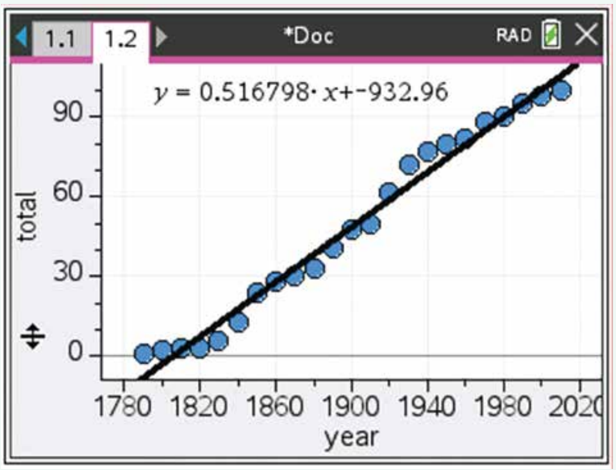
According to records the total number of extinct Australian land animals has increased steadily since European settlement in 1788. This list includes the famous Tasmanian tiger, the desert rat kangaroo and a number of bandicoots.

- **a** Look at the data provided in the given table which shows the accumulated total number of recorded land animal extinctions since colonisation. Which decade recorded the largest increase in the number of extinctions?
- **b** Use the internet to research some of the animals which have become extinct in Australia. List at least five and give the year that they were recorded as being extinct.
- **c** Use the internet to research some of the key reasons why animals in Australia have become extinct since European settlement.

Year	Number of extinctions
1790	1
1800	2
1810	3
1820	3
1830	6
1840	13
1850	24
1860	28
1870	30
1880	33
1890	41
1900	48
1910	50
1920	62
1930	72
1940	77
1950	80
1960	82
1970	88
1980	90
1990	95
2000	98
2010	100

2 Using technology

- **a** Enter the given data into two columns of a list and spreadsheets page on your CAS or inside a spreadsheet.
- **b** Construct a scatter plot of the *Total number of extinctions* vs *Year*.
- **c** Add a line of good fit to your scatter plot. Suggest: least squares regression line.
- **d** Find the equation of your line of good fit.
- **e** Use your equation to estimate the total number of extinctions for the year 2050.
- **f** Use your equation to estimate the year that might record total number of animal extinctions as 130.



$y = 0.516798 \cdot x + -932.96$

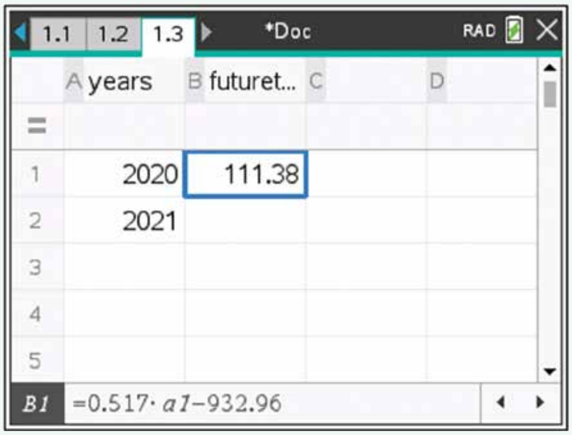
3 Applying an algorithm

a In a new lists and spreadsheets page or in a spreadsheet, create two lists: *years* and *futuretotal*. Make the first *years* data value 2020 and in the *futuretotal* column, enter your regression equation. Use these set of instructions:
  - Step 1: Enter 2020 in cell A1.
  - Step 2: Enter = A1 + 10 into A2.
  - Step 3: Enter your regression equation into B1 but replace x with A1.
  - Step 4: Fill down both columns from cells A2 and B1.

b Fill down to estimate the total number of extinctions until the year 2200.

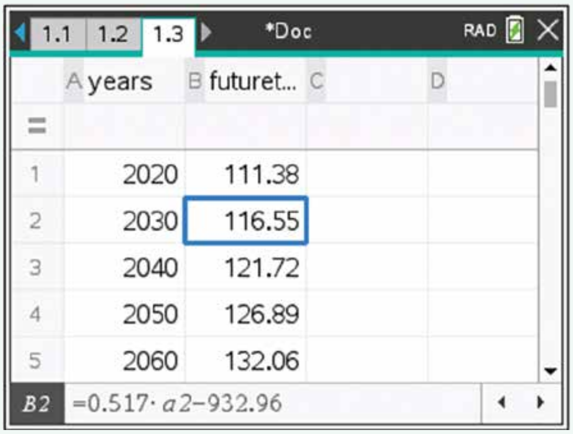
c Use your list to predict the total number of extinct land animals in the following years.
  - **i** 2080
  - **ii** 2250

d Use your list to predict when the total number of extinctions might reach the following numbers.
  - **i** 200
  - **ii** 300



	A years	B futuret...	C	D
=				
1	2020	111.38		
2	2021			
3				
4				
5				

B1 =0.517· a1−932.96



	A years	B futuret...	C	D
=				
1	2020	111.38		
2	2030	116.55		
3	2040	121.72		
4	2050	126.89		
5	2060	132.06		

B2 =0.517· a2−932.96

4 Extension

Consider a subgroup of the above data which includes just mammals.

a Research the extinctions of Australian mammals since European settlement. Collect what data you can find and enter this into a list or spreadsheet.

b Create a scatter plot of the total number of mammals extinct since European settlement and the year beginning at 1790.

c Find a regression equation for your scatter plot.

d Use your regression equation to make predictions about the extinction of Australian mammals in the future.

e Recent reports say that more than 10% of endemic Australian mammal fauna has become extinct since colonisation. Decide if your data supports this claim.

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Indigenous population comparison

The following data were collected by the Australian Bureau of Statistics during the 2016 National Census. The table shows the population of Indigenous and non-Indigenous people in Australia and uses class intervals of 5 years.

	Total	Indigenous	Non-Indigenous	Indigenous status not stated
0–4 years	1 464 779	73 265	1 311 383	80 133
5–9 years	1 502 646	75 755	1 349 301	77 589
10–14 years	1 397 183	71 378	1 255 464	70 340
15–19 years	121 595	66 266	1 281 384	73 945
20–24 years	1 566 793	57 452	1 412 454	96 885
25–29 years	1 664 602	47 934	1 504 558	112 112
30–34 years	1 703 847	40 927	1 554 346	108 575
35–39 years	1 561 679	35 401	1 434 043	92 232
40–44 years	1 583 257	36 994	1 454 855	91 406
45–49 years	1 581 455	36 034	1 452 864	92 552
50–54 years	1 523 551	31381	1 401 237	90 932
55–59 years	1 454 332	25 897	1 339 963	88 476
60–64 years	1 299 397	19 541	1 199 438	80 426
65–69 years	1 188 999	13 722	1 099 519	75 752
70–74 years	887 716	8129	820 805	58 785
75–79 years	652 657	4689	602 518	45 456
80–84 years	460 549	2677	423 557	34 314
85 years and over	486 842	1734	443 533	41 577

Indigenous histogram

a Use the given data to construct a histogram for the population of Indigenous people in Australia in 2016.

b Which age group contained the most Indigenous people?

c Describe the shape of the histogram. Is it symmetrical or skewed?

Non-Indigenous histogram

a Use the given data to construct a histogram for the population of non-Indigenous people in Australia in 2016.

Try to construct this histogram so it is roughly the same width and height as the histogram for the Indigenous population. You will need to rescale the y-axis.

b Which age group contains the most number of non-Indigenous people?

c Describe the shape of the histogram. Is it symmetrical or skewed?

Comparisons

a Explain the main differences in the shapes of the two histograms.

b What do the histograms tell you about the age of Indigenous and non-Indigenous people in Australia in 2016?

c What do the graphs tell you about the difference in life expectancy for Indigenous and non-Indigenous people?

Up for a challenges? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.

1 The mean mass of six boys is 71 kg, and the mean mass of five girls is 60 kg. Find the average mass of all 11 people put together.

2 Sean has a current four-topic average of 78% for Mathematics. What score does he need in the fifth topic to have an overall average of 80%?

3 A single-ordered data set includes the following data.

2, 4, 5, 6, 8, 10, x

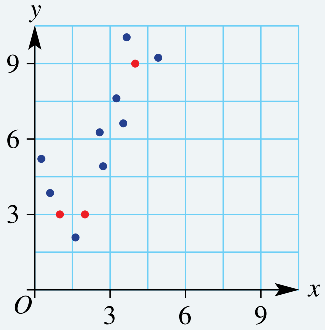
What is the largest possible value of x if it is not an outlier?

4 Find the interquartile range for a set of data if 75% of the data are above 2.6 and 25% of the data are above 3.7.

5 A single data set has 3 added to every value. Describe the change in:

a the mean
b the median
c the range
d the interquartile range
e the standard deviation.

6 Three key points on a scatter plot have coordinates (1, 3), (2, 3) and (4, 9). Find a quadratic equation that fits these three points exactly.



7 Six numbers are written in ascending order: 1.4, 3, 4.7, 5.8, a, 11.
Find all possible values of a if the number 11 is considered to be an outlier.

8 The class mean, $\bar{x}$, and standard deviation, s, for some Year 10 term tests are:
Maths ($\bar{x} = 70\%$, $s = 9\%$); Physics ($\bar{x} = 70\%$, $s = 6\%$); Biology ($\bar{x} = 80\%$, $s = 6.5\%$).
If Emily gained 80% in each of these subjects, which was her best and worst result? Give reasons for your answer.

Chapter summary

Statistics

Cumulative frequency curves and percentiles

Cumulative frequency is the sum of all the frequencies of the categories up to and including that one.

Category	Frequency	Cumulative frequency	% Cumulative frequency
3–	2	2	20
4–	3	5	50
5–	1	6	60
6–	3	9	90
7–8	1	10	100

These points e.g. (4, 20) can be plotted and joined to form a smooth curve.

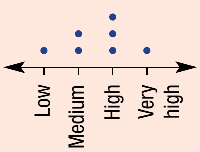
The pth percentile is a data value that sits above p% of the data.

Collecting representative data

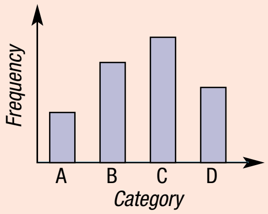
A survey can be used to collect data from a population. The sample of the population chosen to be surveyed should be selected without bias and should be representative of the population.

Graphs for single set of categorical or discrete data

Dot plot



Column graphs



Stem-and-leaf plot

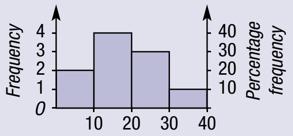
Stem	Leaf
0	1 6
1	2 7 9
2	3 8
3	4

2 | 3 means 23

Grouped data

Class interval	Frequency	Percentage frequency
0–	2	20
10–	4	40
20–	3	30
30–40	1	10
Total	10	100

Histogram



Measures of spread

- Range = max – min
- Interquartile range (IQR) = $Q_3 - Q_1$
- Sample standard deviation (s) for n data values (Opt)

$$= \sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\dots+(x_n-\bar{x})^2}{n-1}}$$

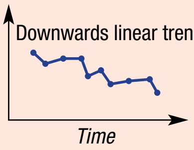
- If s is relatively small the data are concentrated about the mean.
- If s is relatively large the data are spread out from the mean.

Measures of centre

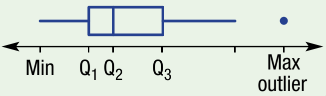
- Mean $(\bar{x}) = \frac{\text{sum of all values}}{\text{number of scores}}$
- Median (Q_2) = middle value

(Mode = most common value)

Time-series data



Box plots



Quartiles

Q_1: above 25% of the data

Q_3: above 75% of the data

e.g. 1: 2 3 5 7 8 | 9 11 12 14 15

$\uparrow Q_1$ $\quad \uparrow Q_2 = 8.5$ $\quad \uparrow Q_3$

e.g. 2: 1 4 7) 8 (12 16 21

$\uparrow Q_1$ $\quad \uparrow Q_2$ $\quad \uparrow Q_3$

Data

Categorical

- Nominal (red, blue, ...)
- Ordinal (low, medium, ...)

Numerical

- Discrete (1, 2, 3, ...)
- Continuous (0.31, 0.481, ...)

Outliers

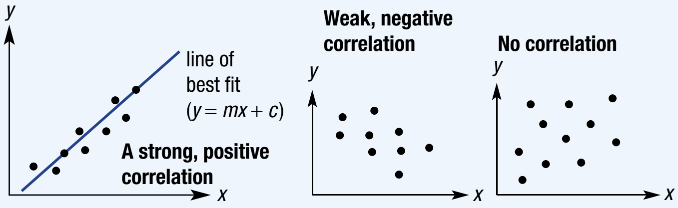
- Single data set
  - less than $Q_1 - 1.5 \times \text{IQR}$ or more than $Q_3 + 1.5 \times \text{IQR}$
- Bivariate
  - not in the vicinity of the rest of the data

Two-way tables

A way of displaying data to help compare two related categorical variables. A Likert scale is a list of ordered options (often a 5-point scale), which can be responded to in a survey question.

Bivariate data

- Two related variables
- Scatter plot



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Chapter checklist with success criteria

A printable version of this checklist is available in the Interactive Textbook ✔

9A **1. I can describe types of data.** ☐
e.g. What type of data would the survey question 'How many pairs of shoes do you own?' generate?

9A **2. I can choose a survey sample.** ☐
e.g. A survey is carried out by calling people listed in the phone book to determine their voting preferences for a state election. Why will this sample not necessarily be representative of the state's views?

9B **3. I can present data in a histogram.** ☐
e.g. Fifteen people were surveyed to find out how many hours they spend on the internet in a week. The data are:
7, 12, 14, 20, 2, 26, 8, 11, 17, 12, 21, 5, 6, 18, 14
Construct a histogram for the data using class intervals of 5, showing both the frequency and percentage frequency on the one graph. Is the data symmetrical, positively skewed or negatively skewed?

9B **4. I can analyse data in a statistical graph.** ☐
e.g. For the stem-and-leaf plot shown below, find the mean correct to one decimal place, the median and the mode.

Stem	Leaf
0	2 5 7
1	1 1 4 6
2	0 3 9
3	2 5

2 | 3 means 23

9C **5. I can interpret a two-way table.** ☐
e.g. This two-way table summarises data from a survey of 40 people using a 5-point Likert scale. The survey asked if they live in Australia or live overseas and are visiting and to what degree they thought Australians were friendly.

	Strongly agree	Agree	Neutral	Disagree	Strongly disagree	Total
Aus	2	4	6	3	3	18
Overseas	8	8	2	3	1	22
Total	10	12	8	6	4	40

i State how many people surveyed were from Australia and responded Agree.
ii Do you think that the data supports the notion that overseas visitors think Australians are more friendly? Give a reason.

9C **6. I can construct a two-way table.** ☐
e.g. A number of people were surveyed to see if they had a mortgage. They were also asked whether they were under or over 50 years of age. 11 of the 15 people under 50 had a mortgage and 5 of the 10 people 50 or over had a mortgage. Represent the information in a two-way table and find what fraction of people surveyed were 50 or over and had a mortgage.

9D **7. I can find the five-figure summary and interquartile range.** ☐
e.g. For the data set below find the minimum, maximum, median, upper and lower quartiles and the range and IQR.
7, 10, 12, 12, 14, 18, 22, 25, 26, 30

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Chapter checklist with success criteria

✓

9D **8. I can find any outliers in a data set.** ☐
e.g. The following data represent the number of aces by a tennis player in 11 grand slam matches for the year:
15, 12, 22, 2, 10, 18, 16, 14, 15, 20, 16
For the data find the upper and lower quartiles and use these to help determine if there are any outliers.

9E **9. I can construct a box plot.** ☐
e.g. For the data set 5, 8, 2, 1, 6, 3, 3, 1, 4, 18, 2, 8, 5, draw a box plot to summarise the data, marking outliers if they exist.

9F **10. I can calculate the standard deviation.** Opt ☐
e.g. For the data set 10, 5, 4, 7, 2, calculate the mean and sample standard deviation correct to one decimal place.

9F **11. I can interpret a standard deviation value.** Opt ☐
e.g. This back-to-back stem-and-leaf plot shows the average monthly maximum temperatures for a year in New York and Melbourne. The mean and standard deviation are given.

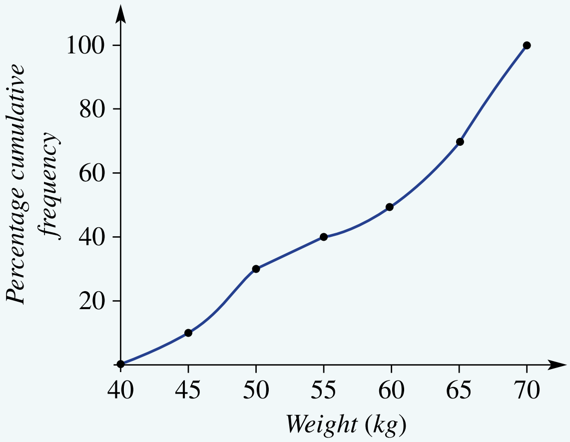
New York Leaf	Stem	Melbourne Leaf
7 5 4	0	
8 7 2 0	1	3 4 5 7 7
9 9 7 5 2	2	0 0 2 4 4 6 6

1|7 means 17°C

Melbourne: $\bar{x} = 19.8$, $s = 4.6$
New York: $\bar{x} = 17.1$, $s = 9.4$

By looking at the stem-and-leaf plot, suggest why New York's mean is less than that of Melbourne and why New York's standard deviation is larger than that of Melbourne.

9G **12. I can interpret a cumulative frequency curve.** ☐
e.g. For the percentage cumulative frequency curve of 20 weights, use the curve to estimate the 40th percentile and the IQR.



Chapter checklist with success criteria

		✓
9G	**13. I can construct a percentage cumulative frequency curve.** e.g. The number of hours of exam study per day by a group of 40 students is summarised in this table. Construct a cumulative frequency curve by completing a percentage cumulative frequency column and use the curve to estimate the 50th percentile.	☐

No. of hours	Frequency
0–	15
2–	6
4–	10
6–	6
8–10	3

9H	**14. I can plot and interpret a time-series plot.** e.g. The approximate number of DVD rental stores in a city over a 10-year period is shown below. Plot the time series and describe the trend in the data over the 10 years.	☐

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019
Number of DVD stores	72	65	56	56	31	22	14	14	8	6

9I	**15. I can construct and interpret a scatter plot.** e.g. For the bivariate data set below, draw a scatter plot and describe the correlation between x and y as positive or negative and strong or weak.	☐

x	5	8	12	4	6	15	11	3
y	4.4	6	11	4.7	5.3	11.6	10.3	2.4

9J	**16. I can fit a line of best fit by eye and use the line to make predictions.** e.g. For the scatter plot from the data set above, fit a line of best fit by eye on the scatter plot and use it to estimate y when $x = 10$ and x when $y = 8$.	☐
9J	**17. I can find the equation of a line of best fit.** e.g. A scatter plot shows a linear relationship between two variables x and y. If the trend line passes through (20, 15) and (40, 25), find the equation of the trend line and use it to estimate x when $y = 50$.	☐

Short-answer questions

9B

1 A group of 16 people was surveyed to find the number of hours of television they watch in a week. The raw data are listed:

6, 5, 11, 13, 24, 8, 1, 12, 7, 6, 14, 10, 9, 16, 8, 3

a Organise the data into a table with class intervals of 5 and include a percentage frequency column.

b Construct a histogram for the data, showing both the frequency and percentage frequency on the graph.

c Would you describe the data as symmetrical, positively skewed or negatively skewed?

d Construct a stem-and-leaf plot for the data, using 10s as the stem.

e Use your stem-and-leaf plot to find the median.

9C

2 This two-way table summarises data from a survey of 30 people using a 5-point Likert scale. The survey asked if they owned a pet or not and whether they thought cats should be kept inside at night.

	Strongly agree	Agree	Neutral	Disagree	Strongly disagree	Total
Pet owner	0	5	3	8	4	20
No pet	2	4	2	1	0	10
Total	2	9	5	9	4	30

a State how many were pet owner's and responded *Disagree*.

b What percentage of those surveyed were not pet owner's and responded *Neutral*?

c Would you suggest that this data supports the notion that those who do not own a pet believe cats should be kept inside at night compared to those who own a pet? Give a reason.

9D/E

3 For each set of data below, complete the following tasks.

i Find the range.

ii Find the lower quartile (Q_1) and the upper quartile (Q_3).

iii Find the interquartile range.

iv Locate any outliers.

v Draw a box plot.

a 2, 2, 3, 3, 3, 4, 5, 6, 12

b 11, 12, 15, 15, 17, 18, 20, 21, 24, 27, 28

c 2.4, 0.7, 2.1, 2.8, 2.3, 2.6, 2.6, 1.9, 3.1, 2.2

9E

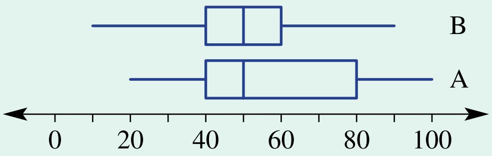
4 Compare these parallel box plots, A and B, and answer the following as true or false.

a The range for A is greater than the range for B.

b The median for A is equal to the median for B.

c The interquartile range is smaller for B.

d 75% of the data for A sit below 80.



9G **5** **a** Complete this cumulative frequency table of times (minutes) in a 5 km fun run for 20 competitors.

Time (mins)	Frequency	Cumulative frequency	Percentage cumulative frequency
30–	3		15
35–		7	
40–	8		
45–	3		
50–55		20	100

b What percentage of competitors finished in less than 40 minutes?

c What percentage of competitors took between 35 minutes and less then 45 minutes?

9I **6** Consider the simple bivariate data set.

x	1	4	3	2	1	4	3	2	5	5
y	24	15	16	20	22	11	5	17	6	8

a Draw a scatter plot for the data.

b Describe the correlation between x and y as positive or negative.

c Describe the correlation between x and y as strong or weak.

d Identify any outliers.

9J **7** The line of best fit passes through the two points labelled on this graph.

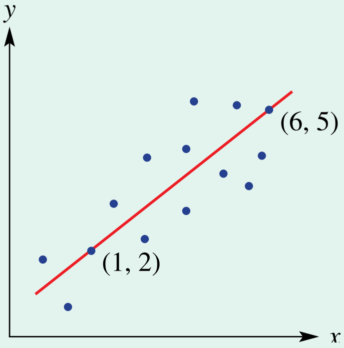
a Find the equation of the line of best fit.

b Use your equation to estimate the value of y when:

- **i** $x = 4$
- **ii** $x = 10$

c Use your equation to estimate the value of x when:

- **i** $y = 3$
- **ii** $y = 12$



9F Opt **8** Calculate the mean and sample standard deviation for these small data sets. Round the standard deviation to one decimal place.

a 4, 5, 7, 9, 10

b 1, 1, 3, 5, 5, 9

9C/F **9** The Wildcats and the JackJumpers basketball teams compare their number of points per match for a season.

The data are presented in this back-to-back stem-and-leaf plot.

JackJumpers Leaf	Stem	Wildcats Leaf
	5	9
2	6	9
8 3	7	0 4 8 9
7 4	8	2 4 7 8 9
9 7 4 1 0	9	2 8
7 6 2	10	0

7|4 means 74

State which team has:

a the higher range

b the higher mean

c the higher median

Opt **d** the higher standard deviation.

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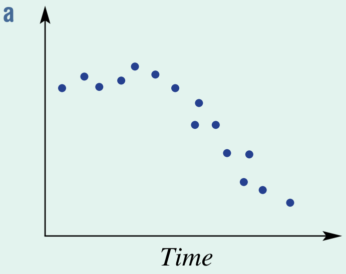
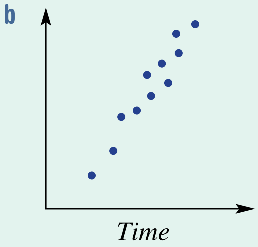
9H **10** Describe the trend in these time-series plots as linear, non-linear or no trend.



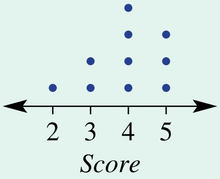


Multiple-choice questions

9A **1** The type of data generated by the survey question *What is your favourite food?* is:

A numerical and discrete
B numerical and continuous
C a sample
D categorical and nominal
E categorical and ordinal

Questions **2**–**4** refer to the dot plot shown at right.



9B **2** The mean of the scores in the data is:

A 3.5 **B** 3.9
C 3 **D** 4
E 5

9B **3** The mode for the data is:

A 3.5 **B** 2 **C** 3 **D** 4 **E** 5

9B **4** The dot plot is:

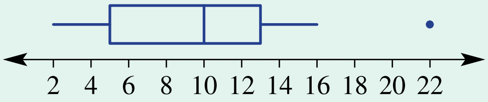
A symmetrical **B** positively skewed **C** negatively skewed
D bimodal **E** correlated

9C **5** From this two-way table, what fraction of people surveyed were retired and play golf?

	Golf	**Not golf**	**Total**
Retired	5	7	12
Not-retired	6	12	18
Total	11	19	30

A $\frac{5}{11}$ **B** $\frac{5}{12}$ **C** $\frac{1}{6}$ **D** $\frac{2}{5}$ **E** $\frac{1}{2}$

Questions **6** and **7** refer to this box plot.



9E **6** The interquartile range is:

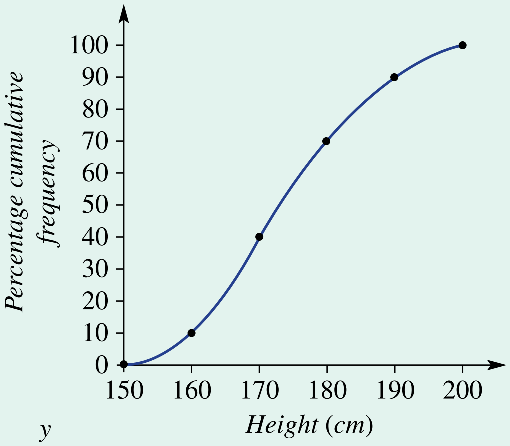
A 8 **B** 5 **C** 3 **D** 20 **E** 14

9E **7** The range is:

A 5 **B** 3 **C** 20 **D** 14 **E** 8

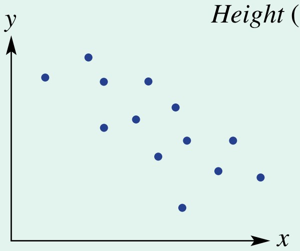
9G **8** This cumulative frequency curve shows the percentage cumulative frequency of heights of 30 people. Using the curve, the IQR is closest to:

A 24 cm
B 16 cm
C 31 cm
D 7 cm
E 10 cm



9I **9** The variables x and y in this scatter plot could be described as having:

A no correlation
B a strong, positive correlation
C a strong, negative correlation
D a weak, negative correlation
E a weak, positive correlation



9J **10** The equation of the line of best fit for a set of bivariate data is given by $y = 2.5x - 3$. An estimate for the value of x when $y = 7$ is:

A -1.4 **B** 1.2 **C** 1.6 **D** 7 **E** 4

Extended-response questions

1 The number of flying foxes taking refuge in two different fig trees was recorded over a period of 14 days. The data collected are given here.

Tree 1	56	38	47	59	63	43	49	51	60	77	71	48	50	62
Tree 2	73	50	36	82	15	24	73	57	65	86	51	32	21	39

a Find the IQR for:
  i tree 1 **ii** tree 2
b Identify any outliers for:
  i tree 1 **ii** tree 2
c Draw parallel box plots for the data.
d By comparing your box plots, describe the difference in the ways the flying foxes use the two fig trees for taking refuge.

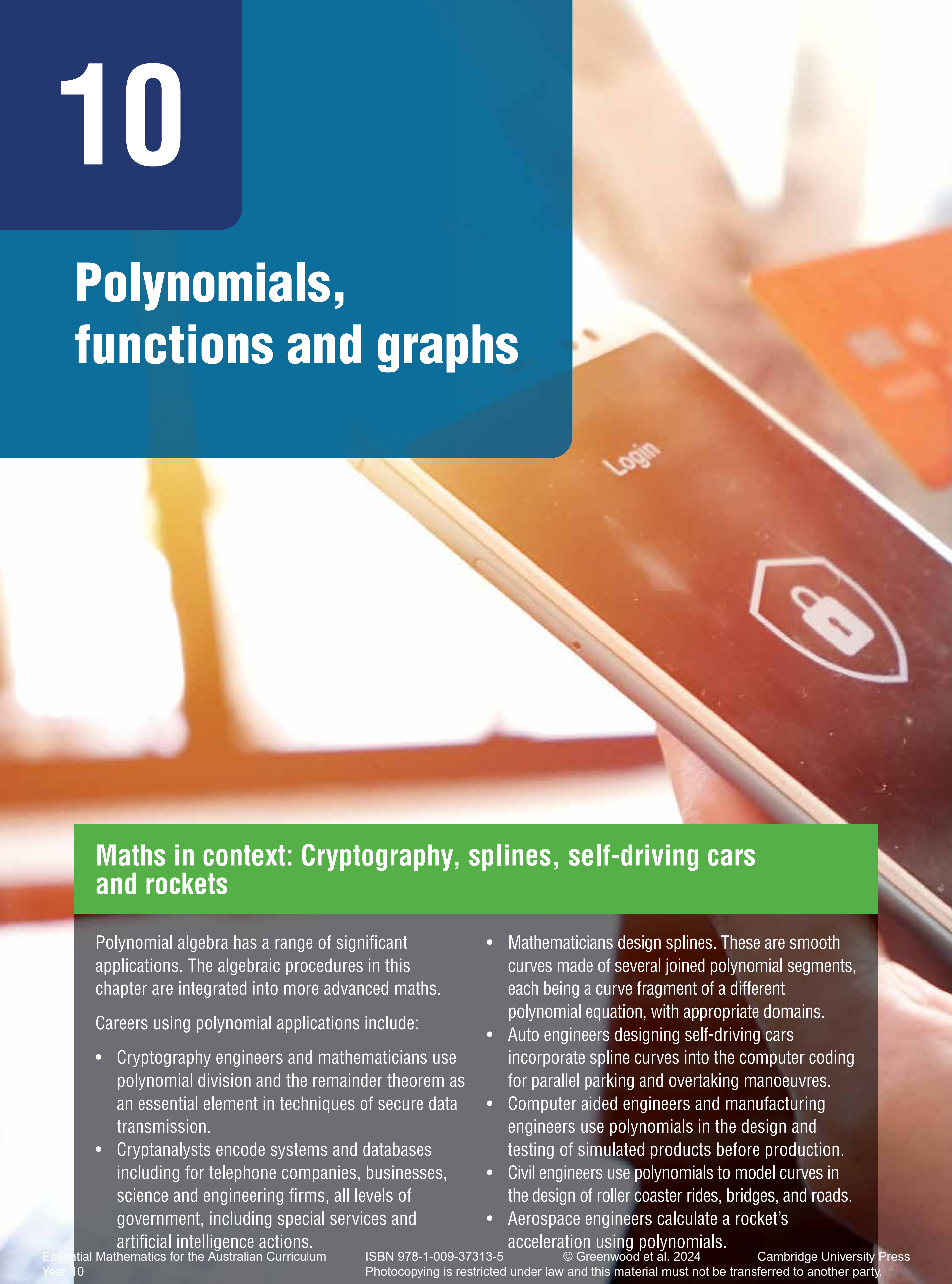
2 A large department store hired an extra 40 workers for the Christmas season. The number of hours they worked in the week before Christmas is shown in the frequency table.

No. of hours	Frequency
20–	3
25–	5
30–	10
35–	14
40–	6
45–50	2

a Construct a cumulative frequency column and percentage cumulative frequency column for the given table.
b Construct a cumulative frequency curve for the data.
c Use the cumulative frequency curve to approximate:
  i the 70th percentile **ii** the median.
d Estimate the percentage cumulative frequency corresponding to working for 32 hours in the week.

10

Polynomials, functions and graphs



Maths in context: Cryptography, splines, self-driving cars and rockets

Polynomial algebra has a range of significant applications. The algebraic procedures in this chapter are integrated into more advanced maths.

Careers using polynomial applications include:

- Cryptography engineers and mathematicians use polynomial division and the remainder theorem as an essential element in techniques of secure data transmission.
- Cryptanalysts encode systems and databases including for telephone companies, businesses, science and engineering firms, all levels of government, including special services and artificial intelligence actions.
- Mathematicians design splines. These are smooth curves made of several joined polynomial segments, each being a curve fragment of a different polynomial equation, with appropriate domains.
- Auto engineers designing self-driving cars incorporate spline curves into the computer coding for parallel parking and overtaking manoeuvres.
- Computer aided engineers and manufacturing engineers use polynomials in the design and testing of simulated products before production.
- Civil engineers use polynomials to model curves in the design of roller coaster rides, bridges, and roads.
- Aerospace engineers calculate a rocket's acceleration using polynomials.

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Chapter contents

Australian Curriculum 9.0

This chapter covers content that is outside the scope of the Australian Curriculum 9.0, but which will be useful for students studying Mathematical Methods or Specialist Mathematics in Year 11 and 12.

Online resources

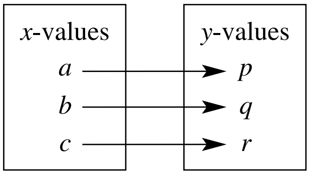
A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

10A Functions and their notation EXTENDING

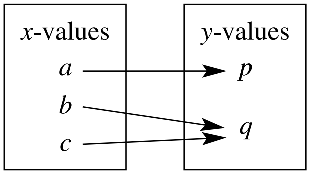
LEARNING INTENTIONS

- To understand what defines a mathematical relation and a function
- To know how to recognise or test for a function
- To be able to use the standard notation for functions
- To know the meaning of the domain and range of a function
- To be able to find the set of allowable x-values (domain) and resulting y-values (range) of a function

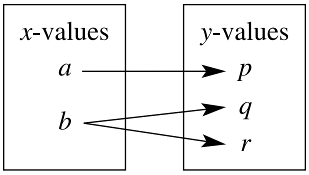
On the Cartesian plane a relationship between the variables x and y can be shown. When this relation has a unique (i.e. only one) y-value for each of its x-values, it is called a function.



Function



Function

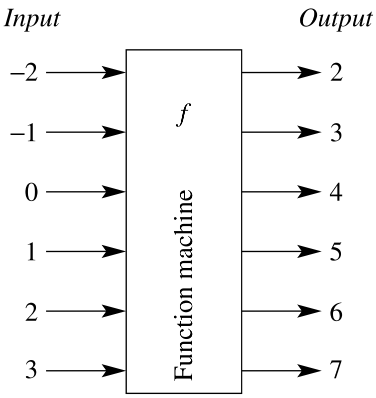


Not a function since $x = b$ gives two y-values.

All functions, including parabolas, use a special notation where the y is replaced by $f(x)$. $y = x^2$ becomes $f(x) = x^2$ (i.e. y is a function of x).

Lesson starter: A function machine

Consider the input and output of the following machine.



This beautiful, old walking bridge in Kromlau, Germany, forms a circle with its reflection. The equation of the full circle is a relation; however, the two semicircle equations, one modelling the bridge and the other its reflection, are functions.

The name f is given to the function and it is written $f(\text{input}) = \text{output}$ i.e. f of input equals output.

- Using the above idea, complete the following.
  $f(-2) =$ _____, $f(-1) =$ _____, $f(0) =$ _____, $f(1) =$ _____, $f(2) =$ _____,
  $f(3) =$ _____ and, hence $f(x) =$ ______.

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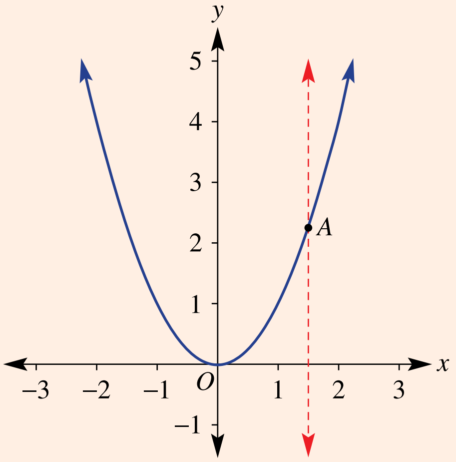
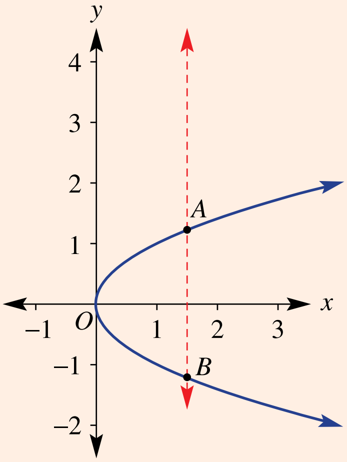
KEY IDEAS

- Any set of ordered pairs is called a **relation**.
- A relation in which each x-value produces only one y-value is called a **function** and can use function notation, $f(x)$, which is read as f of x.
- A relation that is a function has only one y-value for each x-value. Graphically, any vertical line drawn through the graph of a function will cut it only once.
  - For example:



This relation is a function as any vertical line shows that one x-value links to only one y-value.



This relation is *not* a function as a vertical line shows that one x-value links to more than one y-value.

- $f(x)$ is the notation used to replace y if the relation is a function.
  - $f(x) = 3x - 1$ can be written instead of $y = 3x - 1$.
  - The parabola $y = x^2$ is a function, so can be written as $f(x) = x^2$. Also, $f(-2) = 4$ can be written to describe the point $(-2, 4)$.
- The set of allowable x-coordinates (i.e. the input) in a relation is also called the **domain**.
- The **range** is the term given to the set of resulting y-coordinates (output) in the relation.

BUILDING UNDERSTANDING

① Give the following functions using function notation, $f(x)$.

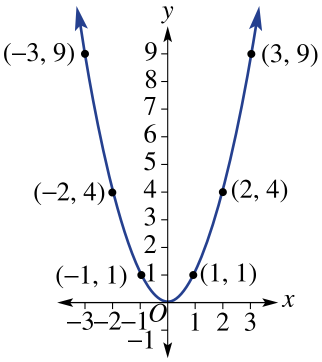
a $y = 8x$ **b** $y = 9 - x^2$ **c** $y = 2^x$

② For each of the following, state whether they are true or false.

a All parabolas are functions.
b Any vertical line will cut $y = 2x - 1$ only once.
c Only positive x-values can be used as the input in $f(x) = x^2$.
d All straight lines are functions.
e A circle is not a function.

③ Use this sketch to decide on the permissible y-values for $y = x^2$, given that the following x-values are allowed.

a $x \geqslant 0$
b $x > 0$
c $x > 3$
d $-1 \leqslant x \leqslant 1$
e All real x-values



Example 1 Using function notation

For $f(x) = x^2 - 3x + 1$, find:

a $f(0)$ **b** $f(-3)$ **c** $f(c)$

SOLUTION	EXPLANATION
a $f(0) = (0)^2 - 3(0) + 1$ $= 1$	The input of 0 is substituted into the rule for each x-value.
b $f(-3) = (-3)^2 - 3(-3) + 1$ $= 9 + 9 + 1$ $= 19$	$x = -3$ is substituted into the rule on the right-hand side. Note that $(-3)^2 \neq -3^2$
c $f(c) = c^2 - 3c + 1$	The x has been replaced by c. Therefore, replace each x with c in the rule.

Now you try

For $f(x) = x^2 - 5x + 2$, find:

a $f(0)$ **b** $f(-2)$ **c** $f(k)$

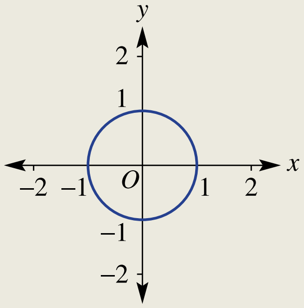
Example 2 Recognising a function

From the following, identify which are functions.

a $\{(1, 4), (2, 8), (4, 16), (5, 20)\}$

b $y = x^2 + 3$

c



SOLUTION

a A function since each x-value has only one y-value.

b $y = x^2 + 3$ is a function since each x-value will produce only one y-value.

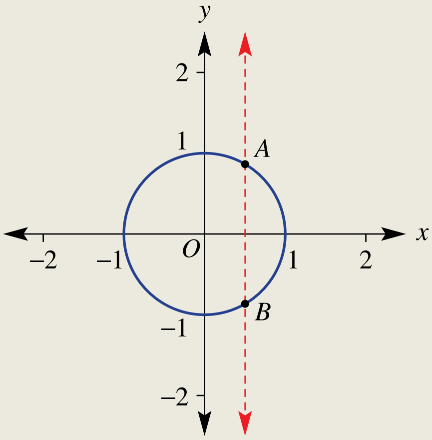
c Not a function because a vertical line shows that one x-value links to more than one y-value.

EXPLANATION

Each of the x-values; i.e. $x = 1, 2, 4$, and 5, occurs only once. So the coordinates represent a function.

As the rule represents a parabola, each x-value will produce only one y-value and so it is a function.

A vertical line drawn anywhere through the graph will cross in more than one place, therefore it is not a function.



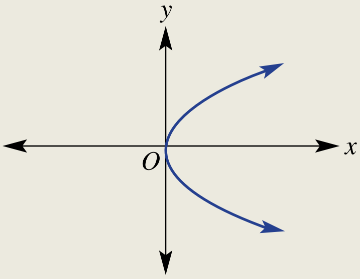
Now you try

From the following, identify which are functions.

a $\{(-1, 4), (0, 2), (1, 0), (2, -2)\}$

b $y = -x^2 + 1$

c



Example 3 Determining domain and range

Write down the allowable x-values (domain) and the resulting y-values (range) for each of these functions.

a $y = 4x - 1$

b $y = x^2 - 4$

SOLUTION	EXPLANATION
a Domain is the set of all real x-values. Range is the set of all real y-values.	The function is a straight line. The input (i.e. x-values) can be any number and will produce any number as an output value.
b Domain is the set of all real x-values. Range is the set of y-values, where $y \geqslant -4$.	It is possible to square any value of x. As squaring a negative number makes it positive, the smallest y-value possible as an output is -4.

Now you try

Write down the allowable x-values (domain) and the resulting y-values (range) for each of these functions.

a $y = -x + 3$

b $y = -x^2 + 2$

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Exercise 10A

FLUENCY | 1–3(½), 5–6(½) | 1–6(½) | 1–6(⅓)

Example 1

1 Given $f(x) = 2x^2 - x + 4$, find:

a $f(0)$ **b** $f(2)$ **c** $f(3)$

d $f(8)$ **e** $f(-1)$ **f** $f(-2)$

g $f\left(\frac{1}{2}\right)$ **h** $f(a)$ **i** $f(k)$

Example 2a,b

2 From the following, identify which are functions.

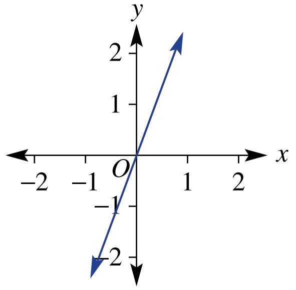
a $\{(1, 2), (2, 4), (3, 6)\}$ **b** $\{(1, 0), (-1, 4), (2, 0)\}$ **c** $\{(0, 4), (1, 3), (2, -1), (1, 5)\}$

d $y = 2x$ **e** $y = -x^2 - 4$ **f** $y = 3^x$

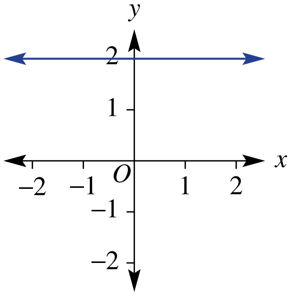
Example 2c

3 Use the vertical line test to decide which of the following graphs represent a function.

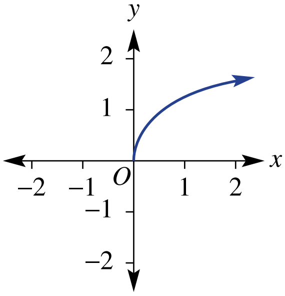
a



b



c



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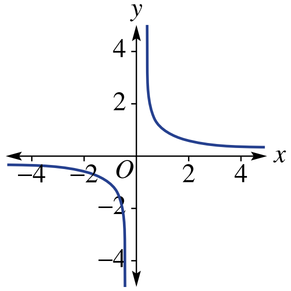
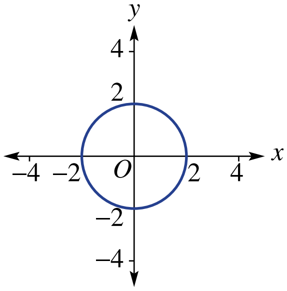


e



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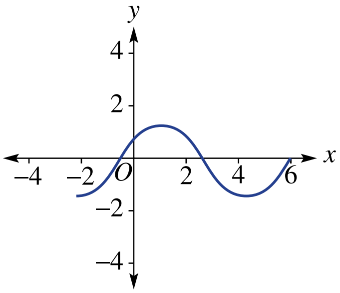

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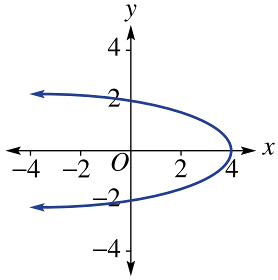

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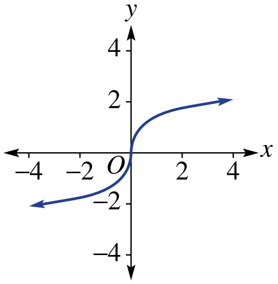
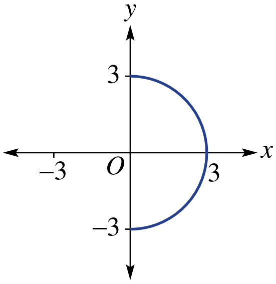


i



4 Find $f(0)$, $f(2)$, $f(-4)$, $f(a)$ and $f(a + 1)$ for each of the following functions.

a $f(x) = 4x$ **b** $f(x) = 1 - x^2$

c $f(x) = (x - 2)(x + 6)$ **d** $f(x) = 4x^2 + 9$

Example 3a

5 Find the set of allowable x-values for each function. (Note: This is the domain of each function.)

a $f(x) = 2 - x$ **b** $f(x) = 2x + 3$ **c** $f(x) = 3x^2$ **d** $f(x) = 2 - x^2$

Example 3b

6 For each function in Question **5**, write down the set of y-values (i.e. the range) that it has as its resulting output.

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PROBLEM–SOLVING	7	7, 8	7–9

7 Given $f(x) = 7x - 9$ and $g(x) = 6 - 2x$:

a Find:

i $f(2)$ **ii** $g(4)$ **iii** $f(2) + g(4)$

iv $f(-2) + 2g(1)$ **v** $f(g(2))$ **vi** $g(f(2))$

b The value of a is such that $f(a) = g(a)$. Explain the significance of $x = a$ in terms of the two graphs $y = f(x)$ and $y = g(x)$.

8 Answer the following as true or false for each of these functions.

i $f(x) = 2x - 2$

ii $f(x) = x^2 + 4$

a $f(3a) = 3f(a)$ **b** $f(a) = f(-a)$ **c** $f(a) + f(b) = f(a + b)$

9 Given the function $f(x) = 2x^2 - 3x - 1$, simplify $\dfrac{f(x + h) - f(x)}{h}$.

REASONING	10	10, 11	10–13

10 **a** Explain why all parabolas of the form $y = ax^2 + bx + c$ are functions.

b What type of straight line is not a function and why is it not a function?

c Why is finding the coordinates of the vertex of a parabola used when finding the range of the function?

d Considering your response to part **c**, find the range of the following quadratic functions.

i $y = x^2 + 4x$

ii $y = x^2 - 5x - 6$

iii $y = 1 - x - 2x^2$

iv $y = x^2 + 6x + 10$

11 Given that $\dfrac{a}{0}$ is undefined, which value of x is not allowable for each of the following functions?

a $f(x) = \dfrac{3}{x - 1}$ **b** $f(x) = \dfrac{3}{2x + 1}$ **c** $f(x) = \dfrac{-2}{1 - x}$

12 Given that you cannot take the square root of a negative number, write down the domain of:

a $y = \sqrt{x}$ **b** $y = \sqrt{x - 2}$ **c** $y = \sqrt{x + 2}$ **d** $y = \sqrt{2 - x}$

13 For $f(x) = x^2 + \dfrac{1}{x^2}$:

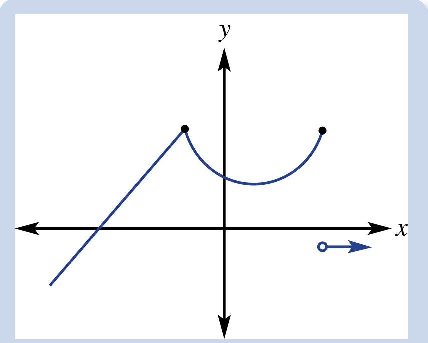
a Find $f(a)$ and $f(-a)$.

b Find the values of $f(-3), f(-2), f(-1)$ and $f(0)$. Hence, sketch the graph of $y = f(x)$, in the domain $-3 \leqslant x \leqslant 3$.

c Comment on any symmetry you notice.

ENRICHMENT: Sketching hybrid functions | – | – | 14

14 Hybrid functions involve more than one rule for different parts of the domain.



A hybrid function includes two or more rules for various parts of the domain.

a Sketch the following functions, using the domain for each part of the graph.

i $f(x) = \begin{cases} 2x & \text{for } x \geqslant 0 \\ -2x & \text{for } x < 0 \end{cases}$

ii $f(x) = \begin{cases} 4 & \text{for } x \geqslant 2 \\ x^2 & \text{for } -2 < x < 2 \\ 4 & \text{for } x \leqslant -2 \end{cases}$

iii $f(x) = \begin{cases} 2x + 4 & \text{for } x > 0 \\ -(x + 4) & \text{for } x \leqslant 0 \end{cases}$

b A function is said to be continuous if its entire graph can be drawn without lifting the pen from the page. Which of the functions in part **a** are discontinuous?

c For the functions sketched in part **a**, write down the range.

d For each of the functions in part **a**, find the value of:

i $f(2) + f(0) + f(-2)$

ii $f(3) - 2f(1) + 4f(-4)$

10B Introducing polynomials EXTENDING

LEARNING INTENTIONS

- To know the general form of a polynomial
- To know the meaning of the degree of a polynomial and the names of common polynomials
- To be able to use function notation for a polynomial

We are familiar with linear expressions such as $3x - 1$ and $4 + \frac{x}{2}$ and with quadratic expressions such as $x^2 - 3$ and $-4x^2 + 2x - 4$. These expressions are in fact part of a larger group called polynomials, which are sums of powers of a variable using whole number powers $\{0, 1, 2, ...\}$. For example, $2x^3 - 3x^2 + 4$ is a cubic polynomial and $1 - 4x^3 + 3x^7$ is a polynomial of degree 7. The study of polynomials opens up many ideas in the analysis of functions and graphing that are studied in many senior mathematics courses.

All calculators perform calculations like log 43, sin 65, etc. by substituting numbers into polynomials. A calculator can't possibly store all potential results, so a specific polynomial from the Taylor series is coded for each calculator function button. (You might encounter Taylor series if you study mathematics at university; it refers to the fact that many different types of functions can be represented by an infinite sum of special terms.)

Lesson starter: Is it a polynomial?

A polynomial is an expression that includes sums of powers of x with whole number powers $\{0, 1, 2, \dots\}$. Decide, with reasons, whether the following are polynomials.

- $5 + 2x + x^2$
- $4x^4 - x^2 - 6$
- $\sqrt{x} + x^2$
- $4x^{\frac{1}{3}} + 2x^2 + 1$
- $\frac{2}{x} + 3$
- 5

KEY IDEAS

- A **polynomial** is an expression of the form $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0x^0$, where:
  - n is a positive integer or zero $\{0, 1, 2, \dots\}$
  - $a_n, a_{n-1}, \dots a_0$ are **coefficients** which can be any real number.
  - $a_0x^0 = a_0$ is the **constant term**
  - a_nx^n is the **leading term**

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- Naming polynomials

  Polynomials are named by the highest power of x. This is called the **degree** of the polynomial.

  - **constant** For example: 2
  - **linear** For example: $3x - 7$
  - **quadratic** For example: $2x^2 - 4x + 11$
  - **cubic** For example: $-4x^3 + 6x^2 - x + 3$
  - **quartic** For example: $\frac{1}{2}x^4 - x^2 - 2$
  - **of degree 8** For example: $3x^8 - 4x^5 + x - 3$

- **Function notation**
  - A polynomial in x can be called $P(x)$.

    For example: $P(x) = 2x^3 - x$ is a cubic polynomial.
  - $P(k)$ is the value of the polynomial at $x = k$.

    For example: If $P(x) = 2x^3 - x$, then:

    $$P(3) = 2(3)^3 - (3) = 51 \quad \text{and} \quad P(-1) = 2(-1)^3 - (-1) = -2 + 1 = -1$$

BUILDING UNDERSTANDING

1 A polynomial expression is given by $3x^4 - 2x^3 + x^2 - x + 2$.

a How many terms does the polynomial have?

b State the coefficient of:

i x^4 ii x^3 iii x^2 iv x

c What is the value of the constant term?

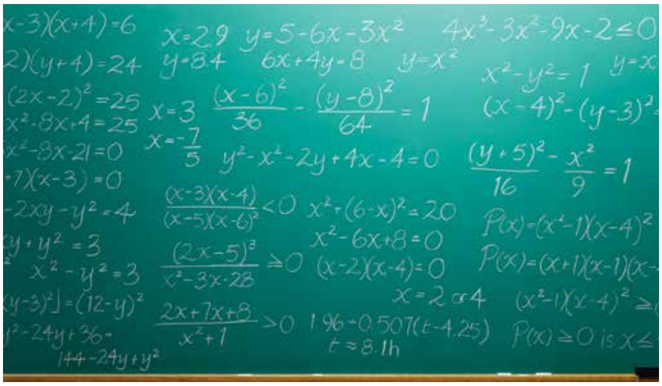
2 Decide if these polynomials are constant, linear, quadratic, cubic or quartic.

a $2x - 5$ b $x^2 - 3$ c $x^4 + 2x^3 + 1$

d $1 + x + 3x^2$ e 6 f $4x - x^3 + x^2$

3 State the degree of each of these polynomials.

a $2x^3 + 4x^2 - 2x + 1$ b $x^4 - 2x^2 - 2$ c $-3x^6 + 2x^4 - 9x^2 + 1$



Example 4 Determining if an expression is a polynomial and the degree of a polynomial

Decide if the following expressions are polynomials and state the degree of any that are polynomials.

a $4x^2 - 1 + 7x^4$

b $2x^2 - \sqrt{x} + \frac{2}{x}$

SOLUTION	EXPLANATION
a Yes and degree is 4	Powers of x are whole numbers $\{0, 1, 2, \ldots\}$. Highest power of x is 4.
b No	$2x^2 - \sqrt{x} + \frac{2}{x} = 2x^2 - x^{\frac{1}{2}} + 2x^{-1}$ Powers include $\frac{1}{2}$ and -1, which are not allowed in the polynomial family.

Now you try

Decide if the following expressions are polynomials and state the degree of any that are polynomials.

a $5x^2 - x + 4x^3$

b $\frac{1}{x} + \sqrt{x} + 1$

Example 5 Evaluating polynomials

If $P(x) = x^3 - 3x^2 - x + 2$, find:

a $P(2)$

b $P(-3)$

SOLUTION	EXPLANATION
a $\begin{aligned} P(2) &= (2)^3 - 3(2)^2 - (2) + 2 \\ &= 8 - 12 - 2 + 2 \\ &= -4 \end{aligned}$	Substitute $x = 2$ and evaluate.
b $\begin{aligned} P(-3) &= (-3)^3 - 3(-3)^2 - (-3) + 2 \\ &= -27 - 27 + 3 + 2 \\ &= -49 \end{aligned}$	Substitute $x = -3$ and note $(-3)^3 = -27$ and $(-3)^2 = 9$.

Now you try

If $P(x) = x^3 + 2x^2 - 3x + 4$, find:

a $P(2)$

b $P(-1)$

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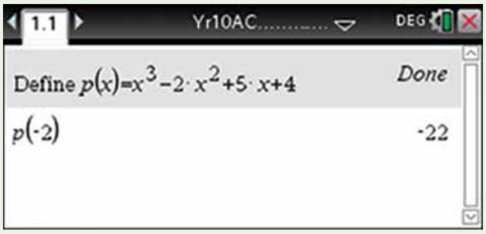
Using calculators to define polynomials

Define the polynomial $P(x) = x^3 - 2x^2 + 5x + 4$ and evaluate at $x = -2$.

Using the TI-Nspire:

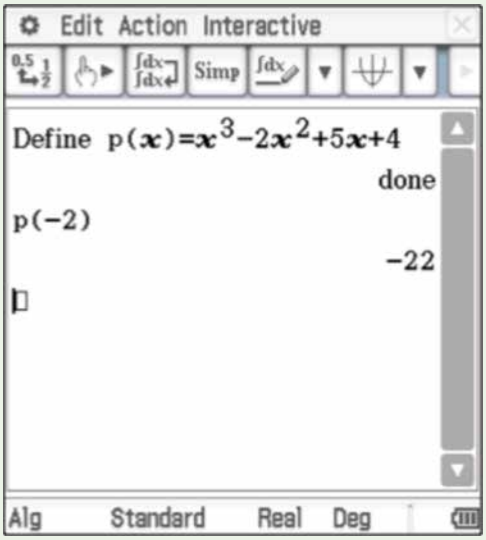
In a **Calculator** page define the polynomial using menu **>Actions>Define** as shown. Evaluate for $x = 2$, $p(-2)$.



```
Define p(x)=x^3-2·x^2+5·x+4          Done
p(-2)                                 -22
```

Using the ClassPad:

In the **Main** application, type and highlight the polynomial expression. Tap **Interactive**, **Define**, **OK**. Evaluate by typing $p(-2)$.



```
Define p(x)=x^3-2x^2+5x+4
                          done
p(-2)
                           -22
```

Exercise 10B

FLUENCY	1, 2, 4	1, 2, 4	1, 3

Example 4

1 Decide if the following are polynomials and state the degree of any that are polynomials.

a $3x^3 + x^2 - x + 3$
b $2x^4 - x^2 - 4$
c $\frac{2}{x} - \frac{3}{x} + 2$
d $\frac{7}{x^2} - \frac{1}{x} + 2$
e $x^4 - x^3 + \frac{2}{x^3}$
f $4 - 7x^8$
g $\sqrt{x} + 2 - x^2$
h $\sqrt[4]{x} + \sqrt[3]{x} + \sqrt{x}$
i $x^3 + \frac{1}{\sqrt{x}}$

Example 5

2 If $P(x) = x^3 - 3x^2 - 2x + 3$, find:

a $P(2)$
b $P(4)$
c $P(-1)$
d $P(-3)$

3 If $P(x) = 2x^4 - 3x^3 + 5x - 4$, find:

a $P(1)$
b $P(3)$
c $P(-1)$
d $P(-2)$

4 Evaluate the quadratic polynomial $x^2 - x + 2$, using:

a $x = 4$
b $x = 10$
c $x = -2$
d $x = -1$

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PROBLEM–SOLVING	5(½), 6	5(½), 6, 7	5(½), 6–8

5 If $P(x) = x^3 - x^2$ and $Q(x) = 4 - 3x$, find:

a $P(1) + Q(2)$ **b** $P(3) + Q(-1)$ **c** $P(-2) - Q(-2)$

d $Q(1) - P(3)$ **e** $(P(2))^2 + (Q(1))^2$ **f** $(P(-1))^3 - (Q(-1))^3$

6 The height (P metres) of a roller coaster track above a platform is given by the equation $P(x) = x^3 - 12x^2 + 35x$, where x metres is the horizontal distance from the beginning of the platform.

a Find the height of the track using:

i $x = 2$ **ii** $x = 3$ **iii** $x = 7$

b Does the track height ever fall below the level of the platform? If so, find a value of x for which this occurs.

Polynomials are used in the design of roller coasters.

7 Find the coefficient of x^2 in these polynomials.

a $P(x) = \dfrac{4 - 2x^2}{4}$ **b** $P(x) = \dfrac{x^3 + 7x^2 + x - 3}{-7}$ **c** $P(x) = \dfrac{x^3 - 4x^2}{-8}$

8 Evaluate $P(-2)$ for these polynomials.

a $P(x) = (x + 2)^2$

b $P(x) = (x - 2)(x + 3)(x + 1)$

c $P(x) = x^2(x + 5)(x - 7)$

REASONING	9	9, 10(½)	10–11(½)

9 **a** What is the maximum number of terms in a polynomial of degree 7?

b What is the maximum number of terms in a polynomial of degree n?

c What is the minimum number of terms in a polynomial of degree 5?

d What is the minimum number of terms in a polynomial of degree n?

10 If $P(x) = x^3 - x^2 - 2x$, evaluate and simplify these without the use of a calculator.

a $P\left(\frac{1}{2}\right)$
b $P\left(\frac{1}{3}\right)$
c $P\left(-\frac{1}{2}\right)$
d $P\left(-\frac{1}{4}\right)$
e $P\left(-\frac{2}{3}\right)$
f $P\left(\frac{4}{5}\right)$
g $P\left(-\frac{1}{2}\right) + P\left(\frac{1}{2}\right)$
h $P\left(-\frac{3}{4}\right) + P\left(\frac{3}{4}\right)$

11 If $P(x) = 2x^3 - x^2 - 5x - 1$, find the following and simplify where possible.

a $P(k)$
b $P(b)$
c $P(2a)$
d $P(-a)$
e $P(-2a)$
f $P(-3k)$
g $P(ab)$
h $P(-ab)$

ENRICHMENT: Finding unknown coefficients	–	–	12

12 If $P(x) = x^3 - 2x^2 + bx + 3$ and $P(1) = 4$, we can find the value of b as follows.

$$\begin{aligned} P(1) &= 4 \\ (1)^3 - 2(1)^2 + b(1) + 3 &= 4 \\ 2 + b &= 4 \\ b &= 2 \end{aligned}$$

a Use this method to find the value of b if $P(x) = x^3 - 4x^2 + bx - 2$ and if:

- **i** $P(1) = 5$
- **ii** $P(2) = -6$
- **iii** $P(-1) = -8$
- **iv** $P(-2) = 0$
- **v** $P(-1) = 2$
- **vi** $P(-3) = -11$

b If $P(x) = x^4 - 3x^3 + kx^2 - x + 2$, find k if:

- **i** $P(1) = 2$
- **ii** $P(-2) = 0$
- **iii** $P(-1) = -15$

c If $P(x) = x^3 + ax^2 + bx - 3$ and $P(1) = -1$ and $P(-2) = -1$, find the values of a and b.

10C Expanding and simplifying polynomials EXTENDING

LEARNING INTENTIONS

- To be able to apply the rules of expanding brackets to multiply polynomials
- To understand that multiplying polynomials results in a polynomial of higher degree

From your work on quadratics, you will remember how to use the distributive law to expand brackets. For example, $(2x - 1)(x + 5)$ expands to $2x^2 + 10x - x - 5$, and after collecting like terms this simplifies to $2x^2 + 9x - 5$. In a similar way we can expand the product of two or more polynomials of any degree. To do this we also multiply every term in one polynomial with every term in the next polynomial.

Polynomial 'secret key' technology enables secure communication between and within groups and is more efficient than standard encryption/decryption systems. Applications include video conferencing, military communications, and between the components of the IoT (Internet of Things).

Lesson starter: The product of two quadratics

The equation $(x^2 - x + 3)(2x^2 + x - 1) = 2x^4 - x^3 + 4x^2 + 4x - 3$ is written on the board.

- Is the equation true for $x = 1$?
- Is the equation true for $x = -2$?
- How can you prove the equation to be true for all values of x?

KEY IDEAS

- Expand products of polynomials by multiplying each term in one polynomial by each term in the next polynomial.
- Simplify by collecting like terms.

BUILDING UNDERSTANDING

1 Expand and simplify these quadratics.

a $x(x + 2)$ **b** $(x - 5)(x + 11)$ **c** $(4x - 3)(2x - 5)$

2 Collect like terms to simplify.

a $2x^4 - 3x^3 + x^2 - 1 - x^4 - 2x^3 + 3x^2 - 2$

b $5x^6 + 2x^4 - x^2 + 5 - 5x^4 + x^3 + 8 - 6x^6$

3 Use substitution to confirm that this equation is true for the given x-values.

$(x^3 - x + 3)(x^2 + 2x - 1) = x^5 + 2x^4 - 2x^3 + x^2 + 7x - 3$

a $x = 1$ **b** $x = 0$ **c** $x = -2$

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Example 6 Expanding polynomials

Expand and simplify.

a $x^3(x - 4x^2)$

b $(x^2 + 1)(x^3 - x + 1)$

SOLUTION

a $x^3(x - 4x^2) = x^4 - 4x^5$

b $(x^2 + 1)(x^3 - x + 1)$

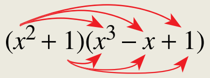
$= x^2(x^3 - x + 1) + 1(x^3 - x + 1)$

$= x^5 - x^3 + x^2 + x^3 - x + 1$

$= x^5 + x^2 - x + 1$

EXPLANATION

$x^3 \times x^1 = x^4$ and $x^3 \times (-4x^2) = -4x^5$ using index law for multiplication.



$(x^2 + 1)(x^3 - x + 1)$

$-x^3$ cancels with x^3.

Now you try

Expand and simplify.

a $x^2(3x^2 - x)$

b $(x - 2)(x^3 + 4x - 3)$

Example 7 Expanding $P(x) \times Q(x)$

If $P(x) = x^2 + x - 1$ and $Q(x) = x^3 + 2x + 3$, expand and simplify the following.

a $P(x) \times Q(x)$

b $(Q(x))^2$

SOLUTION

a $P(x) \times Q(x)$

$= (x^2 + x - 1)(x^3 + 2x + 3)$

$= x^2(x^3 + 2x + 3) + x(x^3 + 2x + 3) - 1(x^3 + 2x + 3)$

$= x^5 + 2x^3 + 3x^2 + x^4 + 2x^2 + 3x - x^3 - 2x - 3$

$= x^5 + x^4 + x^3 + 5x^2 + x - 3$

b $(Q(x))^2$

$= (x^3 + 2x + 3)^2$

$= (x^3 + 2x + 3)(x^3 + 2x + 3)$

$= x^3(x^3 + 2x + 3) + 2x(x^3 + 2x + 3) + 3(x^3 + 2x + 3)$

$= x^6 + 2x^4 + 3x^3 + 2x^4 + 4x^2 + 6x + 3x^3 + 6x + 9$

$= x^6 + 4x^4 + 6x^3 + 4x^2 + 12x + 9$

EXPLANATION

Each term in the first polynomial is multiplied by each term in the second polynomial.

$(Q(x))^2 = Q(x) \times Q(x)$

Expand to gain 9 terms then collect and simplify.

Now you try

If $P(x) = x^3 - x + 3$ and $Q(x) = x^2 + x - 4$, expand and simplify the following.

a $P(x) \times Q(x)$

b $(P(x))^2$

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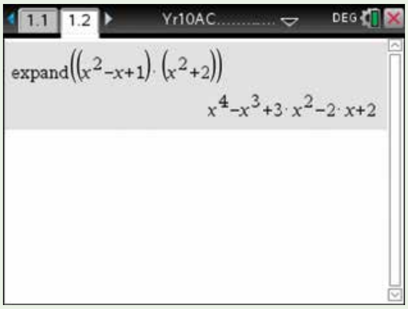
Expanding and simplifying with a calculator

Expand and simplify $(x^2 - x + 1)(x^2 + 2)$.

Using the TI-Nspire:

Use menu **>Algebra>Expand**, then type in the expression and press enter.

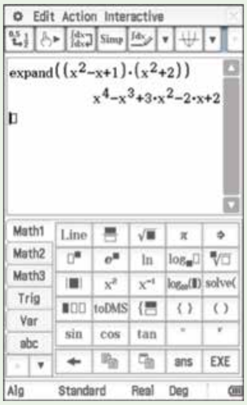


$\text{expand}((x^2-x+1)\cdot(x^2+2))$

$x^4-x^3+3\cdot x^2-2\cdot x+2$

Using the ClassPad:

In the **Main** application, type and highlight expression. Tap **Interactive**, **Transformation**, **expand**. **OK**. **EXE**.



$\text{expand}((x^2-x+1)\cdot(x^2+2))$

$x^4-x^3+3\cdot x^2-2\cdot x+2$

Exercise 10C

FLUENCY	1–2(½), 3	1–2(½), 3	1–2(⅓), 4

Example 6a

1 Expand and simplify.

a $x^2(x - 3)$ **b** $x^2(x^2 - 1)$ **c** $2x^2(1 + 3x)$

d $x^3(1 - x)$ **e** $x^3(x^2 + 3x)$ **f** $-3x^2(x^4 - x)$

g $-2x^3(x^2 + x)$ **h** $-x^2(x^5 - x^2)$ **i** $-4x^3(x^4 - 2x^7)$

Example 6b

2 Expand and simplify.

a $(x^2 + 1)(x^3 + 2)$ **b** $(x^2 - 1)(x^3 + x)$ **c** $(x^2 - x)(x^3 - 3x)$

d $(x^2 - 2)(x^3 + x - 2)$ **e** $(x^3 - x)(x^2 + 2x + 3)$ **f** $(x^3 - x^2)(x^2 - x + 4)$

g $(x^3 - x^2 - 1)(x^3 + x - 2)$ **h** $(x^3 - 5x^2 + 2)(x^3 - x + 1)$ **i** $(x^4 - x^2 + 1)(x^4 + x - 3)$

Example 7

3 If $P(x) = x^2 - 2x + 1$ and $Q(x) = x^3 + x - 1$, expand and simplify.

a $P(x) \times Q(x)$ **b** $(Q(x))^2$ **c** $(P(x))^2$

4 If $P(x) = x^3 + 2x^2 - x - 4$ and $Q(x) = x^2 + x - 2$, expand and simplify.

a $P(x) \times Q(x)$ **b** $(Q(x))^2$ **c** $(P(x))^2$

PROBLEM–SOLVING	5(½)	5(½), 6	5(½), 6, 7

5 If $P(x) = x^2 - 5x + 1$ and $Q(x) = x^3 + x$, simplify.

a $P(x) + Q(x)$ **b** $Q(x) - P(x)$ **c** $5P(x) + 2Q(x)$

d $1 - P(x)Q(x)$ **e** $4 - (Q(x))^2$ **f** $(P(x))^2 - (Q(x))^2$

6 Find the square of $P(x)$ in expanded form when $P(x) = (x^2 + x - 1)^2$.

7 Show that $(x^2 - x - 1)^2 - (x^2 - x + 1)^2 = 4x - 4x^2$.

REASONING	8	8, 9	9, 10

8 If $P(x)$ and $Q(x)$ are polynomials, does $P(x)Q(x) = Q(x)P(x)$ for all values of x?

9 Give the degree of the polynomial $P(x) \times Q(x)$ when:

a $P(x)$ is quadratic and $Q(x)$ is linear

b $P(x)$ is quadratic and $Q(x)$ is cubic

c $P(x)$ is cubic and $Q(x)$ is quartic

d $P(x)$ is of degree 7 and $Q(x)$ is of degree 5.

10 If $P(x)$ is of degree m and $Q(x)$ is of degree n and $m > n$, what is the highest possible degree of the following polynomials?

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $P(x) \times Q(x)$

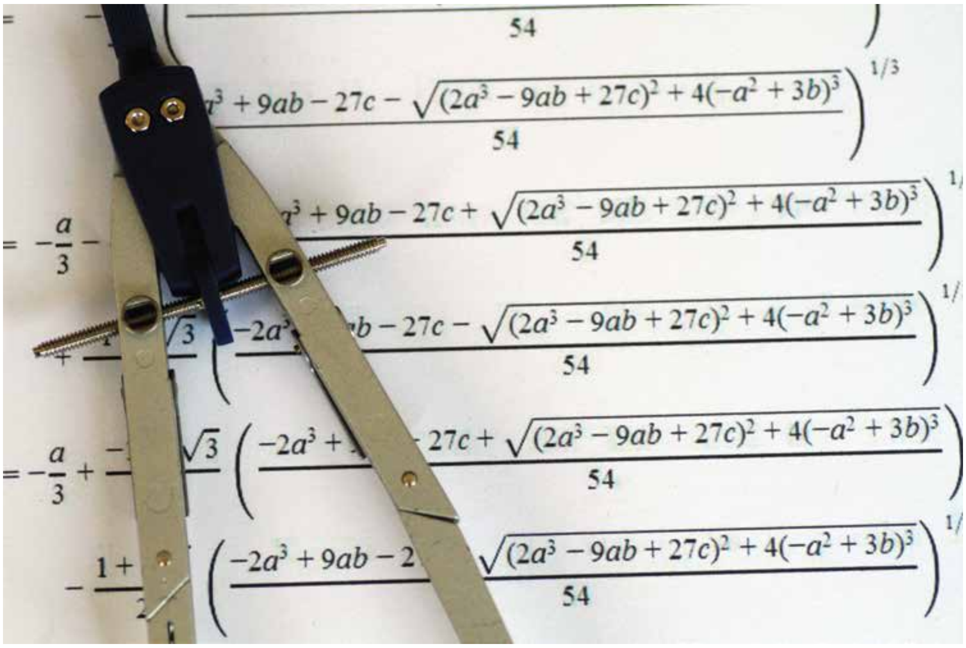
d $(P(x))^2$ **e** $(P(x))^2 - Q(x)$ **f** $(Q(x))^3$

ENRICHMENT: Triple expansions	–	–	11

11 Expand and simplify.

a $x(x^2 + 1)(x - 1)$ **b** $x^3(x + 3)(x - 1)$ **c** $(x + 2)(x - 1)(x + 3)$

d $(x + 4)(2x - 1)(3x + 1)$ **e** $(5x - 2)(x - 2)(3x + 5)$ **f** $(x^2 + 1)(x^2 - 2)(x + 3)$



10D Division of polynomials EXTENDING

LEARNING INTENTIONS

- To know the long division algorithm
- To be able to carry out the long division algorithm to divide polynomials
- To know how to express a polynomial using the quotient, divisor and remainder

Division of polynomials requires the use of the long division algorithm. You may have used this algorithm for the division of whole numbers in primary school.

Recall that 7 divided into 405 can be calculated in the following way.

$$\begin{array}{rr l}
 & 57 & \\
 & 7\overline{)405} & \text{7 into 4 does not go.} \\
5 \times 7 & \underline{35\ \ } & \text{7 into 40 gives 5 and } 5 \times 7 = 35. \\
 & 55 & \text{Then subtract } 405 - 350. \\
7 \times 7 & \underline{49} & \text{7 into 55 gives 7 and } 7 \times 7 = 49. \\
 & 6 & \text{Subtract to give remainder 6.}
\end{array}$$

So $405 \div 7 = 57$ and 6 remainder. The 57 is called the quotient.

Another way to write this is $405 = 7 \times 57 + 6$.

We use this technique to divide polynomials.

Acoustic engineers use complex mathematical procedures, including polynomial division, to analyse and electronically reproduce the vibrations that make sound; for designing headphones and synthesisers; and to analyse the architecture required for a superb concert sound.

Lesson starter: Recall long division

Use long division to find the quotient and remainder for the following.

- $832 \div 3$
- $2178 \div 7$

KEY IDEAS

- We use the long division algorithm to divide polynomials.
- The result is not necessarily a polynomial.
  Example:

$$\text{dividend} \longrightarrow \frac{x^3 - x^2 + x - 1}{\underset{\nearrow\ \text{divisor}}{x + 2}} = \underset{\uparrow\ \text{quotient}}{x^2 - 3x + 7} - \frac{15}{x + 2} \swarrow \text{remainder}$$

We can write this as:

$$\underset{\text{dividend }(P(x))}{\underset{\nearrow}{x^3 - x^2 + x - 1}} = \underset{\text{divisor }(A(x))}{\underset{\uparrow}{(x + 2)}}\underset{\text{quotient }(Q(x))}{\underset{\nwarrow}{(x^2 - 3x + 7)}} - \underset{\text{remainder }(R(x))}{\underset{\nwarrow}{15}}$$

  - $P(x) = Q(x)\,A(x) + R(x)$
- The degree of the remainder must be less than the degree of the divisor.

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BUILDING UNDERSTANDING

1 Use long division to find the remainder.

a $208 \div 9$ **b** $143 \div 7$ **c** $2184 \div 3$

2 Complete the equation with the missing numbers.

a If $182 \div 3 = 60$ remainder 2, then $182 = \_\_\_ \times 60 + \_\_\_$.

b If $2184 \div 5 = 436$ remainder 4, then $2184 = \_\_\_ \times 436 + \_\_\_$.

c If $617 \div 7 = 88$ remainder 1, then $617 = 7 \times \_\_\_ + \_\_\_$.

3 Simplify.

a $x^3 \div x$ **b** $2x^3 \div x$ **c** $4x^2 \div x$ **d** $-7x \div x$

Example 8 Dividing polynomials

a Divide $P(x) = x^3 + 2x^2 - x + 3$ by $(x - 2)$ and write in the form $P(x) = (x - 2)Q(x) + R(x)$, where $R(x)$ is the remainder.

b Divide $P(x) = 2x^3 - x^2 + 3x - 1$ by $(x + 3)$ and write in the form $P(x) = (x + 3)Q(x) + R(x)$.

SOLUTION

a

$$\begin{array}{rr} & x^2 + 4x + 7 \\ x - 2 & \overline{)\,x^3 + 2x^2 - x + 3} \\ x^2(x-2) & \underline{x^3 - 2x^2 } \\ & 4x^2 - x + 3 \\ 4x(x-2) & \underline{4x^2 - 8x } \\ & 7x + 3 \\ 7(x-2) & \underline{7x - 14} \\ & 17 \end{array}$$

$\therefore x^3 + 2x^2 - x + 3 = (x - 2)(x^2 + 4x + 7) + 17$

EXPLANATION

First, divide x from $(x - 2)$ into the leading term (i.e. x^3). So divide x into x^3 to give x^2.

$x^2(x - 2)$ gives $x^3 - 2x^2$ and subtract from $x^3 + 2x^2 - x + 3$.

After subtraction, divide x into $4x^2$ to give $4x$ and repeat the process above.

After subtraction, divide x into $7x$ to give 7.

Subtract to give the remainder 17.

b

$$\begin{array}{rr} & 2x^2 - 7x + 24 \\ x + 3 & \overline{)\,2x^3 - x^2 + 3x - 1} \\ 2x^2(x+3) & \underline{2x^3 + 6x^2 } \\ & -7x^2 + 3x - 1 \\ -7x(x+3) & \underline{-7x^2 - 21x } \\ & 24x - 1 \\ 24(x+3) & \underline{24x + 72} \\ & -73 \end{array}$$

$\therefore 2x^3 - x^2 + 3x - 1 = (x + 3)(2x^2 - 7x + 24) - 73$

First, divide x from $(x + 3)$ into the leading term. So divide x into $2x^3$ to give $2x^2$.

After subtraction, divide x into $-7x^2$ to give $-7x$.

After subtraction, divide x into $24x$ to give 24.

Subtract to give the remainder -73.

Now you try

a Divide $P(x) = x^3 + x^2 - 4x + 3$ by $(x - 1)$ and write in the form $P(x) = (x - 1)Q(x) + R(x)$, where $R(x)$ is the remainder.

b Divide $P(x) = 3x^3 - 2x^2 + 5x - 2$ by $(x + 2)$ and write in the form $P(x) = (x + 2)Q(x) + R(x)$.

Exercise 10D

FLUENCY	1–3	1, 3, 4	2–4

Example 8a

1 Divide $P(x) = x^3 + x^2 - 3x + 2$ by $(x - 1)$ and write in the form $P(x) = (x - 1)Q(x) + R$, where R is the remainder.

2 Divide $P(x) = x^3 + x^2 - 2x + 3$ by $(x - 2)$ and write in the form $P(x) = (x - 2)Q(x) + R$, where R is the remainder.

Example 8b

3 Divide $P(x) = 2x^3 - x^2 + 4x - 2$ by $(x + 2)$ and write in the form $P(x) = (x + 2)(Q(x)) + R$.

4 Divide $P(x) = 3x^3 - x^2 + x + 2$ by $(x + 1)$ and write in the form $P(x) = (x + 1)Q(x) + R$, where R is the remainder.

PROBLEM–SOLVING	5(½)	5–7(½)	6, 7(½)

5 For each of the following, express in this form:

Dividend = divisor × quotient + remainder (as in the examples)

a $(2x^3 - x^2 + 3x - 2) \div (x - 2)$

b $(2x^3 + 2x^2 - x - 3) \div (x + 2)$

c $(5x^3 - 2x^2 + 7x - 1) \div (x + 3)$

d $(-x^3 + x^2 - 10x + 4) \div (x - 4)$

e $(-2x^3 - 2x^2 - 5x + 7) \div (x + 4)$

f $(-5x^3 + 11x^2 - 2x - 20) \div (x - 3)$

6 Divide and write in this form:

Dividend = divisor × quotient + remainder

a $(6x^4 - x^3 + 2x^2 - x + 2) \div (x - 3)$

b $(8x^5 - 2x^4 + 3x^3 - x^2 - 4x - 6) \div (x + 1)$

7 Divide the following and express in the usual form.

a $(x^3 - x + 1) \div (x + 2)$

b $(x^3 + x^2 - 3) \div (x - 1)$

c $(x^4 - 2) \div (x + 3)$

d $(x^4 - x^2) \div (x - 4)$

REASONING	8	8, 9	9–11

8 There are three values of k for which $P(x) = x^3 - 2x^2 - x + 2$ divided by $(x - k)$ gives a remainder of zero. Find the three values of k.

9 Prove that $(6x^3 - 37x^2 + 32x + 15) \div (x - 5)$ leaves remainder 0.

10 Find the remainder when $P(x)$ is divided by $(2x - 1)$ given that:

a $P(x) = 2x^3 - x^2 + 4x + 2$

b $P(x) = -3x^3 + 2x^2 - 7x + 5$

11 Find the remainder when $P(x) = -3x^4 - x^3 - 2x^2 - x - 1$ is divided by these expressions.

a $x - 1$

b $2x + 3$

c $-3x - 2$

ENRICHMENT: When the remainder is not a constant	–	–	12

12 Divide the following and express in the form $P(x) = \text{divisor} \times Q(x) + R(x)$, where $R(x)$ is a function of x.

a $(x^3 - x^2 + 3x + 2) \div (x^2 - 1)$

b $(2x^3 + x^2 - 5x - 1) \div (x^2 + 3)$

c $(5x^4 - x^2 + 2) \div (x^3 - 2)$

10E The remainder and factor theorems EXTENDING

LEARNING INTENTIONS

- To know how to use the remainder theorem to find the remainder when a polynomial is divided by a linear expression
- To understand that a remainder of zero implies that a divisor is a factor
- To be able to use the factor theorem to decide if a linear expression is a factor of a polynomial

Using long division we can show, after dividing $(x-2)$ into $P(x)=x^3-x^2+2x-3$, that $P(x)=x^3-x^2+2x-3=(x-2)(x^2+x+4)+5$, where 5 is the remainder.

Using the right-hand side to evaluate $P(2)$, we have:

$$\begin{aligned}P(2)&=(2-2)\left(2^2+2+4\right)+5\\&=0\times(2^2+2+4)+5\\&=0+5\\&=5\end{aligned}$$

This shows that the remainder when $P(x)$ is divided by $(x-2)$ is $P(2)$.

More generally, when $P(x)$ is divided by $(x-a)$ we obtain:

$$\begin{aligned}P(x)&=(x-a)Q(x)+R\\\text{So}\quad P(a)&=0\times Q(a)+R\\&=R\end{aligned}$$

The polynomial remainder theorem is used in algorithms for detecting and correcting errors that can occur in digital data transmissions, such as from GPS satellites to a phone. Polynomials are exchanged using codes formed from coefficients.

So the remainder is $P(a)$ and this result is called the remainder theorem. This means that we can find the remainder when dividing $P(x)$ by $(x-a)$ simply by evaluating $P(a)$.

We also know that when factors are divided into a number there is zero remainder. So if $P(x)$ is divided by $(x-a)$ and the remainder $P(a)$ is zero, then $(x-a)$ is a factor of $P(x)$. This result is called the factor theorem.

Lesson starter: Which way is quicker?

A polynomial $P(x)=x^3-3x^2+6x-4$ is divided by $(x-2)$.

- Show, using long division, that the remainder is 4.
- Find $P(2)$. What do you notice?
- Explain how you can find the remainder when $P(x)$ is divided by:
  a $x-3$ **b** $x-5$
- Show that when $P(x)$ is divided by $(x+1)$ the remainder is -14.
- What would be the remainder when $P(x)$ is divided by $(x-1)$? What do you notice and what does this say about $(x-1)$ in relation to $P(x)$?

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KEY IDEAS

- **Remainder theorem**: When a polynomial $P(x)$ is divided by $(x-a)$ the remainder is $P(a)$.
  - When dividing by $(x-3)$ the remainder is $P(3)$.
  - When dividing by $(x+2)$ the remainder is $P(-2)$.
- **Factor theorem**: When $P(x)$ is divided by $(x-a)$ and the remainder is zero (i.e. $P(a)=0$), then $(x-a)$ is a factor of $P(x)$.

$$\begin{aligned} P(x) &= x^3-3x^2-3x+10 \\ &= \underbrace{(x-2)}_{\text{factor}}\underbrace{(x^2-x-5)}_{\text{quotient}} \end{aligned}$$

$P(2)=0$

$(x-2)$ is a factor with zero remainder

BUILDING UNDERSTANDING

1. If $P(x)=2x^3-x^2-x-1$, find the value of the following.
   - **a** $P(1)$
   - **b** $P(3)$
   - **c** $P(-2)$
   - **d** $P(-4)$
2. What value of x do you substitute into $P(x)$ to find the remainder when a polynomial $P(x)$ is divided by:
   - **a** $x-3$?
   - **b** $x+2$?
3. What is the remainder when an expression is divided by one of its factors?

Example 9 Using the remainder theorem

Find the remainder when $P(x)=x^3-5x^2-x+4$ is divided by:

a $x-2$

b $x+1$

SOLUTION

a
$$\begin{aligned} P(x) &= x^3-5x^2-x+4 \\ P(2) &= (2)^3-5(2)^2-2+4 \\ &= 8-20-2+4 \\ &= -10 \end{aligned}$$
The remainder is -10.

EXPLANATION

For $(x-2)$ substitute $x=2$.

Using the remainder theorem, $P(2)$ gives the remainder.

SOLUTION

b
$$\begin{aligned} P(x) &= x^3-5x^2-x+4 \\ P(-1) &= -1-5+1+4 \\ &= -1 \end{aligned}$$
The remainder is -1.

EXPLANATION

For $(x+1)$ substitute $x=-1$.

Note: $(-1)^3=-1$, $(-1)^2=1$ and $-(-1)=1$.

Now you try

Find the remainder when $P(x)=x^3-4x^2+6x-1$ is divided by:

a $x-1$

b $x+2$

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Example 10 Finding a linear factor

Decide whether each of the following is a factor of $P(x) = x^3 + x^2 - 3x - 6$.

a $x + 1$

b $x - 2$

SOLUTION	EXPLANATION
a $P(x) = x^3 + x^2 - 3x - 6$ $P(-1) = -1 + 1 + 3 - 6$ $= -3$ $\therefore (x + 1)$ is not a factor.	If $(x + 1)$ is a factor of $P(x)$, then $P(-1) = 0$. This is not true as the remainder is -3.
b $P(x) = x^3 + x^2 - 3x - 6$ $P(2) = 8 + 4 - 6 - 6$ $= 0$ $\therefore (x - 2)$ is a factor.	Substitute $x = 2$ to evaluate $P(2)$. Since $P(2) = 0$, $(x - 2)$ is a factor of $P(x)$.

Now you try

Decide whether each of the following is a factor of $P(x) = 2x^3 - 3x^2 - 11x + 6$.

a $x + 1$

b $x - 3$

Example 11 Applying the remainder theorem

Find the value of k such that $(x^3 - x^2 + 2x + k) \div (x - 1)$ has a remainder of 5.

SOLUTION	EXPLANATION
Let $P(x) = x^3 - x^2 + 2x + k$.	
$P(1) = 5$	The remainder is $P(1)$, which is 5.
$(1)^3 - (1)^2 + 2(1) + k = 5$ $2 + k = 5$ $k = 3$	Substitute $x = 1$ and solve for k.

Now you try

Find the value of k such that $(x^3 + 2x^2 - x + k) \div (x - 2)$ has a remainder of 12.

Exercise 10E

FLUENCY	1–3(½)	1–4(½)	1–4(½)

Example 9

1 Find the remainder when $P(x) = x^3 - 2x^2 + 7x - 3$ is divided by:

a $x - 1$
b $x - 2$
c $x - 3$
d $x - 4$
e $x + 4$
f $x + 2$
g $x + 1$
h $x + 3$

2 Find the remainder when $P(x) = x^4 - x^3 + 3x^2$ is divided by:

a $x - 1$
b $x - 2$
c $x + 2$
d $x + 1$

Example 10

3 Decide which of the following are factors of $P(x) = x^3 - 4x^2 + x + 6$.

a $x - 1$
b $x + 1$
c $x - 2$
d $x + 2$
e $x - 3$
f $x + 3$
g $x - 4$
h $x + 4$

4 Decide which of the following are factors of $P(x) = x^4 - 2x^3 - 25x^2 + 26x + 120$.

a $x - 2$
b $x + 2$
c $x + 3$
d $x - 3$
e $x - 4$
f $x + 4$
g $x - 5$
h $x + 5$

PROBLEM–SOLVING	5	5–6(½)	5–6(½)

5 Use the factor theorem and trial and error to find a linear factor of these polynomials.

a $P(x) = x^3 + 2x^2 + 7x + 6$
b $P(x) = x^3 + 2x^2 - x - 2$
c $P(x) = x^3 + x^2 + x + 6$
d $P(x) = x^3 - 2x - 4$

6 Use the factor theorem to find all three linear factors of these polynomials.

a $P(x) = x^3 - 2x^2 - x + 2$
b $P(x) = x^3 - 2x^2 - 5x + 6$
c $P(x) = x^3 - 4x^2 + x + 6$
d $P(x) = x^3 - 2x^2 - 19x + 20$

REASONING	7	7, 8, 9(½)	8–10

Example 11

7 For what value of k will $(x^3 - 2x^2 + 5x + k) \div (x - 1)$ have the following remainders?

a 0
b 2
c -10
d 100

8 For what value of k will $(x^4 - 2x^3 + x^2 - x + k) \div (x + 2)$ have zero remainder?

9 Find the value of k in these polynomials.

a $P(x) = x^3 + 2x^2 + kx - 4$ and when divided by $(x - 1)$ the remainder is 4.
b $P(x) = x^3 - x^2 + kx - 3$ and when divided by $(x + 1)$ the remainder is -6.
c $P(x) = 2x^3 + kx^2 + 3x - 4$ and when divided by $(x + 2)$ the remainder is -6.
d $P(x) = kx^3 + 7x^2 - x - 4$ and when divided by $(x - 2)$ the remainder is -2.

10 Find the value of k when:

a $(x + 2)$ is a factor of $x^3 - kx^2 - 2x - 4$
b $(x - 3)$ is a factor of $2x^3 + 2x^2 - kx - 3$

ENRICHMENT: Simultaneous coefficients	–	–	11

11 Use simultaneous equations and the given information to find the value of a and b in these cubics.

a $P(x) = x^3 + ax^2 + bx - 3$ and the remainder is -1 when $P(x)$ is divided by $(x - 1)$ and 5 when $P(x)$ is divided by $(x - 2)$
b $P(x) = 2x^3 - ax^2 - bx - 1$ and the remainder is -10 when $P(x)$ is divided by $(x + 1)$ and -37 when $P(x)$ is divided by $(x + 2)$

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10A **1** For $f(x) = x^2 - 2x + 5$, find:

a $f(1)$ **b** $f(-2)$ **c** $f(a)$

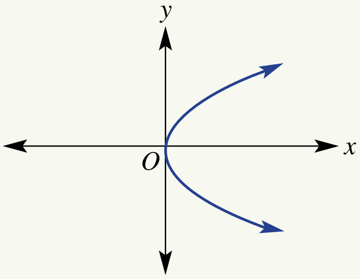
10A **2** Determine which of the following are functions.

a $2x + 3y = 8$

b $y = 2 - x^2$

c $\{(1, 3), (2, 6), (1, -1), (3, 5)\}$

d



10A **3** Write down the allowable x-values (domain) and the resulting y-values (range) for each of these functions.

a $y = 2x + 3$ **b** $y = (x + 1)^2 - 3$

10B **4** Explain why $4x^3 - \sqrt{x} + 6x - 1$ is not a polynomial.

10B **5** Consider the polynomial $P(x) = 3x^4 - 2x^3 + x^2 + 7x + 8$. Find:

a $P(0)$ **b** $P(2)$ **c** $P(-1)$

10B **6** For the polynomial $P(x) = 4x^3 + 3x^2 + 2x + 1$, state:

a the degree of the polynomial

b the constant term

c the coefficent of x^2

d the leading term.

10C **7** Expand and simplify the following.

a $x^4(x^3 - 2x + 1)$ **b** $(x^2 - 1)(x^2 + 2x + 6)$

10C **8** If $P(x) = x^3 + x + 2$ and $Q(x) = x^4 + 2x$, find the following in their simplest forms.

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$

c $(Q(x))^2$ **d** $P(x) \times Q(x)$

10D **9** Divide $P(x)$ by $(x - 2)$ in the following and write in the form $P(x) = (x - 2)Q(x) + R$, where R is the remainder.

a $P(x) = x^3 - 3x^2 + 4x + 7$ **b** $P(x) = 2x^3 + 5x^2 - 3x - 4$

10E **10** Find the remainder when $P(x) = x^3 - 2x^2 + 3x - 1$ is divided by:

a $x - 1$ **b** $x + 2$

10E **11** Find the value of a such that $x + 1$ is a factor of $P(x) = x^3 - ax^2 - 5x - 7$.

10F Solving polynomial equations EXTENDING

LEARNING INTENTIONS
- To know how to find a factor of a polynomial using the factor theorem
- To be able to factorise a polynomial using division by a known factor
- To be able to apply the Null Factor Law to solve a polynomial equation in factorised form

We know from our work with quadratics that the Null Factor Law can be used to solve a quadratic equation in factorised form.

For example:
$$\begin{aligned} x^2 - 3x - 40 &= 0 \\ (x - 8)(x + 5) &= 0 \end{aligned}$$

Using the Null Factor Law:

$$\begin{aligned} x - 8 = 0 &\text{ or } x + 5 = 0 \\ x = 8 &\text{ or } x = -5 \end{aligned}$$

We can also apply this method to solve higher degree polynomials.

Solving complex, realistic polynomial equations occurs in civil, aerospace, electrical, industrial and mechanical engineering. Architects apply polynomial modelling to solve 3D structural problems, such as the curved supports in the Disney Concert Hall, Los Angeles.

If a polynomial is not in a factorised form, we use the remainder and factor theorems to help find its factors. Long division can also be used in this process.

Lesson starter: Solving a cubic

Consider the cubic equation $P(x) = 0$, where $P(x) = x^3 + 6x^2 + 5x - 12$.

- Explain why $(x - 1)$ is a factor of $P(x)$.
- Use long division to find $P(x) \div (x - 1)$.
- Write $P(x)$ in the form $(x - 1)Q(x)$.
- Now complete the factorisation of $P(x)$.
- Show how the Null Factor Law can be used to solve $P(x) = 0$. Why are there three solutions?

KEY IDEAS

- A **polynomial equation** of the form $P(x) = 0$ can be solved by:
  - factorising $P(x)$
  - using the Null Factor Law: If $a \times b \times c = 0$ then $a = 0, b = 0$ or $c = 0$.

- To factorise a polynomial follow these steps.
  - Find one factor using the remainder and factor theorems. Start with $(x - 1)$ using $P(1)$ or $(x + 1)$ using $P(-1)$. If required, move to $(x - 2)$ or $(x + 2)$ etc.
  - A good idea is to first consider factors of the constant term of the polynomial to reduce the number of trials.
  - Use long division to find the quotient after dividing by the factor.
  - Factorise the quotient (if possible).
  - Continue until $P(x)$ is fully factorised.

BUILDING UNDERSTANDING

1 Give a reason why $(x + 1)$ is a factor of $P(x) = x^3 - 7x - 6$. (*Hint*: Find $P(-1)$.)

2 Use the Null Factor Law to solve these quadratic equations.

a $(x - 1)(x + 3) = 0$

b $x^2 - x - 12 = 0$

Example 12 Using the Null Factor Law

Solve for x.

a $(x - 1)(x + 2)(x + 5) = 0$

b $(2x - 3)(x + 7)(3x + 1) = 0$

SOLUTION

a $(x - 1)(x + 2)(x + 5) = 0$

$$\begin{aligned} x - 1 &= 0 & x + 2 &= 0 & x + 5 &= 0 \\ x &= 1 \quad \text{or} & x &= -2 \quad \text{or} & x &= -5 \end{aligned}$$

b $(2x - 3)(x + 7)(3x + 1) = 0$

$$\begin{aligned} 2x - 3 &= 0 & x + 7 &= 0 & 3x + 1 &= 0 \\ 2x &= 3 & x &= -7 & 3x &= -1 \\ x &= \frac{3}{2} \quad \text{or} & x &= -7 \quad \text{or} & x &= -\frac{1}{3} \end{aligned}$$

EXPLANATION

Using the Null Factor Law, if $a \times b \times c = 0$ then $a = 0$ or $b = 0$ or $c = 0$.

Equate each factor to 0 and solve for the three values of x.

Now you try

Solve for x.

a $(x - 2)(x + 1)(x + 6) = 0$

b $(2x - 1)(x + 3)(5x + 2) = 0$

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Example 13 Factorising and solving

Solve $x^3 + 2x^2 - 5x - 6 = 0$.

SOLUTION

$$\begin{aligned} \text{Let} \quad P(x) &= x^3 + 2x^2 - 5x - 6 \\ P(1) &= 1 + 2 - 5 - 6 \neq 0 \\ P(-1) &= -1 + 2 + 5 - 6 = 0 \end{aligned}$$

$\therefore x + 1$ is a factor.

$$\begin{array}{r@{\quad}r}
 & x^2 + x - 6 \\
x + 1 & \overline{)x^3 + 2x^2 - 5x - 6} \\
x^2(x + 1) & \underline{x^3 + x^2 \phantom{{}- 5x - 6}} \\
 & x^2 - 5x - 6 \\
x(x + 1) & \underline{x^2 + x \phantom{{}- 6}} \\
 & -6x - 6 \\
-6(x + 1) & \underline{-6x - 6} \\
 & 0
\end{array}$$

$$\begin{aligned} \therefore P(x) &= (x + 1)(x^2 + x - 6) \\ &= (x + 1)(x + 3)(x - 2) \end{aligned}$$

Solve $P(x) = 0$:

$(x + 1)(x + 3)(x - 2) = 0$

$$\begin{array}{lll} x + 1 = 0 & x + 3 = 0 & x - 2 = 0 \\ \quad x = -1 \text{ or} & \quad x = -3 \text{ or} & \quad x = 2 \end{array}$$

EXPLANATION

Try to find a factor using the remainder and factor theorems. Start with $(x - 1)$ using $P(1)$ or $(x + 1)$ using $P(-1)$. If required, move to $(x - 2)$ or $(x + 2)$ or others using factors of 6.

$P(-1) = 0$ so $(x + 1)$ is a factor.

Divide $(x + 1)$ into $P(x)$ to find the quotient using long division.

Note: The remainder is 0, as expected ($P(-1) = 0$).

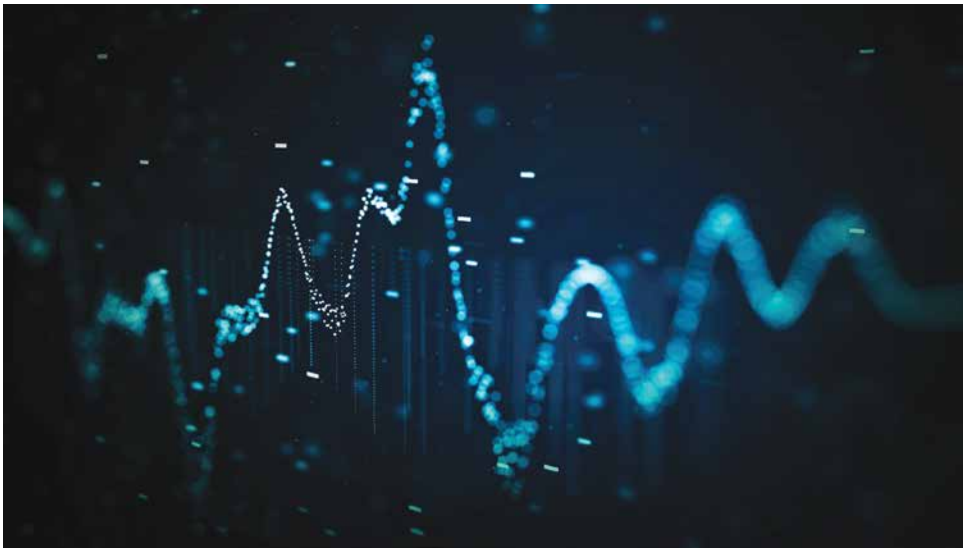
$P(x) = (x + 1)Q(x) + R$ but $R = 0$.

$x^2 + x - 6$ factorises to $(x + 3)(x - 2)$.

Use the Null Factor Law to now solve for x.

Now you try

Solve $x^3 + 2x^2 - 11x - 12 = 0$.



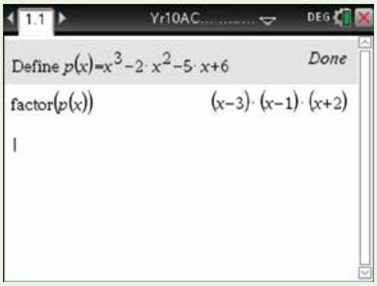
Using calculators to factorise and solve polynomials

1 Define the polynomial $P(x) = x^3 - 2x^2 - 5x + 6$ and factorise.

2 Solve $x^3 - 2x^2 - 5x + 6 = 0$.

Using the TI-Nspire:

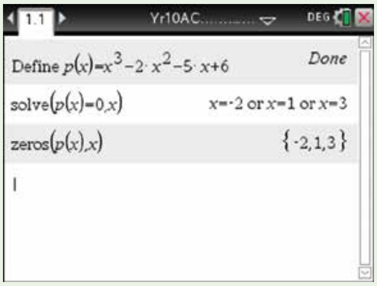
1 In a **Calculator** page define the polynomial using menu **>Actions>Define**. Factor the polynomial using menu **>Algebra>Factor** as shown.



```
Define p(x)=x^3-2·x^2-5·x+6                    Done
factor(p(x))                      (x-3)·(x-1)·(x+2)
```

2 Solve using menu **>Algebra>Solve**. Then type $p(x) = 0$, x as shown.

Alternatively, solve using menu **>Algebra>Zeros** for solving equations equalling zero.



```
Define p(x)=x^3-2·x^2-5·x+6                    Done
solve(p(x)=0,x)                  x=-2 or x=1 or x=3
zeros(p(x),x)                              {-2,1,3}
```

Using the ClassPad:

1 In the **Main** application, type and highlight the polynomial. Tap **Interactive**, **Define**. Type *p* for function name and tap *OK*. Type *p*(*x*) in the next entry line. Highlight and tap **Interactive**, **Transformation**, **factor**, **factor**.

```
Define p(x)=x^3-2x^2-5
                      done
factor(p(x))
           (x+2)·(x-1)·(x-3)
```

2 Type and highlight $p(x) = 0$, Tap **Interactive**, **Advanced**, **Solve**.

```
Define p(x)=x^3-2·x^2-5·x+6
                            done
factor(p(x))
              (x+2)·(x-1)·(x-3)
solve(p(x)=0,x)
                  {x=-2,x=1,x=3}
```

Exercise 10F

FLUENCY	1–2(½)	1(⅓), 2(½)	1–2(⅓)

Example 12

1 Solve for x using the Null Factor Law.

a $(x+3)(x-2)(x-1)=0$

b $(x+2)(x+7)(x-1)=0$

c $(x-4)(x+4)(x-3)=0$

d $\left(x+\frac{1}{2}\right)(x-3)\left(x+\frac{1}{3}\right)=0$

e $(2x+1)(x-3)(3x+2)=0$

f $(4x-1)(5x-2)(7x+2)=0$

g $\left(x+\frac{1}{2}\right)(3x+11)(11x+12)=0$

h $(5x+3)(19x+2)\left(x-\frac{1}{2}\right)=0$

Example 13

2 For each of the following cubic equations, follow these steps as in **Example 13**.

- Use the factor theorem to find a factor.
- Use long division to find the quotient.
- Factorise the quotient.
- Write the polynomial in a fully factorised form.
- Use the Null Factor Law to solve for x.

a $x^3-4x^2+x+6=0$

b $x^3+6x^2+11x+6=0$

c $x^3-6x^2+11x-6=0$

d $x^3-8x^2+19x-12=0$

e $x^3-3x^2-16x-12=0$

f $x^3+6x^2-x-30=0$

PROBLEM–SOLVING	3	3, 4	4, 5

3 Use the quadratic formula to solve for x, expressing your answers in exact form.

a $(x-1)(x^2-2x-4)=0$

b $(x+2)(x^2+6x+10)=0$

4 Solve by first taking out a common factor.

a $2x^3-14x^2+14x+30=0$

b $3x^3+12x^2+3x-18=0$

5 Solve for x.

a $x^3-13x+12=0$

b $x^3-7x-6=0$

REASONING	6	6, 7	6, 7(½), 8, 9

6 State the maximum number of solutions to $P(x)=0$ when $P(x)$ is of degree:

a 3

b 4

c n

7 Show that the following equations can be factorised easily without the use of long division, and then give the solutions.

a $x^3-x^2=0$

b $x^3+x^2=0$

c $x^3-x^2-12x=0$

d $2x^5+4x^4+2x^3=0$

8 Explain why $x^4+x^2=0$ has only one solution.

9 Explain why $(x-2)(x^2-3x+3)=0$ has only one solution.

ENRICHMENT: Quartics with four factors	–	–	10

10 Factorising a quartic may require two applications of the factor theorem and long division. Solve these quartics by factorising the left-hand side first.

a $x^4+8x^3+17x^2-2x-24=0$

b $x^4-2x^3-11x^2+12x+36=0$

c $x^4+x^3-11x^2-9x+18=0$

d $2x^4-3x^3-7x^2+12x-4=0$

10G Graphing cubic functions of the form $y = a(x - h)^3 + k$ EXTENDING

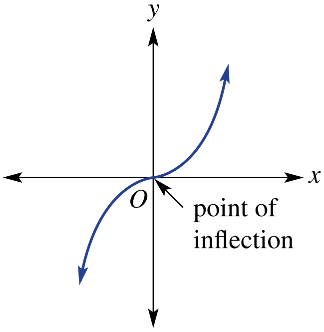
LEARNING INTENTIONS

- To know the shape of the graph of the basic Cubic function $y = x^3$
- To understand how the values of a, h and k in $y = a(x - h)^3 + k$ transform the graph of $y = x^3$.
- To be able to solve simple cubic equations of the form $ax^3 = d$.
- To be able apply transformations including reflections and translations to sketch cubic graphs of the form $y = a(x - h)^3 + k$.

A cubic is a polynomial in which the highest power of the variable is 3 (we say it has degree 3).
In this section, we will consider cubic graphs of the form $y = ax^3$ and their transformations.
When solving a cubic equation of the form $ax^3 = d$, we will always obtain one distinct solution.
Since $(-2)^3 = (-2) \times (-2) \times (-2) = -8$, we have $\sqrt[3]{-8} = -2$, meaning that we can take the cube root of both positive and negative values, thus obtaining one distinct solution.

Cubic functions of the form $y = ax^3$ have particular application in problems relating to volume. For example, the volume of a cube with side length x is $V = x^3$ and the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Lesson starter: How do a and k effect the graph of $y = ax^3 + k$?



The graph of $y = x^3$ is shown. It has a point called the **point of inflection** at the origin, O or (0, 0).

- Use computer graphing software to sketch the graphs in each table on the same set of axes. Complete each table to observe the effect of changing the values of a and d.

Graphs of $y = ax^3$

Rule	Coordinates of point of inflection	Number of x-intercepts	Shape of graph	Is the graph narrower or wider than the graph of $y = x^3$?
$y = x^3$	(0, 0)	1		
$y = 2x^3$				
$y = 5x^3$				
$y = \frac{1}{2}x^3$				
$y = -x^3$				
$y = -3x^3$				
$y = -\frac{1}{2}x^3$				
$y = -\frac{1}{4}x^3$				

- What causes the graph of $y = ax^3$ to reflect in the x-axis?
- Describe the effect on the shape of the graph by different values of a.

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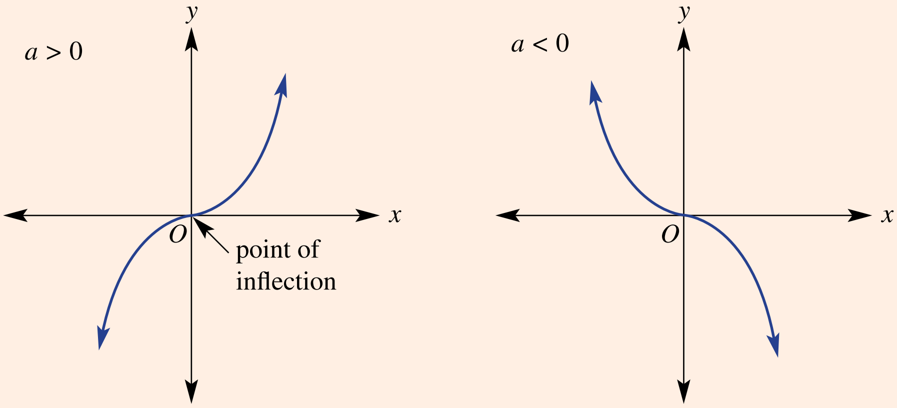
Graphs of $y = x^3 + k$

Rule	Coordinates of point of inflection	Number of x-intercepts	Position of graph compared to $y = x^3$?
$y = x^3$	$(0, 0)$	1	
$y = x^3 + 2$			
$y = x^3 - 3$			
$y = x^3 - 5$			
$y = x^3 + 8$			

- Describe the effect on the graph of $y = x^3 + k$ by different values of k.
- Can you explain why all graphs of the form $y = ax^3 + k$ have exactly one x-intercept?

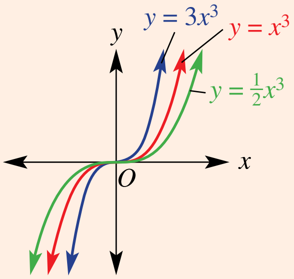
KEY IDEAS

- A cube root of a number, $\sqrt[3]{c}$, is a number a such that $a^3 = c$.
  - All real numbers have exactly one cube root.
  - $\sqrt[3]{8} = 2$ since $2^3 = 8$, and $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$.
- A cubic equation of the form $ax^3 = d$ has exactly one solution.
  - To solve $x^3 = c$, take the cube root of both sides; hence, $x = \sqrt[3]{c}$.
    For example: If $x^3 = 27$, then $x = \sqrt[3]{27} = 3$ since $3^3 = 27$.
  - To solve $ax^3 = d$, first solve for x^3 by dividing both sides by a and then take the cube root of both sides.
- The graphs of $y = ax^3$ are shown.

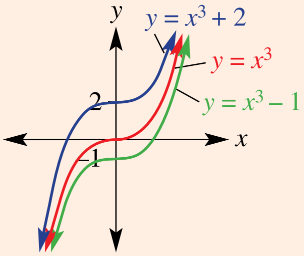


  - Each cubic is a function (i.e. it passes the vertical line test) with allowable values of x and y being all real values.

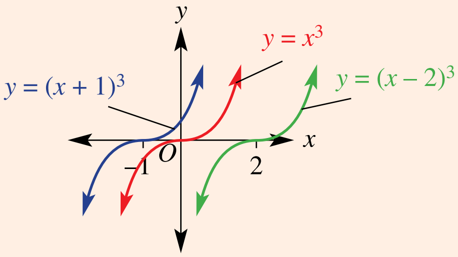
- $y = x^3$ can be written as $f(x) = x^3$. Also, $f(2) = 8$ can be used to represent the point (2, 8).
- The graph has a **point of inflection** (i.e. a point where the gradient of the graph changes from decreasing to increasing or vice versa) at (0, 0).
- Negative values of a cause the graph to reflect in the x-axis. (For cubics this is the same as a reflection in the y-axis.)
- For $a > 0$:
  $a > 1$ causes the graph to rise more quickly and makes it narrower than $y = x^3$.
  $0 < a < 1$ causes the graph to rise more slowly and makes it wider than $y = x^3$.



- The k in the rule $y = ax^3 + k$ translates the graph of $y = ax^3$ in the vertical direction.
  - $k > 0$ translates the graph up k units.
  - $k < 0$ translates the graph down k units.



- The h in the rule $y = a(x - h)^3$ translates the graph of $y = ax^3$ in the horizontal direction.
  - $h > 0$ translates the graph h units to the right.
  - $h < 0$ translates the graph h units to the left.



BUILDING UNDERSTANDING

1 Evaluate:

a 3^3 **b** 5^3 **c** -4^3

d $(-10)^3$ **e** $(-6)^3$ **f** $\sqrt[3]{8}$

g $\sqrt[3]{1}$ **h** $\sqrt[3]{-27}$ **i** $\sqrt[3]{-125}$

2 Find the value $2x^3$ for the following values of x.

a $x = 0$ **b** $x = 3$ **c** $x = -1$ **d** $x = -4$

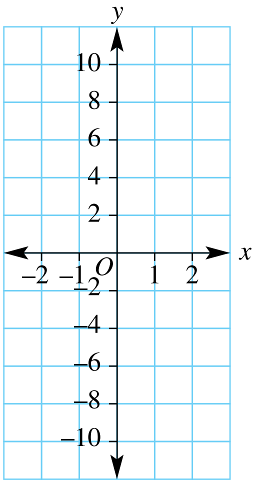
3 Complete the following tables for the given rules and plot on the same Cartesian plane.

a $y = x^3$

x	−2	−1	0	1	2
y					

b $y = -x^3$

x	−2	−1	0	1	2
y					



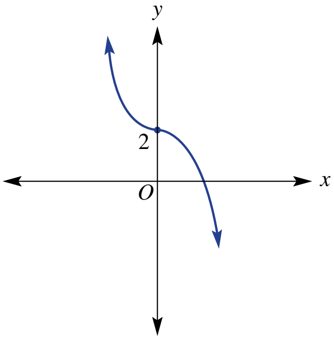
4 The graph of a cubic function is shown.

a Write down the coordinates of the point of inflection of this cubic.

b A possible equation for this cubic function is:

A $y = 2x^3$ **B** $y = (x + 2)^3$ **C** $y = -2x^3$

D $y = -x^3 + 2$ **E** $y = x^3 + 2$



Example 14 Solving simple cubic equations

Solve the following cubic equations.

a $x^3 = 27$

b $-2x^3 = 16$

c $3x^3 + 10 = 193$ (round to one decimal place)

SOLUTION	EXPLANATION
a $\begin{aligned} x^3 &= 27 \\ x &= \sqrt[3]{27} \\ \therefore x &= 3 \end{aligned}$	To solve for x, take the cube root of both sides. $\sqrt[3]{27} = 3$ since $3^3 = 27$.
b $\begin{aligned} -2x^3 &= 16 \\ x^3 &= -8 \\ x &= \sqrt[3]{-8} \\ \therefore x &= -2 \end{aligned}$	Solve for x^3 by dividing both sides by -2. Take the cube root of both sides. The cube root of a negative number will be negative and $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$.
c $\begin{aligned} 3x^3 + 10 &= 193 \\ 3x^3 &= 183 \\ x^3 &= 61 \\ x &= \sqrt[3]{61} \\ \therefore x &= 3.9 \text{ (to 1 d.p.)} \end{aligned}$	Solve for x^3 first by subtracting 10 from both sides and then dividing both sides by 3. Take the cube root of both sides to solve for x. $\sqrt[3]{61}$ is the exact answer as it does not simplify to a whole number. Use a calculator to find $\sqrt[3]{61} = 3.9$, to one decimal place.

Now you try

Solve the following cubic equations.

a $x^3 = 8$

b $-4x^3 = 108$

c $\frac{1}{2}x^3 + 11 = 43$

Example 15 Sketching cubic functions of the form $y = ax^3$

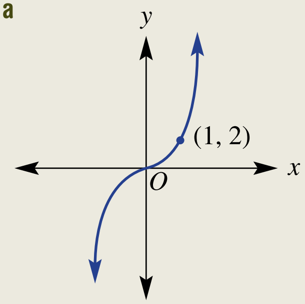
Sketch the following cubic graphs.

a $y = 2x^3$

b $y = -\frac{1}{2}x^3$

SOLUTION

a



EXPLANATION

The graph is a positive cubic graph with point of inflection at (0, 0).

Mark in the point at $x = 1$ to show the effect of the scale factor 2.

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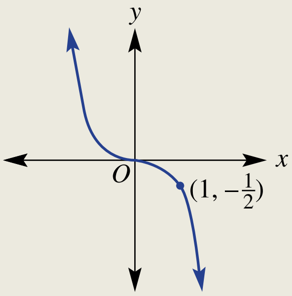
b



The graph is a negative cubic graph with point of inflection at (0, 0). It is the graph of $y = \frac{1}{2}x^3$ and is reflected in the x-axis. Mark in the point at $x = 1$ to show the effect of the scale factor $\frac{1}{2}$.

Now you try

Sketch the following cubic graphs.

a $y = -3x^3$

b $y = \frac{1}{4}x^3$

Example 16 Sketching cubic graphs involving translations

Sketch the following cubic functions, labelling the point of inflection and axes intercepts.

a $y = x^3 - 8$

b $y = -2x^3 + 6$

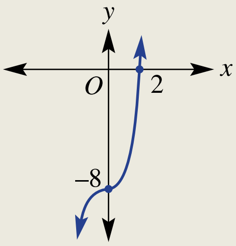
c $y = (x - 1)^3$

SOLUTION

a Point of inflection is (0, −8).

x-intercept (when $y = 0$):

$$\begin{aligned} x^3 - 8 &= 0 \\ x^3 &= 8 \\ x &= 2 \end{aligned}$$



EXPLANATION

Locate the point of inflection at (0, −8) since the graph of $y = x^3 - 8$ is the graph of $y = x^3$ translated down 8 units.

The point of inflection is also the y-intercept.

Determine the x-intercept by substituting $y = 0$.

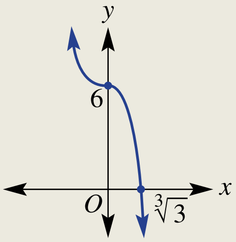
$\sqrt[3]{8} = 2$

Mark in the point of inflection and intercepts and join them to form a positive cubic curve.

b Point of inflection is (0, 6).

x-intercept (when $y = 0$):

$$\begin{aligned} -2x^3 + 6 &= 0 \\ 2x^3 &= 6 \\ x^3 &= 3 \\ x &= \sqrt[3]{3} \end{aligned}$$

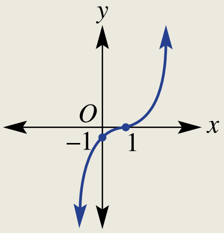


The graph is a negative cubic graph.
The graph is the graph of $y = -2x^3$ translated 6 units up. The point of inflection is also the y-intercept.
Substitute $y = 0$ to find the x-intercept. Solve the remaining equation by first solving for x^3 and then taking the cube root of both sides.
Leave the answer in exact form, although it is useful to consider the decimal approximation when marking on the axis (i.e. $\sqrt[3]{3} \approx 1.4$).
Mark the key points on the graph and join them to form a negative cubic curve.

c Point of inflection is (1, 0).

y-intercept ($x = 0$):

$$\begin{aligned} y &= (0 - 1)^3 \\ y &= -1 \end{aligned}$$



The graph of $y = (x - 1)^3$ is the graph of $y = x^3$ translated 1 unit to the right.
The point of inflection is also the x-intercept.
Locate the y-intercept by substituting $x = 0$.
Recall that $(-1)^3 = -1$.

Mark points on the graph and join them to form a cubic curve.

Now you try

Sketch the following cubic functions, labelling the point of inflection and axes intercepts

a $y = x^3 + 8$ **b** $y = -\frac{1}{2}x^3 - 5$ **c** $y = (x + 2)^3$

Exercise 10G

FLUENCY	1–3(½), 4–6	1–6(½)	1(⅓), 2–6(½)

Example 14a, b

1 Solve the following cubic equations.

a $x^3 = 64$ **b** $x^3 = 125$ **c** $x^3 = -27$

d $x^3 = -8$ **e** $2x^3 = 2000$ **f** $-3x^3 = 81$

g $\frac{x^3}{2} = 108$ **h** $-\frac{x^3}{3} = 9$ **i** $x^3 = \frac{1}{8}$

j $x^3 = \frac{1}{27}$ **k** $25x^3 = \frac{1}{5}$ **l** $-24x^3 = -3$

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Example 14c

2 Solve these cubic equations, rounding your answer to one decimal place where necessary.

a $x^3 + 4 = 5$ **b** $x^3 - 12 = 15$ **c** $2x^3 - 3 = -9$

d $3x^3 + 5 = 86$ **e** $\frac{1}{2}x^3 - 10 = 22$ **f** $\frac{1}{3}x^3 - 43 = 32$

g $4 - x^3 = 28$ **h** $3 - 2x^3 = 165$ **i** $5 - \frac{1}{2}x^3 = 18.5$

3 Give exact solutions to these cubic equations.

a $x^3 - 15 = 6$ **b** $2x^3 + 10 = 22$

c $1 - 2x^3 = 5$ **d** $\frac{x^3}{2} + 9 = 18$

Example 15

4 Sketch a graph of the following cubic functions.

a $y = 3x^3$ **b** $y = 4x^3$ **c** $y = -2x^3$

d $y = \frac{1}{2}x^3$ **e** $y = \frac{1}{10}x^3$ **f** $y = -\frac{1}{4}x^3$

Example 16a, b

5 Sketch these cubic graphs involving vertical translations. Label the axes intercepts with exact values.

a $y = x^3 + 1$ **b** $y = x^3 - 8$ **c** $y = -x^3 - 27$

d $y = \frac{1}{2}x^3 + 32$ **e** $y = 2x^3 + 10$ **f** $y = -\frac{1}{3}x^3 + 8$

Example 16c

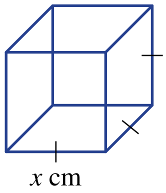
6 Sketch these cubic graphs involving horizontal translations.

a $y = (x + 1)^3$ **b** $y = (x - 2)^3$ **c** $y = (x - 3)^3$

d $y = -(x + 2)^3$ **e** $y = -2(x - 1)^3$ **f** $y = \frac{1}{2}(x + 4)^3$

PROBLEM–SOLVING	7, 8	7–9	8–10

7 A cube with side lengths x cm has a volume of 512 cm^3. Determine its surface area.



8 A seedling is planted and its height, h millimetres, is recorded for a number of weeks. It is found that the height of the plant t weeks after planting can be modelled by the equation $h = 2t^3$.

a What was the height of the plant after 3 weeks?

b After how many weeks was the plant 25 cm tall?

c The plant has an expected maximum height of 1 metre. After how many weeks would it have reached this height?

d Plot a graph of h against t for $0 \le t \le 8$.

9 Solve the following cubic equations.

a $(x + 1)^3 = 8$ **b** $(x - 2)^3 = 27$

c $(x + 3)^3 = -64$ **d** $2(x + 2)^3 = 250$

e $-3(x - 2)^3 = 192$ **f** $-\frac{1}{2}(x + 4)^3 = 500$

10 If the Earth is taken to be spherical with volume $108.321 \times 10^{10}\text{ km}^3$, determine the mean radius of the Earth, to the nearest kilometre. Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.

REASONING

11	11, 12	11–13

11 **a** Explain why $(-x)^2 \neq -x^2$ for $x \neq 0$.

b Explain why $(-x)^3 = -x^3$ for all values of x.

c Generalise your results from parts **a** and **b** and use them to explain for which values of n will $(-x)^n$ equal $-x^n$ for all values of x.

12 A cylindrical tank is such that its height is equal to its radius.

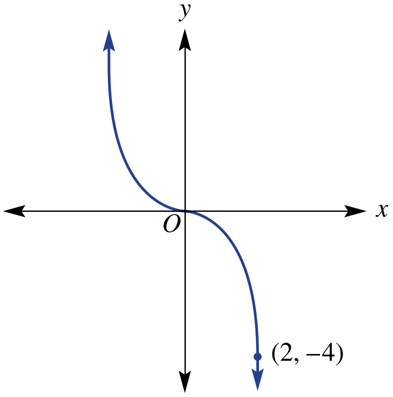
a Plot a graph of the volume, V, of the tank against the radius, r, for suitable values of r.

b Determine the exact radius of the cylinder if the volume is 8000 units^3.

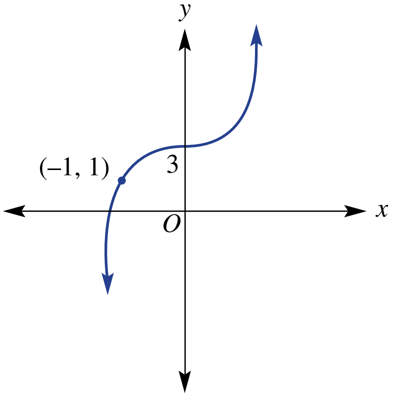
c If the radius of the tank is doubled (and the height is equal to this new radius) what is the resulting change in volume? Can you explain this?

13 The following graphs have rules of the form $y = ax^3 + k$. Use the points given to find the values of a and k in each one.

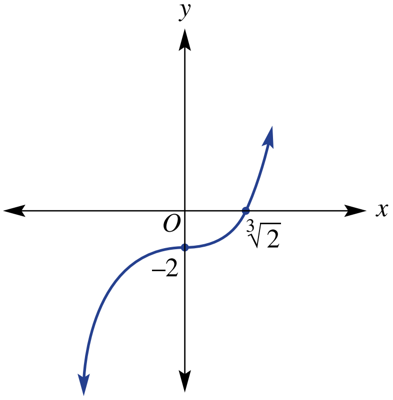
a



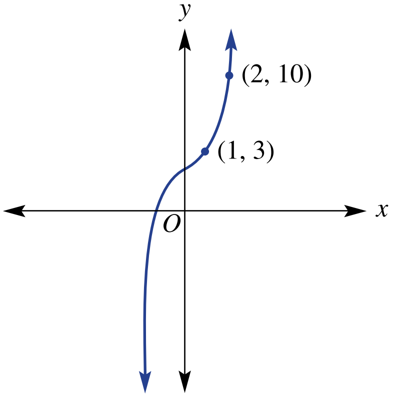
b



c



d



ENRICHMENT: Combining transformations

–	–	14

14 Combine your knowledge of transformations to sketch the following cubic graphs. Label the point of inflection and axes intercepts.

For example, the point of inflection of the graph of $y = (x - 1)^3 - 8$ is at $(1, -8)$.

a $y = (x - 1)^3 - 8$

b $y = (x - 2)^3 + 27$

c $y = 2(x + 3)^3 + 2$

d $y = \frac{1}{2}(x + 4)^3 - 4$

e $y = -(x + 2)^3 + 1$

f $y = -(x + 1)^3 - 8$

g $y = -\frac{1}{3}(x + 3)^3 - 9$

h $y = -2(x - 1)^3 + 16$

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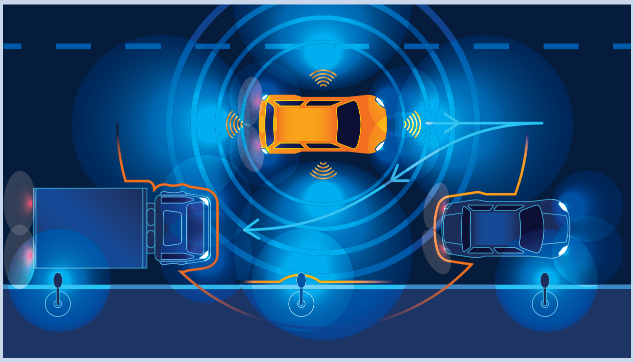
10H Graphs of polynomials EXTENDING

LEARNING INTENTIONS

- To know the shape of a graph of a cubic polynomial with three different factors
- To be able to find the axis intercepts of a cubic graph using the Null Factor Law
- To know how to use the shape and intercepts to sketch a cubic graph

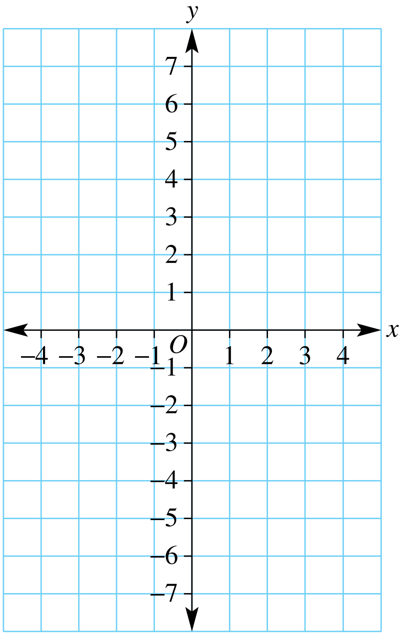
So far in Year 10 we have studied graphs of linear equations (straight lines), graphs of quadratic equations (parabolas) and graphs of cubic functions of the form $y = a(x - h)^3 + k$. We have also looked at graphs of exponentials. In this section we introduce the graphs of cubic polynomials by focusing on those including three linear factors. We start by applying the Null Factor Law to find the x-intercepts.



A spline is a curve formed by joining sections of various polynomial graphs. Splines are used in motion planning algorithms including for robots avoiding obstacles, self-driving cars parallel parking, and industrial robots' joint and hand trajectories.

Lesson starter: Comparing $y = (x + 2)(x - 3)$ and $y = (x + 2)(x - 3)(x - 1)$

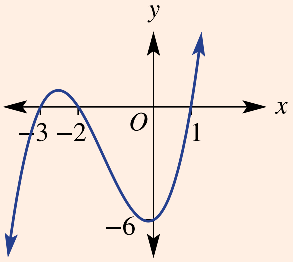
- For the parabola $y = (x + 2)(x - 3)$, find the y-intercept and use the Null Factor Law to find the x-intercepts.
- Use a similar technique to find the y-intercept and x-intercepts for the cubic function $y = (x + 2)(x - 3)(x - 1)$.
- Sketch the above parabola and cubic on the same set of axes.
- Describe the similarities and differences between the two graphs.



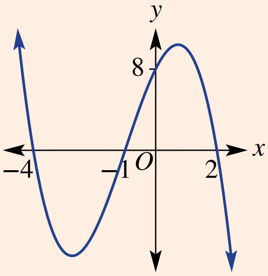
KEY IDEAS

- To sketch cubic graphs in factorised form with three different factors:
  - Find the three x-intercepts using the Null Factor Law.
  - Find the y-intercept.
  - Connect points to sketch a positive or negative cubic graph.

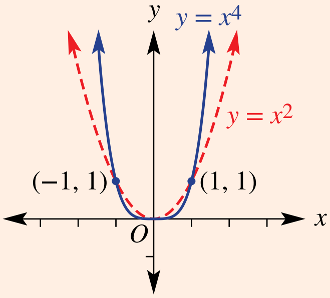
Positive cubic
(The coefficient of x^3 is positive.)
$y = (x - 1)(x + 2)(x + 3)$



Negative cubic
(The coefficient of x^3 is negative.)
$y = -(x + 4)(x - 2)(x + 1)$



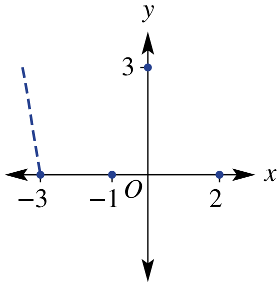
- Further consideration is needed to find turning points of cubics, as they are not located symmetrically between x-intercepts. This will be studied at more senior levels of mathematics.
- The basic quartic $y = x^4$



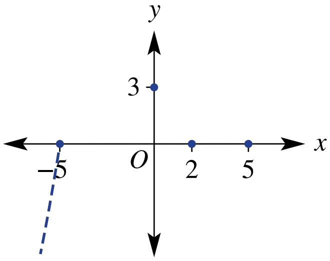
BUILDING UNDERSTANDING

① Join the given x- and y-intercepts to form a smooth cubic curve. Each graph has been started for you on the left-hand side.

a



b



② Find the coordinates of the x- and y-intercepts of the graphs of these cubics.

a $y = (x + 1)(x - 3)(x - 4)$

b $y = -2x(x + 7)(x - 5)$

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Example 17 Sketching cubic graphs

Sketch the graphs of the following by finding the x- and y-intercepts.

a $y = (x + 2)(x - 1)(x - 3)$ **b** $y = -x(x + 3)(x - 2)$

SOLUTION

a $y = (x + 2)(x - 1)(x - 3)$

y-intercept at $x = 0$:

$y = (2)(-1)(-3)$

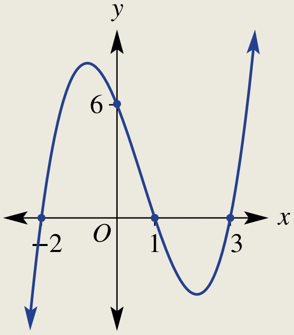
$= 6$

x-intercepts at $y = 0$:

$0 = (x + 2)(x - 1)(x - 3)$

$\therefore x + 2 = 0$ or $x - 1 = 0$ or $x - 3 = 0$

$x = -2$ $x = 1$ $x = 3$



EXPLANATION

Substitute $x = 0$ to find the y-intercept.

Substitute $y = 0$ to find the x-intercepts. Use the Null Factor Law.

Mark the four intercepts and connect to form a positive cubic graph.

The coefficient of x^3 in the expansion of y is positive, so the graph points upwards to the right.

b $y = -x(x + 3)(x - 2)$

y-intercept at $x = 0$:

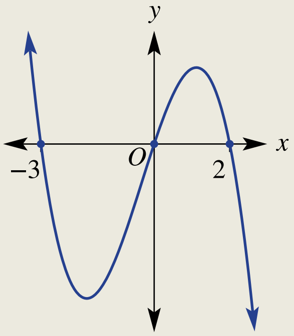
$y = -0(3)(-2) = 0$

x-intercepts at $y = 0$:

$0 = -x(x + 3)(x - 2)$

$\therefore -x = 0$ or $x + 3 = 0$ or $x - 2 = 0$

$x = 0$ $x = -3$ $x = 2$



Find the y-intercept using $x = 0$.

The three factors are $-x$, $x + 3$ and $x - 2$.

The coefficient of x^3 in the expansion of y is negative, so the graph points downwards at the right.

Now you try

Sketch the graphs of the following by finding the x- and y-intercepts.

a $y = (x + 3)(x + 1)(x - 2)$ **b** $y = -x(x + 4)(x - 1)$

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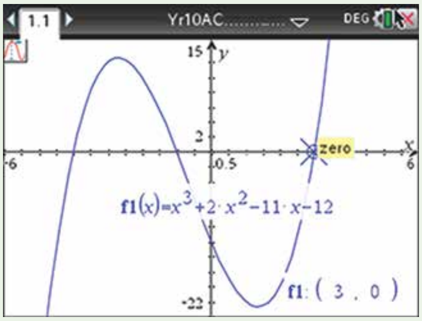
Using calculators to sketch polynomials

Sketch $P(x) = x^3 + 2x^2 - 11x - 12$ and show the x-intercepts.

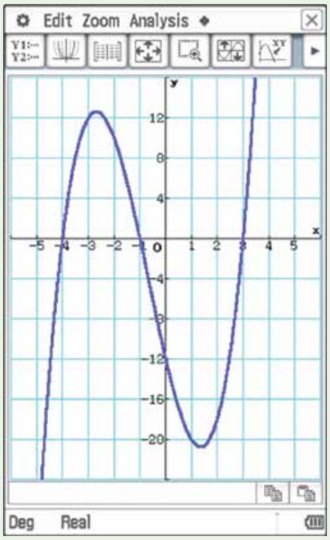
Using the TI-Nspire:

Enter the rule $P(x) = x^3 + 2x^2 - 11x - 12$. Adjust the scale using **Window Settings**. Find the x-intercepts using **Trace>Graph Trace** or using **Analyze Graph>Zero** and set the lower and upper bounds by scrolling left and right.



Using the ClassPad:

Enter the rule $y1 = x^3 + 2x^2 - 11x - 12$. Tap to see the graph. Adjust the scale by tapping on . Tap **Analysis**, **G-Solve**, root to find x-intercepts.



Exercise 10H

FLUENCY	1–2(½), 3	1–2(½), 3	1–3(½)

Example 17a

1 Sketch the graphs of the following by finding x- and y-intercepts.

a $y = (x + 2)(x - 1)(x - 3)$
b $y = (x - 3)(x - 4)(x + 1)$
c $y = (x - 5)(x - 1)(x + 2)$
d $y = \frac{1}{2}(x + 3)(x - 2)(x - 1)$
e $y = x(x - 2)(x + 3)$
f $y = x(x - 5)(x + 1)$

Example 17b

2 Sketch the graphs of the following by finding x- and y-intercepts.

a $y = -x(x + 3)(x - 2)$
b $y = -x(x - 4)(x + 1)$
c $y = -2x(x - 1)(x + 3)$
d $y = -\frac{1}{3}x(x + 1)(x - 3)$
e $y = -(x + 2)(x + 4)(x - 1)$
f $y = -(x + 3)\left(x - \frac{1}{2}\right)(x + 1)$

3 State if the following are positive (P) or negative (N) cubics.

a $y = x(x + 2)(x - 1)$
b $y = 3x(x + 4)(x - 5)$
c $y = -2(x - 2)(x + 1)(x - 6)$
d $y = -7(x + 6)(x - 1)(x - 3)$
e $y = -4x^3 + 2x^2 - x + 4$
f $y = 7x^3 - x + 2$

PROBLEM–SOLVING	4, 5	4, 5	5, 6

4 Which of the following cubics have y-intercept of $(0, -4)$?

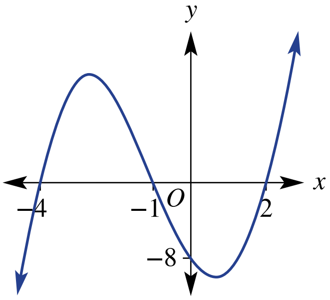
A $y = x(x - 1)(x + 4)$
B $y = x(x + 1)(x - 4)$
C $y = (x + 2)(x - 2)(x + 1)$
D $y = (x + 4)(x - 4)(x + 1)$

5 Which of the following cubics have x-intercepts at -3, 0 and 2?

A $y = x(x - 3)(x + 2)$
B $y = x(x + 3)(x - 2)$
C $y = (x + 3)(x - 1)(x + 2)$
D $y = (x + 3)(x - 2)(x + 2)$

6 Find a cubic rule for these graphs in factorised form e.g. $y = 2(x - 1)(x + 2)(x - 4)$.

a



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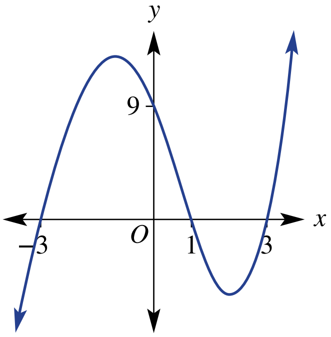
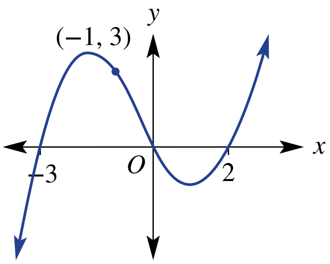
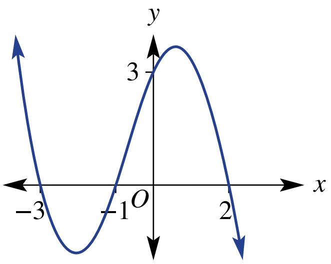


c

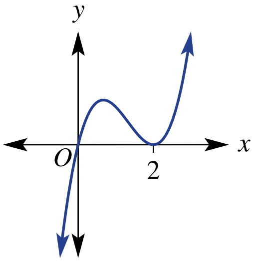


d



REASONING	7	7, 8	7–9

7 If a polynomial has a repeated factor $(x - a)$, then the point at $x = a$ is an x-intercept and also a turning point; e.g. $y = x(x - 2)^2$ as shown. Now sketch these polynomials.



a $y = x(x - 3)^2$
b $y = -2x(x + 1)^2$
c $y = -(x + 2)^2(x - 3)$
d $y = (x + 4)(x + 1)^2$
e $y = (2 - x)(x + 1)^2$
f $y = x^2(x - 2)$

8 We know that the graph of $y = (x - 2)^3 - 1$ is the graph $y = x^3$ translated 2 units to the right and 1 unit down. Use this idea to sketch graphs of the following.

a $y = x^4 - 1$
b $y = (x + 3)^4$
c $y = (x - 2)^4 - 3$

9 Sketch these quartics, making use of the Null Factor Law for x-intercepts.

a $y = (x - 5)(x - 3)(x + 1)(x + 2)$
b $y = -x(x + 4)(x + 1)(x - 4)$

ENRICHMENT: Polynomial with the lot	–	–	10

10 To sketch a graph of a polynomial that is not in factorised form you must factorise the polynomial to help find the x-intercepts.

Complete the following for each polynomial.

- **i** Find the coordinates of the y-intercept.
- **ii** Factorise the polynomial using the factor theorem and long division.
- **iii** Find the coordinates of the x- intercepts.
- **iv** Sketch the graph.

a $y = x^3 + 4x^2 + x - 6$
b $y = x^3 - 7x^2 + 7x + 15$
c $y = x^4 + 2x^3 - 9x^2 - 2x + 8$
d $y = x^4 - 34x^2 + 225$

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10I Graphs of circles EXTENDING

LEARNING INTENTIONS

- To know the form of the Cartesian equation of a circle centred at the origin with radius r
- To be able to sketch a graph of a circle centred at the origin, using the radius to label intercepts
- To know how to find the points of intersection of a line and a circle

We know of the circle as a common shape in geometry, but we can also describe a circle using an equation and as a graph on the Cartesian plane.

Programs that manipulate a robot's arm include circle equations. Working in 2D: with the lower arm secured at its base, the joint can move in a semicircle; each point on this semicircle can be the centre of a circular path that the hand can follow.

Lesson starter: Plotting the circle

A graph has the rule $x^2 + y^2 = 9$.

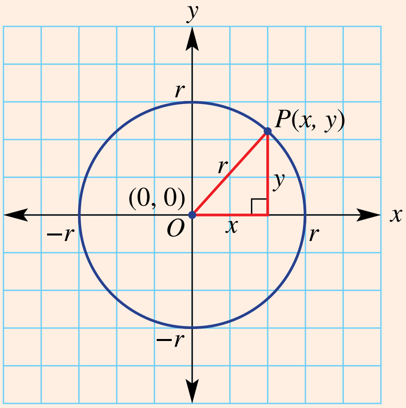
- When $x = 0$, what are the two values of y?
- When $x = 1$, what are the two values of y?
- When $x = 4$, are there any values of y? Discuss.
- Complete this table of values.

x	-3	-2	-1	0	1	2	3
y		$\pm\sqrt{5}$					

- Now plot all your points on a number plane and join them to form a smooth curve.
- What shape have you drawn and what are its features?
- How does the radius of your circle relate to the equation?

KEY IDEAS

- The Cartesian equation of a circle with centre (0, 0) and radius r is given by $x^2 + y^2 = r^2$.
- Making x or y the subject:
  - $y = \pm\sqrt{r^2 - x^2}$
  - $x = \pm\sqrt{r^2 - y^2}$
- To find the intersection points of a circle and a line, use the method of substitution.



Using Pythagoras' theorem, $a^2 + b^2 = c^2$ gives $x^2 + y^2 = r^2$.

BUILDING UNDERSTANDING

① Draw a circle on the Cartesian plane with centre (0, 0) and radius 2.

② Solve these equations for the unknown variable. There are two solutions for each.

a $x^2 + 2^2 = 9$ **b** $x^2 + 3^2 = 25$ **c** $5^2 + y^2 = 36$

③ A circle has equation $x^2 + y^2 = r^2$. Complete these sentences.

a The centre of the circle is _______.

b The radius of the circle is _______.

Example 18 Sketching a circle

For the equation $x^2 + y^2 = 4$, complete the following.

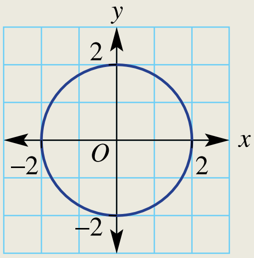
a State the coordinates of the centre.

b State the radius.

c Find the values of y when $x = 1$.

d Find the values of x when $y = \frac{1}{2}$.

e Sketch a graph showing axis intercepts.

SOLUTION	EXPLANATION
a (0, 0)	(0, 0) is the centre for all circles $x^2 + y^2 = r^2$.
b $r = 2$	$x^2 + y^2 = r^2$, so $r^2 = 4$.
c $\begin{aligned} x^2 + y^2 &= 4 \\ 1^2 + y^2 &= 4 \\ y^2 &= 3 \\ y &= \pm\sqrt{3} \end{aligned}$	Substitute $x = 1$ and solve for y. Recall that $(\sqrt{3})^2$ and $(-\sqrt{3})^2$ both equal 3.
d $\begin{aligned} x^2 + \left(\frac{1}{2}\right)^2 &= 4 \\ x^2 + \frac{1}{4} &= 4 \\ x^2 &= \frac{15}{4} \\ x &= \pm\sqrt{\frac{15}{4}} \\ x &= \pm\frac{\sqrt{15}}{2} \end{aligned}$	Substitute $y = \frac{1}{2}$. $4 - \frac{1}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$ $\sqrt{\frac{15}{4}} = \frac{\sqrt{15}}{\sqrt{4}} = \frac{\sqrt{15}}{2}$
e 	Draw a circle with centre (0, 0) and radius 2. Label intercepts.

Now you try

For the equation $x^2 + y^2 = 16$, complete the following.

a State the coordinates of the centre.

b State the radius.

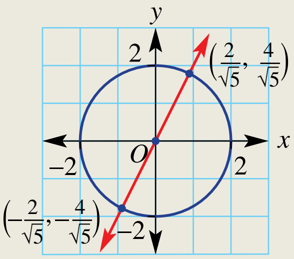
c Find the values of y when $x = 1$.

d Find the values of x when $y = \frac{1}{2}$.

e Sketch a graph showing axis intercepts.

Example 19 Intersecting circles and lines

Find the coordinates of the points where $x^2 + y^2 = 4$ intersects $y = 2x$. Sketch a graph showing the exact intersection points.

SOLUTION	**EXPLANATION**
$x^2 + y^2 = 4$ and $y = 2x$	Substitute $y = 2x$ into $x^2 + y^2 = 4$ and solve for x.
$x^2 + (2x)^2 = 4$ $x^2 + 4x^2 = 4$ $5x^2 = 4$ $x^2 = \frac{4}{5}$	Recall that $(2x)^2 = 2x \times 2x = 4x^2$.
$x = \pm\sqrt{\frac{4}{5}}$ $x = \pm\frac{2}{\sqrt{5}}$	$\sqrt{\frac{4}{5}} = \frac{\sqrt{4}}{\sqrt{5}} = \frac{2}{\sqrt{5}}$
If $y = 2x$: $x = \frac{2}{\sqrt{5}}$ gives $y = 2 \times \frac{2}{\sqrt{5}} = \frac{4}{\sqrt{5}}$ $x = -\frac{2}{\sqrt{5}}$ gives $y = 2 \times \left(-\frac{2}{\sqrt{5}}\right) = -\frac{4}{\sqrt{5}}$	Substitute both values of x into $y = 2x$ to find the y-coordinate.
	For $x^2 + y^2 = 4$, $r = 2$. Graph is centred at (0, 0) with intercepts at 2 and –2. Mark the intersection points and sketch $y = 2x$.

Now you try

Find the coordinates of the points where $x^2 + y^2 = 25$ intersects $y = x$. Sketch a graph showing the exact intersection points.

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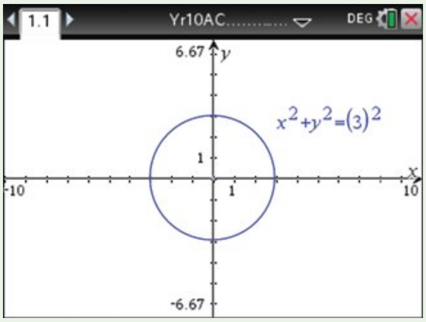
Using calculators to graph circles

Sketch a graph of $x^2 + y^2 = r^2$ using $r = \{1, 3\}$.

Using the TI-Nspire:

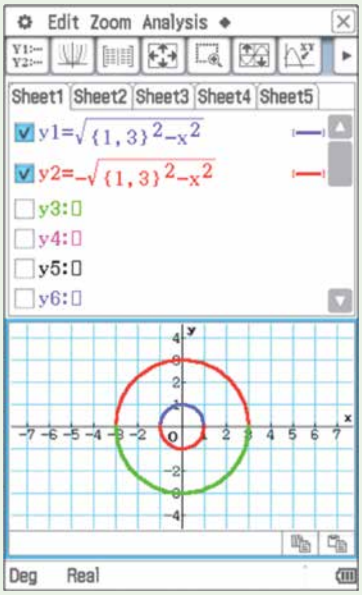
In a **Graphs** page, use menu **> Graph Entry/Edit>Equation>Circle>centre form** and enter $(x - 0)^2 + (y - 0)^2 = 3^2$ for $r = 3$.



Repeat the above for $r = 1$.

Using the ClassPad:

In the **graph&Table** application enter the rule $y1 = (\{1, 3\}^2 - x^2)$ followed by **EXE**. Enter the rule $y2 = -(\{1, 3\}^2 - x^2)$ followed by **EXE**. Tap ⊞ to see the graph. Select **Zoom**, **Square**.



☑ $y1=\sqrt{\{1,3\}^2-x^2}$

☑ $y2=-\sqrt{\{1,3\}^2-x^2}$

☐ y3:□

☐ y4:□

☐ y5:□

☐ y6:□

Exercise 10I

FLUENCY	1, 3–5(½)	1, 3–6(½)	2, 3–6(⅓)

Example 18

1 A circle has equation $x^2 + y^2 = 9$. Complete the following.

- **a** State the coordinates of the centre.
- **b** State the radius.
- **c** Find the values of y when $x = 2$.
- **d** Find the values of x when $y = \frac{3}{2}$.
- **e** Sketch a graph showing intercepts.

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2 For the equation $x^2 + y^2 = 25$ complete the following.

- **a** State the coordinates of the centre.
- **b** State the radius.
- **c** Find the values of y when $x = \frac{9}{2}$.
- **d** Find the values of x when $y = 4$.
- **e** Sketch a graph showing intercepts.

3 Give the radius of the circles with these equations.

a $x^2 + y^2 = 36$ **b** $x^2 + y^2 = 81$ **c** $x^2 + y^2 = 144$
d $x^2 + y^2 = 5$ **e** $x^2 + y^2 = 14$ **f** $x^2 + y^2 = 20$

4 Write the equation of a circle with centre (0, 0) and the given radius.

a 2 **b** 7 **c** 100 **d** 51
e $\sqrt{6}$ **f** $\sqrt{10}$ **g** 1.1 **h** 0.5

5 For the circle with equation $x^2 + y^2 = 4$ find the exact coordinates where:

a $x = 1$ **b** $x = -1$ **c** $x = \frac{1}{2}$
d $y = -\frac{1}{2}$ **e** $y = -2$ **f** $y = 0$

6 Without showing any working steps, write down the x- and y-intercepts of these circles.

a $x^2 + y^2 = 1$ **b** $x^2 + y^2 = 16$
c $x^2 + y^2 = 3$ **d** $x^2 + y^2 = 11$

PROBLEM–SOLVING	7(½), 8	7(½), 8–10	7(½), 9–12

7 Write down the radius of these circles.

a $x^2 + y^2 - 8 = 0$ **b** $x^2 - 4 = -y^2$
c $y^2 = 9 - x^2$ **d** $10 - y^2 - x^2 = 0$
e $3 + x^2 + y^2 = 15$ **f** $17 - y^2 = x^2 - 3$

Example 19

8 Find the coordinates of the points where $x^2 + y^2 = 9$ intersects $y = x$. Sketch a graph showing the intersection points.

9 Find the coordinates of the points where $x^2 + y^2 = 10$ intersects $y = 3x$. Sketch a graph showing the intersection points.

10 Find the coordinates of the points where $x^2 + y^2 = 6$ intersects $y = -\frac{1}{2}x$. Sketch a graph showing the intersection points.

11 Determine the exact length of the chord formed by the intersection of $y = x - 1$ and $x^2 + y^2 = 5$. Sketch a graph showing the intersection points and the chord.

12 For the circle $x^2 + y^2 = 4$ and the line $y = mx + 4$, determine the exact values of the gradient, m, so that the line:

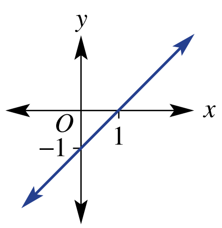
- **a** is a tangent to the circle
- **b** intersects the circle in two places
- **c** does not intersect the circle.

REASONING	13	13, 14	14, 15

13 Match equations **a**–**f** with graphs **A**–**F**.

a $x^2 + y^2 = 7$
b $y = x - 1$
c $y = -2x + 2$
d $y = x^2 - 1$
e $y = -x(x + 2)$
f $x^2 + y^2 = 9$

A



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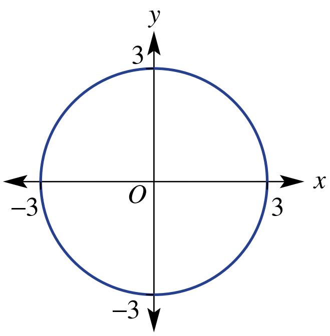

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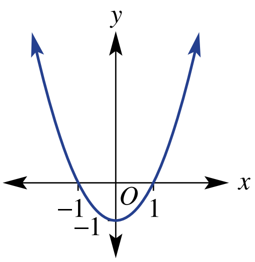

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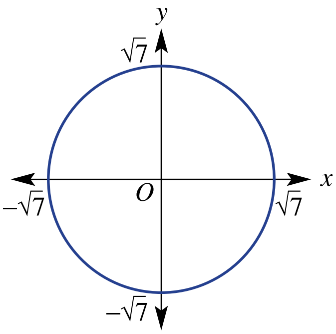
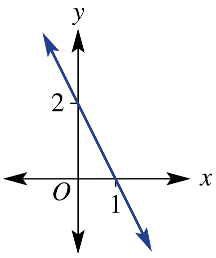
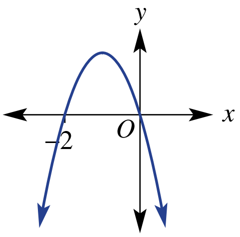


E



F

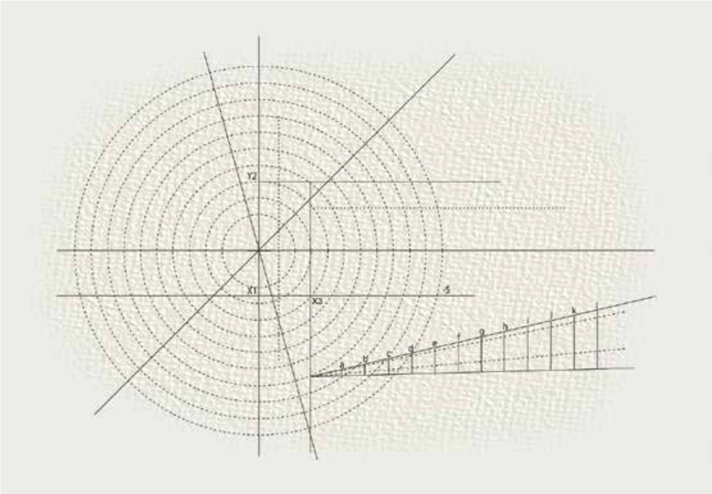


14 **a** Write $x^2 + y^2 = 16$ in the form $y = \pm\sqrt{r^2 - x^2}$.

b Write $x^2 + y^2 = 3$ in the form $x = \pm\sqrt{r^2 - y^2}$.

15 **a** Explain why the graphs of $y = 3$ and $x^2 + y^2 = 4$ do not intersect.

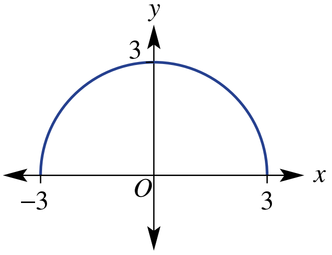
b Explain why the graphs of $x = -2$ and $x^2 + y^2 = 1$ do not intersect.



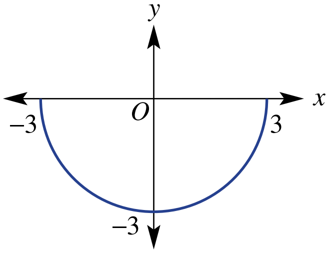
ENRICHMENT: Half circles

– – 16–17(½)

16 When we write $x^2 + y^2 = 9$ in the form $y = \pm\sqrt{9 - x^2}$, we define two circle halves.



$y = \sqrt{9 - x^2}$



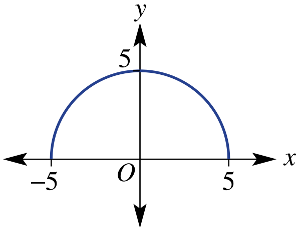
$y = -\sqrt{9 - x^2}$

Sketch the graphs of these half circles.

a $y = \sqrt{4 - x^2}$
b $y = \sqrt{25 - x^2}$
c $y = -\sqrt{1 - x^2}$
d $y = -\sqrt{10 - x^2}$
e $y = \sqrt{16 - x^2}$
f $x = -\sqrt{36 - y^2}$
g $x = -\sqrt{7 - y^2}$
h $x = \sqrt{5 - y^2}$
i $x = \sqrt{12 - y^2}$

17 Write the rules for these half circles.

a



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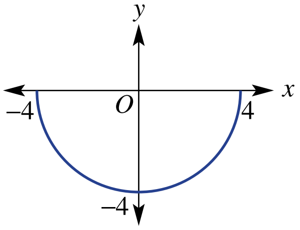

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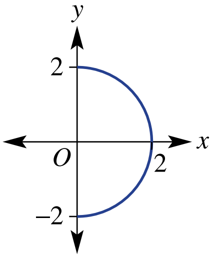

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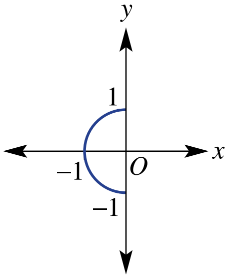

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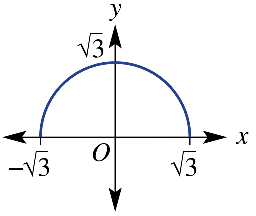

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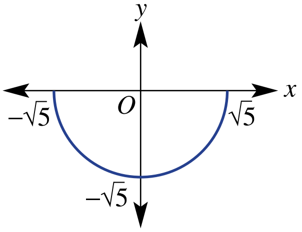

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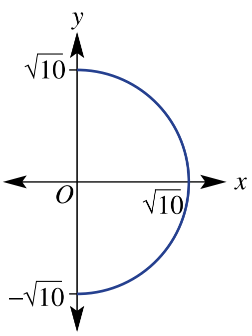

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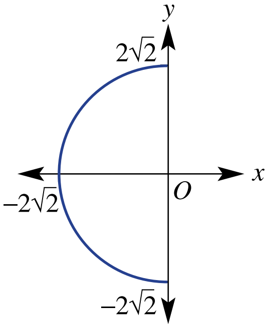
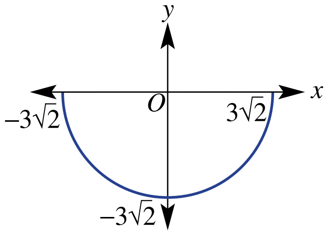


i



The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Dog on patrol

1 A dog is running in a straight line back and forth along a fence. The position of the dog, s metres, relative to a gate on the fence is given by $s(t) = t^3 - 10t^2 + 29t - 20$ where t is time in seconds and $0 \leqslant t \leqslant 5$. A negative value of s represents to the left of the gate and a positive value is to the right of the gate.

To track the position of the dog and how far from the gate he is likely to run, the model for his run along the fence line is investigated.

- **a** Describe the position of where the dog starts its run.
- **b** By factorising, determine at what times the dog is at the gate.
- **c** Use technology or a table of values to determine, to two decimal places, the furthest to the right of the gate the dog is in this run.

Another run along the fence later in the day is such that the dog starts from the gate and is back there after 4 seconds and 6 seconds.

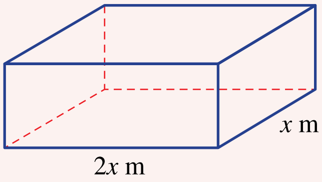
- **d** Give a possible cubic model for the position of the dog, s metres, after t seconds in this run.
- **e** Adjust your model in part **d** so that the furthest the dog is from the gate in either direction in the first six seconds is between 8 m and 9 m.

Rabbit hutch design

2 Parents are designing a rectangular rabbit hutch for their daughter's rabbit.

The parents wish to use a fixed amount of material to construct a special type of hutch and try to maximise its volume. They also want to consider changing the design and the amount of material to maintain a fixed volume and improve the overall conditions for the rabbit.

- **a** They initially have 12 m of wood to make the edges of the frame, the sides of which they will then cover with chicken wire. The base area will be twice as long as it is wide as shown.



  - **i** Find an expression for the allowable height of the hutch in terms of x.
  - **ii** Hence, give a rule for the volume, $V(x)$ m^3, of the hutch in expanded form and state the possible values of x.
  - **iii** Use technology to find the maximum possible volume of this hutch and the dimensions that give this volume. Round values to one decimal place.
  - **iv** Repeat parts **i** and **ii** for p metres of wood to find a rule for the volume in terms of x and p.

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b The parents have decided they may need to purchase extra wood to give the rabbit the space it needs. They redraw their original design so that the base dimensions are $(x + 10)$ cm and $(x + 80)$ cm. The volume of their hutch, in cm^3, is given by $V(x) = x^3 + 60x^2 - 1900x - 24\,000$.

- **i** Determine the height, in cm, of the design in terms of x, using division.
- **ii** If the parents settle on dimensions that use an x-value of 90, how many extra metres of wood will they need?

Logo design

3 Like the famous Olympic rings, many logos involve intersecting circles.

Marketing companies investigate and design logos that represent their company and that also have a visual appeal. Here you will investigate the design of a logo.

a The first circle in the logo when represented on the Cartesian plane is centred at the origin with radius $\sqrt{5}$ units. Draw this circle and give its equation.

b To create the intersecting circle the first circle is translated two units to the right. Add this circle to the Cartesian plane and explain why the equation of this circle is given by $(x - 2)^2 + y^2 = 5$.

c Determine the coordinates of the points of intersection of the two circles.

The marketing company is not convinced by the intended radius of the circles.

d Investigate the effect of changing the radius of the circles on the points of intersection of the circles. Keep the radius of each circle the same and come up with a rule for the points of intersection in terms of the radius r.

The logo will be completed by a tangent line to the translated circle.

e Find the equation of the tangent to the translated circle, which passes through the point (3, 2) on the circumference of the circle. *Recall that a tangent meets the radius of a circle at right angles at the point of contact.*

10J Hyperbolic functions and their graphs EXTENDING

LEARNING INTENTIONS

- To know the general equation form of a rectangular hyperbola: $y = \frac{a}{x}$
- To know how to sketch and label the key features of a rectangular hyperbola, including asymptotes
- To be able to sketch reflections of hyperbolas
- To be able to find point(s) of intersection of a line and a hyperbola

A simple rectangular hyperbola is the graph of the equation $y = \frac{1}{x}$. These types of equations are common in many mathematical and practical situations.

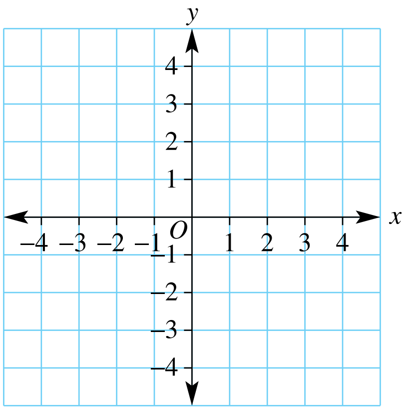
When two stones are thrown into a pond, the resulting concentric ripples intersect at a set of points that together form the graph of a hyperbola. In a similar way, when signals are received from two different satellites, a ship's navigator can map the hyperbolic shape of the intersecting signals and help to determine the ship's position.

Two signal pulses emitted simultaneously will spread in overlapping concentric circles. All circle intersection points with the same difference in distance (or time) from the centres form a hyperbola shape. Using signal time differences and hyperbolic charts, navigators can locate a ship's position more reliably than using GPS.

Lesson starter: How many asymptotes?

Consider the rule for the simple hyperbola $y = \frac{1}{x}$. First, complete the table and graph, and then discuss the points below.

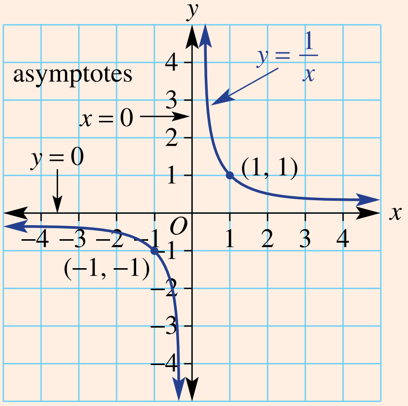
x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y										



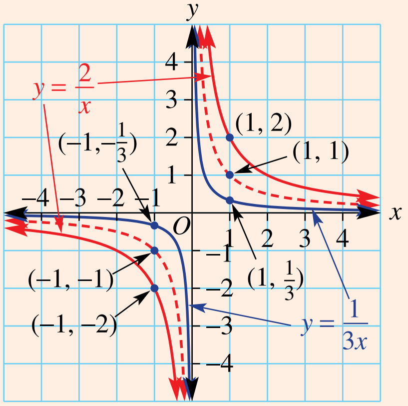
- Discuss the shape of the graph of the hyperbola.
- What would the values of y approach as x increases to infinity or negative infinity?
- What would the values of y approach as x decreases to zero from the left or from the right?
- What are the equations of the asymptotes for $y = \frac{1}{x}$?

KEY IDEAS

- Recall that an **asymptote** is a straight line that a curve approaches more and more closely but never quite reaches. Every hyperbola has two asymptotes.



- A **rectangular hyperbola** is the graph of the rule $y = \frac{a}{x}$, $a \neq 0$.
  - $y = \frac{1}{x}$ is the basic rectangular hyperbola.
  - $x = 0$ (y-axis) and $y = 0$ (x-axis) are its asymptotes.
  - For $a > 1$ the hyperbola will be further out from the asymptotes.
  - For $0 < a < 1$ the hyperbola will be closer in to the asymptotes.
- The graph of $y = -\frac{a}{x}$ is a reflection of the graph of $y = \frac{a}{x}$ in the x-axis or y-axis.
- To find the intersection points of a hyperbola and a line, use the method of substitution.



- All hyperbolas of the form $y = \frac{a}{x}$ are functions because they pass the vertical line test.
  - Note that $y = \frac{1}{x}$ can be written as $f(x) = \frac{1}{x}$
- In the function $f(x) = \frac{a}{x}$:
  - All x values are allowable except $x = 0$
  - All y values result except $y = 0$

BUILDING UNDERSTANDING

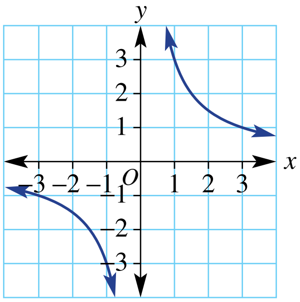
① Match the rules **a**, **b** and **c** with the graphs **A**, **B** and **C**.

a $y = \frac{1}{x}$

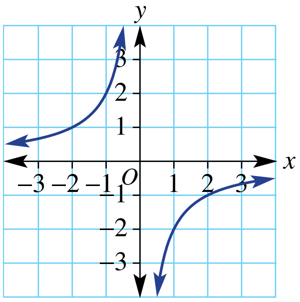
b $y = \frac{3}{x}$

c $y = -\frac{2}{x}$

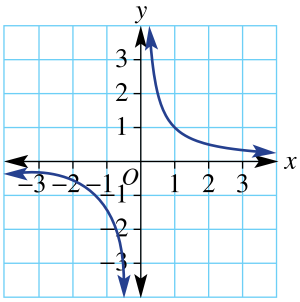
A



B



C



② **a** Give in ascending order: $1 \div 0.1$, $1 \div 0.001$, $1 \div 0.01$, $1 \div 0.00001$.

b For $y = \frac{1}{x}$, which of the following x-values will give the largest value of y: $\frac{1}{5}, \frac{1}{10}, \frac{1}{2}$ or $\frac{1}{100}$?

c For $y = \frac{1}{x}$, calculate the difference in the y-values for $x = 10$ and $x = 1000$.

d For $y = \frac{1}{x}$, calculate the difference in the y-values for $x = -\frac{1}{2}$ and $x = -\frac{1}{1000}$.

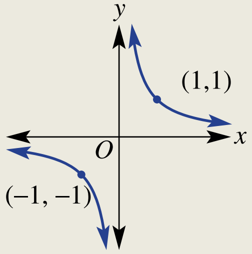
Example 20 Sketching a hyperbola

Sketch the graph of each hyperbola, labelling the points where $x = 1$ and $x = -1$.

a $y = \frac{1}{x}$ **b** $y = \frac{2}{x}$ **c** $y = -\frac{3}{x}$

SOLUTION

a $y = \frac{1}{x}$

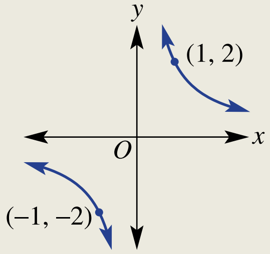


EXPLANATION

Draw the basic shape of a rectangular hyperbola $y = \frac{1}{x}$.

Substitute $x = 1$ and $x = -1$ to find the two points.

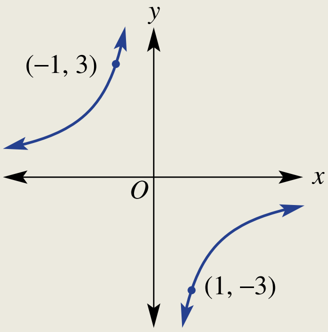
b $y = \frac{2}{x}$



For $x = 1$, $y = \frac{2}{1} = 2$.

For $x = -1$, $y = \frac{2}{-1} = -2$.

c $y = -\frac{3}{x}$



$y = -\frac{3}{x}$ is a reflection of $y = \frac{3}{x}$ in either the x- or y-axis.

When $x = 1$, $y = -\frac{3}{1} = -3$.

When $x = -1$, $y = -\frac{3}{-1} = 3$.

Now you try

Sketch the graph of each hyperbola, labelling the points where $x = 1$ and $x = -1$.

a $y = \frac{1}{x}$ **b** $y = \frac{3}{x}$ **c** $y = -\frac{2}{x}$

Example 21 Finding the intersection points of a line and a hyperbola

Find the coordinates of the points where $y = \frac{1}{x}$ intersects these lines.

a $y = 3$ **b** $y = 4x$

SOLUTION

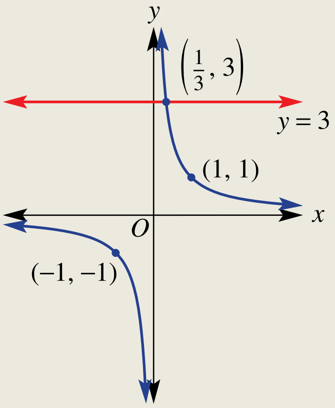
a

$$\begin{aligned} y &= \frac{1}{x} \text{ and } y = 3 \\ 3 &= \frac{1}{x} \\ 3x &= 1 \\ x &= \frac{1}{3} \end{aligned}$$

$\therefore$ Intersection point is $\left(\frac{1}{3}, 3\right)$.

EXPLANATION

Substitute $y = 3$ into $y = \frac{1}{x}$ and solve.



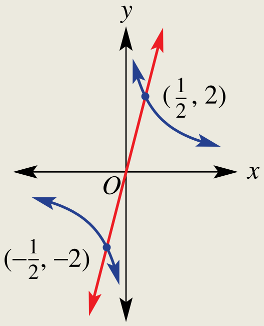
b

$$\begin{aligned} y &= \frac{1}{x} \text{ and } y = 4x \\ 4x &= \frac{1}{x} \\ 4x^2 &= 1 \\ x^2 &= \frac{1}{4} \\ x &= \pm\frac{1}{2} \\ y &= 4 \times \left(\frac{1}{2}\right) = 2 \text{ and } y = 4 \times \left(-\frac{1}{2}\right) = -2 \end{aligned}$$

$\therefore$ Intersection points are $\left(\frac{1}{2}, 2\right)$ and $\left(-\frac{1}{2}, -2\right)$.

Substitute and solve by multiplying both sides by x. Note that $\sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$.

Find the corresponding y-values by substituting into one of the rules.



Now you try

Find the coordinates of the points where $y = \frac{1}{x}$ intersects these lines.

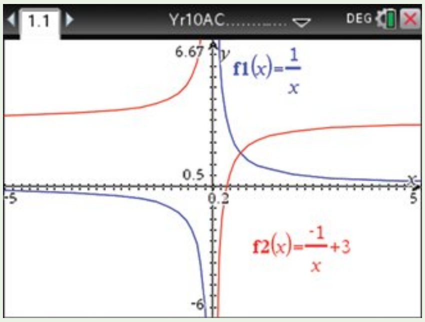
a $y = 4$ **b** $y = 3x$

Using calculators to graph hyperbolas

Sketch a graph of $y = \frac{1}{x}$ and $y = -\frac{1}{x} + 3$ on the same set of axes.

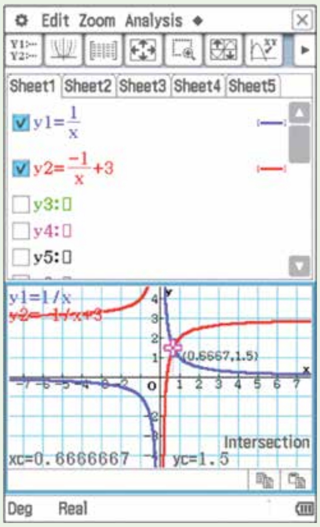
Using the TI-Nspire:

Enter the rules as $f1(x)$ and $f2(x)$. Change the scale using the window setting.
Arrow up or down to toggle between graphs.



Using the ClassPad:

Enter the rules as $y1$ and $y2$. Change the scale by tapping ⊡. Use the **Analysis**, **G-Solve** menu to show significant points. Arrow up or down to toggle between graphs.



xc=0.6666667 yc=1.5

Exercise 10J

FLUENCY	1, 2, 4	1–4(½)	1–4(½)

1 Sketch the graphs of these hyperbolas, labelling the points where $x = 1$ and $x = -1$.

Example 20a, b

a $y = \frac{1}{x}$ **b** $y = \frac{2}{x}$ **c** $y = \frac{3}{x}$

Example 20c

d $y = -\frac{1}{x}$ **e** $y = -\frac{2}{x}$ **f** $y = -\frac{3}{x}$

2 Find the coordinates on the graph of $y = \frac{2}{x}$, where:

a $x = 2$ **b** $x = 4$ **c** $x = -1$ **d** $x = -6$

3 Find the coordinates on the graph of $y = -\frac{5}{x}$, where:

a $x = 10$ **b** $x = -4$ **c** $x = -7$ **d** $x = 9$

4 Find the coordinates on the graph of $y = \frac{3}{x}$, where:

a $y = 3$ **b** $y = 1$ **c** $y = -2$ **d** $y = -6$

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PROBLEM–SOLVING	5–6(½)	5–6(½)	6–7(½)

5 **a** Decide whether the point (1, 3) lies on the hyperbola $y = \frac{3}{x}$.

b Decide whether the point (1, −5) lies on the hyperbola $y = -\frac{5}{x}$.

c Decide whether the point (2, 1) lies on the hyperbola $y = -\frac{2}{x}$.

d Decide whether the point (−3, 6) lies on the hyperbola $y = -\frac{6}{x}$.

Example 21

6 Find the coordinates of the points where $y = \frac{1}{x}$ intersects these lines.

a $y = 2$ **b** $y = 6$ **c** $y = -1$ **d** $y = -10$

e $y = x$ **f** $y = 4x$ **g** $y = 2x$ **h** $y = 5x$

7 Find the coordinates of the points where $y = -\frac{2}{x}$ intersects these lines.

a $y = -3$ **b** $y = 4$ **c** $y = -\frac{1}{2}$ **d** $y = \frac{1}{3}$

e $y = -2x$ **f** $y = -8x$ **g** $y = -\frac{1}{2}x$ **h** $y = -x$

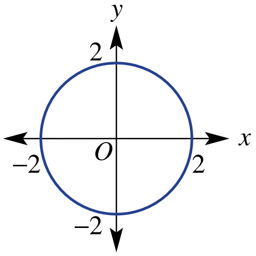
REASONING	8	8, 9	8–11

8 Match equations **a**–**f** with graphs **A**–**F**.

a $y = \frac{4}{x}$ **b** $y = -\frac{1}{x}$ **c** $y = 2^x$

d $y = \frac{1}{2}x + 1$ **e** $x^2 + y^2 = 4$ **f** $y = 3^{-x}$

A



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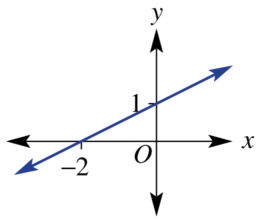

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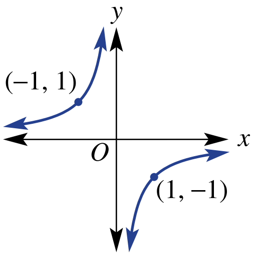
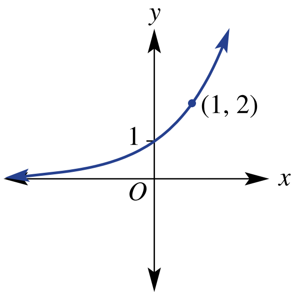
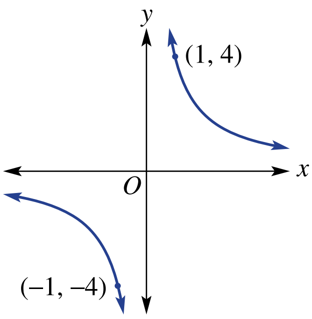


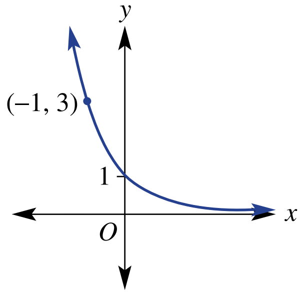
D



E



F



9 Is it possible for a line on a number plane to not intersect the graph of $y = \frac{1}{x}$? If so, give an example.

10 Write the missing word (*zero* or *infinity*) for these sentences.

a For $y = \frac{1}{x}$, when x approaches infinity, y approaches _________.

b For $y = \frac{1}{x}$, when x approaches negative infinity, y approaches _________.

c For $y = \frac{1}{x}$, when x approaches zero from the right, y approaches _________.

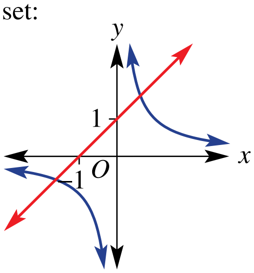
d For $y = \frac{1}{x}$, when x approaches zero from the left, y approaches _________.

11 Compare the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{2x}$. Describe the effect of the coefficient of x in $y = \frac{1}{2x}$.

ENRICHMENT: To intersect or not!	–	–	12

12 The graphs of $y = \frac{1}{x}$ and $y = x + 1$ intersect at two points. To find the points we set:

$$\begin{aligned} \frac{1}{x} &= x + 1 \\ 1 &= x(x + 1) \\ 0 &= x^2 + x - 1 \end{aligned}$$



Using the quadratic formula, $x = \frac{-1 \pm \sqrt{5}}{2}$ and $y = \frac{1 \pm \sqrt{5}}{2}$.

a Find the exact coordinates of the intersection of $y = \frac{1}{x}$ and these lines.

i $y = x - 1$ **ii** $y = x - 2$ **iii** $y = x + 2$

b Try to find the coordinates of the intersection of $y = \frac{1}{x}$ and $y = -x + 1$. What do you notice? What part of the quadratic formula confirms this result?

c Write down the equations of the two straight lines (which have gradient -1) that intersect $y = \frac{1}{x}$ only once.

The shape of the curve formed by the light from the lamp on a wall is a hyperbola!

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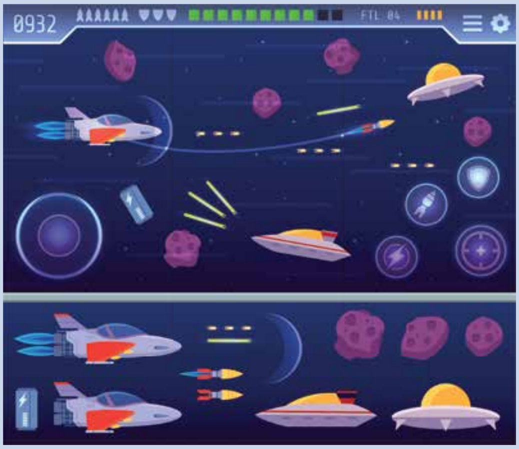
10K Further transformations of graphs EXTENDING

LEARNING INTENTIONS
- To understand the effect of translations on graphs of circles, exponentials and rectangular hyperbolas
- To be able to determine the centre of a circle and asymptotes of exponentials and hyperbolas from the equation of the transformed graph
- To be able to sketch graphs using transformations

In Chapter 7 we considered a wide range of transformations of parabolas. Earlier we have seen a limited number of transformations of circles, exponentials and hyperbolas. We will now look more closely at translations of these relations and the key features of their graphs.



Transformation maths is the basis for coding animations in digital games. Translations, reflections and rotations will move a figure, object or graph without changing its shape. However, dilations change a shape by shrinking or enlarging.

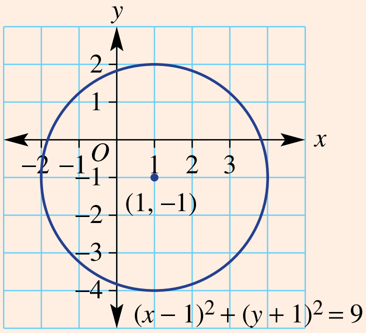
Lesson starter: Translations, translations, translations

Use technology to assist in the discussion of these questions.

- How does the graph of $(x-1)^2+(y+2)^2=9$ compare with that of $x^2+y^2=9$?
- What is the effect of h, k and r in $(x-h)^2+(y-k)^2=r^2$?
- How does the graph of $y=2^{x-2}+1$ compare with that of $y=2^x$?
- What is the effect of h and k in $y=2^{x-h}+k$?
- How does the graph of $y=\frac{1}{x+2}-1$ compare with that of $y=\frac{1}{x}$?
- What is the effect of h and k in $y=\frac{1}{x-h}+k$?

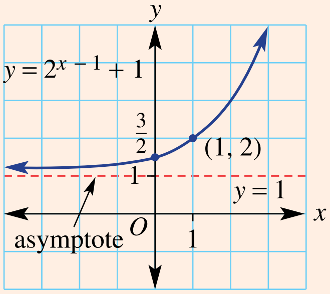
KEY IDEAS

- The equation of a **circle** in standard form is $(x-h)^2+(y-k)^2=r^2$.
  - (h, k) is the centre.
  - r is the radius.



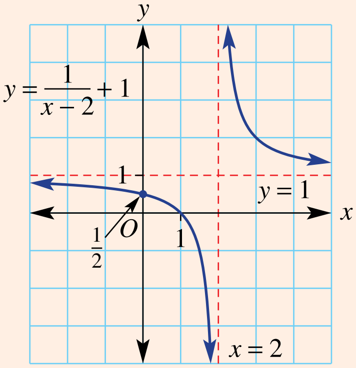
- For the graph of the **exponential** equation $y = a^{x-h} + k$ the graph of $y = a^x$ is:
  - translated h units to the right when $h > 0$.
  - translated h units to the left when $h < 0$.
  - translated k units up when $k > 0$
  - translated k units down when $k < 0$.

  The equation of the asymptote is $y = k$.



- For the graph of the **hyperbola** $y = \frac{1}{x-h} + k$ the graph of $y = \frac{1}{x}$ is:
  - translated h units to the right when $h > 0$.
  - translated h units to the left when $h < 0$.
  - translated k units up when $k > 0$.
  - translated k units down when $k < 0$.

  The asymptotes are $x = h$ and $y = k$.



BUILDING UNDERSTANDING

1 Choose the word *left, right, up* or *down* to complete each sentence.

a The graph of $y = 2^x - 3$ is the translation of the graph of $y = 2^x$ _____ by 3 units.

b The graph of $y = 2^{x-4}$ is the translation of the graph of $y = 2^x$ _____ by 4 units.

c The graph of $y = \frac{1}{x+2}$ is the translation of the graph of $y = \frac{1}{x}$ _____ by 2 units.

d The graph of $y = \frac{1}{x} - 6$ is the translation of the graph of $y = \frac{1}{x}$ _____ by 6 units.

e The graph of $(x + 3)^2 + y^2 = 1$ is the translation of the graph of $x^2 + y^2 = 1$ _____ by 3 units.

f The graph of $x^2 + (y - 2)^2 = 1$ is the translation of the graph of $x^2 + y^2 = 1$ _____ by 2 units.

2 What is the value of k in the equation of the asymptote $y = k$ for the following?

a $y = 2^{x-1} + 3$

b $y = 3^{x+2} - 1$

c $y = -5^{x+2} + 4$

3 What are the values of h and k in the asymptotes $x = h$ and $y = k$ for the following hyperbolas?

a $y = \frac{1}{x} + 2$

b $y = \frac{1}{x-3}$

c $y = \frac{1}{x+2} - 1$

Example 22 Sketching with transformations

Sketch the graphs of the following relations. Label important features.

a $(x-2)^2+(y+3)^2=9$ **b** $y=2^{x+2}-3$ **c** $y=\frac{1}{x+1}+2$

SOLUTION

a $(x-2)^2+(y+3)^2=9$

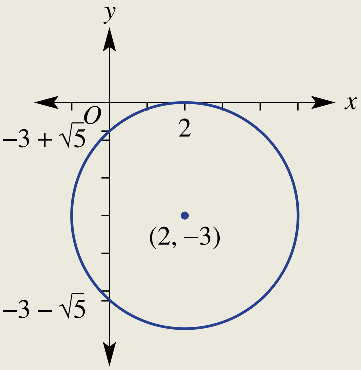
Centre $(2, -3)$

Radius $= 3$

x-intercept is at $x=2$.

y-intercepts at $x=0$:

$$\begin{aligned}(0-2)^2+(y+3)^2 &= 9\\ 4+(y+3)^2 &= 9\\ (y+3)^2 &= 5\\ y+3 &= \pm\sqrt{5}\\ y &= -3\pm\sqrt{5}\end{aligned}$$



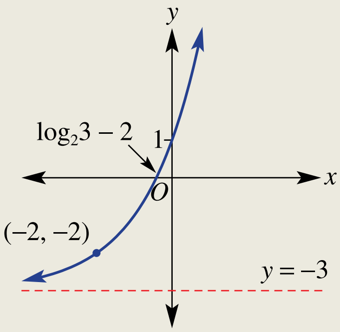
b $y=2^{x+2}-3$

Asymptote is $y=-3$.

y-intercept is $(0, 1)$.

x-intercept $(y=0)$:

$$\begin{aligned}2^{x+2}-3 &= 0\\ 2^{x+2} &= 3\\ x+2 &= \log_2 3\\ x &= \log_2 3-2\end{aligned}$$



EXPLANATION

For $(x-h)^2+(y-k)^2=r^2$, (h, k) is the centre and r is the radius.

Radius is 3 so, from centre $(2, -3)$, point on circle is $(2, 0)$.

Find the y-intercepts by substituting $x=0$ and solving for y.

Label centre and axes intercepts.

For $y=2^{x-h}+k$, $y=k$ is the equation of the asymptote.

Substitute $x=0$ to find the y-intercept.

At $x=0$, $y=2^2-3=1$.

We can use logarithms to locate the x-intercept.

Recall, if $a^x=b$ then $x=\log_a b$

$(0, 1)$ in $y=2^x$ is translated 2 units to the left and 3 units down to $(-2, -2)$ also generates another point.

Alternatively, substitute $x=1$ to label another point.

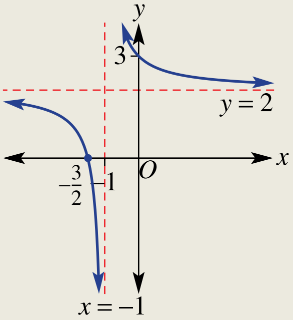
c $y = \dfrac{1}{x+1} + 2$

Asymptotes: $x = -1, y = 2$

At $x = 0$, $y = \dfrac{1}{1} + 2 = 3$.

y-intercept is at $(0, 3)$.

$$\begin{aligned} x\text{-intercept } (y = 0)\text{:}\quad 0 &= \frac{1}{x+1} + 2 \\ -2 &= \frac{1}{x+1} \\ x + 1 &= -\frac{1}{2} \\ x &= -\frac{3}{2} \end{aligned}$$



For $y = \dfrac{1}{x-h} + k$, $h = -1$ and $k = 2$, so the asymptotes are $x = -1$ and $y = 2$.

Substitute to find the x- and y-intercepts.

Now you try

Sketch the graphs of the following relations. Label important features.

a $(x+1)^2 + (y+2)^2 = 9$ **b** $y = 2^{x-1} - 1$ **c** $y = \dfrac{1}{x-1} - 2$

Exercise 10K

FLUENCY 1–3(½) 1–3(½) 1–3(⅓)

Example 22a

1 Sketch the graph of the following circles. Label the coordinates of the centre and find the x- and y-intercepts, if any.

a $(x-3)^2 + (y+1)^2 = 1$ **b** $(x+2)^2 + (y-3)^2 = 4$ **c** $(x+3)^2 + (y-2)^2 = 25$

d $(x-1)^2 + (y+3)^2 = 25$ **e** $(x+2)^2 + (y-1)^2 = 9$ **f** $x^2 + (y-4)^2 = 36$

g $(x+1)^2 + y^2 = 9$ **h** $(x-2)^2 + (y-5)^2 = 64$ **i** $(x+3)^2 + (y-1)^2 = 5$

Example 22b

2 Sketch the graph of these exponentials. Label the asymptote and any intercepts.

a $y = 2^x - 2$ **b** $y = 2^x + 1$ **c** $y = 2^x - 5$

d $y = 2^{x-1}$ **e** $y = 2^{x+3}$ **f** $y = 2^{x+1}$

g $y = 2^{x-1} + 1$ **h** $y = 2^{x+2} - \dfrac{1}{2}$ **i** $y = 2^{x-3} - 4$

Example 22c

3 Sketch the graph of these hyperbolas. Label the asymptotes and find the x-and y-intercepts.

a $y = \frac{1}{x} + 2$ **b** $y = \frac{1}{x} - 1$ **c** $y = \frac{1}{x+3}$

d $y = \frac{1}{x-2}$ **e** $y = \frac{1}{x+1} + 1$ **f** $y = \frac{1}{x-1} - 3$

g $y = \frac{1}{x-3} + 2$ **h** $y = \frac{1}{x+4} - 1$ **i** $y = \frac{1}{x-5} + 6$

PROBLEM–SOLVING	4(½), 5	4(½), 5, 6	4(⅓), 5, 6, 7(½)

4 The graphs of these exponentials involve a number of transformations. Sketch their graphs, labelling any intercepts and the equation of the asymptote.

a $y = -2^x + 1$ **b** $y = -2^x - 3$ **c** $y = -2^{x+3}$

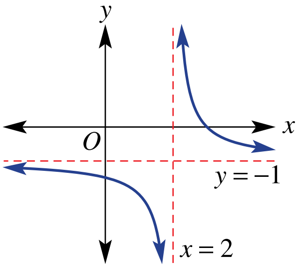
d $y = -2^{x-2}$ **e** $y = -2^{x+1} - 1$ **f** $y = -2^{x+2} + 5$

5 Sketch these hyperbolas, labelling asymptotes and intercepts.

a $y = \frac{-1}{x+1} + 2$ **b** $y = \frac{-2}{x+2} - 1$ **c** $y = \frac{-2}{x-3} - 2$

6 The following hyperbolas are of the form $y = \frac{1}{x-h} + k$. Write the rule for each graph.

a



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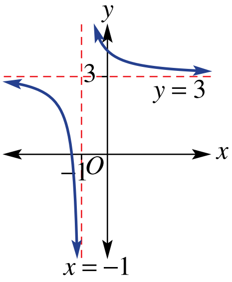
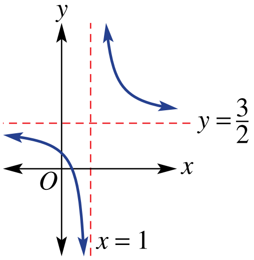


c



7 Find the coordinates of the intersection of the graphs of these equations.

a $y = \frac{1}{x+1}$ and $y = x + 2$

b $y = \frac{1}{x-2} + 1$ and $y = x + 3$

c $y = \frac{-1}{x+2} - 3$ and $y = -2x - 1$

d $(x-1)^2 + y^2 = 4$ and $y = 2x$

e $(x+2)^2 + (y-3)^2 = 16$ and $y = -x - 3$

f $x^2 + (y+1)^2 = 10$ and $y = \frac{1}{3}x - 1$

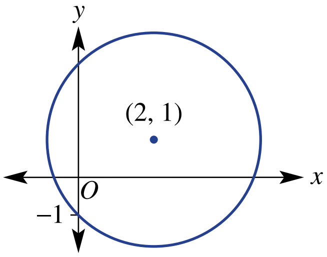
REASONING	8	8, 9	9, 10

8 A circle has equation $(x-3)^2 + (y+2)^2 = 4$. Without sketching a graph, state the minimum and maximum values for:

a x **b** y

9 Find a rule for each graph.

a



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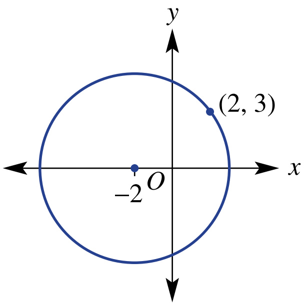

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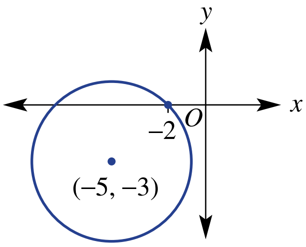
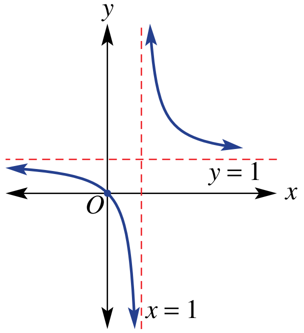
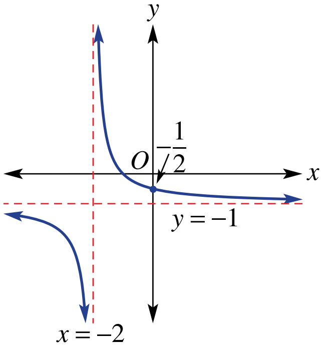


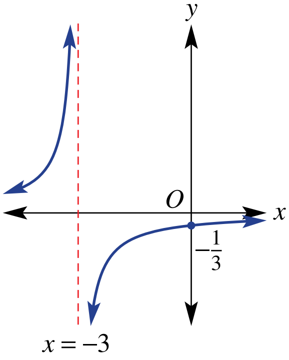
d



e



f



10 Explain why the graphs of the following pairs of relations do not intersect.

a $y = \frac{1}{x}$ and $y = -x$

b $(x - 1)^2 + (y + 2)^2 = 4$ and $y = 1$

c $y = 2^{x-1} + 3$ and $y = x - 3$

d $y = \frac{2}{x + 3} - 1$ and $y = \frac{1}{3x}$

ENRICHMENT: Circles by completing the square

– – 11(½), 12

11 By expanding brackets of an equation in the standard form of a circle, we can write:

$$\begin{aligned} (x-1)^2 + (y+2)^2 &= 4 \quad [1] \\ x^2 - 2x + 1 + y^2 + 4y + 4 &= 4 \\ x^2 - 2x + y^2 + 4y + 1 &= 0 \quad [2] \end{aligned}$$

Note that in equation [2] it is not obvious that the centre is (1, −2) and that the radius is 2. It is therefore preferable to write the equation of a circle in standard form (i.e. as in equation [1]).

If given an equation such as [2], we can complete the square in both x and y to write the equation of the circle in standard form.

$$\begin{aligned} x^2 - 2x + y^2 + 4y + 1 &= 0 \\ (x-1)^2 - 1^2 + (y+2)^2 - 2^2 + 1 &= 0 \\ (x-1)^2 - 1 + (y+2)^2 - 4 + 1 &= 0 \\ (x-1)^2 + (y+2)^2 &= 4 \end{aligned}$$

The radius is 2 and centre (1, −2).

Write these equations of circles in standard form. Then state the coordinates of the centre and the radius.

a $x^2 + 4x + y^2 - 2y + 1 = 0$

b $x^2 + 8x + y^2 + 10y + 5 = 0$

c $x^2 - 6x + y^2 - 4y - 3 = 0$

d $x^2 - 2x + y^2 + 6y - 5 = 0$

e $x^2 + 10x + y^2 + 8y + 17 = 0$

f $x^2 + 6x + y^2 + 6y = 0$

g $x^2 + 3x + y^2 - 6y + 4 = 0$

h $x^2 + 5x + y^2 - 4y - 2 = 0$

i $x^2 - x + y^2 + 3y + 1 = 0$

j $x^2 - 3x + y^2 - 5y - 4 = 0$

12 Give reasons why $x^2 + 4x + y^2 - 6y + 15 = 0$ is not the equation of a circle.

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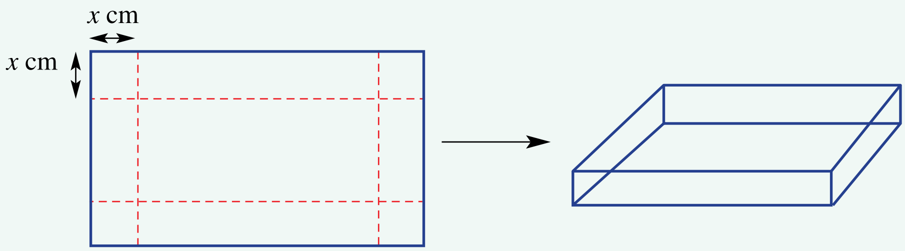
Making a tray of maximum volume

By using a rectangular piece of sheet metal, a square of side length x cm can be cut out and the sides folded up to form an open top tray as shown.



Present a report for the following tasks and ensure that you show clear mathematical workings and explanations where appropriate.

Preliminary task

For this task assume that the width of the metal sheet is 10 cm and the length of the metal sheet is 20 cm.

- **a** If $x = 2$, find the dimensions of the open top tray and its volume.
- **b** Find expressions in terms of x for the length and the width of the base of the tray, as well as the tray's volume.
- **c** What is the maximum value of x? Give a reason.
- **d** Plot a graph of volume against x for x between 0 and 5 using technology or otherwise.
- **e** Determine the value of x which gives a maximum volume correct to one decimal place. Also find the maximum volume of the tray.

Modelling task

Formulate

- **a** The problem is to find the size of the square cut-out to maximise the volume of the tray. Write down all the relevant information that will help solve this problem.
- **b** Choose a length and width of the original metal sheet.
- **c** Determine the allowable values of x.

Solve

- **d** Derive an expression for the length and width of the base of the tray in terms of x.
- **e** Derive an expression for the volume of the tray in terms of x.
- **f** Sketch a graph of volume vs x for your allowable values of x. Graph by hand, plotting points to help.
- **g** Estimate the value of x and hence the size of the cut out which gives a maximum volume and find the maximum volume.

Evaluate and verify

- **h** Use technology to sketch the graph from part **f** above.
- **i** Use this graph to verify your results from part **g** above.

Communicate

- **j** Summarise your results and describe any key findings.

Extension question

- **a** Explore the effect on your results if different lengths and widths are chosen for the metal sheet.

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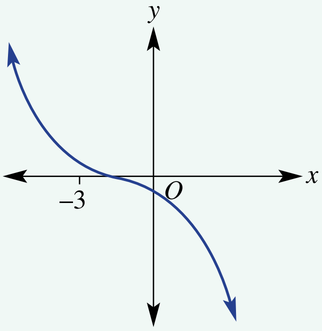
The bisection method

Key technology: Programming

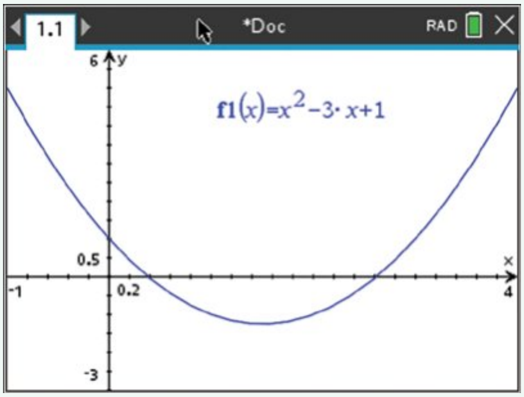
For equations of the form $f(x) = 0$ one common numerical method for finding an approximate solution is the bisection method. Note for the given graph $f(-3) > 0$ and $f(0) < 0$. We therefore know that the solution to $f(x) = 0$ is somewhere between $x = -3$ and $x = 0$.



The bisection method is an algorithm that systematically reduces the size of the interval in which the solution lies until a level of accuracy is reached.

1 Getting started

We will start by looking at the solution to the quadratic equation $x^2 - 3x + 1 = 0$. The graph of $f(x) = x^2 - 3x + 1$ is shown here. Note that the solutions to $x^2 - 3x + 1 = 0$ are the x intercepts of the graph of $f(x) = x^2 - 3x + 1$.



a Evaluate the following.
  - **i** $f(0)$
  - **ii** $f(1)$
  - **iii** $f(2)$
  - **iv** $f(3)$

b What do you notice about the sign (positive or negative) of each of the four answers in part **a** above?

c Between which pairs of integers are the solutions to $f(x) = 0$?

d Consider the two integers a and b either side of the larger solution to $f(x) = 0$. Find the mean of the two integers m. i.e. find $m = \frac{a+b}{2}$.

e Evaluate $f(m)$ and decide if the larger solution is between $x = a$ and $x = m$ or between $x = m$ and $x = b$.

2 Applying an algorithm

A flowchart is shown on the next page which executes the bisection method. We will use this algorithm to find an approximate solution to $x^2 - 3x + 1 = 0$ correct to two decimal places starting with $a = 2$ and $b = 3$.

a Run through the algorithm described in the flowchart step by step and record all values of a, m, b, $f(a)$, $f(m)$ and $f(b)$ using the given table as they update. Continue so that 5 lines are filled out in the table. Some cells have been completed for you.

a	*m*	*b*	*f(a)*	*f(m)*	*f(b)*
2	2.5	3	−1	−0.25	1
2.5	2.75	3	−0.25	0.3125	1
2.5		2.75			

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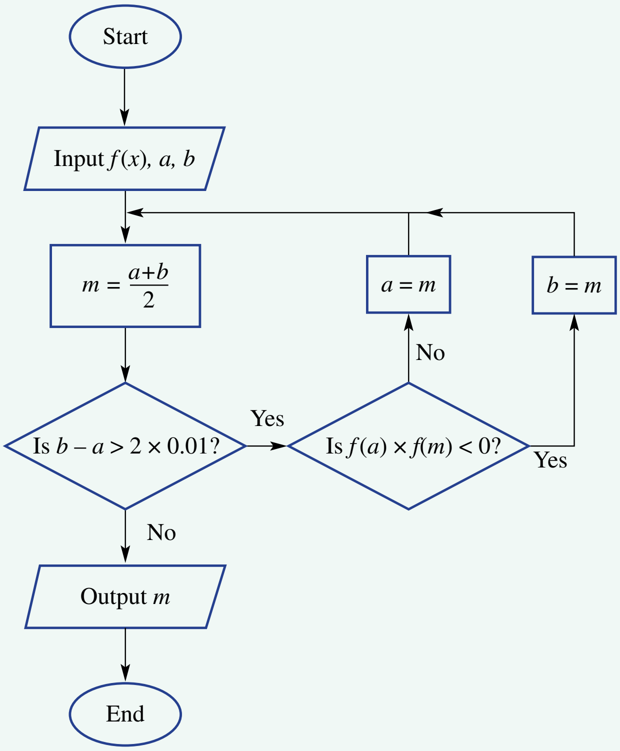


b Note that after the final row in the above table is filled the condition that $b - a > 2 \times 0.01$ is not satisfied and hence the solution to $f(x) = 0$ has not been found to two decimal places. Continue adding rows to the table above using the algorithm until this condition is satisfied.

c Once the $b - a > 2 \times 0.01$ is satisfied the algorithm will output m, the solution to $f(x) = 0$. What is this value of m in this case?

3 Using technology

Here is pseudocode for the bisection algorithm.

```
Define f(x):
    return $x^2 - 3x + 1$
$a \leftarrow 2$
$b \leftarrow 3$
$m \leftarrow \frac{a+b}{2}$
while $b - a > 2 \times 0.01$
    if $f(a) \times f(m) < 0$ then
        $b \leftarrow m$
    else
        $a \leftarrow m$
    end if
    $m \leftarrow \frac{a+b}{2}$
end while
print m
```

a Describe how the condition $f(a) \times f(m) < 0$ correctly allocates the updated values of a and b.

b Describe how the condition $b - a > 2 \times 0.01$ correctly delivers the solution correct to two decimal places.

c Use this algorithm to find the second solution to the equation $x^2 - 3x + 1 = 0$. Complete a table of values similar to the table in section 2 above.

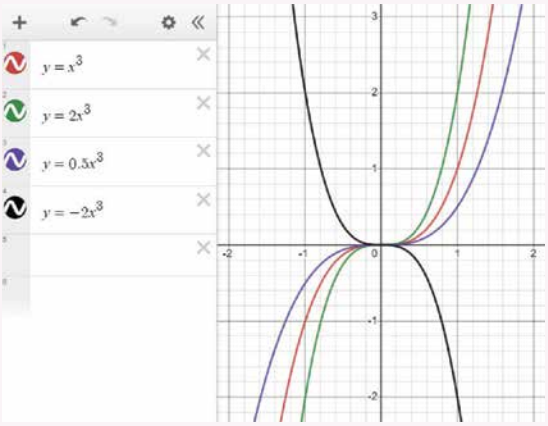
4 Extension

a Try using the bisection method on a different type of equation like $2^x - 5 = 0$.

b Use a language like Python to program the above algorithm and check your solutions to the above problems.

Families of cubic graphs

A family of graphs is a collection of graphs which contain shared properties. The graphs of all the members of the family will have a rule of the same form. For example the rules $y = x^3$, $y = 2x^3$, $y = 0.5x^3$ and $y = -2x^3$ are all simple cubic polynomials belonging to the family with rule $y = ax^3$ where a is a constant. The graphs of the rules listed above are shown here. They are all of the same basic shape, have a gradient of zero at $x = 0$ and pass through the origin.



$y = x^3$

$y = 2x^3$

$y = 0.5x^3$

$y = -2x^3$

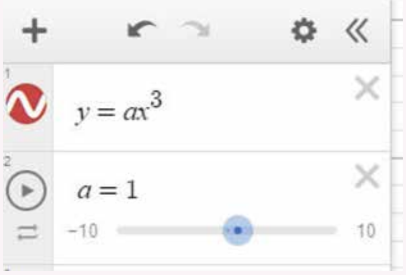
Exploring the cubic family $y = ax^3$

Use a graphing package like desmos to explore the family of cubic polynomials graphs which have the rule $y = ax^3$.

a Enter the rule $y = ax^3$ into your graphing package and add a slider for the value of a.

b Drag the slider to change the value of a.

c Describe the effect of the value of a on the graph of $y = ax^3$. Include a comment about the effect on the graph if the value of a is larger than 1, between -1 and 1 and when a is less than 1.



$y = ax^3$

$a = 1$

Exploring the cubic family $y = x^2(x - b)$

a Enter the rule $y = x^2(x - b)$ into your graphing package and add a slider for the value of b.

b Drag the slider to change the value of b.

c Describe some of the shared properties of all the graphs in this family.

d Describe the effect of the value of b on the graph of $y = x^2(x - b)$. Include a comment about the effect on the graph if the value of b is larger than 0, equal to 0 and when b is less than 0.

Exploring the cubic family $y = ax^3 + bx$

a Enter the rule $y = ax^3 + bx$ into your graphing package and add sliders for the values of a and b. Then drag the sliders to change the values of a and b.

b Describe some of the shared properties of all the graphs in this family. Describe the effect of the values of a and b on the graph of $y = ax^3 + bx$.

Exploring the general cubic polynomial $y = ax^3 + bx^2 + cx + d$

a Enter the rule $y = ax^3 + bx^2 + cx + d$ into your graphing package and add sliders for the values of a, b, c and d. Then drag the sliders to change the values of a, b, c and d.

b Describe the effect of the values of a, b, c and d on the graph of the cubic polynomial.

c Draw some examples of graphs with chosen values of a, b, c and d clearly labelling each graph with their rules.

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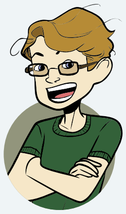
Problems and challenges

Up for a challenges? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



1 Find the remainder when $x^4 - 3x^3 + 6x^2 - 6x + 6$ is divided by $(x^2 + 2)$.

2 $x^3 + ax^2 + bx - 24$ is divisible by $(x + 3)$ and $(x - 2)$. Find the values of a and b.

3 Prove the following, using division.

a $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

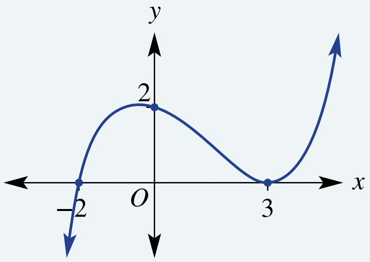
b $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

4 Solve for x.

a $(x + 1)(x - 2)(x - 5) \leqslant 0$

b $x^3 - x^2 - 16x + 16 > 0$

5 A cubic graph has a y-intercept at $(0, 2)$, a turning point at $(3, 0)$ and another x-intercept at $(-2, 0)$. Find the rule for the graph.



6 Given that $x^2 - 5x + 1 = 0$, find the value of $x^4 - 2x^3 - 16x^2 + 13x + 14$ without solving the first equation.

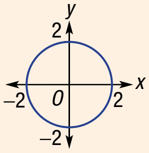
7 A quartic graph has a turning point at $(0, 0)$ and two x-intercepts at $(3, 0)$ and $(-3, 0)$. Find the rule for the graph if it also passes through $(2, 2)$.

8 Sketch the region defined by $x^2 - 4x + y^2 - 6y - 3 \leqslant 0$.

9 Prove that there are no points (x, y) that satisfy $x^2 - 4x + y^2 + 6y + 15 = 0$.

Circles

$x^2 + y^2 = r^2$ is a circle centred at (0, 0) with radius r.



$$x^2 + y^2 = 4$$
$$r^2 = 4$$
$$\therefore r = 2$$

The circle rule can be rearranged to

$y = \pm\sqrt{r^2 - x^2}$ or $x = \pm\sqrt{r^2 - y^2}$

Circle with equation

$(x - h)^2 + (y - k)^2 = r^2$

is centred at (h, k) with radius r.

Relation *vs* function

Any collection of points is a relation.
A function is a special relation.
$y = x^2$ is an example of a function because every x-value gives only one y-value.

Domain

The set of all allowable x-values

Range

The set of all resulting y-values

Other graphs (Ext)

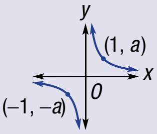
Functions (Ext)

Vertical line test

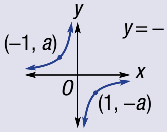
A relation is a function when any vertical line drawn through the graph cuts it only once.

Hyperbolas

$y = \frac{a}{x}$



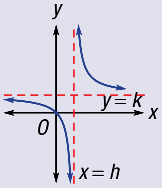
Asymptotes
$x = 0$ and $y = 0$



$y = -\frac{a}{x}$ (reflect in x- or y-axis)

$y = \frac{a}{x - h} + k$

find asymptotes and intercepts to sketch

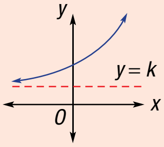


Transformations of exponential graphs

$y = a^{x-h} + k$

Asymptote at $y = k$

Find intercepts and sketch



Function notation

$y = x^2$ can be written as $f(x) = x^2$,

where $f(3) = 3^2$
$= 9$

Expanding polynomials

Example:

$$\begin{aligned} &(x^2 - 1)(x^2 + 2x - 3) \\ &= x^2(x^2 + 2x - 3) - 1(x^2 + 2x - 3) \\ &= x^4 + 2x^3 - 3x^2 - x^2 - 2x + 3 \\ &= x^4 + 2x^3 - 4x^2 - 2x + 3 \end{aligned}$$

Polynomial of degree *n*

$a_n x^n + a_{n-1} x^{n-1} + ... + a_0 x^0$

e.g. cubic $x^3 - 2x^2 + 3x - 4$

Powers of x are whole numbers.

Remainder theorem

If $P(x)$ is divided by $(x - a)$ then the remainder is $P(a)$.

Factor theorem

If $P(x)$ is divided by $(x - a)$ and $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.

Polynomials (Ext)

Division

$(x^3 - 2x^2 + 5x + 1) \div (x - 1)$

$$\begin{array}{rl} & x^2 - x + 4 \\ x-1 & \overline{)x^3 - 2x^2 + 5x + 1} \\ x^2(x-1) & \underline{x^3 - x^2} \\ & -x^2 + 5x + 1 \\ -x(x-1) & \underline{-x^2 + x} \\ & 4x + 1 \\ 4(x-1) & \underline{4x - 4} \\ & 5 \end{array}$$

$$\therefore \underbrace{x^3 - 2x^2 + 5x + 1}_{\text{dividend}} = \underbrace{(x - 1)}_{\text{divisor}}\underbrace{(x^2 - x + 4)}_{\text{quotient}} + \underbrace{5}_{\text{remainder}}$$

Null Factor Law

$(x + 2)(x - 3)(2x + 1) = 0$

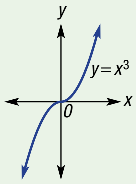
$x + 2 = 0$ or $x - 3 = 0$ or $2x + 1 = 0$

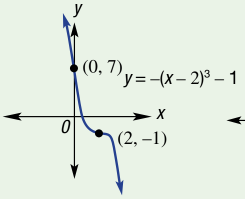
$x = -2$ or $x = 3$ or $x = -\frac{1}{2}$

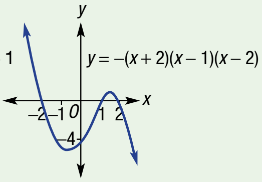
Factorising

- Find one factor using the factor theorem.
- Use long division to find the quotient.
- Factorise the quotient.

Graphs







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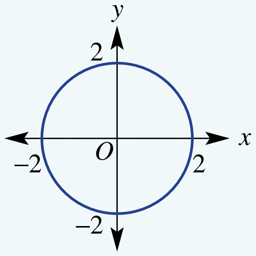
Chapter checklist and success criteria

A printable version of this checklist is available in the Interactive Textbook

			✔
10A	**1. I can use function notation.** e.g. If $f(x) = x^2 - 2x + 4$ find $f(3)$.	Ext	☐
10A	**2. I can recognise if a relation is a function.** e.g. Which of the following are functions? i $y = x^2 - 2x$ ii  iii $\{(1, 3), (2, 5), (4, 3), (2, -1)\}$	Ext	☐
10A	**3. I can determine the domain and range of a function.** e.g. Write down the allowable x-values (domain) and the resulting y-values (range) of $y = x^2 + 6$.	Ext	☐
10B	**4. I can identify a polynomial and state its degree.** e.g. Which of the following expressions are polynomials and state the degree of any polynomials? $2x^2 - \frac{5}{\sqrt{x}}$, $3x^3 - 2x + 7$	Ext	☐
10B	**5. I can evaluate a polynomial for a given value.** e.g. If $P(x) = 2x^3 - x^2 + 3$, find $P(-2)$.	Ext	☐
10C	**6. I can expand and simplify polynomials.** e.g. Expand and simplify $(x^3 + 3x - 2)(2x^2 - x + 4)$.	Ext	☐
10D	**7. I can divide polynomials.** e.g. Divide $P(x) = x^3 - 2x^2 + 3x + 6$ by $(x + 1)$ and write in the form $P(x) = (x + 1)Q(x) + R$, where R is the remainder.	Ext	☐
10E	**8. I can find the remainder using the remainder theorem.** e.g. Find the remainder when $P(x) = x^3 - 2x^2 + 3x - 4$ is divided by $(x - 3)$.	Ext	☐
10E	**9. I can decide whether a linear expression is a factor of a polynomial using the factor theorem.** e.g. Decide if $(x + 2)$ is a factor of $P(x) = x^3 + 3x^2 - x - 6$.	Ext	☐
10E	**10. I can apply the remainder theorem to find a missing value.** e.g. Find the value of k such that $(x^3 - 2x^2 + kx - 2) \div (x - 3)$ has a remainder of 4.	Ext	☐
10F	**11. I can apply the Null Factor Law to solve polynomial equations.** e.g. Solve $(2x + 5)(x - 3)(x + 2) = 0$ for x.	Ext	☐
10F	**12. I can factorise and solve a cubic equation.** e.g. Solve $x^3 - 5x^2 + 2x + 8 = 0$.	Ext	☐
10G	**13. I can solve simple cubic equations.** e.g. Solve $x^3 = -125$ and $\frac{1}{2}x^3 - 6 = 26$.	Ext	☐

Chapter checklist and success criteria

			✔
10G	**14. I can sketch cubic functions of the form $y = a(x - h)^3 + k$.** e.g. Sketch the cubic graphs $y = -2x^3$, $y = x^3 + 2$ and $y = (x + 3)^3$.	Ext	☐
10H	**15. I can sketch a cubic polynomial labelling intercepts.** e.g. Sketch $y = (x - 5)(x + 1)(x - 2)$ labelling x- and y-intercepts.	Ext	☐
10I	**16. I can sketch the graph of a circle by finding its centre and radius.** e.g. For the equation $x^2 + y^2 = 16$, state the coordinates of its centre, state its radius and sketch its graph labelling intercepts.	Ext	☐
10I	**17. I can find the point of intersection of a circle and a line.** e.g. Find the coordinates of the points where $x^2 + y^2 = 9$ intersects $y = 2x$.	Ext	☐
10J	**18. I can sketch the graph of a hyperbola.** e.g. Sketch the graphs of $y = \frac{3}{x}$ and $y = -\frac{2}{x}$, labelling the point where $x = 1$ and $x = -1$.	Ext	☐
10J	**19. I can find the intersection of a line and a hyperbola.** e.g. Find the coordinates of the point where $y = \frac{1}{x}$ meets the line $y = 9x$.	Ext	☐
10K	**20. I can sketch graphs of circles, exponentials and hyperbolas with transformations.** e.g. Sketch the graphs of $(x - 1)^2 + (y + 2)^2 = 4$, $y = 2^{x - 1} - 4$ and $y = \frac{1}{x - 2} + 1$, labelling key features.	Ext	☐

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Ext Short-answer questions

10A **1** If $f(x) = 2x^2 - x + 3$, find:
a $f(2)$ **b** $f(-3)$ **c** $f(0.5)$ **d** $f(k)$

10A **2** For each of the following, state the domain (i.e. the set of allowable x-values) and range (i.e. resulting y-values).
a $y = 2x - 8$ **b** $y = 4$ **c** $x = 1$
d $y = \frac{1}{x}$ **e** $y = x^2 - 3$ **f** $y = x^2 - x$

10B **3** If $P(x) = x^3 - x^2 - x - 1$, find:
a $P(0)$ **b** $P(2)$ **c** $P(-1)$ **d** $P(-3)$

10C **4** Expand and simplify.
a $(x^2 + 2)(x^2 + 1)$ **b** $x^3(x^2 - x - 3)$
c $(x^2 + x - 3)(x^3 - 1)$ **d** $(x^3 + x - 3)(x^3 + x - 1)$

10D **5** Use long division to express each of the following in this form:

Dividend = divisor × quotient + remainder

a $(x^3 + x^2 + 2x + 3) \div (x - 1)$ **b** $(x^3 - 3x^2 - x + 1) \div (x + 1)$
c $(2x^3 - x^2 + 4x - 7) \div (x + 2)$ **d** $(-2x^3 - x^2 - 3x - 4) \div (x - 3)$

10E **6** Use the remainder theorem to find the remainder when $P(x) = 2x^3 - 2x^2 + 4x - 7$ is divided by:
a $x - 1$ **b** $x + 2$ **c** $x + 3$ **d** $x - 3$

10E **7** Using the factor theorem, decide if the following are factors of $P(x) = x^3 - 2x^2 - 11x + 12$.
a $x + 1$ **b** $x - 1$ **c** $x - 4$ **d** $x + 3$

10F **8** Solve these cubic equations.
a $(x - 3)(x - 1)(x + 2) = 0$ **b** $(x - 5)(2x - 3)(3x + 1) = 0$

10F **9** Factorise and solve these cubic equations.
a $x^3 + 4x^2 + x - 6 = 0$ **b** $x^3 - 9x^2 + 8x + 60 = 0$

10G **10** Solve these cubic equations.
a $x^3 = -27$ **b** $2x^3 - 5 = 123$

10G **11** Sketch the following cubic functions, labelling the point of inflection and axes intercepts.
a $y = -x^3 + 1$ **b** $y = (x + 2)^3$

10H **12** Sketch the graphs of these polynomials.
a $y = (x + 1)(x - 1)(x - 4)$ **b** $y = -x(x - 3)(x + 2)$

10I **13** Sketch these circles. Label the centre and axes intercepts.
a $x^2 + y^2 = 25$ **b** $x^2 + y^2 = 7$

10I **14** Find the exact coordinates of the points of intersection of the circle $x^2 + y^2 = 9$ and the line $y = 2x$. Sketch the graphs, showing the points of intersection.

10J **15** Sketch these hyperbolas, labelling the points where $x = 1$ and $x = -1$.
a $y = \frac{2}{x}$ **b** $y = -\frac{3}{x}$

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10J **16** Find the points of intersection of the hyperbola $y = \frac{4}{x}$ and these lines.

a $y = 3$

b $y = 2x$

10K **17** Sketch the following graphs, labelling key features with exact coordinates.

a $(x + 1)^2 + (y - 2)^2 = 4$

b $y = 2^{x-1} + 3$

c $y = \frac{1}{x + 2} - 3$

Ext Multiple-choice questions

10A/H/I **1** Which one of the following is <u>not</u> a function?

A $y = x^2 - 3x$

B $y = 4x - 1$

C $y = -\frac{3}{x}$

D $x^2 + y^2 = 9$

E $y = 4$

10B **2** $x^6 - 2x^2 + 1$ is a polynomial of degree:

A 1

B −2

C 2

D 6

E 0

10B **3** Which of these is a polynomial?

A $x^{\frac{1}{2}} + x^2$

B $\sqrt[3]{x} + x^2$

C $x^3 + x^2 + \frac{1}{x^2}$

D $4x^5 - 2x^3 - 1$

E $\frac{1}{x} - x$

10B **4** If $P(x) = x^3 - x$ and $Q(x) = -2x^2 + 1$, then $P(-1) - Q(1)$ is equal to:

A 1

B −1

C 2

D 3

E −3

10E **5** The remainder when $P(x) = 2x^3 + 4x^2 - x - 5$ is divided by $(x + 1)$ is:

A −4

B −2

C −10

D 0

E −1

10F **6** The three solutions to $(x - 3)(x + 5)(2x - 1) = 0$ are x equals:

A $\frac{1}{3}, -\frac{1}{5}$ and 2

B $5, -3$ and $-\frac{1}{2}$

C $-5, 3$ and $-\frac{1}{2}$

D $-5, 3$ and 1

E $-5, 3$ and $\frac{1}{2}$

10H **7** The equation of this graph could be:

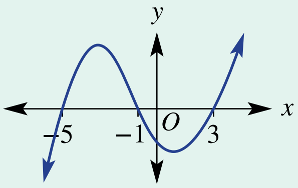
A $y = x(x + 5)(x - 1)$

B $y = (x + 5)(x + 1)(x + 3)$

C $y = (x - 5)(x - 1)(x + 3)$

D $y = (x - 5)(x + 1)(x - 3)$

E $y = (x + 5)(x + 1)(x - 3)$



10I **8** The equation of a circle centred at the origin with radius 4 units is:

A $y = 4x^2$

B $x^2 + y^2 = 4$

C $x^2 + y^2 = 8$

D $y = 4^x$

E $x^2 + y^2 = 16$

10J **9** The point of intersection of the graphs of $y = \frac{2}{x}$ and $y = 8$ is:

A $(2, 8)$

B $(4, 8)$

C $\left(\frac{1}{4}, 8\right)$

D $\left(\frac{1}{8}, 8\right)$

E $(8, 4)$

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10K

10 The graph of $y = \dfrac{1}{x-1} + 2$ has asymptote(s) at:

- **A** $x = 1, y = 2$
- **B** $y = 2$
- **C** $x = -1$
- **D** $x = 2, y = 1$
- **E** $x = -1, y = 2$

Ext Extended-response questions

1 A cubic polynomial has the rule $P(x) = x^3 - 5x^2 - 17x + 21$.

- **a** Find:
  - **i** $P(-1)$
  - **ii** $P(1)$
- **b** Explain why $(x - 1)$ is a factor of $P(x)$.
- **c** Divide $P(x)$ by $(x - 1)$ to find the quotient.
- **d** Factorise $P(x)$ completely.
- **e** Solve $P(x) = 0$.
- **f** Find $P(0)$.
- **g** Sketch a graph of $P(x)$, labelling x- and y-intercepts.

2 A population of butterflies in an enclosure at a Botanical Gardens is modelled by $P(t) = 180 - \dfrac{200}{t+2}$ for $t \geqslant 0$ where P is the number of butterflies t years after monitoring of the butterflies began.

- **a** Find the initial number of butterflies when monitoring began.
- **b** Determine how long until the butterfly population reaches 140 based on this model.
- **c** Sketch a graph of P vs t for $t \geqslant 0$. Label any axis intercepts and the equation of any asymptotes.
- **d** Describe what happens to the population of butterflies over time in this model.

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11

Algorithmic thinking

Maths in context: Driverless cars

The vast majority of accidents in motor vehicles are due to driver error, which is one of the reasons why companies such as Google are experimenting with driverless cars.

Such vehicles are currently being tested in a number of countries. The vehicles map their current position and use a range of sensors to determine the moving and stationary objects in the immediate area. This information is the input for computer-based algorithms which make predictions and test scenarios at a rapid rate. The efficiency of the computer code is critical in the running of the systems, which is why skilled programmers and mathematicians are responsible for their design. If the system detects a pedestrian, for example, the code needs to take into account the probability that this person could cross the road in front of the car. The software must choose the safest possible route at the safest speed.

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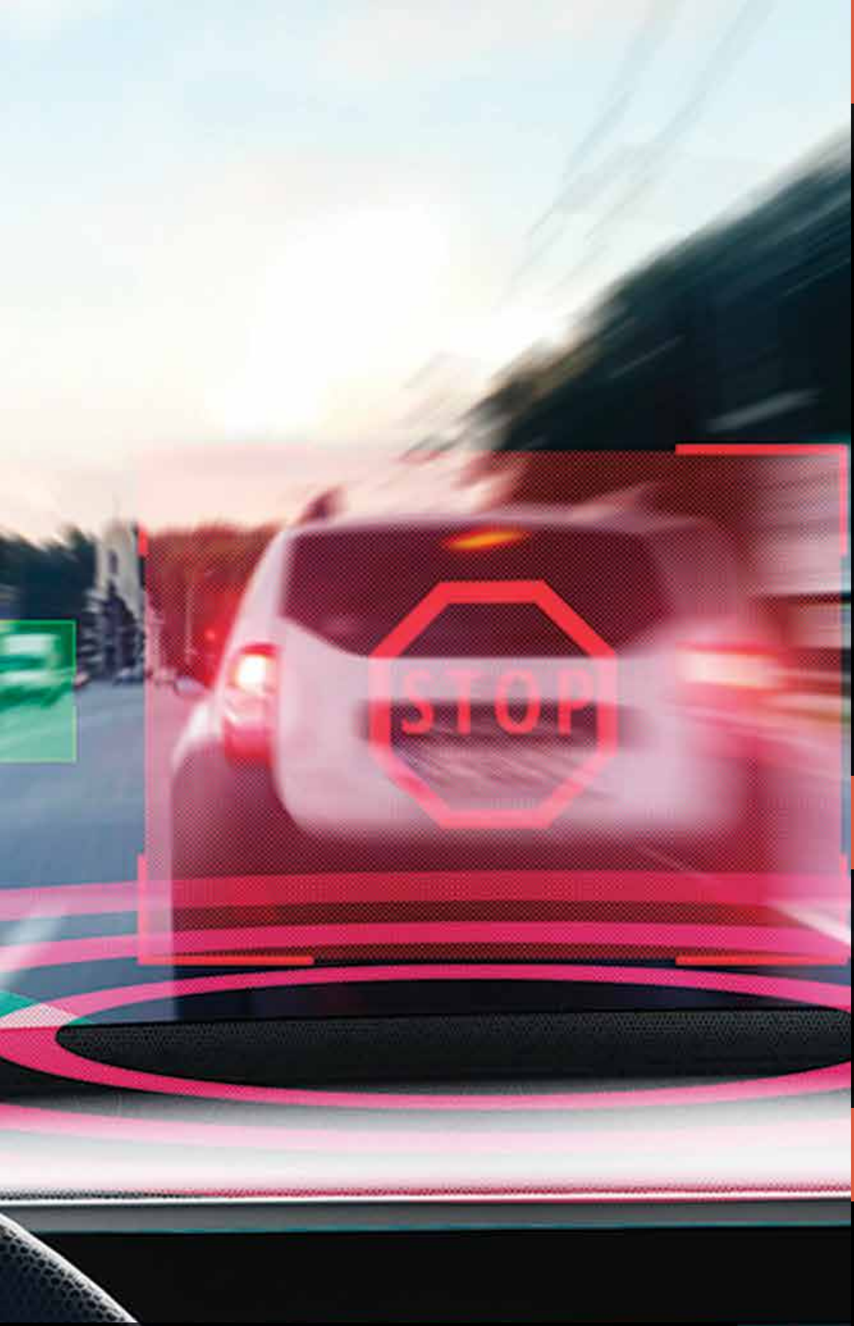


Chapter contents

Australian Curriculum 9.0

Design, test and refine solutions to spatial problems using algorithms and digital tools; communicate and justify solutions (AC9M10SP03)

Online resources

A host of additional online resources are included as part of your Interactive Textbook, including HOTmaths content, video demonstrations of all worked examples, auto-marked quizzes and much more.

Driverless cars have the potential to reduce road fatalities and increase the level of efficiency on our road networks. Parking would become less problematic, with a driverless car dropping you off at work and travelling to a remote parking spot somewhere else, before returning later in the day to pick you up.

Introduction

An **algorithm** is a sequence of steps that when followed, lead to the solution of a problem. It has a defined set of inputs and delivers an output. Each step in the algorithm leads to another step or completes the algorithm.

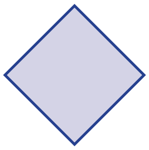
Algorithms occur in mathematics and computing, as well as in simple areas of daily life such as following a recipe. Algorithmic thinking is a type of thinking that involves designing algorithms to solve problems. The algorithms we design can then be written in a way that a computer program will understand, so that the computer does the hard computational work.

In the following activities you will carry out some algorithms as well as think about the design, analysis and implementation of your own algorithms.

The algorithms in these activities will be described through the use of spreadsheets, flow charts (a way of writing an algorithm in the form of a diagram), programming language and simulations. The following symbols will be used in the flow charts with arrows to connect each stage:

for input/output stages

for process stages



for decision stages

Activity 1: Using numerical methods to solve equations

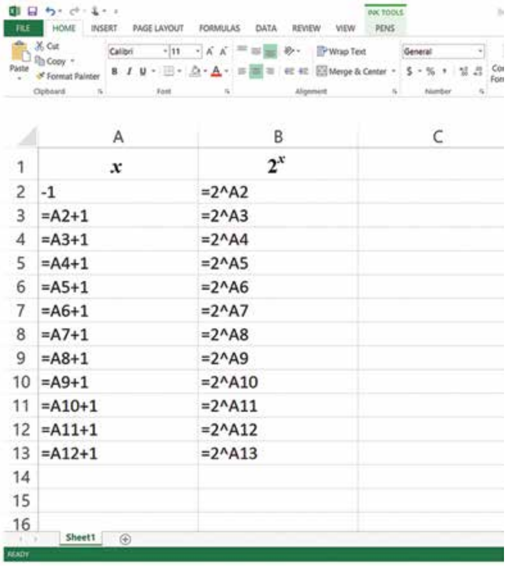
NUMBER AND ALGEBRA

Numerical methods using tables, spreadsheets, graphs and other algorithms can be used to solve both linear and non-linear equations, to find roots of equations and to compute other values. The following activities look at some common approaches.

1.1 Solving equations using tables and spreadsheets

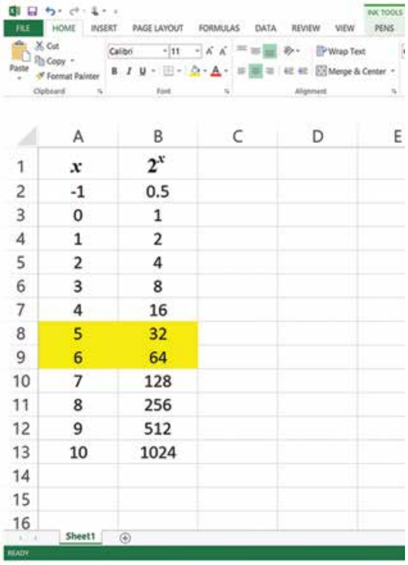
a To solve an equation such as $2^x = 40$, complete the following.

i Set up a spreadsheet like the one shown below, (*Figure 1*), entering the formulas and then filling down. Use an increment of 1 in column A (depending on the equation, you may also need to consider negative values of x) to find which integer values of x make the value of 2^x change from less than 40 to more than 40. In this case (*Figure 2*) you can see this is $x = 5$ and $x = 6$.



	A	B	C
1	x	2^x	
2	-1	=2^A2	
3	=A2+1	=2^A3	
4	=A3+1	=2^A4	
5	=A4+1	=2^A5	
6	=A5+1	=2^A6	
7	=A6+1	=2^A7	
8	=A7+1	=2^A8	
9	=A8+1	=2^A9	
10	=A9+1	=2^A10	
11	=A10+1	=2^A11	
12	=A11+1	=2^A12	
13	=A12+1	=2^A13	
14			
15			
16			

Figure 1



	A	B	C	D	E
1	x	2^x			
2	-1	0.5			
3	0	1			
4	1	2			
5	2	4			
6	3	8			
7	4	16			
8	5	32			
9	6	64			
10	7	128			
11	8	256			
12	9	512			
13	10	1024			
14					
15					
16					

Figure 2

ii Now alter the spreadsheet to only consider x-values between $x = 5$ and $x = 6$, incrementing by 0.1. Fill down and find a solution for x, correct to one decimal place.

iii Now alter the spreadsheet to only consider x-values between $x = 5$ and $x = 6$, incrementing by 0.01. Fill down and find a solution for x, correct to two decimal places.

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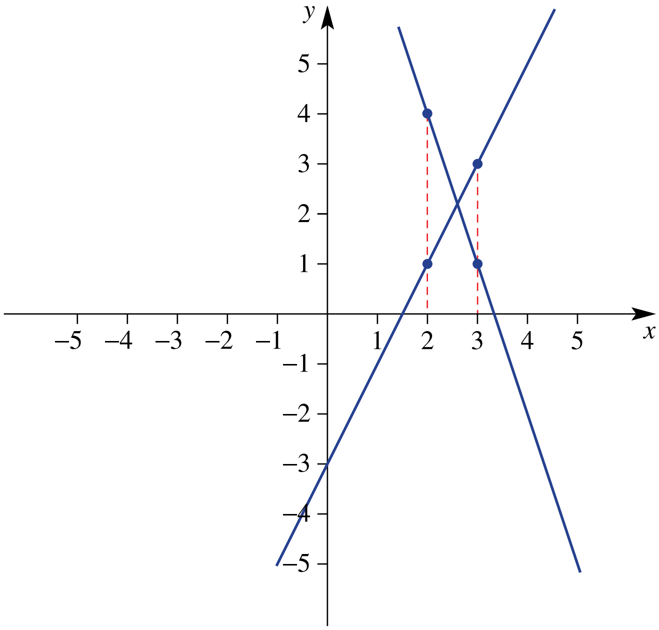
b A similar process can be used to solve a pair of simultaneous equations. To solve the equations $y = 2x - 3$ and $y = -3x + 10$ simultaneously, complete the following.

- **i** Retain the x column like in part **a** and enter the two equations ($= 2^*\mathrm{A2} - 3$ and $= -3^*\mathrm{A2} + 10$) into columns B and C.

  Fill down to look for the x-value that makes the two equations equal. If you first increment the x-values by 1, you are looking for when the column B values overtake the column C values or vice versa, to find the interval containing the point of intersection. This can be seen on the example graph shown.



- **ii** Increase your accuracy by using smaller increments within a chosen interval. See if you can find the exact solution.

c To find the roots (or x-intercepts) of a quadratic equation, you can also use a spreadsheet or table. This time you are looking for the interval(s) (there may be two solutions) where the function value changes from positive to negative or vice versa.

Find the solutions to the quadratic equation $3x^2 - 4x - 2 = 0$, correct to two decimal places, using a spreadsheet or table. You may also need to consider negative values of x.

d The equations in parts **a** and **b** could also be treated as equations equal to 0 by rearranging to avoid having two columns to compare. For example, the solution to $2^x = 40$ is the same as the solution to $2^x - 40 = 0$. What is the equivalent equation equal to 0 that can be solved to find the x-value of the solution to $y = 2x - 3$ and $y = -3x + 10$?

This method will be used further in the next section.

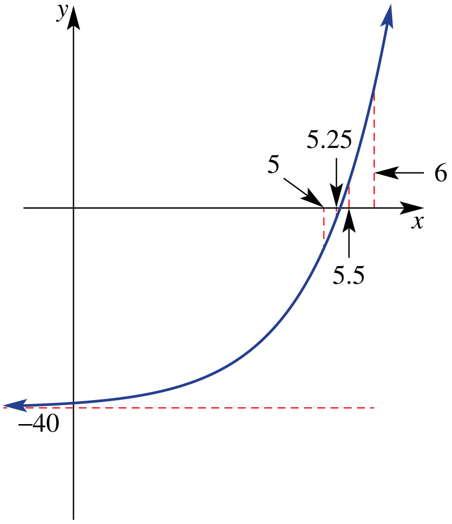
1.2 Solving equations using the bisection method

In the following we will look at the bisection method algorithm for solving the problems that were considered in **Section** 1.1. The process looks at solving an equation $f(x) = 0$ by finding smaller and smaller intervals of x-values within which the solution lies. Since the equation is being solved equal to 0, we will be looking for intervals where the function values are of different sign.

The process involves finding an average value of x between a pair of updated lower and upper bounds. The steps are:

- Find a lower and upper bound which are two x-values that contain the solution (i.e. one function value is positive and one negative).
- Now find the mean of your lower and upper bound.
- Find the value of the function for your mean.
- If the mean function value is zero the solution has been obtained. If it is the same sign as the lower bound, it becomes the new lower bound of the interval, otherwise it becomes the new upper bound.
- Repeating the above steps will deliver smaller and smaller intervals containing the solution.

a Work through the above algorithm to solve $2^x - 40 = 0$ until you are confident you have the answer correct to two decimal places.



The process can take a while by hand, particularly for greater levels of accuracy. We will now consider a spreadsheet approach to this algorithm.

b Set up the following spreadsheet where $f(x) = 2^x - 40$ and the aim is to solve $f(x) = 0$, correct to four decimal places.

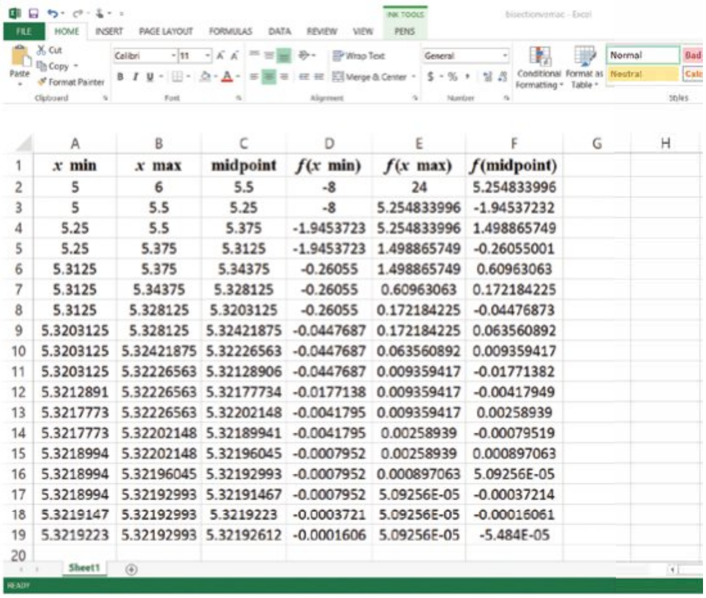
i Start the process by using the interval from $x = 5$ to $x = 6$ obtained in **Section** 1.1.

ii Enter in the formula for finding the midpoint and evaluate the function at the minimum, maximum and midpoint values (row 2).

	A	B	C	D	E	F
1	x min	x max	midpoint	$f(x$ min)	$f(x$ max)	f(midpoint)
2	5	6	=(A2+B2)/2	=2^A2-40	=2^B2-40	=2^C2-40
3	=IF(D2*F2>=0, C2, A2)	=IF(E2*F2>=0,C2,B2)	=(A3+B3)/2	=2^A3-40	=2^B3-40	=2^C3-40
4	=IF(D3*F3>=0, C3, A3)	=IF(E3*F3>=0,C3,B3)	=(A4+B4)/2	=2^A4-40	=2^B4-40	=2^C4-40
5	=IF(D4*F4>=0, C4, A4)	=IF(E4*F4>=0,C4,B4)	=(A5+B5)/2	=2^A5-40	=2^B5-40	=2^C5-40
6	=IF(D5*F5>=0, C5, A5)	=IF(E5*F5>=0,C5,B5)	=(A6+B6)/2	=2^A6-40	=2^B6-40	=2^C6-40
7	=IF(D6*F6>=0, C6, A6)	=IF(E6*F6>=0,C6,B6)	=(A7+B7)/2	=2^A7-40	=2^B7-40	=2^C7-40
8	=IF(D7*F7>=0, C7, A7)	=IF(E7*F7>=0,C7,B7)	=(A8+B8)/2	=2^A8-40	=2^B8-40	=2^C8-40
9	=IF(D8*F8>=0, C8, A8)	=IF(E8*F8>=0,C8,B8)	=(A9+B9)/2	=2^A9-40	=2^B9-40	=2^C9-40
10	=IF(D9*F9>=0, C9, A9)	=IF(E9*F9>=0,C9,B9)	=(A10+B10)/2	=2^A10-40	=2^B10-40	=2^C10-40
11	=IF(D10*F10>=0, C10, A1	=IF(E10*F10>=0,C10,B10)	=(A11+B11)/2	=2^A11-40	=2^B11-40	=2^C11-40
12	=IF(D11*F11>=0, C11, A1	=IF(E11*F11>=0,C11,B11)	=(A12+B12)/2	=2^A12-40	=2^B12-40	=2^C12-40
13	=IF(D12*F12>=0, C12, A1	=IF(E12*F12>=0,C12,B12)	=(A13+B13)/2	=2^A13-40	=2^B13-40	=2^C13-40
14	=IF(D13*F13>=0, C13, A1	=IF(E13*F13>=0,C13,B13)	=(A14+B14)/2	=2^A14-40	=2^B14-40	=2^C14-40
15	=IF(D14*F14>=0, C14, A1	=IF(E14*F14>=0,C14,B14)	=(A15+B15)/2	=2^A15-40	=2^B15-40	=2^C15-40
16	=IF(D15*F15>=0, C15, A1	=IF(E15*F15>=0,C15,B15)	=(A16+B16)/2	=2^A16-40	=2^B16-40	=2^C16-40

iii Use an IF statement (row 3) to determine which value the midpoint replaces in the interval. Here the same sign is determined by seeing if multiplying the function values results in a positive value (indicating both values are negative or both are positive). If the midpoint and minimum value produce function values of the same sign, the midpoint becomes the new minimum, otherwise the current minimum is maintained. The new maximum value is found in the same way.

iv Continue the process by filling down each of the columns until the desired accuracy is obtained. The result should be the spreadsheet values below where the solution of $2^x - 40 = 0$, correct to four decimal places, can be seen as $x = 5.3219$.



	A	B	C	D	E	F	G	H
1	x min	x max	midpoint	$f(x$ min)	$f(x$ max)	f(midpoint)		
2	5	6	5.5	-8	24	5.254833996		
3	5	5.5	5.25	-8	5.254833996	-1.94537232		
4	5.25	5.5	5.375	-1.9453723	5.254833996	1.498865749		
5	5.25	5.375	5.3125	-1.9453723	1.498865749	-0.26055001		
6	5.3125	5.375	5.34375	-0.26055	1.498865749	0.60963063		
7	5.3125	5.34375	5.328125	-0.26055	0.60963063	0.172184225		
8	5.3125	5.328125	5.3203125	-0.26055	0.172184225	-0.04476873		
9	5.3203125	5.328125	5.32421875	-0.0447687	0.172184225	0.063560892		
10	5.3203125	5.32421875	5.32226563	-0.0447687	0.063560892	0.009359417		
11	5.3203125	5.32226563	5.32128906	-0.0447687	0.009359417	-0.01771382		
12	5.3212891	5.32226563	5.32177734	-0.0177138	0.009359417	-0.00417949		
13	5.3217773	5.32226563	5.32202148	-0.0041795	0.009359417	0.00258939		
14	5.3217773	5.32202148	5.32189941	-0.0041795	0.00258939	-0.00079519		
15	5.3218994	5.32202148	5.32196045	-0.0007952	0.00258939	0.000897063		
16	5.3218994	5.32196045	5.32192993	-0.0007952	0.000897063	5.09256E-05		
17	5.3218994	5.32192993	5.32191467	-0.0007952	5.09256E-05	-0.00037214		
18	5.3219147	5.32192993	5.3219223	-0.0003721	5.09256E-05	-0.00016061		
19	5.3219223	5.32192993	5.32192612	-0.0001606	5.09256E-05	-5.484E-05		
20								

c Repeat the process from part **b** to solve the following equations, correct to four decimal places. Recall that to solve $f(x) = g(x)$ you can solve $f(x) - g(x) = 0$.

i $7x - 20 = 0$ **ii** $2x + 15 = 5x - 2$ **iii** $x^2 = x + 1$ (2 solutions)

d Investigate the Excel function *Goal seek* in the *Data* menu under *What-if analysis*. How could this be used to complete the above work?

1.3 Algorithms for finding square roots

The bisection method outlined in **Section** 1.2 can also be used to find the square root of a number to a desired level of accuracy.

To find $\sqrt{5}$, for example, is the same as finding the positive solution $(x > 0)$ of the equation $x^2 = 5$ or $x^2 - 5 = 0$.

a Use the bisection method and a spreadsheet to find $\sqrt{5}$, correct to four decimal places.

While the bisection method is quite an efficient method for finding a square root, other numerical methods also exist.

The Babylonian algorithm, approaches the actual value of the square root very quickly and can achieve a high degree of accuracy within a few run-throughs.

The algorithm is an iterative process whereby each new approximation makes use of the previous approximation.

The basic algorithm for solving $x^2 = S$ is:

- Make an initial guess (x)
- Divide the number S by the guess and average the guess and the quotient $\frac{S}{x}$
- Make this average value the new guess and repeat the above step.

This algorithm can also be expressed as:

$$x_0 \approx \sqrt{S}$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{S}{x_n}\right)$$

where x_0 represents the initial guess and n is the number of iterations. As n increases the solution becomes more accurate.

b Try the method outlined above to evaluate the following square roots, correct to six decimal places.
  i $\sqrt{5}$
  ii $\sqrt{2}$
  iii $\sqrt{45}$

c For $\sqrt{5}$, comment on the efficiency of the two algorithms (parts **a** and **b**) for obtaining an accurate approximation. Efficiency in this context relates to the number of algorithmic steps in a process and how quickly a solution is obtained.

d *Extension*:
  i Show algebraically that $x^2 = S$ can be expressed as $x = \frac{1}{2}\left(x + \frac{S}{x}\right)$, which leads to the iterative formula seen in part **b**.
  ii Use a similar process to part **i** to come up with an iterative formula to solve $x^3 = S$ and hence evaluate $\sqrt[3]{20}$, correct to six decimal places.

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Activity 2: Pythagorean triples

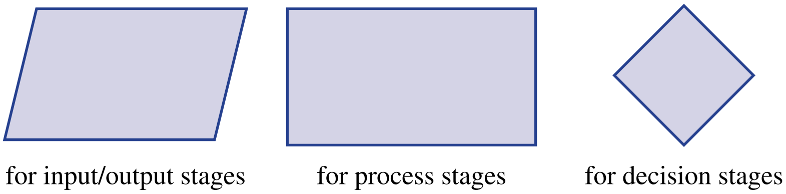
MEASUREMENT AND GEOMETRY

In these activities you will work through some algorithms leading to writing a program to generate the Pythagorean triples. A Pythagorean triple is a set of integers x, y and z such that $x^2 + y^2 = z^2$.

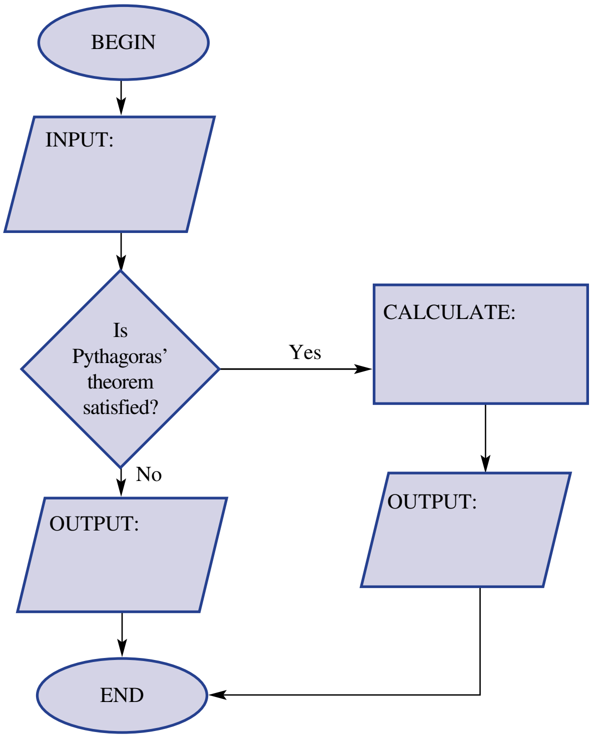
2.1 Is the triangle right-angled?

In this first activity you will design an algorithm to determine if a triangle is right-angled given the three side lengths of the triangle. If the triangle is right-angled, the program should say so and also give the two acute angles of the triangle.

a The flow chart below is set up for the algorithm for this program. Note the use of the following symbols:

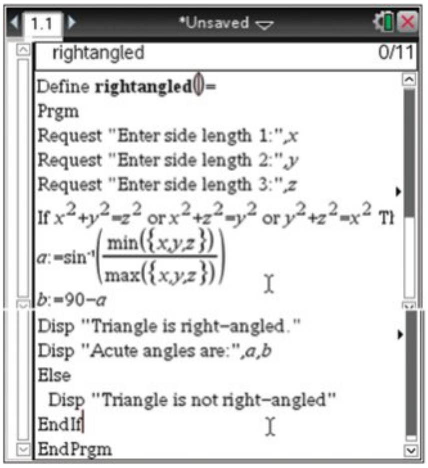


Complete the flow chart by filling in the input, output and calculate boxes.



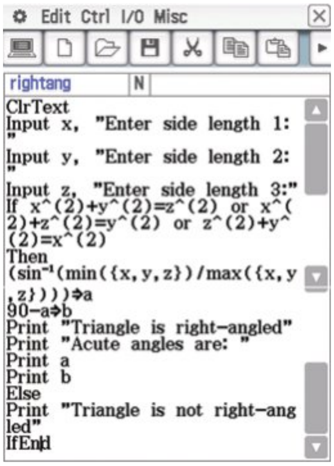
b The screenshot below shows a program for implementing the algorithm in part **a**.

Using the TI-Nspire:



```
Define rightangled()=
Prgm
Request "Enter side length 1:",x
Request "Enter side length 2:",y
Request "Enter side length 3:",z
If x^2+y^2=z^2 or x^2+z^2=y^2 or y^2+z^2=x^2 Th
a:=sin^-1(min({x,y,z})/max({x,y,z}))
b:=90-a
Disp "Triangle is right-angled."
Disp "Acute angles are:",a,b
Else
  Disp "Triangle is not right-angled"
EndIf
EndPrgm
```

Using the ClassPad:

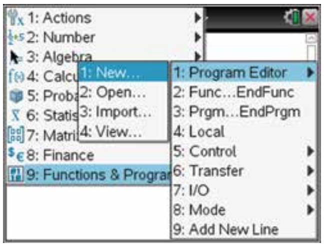


```
ClrText
Input x, "Enter side length 1:"
Input y, "Enter side length 2:"
Input z, "Enter side length 3:"
If x^(2)+y^(2)=z^(2) or x^(2)+z^(2)=y^(2) or z^(2)+y^(2)=x^(2)
Then
(sin⁻¹(min({x,y,z})/max({x,y,z})))⇒a
90−a⇒b
Print "Triangle is right-angled"
Print "Acute angles are: "
Print a
Print b
Else
Print "Triangle is not right-angled"
IfEnd
```

Read over the program and then enter it into your calculator by following these steps:

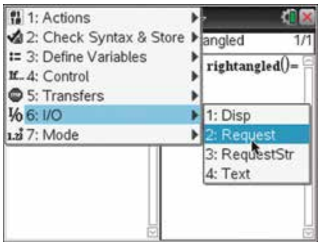
Using the TI-Nspire:

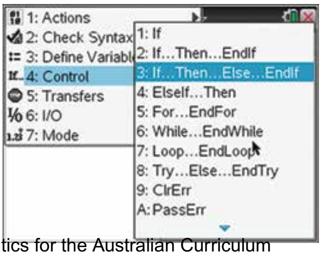
i Open a new document screen and select the **Functions & Programs** menu followed by **Program Editor** and **New**.



ii Give your program a name.

iii Type in the program. The input and output stages and the ‘If Then’ statement can be found in the **I/O** menu and in the **Control** menu or they can be typed directly.



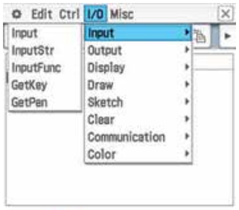


Using the ClassPad:

i From **Menu** choose **Program** followed by **New File**.

ii Give your program a name.

iii Type in the program. The input and output stages and the ‘If … Then’ statement can be found in the **I/O** menu and in the **Ctrl** (Control) menu or they can be typed directly.



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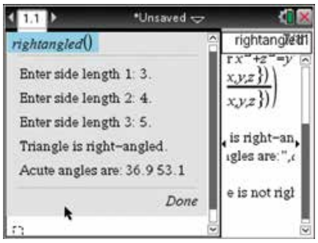
iv To save, from the menu select: **Check syntax and store.**

v To run the program, type the program name on the left-hand panel. You should be prompted for the inputs and the result will be displayed.

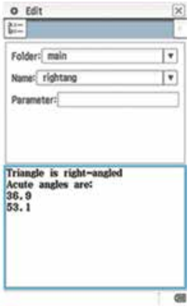


```
Enter side length 1: 3.
Enter side length 2: 4.
Enter side length 3: 5.
Triangle is right-angled.
Acute angles are: 36.9 53.1
                         Done
```

iv To save, from the **Edit** menu select Save or otherwise click on the disk icon.

v To run the program click on the computer icon and then the ‘play’ button:▶. You will be prompted for the inputs and the result will be displayed as below for side lengths 3, 4 and 5.

Enter side length 1: 3



Folder: main
Name: rightang
Parameter:

```
Triangle is right-angled
Acute angles are:
36.9
53.1
```

vi *Extension.* Add to the above program so that it also displays the angles in a non-right-angled triangle.

The above program has a brute force approach to testing if the triangle is right-angled by testing if any of the possible combinations of side lengths satisfies Pythagoras’ theorem. A more efficient algorithm would recognise that the hypotenuse has to be the longest side and that only one possible combination can work.

c The program can be adjusted so that it only tests the one case of Pythagoras’ theorem by determining the longest of the three sides first and using this as the only possible hypotenuse. Complete these steps to achieve this:

i Assign the maximum of the input values as the hypotenuse c using $c := max(\{x, y, z\})$ using the TI-Nspire or $max(\{x, y, z\}) = c$ using the ClassPad.

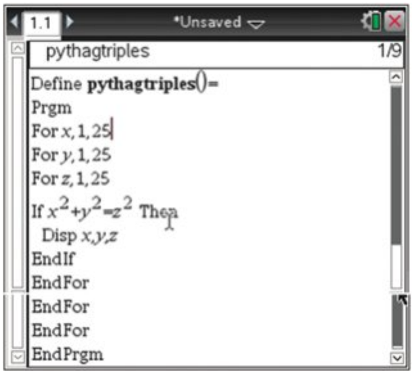
ii Use the *min* and *median* (since there are three lengths) functions to assign the two shorter side lengths a and b.

iii Test one case of Pythagoras’ theorem, if $a^2 + b^2 = c^2$.

2.2 Pythagorean triples

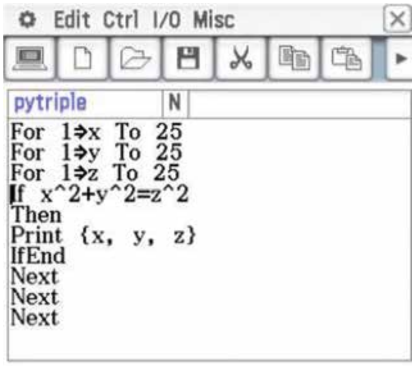
a The program below demonstrates a brute force way of generating a list of Pythagorean triples. The first *For* loop, for instance, runs the part inside the *For* loop for $x = 1$, then $x = 2$ up until $x = 25$.

Using the TI-Nspire:



```
Define pythagtriples()=
Prgm
For x,1,25
For y,1,25
For z,1,25
If x^2+y^2=z^2 Then
 Disp x,y,z
EndIf
EndFor
EndFor
EndFor
EndPrgm
```

Using the ClassPad:



```
For 1⇒x To 25
For 1⇒y To 25
For 1⇒z To 25
If x^2+y^2=z^2
Then
Print {x, y, z}
IfEnd
Next
Next
Next
```

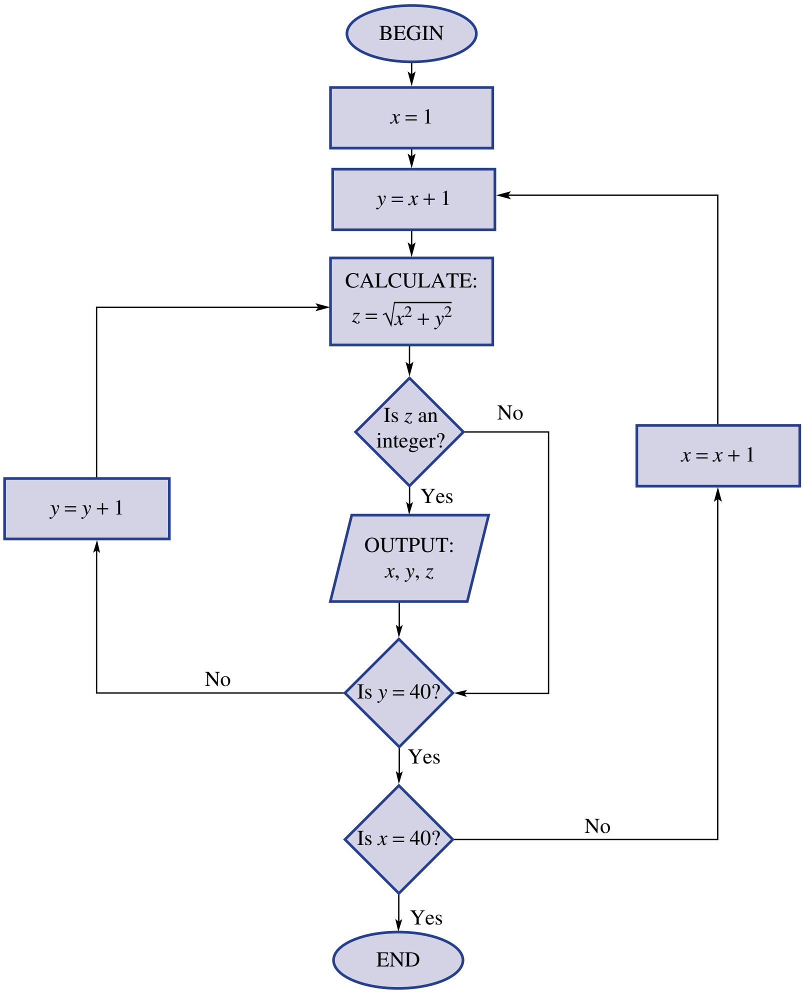
Type in and run the program and comment on any problems you see with the program both from an efficiency and an output point of view.

b Consider possible restrictions on the values of x, y and z to make the algorithm more efficient. We will keep the hypotenuse as having a longest side of 25. If x is to be designated as the shortest side it can be a *For* loop working through the values from 1 to 24.

- **i** If y is now the bigger of the two shorter sides, rewrite the *For* loop for y from its minimum possible value (this will be in terms of x) up to 24.
- **ii** Given z is the longest side, rewrite the *For* loop for z from its minimum possible value up to 25.
- **iii** Rerun your program to generate the Pythagorean triples with hypotenuse length 25 or less. What improvements do you notice from the program you ran in part **a**?

c The algorithm can have its efficiency increased further by taking out the *For* loop for z (the hypotenuse) and instead calculating the value for z based on the x- and y-values and checking if z is an integer. The flow chart for this is shown below. *For* loops can be used to run through the values of x and y and in this flow chart, we are using x- and y-values up to 40.

Implement this flow chart in a new program. (The function *fPart* may be useful as it returns the fraction part of a number. For an integer you would get *fPart* of the number equals 0.)



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2.3 Euclid's algorithm for Pythagorean triples

Euclid's formula is well known and generates the Pythagorean triples. The formula produces all the primitive triples. A primitive triple (or base triad) is one such that a, b and c have no common divisor. All multiples of these triples will also be Pythagorean triples. For example, (3, 4, 5) is a primitive triple while (6, 8, 10) is a Pythagorean triple that is not primitive.

The formula says that for positive integers m, n with $m > n$:

$$a = m^2 - n^2 \qquad b = 2mn \qquad c = m^2 + n^2$$

a Show that $a^2 + b^2 = c^2$ using the formulas for a, b and c above.

To generate the primitive Pythagorean triples, it is required that $m - n$ is odd and that the greatest common divisor of m and n is 1.

b Design a flow chart to implement this algorithm and turn this into a program on your calculator. For $c \leqslant 100$ your program should generate 16 primitive Pythagorean triples. The calculator has a *gcd* function which returns the greatest common divisor of two numbers.

c *Extension.* Extend your program from part **b** to generate all the Pythagorean triples for $c \leqslant 100$.

Statue of Pythagoras on Samos Island.

Activity 3: Using simulations to find probabilities

STATISTICS AND PROBABILITY

In the past you may have seen some simple simulations carried out to determine probabilities. You would have seen that the more trials you run, the closer the experimental probability from the simulations will be to the theoretical probability. This is called the long run proportion.

Simulations can also be used to get an idea of the probability when the theoretical probability is not known or is difficult to find.

The cases below demonstrate some situations where the probability is not immediately obvious and where simulations can be used to see the event in action and the resulting probabilities.

3.1 Walk the plank

In the game of Walk the Plank, a numbered board represents the plank.

- Select a starting position somewhere along the plank.
- Toss a coin to determine whether you take one step forward (heads) or one step backwards (tails).
- Continue tossing the coin until you are either ‘Safe’ and back on board the ship or ‘Overboard’ and into the water.

a Play the game 20 times using a coin. Start from position 3 facing the deck. Count how many times you end up overboard and calculate the proportion of times you end up overboard.

b Play the above game once more. Think about the processes involved if you were going to write this game as an algorithm. Discuss your thoughts with a partner.

c Complete an algorithm flow chart to describe one runthrough of the game. Use the symbols used in **Activity 2**.

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d By considering your flow chart above, fill in the boxes in the program below to complete it. The coin toss is simulated by generating a random number 0 or 1 and assigning 0 as heads and 1 as tails. p is the variable which stores the position on the plank.

Using the TI-Nspire:

```
plankwalk()
                Safe!
                Done
plankwalk()
           Overboard!
                Done
```

```
"plankwalk" stored succ
Define plankwalk()=
Prgm
p:=3
While ☐
 c:=randInt(0,1)
 If c=0 Then
  p:=p+1
 Else
  ☐
 EndIf
EndWhile
If ☐ Then
 Disp "Overboard!"
EndIf
If ☐ Then
 Disp "Safe!"
EndIf
EndPrgm
```

Using the ClassPad:

```
3⇒p
While ☐
rand(0, 1)⇒c
If c=0
Then ☐
IfEnd
WhileEnd
If ☐
Then
Print "Overboard!"
IfEnd
If ☐
Then
Print "Safe!"
IfEnd
```

Output screen

```
Folder: main
Name: plank
Parameter:
```

```
Overboard!
Safe!
```

e Enter your program on your CAS calculator. The *While* loop can be found in the **Control** menu. It ensures the program keeps running while a condition is still being met.
Run your program 50 times and count the number of times overboard and hence calculate the proportion of times you end up overboard.

f Modify the above program to use a *For* loop to play the game multiple times and have a counter to count the number of times you end up overboard.

g Modify your program to cater for different start positions. For each start position, have 100 simulations of the game and record the number of times overboard in a table like the one below.

Start position, p	Overboard tally	Overboard frequency
1		
2		
3		
4		
5		

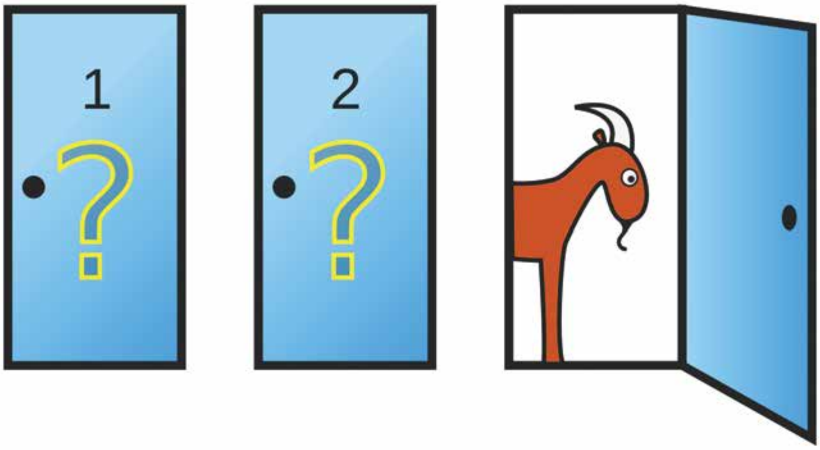
Construct a frequency histogram from your table. Comment on your results.

h *Extension*: Consider ways you could add extra elements to this game. Once you have some ideas, design the algorithm and implement the program to simulate your game.

3.2 The Monty Hall Problem

The Monty Hall Problem is a probability puzzle based on a game show and is named after its original host.

In the game show you have three doors with a car prize behind one door and a goat behind each of the other two doors.



The contestant selects a door. The host (who knows what is behind each door) then opens one of the other two doors, always opening a door with a goat behind it. The host then asks the contestant if they want to stay with the door they have chosen or switch to the other unopened door.

The question is: Is it to your advantage to switch from your original choice or stay? Or, does it matter at all?

a With a partner, run the game above 10 times each where one of you is the host and one the contestant.

- **i** Record in a frequency table (like the one below) the number of times staying with the same door would have won and the number of times switching would have won out of the 20 games.

	Stay with choice		**Switch choice**	
	Tally	Frequency	Tally	Frequency
Winning move				

- **ii** From your results, come up with a hypothesis to test whether you are better off staying with your choice or switching.
- **iii** Combine your results with two or three other of pairs. Does your hypothesis hold up?

b To carry out the simulation 100 or more times, we can write a program to do this. Think about the processes involved in the game above. Use this to design an algorithm flow chart that runs the simulation a number of times and counts and displays how many times staying wins and how many times switching wins. Use the symbols from **Activity 2**.

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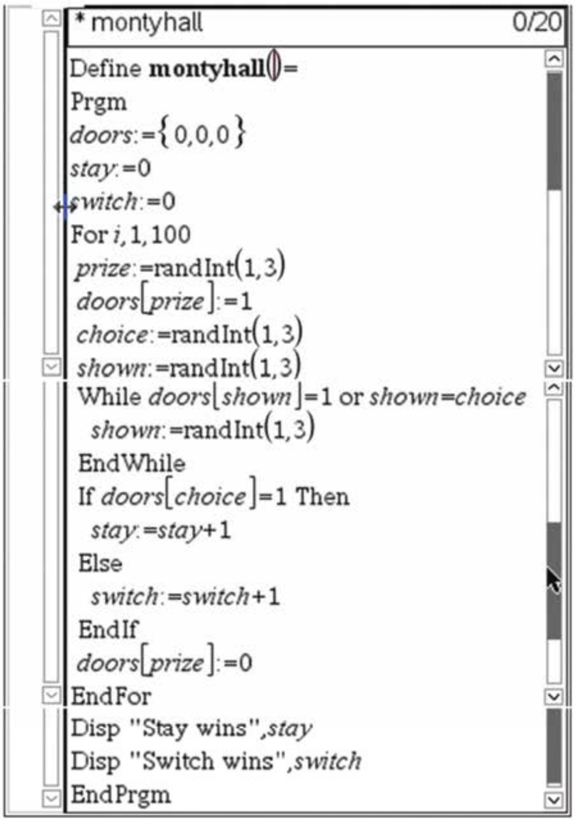
c A program for the Monty Hall simulation is shown below. The three doors are stored in a list. A 0 for a door represents a goat standing behind the door while a 1 indicates the prize is behind that door. Each door is initialised with a 0 in line 3 for the TI-Nspire and in the first line for the ClassPad.

Using the TI-Nspire:



```
Define montyhall()=
Prgm
doors:={0,0,0}
stay:=0
switch:=0
For i,1,100
 prize:=randInt(1,3)
 doors[prize]:=1
 choice:=randInt(1,3)
 shown:=randInt(1,3)
 While doors[shown]=1 or shown=choice
   shown:=randInt(1,3)
 EndWhile
 If doors[choice]=1 Then
   stay:=stay+1
 Else
   switch:=switch+1
 EndIf
 doors[prize]:=0
EndFor
Disp "Stay wins",stay
Disp "Switch wins",switch
EndPrgm
```

Using the ClassPad:

```
{0,0,0}⇒doors
0⇒stay
0⇒swit
For 1⇒j To 100
rand(1,3)⇒prize
1⇒doors[prize]
rand(1,3)⇒choice
rand(1,3)⇒shown
While doors[shown] =1 or sho
wn=choice
rand(1,3)⇒shown
WhileEnd
If doors[choice] =1
Then
stay+1⇒stay
Else
swit+1⇒swit
IfEnd
0⇒doors[prize]
Next
Print "Stay wins"
Print stay
stay+1⇒stay
Else
swit+1⇒swit
IfEnd
0⇒doors[prize]
Next
Print "Stay wins"
Print stay
Print "Switch wins"
Print swit
```

i How does it compare to the algorithm flow chart you prepared in part **b**?

ii Analyse the program and comment on the following:

- How many times does the simulation run?
- What is happening in line 8 for the TI-Nspire and in line 6 for the ClassPad?
- What is the purpose of the While loop (lines 11 and 12 for the TI-Nspire and lines 9–11 for the ClassPad) in this program?
- How are the stay and switch (swit) counters controlled?

iii Enter and run the program on a CAS calculator. You can alter the number of simulations to see how the results vary.

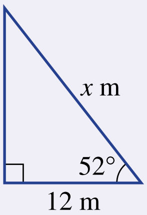
iv How do the results compare with part **a**? Does your hypothesis still hold? What would you now say in answer to the questions: Are you better off staying or switching? What is your chance of winning if you switch?

Trigonometry

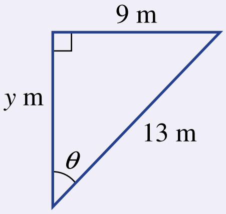
Short-answer questions

1 Find the value of the pronumeral in each right-angled triangle, correct to one decimal place.

a



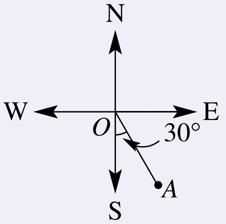
b



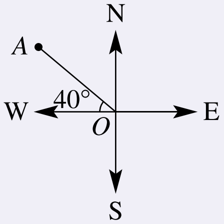
2 For the following bearings, give the true bearing of:

i A from O **ii** O from A

a



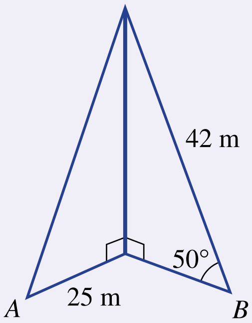
b



3 Two wires reach from the top of an antenna to points A and B on the ground, as shown. Point A is 25 m from the base of the antenna, and the wire from point B is 42 m long and makes an angle of 50° with the ground.

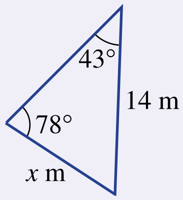
a Find the height of the antenna, to three decimal places.

b Find the angle the wire at point A makes with the ground, to one decimal place.

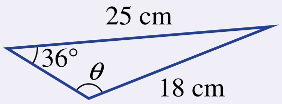


Ext **4** Find the value of the pronumeral, correct to one decimal place.

a



b θ is obtuse.



Ext **5** Find the largest angle, correct to one decimal place, in a triangle with side lengths 8 m, 12 m and 15 m.

Opt **6** **a** If $\theta = 223°$, state which of $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive?

b Choose the angle θ to complete each statement.

i $\sin 25° = \sin\theta$, where θ is obtuse.

ii $\tan 145° = -\tan\theta$, where θ is acute.

iii $\cos 318° = \cos\theta$, where θ is the reference angle.

c State the exact value of:

i $\cos 60°$ **ii** $\sin 135°$ **iii** $\tan 330°$ **iv** $\cos(-30°)$

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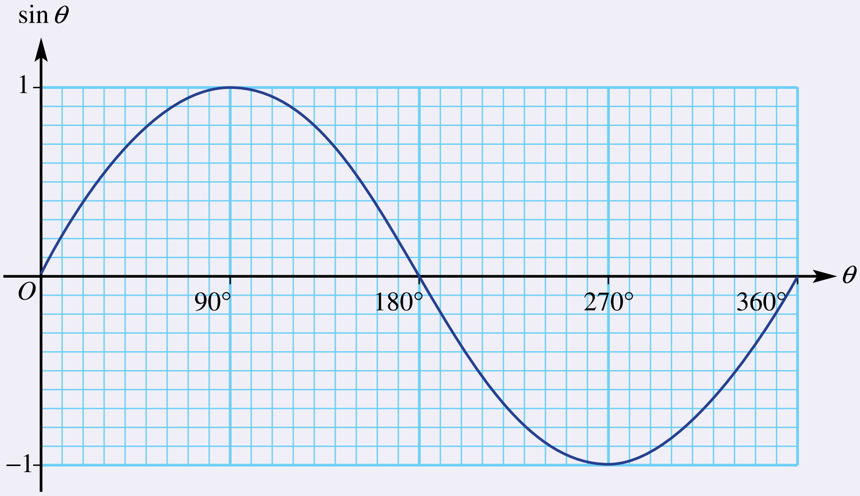
Opt **7** Use the graph of $\sin\theta$ shown to answer the following.



a Estimate the value of $\sin\theta$ for $\theta = 160°$.

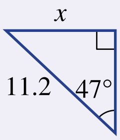
b Estimate the two values of θ for which $\sin\theta = -0.8$.

c Is $\sin 40° < \sin 120°$?

Multiple-choice questions

1 The value of x in the triangle shown is approximately:

A 7.6 **B** 12.0 **C** 10.4
D 6.5 **E** 8.2



2 A bird 18 m up in a tree spots a worm on the ground 12 m from the base of the tree. The angle of depression from the bird to the worm is closest to:

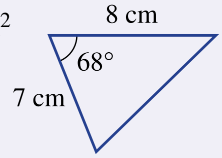
A 41.8° **B** 56.3° **C** 61.4°
D 33.7° **E** 48.2°

3 A walker travels due south for 10 km and then on a true bearing of 110° for 3 km. The total distance south from the starting point to the nearest kilometre is:

A 11 km **B** 1 km **C** 9 km
D 13 km **E** 15 km

Ext **4** The area of the triangle shown is closest to:

A 69 cm^2 **B** 52 cm^2 **C** 28 cm^2
D 26 cm^2 **E** 10 cm^2



Opt **5** Choose the *incorrect* statement.

A $\theta = 290°$ is in quadrant 4. **B** $\sin 120° = \frac{\sqrt{3}}{2}$
C $\cos 110° = -\cos 20°$ **D** $\tan\theta$ is positive for $200° < \theta < 250°$.
E $\sin 230° = -\sin 50°$

Extended-response question

A group of walkers set out on a trek to get to the base of a mountain range. The mountains have two peaks, which are 112 m and 86 m above ground level from the base. The angle of elevation from the peak of the smaller mountain to the peak of the taller mountain is 14°.

a Find the horizontal distance between the two mountain peaks, correct to one decimal place.

Ext

To get to the base of the mountain range, the walkers set out from the national park entrance on a bearing of 52°T for a distance of 13 km and then turn on a bearing of 340°T for the last 8 km of the trek.

b Draw a diagram representing the trek. Label all known measurements.

c If the walkers are able to trek directly from their start location to their endpoint, what distance would they cover? Round your answer to three decimal places.

d After they have explored the mountains, the group will be taken by bus back along the direct path from their end location to the park entrance. Determine the true bearing on which they will travel. Round your answer to the nearest degree.

Parabolas and rates of change

Short-answer questions

1 Sketch the following parabolas and state the transformations from $y = x^2$.

a $y = 3x^2$ **b** $y = -(x + 2)^2$ **c** $y = x^2 + 5$

2 Consider the quadratic $y = x^2 + 4x - 5$.

a Find the coordinates of the y-intercept.
b Find the coordinates of the x-intercepts by factorising.
c Use symmetry to find the turning point.
d Sketch the graph.

3 Consider the quadratic $y = -2(x - 3)^2 + 8$.

a State the coordinates of the turning point and whether it is a maximum or minimum.
b Find the coordinates of the y-intercept.
c Find the coordinates of the x-intercepts.
d Sketch the graph.

4 Sketch the following quadratics by first completing the square.

a $y = x^2 + 6x + 2$ Opt **b** $y = x^2 - 5x + 8$

Opt

5 Consider the quadratic $y = 2x^2 - 4x - 7$.

a Use the discriminant to determine the number of x-intercepts of the graph.
b Sketch its graph using the quadratic formula. Round x-intercepts to one decimal place.
c Find the points of intersection of the quadratic and the line $y = -6x - 3$ by solving simultaneously.

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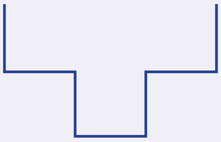
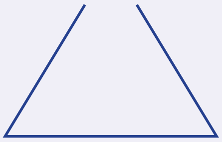
Opt **6** Water is being poured into these containers at a constant rate. Draw a graph showing the relationship between the height, h, of water in the container at time, t.

a

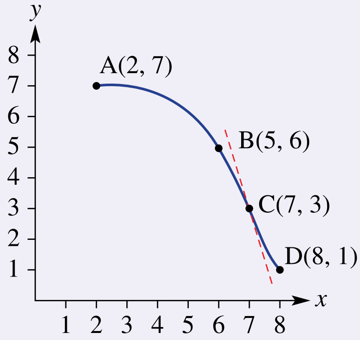


b



Opt **7** **a** Consider the relation with rule $y = x^2 + 2x + 3$. Find the average rate of change of y as x changes from 1 to 3.

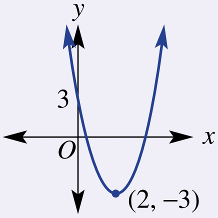
b The graphs shows a part of a curve with tangent at C to show the instantaneous rate of change at that point. Find the gradient of the line segments AC and BC and explain which one you think gives a better approximation of the instantaneous rate of change at point C.



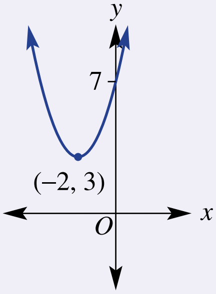
Multiple-choice questions

1 The graph of $y = (x - 2)^2 + 3$ could be:

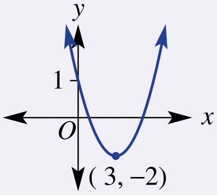
A



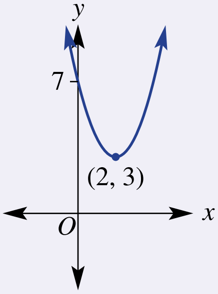
B



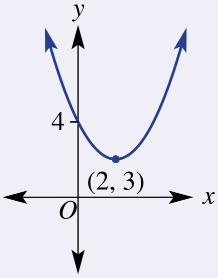
C



D



E



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2 The graph of $y = x^2 - 4x$ has a turning point with coordinates:

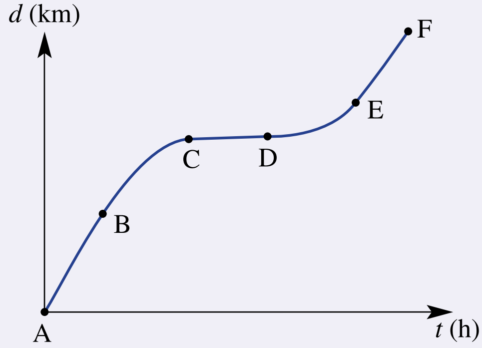
A $(2, -4)$ **B** $(0, -4)$ **C** $(4, 0)$ **D** $(-2, 12)$ **E** $(1, -3)$

Opt **3** For the quadratic $y = ax^2 + bx + c, b^2 - 4ac < 0$, we know that the graph has:

A a maximum turning point
B two x-intercepts
C no y-intercept
D no x-intercepts
E a minimum turning point

Opt **4** The distance–time graph shows a long distance journey by train over a number of hours. Between what two points does the graph show a speed which is decreasing?

A A to B
B B to C
C C to D
D D to E
E E to F



Ext **5** For which of the following rules are the variables in direct proportion?

A $y = 2x + 1$ **B** $y = x^2$ **C** $xy = 4$
D $y - 3x = 0$ **E** $y = 2^x$

Extended-response question

A rollercoaster has a section modelled by the equation $h = \frac{1}{40}(x^2 - 120x + 1100)$, where h is the height above the ground and x is the horizontal distance from the start of the section. All distances are measured in metres and x can take all values between 0 and 200 metres.

a Sketch the graph of h vs x for $0 \leqslant x \leqslant 200$, labelling the endpoints.

b What is the height above ground at the start of the section?

c The rollercoaster travels through an underground tunnel. At what positions from the start will it enter and leave the tunnel?

d What is the maximum height the rollercoaster reaches?

e What is the maximum depth the rollercoaster reaches?

Probability and counting techniques

Short-answer questions

1 Consider events A and B. Event A is the set of letters in the word 'grape' and event B is the set of letters in the word 'apricot':

$A = \{g, r, a, p, e\}$ $B = \{a, p, r, i, c, o, t\}$

a Represent the two events A and B in a Venn diagram, if the sample space is all the letters in the alphabet.

b If a letter is randomly selected from the alphabet, find:

i $\Pr(A)$ **ii** $\Pr(A \cap B)$ **iii** $\Pr(A \cup B)$ **iv** $\Pr(B')$

c Are the events A and B mutually exclusive? Why or why not?

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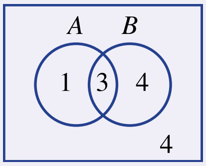
2 The Venn diagram shows the distribution of elements in two sets, A and B.



a Transfer the information in the Venn diagram to a two-way table.

b Find:

i $n(A \cap B)$ **ii** $n(A' \cap B)$ **iii** $n(B')$ **iv** $n(A \cup B)$

c Find:

i $\Pr(A \cap B)$ **ii** $\Pr(A \cap B')$ **iii** $\Pr(B)$ **iv** $\Pr(B \mid A)$

3 Two events, A and B, are such that $\Pr(A) = 0.24$, $\Pr(B) = 0.57$ and $\Pr(A \cup B) = 0.63$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A' \cap B')$

4 Two fair 4-sided dice numbered 1 to 4 are rolled and the total is noted.

a List the sample space as a table. **b** State the total number of outcomes.

c Find the probability of obtaining:

i a sum of 4

ii a sum of at least 5

iii a sum of 7, given the sum is at least 5.

5 In a group of 12 friends, 8 study German, 4 study German only and 2 study neither German nor Mandarin. Let A be the event 'studies German' and let B be the event 'studies Mandarin'.

a Summarise the information in a Venn diagram.

b Find:

i $\Pr(A)$ **ii** $\Pr(A \mid B)$

c State whether or not the events A and B are independent.

Opt **6** Evaluate:

a $4!$ **b** $\frac{7!}{4!}$ **c** ${}^6C_4$ **d** $\binom{8}{0}$

Opt **7** Six pieces of student artwork have been chosen to be displayed around the school.

a Three of the pieces will be hung in a row on a wall. How many arrangements are possible?

b If all six will be displayed in a row but two must be next to each other as they go together, how many arrangements are possible?

Opt **8** Three chocolate bars are to be selected from a choice of ten. Six of which are dark chocolate and four milk chocolate.

a How many possible selections are there?

b If 1 dark chocolate must be chosen and 2 milk chocolate bars, how many possible selections are there?

c What is the probability that one particular combination from part **b** is chosen?

Multiple-choice questions

1 The number of tails obtained from 100 tosses of two fair coins is shown in the table.

Number of tails	0	1	2
Frequency	23	57	20

From this table, the experimental probability of obtaining two tails is:

A 0.23 **B** 0.25 **C** 0.2 **D** 0.5 **E** 0.77

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2 From the given two-way table $\Pr(A \cap B')$ is:

	A	**A′**	
B	2		
B′			5
	6		12

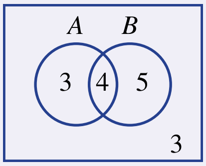
A $\frac{1}{2}$ **B** $\frac{2}{3}$ **C** $\frac{1}{4}$

D $\frac{4}{5}$ **E** $\frac{1}{3}$

3 Two events, A and B, are such that $\Pr(A) = 0.7$, $\Pr(B) = 0.4$ and $\Pr(A \cap B) = 0.3$. $\Pr(A \cup B)$ is equal to:

A 1.4 **B** 0.8 **C** 0.6 **D** 0 **E** 0.58

4 From the information in the Venn diagram, $\Pr(A|B)$ is:



A $\frac{5}{12}$ **B** $\frac{4}{5}$ **C** $\frac{4}{7}$

D $\frac{4}{9}$ **E** $\frac{1}{3}$

Opt **5** A restaurant has a 2 course menu with 5 entrees and 8 main courses. How many meal options are there if you must choose one of each course?

A 18 **B** 40 **C** 13 **D** 26 **E** 21

Extended-response question

Lindiana Jones selects two weights from her pocket to sit on a weight-sensitive trigger device after removing the goblet of fire. Her pocket contains three weights, each weighing 200 g, and five weights, each weighing 250 g. The two weights are selected randomly without replacement. Use a tree diagram to help answer the following.

a Find the probability that Lindiana selects two weights totalling:
  i 400 g **ii** 450 g **iii** 500 g

b If the total weight selected is less than 480 g, a poison dart will shoot from the wall. Find the probability that Lindiana is at risk from the poison dart.

c By feeling the weight of her selection, Lindiana knows that the total weight is more than 420 g. Given this information, what is the probability that the poison dart will be fired from the wall?

Statistics

Short-answer questions

1 Twenty people are surveyed to find out how many days in the past completed month they used public transport. The results are as follows.

7, 16, 22, 23, 28, 12, 18, 4, 0, 5, 8, 19, 20, 22, 14, 9, 21, 24, 11, 10

a Organise the data into a frequency table with class intervals of 5 and include a percentage frequency column.

b Construct a histogram for the data, showing both the frequency and the percentage frequency on the one graph.

c **i** State the frequency of people who used public transport on 10 or more days.
  ii State the percentage of people who used public transport on fewer than 15 days.
  iii State the most common interval of days for which public transport was used. Can you think of a reason for this?

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2 By first finding quartiles and checking for outliers, draw box plots for the following data sets.

a 8, 10, 2, 17, 6, 25, 12, 7, 12, 15, 4

b 5.7, 4.8, 5.3, 5.6, 6.2, 5.7, 5.8, 5.1, 2.6, 4.8, 5.7, 8.3, 7.1, 6.8

3 This two-way table summarises data collected from a survey using a 5-point Likert scale. The survey asked 25 sport fans if they thought watching their team live in person was better than watching on TV. Some of those surveyed were adults and some were children.

	Strongly agree	**Agree**	**Neutral**	**Disagree**	**Strongly disagree**	**Total**
Adult	4	3	2	6	0	15
Child	5	2	1	2	0	10
Total	9	5	3	8	0	25

a State how many of those surveyed were children.

b State what percentage of those surveyed were adults and responded *Agree* to the question?

c Would you say this data supports the notion that compared to children, adults feel it is just as good to watch their team on television? Give a reason.

4 Farsan's bank balance over 12 months is recorded below.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Balance ($)	1500	2100	2300	2500	2200	1500	1200	1600	2000	2200	1700	2000

a Plot the time-series for the 12 months.

b Describe the way in which the bank balance has changed over the 12 months.

c Between which consecutive months did the biggest change in the bank balance occur?

d What is the overall change in the bank balance over the year?

5 Consider the variables x and y and the corresponding bivariate data below.

x	4	5	6	7	8	9	10
y	2.1	2.5	3.1	2.8	4	3.6	4.9

a Draw a scatter plot for the data.

b Describe the correlation between x and y as either positive, negative or none.

c Fit a line of good fit by eye to the data on the scatter plot.

d Use your line of good fit to estimate:

- **i** y when $x = 7.5$
- **ii** x when $y = 5.5$

6 The back-to-back stem-and-leaf plot below shows the number of fantasy books owned by people in two different age groups.

40 and over Leaf	**Stem**	**Under 40 Leaf**
9 7 6 6 4 3 2	0	7 8
6 4 3 2 2 0	1	2 5 5 7 8
8 3	2	4 4 6
	3	2 6 9
	4	1 8

2 | 4 means 24

a By considering the centre and spread of the data, state with reasons:

- **i** which data set will have the higher mean?
- **ii** which data set will have the smaller standard deviation?

b Calculate the mean and sample standard deviation for each data set. Round your answers to one decimal place where necessary.

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Multiple-choice questions

1 For the given stem-and-leaf plot, the range and median, respectively, of the data are:

Stem	Leaf
0	2 2 6 7
1	0 1 2 3 5 8
2	3 3 5 7 9

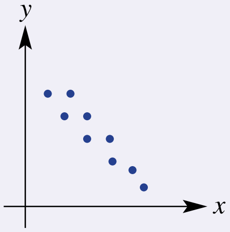
1 | 5 means 15

A 20, 12.5 **B** 7, 12 **C** 27, 12.5 **D** 29, 3 **E** 27, 13

2 The interquartile range (IQR) for the data set 2, 3, 3, 7, 8, 8, 10, 13, 15 is:

A 5 **B** 8.5 **C** 7 **D** 13 **E** 8

3 The best description of the correlation between the variables for the scatter plot shown is:



- **A** weak, negative
- **B** strong, positive
- **C** strong, negative
- **D** weak, positive
- **E** no correlation

4 A line of best fit passes through the points (10, 8) and (15, 18). The equation of the line is:

A $y = \frac{2}{3}x + 8$ **B** $y = 2x - 12$ **C** $y = -\frac{1}{2}x + 13$

D $y = 2x + 6$ **E** $y = \frac{1}{2}x + 3$

Opt

5 The mean and sample standard deviation of the small data set 2, 6, 7, 10 and 12, correct to one decimal place, are:

A $\bar{x} = 7.4$ and $s = 3.8$ **B** $\bar{x} = 7$ and $s = 3.7$ **C** $\bar{x} = 7.4$ and $s = 3.4$

D $\bar{x} = 7$ and $s = 7.7$ **E** $\bar{x} = 27.1$ and $s = 9.9$

Extended-response question

This frequency table shows information on flight delays in minutes of 30 aircraft on a particular day.

Delay (mins)	Frequency
0–	5
15–	6
30–	8
45–	5
60–	4
75–90	2

a Construct a percentage cumulative frequency curve for the data by first adding a cumulative frequency column to the table.

b Use the curve to approximate:

- **i** the 70th percentile
- **ii** the 50th percentile and interpret the result.

c Estimate the percentage cumulative frequency corresponding to a 40-minute flight delay.

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Polynomials, functions and graphs

Ext Short-answer questions

1 Consider the function $f(x) = (x - 2)^2 + 3$.

a Explain why $f(x)$ is a function.

b Evaluate:

i $f(2)$ **ii** $f(-1)$ **iii** $f(a)$

c Give the allowable x-values (domain) and resulting y-values (range) of this function.

2 Consider the polynomials $P(x) = x^3 + 3x^2 - 4x - 6$ and $Q(x) = 2x^3 - 3x - 4$.

a Find:

i $P(2)$ **ii** $P(-1)$ **iii** $Q(-3)$ **iv** $Q(1)$

b Expand and simplify.

i $P(x) \times Q(x)$ **ii** $(Q(x))^2$

3 Divide $P(x) = x^3 - 4x^2 + 2x + 7$ by $(x - 3)$ and write in the form $P(x) = (x - 3)\,Q(x) + R$, where $Q(x)$ is the quotient and R is the remainder.

4 Find the remainder when $P(x) = x^3 - 2x^2 - 13x - 10$ is divided by each of the following and, hence, state if it is a factor.

a $x - 1$ **b** $x + 2$ **c** $x - 3$

5 Solve for x.

a $(x + 1)(x - 3)(x + 6) = 0$ **b** $-x(2x - 5)(3x + 2) = 0$

c $x^3 + 5x^2 + 2x - 8 = 0$ **d** $2x^3 - 3x^2 - 3x + 2 = 0$

6 Sketch the following graphs, labelling key features.

a $x^2 + y^2 = 4$ **b** $x^2 + y^2 = 10$ **c** $y = \frac{2}{x}$ **d** $y = -\frac{6}{x}$

7 Find the coordinates of the points of intersection of the graphs of the following.

a $x^2 + y^2 = 15$ and $y = 2x$ **b** $y = \frac{2}{x}$ and $y = 8x$

8 Sketch the graphs of the following relations. Label important features.

a $(x - 2)^2 + (y + 1)^2 = 16$ **b** $y = 2^{x+3} + 1$ **c** $y = \frac{1}{x + 2} - 3$

Ext Multiple-choice questions

1 The expression that is *not* a polynomial is:

A $3x^2 + 1$ **B** $2 - 5x^5 + x$ **C** $7x - 5$

D $4x^3 + 2x - \frac{1}{x}$ **E** $5x^6 + 2x^4 - 3x^2 - x$

2 When $P(x) = x^4 - 3x^3 + 2x + 1$ is divided by $(x - 2)$, the remainder is:

A 37 **B** 2 **C** -3 **D** 13 **E** 5

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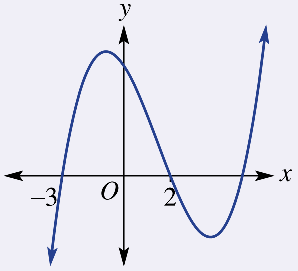
3 A possible graph of $y = -x(x + 3)(x - 2)$ is:

A



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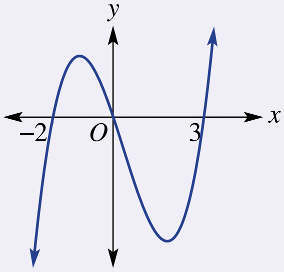
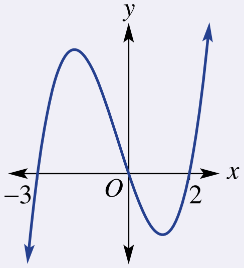
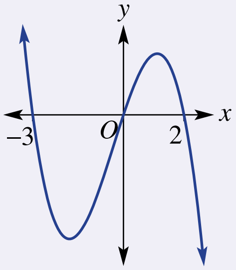


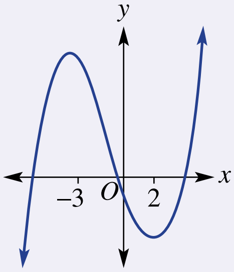
C



D



E



4 The graph with equation $x^2 + y^2 = 9$ is:

A a circle with radius 9

B a parabola with turning point (0, 9)

C a circle with radius 3

D a hyperbola with asymptote at $x = 3$

E an exponential curve with y-intercept (0, 3)

5 The equations of the asymptotes of $y = \frac{1}{x} + 3$ are:

A $x = 0, y = 0$

B $x = 0, y = 3$

C $x = 3, y = 0$

D $x = 0, y = -3$

E $x = -3, y = 3$

Ext

Extended-response question

A section of a train track that heads through a valley and then over a mountain is modelled by the equation $P(x) = -2x^3 + 3x^2 + 23x - 12$ for $-5 \leqslant x \leqslant 6$.

a Show that $(x + 3)$ is a factor of $P(x)$.

b Hence, factorise $P(x)$ using division.

c Sketch a graph of this section of the track, labelling axes intercepts and endpoints.

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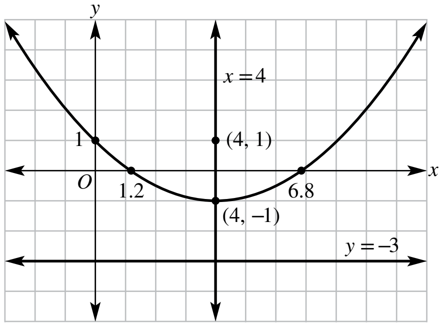
Working with unfamiliar problems: Part 1

1 $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$
2 99
3 a $x = 15$ b $x = 450$ c $x = 6$
4 $\frac{1}{8}$, \$56
5 24 cm
6 a i 11.8 seconds ii 6.5 seconds
b $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$
7 $XY = 5.6$ cm
8 35
9 72
10 Charlie 23 years, Bob 68 years
11 $\frac{D}{2}$
12 $b = 1\frac{1}{3}$
13 $k = 11$
14 $V = 27\text{ cm}^3$, TSA $= 54\text{ cm}^2$
15 4 cm < third side < 20 cm. Its length is between the addition and subtraction of the other two sides.
16 785
17 $n + 1$
18 10
19 3 : 5

Working with unfamiliar problems: Part 2

1 a i $P = 3\times 4^5\times\frac{x}{3^5}$ or $P = 3x\left(\frac{4}{3}\right)^5$;
$P = 3\times 4^n\times\frac{x}{3^n}$
ii $A = \frac{\sqrt{3}}{4}x^2 + 3\times\frac{\sqrt{3}}{4}\left(\frac{x}{3}\right)^2 + 3\times 4\times\frac{\sqrt{3}}{4}\left(\frac{x}{3^2}\right)^2 + 3\times 4^2\times\frac{\sqrt{3}}{4}\left(\frac{x}{3^3}\right)^2$
Area change $= 3\times 4^{n-1}\times\frac{\sqrt{3}}{4}\left(\frac{x}{3^n}\right)^2$
b The perimeter increases indefinitely as $3x\left(\frac{4}{3}\right)^n \to \infty$ as $n \to \infty$. The area approaches a finite value as area change $\frac{\sqrt{3}}{4}x^2\times\frac{3}{4}\left(\frac{4}{9}\right)^n \to 0$ as $n\to\infty$.
2 77 cm, 181 cm
3 $2k(2\sqrt{3} - 3)$
4 22°
5 $y = 3\frac{1}{4}$
6 $x = 0.9$, $y = 3.3$
7 $16\sqrt{10}$ cm
8 20 students; 6 with 100%, 7 with 75%, 7 with 76%, mean = 82.85%.
9 $9\frac{1}{8}$ units
10 a 11 b 28 c 20 d $\frac{3}{2}$
11 640
12 P to R: 145°, 1606 m; R to S: 295°, 789 m; S to Q: 051°, 1542 m; Q to P: 270°, 1400 m
13 $y = \frac{1}{8}x^2 - x + 1$ or $y = \frac{1}{8}(x - 4)^2 - 1$



14 a

Hours	1st and 6th	2nd and 5th	3rd and 4th
% change from equation	5.8%	16.0%	25.9%
% change from 'rule of thumb'	8.3%	16.7%	25%

The percentage change per hour for the 'rule of thumb' is 2.5 points higher for the 1st and 6th hours, 0.7 points higher in the 2nd and 5th hours and 0.9 points lower in the middle two hours. Overall, this is quite an accurate 'rule of thumb'.
b The proportion of tide height change is $\cos(30t_1) - \cos(30t_2)$
15 $\frac{1 + \sqrt{5}}{2}$, 1.618034

Chapter 1

1A

Building understanding

1 C
2 D
3 a 1 b -4 c $-\frac{1}{5}$ d $-\frac{7}{3}$
4 a Yes b Yes c No
5 a 9 b -8 c -8 d -9

Now you try

Example 1
a $17a$ b $3ab^2$ c $2xy + 6x^2y$
Example 2
a $18ab$ b $-10x^2y$ c $-\frac{b}{2}$
Example 3
a $3x + 6$ b $-2x^2 + 2xy$ c $-4x + 9$
Example 4
a $2(x - 5)$ b $3x(x + 3)$
Example 5
13

Exercise 1A

1 **a** $10a$ **b** $15d$ **c** 0
d $5xy$ **e** $4ab$ **f** $9t$
g $2a^2b$ **h** x^2y **i** $-st^2$
j $-3m^2n$ **k** $-0.7a^2b$ **l** $-1.8ab^2$
m $2gh+5$ **n** $12xy-3y$ **o** $3a+7b$
p $8jk-7j$ **q** ab^2+10a^2b **r** $2mn-m^2n$

2 **a** $12ab$ **b** $25ab$ **c** $-6ad$
d $-10hm$ **e** $30ht$ **f** $30bl$
g $12s^2t$ **h** $-21b^2d^5$ **i** $8a^2b$
j $24p^2q$ **k** $-15ab^2$ **l** $63m^2pr$
m x **n** $3ab$ **o** $-\frac{a}{3}$
p $-\frac{ab}{4}$ **q** $2b$ **r** $-3x$
s $-\frac{y}{2}$ **t** $-\frac{a}{2}$

3 **a** $5x+5$ **b** $2x+8$
c $3x-15$ **d** $-20-5b$
e $-2y+6$ **f** $-7a-7c$
g $6m+18$ **h** $4m-12n+20$
i $-2p+6q+4$ **j** $2x^2+10x$
k $6a^2-24a$ **l** $-12x^2+16xy$
m $15y^2+3yz-24y$ **n** $36g-18g^2-45gh$
o $-8ab+14a^2-20a$ **p** $14y^2-14y^3-28y$
q $-6a^3+3a^2+3a$ **r** $-5t^4-6t^3-2t$
s $6m^4-2m^3+10m^2$ **t** x^4-x
u $3s^4-6st$

4 **a** $5x+23$ **b** $10a+26$
c $21y+3$ **d** $15m+6$
e 10 **f** $11t-1$
g $3x^2+15x$ **h** $15z-7$

5 **a** $3(x-3)$ **b** $4(x-2)$
c $10(y+2)$ **d** $6(y+5)$
e $x(x+7)$ **f** $2a(a+4)$
g $5x(x-1)$ **h** $9y(y-7)$
i $xy(1-y)$ **j** $x^2y(1-4y)$
k $8a^2(b+5)$ **l** $ab(7a+1)$
m $-5t(t+1)$ **n** $-6mn(1+3n)$
o $-y(y+8z)$

6 **a** -32 **b** 7 **c** 61 **d** 12
e $-\frac{1}{2}$ **f** $\frac{13}{5}$ **g** $-\frac{7}{5}$ **h** 1

7 **a** $2x^2+6x$ **b** x^2-5x

8 **a** $P=4x-4, A=x^2-2x-4$
b $P=4x+2, A=3x-1$
c $P=4x+14, A=7x+12$

9 **a** $(-2)(-2)=4$, negative signs cancel
b $a^2>0 \therefore -a^2<0$
c $(-2)^3=(-2)(-2)(-2)=-8$

10 **a** True
b False, $1-2 \neq 2-1$
c True
d False, $\frac{1}{2} \neq \frac{2}{1}$
e True
f False, $3-(2-1) \neq (3-2)-1$
g True
h False, $8 \div (4 \div 2) \neq (8 \div 4) \div 2$

11 **a** $\frac{x+y}{2}$ or $x+\frac{y}{2}$
b It could refer to either of the above, depending on interpretation.
c 'Half of the sum of a and b' or 'a plus b all divided by 2'.

12 **a** $P=\left(4+\frac{\pi}{2}\right)x+2, A=\left(1+\frac{\pi}{4}\right)x^2+x$
b $P=\left(6+\frac{\pi}{2}\right)x-6, A=\left(3-\frac{\pi}{4}\right)x^2-3x$
c $P=2\pi x, A=\left(1+\frac{\pi}{2}\right)x^2$

1B

Building understanding

1 **a** No **b** No **c** Yes **d** Yes
2 **a** True **b** False **c** False
3 **a** False **b** True **c** True
4 **a** $x=5$ **b** $x=8$ **c** $x=-3$ **d** $x=4$
5 **a** $\frac{5}{6}$ **b** $\frac{17}{15}$ **c** $\frac{5}{14}$ **d** $\frac{17}{6}$

Now you try

Example 6
a $x=3$ **b** $x=-\frac{2}{3}$

Example 7
a $x=11$ **b** $x=-20$ **c** $x=-\frac{1}{2}$

Example 8
a $\frac{13a}{24}$ **b** $a=\frac{48}{13}$

Exercise 1B

1 **a** $x=3$ **b** $x=3$ **c** $x=-\frac{1}{3}$
d $x=-\frac{11}{6}$ **e** $x=-4$ **f** $x=\frac{3}{2}$
g $x=-\frac{9}{2}$ **h** $x=-\frac{4}{3}$ **i** $x=-2$
j $x=7$ **k** $x=-2$ **l** $x=\frac{11}{9}$

2 **a** $x=1$ **b** $x=9$ **c** $x=\frac{23}{2}$
d $x=-\frac{5}{6}$ **e** $x=-\frac{9}{11}$ **f** $x=\frac{2}{3}$
g $x=1$ **h** $x=2$ **i** $x=4$
j $x=7$ **k** $x=-9$ **l** $x=5$
m $x=19$ **n** $x=23$ **o** $x=1$

3 **a** $x=10$ **b** $x=13$ **c** $x=-22$
d $x=4$ **e** $x=-5$ **f** $x=6$
g $x=16$ **h** $x=4$ **i** $x=-9$
j $x=8$ **k** $x=6$ **l** $x=-7$

4 **a** $x=-\frac{1}{4}$ **b** $x=\frac{7}{5}$ **c** $x=\frac{4}{3}$
d $x=-17$ **e** $x=20$ **f** $x=15$
g $x=-9$ **h** $x=5$ **i** $x=-11$

5 **a** $x+3=7, x=4$ **b** $x+8=5, x=-3$
c $x-4=5, x=9$ **d** $15-x=22, x=-7$
e $2x+5=13, x=4$ **f** $2(x-5)=-15, x=-\frac{5}{2}$
g $3x+8=23, x=5$ **h** $2x-5=x-3, x=2$

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6 **a** **i** $\frac{7x}{12}$ **ii** 12

b **i** $\frac{17x}{30}$ **ii** $\frac{60}{17}$

c **i** $\frac{x}{8}$ **ii** 40

d **i** 6 **ii** −15 **iii** 20

7 **a** $a = 1$ **b** $u = 6$ **c** $b = 2$ **d** $r = 25$

8 **a** \$214 **b** \$582

c **i** 1 **ii** 10.5 **iii** 21

9 17 cm

10 17 and 18

11 24 km

12 **a** 41 L

b 90 s = 1 min 30 s

c 250 s = 4 min 10 s

13 $x = 9$. Method 2 is better, expanding the brackets is unnecessary, given 2 is a factor of 8.

14 **a** $x = 5 - a$ **b** $x = \frac{a}{6}$ **c** $x = \frac{5}{a}$

d $x = \frac{2a + 1}{a}$ **e** $x = \frac{3a + 1}{a}$ **f** $x = \frac{c - b}{a}$

15 **a** $a = \frac{c}{b + 1}$ **b** $a = \frac{b}{b + 1}$ **c** $a = \frac{1}{c - b}$

d $a = \frac{b}{b - 1}$ **e** $a = -b$ **f** $a = \frac{bc}{b - c}$

16 **a** $6a$ **b** $\frac{ab}{a + b}$ **c** $\frac{abc}{b - a}$

1C

Building understanding

1 **a** 3, 6, 10 (Answers may vary.)

b −4, −3, −2 (Answers may vary.)

c 5, 6, 7 (Answers may vary.)

d −8.5, −8.4, −8.3 (Answers may vary.)

2 **a** B **b** C **c** A

3 11, 12 or 13 rabbits

4 **a** > **b** <

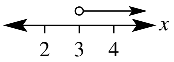
Now you try

Example 9

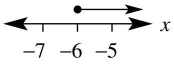
a $x < 2$ **b** $-4 < x \leqslant -1$

Example 10

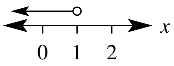
a $x > 3$



b $x \geqslant -6$



c $x < 1$



Exercise 1C

1 **a** $x \geqslant 1$ **b** $x < 7$

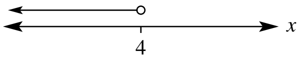
c $x \leqslant 4$ **d** $x > -9$

e $-2 < x \leqslant 1$ **f** $8 < x \leqslant 11$

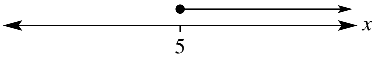
g $-9 < x < -7$ **h** $1.5 \leqslant x \leqslant 2.5$

i $-1 \leqslant x < 1$

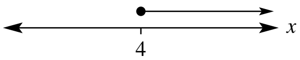
2 **a** $x < 4$



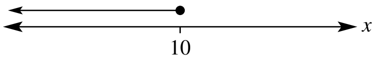
b $x \geqslant 5$



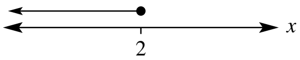
c $x \geqslant 4$



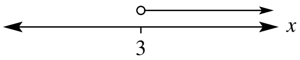
d $x \leqslant 10$



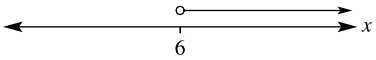
e $x \leqslant 2$



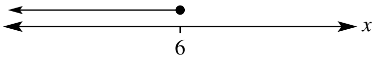
f $x > 3$



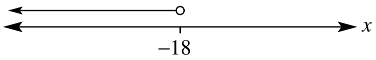
g $x > 6$



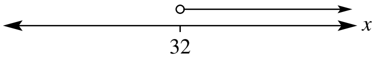
h $x \leqslant 6$



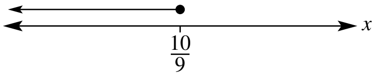
i $x < -18$



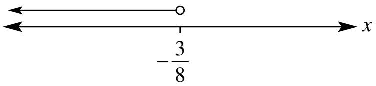
j $x > 32$



k $x \leqslant \frac{10}{9}$



l $x < -\frac{3}{8}$



3 **a** $x \geqslant -1$ **b** $x < 2$ **c** $x \leqslant -5$

d $x \leqslant -7$ **e** $x < -8$ **f** $x \geqslant 4$

g $x \geqslant -10$ **h** $x < -21$

4 **a** $x > 6$ **b** $x \leqslant 2$ **c** $x < \frac{5}{2}$

d $x \geqslant 10$ **e** $x \leqslant \frac{1}{16}$ **f** $x < \frac{11}{4}$

5 **a** $2x + 7 < 12, x < \frac{5}{2}$

b $4 - \frac{x}{2} \geqslant -2, x \leqslant 12$

c $3(x + 1) \geqslant 2, x \geqslant -\frac{1}{3}$

d $x + (x + 2) \leqslant 24, x \leqslant 10$ since x must be even

e $(x - 6) + (x - 4) + (x - 2) + x \leqslant 148, x \leqslant 40$

6 **a** **i** $C < \$1.30$ **ii** $C > \$2.30$

b **i** Less than 9 min **ii** 16 min or more

7 **a** $x < -6$ **b** $x > \frac{16}{3}$

c $x < \frac{48}{7}$ **d** $x > 0$

8 **a** An infinite number of whole numbers (all the ones greater than 8).

b 1, 3 is the only whole number.

9 **a** $x \geqslant \frac{a+3}{10}$

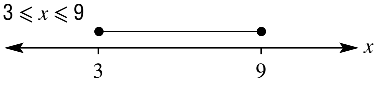
b $x < 2 - 4a$ if $a > 0$ and $x > 2 - 4a$ if $a < 0$

c $x < 1 - \frac{7}{a}$ or $x < \frac{a-7}{a}$ if $a > 0$ and $x > 1 - \frac{7}{a}$ or $x > \frac{a-7}{a}$ if $a < 0$.

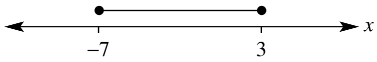
10 **a** $-4 \leqslant x < 5$ **b** $-9.5 < x \leqslant -7$

c $x = 10$

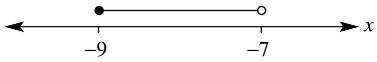
11 **a** $3 \leqslant x \leqslant 9$



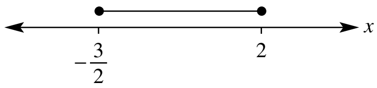
b $-7 \leqslant x \leqslant 3$



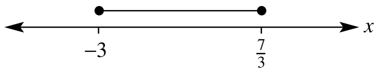
c $-9 \leqslant x < -7$



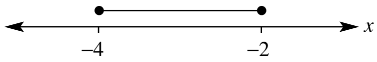
d $-\frac{3}{2} \leqslant x \leqslant 2$



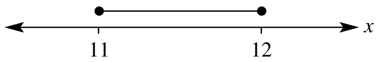
e $-3 \leqslant x \leqslant \frac{7}{3}$



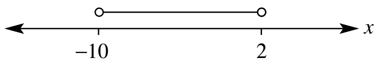
f $-4 \leqslant x \leqslant -2$



g $11 \leqslant x \leqslant 12$



h $-10 < x < 2$



12 **a** $-4 < x \leqslant -2$ **b** $-4 \leq x \leqslant 5$

c $0 < x < 5$ **d** $-3 < x < 2$

e $8 < x < 12$ **f** $\frac{5}{3} \leqslant x \leqslant 5$

1D

Building understanding

1 **a** $2x - 4$ **b** $-x - 6$

c $-6x + 12$ **d** $2x - 1$

2 **a** 12 **b** 6 **c** 14 **d** $2x$

3 **a** $x = 10$ **b** $x = 8$ **c** $x = \frac{11}{2}$

4 **a** $x + 1 = \frac{2(x+5)}{3}$

b $3(x + 1) = 2(x + 5)$

c $x = 7$

Now you try

Example 11

a $\frac{5x-4}{6}$ **b** $\frac{13x-2}{10}$

Example 12

a $x = -1$ **b** $x = -7$

Exercise 1D

1 **a** $\frac{9x+23}{20}$ **b** $\frac{7x+11}{12}$ **c** $\frac{3x+1}{4}$

d $\frac{4x+9}{9}$ **e** $\frac{8x-1}{6}$ **f** $\frac{8x+3}{10}$

g $\frac{7x+2}{24}$ **h** $\frac{5x-1}{5}$ **i** $\frac{x+1}{14}$

2 **a** $\frac{x+5}{6}$ **b** $\frac{6x+5}{12}$

c $\frac{-2x+38}{15}$ **d** $\frac{3x-23}{14}$

e $\frac{14x-8}{21}$ **f** $\frac{18x-9}{6} = \frac{6x-3}{2}$

g $\frac{x+14}{30}$ **h** $\frac{-14x-7}{15}$

i $\frac{-3x+10}{4}$

3 **a** 1 **b** 0 **c** −17

d $\frac{7}{2}$ **e** $\frac{27}{23}$ **f** $\frac{28}{5}$

g $\frac{13}{14}$ **h** $\frac{2}{5}$ **i** $\frac{86}{19}$

4 **a** 3 **b** 6 **c** 1

d −26 **e** −10 **f** $\frac{-10}{11}$

5 Matthew 6 km, Zoe 10 km

6 **a** $x > \frac{19}{5}$ **b** $x > 19$

7 **a** $\frac{-45}{29}$ **b** $\frac{-2}{5}$ **c** $\frac{23}{34}$ **d** $\frac{-98}{9}$

8 The 2 in the second numerator needs to be subtracted, $\frac{x-2}{6}$.

9 **a** $-(3 - 2x) = -3 + 2x \; (-1 \times (-2x) = 2x)$

b **i** $\frac{2}{x-1}$ **ii** $\frac{2x}{3-x}$ **iii** $\frac{x+3}{7-x}$

10 **a** $\frac{7x+22}{(x+1)(x+4)}$ **b** $\frac{7x-13}{(x-7)(x+2)}$

c $\frac{3x-1}{(x-3)(x+5)}$ **d** $\frac{x-18}{(x+3)(x-4)}$

e $\frac{-21}{(2x-1)(x-4)}$ **f** $\frac{14x-26}{(x-5)(3x-4)}$

g $\frac{41-7x}{(2x-1)(x+7)}$ **h** $\frac{3x+17}{(x-3)(3x+4)}$

i $\frac{14-17x}{(3x-2)(1-x)}$

11 **a** 2 **b** 1

1E

Building understanding

1 a $y = -2x + 5$, $m = -2$, $c = 5$
  b $y = 2x - 3$, $m = 2$, $c = -3$
  c $y = x - 7$, $m = 1$, $c = -7$
  d $y = -\frac{2x}{5} - \frac{3}{5}$, $m = -\frac{2}{5}$, $c = -\frac{3}{5}$

2 a i 3 ii 6 iii $\frac{21}{2}$
  b i 2 ii 6 iii $\frac{8}{3}$

3 a A b D c B
  d C e E f F

4 a $x = 2$ b $y = 2$

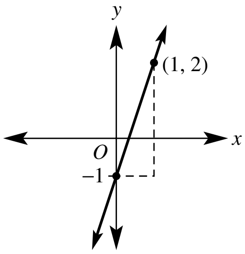
Now you try

Example 13

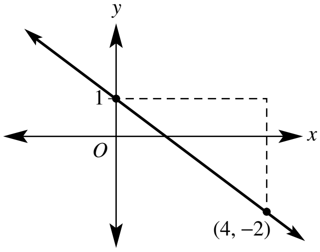
a Yes b No

Example 14

a Gradient = 3, y-intercept = −1

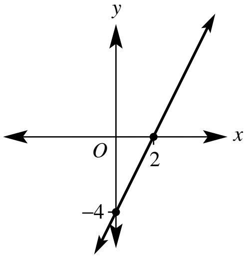


b Gradient = $-\frac{3}{4}$, y-intercept = 1

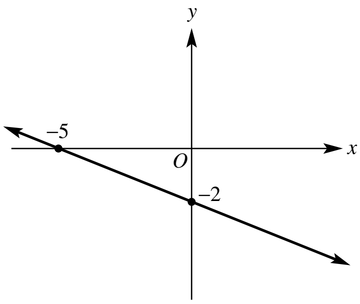


Example 15

a

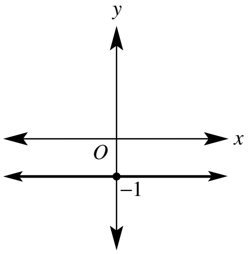


b

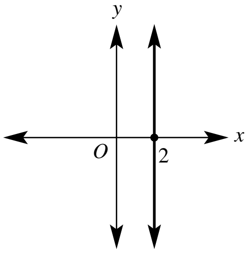


Example 16

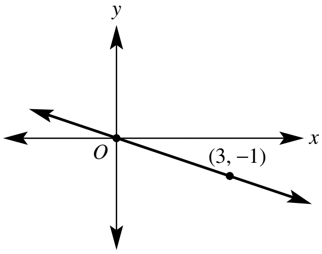
a



b



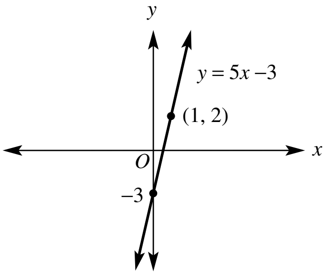
c



Exercise 1E

1 a i No ii No iii Yes
  b i No ii Yes iii Yes

2 a $m = 5$, $c = -3$



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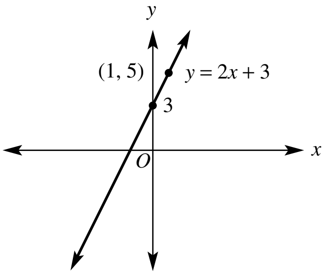
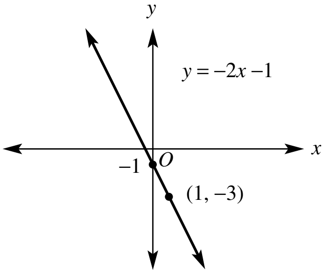
Cambridge University Press

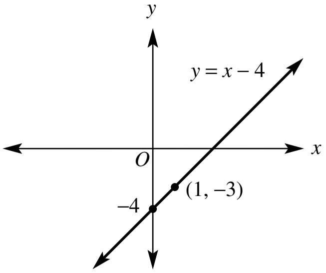
b $m = 2, c = 3$



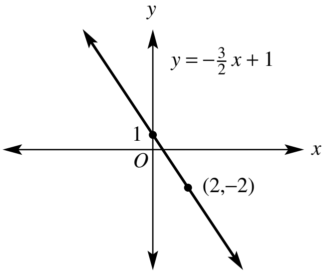
c $m = -2, c = -1$



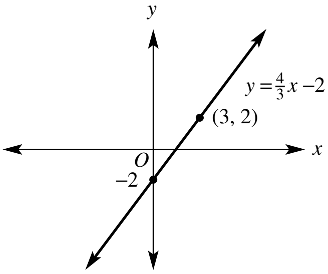
d $m = 1, c = -4$



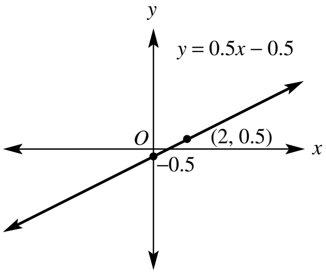
e $m = -\frac{3}{2}, c = 1$



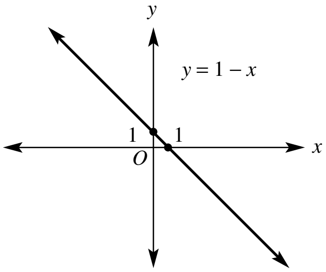
f $m = \frac{4}{3}, c = -2$



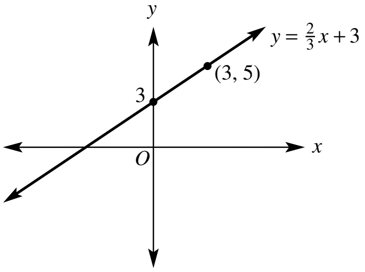
g $m = 0.5, c = -0.5$



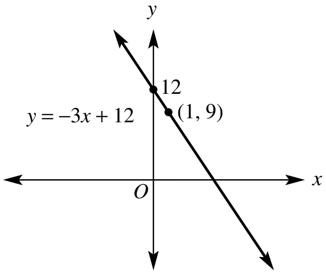
h $m = -1, c = 1$



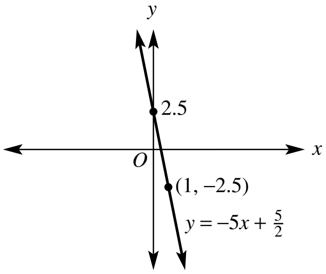
i $m = \frac{2}{3}, c = 3$



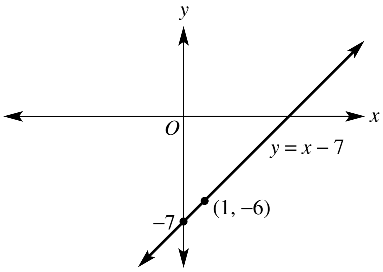
3 **a** $m = -3, c = 12$



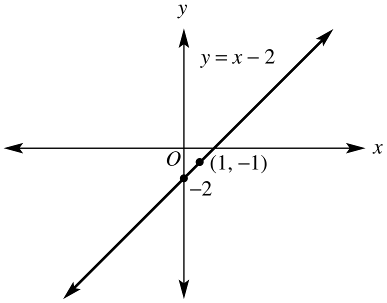
b $m = -5, c = \frac{5}{2}$



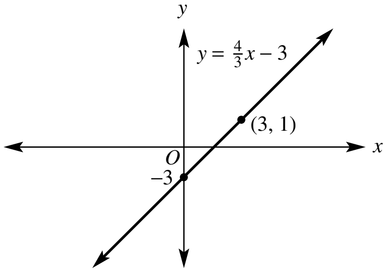
c $m = 1, c = -7$



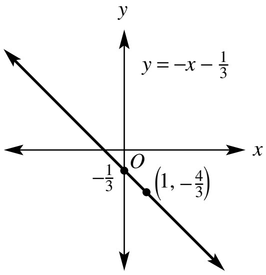
d $m = 1, c = -2$



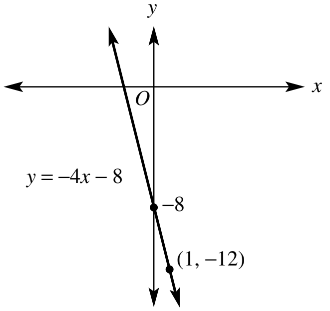
e $m = \frac{4}{3}, c = -3$



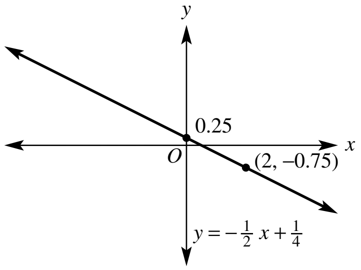
f $m = -1, c = -\frac{1}{3}$



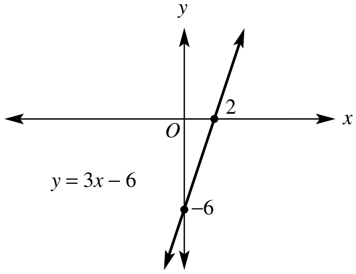
g $m = -4, c = -8$



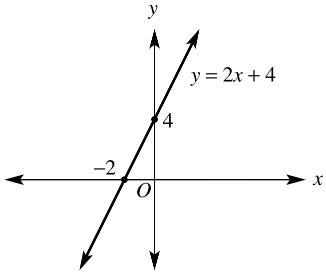
h $m = -\frac{1}{2}, c = \frac{1}{4}$



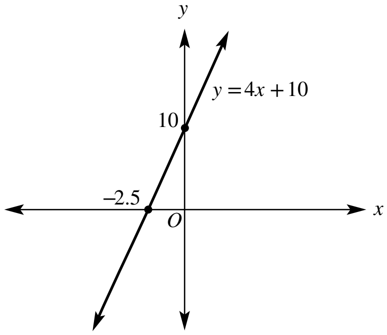
4 a $x = 2, y = -6$



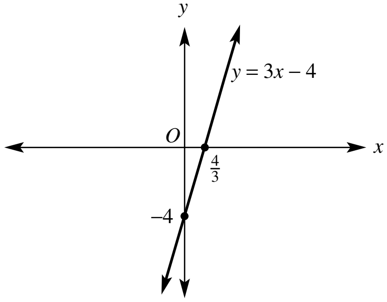
b $x = -2, y = 4$



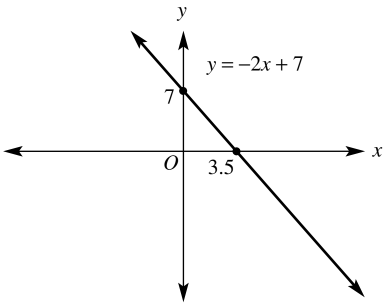
c $x = -2.5, y = 10$



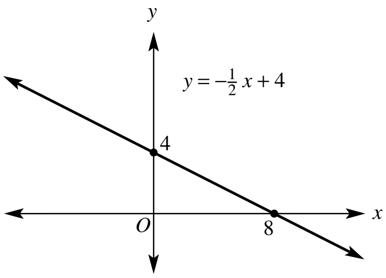
d $x = \frac{4}{3}, y = -4$



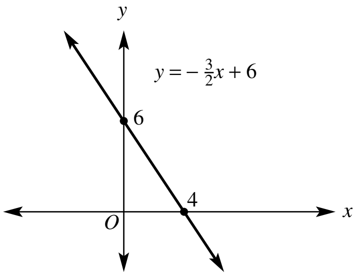
e $x = 3.5, y = 7$



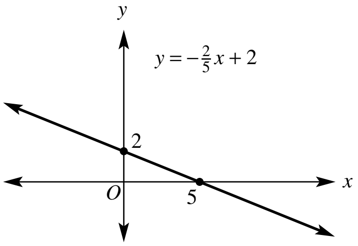
f $x = 8, y = 4$



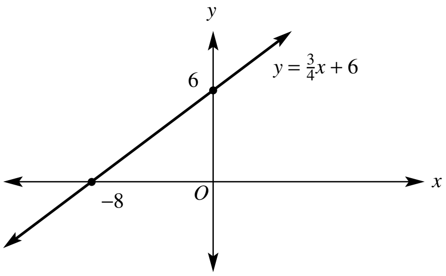
g $x = 4, y = 6$



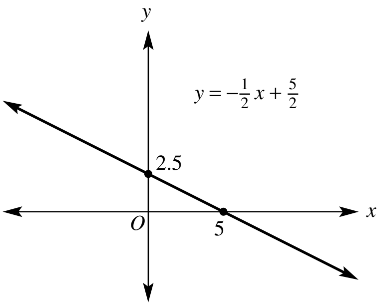
h $x = 5, y = 2$



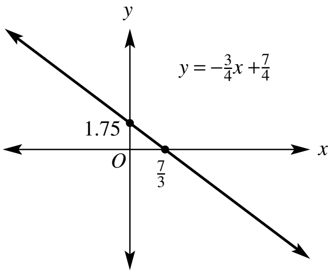
i $x = -8, y = 6$



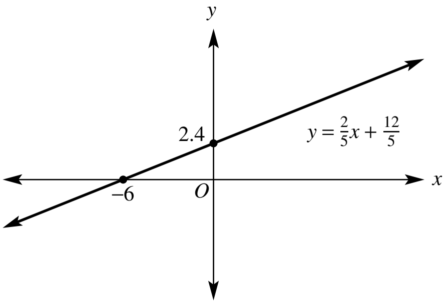
j $x = 5, y = 2.5$



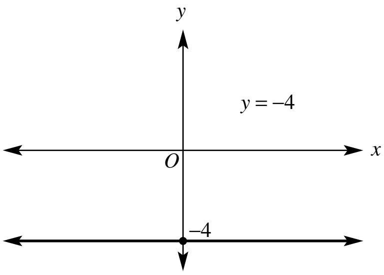
k $x = \frac{7}{3}, y = \frac{7}{4}$



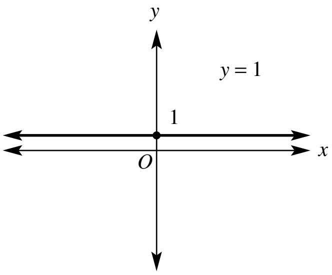
l $x = -6, y = \frac{12}{5}$



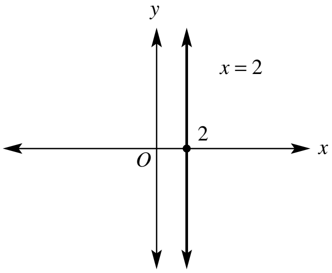
5 **a**



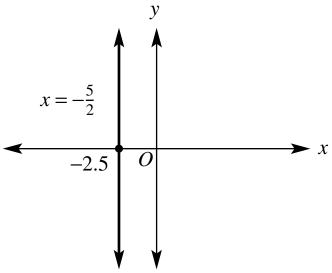
b



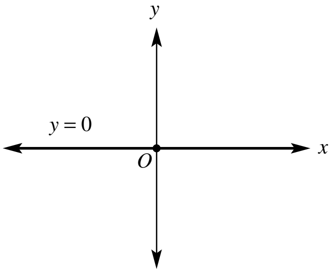
c


y
x = 2
2
O
x

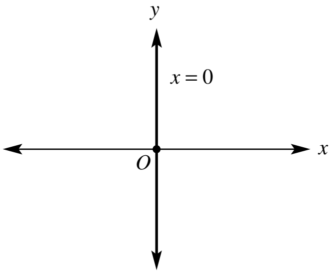
d


y
$x = -\frac{5}{2}$
−2.5
O
x

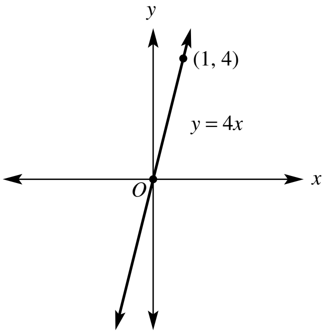
e


y
y = 0
O
x

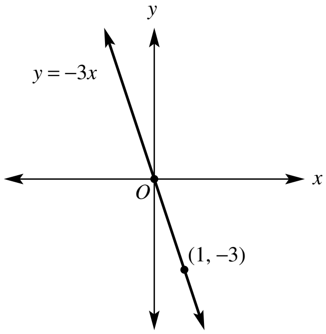
f


y
x = 0
O
x

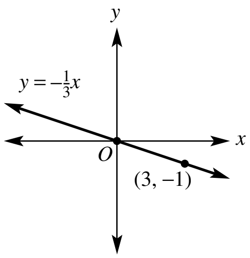
g


y
(1, 4)
y = 4x
O
x

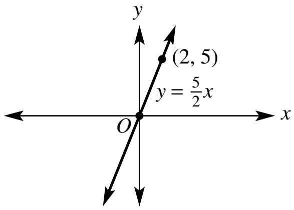
h


y
y = −3x
O
x
(1, −3)

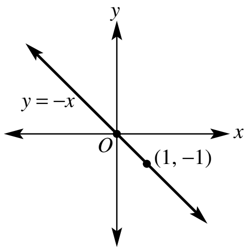
i


y
$y = -\frac{1}{3}x$
O
x
(3, −1)

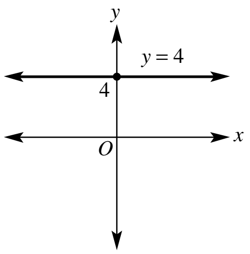
j


y
(2, 5)
$y = \frac{5}{2}x$
O
x

k

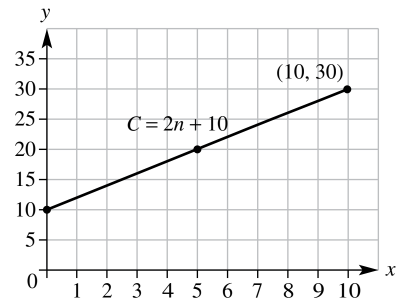

y
y = −x
O
x
(1, −1)

l


y
y = 4
4
O
x

6 **a** $C = 2n + 10$

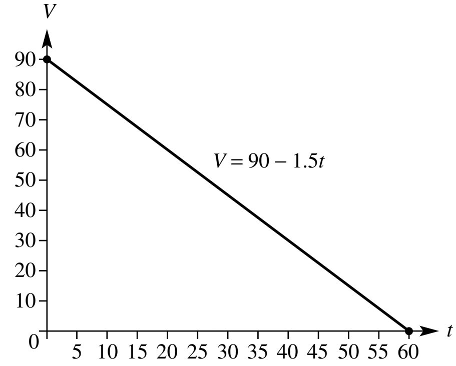
b



c **i** $28 **ii** 23.5 kg

7 **a** $V = 90 - 1.5t$

b



c **i** 82.5 L **ii** 60 hours

8 **a** $0.05/km **b** $C = 0.05k$

c $C = 1200 + 0.05k$

9 **a** $m = 25$, 25 km per hour i.e. speed

b The cyclist started 30 km from home.

c (0, 30)

10 **a** $y = x + \frac{1}{2}$, gradient = 1

b $y = 0.5x + 1.5$, y-intercept is at (0, 1.5)

c $y = -3x + 7$, gradient = −3

d $y = \frac{1}{2}x - 2$, gradient = $\frac{1}{2}$

11 **a** Gradient = $\frac{3}{a}$, y-intercept $\left(0, \frac{7}{a}\right)$

b Gradient = a, y-intercept $(0, -b)$

c Gradient = $-\frac{a}{b}$, y-intercept $\left(0, \frac{3}{b}\right)$

12 **a** $\left(\frac{d}{a}, 0\right)$ **b** $\left(0, \frac{d}{b}\right)$ **c** $-\frac{a}{b}$

13 **a** 12 sq. units **b** 9 sq. units **c** $\frac{121}{4}$ sq. units

d $\frac{121}{5}$ sq. units **e** $\frac{32}{3}$ sq. units

1F

Building understanding

1 **a** 2 **b** 3 **c** 0

d −4 **e** −3 **f** Undefined

2 **a** $c = 8$ **b** $c = -6$ **c** $c = 12$

Now you try

Example 17

$-\frac{7}{5}$

Example 18

$y = 2x + 1$

Example 19

$y = -\frac{5}{4}x + 12$

Exercise 1F

1 **a** $\frac{1}{4}$ **b** 2 **c** $\frac{5}{2}$

d 3 **e** 0 **f** 0

g −1 **h** $\frac{5}{2}$ **i** $-\frac{5}{7}$

j Undefined **k** $\frac{3}{2}$ **l** $-\frac{3}{2}$

2 **a** $y = x + 3$ **b** $y = x - 2$ **c** $y = 3x + 6$

d $y = -3x + 4$ **e** $y = 4$ **f** $y = -7x - 10$

3 **a** $y = 2x + 4$ **b** $y = 4x - 5$

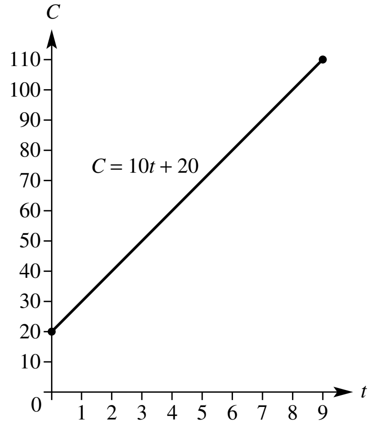
c $y = x - 4$ **d** $y = -2x + 12$

e $y = -3x - 4$ **f** $y = -3x - 2$

4 **a** $y = 3x + 5$ **b** $y = -2x + 4$

c $y = \frac{1}{2}x - \frac{3}{2}$ **d** $y = -2x - 2$

5 **a**



b $C = 10t + 20$

c **i** $10 per hour **ii** $20 up-front fee

6 **a** $A = 500t + 15\,000$

b $15 000

c 4 years more, i.e 10 years from investment

d $21 250

7 **a** **i** $V = 4t$ **ii** $V = 3t$

iii $V = t + 1$ **iv** $V = 1.5t + 2$

b 1 L, 2L

c Initially the flask contains b litres and it is losing 1 litre per minute.

8 **a** $m = \frac{-5}{5} = -1$

b $m = \frac{5}{-5} = -1$

c It doesn't matter in which pair of points is (x_1, y_1) and which is (x_2, y_2).

9 **a** $-\frac{4}{3}$

b $y = -\frac{4x}{3} + \frac{13}{3}$

c $y = -\frac{4x}{3} + \frac{13}{3}$

d The results from parts **b** and **c** are the same (when simplified). So it doesn't matter which point on the line is used in the formula $y - y_1 = m(x - x_1)$.

10 **a** **i** $\frac{1}{50} = 0.02$ **ii** $\frac{2}{50} = 0.04$

b **i** $y = 0.02x + 1.5$ **ii** $y = 0.04x + 1.5$

c The archer needs m to be between 0.02 and 0.04 to hit the target.

Progress quiz

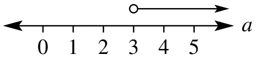
1 **a** $9a^2b + 2ab + 8b$ **b** $-12x^2y$

c $13m + 14$

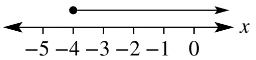
2 **a** $x = 5$ **b** $k = -\frac{3}{2}$ **c** $m = 30$

3 **a** $\frac{6 + m}{8}$ **b** $m = 10$

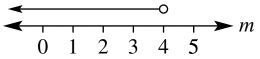
4 **a** $a > 3$



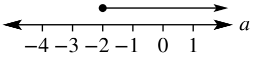
b $x \geqslant -4$



c $m < 4$



d $a \geqslant -2$



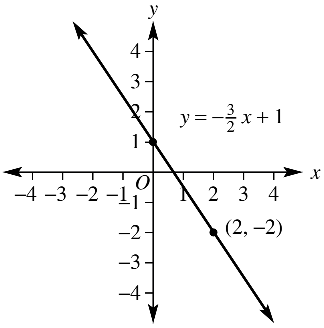
5 **a** $\frac{14 - 3a}{24}$ **b** $\frac{4x + 16}{15}$

6 **a** $y = \frac{23}{8}$ **b** $a = -\frac{9}{2}$

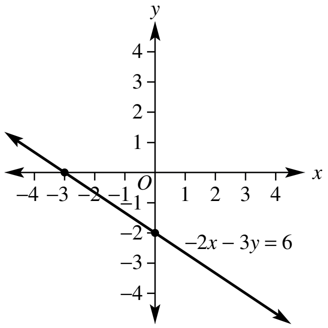
7 **a** (−3, 2) is not on the line.

b (−3, 2) is on the line.

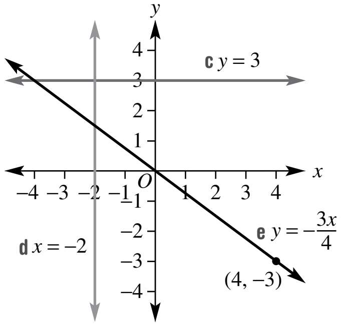
8 **a** Gradient = $-\frac{3}{2}$, y-intercept = 1



b



c, d, e



9 **a** $m = 3$, y-intercept is (0, −2)

b $m = -\frac{3}{5}$, y-intercept is (0, 3)

10 **a** $y = 2x + 3$ **b** $y = -\frac{3}{2}x + 8$ **c** $x = 5$

1G

Building understanding

1 **a** 4 **b** 5 **c** $\sqrt{41}$ **d** $\left(3, \frac{9}{2}\right)$

2 **a** 4 **b** 4

c $\sqrt{32} = 4\sqrt{2}$ **d** (0, −3)

3 **a** 3 **b** $\frac{11}{2}$ or 5.5

c 3 **d** −4

Now you try

Example 20

a $\sqrt{5}$ **b** $\sqrt{89}$

Example 21

$\left(\frac{1}{2}, -4\right)$

Example 22

$a = 4$ or $a = 10$

Exercise 1G

1 **a** $\sqrt{29}$ **b** $\sqrt{53}$ **c** $\sqrt{13}$

d $\sqrt{65}$ **e** $\sqrt{37}$ **f** 15

g $\sqrt{101}$ **h** $\sqrt{193}$ **i** $\sqrt{37}$

2 **a** (1, 6.5) **b** (1, 6.5) **c** (3.5, 6)

d (−1, 4.5) **e** (1, −1.5) **f** (−3.5, 3)

g (−3, −0.5) **h** (2, 2.5) **i** (−7, 10.5)

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3 B and C are both 5 units away from (2, 3).

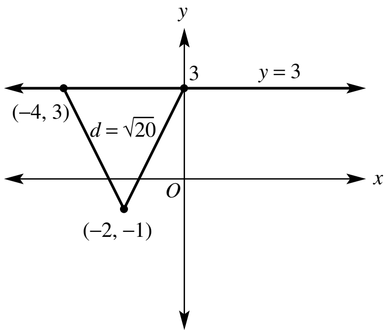
4 a $a = 3, b = 5$ b $a = -4, b = 5$
c $a = -2, b = 2$ d $a = 11, b = 2$

5 a 3, 7 b −1, 3 c −1, 9 d −6, 0

6 a 1478 m b 739 m

7 a (−0.5, 1) b (−0.5, 1)
c These are the same. The order of the points doesn't matter since addition is commutative $\frac{x_1 + x_2}{2} = \frac{x_2 + x_1}{2}$.
d 5 e 5
f The order of the points doesn't matter $(x - y)^2 = (y - x)^2$, as $(-3)^2 = (3)^2$.

8 $a = -4, 0$



9 a $\left(\frac{1}{2}, 2\right)$ b $\left(-\frac{1}{3}, \frac{4}{3}\right)$ c $\left(\frac{4}{3}, \frac{8}{3}\right)$
d $\left(2, \frac{16}{5}\right)$ e $\left(-\frac{3}{4}, 1\right)$ f $\left(0, \frac{8}{5}\right)$

10 a $\sqrt{(x - 7)^2 + y^2}$
b $\sqrt{(x - 7)^2 + (x + 3)^2}$
c i 721 m ii 707 m
iii 721 m iv 762 m
d $x = 2$
e The distance will be a minimum when the dotted line joining Sarah to the fence is perpendicular to the fence (when it has gradient −1). The closest point is (2, 5).

1H

Building understanding

1 a 4 b −7 c $-\frac{3}{4}$ d $\frac{8}{7}$

2 a $-\frac{1}{3}$ b $\frac{1}{2}$ c $-\frac{8}{7}$ d $\frac{9}{4}$

3 a 5 b 4 c $y = 5x + 4$

4 a True b False

Now you try

Example 23
a Perpendicular b Neither c Parallel

Example 24
a $y = -3x + 8$ b $y = -\frac{3}{2}x + 2$

Exercise 1H

1 a Parallel b Parallel c Neither
d Neither e Perpendicular f Perpendicular
g Parallel h Parallel i Perpendicular
j Perpendicular

2 a $y = x + 4$ b $y = -x - 6$
c $y = -4x - 1$ d $y = \frac{2}{3}x - 6$
e $y = -\frac{4}{5}x + 7$

3 a $y = -\frac{1}{2}x + 6$ b $y = \frac{1}{4}x - 2$
c $y = -\frac{3}{2}x + 5$ d $y = -\frac{3}{4}x - 5$
e $y = \frac{7}{2}x + 31$

4 a $x = 6$ b $x = 0$ c $y = 11$
d $y = 8.4$ e $y = 3$ f $y = -3$
g $x = \frac{2}{3}$ h $x = -\frac{4}{11}$

5 a $y = \frac{2}{3}x + 5$ b $y = -\frac{5}{7}x + \frac{54}{7}$
c $y = \frac{2}{3}x + \frac{16}{3}$ d $y = 7x + 20$

6 a $y = -\frac{3}{2}x + 5$ b $y = \frac{7}{5}x + \frac{28}{5}$
c $y = -\frac{3}{2}x + 1$ d $y = -\frac{1}{7}x - \frac{10}{7}$

7 The second line has equation $y = -\frac{2}{3}x - \frac{5}{3}$. It cuts the x-axis at $x = -\frac{5}{2}$.

8 a m b $-\frac{a}{b}$ c $-\frac{1}{m}$ d $\frac{b}{a}$

9 a 14 b −2 c 5 d $\frac{9}{7}$

10 a $y = 2x + b - 2a$ b $y = mx + b - ma$
c $y = x + b - a$ d $y = -\frac{1}{m}x + b + \frac{a}{m}$

11 a i 1 ii −1 iii 1 iv −1
b AB is parallel to CD, BC is parallel to DA, AB and CD are perpendicular to BC and DA; i.e. opposite sides are parallel and adjacent sides are perpendicular.
c Rectangle.

12 a i $\frac{4}{3}$ ii $-\frac{3}{4}$ iii 0
b Right-angled triangle (AB is perpendicular to BC).
c 20 units

13 $y = -\frac{1}{2}x + 4$, x-intercept = 8

1I

Building understanding

1 a Yes b Yes c No d Yes

2 a i Joe's: \$60, Paul's: \$150
ii Joe's: \$0.20 per km, Paul's: \$0.10 per km
iii Joe's: $C = 0.2k + 60$, Paul's: $C = 0.1k + 150$
iv 900 km
b Joe's Car Rental
c Paul's Motor Mart

Now you try

Example 25

a (2, −2) **b** (−1, 3)

Example 26

(2, −1)

Exercise 1I

1 **a** $x = 2, y = 7$ **b** $x = 2, y = 5$
c $x = 1, y = 1$ **d** $x = 1, y = 1$
e $x = 2, y = 1$ **f** $x = 3, y = 1$
g $x = 5, y = 1$ **h** $x = 10, y = 4$
i $x = 1, y = 2$ **j** $x = 9, y = 2$
2 **a** $x = 2, y = 10$ **b** $x = 1, y = -5$
c $x = -3, y = 3$ **d** $x = 13, y = -2$
e $x = 3, y = 1$ **f** $x = 2, y = 1$
g $x = 1, y = 4$ **h** $x = 1, y = 3$
3 **a** **i** $E = 20t$ **ii** $E = 15t + 45$
b $t = 9, E = 180$
c **i** 9 hours **ii** $180
4 18 years
5 **a** **i** $V = 62\,000 - 5000t$ **ii** $V = 40\,000 - 3000t$
b $t = 11, V = 7000$
c **i** 11 years **ii** $7000
6 197 600 m^2
7 **a** No **b** No **c** Yes **d** Yes
e No **f** Yes **g** Yes **h** No
8 **a** −4 **b** $\frac{3}{2}$ **c** 12
9 **a** $\left(\frac{k}{3}, \frac{2k}{3}\right)$ **b** $\left(\frac{k}{2}, -\frac{k}{2}\right)$
c $(-1 - k, -2 - k)$ **d** $\left(\frac{-2k - 1}{3}, \frac{-2k - 4}{3}\right)$
10 **a** $x = \frac{b}{a - b}, y = \frac{b^2}{a - b}$ **b** $x = \frac{-b}{a + b}, y = \frac{a}{a + b}$
c $x = \frac{a}{1 + b}, y = \frac{-a}{1 + b}$ **d** $x = \frac{b}{b - a}, y = \frac{b^2}{b - a}$
e $x = \frac{1}{a - 2b}, y = \frac{a - b}{a - 2b}$ **f** $x = \frac{c(1 - b)}{a(b + 1)}, y = \frac{2c}{b + 1}$
g $x = \frac{ab}{a^2 + b}, y = \frac{a^2 b}{a^2 + b}$ **h** $x = \frac{ab}{a^2 + 1}, y = \frac{b}{a^2 + 1}$
11 Answers will vary.

1J

Building understanding

1 **a** 0 **b** 0 **c** 0 **d** 0
2 **a** Subtract **b** Add
c Add **d** Subtract
3 **a** $4x - 6y = 8$ **b** $6x - 9y = 12$
c $8x - 12y = 16$ **d** $20x - 30y = 40$

Now you try

Example 27

$x = 3, y = 1$

Example 28

a $x = 1, y = 3$
b $x = 3, y = -2$

Exercise 1J

1 **a** $x = 2, y = 5$ **b** $x = 2, y = 3$
c $x = 4, y = 2$ **d** $x = 2, y = 2$
e $x = 1, y = 1$ **f** $x = 2, y = 1$
g $x = 2, y = -1$ **h** $x = 2, y = 2$
i $x = 1, y = 2$ **j** $x = 2, y = 1$
k $x = 2, y = 1$ **l** $x = -1, y = 2$
2 **a** $x = 1, y = 1$ **b** $x = 4, y = 2$
c $x = 2, y = 1$ **d** $x = 4, y = -3$
e $x = \frac{1}{2}, y = 1$ **f** $x = -\frac{1}{2}, y = -\frac{1}{2}$
3 **a** $x = 4, y = -3$ **b** $x = 1, y = 1$
c $x = 3, y = 4$ **d** $x = 2, y = 2$
e $x = \frac{1}{2}, y = -1$ **f** $x = -3, y = \frac{1}{3}$
4 799 and 834
5 $0.60
6 $A = \$15$, $C = \$11$
7 Should have been [1]–[2], to eliminate $y: -2y - (-2y) = 0$.
The correct solution is (1, −1).
8 **a** $x = \frac{1}{a}, y = -1$ **b** $x = \frac{13}{3}, y = \frac{1}{3b}$
c $x = \frac{-2}{a}, y = \frac{2}{b}$ **d** $x = \frac{a + b}{2a}, y = \frac{a - b}{2b}$
e $x = \frac{c}{a + b}, y = \frac{c}{a + b}$
9 The two lines are parallel, they have the same gradient.
10 **a** $\frac{2}{x - 1} - \frac{2}{x + 1}$ **b** $\frac{2}{2x - 3} - \frac{1}{x + 2}$
c $\frac{3}{3x + 1} - \frac{2}{2x - 1}$ **d** $\frac{3}{3x - 1} + \frac{2}{x + 2}$
e $\frac{1}{x + 3} + \frac{1}{x - 4}$ **f** $\frac{1}{7(2x - 1)} - \frac{3}{7(4 - x)}$

1K

Building understanding

1 **a** $x + y = 16, x - y = 2$
b $x + y = 7, 2x + y = 12$
c $2x + 3y = 11, 4x - 3y = 13$
2 $l = 3w, 2l + 2w = 56$ or $l + w = 28$
3 **a** $5x$ dollars
b $15y$ dollars
c $3d + 4p$ dollars

Now you try

Example 29

Tim is 14, Tina is 6.

Example 30

A coffee is $4 and a muffin is $3.

Exercise 1K

1 Nikki is 16, Travis is 8.
2 Cam is 33, Lara is 30.
3 Bolts cost $0.10, washers cost $0.30.
4 There were 2500 adults and 2500 children.
5 Thickshakes cost $5, juices cost $3.
6 There are 36 ducks and 6 sheep.

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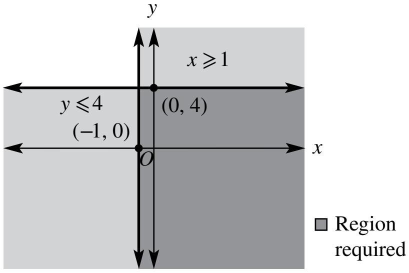
Cambridge University Press

7 43
8 \$6.15 (mangoes cost \$1.10, apples cost \$0.65)
9 70
10 1 hour and 40 minutes
11 $\frac{1}{7}$ of an hour
12 200 m
13 $\frac{4}{17}$ L
14 $\frac{210}{19}$ L

1L

Building understanding

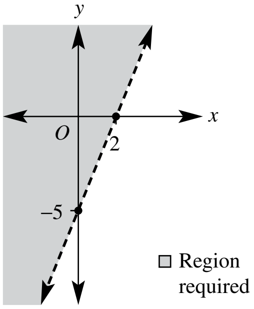
1 **a** No **b** Yes **c** No
d No **e** No **f** Yes
2 **a** B **b** C **c** A
3 **a–d** $x \geqslant -1, y \leqslant 4$



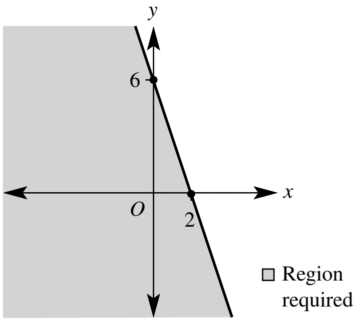
Now you try

Example 31

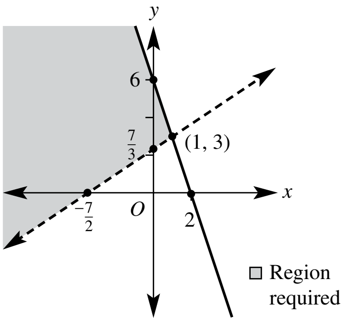
a



b

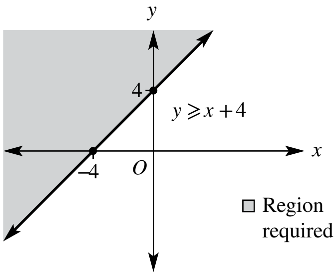


Example 32

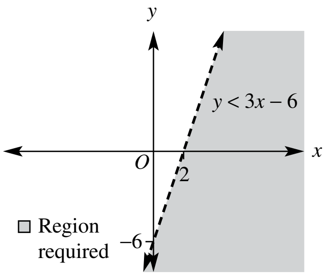


Exercise 1L

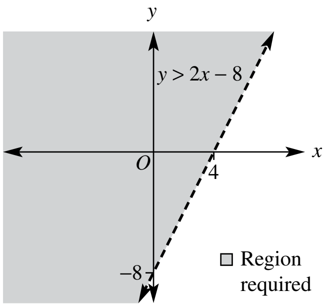
1 **a** $y \geqslant x + 4$



b $y < 3x - 6$



c $y > 2x - 8$



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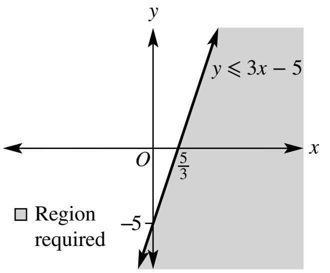
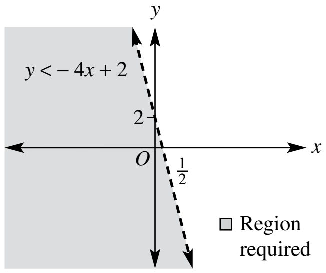
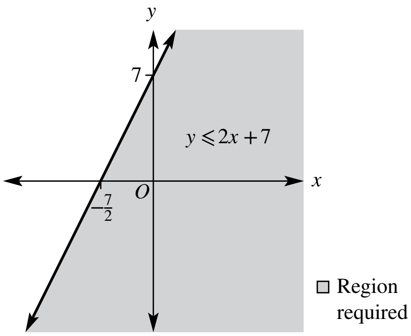
d $y \leqslant 3x - 5$



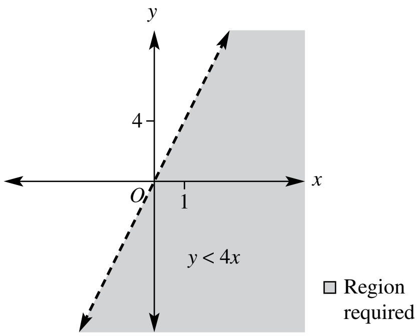
e $y < -4x + 2$



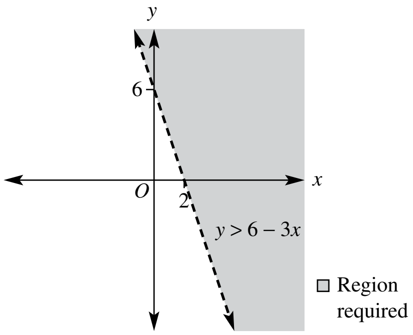
f $y \leqslant 2x + 7$



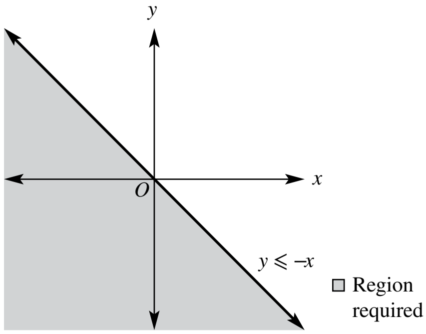
g $y < 4x$



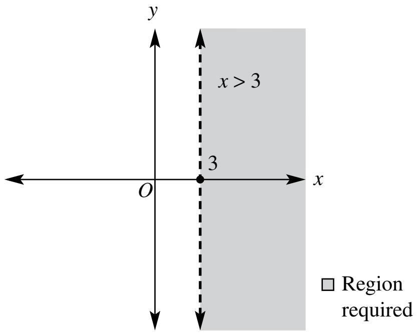
h $y > -3x + 6$



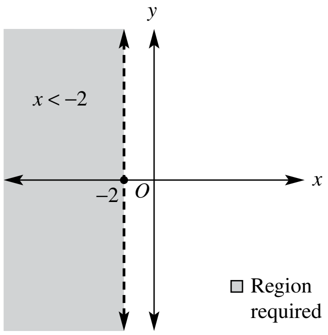
i $y \leqslant -x$



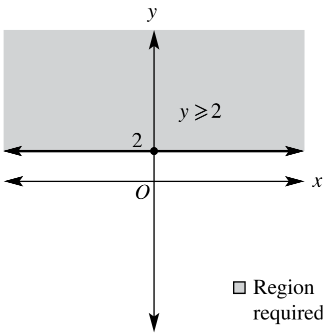
j $x > 3$



k $x < -2$



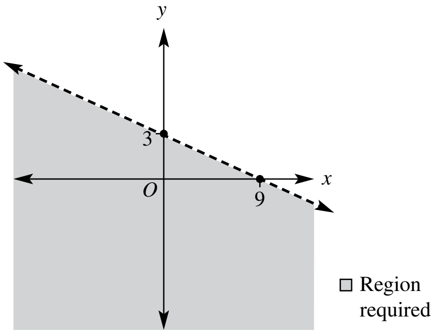
l $y \geqslant 2$



2 **a** Yes **b** No **c** No **d** Yes

3 **a** Yes **b** Yes **c** No **d** Yes

4 **a**



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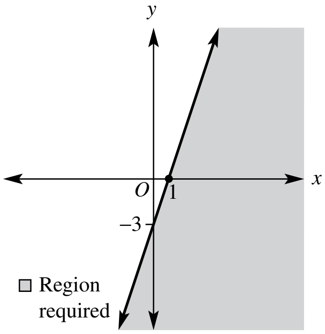

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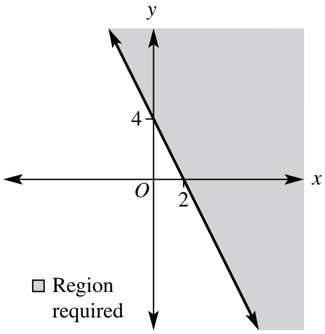

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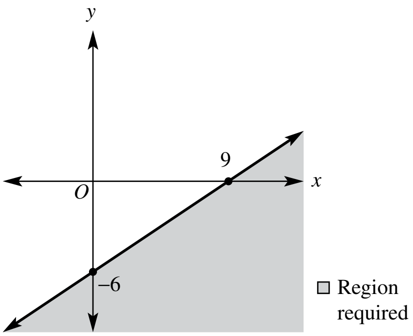

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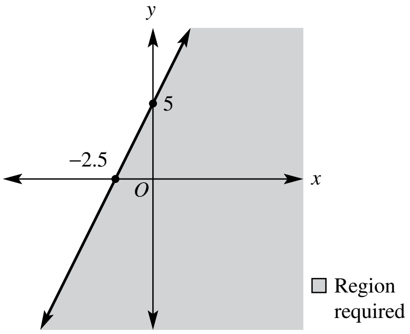

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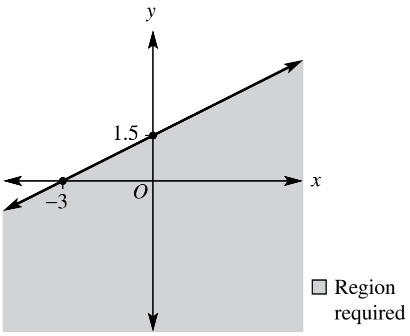

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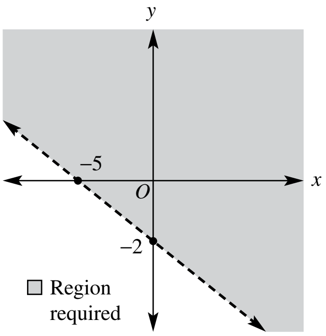

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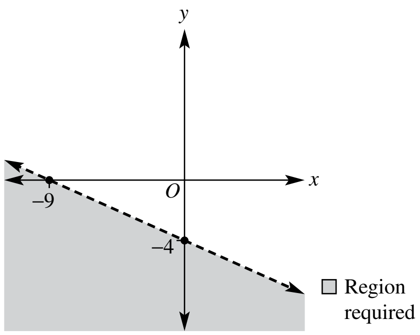


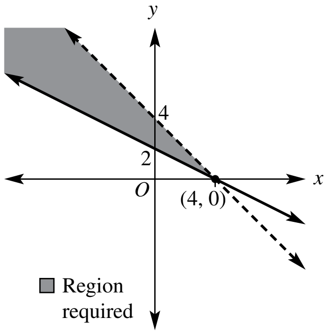
5 **a** $y \leqslant x + 3$

b $y \geqslant -2x + 2$

c $y < -\frac{3}{2}x - 3$

d $y > \frac{2}{5}x - 2$

6 **a**



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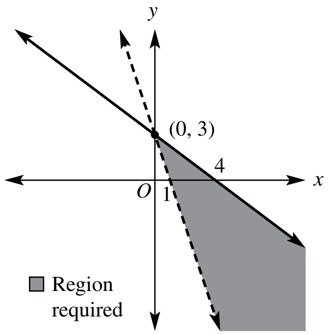

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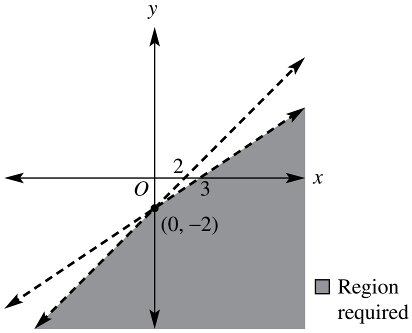

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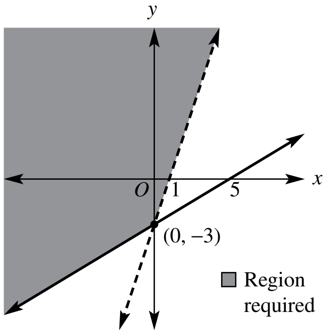

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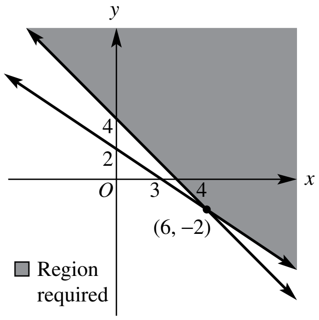

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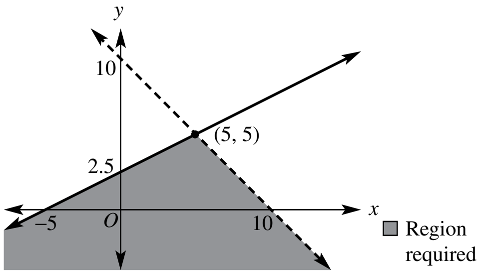

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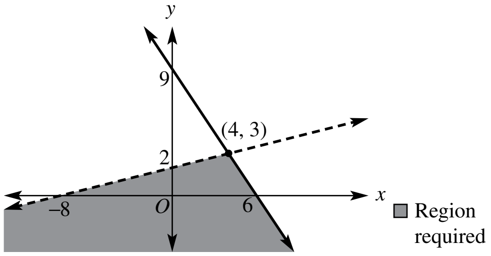

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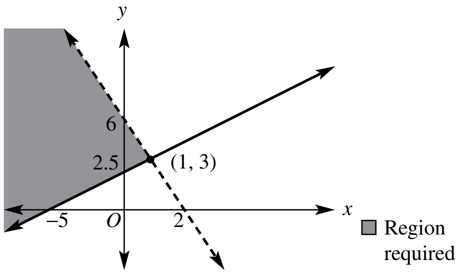
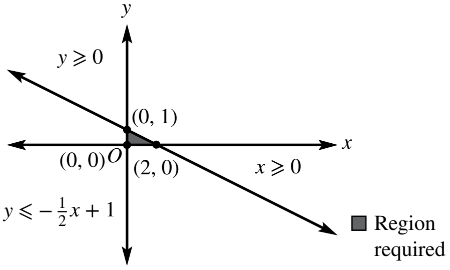


7 a



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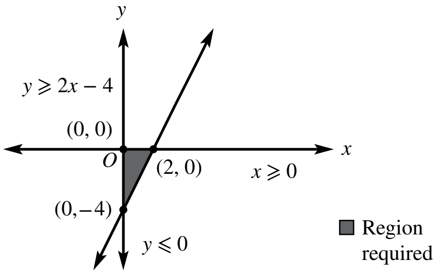

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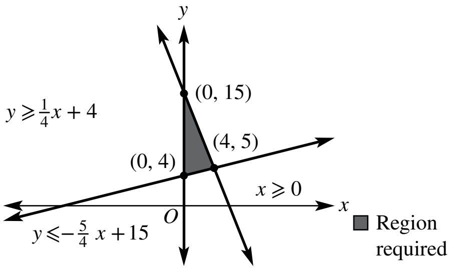

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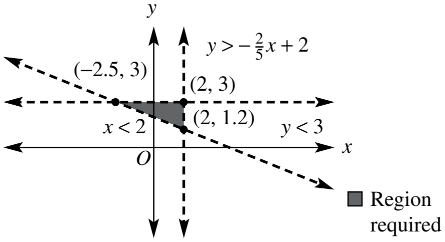

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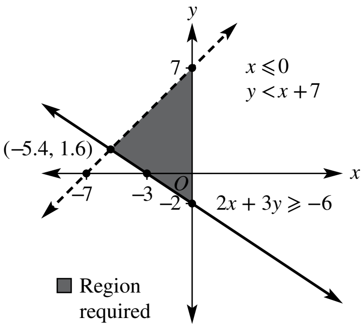

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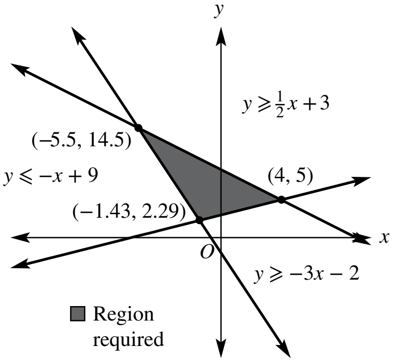


8 a $y \geqslant 0$, $y < 2x + 4$, $y \leqslant -x + 7$

b $y > -\frac{1}{2}x + 6$, $y \leqslant x + 3$, $x < 8$

9 a 1 units^2 **b** 4 units^2

c 22 units^2 **d** $\frac{81}{20}$ units^2

10 a i $\frac{115}{6}$ **ii** $\frac{578}{15}$

b Answers may vary; e.g. $x > 0$, $x < 3$, $y > 0$, $y < 2$

Problems and challenges

1 0.75 km

2 $\frac{6}{8}$

3 **a** The gradient from (2, 12) to (−2, 0) = the gradient from (−2, 0) to (−5, −9) = −3.

b The gradient from $(a, 2b)$ to $(2a, b)$ = the gradient from $(2a, b)$ to $(-a, 4b) = -\frac{b}{a}$.

4 The gradient of AC is $\frac{3}{5}$ and the gradient of AB is $-\frac{5}{3}$. So ΔABC is a right-angled triangle, as AC is perpendicular to AB. Can also show that side lengths satisfy Pythagoras' theorem.

5 The missiles are travelling at $\frac{4840}{9}$ km/h and $\frac{9680}{9}$ km/h.

6 The distance between the two points and (2, 5) is 5 units.

7 The diagonals have equations $x = 0$ and $y = 3$. These lines are perpendicular and intersect at the midpoint (0, 3) of the diagonals. It is not a square since the angles at the corners are not 90°. In particular, AB is not perpendicular to BC $\left(\mathrm{m}_{\mathrm{AB}} \neq \frac{-1}{m_{BC}}\right)$.

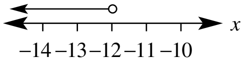
8 $x = 2$, $y = -3$, $z = -1$

9 24 units^2

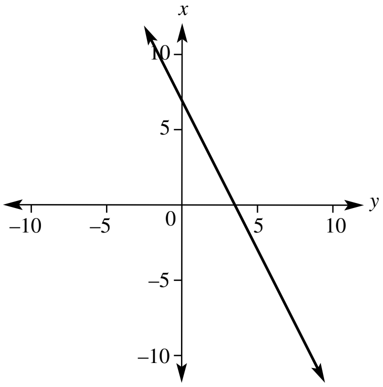
10 24, 15 years

Answers to success criteria example questions

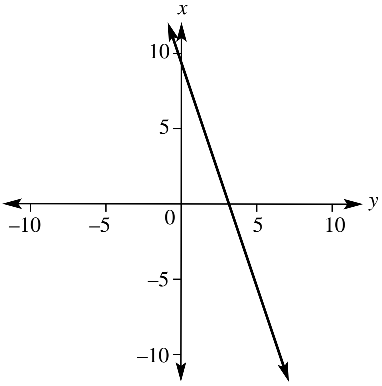
1 $8xy - x^2y$
2 $-8y^2 - 12y$
3 $3x(2x - 5)$
4 -3
5 $x = 4$
6 $x = 11$; $x = 20$
7 $\frac{11x}{12}$; $x = \frac{24}{11}$
8 $x > -1$
9 $x < -12$



10 $\frac{2x - 6}{35}$
11 $x = \frac{31}{8}$; $x = -\frac{1}{7}$
12 Yes, the point is on the line.
13 Gradient: $\frac{2}{3}$; y-intercept: (0, 2)
14 Gradient: −2; y-intercept: (0, 7)



15 x-intercept is (3, 0)
y-intercept is (0, 9)



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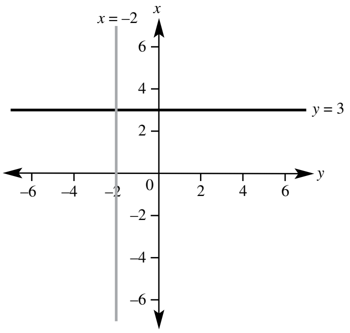

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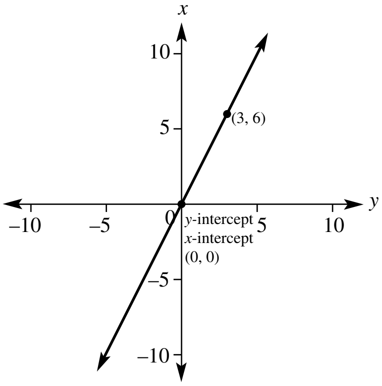
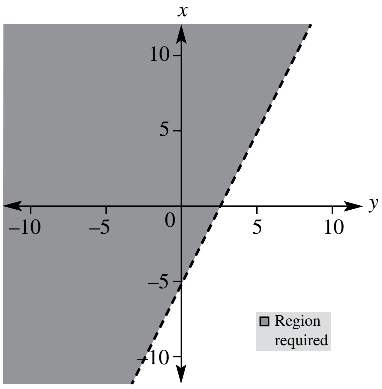


18 $-\frac{3}{5}$
19 $y = 2x + 2$
20 $y = \frac{5}{4}x + \frac{1}{2}$
21 $\sqrt{13}$
22 (2, 3.5)
23 $a = 5$ or $a = 15$
24 Perpendicular
25 $y = 3x - 2$
26 $y = \frac{1}{2}x + 8$
27 $x = 2$, $y = -1$
28 $x = 6$, $y = -2$
29 $x = -2$, $y = 3$
30 Chocolate bars cost \$2 and ice-creams cost \$4
31



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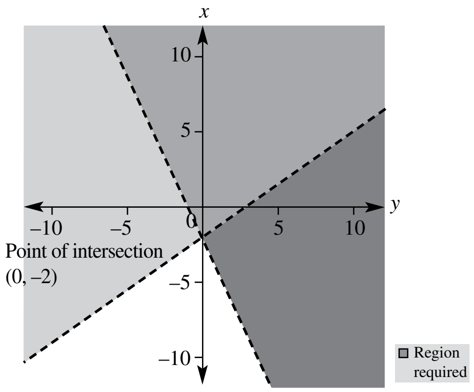


Short-answer questions

1 **a** $5xy + 6x$ **b** $12a^2b$ **c** $\frac{3}{2}x$

d $3b + 21$ **e** $-2m^2 + 12m$ **f** $x + 2$

2 **a** $x = -3$ **b** $x = -\frac{3}{4}$ **c** $x = 1$

3 **a** $x < 1$ **b** $x \geqslant -4$ **c** $-1 < x \leqslant 3$

4 **a** $x > 5$ **b** $x \geqslant 10$

c $x > -3$ **d** $x \leqslant \frac{2}{7}$

5 **a** $V = 2 - 0.4t$

b 1.4 L

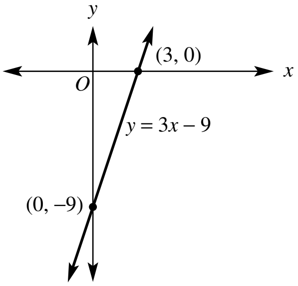
c 5 minutes **d** $\geqslant$3.5 minutes

6 **a** $\frac{6x+4}{9}$ **b** $\frac{3x+4}{8}$ **c** $\frac{2x+22}{15}$

7 **a** $x = \frac{21}{5}$ **b** $x = -2$

c $x = \frac{-20}{7}$ **d** $x = \frac{11}{2}$

8 **a**



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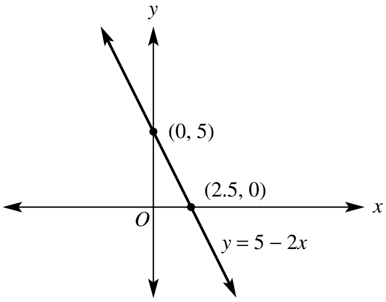

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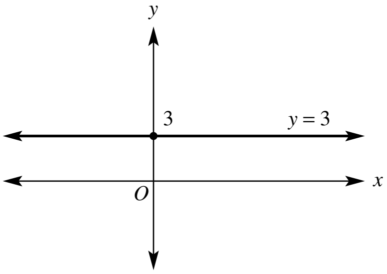

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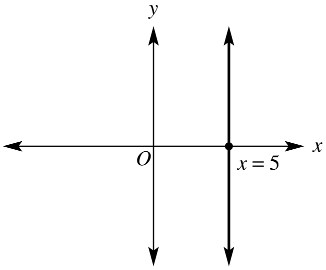

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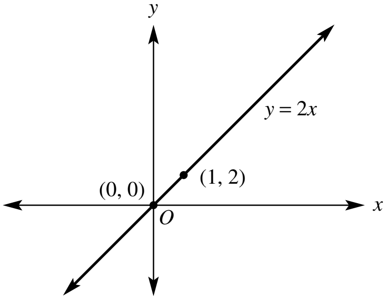

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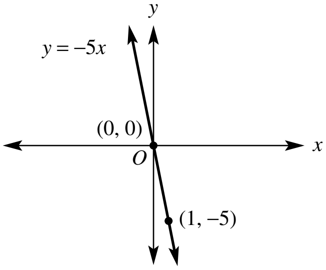

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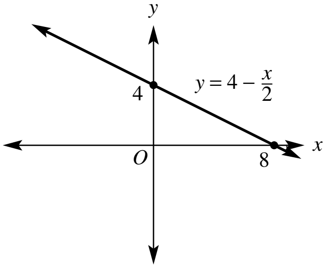

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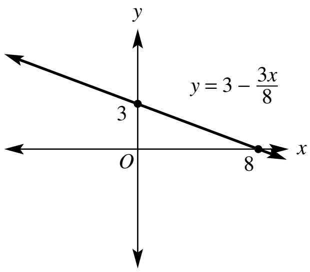


9 a $y = \frac{1}{2}x + 3$ **b** $y = \frac{5}{2}x + 5$

c $y = -\frac{3}{2}x + \frac{15}{2}$ **d** $y = 2x - 3$

10 a $m = -\frac{3}{5}$ **b** $y = -\frac{3}{5}x + \frac{34}{5}$

11 a $M = (4, 8)$, $d = \sqrt{52} = 2\sqrt{13}$

b $M = \left(\frac{11}{2}, 1\right)$, $d = \sqrt{61}$

c $M = \left(\frac{1}{2}, -\frac{5}{2}\right)$, $d = \sqrt{18} = 3\sqrt{2}$

12 a $y = 3x - 2$ **b** $y = -1$

c $y = -\frac{1}{2}x + 5$ **d** $y = 3x - 1$

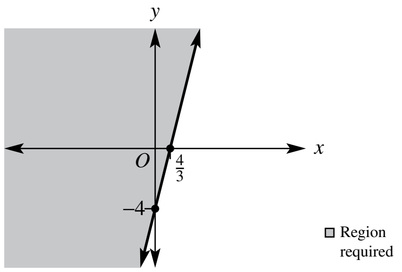
13 a $a = 7$ **b** $b = -8$ **c** $c = 0$ or 4

14 a $(-3, -1)$ **b** $(-28, -51)$

15 a $(-3, -1)$ **b** $(0, 2)$

16 A regular popcorn costs \$4 and a small drink costs \$2.50.

17 a



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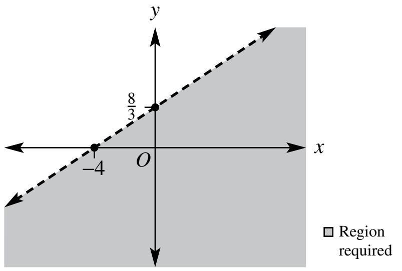
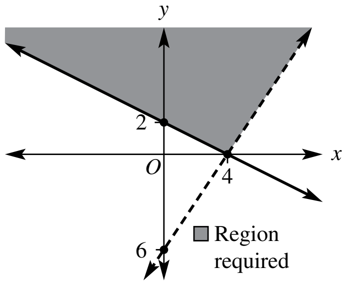


18 The point of intersection is $(4, 0)$.



Multiple-choice questions

1 E
2 C
3 D
4 $\frac{13x+7}{21}$
5 B
6 C
7 A
8 C
9 D
10 E
11 B
12 A
13 A
14 D
15 C

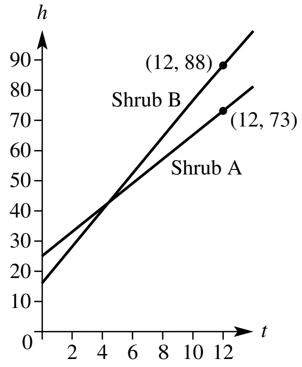
Extended-response questions

1 a i $h = 4t + 25$ **ii** $h = 6t + 16$

b 16 cm

c Shrub B because its gradient is greater.

d



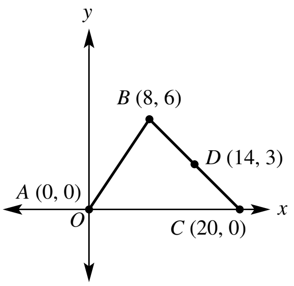
e After 4.5 months

f i 1.24 m

ii 26.25 months

iii Between 8.75 and 11.25 months

2 a $A(0, 0)$, $B(8, 6)$, $C(20, 0)$



b 43.4 km

c The drink station is at $(14, 3)$.

d i $y = \frac{3}{4}x$ **ii** $y = -\frac{1}{2}x + 10$

iii $y = 0$

e $y \geqslant 0$, $y \leqslant \frac{3}{4}x$, $y \leqslant -\frac{1}{2}x + 10$

f $y = -\frac{4}{3}x + \frac{80}{3}$

Chapter 2

2A

Building understanding

1 Triangle, quadrilateral, pentagon, hexagon, heptagon, octagon, nonagon, decagon
2 a False b True c True
d True e False f False
g True h False i True
3 a b b c c d d a
4 a $a = 110$ (angles on a line), $b = 70$ (vertically opposite)
b $a = 140$ (angles in a revolution)
c $a = 19$ (complementary)
d $a = 113$ (cointerior angles in $\|$ lines), $b = 67$ (alternate angles in $\|$ lines), $c = 67$ (vertically opposite to b)
e $a = 81$ (isosceles triangle), $b = 18$ (angles in a triangle)
f $a = 17$ (angles in a triangle), $b = 102$ (angles at a line)

Now you try

Example 1
a $x = 65$ b $x = 70$

Example 2
a $x = 100$ b $a = 108$

Example 3
a $a = 80$ b $a = 80$

Exercise 2A

1 a $x = 80$ b $x = 35$ c $x = 110$
d $x = 72$ e $x = 60$ f $x = 56$
2 a $x = 50$ (angle sum in a quadrilateral)
b $x = 95$ (angle sum in a quadrilateral)
c $x = 125$ (angle sum in a pentagon)
d $x = 30$ (angle sum in a pentagon)
e $x = 45$ (angle sum in a hexagon)
f $x = 15$ (angle sum in a quadrilateral)
3 a 108° b 135° c 144°
4 a $x = 121$ (vertically opposite to cointerior angle in $\|$ lines)
b $x = 109$ (alternate and cointerior angles in $\|$ lines)
c $x = 60$ (isosceles, cointerior angles in $\|$ lines)
d $x = 85$ (cointerior and corresponding angles in $\|$ lines)
e $x = 71$ (isosceles, cointerior angles in $\|$ lines)
f $x = 130$ (cointerior angles in $\|$ lines and revolution)
5 a $a = 95$ (alternate + cointerior)
b $a = 113$ (alternate + alternate)
c $a = 85$ (alternate + cointerior)
d $a = 106$ (cointerior)
e $a = 147$, (cointerior, angles in a revolution)
f $a = 292$, (angles in a revolution, alternate + cointerior)
6 a 176.4° b 3.6°
7 a 12 b 20 c 48
8 $x = 36$, $y = 144$
9 115, equilateral and isosceles triangle 60 + 55
10 a Expand the brackets. b $n = \dfrac{S + 360}{180}$
c $I = \dfrac{S}{n} = \dfrac{180(n - 2)}{n}$ d $E = 180 - I = \dfrac{360}{n}$
11 a $\angle BCA = 180° - a° - b°$ (angles in a triangle)
b $c° = 180° - \angle BCA = a° + b°$ (angles at a line)
12 a Alternate angles ($BA \parallel CD$)
b $\angle ABC + \angle BCD = 180°$ (cointerior), so $a + b + c = 180$.
c Angle sum of a triangle is 180°.
13 $\angle ACB = \angle ECD$ (vertically opposite), so $\angle CAB = \angle CBA = \angle CDE = \angle CED$ (isosceles) since $\angle CAB = \angle CED$ (alternate) $AB \parallel DE$.
14 Answers may vary.
15 a 15 (alternate angles in parallel lines)
b 315 (angle sum in an octagon)
16 Let M be the midpoint of AC. Then $\angle AMB = 60°$ ($\triangle ABM$ is equilateral). $\angle BMC = 120°$ (supplementary). Therefore, $\angle MBC = 30°$ ($\triangle MBC$ is isosceles). So $\angle ABC = \angle ABM + \angle MBC = 60° + 30° = 90°$.
17 Let $\angle AOB = x$ and $\angle COD = y$. $2x + 2y = 180°$ (angles at a line). So $\angle BOD = x + y = 90°$.

2B

Building understanding

1 a SAS b SSS c AAS
d SAS e RHS f RHS
2 a $x = 5$ b $x = 4$ c $x = 3$ d $x = 5$

Now you try

Example 4
a $AB = DE$ (given) S
$\angle ABC = \angle DEF$ (given) A
$BC = EF$ (given) S
$\therefore \triangle ABC \equiv \triangle DEF$ (SAS)
b $\angle ABC = \angle DEF$ (given) A
$\angle BAC = \angle EDF$ (given) A
$AC = DF$ (given) S
$\therefore \triangle ABC \equiv \triangle DEF$ (AAS)

Example 5
a $\angle A = \angle C = 90°$ (given) R
BD is common H
$AB = CB$ (given) S
$\therefore \triangle ABD \equiv \triangle CBD$ (RHS)
b $\triangle ABD \equiv \triangle CBD$ so $AD = CD$
c $AD = 4$ m

Exercise 2B

1 a $AB = DE$ (given) S
$\angle BAC = \angle EDF$ (given) A
$AC = DF$ (given) S
$\therefore \triangle ABC \equiv \triangle DEF$ (SAS)

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b $\angle ABC = \angle DEF$ (given) A
$\angle BAC = \angle EDF$ (given) A
$BC = EF$ (given) S
$\therefore \triangle ABC \equiv \triangle DEF$ (AAS)

c $AB = DE$ (given) S
$\angle ABC = \angle DEF$ (given) A
$BC = EF$ (given) S
$\therefore \triangle ABC \equiv \triangle DEF$ (SAS)

d $\angle FED = \angle CBA = 90^\circ$ (given) R
$FD = CA$ (given) H
$FE = CB$ (given) S
$\therefore \triangle FED \equiv \triangle CBA$ (RHS)

e $AC = DF$ (given) S
$BC = EF$ (given) S
$AB = DE$ (given) S
$\therefore \triangle ACB \equiv \triangle DFE$ (SSS)

f $\angle EDF = \angle BAC$ (given) A
$\angle DFE = \angle ACB$ (given) A
$EF = BC$ (given) S
$\therefore \triangle EDF \equiv \triangle BAC$ (AAS)

2 a $x = 7.3, y = 5.2$
b $x = 12, y = 11$
c $a = 2.6, b = 2.4$
d $x = 16, y = 9$

3 a $AD = CB$ (given) S
$DC = BA$ (given) S
AC is common; S
$\therefore \triangle ADC \equiv \triangle CBA$ (SSS)

b $\angle ADB = \angle CBD$ (given) A
$\angle ABD = \angle CDB$ (given) A
BD is common; S
$\therefore \triangle ADB \equiv \triangle CBD$ (AAS)

c $\angle BAC = \angle DEC$ (alternate, $AB \| DE$) A
$\angle CBA = \angle CDE$ (alternate, $AB \| DE$) A
$BC = DC$ (given) S
$\therefore \triangle BAC \equiv \triangle DEC$ (AAS)

d $DA = DC$ (given) S
$\angle ADB = \angle CDB$ (given) A
DB is common; S
$\therefore \triangle ADB \equiv \triangle CDB$ (SAS)

e $OA = OC$ (radii) S
$OB = OD$ (radii) S
$AB = CD$ (given) S
$\therefore \triangle OAB \equiv \triangle OCD$ (SSS)

f $\angle ADC = \angle ABC = 90^\circ$ (given) R
AC is common; H
$DC = BC$ (given) S
$\therefore \triangle ADC \equiv \triangle ABC$ (RHS)

4 a $OA = OC$ (radii) S
$\angle AOB = \angle COB$ (given) A
OB is common; S
$\therefore \triangle AOB \equiv \triangle COB$ (SAS)

b $AB = BC$ (corresponding sides in congruent triangles)
c 10 mm

5 a $BC = DC$ (given) S
$\angle BCA = \angle DCE$ (vertically opposite) A
$AC = EC$ (given) S
$\therefore \triangle ABC \equiv \triangle EDC$ (SAS)

b $AB = DE$ (corresponding sides in congruent triangles)

c $\angle ABC = \angle CDE$ (corresponding angles in congruent triangles).
$\angle ABC$ and $\angle EDC$ are alternate angles (and equal).
$\therefore AB \| DE$.

d 5 cm

6 a $AB = CD$ (given) S
$AD = CB$ (given) S
BD is common; S
$\therefore \triangle ABD \equiv \triangle CDB$ (SSS)

b $\angle DBC = \angle BDA$ (corresponding angles in congruent triangles)

c $\angle DBC$ and $\angle BDA$ are alternate angles (and equal).
$\therefore AD \| BC$.

7 a $CB = CD$ (given) S
$\angle BCA = \angle DCE$ (vertically opposite) A
$CA = CE$ (given) S
$\therefore \triangle BCA \equiv \triangle DCE$ (SAS)
$\angle BAC = \angle DEC$ (corresponding angles in congruent triangles)
$\therefore$ Alternate angles are equal, so $AB \| DE$.

b $\angle OBC = \angle OBA = 90^\circ$ (given) R
$OA = OC$ (radii) H
OB is common; S
$\therefore \triangle OAB \equiv \triangle OCB$ (RHS)
$AB = BC$ (corresponding sides in congruent triangles)
$\therefore OB$ bisects AC.

c $AB = CD$ (given) S
AC is common; S
$AD = CB$ (given) S
$\therefore \triangle ACD \equiv \triangle CAB$ (SSS)
$\angle DAC = \angle BCA$ (corresponding angels in congruent triangles)
$\therefore$ Alternate angles are equal, so $AD \| BC$.

d $AB = AE$ (given) S
$\angle ABC = \angle AED$ ($\triangle ABE$ is isosceles) A
$ED = BC$ (given) S
$\therefore \triangle ABC \equiv \triangle AED$ (SAS)
$AD = AC$ (corresponding sides in congruent triangles)

e $OD = OC$ (given) S
$\angle AOD = \angle BOC$ (vertically opposite) A
$OA = OB$ (given) S
$\therefore \triangle AOD \equiv \triangle BOC$ (SAS)
$\angle OAD = \angle OBC$ (corresponding angels in congruent triangles)

f $AD = AB$ (given) S
$\angle DAC = \angle BAC$ (given) A
AC is common; S
$\therefore \triangle ADC \equiv \triangle ABC$ (SAS)
$\angle ACD = \angle ACB$ (corresponding angels in congruent triangles)
$\angle ACD = \angle ACB$ are supplementary.
$\therefore \angle ACD = \angle ACB = 90^\circ$
$\therefore AC \perp BD$

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8 **a** $OA = OB$ (radii) S
OM is common; S
$AM = BM$ (M is midpoint) S
$\therefore \triangle OAM \equiv \triangle OBM$ (SSS)
$\angle OMA = \angle OMB$ (corresponding angels in congruent triangles)
$\angle OMA$ and $\angle OMB$ are supplementary.
$\therefore \angle OMA = \angle OMB = 90°$
$\therefore OM \perp AB$

b $OA = OB$ (radii of same circle) S
$CA = CB$ (radii of same circle) S
OC is common; S
$\therefore \triangle OAC \equiv OBC$ (SSS)
$\angle AOC = \angle BOC$ (corresponding angels in congruent triangles)

c $\angle CAB = \angle CBA = x$ ($\triangle ABC$ is isosceles)
$\angle EAB = \angle DBA = \frac{x}{2}$
$\therefore \triangle AFB$ is isosceles, so $AF = BF$.

2C

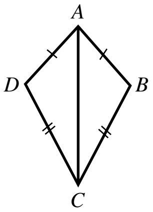
Building understanding

1 **a**	Rectangle	**b**	Parallelogram
c	Square	**d**	Rhombus
2 **a**	Rectangle, square	**b**	Rectangle, square
c	Parallelogram, rhombus, rectangle, square		
d	Rhombus, square	**e**	Rhombus, square

3 **a** A trapezium does not have both pairs of opposite sides parallel.
b A kite does not have two pairs of opposite sides parallel.

Now you try

Example 6



$AB = AD$ (given) S
$BC = DC$ (given) S
AC is common S
$\therefore \triangle ABC \equiv \triangle ADC$ (SSS)
$\therefore \angle ABC = \angle ADC$ (corresponding angles in congruent triangles)

Example 7
$AB = CD$ (given) S
$BC = DA$ (given) S
AC is common S
$\therefore \triangle ABC \equiv \triangle CDA$ (SSS)
$\therefore \angle BAC = \angle DCA$ so $AB \parallel CD$
and $\angle ACB = \angle CAD$ so $BC \parallel DA$
$\therefore ABCD$ is a parallelogram.

Exercise 2C

1 **a** $\angle BAC = \angle DCA$ (alternate angles)
$\angle BCA = \angle DAC$ (alternate angles)
AC is common.
$\therefore \triangle ABC \equiv \triangle CDA$ (AAS)
b As $\triangle ABC \equiv \triangle CDA$, $AD = CB$, $AB = CD$ (corresponding sides).

2 **a** $\angle ABE = \angle CDE$ (alternate angles)
$\angle BAE = \angle DCE$ (alternate angles)
$AB = CD$ (opposite sides of parallelogram)
$\therefore \triangle ABE \equiv \triangle CDE$ (AAS)
b $AE = CE$ (corresponding sides), $BE = DE$ (corresponding sides).

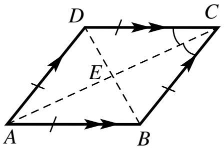
3 **a** $AB = CB$ (given)
$AD = CD$ (given)
BD is common.
$\therefore \triangle ABD \equiv \triangle CDB$ (SSS)
b $\angle ABD = \angle ADB = \angle CBD = \angle CDB$ (equal angels in congruent isosceles triangles). Therefore, BD bisects $\angle ABC$ and $\angle CDA$.

4 **a** $AE = CE$ (given)
$BE = DE$ (given)
$\angle AEB = \angle CED$ (vertically opposite angels)
$\therefore \triangle ABE \equiv \triangle CDE$ (SAS)
b $\angle ABE = \angle CDE$ (corresponding angles), $\angle BAE = \angle DCE$ (corresponding angles). Therefore, $AB \parallel DC$ (alternate angles are equal). $\angle ADE = \angle CBE$ (corresponding angles), $\angle DAE = \angle BCE$ (corresponding angles). Therefore, $AD \parallel BC$ (alternate angles are equal).

5 **a** $AD = CB$ (given)
$\angle DAC = \angle BCA$ (alternate angles)
AC is common.
$\therefore \triangle ABC \equiv \triangle CDA$ (SAS)
b $\angle BAC = \angle DCA$ (corresponding angles), therefore $AB \parallel DC$ (alternate angles are equal).

6 **a** $\triangle ABE \equiv \triangle CBE \equiv \triangle ADE \equiv \triangle CDE$ (SAS)
b $\angle ABE = \angle CDE$ (corresponding angles), $\angle BAE = \angle DCE$ (corresponding angles), therefore $AB \parallel CD$. $\angle ADE = \angle CBE$ (corresponding angles), $\angle DAE = \angle BCE$ (corresponding angles), therefore $AD \parallel CB$. Also, $AB = AD = CB = CD$ (corresponding sides). Therefore, $ABCD$ is a rhombus.

7 **a**



$\angle CAB = \angle ACD$ and $\angle CAD = \angle ACB$ (alternate angles).
So $\angle ECB = \angle ECD$ since $\triangle ABC$ and $\triangle ADC$ are isosceles.
$DC = BC$ (given)
EC is common.
$\therefore \triangle CDE \equiv \triangle CBE$ (SAS)
So $\angle CED = \angle CEB = 90°$.
b From part **a**, $\angle ECD = \angle ECB$.

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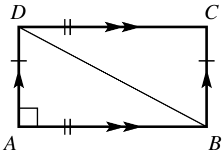
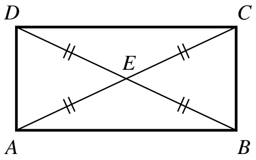
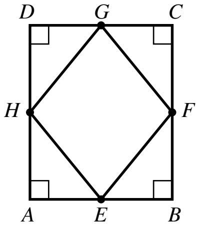
8 

As $ABCD$ is a parallelogram, $\angle BDC = \angle DBA$ (alternate angles) and $\angle DBC = \angle BDA$ (alternate angles).
BD is common.
So $\triangle CBD \equiv \triangle ADB$ (AAS).
So $\angle BAD = \angle DCB = 90°$.
Similarly, $\angle ADC = \angle 180° - \angle BAD$ (cointerior angles)
$= 90°$ and similarly for $\angle ABC$.

9 

First, prove $\triangle AED \equiv \triangle BEC$ (SAS).
Hence, corresponding angles in the isosceles triangles are equal and $\triangle CED \equiv \triangle BEA$ (SAS).
Hence, corresponding angles in the isosceles triangles are equal.
So $\angle ADC = \angle DCB = \angle CBA = \angle BAD$, which sum to 360°.
Therefore, all angles are 90° and $ABCD$ is a rectangle.

10 

First, prove all four corner triangles are congruent (SAS).
So $EF = FG = GH = HE$, so $EFGH$ is a rhombus.

2D

Building understanding

1 a Yes, both squares have all angles 90° and all sides of equal length.
b 3 c 15 cm
2 a 2 b $\frac{8}{5}$ c $\frac{4}{3}$ d $\frac{3}{2}$
3 a A b $\angle C$
c FD d $\triangle ABC \mathbin{|||} \triangle EFD$

Now you try

Example 8
a $ABCD \mathbin{|||} EFGH$ b $\frac{EH}{AD} = \frac{FG}{BC}$ c 2
d 6 e 4

Exercise 2D

1 a $ABCD \mathbin{|||} EFGH$ b $\frac{EH}{AD} = \frac{FG}{BC}$
c 2 d 8
e 4
2 a $ABCDE \mathbin{|||} FGHIJ$ b $\frac{AB}{FG} = \frac{DE}{IJ}$
c $\frac{3}{2}$ d $\frac{3}{2}$ cm
e $\frac{4}{3}$ cm
3 a $ABCD \mathbin{|||} EFGH$ b $\frac{EF}{AB} = \frac{GH}{CD}$
c $\frac{4}{3}$ d 12 m
e 10.5 m
4 a $x = 1.2$ b $x = 12.5$ c $x = 4.8$
d $x = 3.75$ e $x = 11.5$ f $x = 14.5$
5 1.7 m
6 a 1.6 b 62.5 cm
7 a 2 b 1 c 1.875 d 4.3
8 a BC b $\triangle ABC \mathbin{|||} \triangle EDC$
c 1 d 4.5
9 a True b True c False d False
e False f False g False h False
i True j True
10 Yes, the missing angle in the first triangle is 20° and the missing angle in the second triangle is 75°, so all three angles are equal.
11 a $\frac{3}{2}$
b i 4 ii 9
c i 8 ii 27
d

Cube	Length	Area	Volume
Small	2	4	8
Large	3	9	27
Scale factor (fraction)	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{27}{8}$

e Scale factor for area = (scale factor for length)^2;
Scale factor for volume = (scale factor for length)^3.
f i $\frac{b^2}{a^2}$ ii $\frac{b^3}{a^3}$

2E

Building understanding

1 a E b $\angle C$
c AB d $\triangle ABC \mathbin{|||} \triangle DEF$
2 a $\angle D$ (alternate angles) b $\angle A$ (alternate angles)
c $\angle ECD$ d CA
e $\triangle ABC \mathbin{|||} \triangle EDC$
3 a SAS b AAA c SAS d SSS

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Now you try

Example 9

a $\frac{DE}{AB} = 1.5$ (ratio of corresponding sides) S
$\frac{DF}{AC} = 1.5$ (ratio of corresponding sides) S
$\angle BAC = \angle EDF$ (given) A
$\therefore \triangle ABC\ |||\ \triangle DEF$ (SAS)

b $\angle ABC = \angle DEF$ (given) A
$\angle ACB = \angle DFE$ (given) A
$\therefore \triangle ABC\ |||\ \triangle DEF$ (AAA)

Example 10

a $\angle BAC$ is common A
$\angle ABC = \angle ADE$ (corresponding angles in parallel lines) A
$\therefore \triangle ADE\ |||\ \triangle ABC$ (AAA)

b $DE = 1$ m

Exercise 2E

1 **a** $\frac{DE}{AB} = \frac{4}{2} = 2$ (ratio of corresponding sides) S
$\frac{EF}{BC} = \frac{2}{1} = 2$ (ratio of corresponding sides) S
$\angle ABC = \angle DEF$ (given corresponding angles) A
$\therefore \triangle ABC\ |||\ \triangle DEF$ (SAS)

b $\angle ABC = \angle DEF$ (given corresponding angles) A
$\angle ACB = \angle DFE$ (given corresponding angles) A
$\therefore \triangle ABC\ |||\ \triangle DEF$ (AAA)

c $\angle ABC = \angle DEF = 65°$ A
$\angle BAC = \angle EDF = 70°$ A
$\therefore \triangle ABC\ |||\ \triangle DEF$ (AAA)

d $\frac{DE}{AB} = \frac{2}{1} = 2$ (ratio of corresponding sides) S
$\frac{EF}{BC} = \frac{6}{3} = 2$ (ratio of corresponding sides) S
$\angle ABC = \angle DEF = 120°$ A
$\therefore \triangle ABC\ |||\ \triangle DEF$ (SAS)

e $\frac{DF}{CA} = \frac{10}{5} = 2$ (ratio of corresponding sides) H
$\frac{DE}{CB} = \frac{8}{4} = 2$ (ratio of corresponding sides) S
$\angle ABC = \angle FED = 90°$ R
$\therefore \triangle ABC\ |||\ \triangle FED$ (RHS)

f $\frac{AB}{DE} = \frac{28}{7} = 4$ (ratio of corresponding sides) S
$\frac{BC}{EF} = \frac{16}{4} = 4$ (ratio of corresponding sides) S
$\frac{AC}{DF} = \frac{32}{8} = 4$ (ratio of corresponding sides) S
$\therefore \triangle ABC\ |||\ \triangle DEF$ (SSS)

2 **a** $x = 1.5$ **b** $x = 19.5$ **c** $x = 2.2$
d $a = 4$, $b = 15$
e $x = 0.16$, $y = 0.325$
f $a = 43.2$, $b = 18$

3 **a** $\angle ABC = \angle EDC$ (alternate angles)
$\angle BAC = \angle DEC$ (alternate angles)
$\angle ACB = \angle ECD$ (vertically opposite angles)
$\therefore \triangle ABC\ |||\ \triangle EDC$ (AAA)

b $\angle ABE = \angle ACD$ (corresponding angles)
$\angle AEB = \angle ADC$ (corresponding angles)
$\angle BAE = \angle CAD$ (common)
$\therefore \triangle ABE\ |||\ \triangle ACD$ (AAA)

c $\angle DBC = \angle AEC$ (given)
$\angle BCD = \angle ECA$ (common)
$\therefore \triangle BCD\ |||\ \triangle ECA$ (AAA)

d $\frac{AB}{CB} = \frac{3}{7.5} = 0.4$
$\frac{EB}{DB} = \frac{2}{5} = 0.4$ (ratio of corresponding sides)
$\angle ABE = \angle CBD$ (vertically opposite angles)
$\therefore \triangle AEB\ |||\ \triangle CDB$ (SAS)

4 **a** $\angle EDC = \angle ADB$ (common)
$\angle CED = \angle BAD = 90°$
$\therefore \triangle EDC\ |||\ \triangle ADB$ (AAA)

b $\frac{4}{3}$ cm

5 **a** $\angle ACB = \angle DCE$ (common)
$\angle BAC = \angle EDC = 90°$
$\therefore \triangle BAC\ |||\ \triangle EDC$ (AAA)

b 1.25 m

6 1.90 m

7 4.5 m

8 **a** Yes, AAA for both.
b 20 m
c 20 m
d Less working required for May's triangles.

9 The missing angle in the smaller triangle is 47°, and the missing angle in the larger triangle is 91°. Therefore the two triangles are similar (AAA).

10 **a** $\angle AOD = \angle BOC$ (common)
$\angle OAD = \angle OBC$ (corresponding angles)
$\angle ODA = \angle OCB$ (corresponding angles)
So $\triangle OAD\ |||\ \triangle OBC$ (AAA).
$\frac{OC}{OD} = \frac{3}{1} = 3$ (ratio of corresponding sides),
therefore $\frac{OB}{OA} = 3$
$OB = 3OA$

b $\angle ABC = \angle EDC$ (alternate angles)
$\angle BAC = \angle DEC$ (alternate angles)
$\angle ACB = \angle ECD$ (vertically opposite)
So $\triangle ABC\ |||\ \triangle EDC$ (AAA).
$\frac{CE}{AC} = \frac{CD}{BC} = \frac{2}{5}$, therefore $\frac{AC + CE}{AC} = \frac{5+2}{5} = \frac{7}{5}$.
But $AC + CE = AE$, so $\frac{AE}{AC} = \frac{7}{5}$ and $AE = \frac{7}{5}AC$.

11 **a** $\angle BAD = \angle BCA = 90°$
$\angle ABD = \angle CBA$ (common)
So $\triangle ABD\ |||\ \triangle CBA$ (AAA).
Therefore, $\frac{AB}{CB} = \frac{BD}{AB}$.
$AB^2 = CB \times BD$

b $\angle BAD = \angle ACD = 90°$
$\angle ADB = \angle CDA$ (common)
So $\triangle ABD\ |||\ \triangle CAD$ (AAA)
Therefore, $\frac{AD}{CD} = \frac{BD}{AD}$.
$AD^2 = CD \times BD$

c Adding the two equations:

$$\begin{aligned} AB^2 + AD^2 &= CB \times BD + CD \times BD \\ &= BD(CB + CD) \\ &= BD \times BD \\ &= BD^2 \end{aligned}$$

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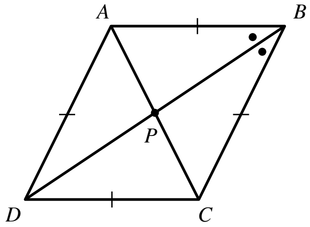
Progress quiz

1 a $w = 94$ (angle sum of an isosceles triangle)
  b $x = 78$ (exterior angle of a triangle)
  c $w = 118$ (angle sum of a pentagon)
  d $x = 120$ (interior angle of a regular hexagon)
  e $x = 97$ (cointerior angles in parallel lines, vertically opposite angles equal)
  f $x = 35$ (alternate angles in parallel lines)

2 $AB = DE$ (given)
$CB = FE$ (given)
$\angle ABC = \angle DEF$ (given)
$\therefore \triangle ABC \equiv \triangle DEF$ (SAS)

3 a $AB = QB$ (given)
  $\angle ABC = \angle QBP$ (vertically opposite)
  $\angle CAB = \angle PQB$ (alternate angles $AC \parallel PQ$)
  $\therefore \triangle ABC \equiv \triangle QBP$ (AAS)
  b $CB = PB$ corresponding sides of congruent triangles and B is the midpoint of CP.

4 Let $ABCD$ be any rhombus with diagonals intersecting at P.
$AB = BC$ (sides of a rhombus equal)
$\angle ABP = \angle CBP$ (diagonals of a rhombus bisect the interior angles through which they cross)
$\therefore \triangle ABP \equiv \triangle CBP$ (SAS)
and $\angle APB = \angle CPB$ (corresponding angles of congruent triangles).
And $\angle APB + \angle CPB = 180°$ (straight line)
$\therefore$ diagonal $AC \perp$ diagonal DB.



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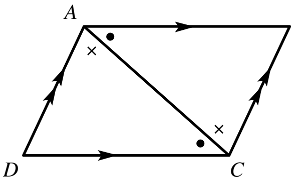


Let $ABCD$ be any parallelogram with opposite sides parallel.
AC is common.
$\angle BAC = \angle DCA$ (alternate angles $AB \parallel CD$)
$\angle BCA = \angle DAC$ (alternate angles $AD \parallel CB$)
$\therefore \triangle ABC \equiv \triangle CDA$ (AAS)
and $AB = DC$ as well as $AD = BC$ (corresponding sides in congruent triangles).

6 a $ABCD \mid\mid\mid FEHG$
  b 1.5
  c $x = 12$
  d $y = 10$

7 a $\angle CAB = \angle FDE$ (given)
  $\frac{AC}{DF} = \frac{AB}{DE} = \frac{1}{3}$ (ratio of corresponding sides)
  $\therefore \triangle CAB \mid\mid\mid \triangle FDE$ (SAS)
  b $\angle BAO = \angle CDO$ (alternate angles $AB \parallel DC$)
  $\angle AOB = \angle DOC$ (vertically opposite)
  $\therefore \triangle ABO \mid\mid\mid \triangle DCO$ (AAA)

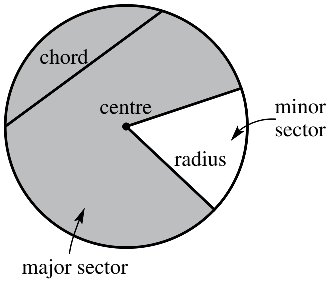
8 a $\angle D$ is common
  $\angle ABD = \angle ECD$ (corresponding angles equal since $AB \parallel EC$)
  $\therefore \triangle ABD \mid\mid\mid \triangle ECD$ (AAA)
  b 3 cm

9 $\angle A$ is common,
as Q and P are both midpoints.
$\frac{AP}{AB} = \frac{1}{2}$ and $\frac{AQ}{AC} = \frac{1}{2}$
$\therefore \triangle AQP \mid\mid\mid \triangle ACB$ (SAS)
and $\frac{QP}{CB} = \frac{1}{2}$ (corresponding sides in the same ratio).
$\therefore QP = \frac{1}{2}CB$

2F

Building understanding

1 a–e



2 a 55° b 90° c 75° d 140°

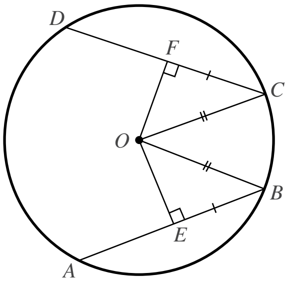
3 a 85° each
  b $\angle AOB = \angle COD$ (chord theorem 1)
  c 0.9 cm each
  d $OE = OF$ (chord theorem 2)

4 a 1 cm each b 52° each
  c $AM = BM$ and $\angle AOM = \angle BOM$ (chord theorem 3)

Now you try

Example 11
a $OF = 2$ m (chord theorem 2)
b $AM = 3$ m, $\angle AOM = 60°$ (chord theorem 3)

Example 12



Since $CD = AB$ and E and F are midpoints (from theorem 3) then $CF = BE$ (S).
Also $OC = OB$ (radii) H

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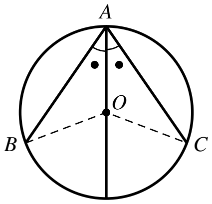
Also $\angle OFC = \angle OEB = 90^\circ$ (from theorem 3) R
$\therefore \triangle OCF \equiv \triangle OBE$ (RHS).
$\therefore OF = OE$

Exercise 2F

1 **a** $OF = 4$ cm (using chord theorem 2)
  b $AM = 3$ m
  $\angle AOM = 50^\circ$ (using chord theorem 3)
2 **a** $\angle DOC = 70^\circ$ (chord theorem 1)
  b $OE = 7.2$ cm (chord theorem 2)
  c $XZ = 4$ cm and $\angle XOZ = 51^\circ$ (chord theorem 3)
3 The perpendicular bisectors of two different chords of a circle intersect at the centre of the circle.
4 **a** 3.5 m **b** 9 m **c** 90° **d** 90°
5 **a** $a = 140$ **b** $a = 40$ **c** $a = 19$
  d $a = 72$ **e** $a = 30$ **f** $a = 54$
6 6 m
7 $3 + \sqrt{128}$ mm $= 3 + 8\sqrt{2}$ mm
8 **a** Triangles are congruent (SSS), so angles at the centre of the circle are corresponding, and therefore equal.
  b Triangles are congruent (SAS), so chords are corresponding sides, and therefore equal.
9 **a** Triangles are congruent (SSS), so the angles formed by the chord and radius are corresponding, and therefore equal. Since these angles are also supplementary, they must be 90°.
  b Triangles are congruent (SAS), so the angles formed by the chord and radius are corresponding, and therefore equal. Since these angles are also supplementary, they must be 90°.
10



First, prove $\triangle OAB \equiv \triangle OAC$ (AAS), which are isosceles. So $AB = AC$, corresponding sides in congruent triangles.

11 **a** $AD = BD$ (radii of same circle)
  $AC = BC$ (radii of same circle)
  CD is common.
  $\therefore \triangle ACD \equiv \triangle BCD$ (SSS).
  b $AC = BC$ (radii of same circle)
  $\angle ACE = \angle BCE$ (corresponding angles in congruent triangles)
  CE is common.
  $\therefore \triangle ACE \equiv \triangle BCE$ (SAS).
  c $\angle AEC = \angle BEC$ (corresponding angles in congruent triangles)
  $\angle AEC + \angle BEC = 180^\circ$, so $\angle AEC = 90^\circ$
  $\therefore CD \perp AB$

2G

Building understanding

1 **a** $\angle ADC$ **b** $\angle ADC$ **c** $\angle ADC$
  d $\angle AFC$ **e** $\angle AEC$ **f** $\angle AEC$
2 **a** $\angle AOB$ **b** $\angle ACB$ **c** 80° **d** 61°
3 **a** 180° **b** 90° **c** 60° **d** 7°

Now you try

Example 13
a $a = 50$ **b** $a = 25$

Example 14
60°

Exercise 2G

1 **a** $x = 50$ **b** $x = 40$ **c** $x = 36$
  d $x = 80$ **e** $x = 60$ **f** $x = 38$
  g $x = 250$ **h** $x = 112.5$ **i** $x = 120$
2 **a** $x = 70$ **b** $x = 25$ **c** $x = 10$
3 **a** $\angle ABC = 72^\circ$, $\angle ABD = 22^\circ$
  b $\angle ABC = 70^\circ$, $\angle ABD = 45^\circ$
  c $\angle ABC = 72^\circ$, $\angle ABD = 35^\circ$
4 **a** $\angle ADC = 75^\circ$, $\angle ABC = 75^\circ$
  b $\angle ABC = 57.5^\circ$, $\angle ADC = 57.5^\circ$
  c $\angle AOD = 170^\circ$, $\angle ABD = 85^\circ$
5 **a** 100° **b** 94.5° **c** 100°
  d 119° **e** 70° **f** 66°
6 **a** 58° **b** 53° **c** 51°
  d 45° **e** 19° **f** 21°
7 **a** 70° **b** 90°
  c The angle in a semicircle is 90°.
  d Theorem 2 is the specific case of theorem 1 when the angle at the centre is 180°.
8 **a** **i** False **ii** True **iii** True **iv** False
  b **i** False **ii** True **iii** True **iv** False
9 **a** $2x^\circ$ **b** $360 - 2x$
10 **a** $\angle AOC = 180^\circ - 2x^\circ$ ($\triangle AOC$ is isosceles)
  b $\angle BOC = 180^\circ - 2y^\circ$ ($\triangle BOC$ is isosceles)
  c $\angle AOB = 360^\circ - \angle AOC - \angle BOC = 2x^\circ + 2y^\circ$
  d $\angle AOB = 2(x^\circ + y^\circ) = 2\angle ACB$
11 **a** $\angle BOC = 180^\circ - 2x^\circ$ ($\triangle BOC$ is isosceles).
  $\angle AOB = 180^\circ - \angle BOC = 180^\circ - (180^\circ - 2x^\circ) = 2x^\circ$
  b $\angle AOC = 180^\circ - 2x^\circ$ ($\triangle AOC$ is isosceles)
  $\angle BOC = 180^\circ - 2y^\circ$ ($\triangle BOC$ is isosceles)
  Reflex $\angle AOB = 360^\circ - \angle AOC - \angle BOC$
  $= 360^\circ - (180^\circ - 2x^\circ) - (180^\circ - 2y^\circ)$
  $= 2x^\circ + 2y^\circ = 2(x + y)^\circ = 2\angle ACB$
  c $\angle OBC = x^\circ + y^\circ$ ($\triangle OCB$ is isosceles)
  $\angle COB = 180^\circ - 2(x + y)^\circ$
  $\angle AOB = 180^\circ - 2x^\circ - (180^\circ - 2(x + y)^\circ)$
  $= 2y^\circ$

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12 $\angle AOB = 180° - 2x°$ ($\triangle AOB$ is isosceles)
$\angle BOC = 180° - 2y°$ ($\triangle BOC$ is isosceles)
$\angle AOB + \angle BOC = 180°$ (supplementary angles),
therefore $(180 - 2x) + (180 - 2y) = 180$
$$\begin{aligned} 360 - 2x - 2y &= 180 \\ 2x + 2y &= 180 \\ 2(x+y) &= 180 \\ x + y &= 90 \end{aligned}$$

2H

Building understanding

1 a $\angle ACD$ b $\angle ACD$ c $\angle ACD$
2 a $\angle ABD$ and $\angle ACD$ b $85°$
c $\angle BAC$ and $\angle BDC$ d $17°$
3 a Supplementary angles sum to $180°$.
b $117°$
c $109°$
d Yes, $117° + 109° + 63° + 71° = 360°$

Now you try

Example 15
a $a = 37$ b $a = 80, b = 95$

Exercise 2H

1 a $x = 37$ b $x = 20$ c $x = 110$
d $x = 40$ e $x = 22.5$ f $x = 55$
2 a $x = 60$ b $x = 90$
c $x = 30$ d $x = 88$
e $x = 72, y = 108$ f $x = 123$
3 a $x = 72$ b $x = 43$ c $x = 69$
d $x = 57$ e $x = 52$ f $x = 48$
g $x = 30$ h $x = 47$ i $x = 108$
4 a $a = 30, b = 100$ b $a = 54, b = 90$
c $a = 105, b = 105, c = 75$ d $a = 55, b = 70$
e $a = 118, b = 21$ f $a = 45, b = 35$
5 a $80°$ b $71°$
c $\angle CBE + \angle ABE = 180°$ (supplementary angles)
$\angle CBE + \angle CDE = 180°$ (circle theorem 4)
$\therefore \angle CBE + \angle ABE = \angle CBE + \angle CDE$
$\therefore \angle ABE = \angle CDE$
6 a $\angle ACD = \angle ABD = x°$ and $\angle DAC = \angle DBC = y°$ (circle theorem 3)
b Using angle sum of $\triangle ACD$, $\angle ADC = 180° - (x° + y°)$.
c $\angle ABC$ and $\angle ADC$ are supplementary.
7 a i $80°$ ii $100°$ iii $80°$
b $\angle BAF + \angle DCB = 180°$, therefore $AF \parallel CD$ (cointerior angles are supplementary).
8 a $\angle PCB = 90°$ (circle theorem 2)
b $\angle A = \angle P$ (circle theorem 3)
c $\sin P = \frac{a}{2r}$
d As $\angle A = \angle P$, $\sin A = \frac{a}{2r}$, therefore $2r = \frac{a}{\sin A}$.

2I

Building understanding

1 a False b True c True
2 a $\angle BAP$ b $\angle BPX$ c $\angle ABP$ d $\angle APY$

Now you try

Example 16
a $a = 20$ b $a = 30$

Example 17
a $50°$ b $70°$

Exercise 2I

1 a $a = 19$ b $a = 62$ c $a = 70$
2 a $a = 50$ b $a = 28$ c $a = 25$
3 a $50°$ b $59°$
4 a $a = 73, b = 42, c = 65$
b $a = 26, b = 83, c = 71$
c $a = 69, b = 65, c = 46$
5 a 5 cm b 11.2 cm
6 a $a = 115$ b $a = 163$ c $a = 33$
d $a = 28$ e $a = 26$ f $a = 26$
g $a = 36$ h $a = 26$ i $a = 30$
7 a $a = 70$ b $a = 50$ c $a = 73$
d $a = 40$ e $a = 19$ f $a = 54$
8 4 cm
9 a OA and OB are radii of the circle.
b $\angle OAP = \angle OBP = 90°$
c $\angle OAP = \angle OBP = 90°$
OP is common
$OA = OB$
$\therefore \triangle OAP \equiv \triangle OBP$ (RHS)
d AP and BP are corresponding sides in congruent triangles.
10 a $\angle OPB = 90° - x°$, tangent meets radii at right angles
b $\angle BOP = 2x°$, using angle sum in an isosceles triangle
c $\angle BAP = x°$, circle theorem 1
11 $\angle BAP = \angle BPY$ (alternate segment theorem)
$\angle BPY = \angle DPX$ (vertically opposite angles)
$\angle DPX = \angle DCP$ (alternate segment theorem)
$\therefore \angle BAP = \angle DCP$, so $AB \parallel DC$ (alternate angles are equal).
12 $AP = TP$ and $TP = BP$, hence $AP = BP$.
13 a Let $\angle ACB = x°$, therefore $\angle ABC = 90° - x°$. Construct OP. $OP \perp PM$ (tangent). $\angle OPC = x°$ ($\triangle OPC$ is isosceles). Construct OM. $\triangle OAM \equiv \triangle OPM$ (RHS), therefore $AM = PM$. $\angle BPM = 180° - 90° - x° = 90° - x°$. Therefore, $\triangle BPM$ is isosceles with $PM = BM$. Therefore, $AM = BM$.
b Answers may vary.

2J

Building understanding

1 a 3 b 6 c 7 d 8
2 a $\frac{21}{2}$ b $\frac{5}{2}$ c $\frac{33}{7}$ d $\frac{27}{7}$

3 a $AP \times CP = BP \times DP$ b $AP \times BP = DP \times CP$
c $AP \times BP = CP^2$

Now you try

Example 18
a $\frac{4}{5}$ b $\frac{39}{5}$ c $\frac{5}{2}$

Exercise 2J

1 a 5 b 10 c $\frac{112}{15}$
2 a $\frac{143}{8}$ b $\frac{178}{9}$ c $\frac{161}{9}$
3 a $\frac{32}{3}$ b $\frac{16}{3}$ c $\frac{35}{2}$
4 a $\sqrt{65}$ b $\sqrt{77}$
5 a $\frac{64}{7}$ b $\frac{209}{10}$ c $\frac{81}{7}$
d $\frac{74}{7}$ e $\frac{153}{20}$ f $\sqrt{65} - 1$
6 a $x(x + 5) = 7 \times 8, x^2 + 5x = 56, x^2 + 5x - 56 = 0$
b $x(x + 11) = 10 \times 22, x^2 + 11x = 220,$
$x^2 + 11x - 220 = 0$
c $x(x + 23) = 31^2, x^2 + 23x = 961, x^2 + 23x - 961 = 0$
7 For this diagram, the third secant rule states:
$AP^2 = DP \times CP$ and $BP^2 = DP \times CP$, so $BP = AP$.
8 $AP \times BP = DP \times CP$
$AP \times BP = AP \times CP$ since $AP = DP$.
$BP = CP$
9 a $\angle A = \angle D$ and $\angle B = \angle C$ (circle theorem 3)
b $\angle P$ is the same for both triangles (vertically opposite), so $\triangle ABP \mathrel{|||} \triangle DCP$ (AAA).
c $\frac{AP}{DP} = \frac{BP}{CP}$
d $\frac{AP}{DP} = \frac{BP}{CP}$, cross-multiplying gives $AP \times CP = BP \times DP$.
10 a $\angle B = \angle C$ (circle theorem 3)
b $\angle P$ is the same for both triangles (common), so $\triangle PBD \mathrel{|||} \triangle PCA$ (AAA)
c $\frac{AP}{DP} = \frac{CP}{BP}$, so $AP \times BP = DP \times CP$.
11 a Yes
b Alternate segment theorem
c $\triangle BPC \mathrel{|||} \triangle CPA$ (AAA)
d $\frac{BP}{CP} = \frac{CP}{AP}$, so $CP^2 = AP \times BP$.
12 $d = \sqrt{4r_1r_2} = 2\sqrt{r_1r_2}$

2K

Building understanding

1 a 4
b 3
c i 3 ii 3 iii 3
d i 3 ii 3 iii 2
2 a 3 b 3 c 4 d 8
e The sum of degrees is twice the total number of edges.

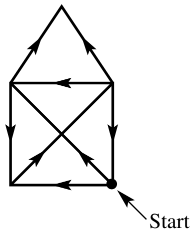
Now you try

Example 19
a i 5 ii 6
b i 3 ii 2 iii 3
iv 3 v 1
c 12
d i A, C, D and E ii B

Example 20
a 3 b 4

Exercise 2K

1 a i 3 ii 4
b i 2 ii 3 iii 3
c 8
d i B and C ii A
2 a i 4 ii 5
b i 3 ii 1 iii 3 iv 3
c 10
d i A, B, C and D ii None
3 a i 4 ii 5
b i 2 ii 3 iii 3 iv 2
c 10
d i B and C ii A and D
4 C
5 B
6 a 2 b 3 c 3
7 C
8 a No b No c No
9 B
10 Yes
11 a 8
b 16
c The sum of degrees is twice the number of edges.
12 a No b No
c Yes
13 a 2 b 5
14 a 6 b 4



2L

Building understanding

1 a Answers will vary, a rectangle of ABCD would be an example.
b Yes
2 a i 5 ii 7 iii 4
b i 9 ii 9
c They are equal.

Now you try

Example 21
No

Example 22

a Yes b No

Example 23

a i 4 ii 6 iii 4

b $v + f = 8$ and $e + 2 = 8$

Exercise 2L

1 a Yes b No c No
  d Yes e Yes

2 a Planar b Planar c Non-planar
  d Non-planar e Planar f Planar

3 a i 4 ii 6 iii 4
  b $v + f = 8$ and $e + 2 = 8$

4 a i 5 ii 7 iii 4
  b $v + f = 9$ and $e + 2 = 9$

5 A and C

6 A and C

7 A, B, C, D and E

8 a 12 b 6 c 4

9 a $v = 4,\ e = 6$ and $f = 4$
  b $v + f = 8$ and $e + 2 = 8$
  c Answers will vary.
  d $v + f = 8$ and $e + 2 = 8$

10 a $v = 6, e = 9$ and $f = 5$
  $v + f = 11$ and $e + 2 = 11$
  b $v = 8, e = 12$ and $f = 6$
  $v + f = 14$ and $e + 2 = 14$

11 $v = 12, e = 18$ and $f = 8$
  $v + f = 20$ and $e + 2 = 20$

12 a i $n = 3, e = 6, d = 3$
  ii $\frac{1}{n} + \frac{1}{d} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$
  $\frac{1}{2} + \frac{1}{e} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$
  b Answers will vary.
  c $v + f = 6 + 5 = 11$ and $e + 2 = 9 + 2 = 11$. The solid is not regular so n and d cannot be determined.

2M

Building understanding

1 a Walk b Circuit
  c Cycle d Trail
  e Path

2 a B and C b e.g. B-A-C-B-D-C
  c Eulerian trail d Yes
  e 2 f 0 or 2

Now you try

Example 24

a i No ii Yes

b i No ii Yes

c i Yes ii No

d i No ii Yes

Example 25

Yes, Eulerian trails will be circuits.

Exercise 2M

1 a i Yes ii Yes
  b i Yes ii No
  c i No ii Yes
  d i Yes ii No

2 a i Yes ii No
  b i Yes ii No
  c i No ii Yes
  d i Yes ii No

3 a i Yes ii No
  b i Yes ii No
  c i Yes ii No
  d i Yes ii No

4 a Yes, yes b Yes, no
  c Yes, no d Yes, yes
  e No f No

5 a Yes b No c Yes d No

6 a B-A-E-D-B-C-D
  B-A-E-D-C-B-D
  B-C-D-E-A-B-D
  B-C-D-B-A-E-D
  B-D-C-B-A-E-D
  B-D-E-A-B-C-D
  b D-C-B-D-E-A-B
  D-C-B-A-E-D-B
  D-B-C-D-E-A-B
  D-B-A-E-D-C-B
  D-E-A-B-C-D-B
  D-E-A-B-D-C-B

7 a Yes, 4 b No

8 6

9 Yes, B-C-D-E-C-G-E-F-G-B-F-A-B

10 a Graph 2
  b Bridges
  c 4
  d Does not have an Eulerian trail

11 a T b T c F
  d F e T f F

12 Has two vertices of odd degree.

13 a i No ii No
  b Yes, add another DE edge.
  c No
  d 2

2N

Building understanding

1 a 2 b 4 c 3
  d 12 km e 27 km

2 a i 15 km ii 16 km
  b 9 km
  c A, B, D and F

Now you try

Example 26

a 13 m b 14 m

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Example 27

a i 19 m ii 15 m

b i 29 m ii 25 m

Exercise 2N

1 **a** 26 km **b** 24 km

2 **a** 26 m **b** 39 m

3 **a** 15 cm **b** 19 cm

4 **a** i 4 km ii 12 km

b i 6 km ii 5 km

5 **a** 11 km **b** 9 km **c** 6 km **d** 7 km

6 8 km

7 **a** 7 cm **b** 7 cm

8 A-B-C is shorter than A-C and C-D-E is shorter than C-E.

9 **a** A-B-C-D-E-G **b** $270

10 **a** i No ii No iii Yes

b True

11 **a** i No ii No iii No iv Yes

b 15 km **c** 12 km **d** No

Problems and challenges

1 21 units^2

2 $BD = 5$ cm, $CE = 19$ cm

3 $\angle ADE = \angle ABE, \angle EFD = \angle BFA, \angle DEB = \angle DAB,$
$\angle DFB = \angle EFA, \angle CDB = \angle CAE, \angle DAE = \angle DBE,$
$\angle ADB = \angle AEB, \angle ABD = \angle AED = \angle CBD = \angle CEA$

4 42.5%

5 Check with your teacher.

6 **a** $\angle FDE = \angle DFC = \angle ABC$ (alternate and corresponding angles in parallel lines)

$\angle FED = \angle EFB = \angle ACB$ (alternate and corresponding angles in parallel lines)

$\angle DFE = \angle BAC$ (angle sum of a triangle)

$\triangle ABC \;|||\; \triangle FDE$ (AAA)

b i 4 : 1 ii 16 : 1

c $4^{n-1} : 1$

Answers to success criteria example questions

1 $x = 100$ (isosceles triangle and angle sum of a triangle)

2 $x = 85$ (exterior angle theorem)

3 $x = 125$

4 **a** $x = 68$ (cointerior angles in || lines and vertically opposite angles)

b $x = 70$ (alternate and cointerior angles in || lines)

5 $AB = EF$ (S)
$\angle BAC = FED$ (A)
$\therefore \Delta ABC \equiv \Delta EFD$ (SAS)

6 $\angle CAD = \angle CBD$ (A)
$\angle CDA = \angle CDB$ (A)
CD is common (S)
$\therefore \Delta ADC \equiv \Delta BDC$ (AAS)
$\therefore AC = 8$ cm (corresponding sides in congruent triangles)

7 $\angle ABD = \angle CDB$ (alternate angels in || lines) (A)
$\angle ADB = \angle CBD$ (alternate angels in || lines) (A)
BD is common (S)
$\therefore \Delta ABD = \Delta CDB$ (AAS)
$\therefore AD = CD$ and $AB = CD$ (corresponding sides in congruent triangles)

8 $\angle ADB = \angle CBD$ (alternate angles in || lines)
$AD = CB$ (given) (S)
BD is common
$\therefore \Delta ABD \equiv \Delta CDB$ (SAS)
$\therefore \angle ABD \equiv \angle CDB$ (corresponding angles in congruent triangles)
So $AB \| CD$ (since alternate angles are equal).
$\therefore ABCD$ is a parallelogram.

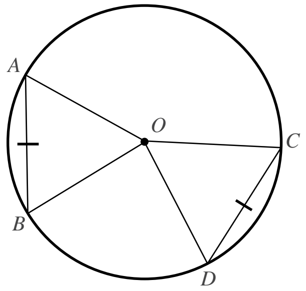
9 $\frac{3}{2}$; $x = 5$; $y = 4.5$

10 $\frac{EF}{BC} = \frac{12}{6} = 2$ (S)
$\frac{ED}{BA} = \frac{5}{2.5} = 2 = \frac{EF}{BC}$ (S)
$\angle ABC = \angle DEF$ (given) (A)
$\therefore \Delta ABC|||DEF$ (SAS)

11 $\angle BAC = \angle DAE$ (common) (A)
$\angle BCA = \angle DEA$ (corresponding angles in parallel lines) (A)
$\therefore \Delta ADE|||\Delta ABC$ (AAA)
$DE = 1$ cm

12 $\angle AOM = 50°, \angle OMB = 90°$

13



$OA = OB = OC = OD$ (radii)
$AB = CD$ (given)
$\Delta AOB \equiv \Delta COD$ (SSS)
$\therefore \angle AOB = \angle COD$ (corresponding angles in congruent triangles)

14 $y = 41$

15 $y = 25$

16 68°

17 $a = 75$

18 $a = 85$

19 $b = 75$

20 65°

21 $x = \frac{20}{3}$

22 i $x = 5$
ii $x = \frac{39}{5}$

23 i 5 ii 3

24 4

25 Yes, isomorphic

26 Planar

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27 $v = 4,\ e = 6,\ f = 4$; thus $v + f = 8 = e + 2$ or $v - e + f = 4 - 6 + 4 = 2$ as required

28 i Yes
 ii No

29 Yes, Eulerian trail, e.g. $C–D–E–A–B–E–C–B$ but no Eulerian circuits as there are vertices of odd degree

30 33 km

31 260 m

Short-answer questions

1 a $x = 65$
 b $y = 120$
 c $x = 62, y = 118$
 d $x = 46$

2 a 148° b 112°

3 a $AB = DE$ (given)
 $\angle ABC = \angle DEF$ (given)
 $\angle BAC = \angle EDF$ (given)
 $\therefore \triangle ABC \equiv \triangle DEF$ (AAS)
 b $AB = AD$ (given)
 $\angle BAC = \angle DAC$ (given)
 AC is common.
 $\therefore \triangle ABC \equiv \triangle ADC$ (SAS)
 c $AB = CD$ (given)
 $AD = CB$ (given)
 BD is common.
 $\therefore \triangle ABD \equiv \triangle CDB$ (SSS)

4 a $AB = CD$ (given)
 $\angle BAC = \angle DCA$ (alternate angles)
 AC is common.
 $\therefore \triangle ABC \equiv \triangle CDA$ (SAS)
 b $\angle BCA = \angle DAC$ (alternate angles), therefore $AD \parallel BC$ (alternate angles are equal).

5 a $\frac{DE}{AB} = \frac{10.5}{7} = 1.5$
 $\frac{EF}{BC} = \frac{14.7}{9.8} = 1.5$ (ratio of corresponding sides)
 $\angle ABC = \angle DEF$ (given)
 $\therefore \triangle ABC \mathbin{|||} \triangle DEF$ (SAS)
 $x = 19.5$
 b $\angle EAB = \angle DAC$ (common)
 $\angle EBA = \angle DCA$ (corresponding with $EB \parallel DC$)
 $\therefore \triangle ABE \mathbin{|||} \triangle ACD$ (AAA)
 $x = 6.25$
 c $\angle BAC = \angle EDC$ (given)
 $\angle ACB = \angle DCE$ (vertically opposite)
 $\therefore \triangle ABC \mathbin{|||} \triangle DEC$ (AAA)
 x = 8.82
 d $\angle ABD = \angle DBC$ (given)
 $\angle DAB = \angle CDB = 35°$ (angle sum of triangle)
 $\therefore \triangle ABD \mathbin{|||} \triangle DBC$ (AAA)
 $x = \frac{100}{7}$

6 a $a = 65$ (chord theorem 1)
 b $x = 7$ (chord theorem 2)
 c $x = 6$ (chord theorem 3)

7 a $a = 25$ b $a = 50, b = 40$
 c $a = 70$ d $b = 54$
 e $a = 115$ f $a = 30, b = 30$

8 a $x = 26, y = 58, z = 64$
 b $a = 65, b = 130, c = 50, d = 8$
 c $t = 63$

9 a $x = 5$ b $x = 6$ c $x = \frac{40}{3}$

10 a No, BC not joined by an edge.
 b i Planar ii Planar

11 a $e = 6, v = 4$
 b $f = 4$
 c Yes, $v + f = e + 2$

12 a No, AC repeated.
 b It has more than 2 odd vertices.
 c DF or DB or DA
 d Answer will vary. It will not be an Eulerian circuit as it will not have no odd vertices.

13 a 15 cm b 12 cm

Multiple-choice questions

1	C	2	B	3	B	4	C	5	B
6	A	7	E	8	C	9	D	10	B
11	C	12	C	13	D	14	A		

Extended-response questions

1 a $\angle BAC = \angle BDE = 90°$
 $\angle B$ is common.
 $\therefore \triangle ABC \mathbin{|||} \triangle DBE$ (AAA)
 b 1.2 km
 c i $\frac{AC}{DE} = \frac{3}{2}$
 $\therefore \frac{AB}{DB} = \frac{3}{2}$ (ratio of corresponding sides in
 $\frac{x+1}{x} = \frac{3}{2}$ (similar triangles)
 $\therefore 2(x + 1) = 3x$
 ii 2
 d 44.4%

2 a 14
 b 9 km
 c $v + f = 5 + 4 = 9$ and $e + 2 = 7 + 2 = 9$
 d No, every edge is not included exactly once.
 e i 9 km
 ii 16 km

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Chapter 3

3A

Building understanding

1 a 3^4 b 7^6 c $6x^3$ d $8a^3b^2$

2

x	4	3	2	1	0
2^x	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$

3 a $2^2 \times 2^3 = 2 \times 2 \times 2 \times 2 \times 2$
$= 2^5$

b $\frac{x^5}{x^3} = \frac{x \times x \times x \times x \times x}{x \times x \times x}$
$= x^2$

c $(a^2)^3 = a \times a \times a \times a \times a \times a$
$= a^6$

d $(2x)^0 \times 2x^0 = 1 \times 2$
$= 2$

Now you try

Example 1

a x^7 b $14a^3b^5$ c m^2 d $\frac{1}{2}xy^2$

Example 2

a a^6 b $27y^9$ c $\frac{x^6}{25y^2}$

Example 3

a 2 b 6

Example 4

a $2ab^5$ b $4x^5y^4$

Exercise 3A

1 a a^9 b x^5 c b^6
d $14m^5$ e $6s^7$ f $2t^{16}$
g $\frac{p^3}{5}$ h $\frac{c^7}{6}$ i $\frac{9}{25}s^2$
j $6x^3y^3$ k $15a^3b^6$ l $18v^9w^2$
m $150x^5y^6$ n $12r^7s^6$ o $20m^8n^{10}$

2 a x^3 b a c q^3
d b^4 e y^5 f d^5
g j h m^6 i $2xy^3$
j $3r^2s$ k $2p^2$ l $2m^4x$
m $5b^3$ n $4st$ o $\frac{1}{4}v^2$
p $\frac{a}{2}$ q $-\frac{x}{3}$ r $-\frac{y^2}{2}$

3 a x^{10} b t^6
c $4a^6$ d $5y^{15}$
e $64t^6$ f $4u^4$
g $27r^9$ h $81p^{16}$
i $\frac{a^4}{b^6}$ j $\frac{x^9}{y^{12}}$
k $\frac{x^4y^6}{z^8}$ l $\frac{u^{16}w^8}{v^8}$
m $\frac{27f^6}{125g^3}$ n $\frac{9a^4b^2}{4p^2q^6}$
o $\frac{a^3t^9}{27g^{12}}$ p $\frac{256p^8q^{12}}{81r^4}$

4 a 8 b 3 c 1 d 1
e 5 f 3 g −5 h 3

5 a x^8 b x^2y^2 c x^6n^8
d xy^2 e m f r^4s^7
g $\frac{9x^8y^2}{2}$ h $2y^4$ i $2a^2b^2$
j $27m^7n^{14}$ k $-45a^8b^5$ l $\frac{16}{3}f^3$
m $2m^6n^3$ n $21y^3z^2$ o 1
p $-6m^2n^7$

6 a −27 b −27 c 81 d −81

7 a x^{12} b a^{105} c $\frac{a^{30}}{b^{15}}$

8 a 13 b 18 c 81 d 64
e 1 f 1 g 9 h 8

9 He has not included the minus sign inside the brackets, i.e. has only applied it afterwards. Need $(-2)^4$ not -2^4.

10 a 3 b 4 c 1
d 3 e 4 f 1

11 a 9 b 2 c 162 d −18

12 a ±2 b 5 c 2 d $\frac{7}{2}$

13 a $x = 2$, $y = 4$ or $x = 4$, $y = 2$ or $x = 16$, $y = 1$
b $x = 8$, $y = 2$ or $x = 4$, $y = 3$ or $x = 64$, $y = 1$, or $x = 2$, $y = 6$
c $x = 9$, $y = 2$, or $x = 3$, $y = 4$ or $x = 81$, $y = 1$
d $x = 1$, $y =$ any positive integer

3B

Building understanding

1 a $2^{-2}, 2^{-3}, 2^{-4}$ b x^{-1}, x^{-2}, x^{-3}

2 a $\frac{1}{3^2}$ b $\frac{1}{5^2}$ c $\frac{5}{4^2}$ or $\frac{5}{2^4}$ d $\frac{-2}{3^3}$

3 a $\frac{1}{a^b}$ b a^b

4 a $\frac{1}{25}$ b $\frac{1}{27}$ c $\frac{4}{49}$

Now you try

Example 5

a $\frac{1}{b^3}$ b $\frac{2y^3}{x^2}$ c $2x^4$

Example 6

a $\frac{5}{a^2b^4}$ b $\frac{n^8}{3m^7}$

Example 7

a $\frac{x^{13}}{y^{11}}$ b $\frac{1}{2p^3q}$

Exercise 3B

1. a $\frac{1}{x^5}$ b $\frac{1}{a^4}$ c $\frac{2}{m^4}$ d $\frac{3}{y^7}$
e $\frac{3a^2}{b^3}$ f $\frac{4m^3}{n^3}$ g $\frac{10y^5z}{x^2}$ h $\frac{3z^3}{x^4y^2}$
i $\frac{q^3r}{3p^2}$ j $\frac{d^2f^5}{5e^4}$ k $\frac{3u^2w^7}{8v^6}$ l $\frac{2b^3}{5c^5d^2}$

2 a x^2 b $2y^3$
c $4m^7$ d $3b^5$
e $2b^4d^3$ f $3m^2n^4$
g $\frac{4b^4a^3}{3}$ h $\frac{5h^3g^3}{2}$

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3 a x b a^3 c $\frac{2}{b^4}$ d $\frac{3}{y^3}$
e $\frac{1}{xy}$ f $\frac{4y^2}{a^3}$ g $\frac{6}{a^5b^2}$ h $\frac{18b^4}{a^2}$
i $\frac{a^2}{b^2}$ j $\frac{m^2}{n}$ k $\frac{x}{2y}$ l $\frac{4m}{7n^3}$
m $\frac{a}{b^3}$ n $\frac{q}{p^5}$ o $\frac{p}{q^2r}$ p $\frac{4r^3s^{14}}{3}$

4 a $\frac{2x}{3}$ b $\frac{7d^2}{10}$ c $\frac{5}{3s^3}$ d $\frac{4}{3f^2}$
e $\frac{f^5}{g^5}$ f $\frac{1}{r^6s^2}$ g $\frac{wx^5}{2}$ h $\frac{5c^5d^4}{4}$

5 a $\frac{16}{x^4}$ b $\frac{1}{64m^6}$ c $\frac{2}{x^{21}}$ d $\frac{4}{d^6}$
e $\frac{9}{t^8}$ f $\frac{5}{x^4}$ g $\frac{81}{x^{20}}$ h $\frac{-8}{x^{15}}$
i $\frac{y^4}{16}$ j $\frac{h^{12}}{81}$ k $7j^8$ l $2t^6$

6 a a^7b^2 b $\frac{16p^4}{9q^2}$ c $54x^7y^{10}$
d $4a^8b^3$ e $\frac{324r^{11}}{s}$ f $\frac{2y^{14}}{x^3}$
g a^2b^{18} h $\frac{m^{14}}{n^8}$ i $\frac{27x}{2y}$

7 a $\frac{1}{25}$ b $\frac{1}{64}$ c $\frac{2}{49}$ d $\frac{-5}{81}$
e $\frac{1}{9}$ f 1 g 98 h −48
i $\frac{9}{4}$ j $\frac{-64}{125}$ k $\frac{1}{16}$ l 100

8 0.0041 cm

9 The negative index should only be applied to x not to 2:
$2x^{-2} = \frac{2}{x^2}$

10 a i $\frac{3}{2}$ ii $\frac{7}{5}$ iii $\frac{y}{2x}$
b $\frac{b}{a}$

11 a $\frac{5}{6}$ b $\frac{5}{18}$ c $\frac{1}{3}$
d $-\frac{7}{12}$ e $\frac{71}{48}$ f $\frac{106}{9}$

12 Proof: $\left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-1\times x} = 2^{-x}$

13 a −2 b −5 c −3 d −1
e −2 f −3 g −3 h −4
i 0 j 0 k 1 l 2
m −2 n 1 o 2

3C

Building understanding

1 a 3 b 3 c 3 d 3
2 a 10^3 b 10^7 c 10^{-6} d 10^{-3}
3 a 4 b 5 c 3
d −4 e −3 f −5

Now you try

Example 8
a 20 480 b 0.000047

Example 9
a 7.94×10^6 b 2.71×10^{-4}

Example 10
a 1.13×10^2

Exercise 3C

1 a 3120 b 54 293
c 710 500 d 8 213 000
e 59 500 f −800 200
g −10 120 h 9 990 000
i 210 500 000 j −55 000
k 2 350 000 000 l 1 237 000 000 000

2 a 0.0045 b 0.0272
c 0.0003085 d 0.00783
e −0.000092 f 0.265
g 0.0001002 h −0.000006235
i 0.98 j −0.000000000545
k 0.000000000003285 l 0.000000875

3 a 6.24×10^3 b -5.73×10^5 c 3.02×10^4
d 4.24×10^5 e -1.01×10^4 f 3.50×10^7
g 7.25×10^4 h 3.56×10^5 i 1.10×10^8
j 9.09×10^5 k -4.56×10^6 l 9.83×10^9

4 a 2.42×10^{-3} b -1.88×10^{-2} c 1.25×10^{-4}
d 7.87×10^{-3} e 7.08×10^{-4} f 1.14×10^{-1}
g 6.40×10^{-6} h 7.89×10^{-5} i 1.30×10^{-4}
j 7.01×10^{-7} k 9.89×10^{-9} l -5.00×10^{-4}

5 a -2.4×10^4 b 5.71×10^6 c 7.0×10^8
d 4.88×10^3 e 1.9×10^{-3} f -7.05×10^{-4}
g 9.8×10^{-6} h -3.571×10^{-1} i 5.00×10^{-5}

6 a 4.41×10^{-8} b 6.38×10^{-3} c 8.00×10^7
d 3.63×10^8 e 1.80×10^{-3} f 1.02×10^2

7 a 7.7×10^6 km^2 b 2.5×10^6
c 7.4×10^9 km d 1×10^{-2} cm
e 1.675×10^{-27} kg f 9.5×10^{-13} g

8 328 minutes

9 38 is larger than 10.

10 a 2.1×10^4 b 3.94×10^9 c 6.004×10^1
d 1.79×10^{-4} e 2×10^3 f 7×10^{-1}
g 1×10^7 h 6×10^6 i 4×10^{-3}
j 3.1×10^{-14} k 2.103×10^{-4} l 9.164×10^{-21}

11 a 9×10^4 b 8×10^9 c 6.4×10^9
d 1.44×10^{-8} e 4×10^4 f 6.25×10^{-12}
g 2.25×10^{-6} h 1.25×10^7 i 1×10^{-5}
j 1.275×10^{-4} k 1.8×10^{-1} l 2×10^2
m 8×10^{-1} n 4×10^{-14} o 2.5×10^4

12 $3 \times 10^{-4} = 3 \div 10\,000$

13 a i 9×10^{17} J ii 2.34×10^{21} J
iii 2.7×10^{15} J iv 9×10^{11} J
b i 1.11×10^8 kg ii 4.22×10^{-1} kg
iii 9.69×10^{-13} kg iv 1.89×10^{-19} kg
c 5.4×10^{41} J

3D

Building understanding

1 a 3, 2 b 5, 5 c 4, 4 d 5, 5
2 a 3 b 11 c 3 d 4
e 2 f 3 g 2 h 10
3 a 1.91, 1.91 b 1.58, 1.58 c 1.43, 1.43

Now you try

Example 11
a $11^{\frac{1}{2}}$ b $3^{\frac{1}{2}}x^{\frac{7}{2}}$ c $2x^{\frac{9}{4}}$ d $7^{\frac{3}{2}}$

Example 12
a $\sqrt[3]{5}$ b $\left(\sqrt[3]{11}\right)^2$ or $\sqrt[3]{121}$

Example 13
a 5 b $\frac{1}{2}$ c 16

Exercise 3D

1 a $29^{\frac{1}{2}}$ b $35^{\frac{1}{3}}$ c $x^{\frac{2}{5}}$ d $b^{\frac{3}{4}}$
e $2^{\frac{1}{2}}a^{\frac{1}{2}}$ f $4^{\frac{1}{3}}t^{\frac{7}{3}}$ g $10^{\frac{1}{5}}t^{\frac{2}{5}}$ h $8^{\frac{1}{8}}m^{\frac{1}{2}}$
2 a $7x^{\frac{5}{2}}$ b $6n^{\frac{7}{3}}$ c $3y^3$
d $5p^{\frac{2}{3}}r^{\frac{1}{3}}$ e $2a^{\frac{4}{3}}b^{\frac{2}{3}}$ f $2g^{\frac{3}{4}}h^{\frac{5}{4}}$
g $5^{\frac{3}{2}}$ or $125^{\frac{1}{2}}$ h $7^{\frac{3}{2}}$ or $343^{\frac{1}{2}}$ i $4^{\frac{4}{3}}$ or $256^{\frac{1}{3}}$
3 a $\sqrt[5]{2}$ b $\sqrt[7]{8}$ c $\sqrt[3]{6}$ d $\sqrt[10]{11}$
e $\sqrt[3]{9}$ f $\sqrt[3]{49}$ g $\sqrt[5]{8}$ h $\sqrt[7]{81}$
4 a 6 b 3 c 4 d 7
e 2 f 5 g $\frac{1}{3}$ h $\frac{1}{2}$
i $\frac{1}{3}$ j $\frac{1}{10}$ k $\frac{1}{20}$ l $\frac{1}{10}$
5 a 4 b 8 c 216 d 32
e $\frac{1}{8}$ f $\frac{1}{9}$ g $\frac{1}{16}$ h $\frac{1}{125}$
i 125 j $\frac{1}{16}$ k $\frac{1}{81}$ l $\frac{1}{100}$
6 a a^2 b m^3 c x d $b^{\frac{1}{2}}$
e $s^{\frac{6}{7}}$ f $y^{\frac{1}{9}}$ g 1 h $\frac{a^{\frac{1}{2}}}{b}$
7 a $5s^2$ b $3t^2$ c $2t^2$ d $5t^4$
e x f b^4 g t^3 h m^2
i $4ab^4$ j $6m^2n$ k $2x^2y^3$ l $7r^3t^2$
m $\frac{5}{7}$ n $\frac{2x}{3}$ o $\frac{2}{x^2}$ p $10x$
8 a Method B
b i 32 ii 216 iii 128
iv 81 v 625 vi $\frac{1}{27}$
vii $\frac{32}{3125}$ viii $\frac{81}{10\,000}$
9 It equals 2 since $2^6 = 64$.
10 a i -3 ii -10
iii -2 iv -3
b i No ii Yes
iii Yes iv No
c y is a real number when n is odd, for $x < 0$.

3E

Building understanding

1 a i 4 ii 8 iii 16 iv 32
b i 3 ii 5 iii 6
2 a 16, 32, 64, 128, 256, 512
b 81, 243, 729, 2187, 6561
c 64, 256, 1024, 4096
d 125, 625, 3125
e 216, 1296
3 a 3^2 b 5^3 c 3^5
d 2^7 e 3^6

Now you try

Example 14
a $x = 3$ b $x = -3$ c $x = \frac{3}{2}$

Example 15
$x = 1$

Exercise 3E

1 a $x = 3$ b $x = 3$ c $x = 2$ d $x = 2$
e $x = 3$ f $x = 3$ g $x = 4$ h $x = 3$
i $x = 4$ j $x = 5$ k $x = 4$ l $x = 3$
2 a $x = -2$ b $x = -2$ c $x = -2$
d $x = -4$ e $x = -5$ f $x = 3$
g $x = 2$ h $x = 6$ i $x = 3$
3 a $x = \frac{3}{2}$ b $x = \frac{4}{3}$ c $x = \frac{3}{2}$ d $x = \frac{3}{2}$
e $x = \frac{1}{2}$ f $x = \frac{1}{3}$ g $x = \frac{1}{5}$ h $x = \frac{1}{4}$
i $x = -2$ j $x = -4$ k $x = -\frac{3}{2}$ l $x = -\frac{3}{2}$
4 a 1
b i 2 ii 32 iii 2^{60} iv 2^{1440}
c i 3 min ii 8 min iii 10 min
5 a $x = \frac{1}{2}$ b $x = 1$ c $x = 3$ d $x = 1$
e $x = \frac{3}{4}$ f $x = 2$ g $x = 9$ h $x = \frac{6}{7}$
i $x = \frac{15}{4}$ j $x = -\frac{11}{2}$ k $x = 4$ l $x = -\frac{3}{2}$
6 1 cent doubled every second for 30 seconds. Receive 2^{30} cents, which is more than 1 million dollars.
7 a i 1 ii 1 iii 1
b No solutions. If $a = 1$, then $a^x = 1$ for all values of x.
8 a 2 b 1 c $\frac{2}{3}$ d $\frac{3}{4}$
e $\frac{5}{4}$ f $\frac{1}{3}$ g $\frac{3}{10}$ h $-\frac{1}{2}$
9 a i 0.25 ii 0.125
iii 0.001 iv 0.0016
b i 5^{-2} ii 2^{-4}
iii 2^{-1} iv 5^{-4}
10 a -4 b -6 c -5
d $\frac{1}{2}$ e $-\frac{3}{2}$ f $-\frac{3}{4}$
11 a 1 b -1 c 8 d $-\frac{3}{2}$
e $-\frac{5}{2}$ f -2 g 3 h $\frac{2}{3}$
i $\frac{1}{5}$ j 2 k 0 l -2

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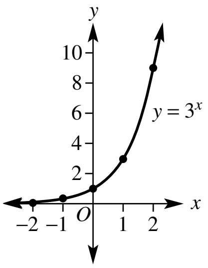
3F

Building understanding

1 a

x	-2	-1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

b

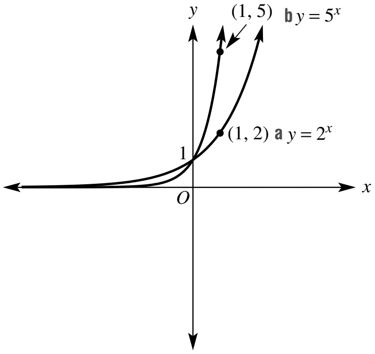


2 a asymptote b (0, 1)
c y-axis d x-axis

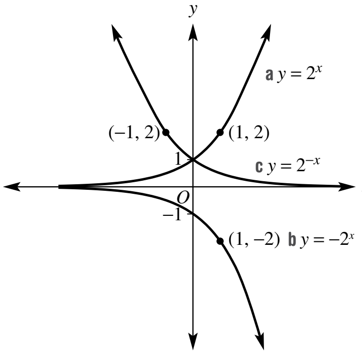
3 a $a^{-2} = \frac{1}{a^2} \neq -a^2$ b False since $3^{-2} = \frac{1}{3^2}$.
c $5^{-3}, 3^{-2}, 2^{-1}$ d $-9, -125, -\frac{1}{4}$

Now you try

Example 16



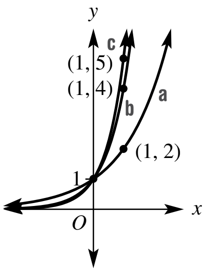
Example 17



Example 18
(3, 27)

Exercise 3F

1



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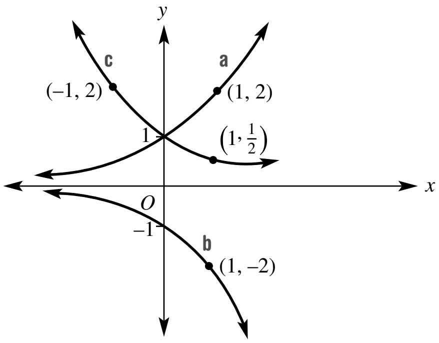

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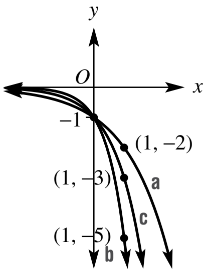
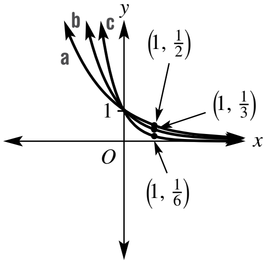


4



5 a (2, 4) b (2, 9) c (2, 16)
d (1, −4) e (0, −1) f (−3, 8)

6 a 1000
b i 2000 ii 8000
c i 2 years ii 4 years

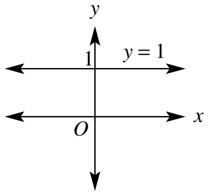
7 a $N = 2^t$
b $N = 2^{10} = 1024$
c 14 seconds

8 $x = 2.322$

9 **a** C **b** A **c** D
d E **e** F **f** B

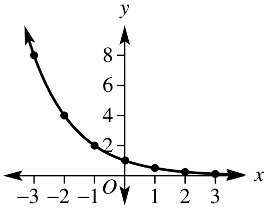
10 Substitute (2, 5) into the equation $y = 2^2 = 4 \neq 5$.

11 $y = 1$



12 It is the asymptote.

13 **a**



They are the same graph.

b **i** $y = \left(\frac{1}{3}\right)^x$ **ii** $\left(\frac{1}{5}\right)^x$ **iii** $y = \left(\frac{1}{10}\right)^x$

c **i** $y = 4^{-x}$ **ii** $y = 7^{-x}$ **iii** $y = 11^{-x}$

d $\frac{1}{a} = a^{-1}$, thus $\left(\frac{1}{a}\right)^x = \left(a^{-1}\right)^x = a^{-x}$ as required (or similar)

Progress quiz

1 **a** a^5 **b** $12x^3y^4$ **c** h^4 **d** $\frac{m^6n^3}{2}$
e a^6 **f** $9m^{10}$ **g** $\frac{m^6}{64}$ **h** 6

2 **a** $2xy^4$ **b** $\frac{4p^8}{49q^2r^4}$

3 **a** $\frac{1}{x^3}$ **b** $\frac{2b^4}{a^2c^3}$ **c** $7m^2$ **d** $\frac{4}{5d^2}$
e $\frac{16}{k^8}$ **f** $\frac{a^6}{8}$ **g** $\frac{12m}{a^5}$ **h** $\frac{4d^5}{3c^2}$

4 **a** $\frac{x^{10}}{y^{18}}$ **b** $\frac{a^{20}b^{19}}{5}$

5 **a** 32 040 000 **b** 0.00047

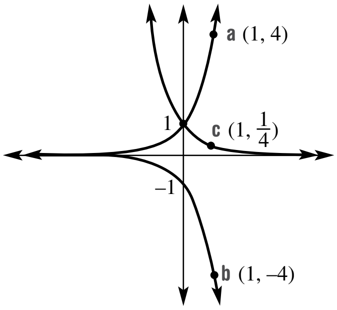
6 **a** 3.47×10^4 **b** 4.57×10^{-4} **c** 1.09×10^9

7 **a** $10^{\frac{1}{2}}$ **b** $4^{\frac{1}{3}}x^{\frac{2}{3}}$ **c** $6^{\frac{3}{2}}$

8 **a** $\sqrt[7]{4}$ **b** 5 **c** $\frac{1}{2}$

9 **a** $x = 3$ **b** $x = -3$ **c** $x = 2$

10



3G

Building understanding

1 **a** \$50 **b** \$1050 **c** \$52.50
d \$55.13 **e** \$1276.28

2 **a** 4.9 kg **b** $\frac{2}{100}$, 0.98 **c** 4.52 kg

3 **a** Growth **b** Decay **c** Growth **d** Decay

Now you try

Example 19
a $A = 50\,000(1.16)^n$ **b** $P = 10\,000(0.91)^n$

Example 20
a $V = 400\,000(1.07)^n$
b **i** \$428 000 **ii** \$490 017.20
c 3.3 years

Exercise 3G

1 **a** $V = 6000(1.12)^n$ **b** $P = 2000(0.92)^n$

2 **a** A = amount of money at any time, n = number of years of investment
$A = 200\,000 \times 1.17^n$
b A = house value at any time, n = number of years since initial valuation
$A = 530\,000 \times 0.95^n$
c A = car value at any time, n = number of years since purchase
$A = 14\,200 \times 0.97^n$
d A = size of oil spill at any time, n = number of minutes elapsed
$A = 2 \times 1.05^n$
e A = litres in tank at any time, n = number of hours elapsed
$A = 1200 \times 0.9^n$
f A = cell area at any time, n = number of minutes elapsed
$A = 0.01 \times 2^n$
g A = population at any time, n = number of years since initial census
$A = 172\,500 \times 1.15^n$
h A = mass of substance at any time, n = number of hours elapsed
$A = 30 \times 0.92^n$

3 **a** 1.1
b **i** \$665 500
ii \$1 296 871.23
iii \$3 363 749.98
c 7.3 years

4 **a** 300 000
b **i** \$216 750 **ii** \$96 173.13 **iii** \$42672.53
c 3.1 years

5 **a** $V = 15\,000 \times 0.94^t$
b **i** 12 459 L **ii** 9727 L
c 769.53 L
d 55.0 hours

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6 a $V = 50\,000 \times 1.11^n$
b i $75 903.52   ii $403 115.58
c 6.64 years
7 a 3000
b i 7800   ii 20 280   iii 243 220
c 10 hours 11 minutes
8 a $D = 10 \times 0.875^t$, where t = number of 10 000 km travelled
b 90 000
c Yes
9 a $T = 90 \times 0.92^t$
b i 79.4° C   ii 76.2° C
c 3.22 minutes $\approx$ 3 minutes 13 seconds
10 a i $1610.51   ii $2143.59   iii $4177.25
b i $1645.31   ii $2218.18   iii $4453.92
11 a $2805.10   b $2835.25   c $2837.47
12 a i 90 g   ii 72.9 g   iii 53.1 g
b 66 years
13 a 60 L   b 22.8 minutes
14 0.7%

3H

Building understanding

1 a $50   b $550   c $55   d $605   e $605

2

2	4200	210	4410
3	4410	220.50	4630.50
4	4630.50	231.53	4862.03
5	4862.03	243.10	5105.13

3 a $1102.50   b $1102.50
c $1157.63   d $1157.63
4 a $700(1.08)^2$ b $1000(1.15)^6$ c $850(1.06)^4$

Now you try

Example 21
$4502.19

Example 22
a $n = 60, r = \frac{5}{12}$ b $n = 12, r = 3.5$

Example 23
$8964.49

Exercise 3H

1 a $5105.13   b $11 946.33
c $13 652.22   d $9550.63
2 a $106 000   b $112360   c $119 101.60
d $133 822.56   e $179 084.77   f $239 655.82
3 a 6, 3%   b 60, 1%   c 52, 0.173%
d 14, 2.625%   e 32, 3.75%   f 120, 0.8%
4 a $2254.32   b $87 960.39   c $1461.53
d $789.84   e $591.63   f $1407.76
5 a $5075   b $5228.39   c $5386.42
6 $11 651.92

7 a i $3239.42   ii $3348.15   iii $3446.15
iv $3461.88   v $3465.96
b $226.54
8 a $P = 300, n = 12, r = 7\%, R = 14\%, t = 6$ years
b $P = 5000, n = 24, r = 2.5\%, R = 30\%, t = 2$ years
c $P = 1000, n = 65, r = 0.036\%, R = 0.936\%, t = 2.5$ years
d $P = 3500, n = 30, r = 0.0053\%, R = 1.9345\%, t = 30$ days
e $P = 10\,000, n = 10, r = 7.8\%, R = 7.8\%, t = 10$ years
f $P = 6000, n = 91, r = 0.22\%, R = 5.72\%, t = 3.5$ years
9 5.3% compounded bi-annually
10 a i Approx. 6 years   ii Approx. 12 years
iii Approx. 9 years   iv Approx. 5 years
v Approx. 7 years   vi Approx. 4 years
b Same answer as part **a**   c Yes

3I

Building understanding

1

x	0	1	2	3	4	5
2^x	1	2	4	8	16	32
3^x	1	3	9	27	81	243
4^x	1	4	16	64	256	1024
5^x	1	5	25	125	625	3125
10^x	1	10	100	1000	10 000	100 000

2 a 4   b 4   c 3   d 4
3 a $\frac{1}{10\,000}$ b $\frac{1}{2}$ c $\frac{1}{4}$ d $\frac{1}{27}$

Now you try

Example 24
a $10^2 = 100$ b $\log_3(81) = 4$

Example 25
a i 4   ii 5
b i -3 ii -2
c i 0.699   ii -0.347

Example 26
a 4   b 125

Exercise 3I

1 a $2^4 = 16$ b $10^2 = 100$
c $3^3 = 27$ d $2^{-2} = \frac{1}{4}$
e $10^{-1} = 0.1$ f $3^{-2} = \frac{1}{9}$
2 a $\log_2(8) = 3$ b $\log_3(81) = 4$
c $\log_2(32) = 5$ d $\log_4(16) = 2$
e $\log_{10}\left(\frac{1}{10}\right) = -1$ f $\log_5\left(\frac{1}{125}\right) = -3$

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3 a 4 b 2 c 6
d 3 e 1 f 2
g 3 h 3 i 2
j 2 k 5 l 3
m 0 n 0 o 0
p Undefined

4 a -3 b -2 c -2 d -3
e -2 f -4 g -4 h -1
i -1 j -3 k -5 l -1
m -3 n -1 o -2 p -2

5 a 0.699 b 1.672 c 2.210
d -0.097 e -0.770 f -1.431

6 a 3 b 5 c 6 d 4
e 3 f 2 g 16 h 81
i 1000 j $\frac{1}{9}$ k $\frac{1}{4}$ l $\frac{1}{343}$
m 3 n 2 o 4 p 8
q 3 r 10 s 2 t -1

7 a

Time (min)	0	1	2	3	4	5
Population	1	2	4	8	16	32

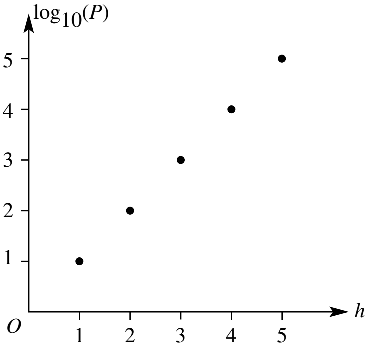
b $P = 2^t$
c 256
d 13 min
e $\log_2(10\,000)$

8 a 16 b 26 c 6

9 a

h	0	1	2	3	4	5
P	1	10	100	1000	10 000	100 000
$\log_{10}(P)$	0	1	2	3	4	5

b



Graph is a straight line.

c $\log_{10}(P) = h$

10 a i 10 ii 100 iii 10 000
b i 10 ii 6 iii 3

11 Yes. $0 < b < 1 \Rightarrow \log_a(b) < 0$, when $a > 1$; e.g. $\log_2\left(\frac{1}{4}\right) = -2$.

12 a $\frac{1}{4}$ b $\frac{1}{5}$ c $\frac{1}{2}$
d $\frac{1}{3}$ e $\frac{1}{2}$ f $\frac{1}{3}$
g $\frac{2}{3}$ h $\frac{4}{3}$ i $\frac{1}{2}$
j $\frac{1}{2}$ k $\frac{6}{5}$ l $\frac{4}{7}$

3J

Building understanding

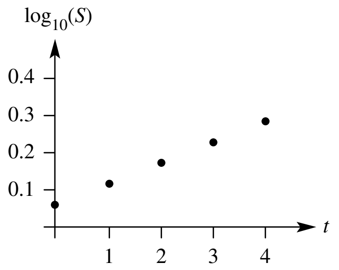
1 a $10^0, 10^1, 10^2, 10^3, 10^4, 10^5$
b i 100 ii 1000
c 0, 1, 2, 3, 4, 5
d 0, 1, 2, 3, 4, 5
e They are equal.

2 a The values are approximately 2, 1.5, 1.0, 0.85 and 0.6.
b $\log_{10}$(Profit)
c i 10 ii 1

Now you try

Example 27
a i 1
ii 3
b i 10
ii 1000

Example 28
a 0.08, 0.11, 0.16, 0.22, 0.30
b



c Linear (straight line). An exponential relationship between S and t.

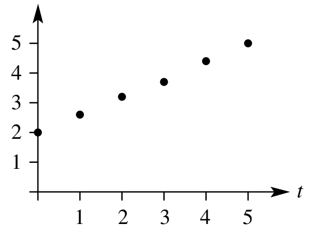
Exercise 3J

1 a i 2 ii 3
b i 100 ii 1000

2 a i 3 ii 2
b i 1000 ii 100

3 a i 3 ii 6
b i 1000 ii 1 000 000

4 a 2.0, 2.6, 3.2, 3.8, 4.4. 5.0
b $\log_{10}(P)$



c Linear (straight line), so an exponential relationship between P and t.

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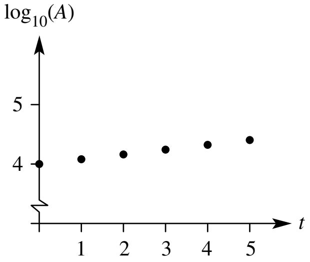
5 a 4.00, 4.08, 4.16, 4.24, 4.32, 4.40

b



c Linear (straight line), so an exponential relationship between A and t.

6 a i 2 ii 3

b \$10 000 000, \$100 000 000, \$1 000 000 000, \$100 000 000 000

7 a i 0.4 ii 1.1

b i \$50 000 ii \$130 000

iii \$20 000 iv \$10 000

c \$13 000

8 500 000 000 ML

9 a i 4 ii 8

b i 10^1 ii 10^3 iii 10^5

iv 10^7 v 10^9

c The given graph uses $\log_{10}(C)$ not C on the y-axis.

d i 11 ii 17

10 a i 100 ii 10 000

b $y = 10^{x+1}$ or $y = 10 \times 10^x$

11 a $P = 2^{x+7}$ or $P = 128 \times 2^x$

b $P = 3^{x+4}$ or $P = 81 \times 3^x$

12 a i 3 ii 6 iii 11

b 0.0000001

c 100

d Answers will vary.

3K

Building understanding

1 a $\log_b(xy) = \log_b(x) + \log_b(y)$

b $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

c $\log_a(b^m) = m \times \log_a(b)$

d $\log_a(a) = 1$

e $\log_c(1) = 0$

f $\log_a\left(\frac{1}{b}\right) = -\log_a(b)$

2 a 2 b 1

c 4 d 12

e 4 f −1

3 a 2 b 5

c 3 d −4

e 12 f 0

Now you try

Example 29

a $\log_a(24)$ b $\log_a(2)$ c $\log_a(16)$

Example 30

a 0 b 1 c −2 d 1

Exercise 3K

1 a $\log_a(6)$ b $\log_a(15)$ c $\log_a(28)$

d $\log_b(18)$ e $\log_b(15)$ f $\log_b(17)$

2 a $\log_a(2)$ b $\log_a(3)$ c $\log_a(10)$

d $\log_b(2)$ e $\log_b\left(\frac{3}{2}\right)$ f $\log_b\left(\frac{7}{5}\right)$

3 a $\log_a(9)$ b $\log_a(25)$ c $\log_a(27)$

d $\log_a(16)$ e $\log_a(32)$ f $\log_a(1000)$

4 a 0 b 0 c 0 d 1

e 1 f 1 g 0 h 3

i $\frac{1}{3}$ j $\frac{2}{3}$ k 1 l $\frac{1}{2}$

5 a −2 b −3 c −3

d −1 e −2 f −5

6 a 1 b 1 c 3

d 2 e 2 f 2

7 a $\log_3(20)$ b $\log_{10}(48)$ c $\log_{10}(2)$

d $\log_7(2)$ e $\log_3(8)$ f 0

g $\log_2\left(\frac{3}{4}\right)$ h $\log_5(6)$

8 a $\frac{3}{2}$ b $\frac{5}{2}$ c $\frac{4}{3}$

d $\frac{3}{2}$ e $\frac{1}{3}$ f $\frac{4}{5}$

9 a $\log_a\left(\frac{1}{x}\right) = \log_a(1) - \log_a(x) = 0 - \log_a(x) = -\log_a(x)$ as required (using 2nd log law)

b $\log_a\left(\frac{1}{x}\right) = \log_a(x^{-1}) = -\log_a(x)$ as required (using 3rd log law)

10 $\log_a(\sqrt[n]{x}) = \log_a\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\log_a(x) = \frac{\log_a(x)}{n}$ as required (using 3rd log law)

11 a Recall index law 1: $a^m \times a^n = a^{m+n}$

Now let $x = a^m$ and $y = a^n$ (1)

so $m = \log_a(x)$ and $n = \log_a(y)$ (2)

From (1), $xy = a^m \times a^n = a^{m+n}$

So $m + n = \log_a(xy)$

From (2), $m + n = \log_a(x) + \log_a(y)$

So $\log_a xy = \log_a(x) + \log_a(y)$, as required.

b Recall index law 2: $a^m \div a^n = a^{m-n}$

Now let $x = a^m$ and $y = a^n$ (1)

so $m = \log_a(x)$ and $n = \log_a(y)$ (2)

From (1), $x \div y = \frac{x}{y} = a^m \div a^n = a^{m-n}$

So $m - n = \log_a\left(\frac{x}{y}\right)$

From (2) $m - n = \log_a(x) - \log_a(y)$

So $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$, as required.

c Recall index law 3: $(am)^n = a^{mn}$

Let $x = a^m$

So $m = \log_a(x)$ (1)

$x^n = a^{mn}$ using index law 3

So $mn = \log_a(x^n)$

From (1): $n\log_a x = \log_a(x^n)$, as required.

3L

Building understanding

1 a $\log_2 8 = 3$ b $\log_4 2 = \frac{1}{2}$ c $\log_3 10 = x$

2 a 3 b 4 c 8 d 2

3 a 0.845 b −0.222 c −0.125 d 1.277

Now you try

Example 31

a 2.096 **b** 6.026

Example 32

a 3.459 **b** 8.530

Exercise 3L

1 **a** 1.465 **b** 3.459 **c** 1.594
  d 6.871 **e** 1.177 **f** 2
2 **a** 1 **b** 1 **c** 3.969
  d 1.727 **e** 6.579 **f** 1.528
3 **a** 2.585 **b** 1.893 **c** 1.209
  d 1.129 **e** 1.559 **f** 6.579
  g 3.322 **h** 1.262 **i** 0.356
  j 3.969 **k** 3.106 **l** 1.137
4 **a** 2 days **b** 2.548 days **c** 3.322 days
5 **a** 14.21 years **b** 23.84 years **c** 47.19 years
6 **a** 10.48 years **b** 22.20 years **c** 91.17 years
7 **a** $A = 2000 \times 1.1^n$ **b** 7 years
8 **a** $F = 300\,000 \times 0.92^n$ **b** 8.3 years
9 **a** 69 years **b** 1386 years
10 **a** **i** $\frac{\log_{10}(7)}{\log_{10}(2)}$ **ii** $\frac{\log_{10}(16)}{\log_{10}(3)}$ **iii** $\frac{\log_{10}(1.3)}{\log_{10}(5)}$
  b **i** $\frac{1}{\log_{10}(5)}$ **ii** $\frac{3}{\log_{10}(2)}$ **iii** $\frac{-1}{\log_{10}(3)}$
  c **i** 1.631 **ii** 1.167 **iii** −0.196

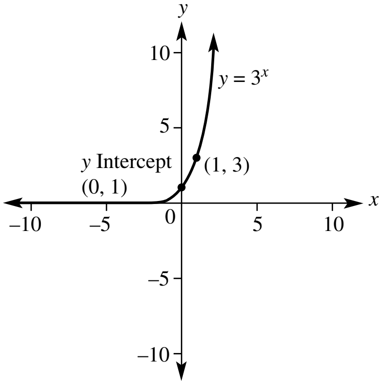
Problems and challenges

1 3^n
2 **a** 5 **b** $\left(\frac{4}{9}\right)^x$
3 $\frac{1}{5}$
4 **a** −8 **b** 2^{2-a}
5 **a** $\frac{x-y}{xy}$ **b** $\sqrt{xy}(x-y)$
6 $x = 3.5$
7 **a** 2 **b** 8 **c** $\frac{1}{2}$ **d** 3
8 **a** 1.43 **b** −1.43 **c** −2.71
9 $x = 6$
10 $2p - q + 2$
11 **a** $x > 2.10$ **b** $x \geqslant -2.81$
12 $a = 2 \times 3^{\frac{1}{4}}, b = \frac{1}{4}\log_2(3)$
13 $\frac{\log_{10}(2)}{\log_{10}(1.1)} = \log_{1.1}(2)$

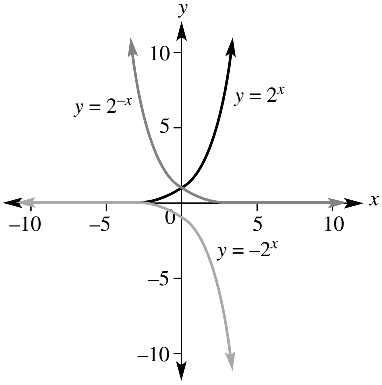
Answers to success criteria example questions

1 $6x^3y^7$ **2** $\frac{x^3y^2}{4}$
3 $4m^6$; $\frac{x^8}{y^4}$ **4** $4x^4y^3$
5 −4 **6** $\frac{3x^2}{y^3}$
7 $3y^4$ **8** $\frac{x^5}{4y^6}$

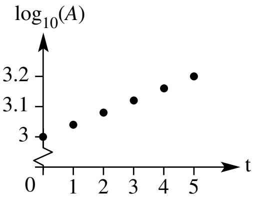
9 30 700; 0.0041 **10** 3.59×10^{-3}
11 1.74×10^{12} **12** x^2
13 $\sqrt{x^3}$ **14** $\frac{1}{5}$
15 $x = 3$ **16** $x = 6$
17



18



19 (3, 27)
20 $V = 35\,000(0.85)^n$
21 \$744 861.41; 9.06 years (correct to 2 decimal places)
22 \$6749.18
23 60 periods and 0.25% per period
24 \$2309.10
25 $2^3 = 8$; $\log_3(81) = 4$
26 3
27 0.903
28 243
29 100 times larger
30



The relationship between A and t is linear
31 $\log_2(15)$; $\log_2(3)$
32 1
33 $x = 2.727$
34 $x = 11.527$

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Short-answer questions

1 a $25y^6$ b 6 c $20x^7y^{10}$
d $\frac{3x^2}{y^4}$ e $\frac{3y^4}{2x^3}$ f $\frac{27}{4b^8}$

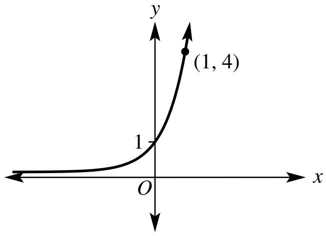
2 a i 3210 ii 4 024 000
iii 0.00759 iv 0.0000981
b i 3.08×10^{-4} ii 7.18×10^{-6}
iii 5.68×10^6 iv 1.20×10^8

3 a $21^{\frac{1}{2}}$ b $x^{\frac{1}{3}}$ c $m^{\frac{5}{3}}$
d $2^{\frac{1}{3}}a^3b^{\frac{1}{3}}$ e $7^{\frac{3}{2}}$

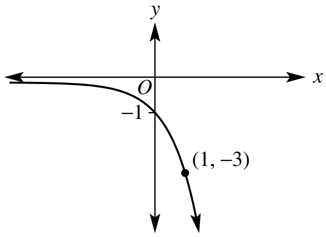
4 a 5 b 4 c $\frac{1}{7}$
d $\frac{1}{10}$ e 25 f 1000

5 a 3 b 2 c 1
d −2 e −3 f $\frac{3}{2}$
g 4 h −4 i 0

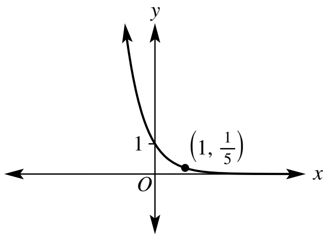
6 a



b



c



7 a $V = 800 \times 1.07^t$ b $V = 3000 \times 0.82^t$

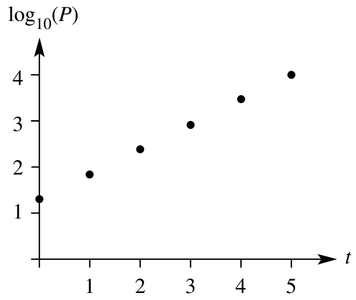
8 a $1215.51 b $3519.60 c $5637.46

9 a $\log_2(16) = 4$ b $\log_{10}(1000) = 3$
c $\log_3\left(\frac{1}{9}\right) = -2$

10 a $3^4 = 81$ b $4^{-2} = \frac{1}{16}$ c $10^{-1} = 0.1$

11 a 3 b 4 c 0
d −3 e −3 f −1

12 a



b Graph is linear and therefore relationship between P and t is exponential.

13 a $\log_a(8)$ b $\log_b(21)$ c $\log_a(10)$
d $\log_a(4)$ e 1 f $\frac{3}{2}$

14 a $x = \log_3(6)$ b $x = \log_{1.2}(2)$

15 a $\frac{\log_{10}(13)}{\log_{10}(2)}$ b $\frac{\log_{10}(2)}{\log_{10}(0.8)}$

Multiple-choice questions

1 A 2 D 3 C 4 B 5 D
6 C 7 C 8 D 9 B 10 D
11 C 12 B 13 C 14 A

Extended-response questions

1 a $V = 10\,000 \times 1.065^n$
b i $11 342.25 ii $13 700.87
c 11.0 years
d i $14 591 ii $V = 14\,591 \times 0.97^t$
iii $12917; profit of $2917

2 a i 120 dB
ii 10^{-6} watt/cm^2
iii 10^{-16} watt/cm^2
b i 10^{-11} watt/cm^2 to 10^{-9} watt/cm^2
ii An increase of 20 decibels increases intensity by a factor of 100.
c i $c = 10\log_{10}\left(\frac{P_2}{P_1}\right)$ ii 20

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Chapter 4

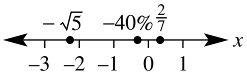
4A

Building understanding

1 a irrational b root
c non-recurring d rational number
2 a 4 b 25
c 16 d 36

Now you try

Example 1
a Irrational b Rational c Rational



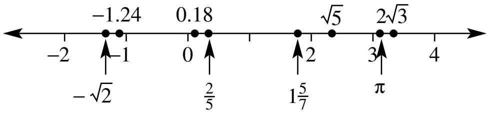
$-\sqrt{5} \approx -2.2, -40\% = -0.4, \frac{2}{7} \approx 0.29$

Example 2
a $2\sqrt{5}$ b $20\sqrt{3}$ c $2\sqrt{3}$ d $\frac{5\sqrt{5}}{4}$

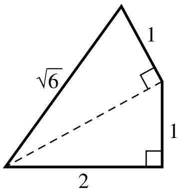
Example 3
a $\sqrt{18}$ b $\sqrt{75}$

Exercise 4A

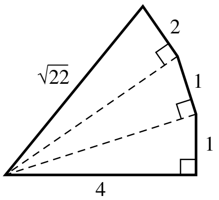
1 a Irrational b Rational
c Rational d Rational
e Rational f Irrational
g Irrational h Irrational



2 a Yes b Yes c No d No
e No f Yes g Yes h No
3 a $2\sqrt{3}$ b $3\sqrt{5}$ c $2\sqrt{6}$ d $4\sqrt{3}$
e $5\sqrt{3}$ f $10\sqrt{5}$ g $7\sqrt{2}$ h $3\sqrt{10}$
i $8\sqrt{2}$ j $6\sqrt{10}$ k $9\sqrt{2}$ l $4\sqrt{5}$
4 a $6\sqrt{2}$ b $6\sqrt{5}$ c $16\sqrt{3}$ d $6\sqrt{7}$
e $21\sqrt{2}$ f $20\sqrt{5}$ g $\sqrt{5}$ h $\sqrt{7}$
i $\frac{\sqrt{6}}{2}$ j $\frac{\sqrt{6}}{4}$ k $\frac{\sqrt{5}}{5}$ l $\frac{\sqrt{11}}{6}$
m $3\sqrt{11}$ n $2\sqrt{2}$ o $2\sqrt{2}$ p $\frac{2\sqrt{17}}{7}$
q $\frac{3\sqrt{3}}{2}$ r $4\sqrt{6}$ s $\frac{2\sqrt{3}}{3}$ t $\frac{3\sqrt{3}}{2}$
5 a $\frac{2\sqrt{2}}{3}$ b $\frac{2\sqrt{3}}{7}$ c $\frac{3\sqrt{2}}{5}$ d $\frac{\sqrt{11}}{5}$
e $\frac{\sqrt{10}}{3}$ f $\frac{\sqrt{3}}{6}$ g $\frac{\sqrt{13}}{4}$ h $\frac{1}{\sqrt{2}}$
i $\frac{\sqrt{5}}{3}$ j $\frac{3\sqrt{3}}{2}$ k $\frac{\sqrt{5}}{2\sqrt{2}}$ l $\frac{\sqrt{14}}{\sqrt{19}}$
6 a $\sqrt{12}$ b $\sqrt{32}$ c $\sqrt{50}$ d $\sqrt{27}$
e $\sqrt{45}$ f $\sqrt{108}$ g $\sqrt{128}$ h $\sqrt{700}$
i $\sqrt{810}$ j $\sqrt{125}$ k $\sqrt{245}$ l $\sqrt{363}$
7 a $4\sqrt{2}$ m b $2\sqrt{30}$ cm c $4\sqrt{15}$ mm
8 a Radius = $2\sqrt{6}$ cm, diameter = $4\sqrt{6}$ cm
b Radius = $3\sqrt{6}$ m, diameter = $6\sqrt{6}$ m
c Radius = $8\sqrt{2}$ m, diameter = $16\sqrt{2}$ m
9 a $15\sqrt{3}$ b $13\sqrt{7}$ c $19\sqrt{5}$ d $31\sqrt{3}$
10 a $2\sqrt{5}$ cm b $3\sqrt{5}$ m c $\sqrt{145}$ mm
d $\sqrt{11}$ m e $\sqrt{11}$ mm f $2\sqrt{21}$ cm
11 $\sqrt{72} = \sqrt{36} \times 2$ (i.e. 36 is highest square factor of 72)
$= 6\sqrt{2}$
12 a 9, 25, 225
b $15\sqrt{2}$
13 a Draw triangle with shorter sides length 1 cm and 3 cm.
b Draw triangle with shorter sides length 2 cm and 5 cm.
c



d



14 Check with your teacher.

4B

Building understanding

1 a Yes b No c No d Yes
e Yes f No g Yes h Yes
2 a $6x$ b $-5x$ c $17a$ d t
3 a $4\sqrt{3}$
b i $5\sqrt{3}$ ii $-3\sqrt{3}$ iii $17\sqrt{3}$

Now you try

Example 4
a $5\sqrt{5}$ b $\sqrt{7} + 7\sqrt{3}$

Example 5
a $5\sqrt{2}$ b $2\sqrt{3}$

Exercise 4B

1 a $6\sqrt{5}$ b $7\sqrt{3}$ c $4\sqrt{2}$
d $3\sqrt{2}$ e $11\sqrt{5}$ f $\sqrt{3}$
g $6\sqrt{10}$ h $5\sqrt{2}$ i $-2\sqrt{21}$
j $-6\sqrt{11}$ k $-\sqrt{13}$ l $-7\sqrt{30}$
2 a $\sqrt{3} + 5\sqrt{2}$ b $3\sqrt{6} + 7\sqrt{11}$
c $4\sqrt{5} - 7\sqrt{2}$ d $-2\sqrt{2} + \sqrt{5}$
e $4\sqrt{3}$ f 0
g $-3\sqrt{2} - 3\sqrt{10}$ h $-2\sqrt{5} + 3\sqrt{15}$
3 a $\sqrt{2}$ b $5\sqrt{2}$ c $4\sqrt{3}$ d $\sqrt{5}$
e $7\sqrt{2}$ f $12\sqrt{3}$ g $8\sqrt{11}$ h $3\sqrt{2}$
i $5\sqrt{6}$ j $\sqrt{5}$ k $32\sqrt{2}$ l $20\sqrt{2}$
4 a $13\sqrt{2}$ b $9\sqrt{6}$
c $2\sqrt{5} - \sqrt{7}$ d $5\sqrt{5} + 6\sqrt{7}$
e $9\sqrt{3} + 2\sqrt{2}$ f $2\sqrt{3} + 11\sqrt{5} - 5\sqrt{2}$
g $\sqrt{6} - 3\sqrt{2}$ h $11 - 9\sqrt{3}$
i $9 + 18\sqrt{2}$ j $-9\sqrt{2} + 15\sqrt{5}$
5 a $\frac{5\sqrt{3}}{6}$ b $\frac{7\sqrt{5}}{12}$ c $\frac{\sqrt{2}}{30}$

d $\frac{\sqrt{7}}{6}$ e $\frac{-\sqrt{2}}{10}$ f $\frac{13\sqrt{3}}{14}$
g $\frac{13\sqrt{5}}{18}$ h $\frac{-7\sqrt{3}}{30}$ i $\frac{-11\sqrt{10}}{24}$

6 a $4\sqrt{3} + 2\sqrt{5}$ cm b $14\sqrt{2}$ cm
c $\sqrt{10} + 3\sqrt{2}$ cm d $2\sqrt{10} + 4\sqrt{5}$ cm
e $4\sqrt{3} + \sqrt{30}$ m f $12\sqrt{3}$ cm

7 a $\sqrt{20} = 2\sqrt{5}$
b $3\sqrt{72} = 18\sqrt{2}, \sqrt{338} = 13\sqrt{2}$

8 a $5\sqrt{3} - 6\sqrt{3} + \sqrt{3} = 0$
b $\sqrt{6} + 2\sqrt{6} - 3\sqrt{6} = 0$
c $6\sqrt{2} - 8\sqrt{2} + 2\sqrt{2} = 0$
d $2\sqrt{2} - 3\sqrt{2} + \sqrt{2} = 0$
e $4\sqrt{5} - 7\sqrt{5} + 3\sqrt{5} = 0$
f $3\sqrt{2} - 6\sqrt{3} - 5\sqrt{2} + 6\sqrt{3} + 2\sqrt{2} = 0$

9 a $6\sqrt{3} - 3\sqrt{2}$, unlike surds
b $8\sqrt{2} + 2\sqrt{5}$, unlike surds
c $5\sqrt{2} - 6\sqrt{5}$, unlike surds
d $10\sqrt{10} + 10\sqrt{3}$, unlike surds
e $20\sqrt{2} + 30\sqrt{3}$, unlike surds
f $4\sqrt{5} - 6\sqrt{6}$, unlike surds

10 a $\frac{7\sqrt{2}}{15}$ b $\frac{2\sqrt{3}}{3}$ c $\frac{\sqrt{5}}{12}$ d $\frac{-3\sqrt{2}}{4}$
e $\frac{\sqrt{3}}{2}$ f $\frac{-7\sqrt{7}}{15}$ g $-\sqrt{2}$ h $\frac{29\sqrt{6}}{28}$
i 0 j $8\sqrt{3}$ k $\frac{6\sqrt{6}}{35}$ l $\frac{29\sqrt{5}}{42}$

4C

Building understanding

1 a $\sqrt{\frac{15}{3}} = \sqrt{5}$ b $\sqrt{\frac{42}{7}} = \sqrt{6}$
c $\sqrt{6 \times 5} = \sqrt{30}$ d $\sqrt{11 \times 2} = \sqrt{22}$

2 a 6 b 7 c 5

3 a $2x + 6$ b $10x - 5$ c $30 - 24x$

Now you try

Example 6
a $\sqrt{15}$ b $24\sqrt{3}$ c 63

Example 7
a $-\sqrt{3}$ b $2\sqrt{2}$

Example 8
a $5\sqrt{6} - \sqrt{14}$ b $30\sqrt{2} - 45$

Exercise 4C

1 a $\sqrt{15}$ b $\sqrt{21}$ c $\sqrt{26}$
d $\sqrt{35}$ e $-\sqrt{30}$ f $-\sqrt{30}$
g $\sqrt{66}$ h $\sqrt{6}$ i $\sqrt{70}$

2 a $\sqrt{10}$ b $\sqrt{6}$ c $-\sqrt{3}$
d $\sqrt{5}$ e $\sqrt{3}$ f $\sqrt{10}$
g $\sqrt{5}$ h $-\sqrt{13}$ i $-\sqrt{5}$

3 a 3 b 5 c 9
d $7\sqrt{2}$ e $2\sqrt{11}$ f $3\sqrt{6}$
g $5\sqrt{2}$ h $4\sqrt{6}$ i 10

4 a $6\sqrt{35}$ b $21\sqrt{2}$ c $8\sqrt{30}$
d $-50\sqrt{3}$ e $-18\sqrt{3}$ f $15\sqrt{5}$
g $42\sqrt{6}$ h $-60\sqrt{10}$ i $-20\sqrt{10}$
j $42\sqrt{2}$ k $24\sqrt{30}$ l $216\sqrt{7}$

5 a 11 b 13 c 12
d 125 e 147 f 162

6 a $2\sqrt{2}$ b $3\sqrt{6}$ c $\frac{\sqrt{5}}{2}$
d $\frac{-4}{\sqrt{13}}$ e $\frac{-1}{3\sqrt{7}}$ f $\frac{2\sqrt{5}}{3}$

7 a $\sqrt{6} + \sqrt{15}$ b $\sqrt{14} - \sqrt{10}$
c $-\sqrt{55} - \sqrt{65}$ d $-2\sqrt{15} - 2\sqrt{21}$
e $6\sqrt{26} - 3\sqrt{22}$ f $20 - 20\sqrt{2}$
g $30\sqrt{2} + 15\sqrt{30}$ h $-12\sqrt{3} + 12\sqrt{2}$
i $42 + 63\sqrt{2}$ j $90\sqrt{3} - 24\sqrt{10}$
k $-16 + 24\sqrt{10}$ l $42\sqrt{2} + 30$

8 a $2\sqrt{6}$ b $\sqrt{30}$ c 6

9 a 28 b 18 c 75
d $\sqrt{2} - \sqrt{6}$ e $3\sqrt{3} + 4$ f $-\sqrt{10} + \sqrt{5}$
g 2 h $8\sqrt{2}$ i $\sqrt{2} - 6$

10 a $\frac{3}{4}$ cm^2 b $2\sqrt{6}$ cm

11 a $\sqrt{6} \times \sqrt{6} = \sqrt{6 \times 6} = \sqrt{36} = 6$
b $-\sqrt{8} \times \sqrt{8} = -\sqrt{8 \times 8} = -\sqrt{64} = -8$
c $-\sqrt{5}(-\sqrt{5}) = 1\sqrt{5 \times 5} = \sqrt{25} = 5$

12 a Simplify each surd before multiplying.
b Allows for the multiplication of smaller surds, which is simpler.
c i $3\sqrt{2} \times 3\sqrt{3} = 9\sqrt{6}$
ii $2\sqrt{6} \times 2\sqrt{5} = 4\sqrt{30}$
iii $5\sqrt{2} \times 3\sqrt{5} = 15\sqrt{10}$
iv $3\sqrt{6} \times 5\sqrt{3} = 45\sqrt{2}$
v $6\sqrt{2} \times 4\sqrt{3} = 24\sqrt{6}$
vi $6\sqrt{3}(-10\sqrt{5}) = -60\sqrt{15}$
vii $-12\sqrt{3} \times 22\sqrt{7} = 24\sqrt{21}$
viii $7\sqrt{2} \times 10\sqrt{3} = 70\sqrt{6}$
ix $12\sqrt{2} \times 12\sqrt{5} = 144\sqrt{10}$

13 a 3 b 2 c -9
d $-\frac{1}{5}$ e $\frac{2}{5}$ f 3

14 a $54\sqrt{2}$ b $375\sqrt{3}$ c $162\sqrt{3}$
d 25 e 9 f $128\sqrt{2}$
g $-120\sqrt{5}$ h $-108\sqrt{2}$ i 720
j $14\sqrt{7}$ k $\frac{27\sqrt{2}}{2}$ l 81
m $100\sqrt{3}$ n 144 o $-96\sqrt{15}$
p $\frac{81\sqrt{3}}{25}$ q $\frac{5}{3\sqrt{3}}$ r $\frac{9\sqrt{6}}{2}$

15 a $19 - 2\sqrt{6}$ b 16
c $2\sqrt{15} - 85$ d $10\sqrt{3} - 37$
e $30 - 10\sqrt{2}$ f 0
g $4\sqrt{3} - 14$ h $47\sqrt{2} - 10\sqrt{30} + 11$

4D

Building understanding

1 a 1 b $\frac{1}{2}$ c -2 d 6

2 a $\sqrt{3}$ b 10 c $\sqrt{5}$ d $\sqrt{7}$

3 a 0.377... b 2.886... c 16.31...
All pairs of numbers are equal.

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Now you try

Example 9

a $\frac{3\sqrt{2}}{2}$ b $\frac{4\sqrt{21}}{7}$ c $\frac{\sqrt{10}}{3}$ d $\frac{2\sqrt{7}-7}{7}$

Exercise 4D

1 a $\frac{\sqrt{2}}{2}$ b $\frac{\sqrt{7}}{7}$ c $\frac{3\sqrt{11}}{11}$ d $\frac{4\sqrt{5}}{5}$
e $\frac{5\sqrt{3}}{3}$ f $4\sqrt{2}$ g $\frac{\sqrt{15}}{3}$ h $\frac{\sqrt{14}}{7}$

2 a $\frac{\sqrt{6}}{3}$ b $\frac{\sqrt{35}}{7}$ c $\frac{\sqrt{66}}{11}$ d $\frac{\sqrt{10}}{5}$
e $\frac{\sqrt{21}}{3}$ f $\frac{\sqrt{42}}{7}$ g $\frac{\sqrt{30}}{3}$ h $\frac{\sqrt{34}}{2}$

3 a $\frac{4\sqrt{14}}{7}$ b $\frac{5\sqrt{6}}{3}$ c $\frac{3\sqrt{10}}{2}$
d $\frac{3\sqrt{42}}{7}$ e $\frac{7\sqrt{30}}{10}$ f $\frac{2\sqrt{105}}{15}$

4 a $\frac{4\sqrt{21}}{15}$ b $\frac{\sqrt{6}}{3}$ c $\frac{\sqrt{35}}{3}$ d $\frac{2\sqrt{2}}{5}$
e $\frac{2\sqrt{5}}{15}$ f $\frac{10}{9}$ g $\frac{9\sqrt{2}}{2}$ h $\frac{3\sqrt{7}}{2}$

5 a $\frac{\sqrt{3}+\sqrt{6}}{3}$ b $\frac{3\sqrt{7}+\sqrt{35}}{7}$ c $\frac{2\sqrt{5}-\sqrt{15}}{5}$
d $\frac{\sqrt{6}-\sqrt{10}}{2}$ e $\frac{\sqrt{35}+\sqrt{14}}{7}$ f $\frac{\sqrt{30}-\sqrt{21}}{3}$
g $\frac{2\sqrt{3}+\sqrt{42}}{6}$ h $\frac{5\sqrt{2}+2\sqrt{5}}{10}$ i $\frac{\sqrt{30}-5\sqrt{2}}{5}$
j $\frac{8\sqrt{3}-15\sqrt{2}}{6}$ k $\frac{3\sqrt{2}+2\sqrt{5}}{2}$ l $\frac{6\sqrt{5}+5\sqrt{6}}{2}$

6 a $\frac{5\sqrt{3}}{3}$ cm^2 b $\frac{2}{3}$ m^2
c $\frac{\sqrt{10}+\sqrt{15}}{10}$ mm^2

7 a $\frac{2\sqrt{3}+3\sqrt{2}}{6}$ b $\frac{6\sqrt{5}+5\sqrt{2}}{10}$ c $\frac{9\sqrt{7}-14\sqrt{3}}{21}$
d $\frac{5\sqrt{3}-2\sqrt{2}}{6}$ e $\frac{2\sqrt{2}+5\sqrt{3}}{12}$ f $\frac{9\sqrt{5}+4\sqrt{3}}{30}$
g $\frac{-2\sqrt{14}}{15}$ h $\frac{6\sqrt{30}+4\sqrt{6}}{9}$ i $\frac{3\sqrt{10}-2\sqrt{42}}{9}$

8 As $\frac{\sqrt{x}}{\sqrt{x}}$ is equal to 1.

9 a $\frac{\sqrt{21}+\sqrt{7}\,a}{7}$ b $\frac{\sqrt{30}+\sqrt{5}\,a}{5}$
c $\frac{2\sqrt{3}+\sqrt{6}\,a}{6}$ d $1-\sqrt{3}\,a$
e $1-\sqrt{5}\,a$ f $1-\sqrt{7}\,a$
g $\frac{4\sqrt{10}\,a+5\sqrt{2}}{10}$ h $\frac{\sqrt{6}\,a+\sqrt{2}}{2}$
i $\frac{2\sqrt{14}\,a+7\sqrt{2}}{14}$

10 a i 14 ii 2 iii 47
b Each question is a difference of perfect squares, and each answer is an integer.
c $\frac{4+\sqrt{2}}{4+\sqrt{2}}$
d i $\frac{12+3\sqrt{2}}{14}$ ii $\frac{-3\sqrt{3}-3}{2}$
iii $2\sqrt{2}+\sqrt{6}$ iv $\frac{-(6+2\sqrt{30})}{7}$

11 a $\frac{5\sqrt{3}-5}{2}$ b $2\sqrt{3}+2$
c $3\sqrt{5}+6$ d $-4-4\sqrt{2}$
e $\frac{-3-3\sqrt{3}}{2}$ f $\frac{42+7\sqrt{7}}{29}$
g $-12-4\sqrt{10}$ h $-14-7\sqrt{5}$
i $\frac{2\sqrt{11}+2\sqrt{2}}{9}$ j $2\sqrt{5}-2\sqrt{2}$
k $\sqrt{7}-\sqrt{3}$ l $\frac{\sqrt{14}-\sqrt{2}}{6}$
m $\frac{6+\sqrt{6}}{5}$ n $\sqrt{14}+2\sqrt{2}$
o $10-4\sqrt{5}$ p $\frac{b\sqrt{a}-b\sqrt{b}}{a-b}$
q $\frac{a\sqrt{a}+a\sqrt{b}}{a-b}$ r $\frac{a+b-2\sqrt{ab}}{a-b}$
s $\frac{a-\sqrt{ab}}{a-b}$ t $\frac{a\sqrt{b}+b\sqrt{a}}{a-b}$

4E

Building understanding

1 a 40 mm b 9.6 cm c 1 m
d 8 m e 0.297 km f 510.2 cm

2 a $\frac{1}{4}$ b $\frac{1}{2}$ c $\frac{3}{4}$
d $\frac{1}{9}$ e $\frac{7}{24}$ f $\frac{11}{12}$

3 a 810 m b 9.4 km c 180 cm

Now you try

Example 10
a 20.2 m b 8.1 c $x = P - 13.2$

Example 11
a 21.99 cm b $r = \frac{C}{2\pi}$ c 1.91 cm

Example 12
a $10 + \frac{70\pi}{9}$ cm b 34.4 cm

Exercise 4E

1 a 36.6 cm b 5.1 cm c $x = P - 28.6$
2 a 21.8 m b 3.2 m c $x = P - 16.8$
3 a 43.98 cm b 7.54 m
c 89.22 mm d 3.46 km
4 a 75.40 m b $r = \frac{C}{2\pi}$ c 5.57 m
5 a i $8 + 4\pi$ m ii 20.6 m
b i $4 + \pi$ m ii 7.1 m
c i $2 + \frac{\pi}{3}$ km ii 3.0 km
d i $12 + 10\pi$ cm ii 43.4 cm
e i $10 + \frac{70\pi}{9}$ mm ii 34.4 mm
f i $6 + \frac{31\pi}{12}$ cm ii 14.1 cm
6 a $x = 3$ b $x = 8.8$ c $x = 0.009$
d $x = 2.65$ e $x = 3.87$ f $x = 2.4$
7 57.6 m
8 a 12.25 b 53.03 c 1.37
d 62.83 e 19.77 f 61.70
9 a 6π m b $10 + \frac{5\pi}{2}$ m c $\pi + 1$ km

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10 **a** **i** 201 cm **ii** 1005 cm
b 4974
c $\dfrac{1\,000\,000}{\pi d}$

11 $r = \dfrac{2n}{\pi}$

12 $\pi\sqrt{2}\,x$

13 **a** $l = \dfrac{P - 2w}{2}$ or $\dfrac{1}{2}P - w$ **b** $l = 5 - w$
c $0 < w < 5$ **d** $0 < l < 5$

14 **a** 720° **b** 1080° **c** 540° **d** 1440°

4F

Building understanding

1 **a** $a = \sqrt{55}$ **b** $a = \sqrt{77}$
c $a = \sqrt{2}$ **d** $a = \sqrt{50} = 5\sqrt{2}$

2 **a** $x^2 + y^2 = z^2$ **b** $a^2 + d^2 = b^2$ **c** $2x^2 = c^2$

Now you try

Example 13
a 8.06 cm **b** 1.92 m

Example 14
a $\sqrt{61}$ **b** 10.49

Exercise 4F

1 **a** 5 cm **b** 11.18 m **c** 16.55 km
d 1.81 mm **e** 0.43 km **f** 77.10 cm

2 **a** 4.58 m **b** 7.94 m **c** 0.63 m
d 1.11 cm **e** 14.60 cm **f** 0.09 cm

3 **a** **i** $\sqrt{34}$ **ii** 6.16
b **i** $\sqrt{80}$ (or $4\sqrt{5}$) **ii** 16.61
c **i** $\sqrt{10}$ **ii** 7.68
d **i** $\sqrt{89}$ **ii** 13.04

4 **a** No **b** Yes **c** No
d No **e** Yes **f** Yes

5 **a** 2.86 m **b** 2.11 cm **c** 26.38 m
d 4.59 cm **e** 0.58 km **f** 1.81 km

6 8.3 cm

7 **a** 13.19 mm **b** 13.62 m **c** 4.53 cm
d 2.61 m **e** 12.27 km **f** 5.23 cm

8 **a** $2\sqrt{13}$ **b** $4\sqrt{2}$ **c** $\sqrt{181}$

9 **a** **i** 22.4 cm **ii** 24.5 cm
b Investigation required.

10 $\dfrac{\sqrt{5}}{2}$ cm, using Pythagoras' theorem given that an angle in a semicircle is 90°.

11 **a** $4\sqrt{5}$ cm by $2\sqrt{5}$ cm
b $3\sqrt{10}$ cm by $\sqrt{10}$ cm
c $\sqrt{\dfrac{100}{101}}$ cm by $10\sqrt{\dfrac{100}{101}}$ cm $= \dfrac{10\sqrt{101}}{101}$ cm by $\dfrac{100\sqrt{101}}{101}$ cm

12 **a** **i** 5.41 m **ii** 4.61 m **iii** 5.70 m
iv 8.70 m **v** 8.91 m **vi** 6.44 m
b 7.91 m

13 Research required.

4G

Building understanding

1 **a** πr^2 **b** $\dfrac{\theta}{360^\circ} \times \pi r^2$ **c** l^2
d lw
e $\dfrac{1}{2}xy$, where x and y are the diagonals.
f $\dfrac{1}{2}(a + b)h$ **g** $\dfrac{1}{2}bh$ **h** $\dfrac{1}{2}xy$
i bh **j** $\dfrac{1}{2}\pi r^2$ **k** $\dfrac{1}{4}\pi r^2$

2 **a** **i** 10 **ii** 100
b **i** 100 **ii** 10 000
c **i** 1000 **ii** 1 000 000
d 10 000

Now you try

Example 15
a 35 000 cm^2 **b** 0.05 m^2

Example 16
a 12 m^2 **b** 1.43 cm^2 **c** 12.01 km^2

Example 17
a $l = 2.7$ **b** $a = 2.16$

Example 18
a $\dfrac{26\pi}{9} \approx 9.08$ cm^2 **b** $60 - \dfrac{9\pi}{2} \approx 45.86$ m^2

Exercise 4G

1 **a** 150 mm^2 **b** 50 000 cm^2
c 200 000 m^2 **d** 0.0007 km^2
e 0.45 m^2 **f** 0.006 km^2
g 230 cm^2 **h** 53 700 mm^2
i 2700 m^2 **j** 10 000 000 mm^2
k 2 200 000 cm^2 **l** 0.000 145 km^2

2 **a** 25 cm^2 **b** 54.6 m^2 **c** 1.82 km^2
d 0.025 mm^2 **e** 153.94 m^2 **f** 75 cm^2
g 1472 m^2 **h** 0.05 mm^2 **i** 0.17 km^2
j 2.36 km^2 **k** 1.1234 m^2 **l** 3.97 cm^2

3 **a** $w = 2.88$ **b** $l = 14.35$ **c** $h = 1.44$
d $a = 1.05$ **e** $h = 1.91$ **f** $z = 8.89$
g $r = 1.26$ **h** $d = 0.52$ **i** $d = 5753.63$

4 **a** 9π cm^2, 28.27 cm^2 **b** $\dfrac{25}{2}\pi$ m^2, 39.27 m^2
c $\dfrac{49}{3}\pi$ m^2, 51.31 m^2 **d** $\dfrac{26}{9}\pi$ m^2, 9.08 m^2
e 21π km^2, 65.97 km^2 **f** $\dfrac{7}{8}\pi$ mm^2, 2.75 mm^2

5 43.2 m^2

6 **a** $\left(\dfrac{25}{8}\pi + 25\right)$ cm^2, 34.82 cm^2
b 49 m^2
c $\left(\dfrac{289}{200}\pi + \dfrac{104}{25}\right)$ m^2, 8.70 m^2
d $\dfrac{(3969 - 441\pi)}{25}$ mm^2, 103.34 mm^2
e $(81\pi + 324)$ km^2, 578.47 km^2
f $\left(\dfrac{49}{200}\pi - \dfrac{99}{400}\right)$ m^2, 0.52 m^2

7 **a** 66 m^2 **b** 27 bags

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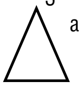
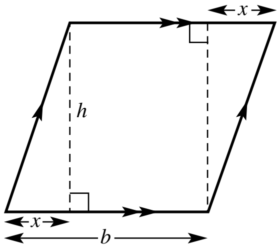
8 **a** 100 ha **b** 200 000 m^2
c 0.4 ha **d** 2.5 acres

9 **a** $a = \frac{2A}{h} - b$
b **i** $3\frac{1}{3}$ **ii** 4.7 **iii** 0
c  a triangle

10 **a**



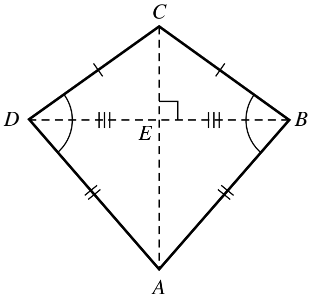
Let x be the base of each triangle.

$$A = (b - x) \times h + \frac{1}{2}xh + \frac{1}{2}xh$$

(i.e. rectangle and two triangles)

$$A = bh - xh + xh$$
$$A = bh$$

b



Let $x = AC$ and $y = BD$.

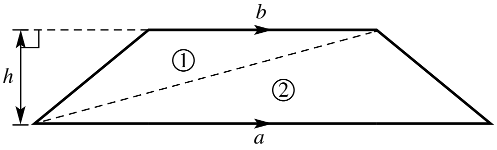
AC bisects BD, hence $DE = EB = \frac{1}{2}y$.

$$A = \frac{1}{2} \times x \times \frac{1}{2}y + \frac{1}{2} \times x \times \frac{1}{2}y$$

(i.e. area of $\triangle ACD$ plus area of $\triangle ABC$)

$$A = \frac{1}{4}xy + \frac{1}{4}xy$$
$$A = \frac{1}{2}xy$$

c Consider the following trapezium.



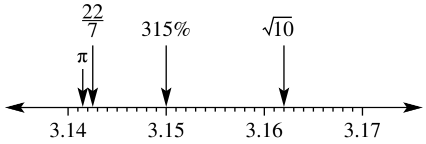
$A =$ Area① + Area②

$$A = \frac{1}{2} \times a \times h + \frac{1}{2} \times a \times h$$
$$A = \frac{1}{2}(a + b)h$$

11 **a** 63.7% **b** 78.5% **c** 50% **d** 53.9%

Progress quiz

1 **a** 3.1622776… irrational
b $3.\overline{142857}$ rational
c 3.141592653… irrational
d 3.15 rational



2 **a** $7\sqrt{2}$ **b** $10\sqrt{3}$ **c** $\frac{5\sqrt{2}}{2}$ **d** $\frac{5\sqrt{5}}{4}$
3 $\sqrt{192}$
4 **a** $3\sqrt{3}$ **b** $6\sqrt{2} + 2\sqrt{5}$
c $16\sqrt{3}$ **d** $17\sqrt{5} - 6\sqrt{3}$
5 **a** $-\sqrt{15}$ **b** $20\sqrt{7}$ **c** $\frac{2\sqrt{5}}{3}$ **d** 6
e 52
6 $66\sqrt{2}$
7 **a** $\frac{3\sqrt{7}}{7}$ **b** $\frac{2\sqrt{15}}{5}$ **c** $\frac{6\sqrt{2} - 3\sqrt{10}}{2}$
8 **a** 36 cm **b** 26.85 cm **c** 30 cm
9 **a** **i** 5 **ii** 30 cm^2
b **i** 41 **ii** 180 cm^2
c **i** 6.40 **ii** 50 cm^2
10 **a** $C = 25.13$ mm, $A = 50.27$ mm^2
b $C = 55.29$ mm, $A = 243.28$ mm^2
11 13.75 cm
12 **a** 17.45 cm^2 **b** 29.32 cm^2
13 **a** 450 **b** 0.00045
14 **a** 3.86 cm^2 **b** 82.27 m^2

4H

Building understanding

1 Some examples are 3.35, 3.37, 3.40 and 3.42.
2 **a** 347 cm **b** 3 m
3 6.65
4 **a** **i** 31.0 **ii** 55.8
b **i** 30.972 **ii** 55.7
c Answers differ by 0.1, 55.7 is more accurate due to no prior rounding.

Now you try

Example 19

a 6.5 m **b** 16.6 m^2 **c** 9.1 m^2
d 25.7 m^2 **e** 25.9 m^2
f Differ by 0.2 m^2, due to accumulated error from rounding in parts **a**–**c**.

Example 20

a 44.5 cm to 45.5 cm
b 15.65 mm to 15.75 mm

Example 21

a 8.5 cm to 9.5 cm
b 34 cm to 38 cm
c 72.25 cm^2 to 90.25 cm^2

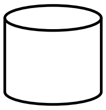
Exercise 4H

1 a 4.9 m b 9.4 m^2 c 5.2 m^2
d 14.6 m^2 e 14.7 m^2
f Differ by 0.1 m^2 due to accumulated rounding error from parts **a**–**c**.
2 a 13.7 m b 147.4 m^2 c 43.3 m^2
d 190.7 m^2 e 190.8 m^2
f Differ by 0.1 m^2 due to accumulated rounding error from parts **a**–**c**.
3 a i 1 cm ii 44.5 cm to 45.5 cm
b i 0.1 mm ii 6.75 mm to 6.85 mm
c i 1 m ii 11.5 m to 12.5 m
d i 0.1 kg ii 15.55 kg to 15.65 kg
e i 0.1 g ii 56.75 g to 56.85 g
f i 1 m ii 9.5 m to 10.5 m
g i 1 h ii 672.5 h to 673.5 h
h i 0.01 m ii 9.835 m to 9.845 m
i i 0.01 km ii 12.335 km to 12.345 km
j i 0.001 km ii 0.9865 km to 0.9875 km
k i 0.01 L ii 1.645 L to 1.655 L
l i 0.01 mL ii 9.025 mL to 9.035 mL
4 a $4450 to $4550
b $4495 to $4505
c $4499.50 to $4500.50
5 a 30 m b 15 g
c 4.6 km d 9.0 km
e 990 g f 990 g (nearest whole)
6 a 149.5 cm to 150.5 cm b 145 cm to 155 cm
c 149.95 cm to 150.05 cm
7 a 24.5 cm to 25.5 cm b 245 cm
c 255 cm
8 a 9.15 cm
b 9.25 cm
c 36.6 cm to 37 cm
d 83.7225 cm^2 to 85.5625 cm^2
9 a 9.195 cm
b 9.205 cm
c 36.78 cm to 36.82 cm
d 84.548025 cm^2 to 84.732025 cm^2
e Increasing the level of accuracy lowers the difference between the upper and lower limits of any subsequent working.
10 a Different rounding (level of accuracy being used).
b Cody used to the nearest kg, Jacinta used to the nearest 100 g and Luke used to the nearest 10 g.
c Yes
11 a Distances on rural outback properties, distances between towns, length of wires and pipes along roadways.
b Building plans, measuring carpet and wood.
c Giving medicine at home to children, paint mixtures, chemical mixtures by students.
d Buying paint, filling a pool, recording water use.
12 a ±1.8% b ±5.6% c ±0.56%
d ±0.056% e ±0.28% f ±0.056%
g ±0.12% h ±0.071%

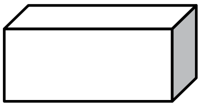
4I

Building understanding

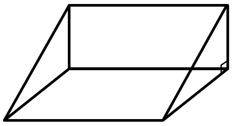
1 a



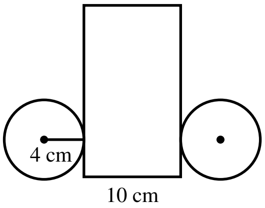
b



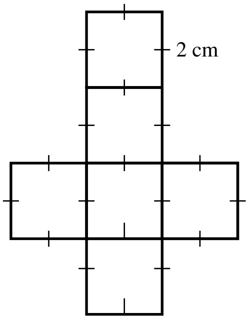
c



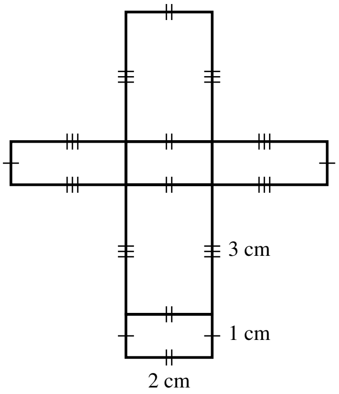
2 a



b



c



Now you try

Example 22
a 94 m^2 b 95.06 cm^2

Example 23
a 182.5 m^2 b 101.7 m^2

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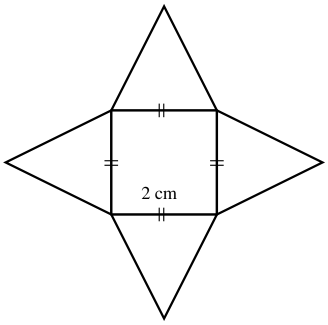
Exercise 4I

1 a 144 m^2 b 47.82 mm^2 c 111.3 cm^2
2 a 72.57 cm^2 b 502.91 m^2 c 168.89 m^2
3 a 8.64 cm^2 b 96 mm^2 c 836.6 m^2
d 688 mm^2 e 4.74 cm^2 f 43.99 m^2
4 24.03 m^2
5 3880 cm^2
6 a 121.3 cm^2 b 10.2 m^2
c 236.5 m^2 d 2437.8 cm^2
7 a 66.2 b 17.9 c 243.1
d 207.3 e 2308.7 f 65.0
8 a 144.5 cm^2 b 851.3 m^2
c 1192.7 cm^2 d 4170.8 m^2
9 33.5 m^2
10 a $6x^2$
b $2(ab + ac + bc)$
c $\pi\left(\frac{1}{2}d\right)^2 + \frac{1}{2}\pi dh + dh$
d $\frac{1}{2}\pi r^2 + 2rh + \frac{1}{2}\pi rh$
11 a 6π b $\frac{11\pi}{2}$
12 a 0.79 m b 7.71 m
13 1 cm
14 a $4\pi r^2$
b $2x(x + 2y)$
c $2rh + \pi r(h + r)$
d $2rh + \frac{\theta}{180°}\pi r(h + r)$

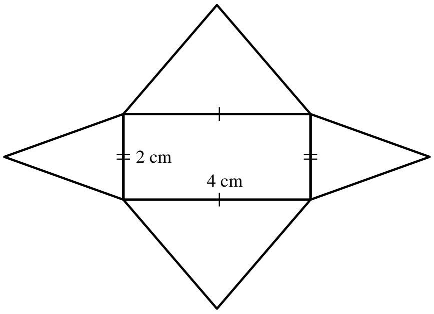
4J

Building understanding

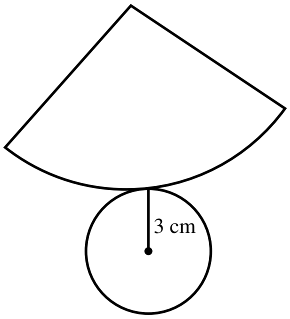
1 a $\frac{1}{2}bh$ b πr^2 c πrs
2 a $\sqrt{29}$ cm b $\sqrt{221}$ m c $\sqrt{109}$ cm
3 a



b



c



Now you try

Example 24
a 81.05 m^2 b 1160 mm^2

Example 25
a 12.7 cm b 12.6 cm

Exercise 4J

1 a 144.51 m^2 b 593.76 mm^2 c 0.82 m^2
2 a 64 m^2 b 105 cm^2 c 0.16 m^2
3 a 62.83 m^2 b 5.18 cm^2 c 1960.35 mm^2
4 a 10.44 cm b 126.7 cm^2
5 a 25.5 cm b 25.0 cm
6 a 18.9 cm b 17.8 cm
7 Hat B
8 a 6.3 m b 66.6 m^2
9 a 105 cm^2 b 63 cm^2 c 163.3 cm^2
d 299.4 m^2 e 502.8 mm^2 f 76.6 m^2
10 Slant height, $s = \sqrt{r^2 + h^2}$, so $\pi r(r + s) = \pi r\left(r + \sqrt{r^2 + h^2}\right)$.
11 Substitute $h = r$ into the equation given in Question 11.

$$\begin{aligned}\pi r\left(r + \sqrt{r^2 + h^2}\right) &= \pi r\left(r + \sqrt{r^2 + r^2}\right)\\ &= \pi r\left(r + \sqrt{2r^2}\right)\\ &= \pi r(r + \sqrt{2}\,r)\\ &= \pi r^2(1 + \sqrt{2}) \text{ as required}\end{aligned}$$

12 182.3 cm^2
13 a $4\sqrt{26}$ cm b 306.57 cm^2
c $4\sqrt{2}$ cm d 20.199 cm
e 260.53 cm^2 f 85%

4K

Building understanding

1 a 80 cm^3 b 32 m^3
2 a $V = abc$ b $V = y^2h$ c $V = \pi r^2h$

Now you try

Example 26
a 42 500 mm^3 b 0.124 ML

Example 27
a 240 cm^3 b 50.27 m^3

Example 28
393 mL

Exercise 4K

1 **a** 2000 mm^3 **b** 200 000 cm^3 **c** 15 000 000 m^3
d 5.7 cm^3 **e** 0.0283 km^3 **f** 0.762 m^3
g 130 000 cm^3 **h** 1000 m^3 **i** 2094 mm^3
j 2700 mL **k** 0.342 ML **l** 0.035 kL
m 5720 kL **n** 74.25 L **o** 18 440 L

2 **a** 40 cm^3 **b** 10 500 m^3 **c** 259.7 mm^3

3 **a** 785.40 m^3 **b** 18.85 cm^3 **c** 1583.36 m^3

4 **a** 30 km^3 **b** 196 cm^3 **c** 30 m^3
d 10 cm^3 **e** 0.002 m^3 **f** 4752.51 cm^3
g 0.157 m^3 **h** 1357.168 cm^3 **i** 24 m^3

5 1000

6 480 L

7 **a** 379.33 cm^3 **b** 223.17 m^3 **c** 6.808 m^3
d 716 mL **e** 142 mL **f** 43 mL

8 **a** 27 cm^3 **b** $3\sqrt{3}$ m^3

9 0.5 cm

10 **a** 12 cm^3 **b** 1570.8 m^3 **c** 2.448 mm^3

11 $V = \frac{\theta}{360^\circ}\pi r^2 h$

12 Yes; 69.3 m^3

13 **a** $\frac{1}{\sqrt{2}}$ m **b** 5.8 m^3

4L

Building understanding

1 4 cm^3

2 15 m^3

3 **a** 10 m^3 **b** $\frac{8}{3}$ cm^3 **c** $58\frac{1}{3}$ mm^3

Now you try

Example 29
a 19.25 m^3 **b** 1119.19 mm^3

Exercise 4L

1 **a** 4 cm^3 **b** 77 m^3 **c** 50 km^3 **d** 585 m^3

2 **a** 1588.86 mm^3 **b** 0.82 m^3
c 9.38 mm^3 **d** 25 132.74 m^3

3 47 mL

4 **a** 282.74 m^3 **b** 276 cm^3 **c** 48 m^3
d 56.88 mm^3 **e** 10.3488 m^3 **f** 70.79 m^3

5 4.76 cm

6 $\frac{2}{3}$

Wood wasted = volume of cylinder – volume of cone

Wood wasted $= \pi r^2 h - \frac{1}{3}\pi r^2 h$

Wood wasted $= \frac{2}{3}\pi r^2 h$

Wood wasted $= \frac{2}{3}$ of the volume of cylinder

7 **a** **i** $V = \frac{1}{3}x^2 h$ **ii** $V = \frac{1}{12}\pi x^2 h$
b $\frac{\pi}{4}$

8 **a** 3.7 cm
b **i** $h = \frac{3V}{\pi r^2}$ **ii** $r = \sqrt{\frac{3V}{\pi h}}$

9 **a** Similar triangles are formed so corresponding sides are in the same ratio.
b $\frac{1}{3}\pi\left(r_1^2 h_1 - r_2^2 h_2\right)$
c **i** 18.3 cm^3 **ii** 14.7 cm^3

4M

Building understanding

1 **a** 314.16 **b** 3.14
c 91.95 **d** 1436.76

2 $r = \sqrt{\frac{3}{\pi}}$

3 **a** $\frac{1}{2}$ **b** $\frac{1}{8}$ **c** $\frac{1}{4}$

Now you try

Example 30
TSA = 314.16 cm^2, V = 523.60 cm^3

Example 31
1.13 m

Example 32
a TSA = 414.69 cm^2
b 753.98 cm^3

Exercise 4M

1 **a** 50.27 cm^2, 33.51 cm^3
b 3.14 m^2, 0.52 m^3
c 18145.84 mm^2, 229 847.30 mm^3
d 1017.88 cm^2, 3053.63 cm^3
e 2.66 km^2, 0.41 km^3
f 5.81 m^2, 1.32 m^3

2 **a** 113.10 cm^2, 113.10 cm^3 **b** 201.06 m^2, 268.08 m^3
c 688.13 m^2, 1697.40 m^3 **d** 15.71 mm^2, 5.85 mm^3
e 21.99 m^2, 9.70 m^3 **f** 15.21 km^2, 5.58 km^3

3 **a** **i** 1.53 cm **ii** 3.50 cm **iii** 0.50 km
b **i** 0.89 m **ii** 3.09 cm **iii** 0.18 mm

4 **a** 113.10 cm^3 **b** 5654.9 cm^3 **c** 21 345.1 cm^3

5 11.5 cm

6 **a** 32.72 cm^3 **b** 67.02 cm^3 **c** 0.52 m^3

7 **a** 4 m **b** 234.6 m^3

8 **a** 235.62 m^2 **b** 5.94 cm^2 **c** 138.23 mm^2
d 94.25 m^2 **e** 27.14 m^2 **f** 26.85 cm^2

9 **a** 5.24 m^3 **b** 942.48 m^3 **c** 10.09 cm^3
d 1273.39 cm^3 **e** 4.76 m^3 **f** 0.74 cm^3

10 52%

11 61.2 cm

12 **a** 5 cm **b** $5\sqrt{5}$ cm **c** 332.7 cm^2

13 **a** $r = \sqrt{\frac{S}{4\pi}}$ **b** $r = \sqrt[3]{\frac{3V}{4\pi}}$

14 **a** 4 times **b** 8 times

15 $V = \frac{4}{3} \times \pi r^3$

Substitute $\frac{d}{2}$ into r, giving:

$$V = \frac{4}{3} \times \pi\left(\frac{d}{2}\right)^3$$

$$V = \frac{4}{3} \times \frac{\pi d^3}{8} = \frac{1}{3} \times \frac{\pi d^3}{2}$$

$$V = \frac{1}{6}\pi d^3$$

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16 $h = \frac{4}{3}r$

17 **a** **i** $\sqrt[3]{\frac{3}{4\pi}}$ **ii** $\sqrt[3]{36\pi}$ **iii** 1

iv 6 **v** 80.6%

b **i** $4\pi r^2$ **ii** $x = \sqrt[3]{\frac{4\pi}{3}}r$ **iii** $6\left(\frac{4\pi}{3}\right)^{\frac{2}{3}}r^2$

c Proof required. Example:

$$\frac{4\pi r^2}{6\left(\frac{4\pi}{3}\right)^{\frac{2}{3}}r^2} = \frac{2\pi}{3^{\frac{1}{3}}(4\pi)^{\frac{2}{3}}} = \frac{2\pi^{\frac{1}{3}}}{8^{\frac{1}{3}} \times 6^{\frac{1}{3}}} = \sqrt[3]{\frac{\pi}{6}}, \text{ as required.}$$

d They are the same.

Problems and challenges

1 6

2 1.3 m

3 **a** As the sphere touches the top, bottom and curved surface, the height of the cylinder is $2r$, and the radius of the base is r. So the curved surface area $= 2 \times \pi \times r \times h$ and $h = 2r$, therefore this equals $4\pi r^2$, which is equal to the surface area of the sphere.

b 67%

4 $h = 4r$

5 Length $= 10\sqrt{2}$ cm, width $= 10$ cm

6 $\frac{-3 - \sqrt{2} + 7\sqrt{3}}{7}$

7 $12 + 8\sqrt{2}$

8 $(4 - \pi)r^2$

9 $\sqrt{2}:1$

Answers to success criteria example questions

1 $\sqrt{6} = 2.4494\ldots$; irrational

2 $5\sqrt{3}$ **3** $\sqrt{63}$ **4** $4\sqrt{3} + 3\sqrt{2}$

5 $12\sqrt{3}$ **6** $30\sqrt{2}$ **7** $4\sqrt{5}$

8 $6\sqrt{15} - 10$ **9** $\frac{4\sqrt{14}}{7}$ **10** $x = 8.4$

11 $r = 1.91$ **12** $\frac{10\pi}{3} + 10$ cm **13** $x = 7.21$

14 $BE = \sqrt{125}$; $BH = 11.87$

15 0.12 m^2

16 30 m^2; 20 cm^2

17 $h = 6$

18 $(9.66 + 1.3225\pi)$ m^2; 13.81 m^2

19 **a** **i** 77.4 m^2

ii 77.9 m^2

b The difference is due to the accumulated rounding error in part **i**. by rounding off at every stage of the calculation.

20 84.5 cm to 85.5 cm

21 Lower limit = 38 cm; upper limit = 42 cm

22 304 m^2 **23** 106.88 m^2 **24** 496 cm^2

25 175.93 cm^2 **26** 865.80 cm^2 **27** 5.8 cm

28 **a** **i** 1 200 000 cm^3

ii 1.49 kL

29 125.66 cm^3 **30** 685.6 cm^3

31 1.764 m^3 **32** 678.58 cm^3

33 Volume = 113.10 cm^3; surface area = 113.10 cm^2

34 1.42

35 Surface area = 103.67 cm^2; volume = 94.25 cm^3

Short-answer questions

1 **a** $2\sqrt{6}$ **b** $30\sqrt{2}$ **c** $\frac{2\sqrt{2}}{3}$ **d** $\frac{2\sqrt{5}}{5}$

2 **a** $4 + 7\sqrt{3}$ **b** $2\sqrt{5} + 2\sqrt{7}$ **c** $5\sqrt{2}$

d $4\sqrt{3} + 2\sqrt{2}$ **e** $2\sqrt{30}$ **f** $-12\sqrt{5}$

g $2\sqrt{5}$ **h** $\frac{\sqrt{7}}{3}$ **i** 0

3 **a** $2\sqrt{6} + 4\sqrt{2}$ **b** $12\sqrt{5} - 6$

c 11 **d** 48

4 **a** $\frac{\sqrt{6}}{6}$ **b** $3\sqrt{6}$

c $\frac{3\sqrt{2}}{4}$ **d** $\frac{4\sqrt{6} - 3}{3}$

5 **a** 23 cm **b** 2.7 cm^2

c 2 600 000 cm^3 **d** 8372 mL

e 0.63825 m^2 **f** 3 000 000 cm^2

6 **a** 32 m **b** 28.6 m **c** 20.4 cm

7 **a** $\frac{7}{\pi}$ m **b** 15.60 m^2

8 **a** $\sqrt{65}$ **b** 8.31

9 **a** 16.12 m^2 **b** 216 m^2 **c** 38.5 m^2

d 78.54 cm^2 **e** 100.43 m^2 **f** 46.69 m^2

10 **a** 4.8 m **b** 25.48 m

11 **a** 314.2 m^3 **b** 243.2 m^3

c 557.4 m^3 **d** 557.3 m^3

e Differ by 0.1 m^3 due to accumulated rounding error from part **a** and **b** answers.

12 **a** 7.5 m to 8.5 m

b 10.25 kg to 10.35 kg

c 4.745 L to 4.755 L

13 **a** **i** 236 m^2 **ii** 240 m^3

b **i** 184 cm^2 **ii** 120 cm^3

c **i** 1407.43 cm^2 **ii** 4021.24 cm^3

d **i** 360 cm^2 **ii** 400 cm^3

e **i** 201.06 m^2 **ii** 268.08 m^3

f **i** 282.74 cm^2 **ii** 314.16 cm^3

14 **a** $\frac{175}{3\pi}$ cm **b** 17.6 cm

15 **a** **i** 414.25 cm^2 **ii** 535.62 cm^3

b **i** 124 m^2 **ii** 88 m^3

c **i** 19.67 mm^2 **ii** 6.11 mm^3

16 **a** **i** 117.27 cm^2 **ii** 84.94 cm^3

b **i** 104 cm^2 **ii** 75 cm^3

c **i** 25.73 cm^2 **ii** 9.67 cm^3

17 **a** 4950π cm^3 **b** 1035π cm^2

Multiple-choice questions

1 C **2** B **3** A **4** D **5** D

6 E **7** A **8** D **9** C **10** A

11 B **12** D **13** E **14** D **15** E

Extended-response questions

1 **a** 72 m^3 **b** $\sqrt{37}$ m

c 138.7 m^2 **d** 6 L, \$120

2 **a** 100 m **b** $50\sqrt{2}$ m **c** 5000 m^2 **d** 36%

e Athlete A, 0.01 seconds

3 **a** $36\sqrt{15} + 3\sqrt{45} = 36\sqrt{15} + 9\sqrt{5}$ cm^2

b $4\sqrt{3} + 1$

c **i** 10 000 cm^2 **ii** 1.6%

Chapter 5

5A

Building understanding

1 a $x^2 + 2x$ b $x^2 + 4x + 3$ c $x^2 + 8x + 16$
2 a $6x$ b $-20x$ c $-4x^2$ d $\frac{x}{2}$
e $-3x$ f $-18x$ g $5x$ h $-13x$

Now you try

Example 1
a $-2x + 8$ b $20x - 5x^2$ c $8x^2 - 3x$

Example 2
a $x^2 + 7x + 10$ b $x^2 - 4x + 4$ c $9x^2 - 4$

Example 3
a $6x^2 + 19x - 7$ b $3x^2 - 15x + 12$ c $11x - 9$

Exercise 5A

1 a $2x + 10$ b $3x - 12$ c $-5x - 15$
d $-4x + 8$ e $6x - 3$ f $12x + 4$
g $-10x + 6$ h $-20x - 15$ i $2x^2 + 5x$
j $3x^2 - x$ k $2x - 2x^2$ i $6x - 3x^2$
m $-6x^2 - 4x$ n $-18x^2 + 6x$ o $-10x + 10x^2$
p $-4x + 16x^2$ q $4x + \frac{8}{5}$ r $6x - \frac{15}{4}$
s $-2x - \frac{1}{3}$ t $-2x + \frac{3}{2}$
2 a $2x^2 + 3x$ b $6x^2 - 3x$ c $2x^2 + 7x$
d $8x^2 + 7x$ e $2x^2 - 2x$ f $25x - 12x^2$
3 a $x^2 + 10x + 16$ b $x^2 + 7x + 12$
c $x^2 + 12x + 35$ d $x^2 + 5x - 24$
e $x^2 + x - 30$ f $x^2 + x - 6$
g $x^2 - 4x - 21$ h $x^2 - 10x + 24$
i $x^2 - 13x + 40$
4 a $x^2 + 10x + 25$ b $x^2 + 14x + 49$
c $x^2 + 12x + 36$ d $x^2 - 6x + 9$
e $x^2 - 16x + 64$ f $x^2 - 20x + 100$
g $x^2 - 16$ h $x^2 - 81$
i $4x^2 - 9$ j $9x^2 - 16$
k $16x^2 - 25$ l $64x^2 - 49$
5 a $6x^2 + 13x + 5$ b $12x^2 + 23x + 10$
c $10x^2 + 41x + 21$ d $9x^2 - 9x - 10$
e $20x^2 + 2x - 6$ f $6x^2 + 5x - 25$
g $16x^2 - 25$ h $4x^2 - 81$
i $25x^2 - 49$ j $14x^2 - 34x + 12$
k $25x^2 - 45x + 18$ l $56x^2 - 30x + 4$
m $4x^2 + 20x + 25$ n $25x^2 + 60x + 36$
o $49x^2 - 14x + 1$
6 a 3 b 3 c 3 d 8 e 1 f 2
7 a $2x^2 + 14x + 24$ b $3x^2 + 27x + 42$
c $-2x^2 - 20x - 32$ d $-4x^2 - 44x - 72$
e $5x^2 + 5x - 60$ f $3x^2 + 6x - 45$
g $-3a^2 + 15a + 42$ h $-5a^2 + 30a + 80$
i $4a^2 - 36a + 72$ j $3y^2 - 27y + 60$
k $-2y^2 + 22y - 48$ l $-6y^2 + 42y - 72$
m $12x^2 + 48x + 45$ n $18x^2 + 12x - 48$
o $-6x^2 - 10x + 56$ p $2x^2 + 12x + 18$
q $4m^2 + 40m + 100$ r $2a^2 - 28a + 98$
s $-3y^2 + 30y - 75$ t $12b^2 - 12b + 3$
u $-12y^2 + 72y - 108$
8 a $2x^2 + 10x + 11$ b $2x^2 + 20x + 44$
c $2y^2 - 4y + 5$ d $2y^2 - y - 43$
e $-24a - 45$ f $b^2 + 54b + 5$
g $x^2 + 10x + 18$ h $x^2 - 14x + 40$
i $-4x^2 + 36x - 78$ j $-25x^2 - 30x + 5$
9 a $x^2 - 12x + 36\,\text{cm}^2$ b $x^2 + 10x - 200\,\text{cm}^2$
10 a 618 b 220 c 567 d 1664
e 1386 f 891 g 3960 h 3480
11 a $-x^2 + 7x$ b $10a - 28$
c $4x^2 + 12x + 9$ d $4x + 8$
12 a $(a + b)(a - b) = a^2 - ab + ba - b^2 = a^2 - b^2$
b $$\begin{aligned}(a + b)^2 &= (a + b)(a + b) = a^2 + ab + ba + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$$
c $$\begin{aligned}(a - b)^2 &= (a - b)(a - b) = a^2 - ab - ba + b^2\\ &= a^2 - 2ab + b^2\end{aligned}$$
d $$\begin{aligned}(a + b)^2 - (a - b)^2 &=\\ a^2 + ab + ba + b^2 - \left(a^2 - ab - ba + b^2\right) &=\\ 2ab + 2ab &= 4ab\end{aligned}$$
13 a $x^3 + 6x^2 + 11x + 6$ b $x^3 + 11x^2 + 38x + 40$
c $x^3 + 2x^2 - 15x - 36$ d $2x^3 - 13x^2 + 17x + 12$
e $2x^3 - x^2 - 63x + 90$ f $6x^3 - 35x^2 + 47x - 12$
14 a $2ab$ b $(a + b)^2 - c^2$
c $$\begin{aligned}(a + b)^2 - c^2 &= 2ab\\ c^2 &= a^2 + 2ab + b^2 - 2ab\\ c^2 &= a^2 + b^2\end{aligned}$$

5B

Building understanding

1 a 7 b −5 c $3a$ d $-3xy$
2 a If $x(x - 1) = x^2 - x$, then $x^2 - x = x(x - 1)$.
b If $2(1 - x) = 2 - 2x$, then $2 - 2x = 2(1 - x)$.
c If $(x + 2)(x - 2) = x^2 - 4$, then $x^2 - 4 = (x + 2)(x - 2)$.
d If $(3x - 7)(3x + 7) = 9x^2 - 49$, then $9x^2 - 49 = (3x - 7)(3x + 7)$.

Now you try

Example 4
a $-2(x + 4)$ b $5a(3a + 4)$ c $(x + 2)(3 - a)$

Example 5
a $(x + 5)(x - 5)$ b $(4a + 3b)(4a - 3b)$
c $2(y + 7)(y - 7)$ d $(x + 8)(x - 4)$

Example 6
a $(x + \sqrt{7})(x - \sqrt{7})$ b $(x - 5 + \sqrt{2})(x - 5 - \sqrt{2})$

Example 7
$(x - 2)(x + a)$

Exercise 5B

1 a $3(x - 6)$ b $4(x + 5)$
c $7(a + b)$ d $3(3a - 5)$

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e $-5(x+6)$
f $-2(2y+1)$
g $-3(4a+1)$
h $-b(2a+c)$
i $x(4x+1)$
j $x(5x-2)$
k $6b(b-3)$
l $7a(2a-3)$
m $5a(2-a)$
n $6x(2-5x)$
o $-x(2+x)$
p $-4y(1+2y)$
q $ab(b-a)$
r $2xy(xz-2)$
s $-12mn(m+n)$
t $3z^2(2xy-1)$

2 **a** $(x-1)(5-a)$
b $(x+2)(b+3)$
c $(x+5)(a-4)$
d $(x+2)(x+5)$
e $(x-4)(x-2)$
f $(x+1)(3-x)$
g $(x+3)(a+1)$
h $(x-2)(x-1)$
i $(x-6)(1-x)$

3 **a** $(x+3)(x-3)$
b $(x+5)(x-5)$
c $(y+7)(y-7)$
d $(y+1)(y-1)$
e $(2x-3)(2x+3)$
f $(6a-5)(6a+5)$
g $(1+9y)(1-9y)$
h $(10-3x)(10+3x)$
i $(5x-2y)(5x+2y)$
j $(8x-5y)(8x+5y)$
k $(3a+7b)(3a-7b)$
l $(12a-7b)(12a+7b)$

4 **a** $2(x+4)(x-4)$
b $5(x+3)(x-3)$
c $6(y+2)(y-2)$
d $3(y+4)(y-4)$
e $3(x+5y)(x-5y)$
f $3(a+10b)(a-10b)$
g $3(2x+3y)(2x-3y)$
h $7(3a+4b)(3a-4b)$
i $(x+9)(x+1)$
j $(x-7)(x-1)$
k $(a+5)(a-11)$
l $(a-8)(a-6)$
m $(4x+5)(2x+5)$
n $(y+7)(3y+7)$
o $(3x+11)(7x+11)$
p $3x(3x-10y)$

5 **a** $(x+\sqrt{7})(x-\sqrt{7})$
b $(x+\sqrt{5})(x-\sqrt{5})$
c $(x+\sqrt{19})(x-\sqrt{19})$
d $(x+\sqrt{21})(x-\sqrt{21})$
e $(x+\sqrt{14})(x-\sqrt{14})$
f $(x+\sqrt{30})(x-\sqrt{30})$

6 **a** $(x+4)(x+a)$
b $(x+7)(x+b)$
c $(x-3)(x+a)$
d $(x+2)(x-a)$
e $(x+5)(x-b)$
f $(x+3)(x-4a)$
g $(x-a)(x-4)$
h $(x-2b)(x-5)$
i $(x-2a)(3x-7)$

7 **a** $(x+2)(y-3)$
b $(a-4)(x+3)$
c $(a+5)(x-2)$
d $(y-4)(x-3)$
e $(a-3)(2x-1)$
f $(2a-5)(x+4)$

8 **a** $\left(x+\frac{1}{3}\right)\left(x-\frac{1}{3}\right)$
b $\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$
c $\left(x-\frac{3}{4}\right)\left(x+\frac{3}{4}\right)$
d $\left(x-\frac{5}{6}\right)\left(x+\frac{5}{6}\right)$
e $(\sqrt{3}x+2)(\sqrt{3}x-2)$
f $(\sqrt{5}x+3)(\sqrt{5}x-3)$
g $(\sqrt{2}x+3)(\sqrt{2}x-3)$
h $(\sqrt{5}x+4)(\sqrt{5}x-4)$

9 **a** $(x+3\sqrt{2})(x-3\sqrt{2})$
b $(x+3\sqrt{5})(x-3\sqrt{5})$
c $(x+2\sqrt{5})(x-2\sqrt{5})$
d $(x+4\sqrt{2})(x-4\sqrt{2})$
e $(x+4\sqrt{3})(x-4\sqrt{3})$
f $(x+5\sqrt{2})(x-5\sqrt{2})$

10 **a** $5(x+2\sqrt{6})(x-2\sqrt{6})$
b $3(x+3\sqrt{6})(x-3\sqrt{6})$
c $7(x+3\sqrt{2})(x-3\sqrt{2})$
d $2(x+4\sqrt{3})(x-4\sqrt{3})$
e $2(x+3+\sqrt{5})(x+3-\sqrt{5})$
f $3(x-1+\sqrt{7})(x-1-\sqrt{7})$
g $4(x-4+2\sqrt{3})(x-4-2\sqrt{3})$
h $5(x+6+3\sqrt{2})(x+6-3\sqrt{2})$

11 **a** 60 **b** 35 **c** 69 **d** 104
e 64 **f** 40 **g** 153 **h** 1260

12 **a** $4-(x+2)^2=(2-(x+2))(2+(x+2))=-x(x+4)$
b **i** $-x(x+6)$ **ii** $-x(x+8)$
iii $x(10-x)$ **iv** $(3-x)(7+x)$
v $(8-x)(6+x)$ **vi** $(6-x)(14+x)$

13 **a** $(x+a)^2=x^2+2ax+a^2\neq x^2+a^2$
b If $x=0$, then $(x+a)^2=x^2+a^2$. Or if $a=0$, then $(x+a)^2=x^2+a^2$ is true for all real values of x.

14 $x^2-\frac{4}{9}=\frac{1}{9}(9x^2-4)=\frac{1}{9}(3x+2)(3x-2)$

$$x^2-\frac{4}{9}=\left(x+\frac{2}{3}\right)\left(x-\frac{2}{3}\right)$$

or: $$=\frac{1}{3}(3x+2)\frac{1}{3}(3x-2)$$

$$=\frac{1}{9}(3x+2)(3x-2)$$

15 **a** $-(2x+5)$
b $-11(2y-3)$
c $16(a-1)$
d $20b$
e $-12s$
f $-28y$
g $(5w+7x)(-w-x)$
h $(4d+3e)(-2d+7e)$
i $12f(f+3j)$
j 0

16 **a** x^2+5y-y^2+5x
$=x^2-y^2+5x+5y$
$=(x-y)(x+y)+5(x+y)$
$=(x+y)(x-y+5)$
b **i** $(x+y)(x-y+7)$
ii $(x+y)(x-y-2)$
iii $(2x+3y)(2x-3y+2)$
iv $(5y+2x)(5y-2x+3)$

5C

Building understanding

1 **a** $\frac{2}{3}$
b $\frac{3}{7a}$
c $-\frac{7t}{4xy}$
d $-\frac{b^2c}{8x^2a}$

2 **a** 1
b $\frac{5}{6}$
c 2
d $\frac{3}{4}$

3 **a** $5x$ **b** $4x$ **c** $\frac{a}{4}$
d $\frac{1}{3a}$ **e** $\frac{1}{3}$ **f** $\frac{1}{4}$

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Now you try

Example 8
a $3ab$ b $1 - 2x$

Example 9
a $\frac{a}{2}$ b 6 c $\frac{x}{2}$

Exercise 5C

1 a $5x$ b $-2x$ c $-9b$ d $-2y$
e $-\frac{1}{2p}$ f $-\frac{4}{9st}$ g $-\frac{3x}{y}$ h $\frac{6b}{7}$

2 a $x + 2$ b $a - 5$ c $3x - 9$ d $1 - 3y$
e $1 + 6b$ f $1 - 3x$ g $3 - t$ h $x - 4$
i $x + 2$ j $3 - 2x$ k $a - 1$ l $\frac{1 + 2a}{3}$

3 a $\frac{4}{9}$ b 5 c $\frac{5a}{2}$
d $\frac{x - 1}{2x}$ e $\frac{x + 4}{5x}$ f $\frac{2(x - 2)}{x}$
g 2 h 15 i $-\frac{1}{2}$

4 a 3 b 3 c $\frac{18}{5}$
d $\frac{3}{4}$ e $\frac{4}{3}$ f $\frac{1}{25}$
g $-\frac{5}{3}$ h $\frac{2}{5}$ i $-\frac{1}{3}$

5 a $x + 1$ m^2 b 2 cm^2 c 4 mm^2

6 a $\frac{x - 3}{2}$ b $\frac{1}{x - 1}$ c $3(x - 2)$
d $\frac{2x^2}{x + 3}$ e $\frac{x}{2}$ f $\frac{3}{2}$

7 a $3x$ b $\frac{4}{3a}$ c $\frac{x + 3}{5}$
d $\frac{4}{x}$ e $\frac{4}{7x}$ f $\frac{2b^2}{b - 1}$

8 a $\frac{10}{x + 3}$ b $\frac{3x}{1 - x}$
c $\frac{3(x + 2)}{2}$ d $\frac{10x}{3}$
e $\frac{x - 1}{2x}$ f $\frac{35x^2}{(2 - x)(x - 1)}$

9 a $x - 1$ b $3(x + 2)$
c $2(x - 3)$ d $\frac{4}{x + 2}$
e $\frac{-5}{1 - x}$ f $4(x - 1)$

10 a–c Factorise and cancel to 1.

11 a $1 - x = -(x - 1)$
b i $\frac{-7}{3}$ ii $\frac{-12}{x}$ iii $\frac{7}{2}$

12 a They are the same, $x^2 - 6x + 9$
b
$$\begin{aligned}(b - a)^2 &= (-(a - b))^2\\ &= (-1)^2(a - b)^2\\ &= (a - b)^2\end{aligned}$$
c i 3 ii $2x - 3$
iii $\frac{2}{x - 5}$ iv $\frac{3}{2(x - 6)}$
v $\frac{1}{2x}$ vi $\frac{y - x}{xy}$

5D

Building understanding

1 a $9, 2$ b $10, 2$ c $5, -3$
d $4, -3$ e $-8, 3$ f $-10, 3$
g $-2, -5$ h $-12, -3$

2 a 2 b 6 c $2, 5$

Now you try

Example 10
a $(x + 3)(x + 4)$ b $(x - 6)(x - 4)$
c $2(x - 3)(x + 2)$ d $(x - 3)^2$

Example 11
a $x - 4$ b $\frac{x + 4}{2}$

Exercise 5D

1 a $(x + 6)(x + 1)$ b $(x + 3)(x + 2)$
c $(x + 3)^2$ d $(x + 5)(x + 2)$
e $(x + 4)(x + 3)$ f $(x + 9)(x + 2)$
g $(x - 1)(x + 6)$ h $(x + 3)(x - 2)$
i $(x + 4)(x - 2)$ j $(x - 1)(x + 4)$
k $(x + 10)(x - 3)$ l $(x + 11)(x - 2)$
m $(x - 2)(x - 5)$ n $(x - 4)(x - 2)$
o $(x - 4)(x - 3)$ p $(x - 1)^2$
q $(x - 6)(x - 3)$ r $(x - 2)(x - 9)$
s $(x - 6)(x + 2)$ t $(x - 5)(x + 4)$
u $(x - 7)(x + 2)$ v $(x - 4)(x + 3)$
w $(x + 8)(x - 4)$ x $(x - 5)(x + 2)$

2 a $2(x + 5)(x + 2)$ b $3(x + 4)(x + 3)$
c $2(x + 9)(x + 2)$ d $5(x - 2)(x + 1)$
e $4(x - 5)(x + 1)$ f $3(x - 5)(x + 2)$
g $-2(x + 4)(x + 3)$ h $-3(x - 2)(x - 1)$
i $-2(x - 7)(x + 2)$ j $-4(x - 2)(x + 1)$
k $-5(x + 3)(x + 1)$ l $-7(x - 6)(x - 1)$

3 a $(x - 2)^2$ b $(x + 3)^2$
c $(x + 6)^2$ d $(x - 7)^2$
e $(x - 9)^2$ f $(x - 10)^2$
g $2(x + 11)^2$ h $3(x - 4)^2$
i $5(x - 5)^2$ j $-3(x - 6)^2$
k $-2(x - 7)^2$ l $-4(x + 9)^2$

4 a $x + 6$ b $x - 3$ c $x - 3$
d $\frac{1}{x + 7}$ e $\frac{1}{x - 5}$ f $\frac{1}{x - 6}$
g $\frac{2}{x - 8}$ h $\frac{x + 4}{3}$ i $\frac{x - 7}{5}$

5 a $\frac{5}{x + 6}$ b $\frac{x - 3}{3}$ c $\frac{2(x - 1)}{x + 5}$
d $\frac{4}{x + 5}$ e $\frac{4}{x + 7}$ f $\frac{6}{x - 2}$
g $\frac{x + 2}{x - 1}$ h $\frac{x - 4}{x + 6}$

6 a $x - \sqrt{7}$ b $x + \sqrt{10}$
c $\frac{1}{\sqrt{5}\,x - 3}$ d $\frac{1}{\sqrt{3}\,x + 4}$
e $x + 1 - \sqrt{2}$ f $x - 3 + \sqrt{5}$

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7 **a** $\frac{2(x+3)}{3(x-5)}$ **b** $\frac{x-3}{4}$ **c** $\frac{3}{x-3}$

d $\frac{3}{2}$ **e** $\frac{x-2}{x+3}$ **f** $\frac{x+3}{x-1}$

8 $\frac{t^2-49}{5t-40} \times \frac{t^2-5t-24}{2t^2-8t-42} =$

$\frac{\cancel{(t-7)}(t+7)}{5\cancel{(t-8)}} \times \frac{\cancel{(t-8)}\cancel{(t+3)}}{2\cancel{(t-7)}\cancel{(t+3)}} = \frac{t+7}{10}$

9 **a** $x-3$ **b** $x+1$ **c** $x-8$

d $\frac{6}{x-2}$ **e** $\frac{4}{x+5}$ **f** $\frac{x-7}{5}$

10 **a** $\frac{a^2+2ab+b^2}{a^2+ab} \times \frac{a^2-ab}{a^2-b^2}$

$= \frac{\cancel{(a+b)^2}}{a\cancel{(a+b)}} \times \frac{a\cancel{(a+b)}}{\cancel{(a+b)}\cancel{(a+b)}}$

$= 1$

b Answers will vary.

11 **a** $\frac{a-b}{a}$ **b** 1

c $\frac{(a+b)^2}{(a-b)^2}$ **d** $\frac{(a+b)(a-b)}{a^2}$

12 **a** $\frac{3x-8}{(x+3)(x-4)}$ **b** $\frac{7x-36}{(x+2)(x-9)}$

c $\frac{x-12}{(x+4)(x-4)}$ **d** $\frac{3x-23}{(x+3)(x-3)(x-5)}$

e $\frac{x-14}{(x-3)(x+2)(x-6)}$ **f** $\frac{14x+9}{(x+3)(x+4)(x-8)}$

g $\frac{9-3x}{(x+5)(x-5)(x-1)}$ **h** $\frac{4x+11}{(x-1)^2(x+4)}$

5E

Building understanding

1

ax^2+bx+c	$a \times c$	Two numbers which multiply to give $a \times c$ and add to give b
$6x^2+13x+6$	36	9 and 4
$8x^2+18x+4$	32	16 and 2
$12x^2+x-6$	−72	−8 and 9
$10x^2-11x-6$	−60	−15 and 4
$21x^2-20x+4$	84	−6 and −14
$15x^2-13x+2$	30	−3 and −10

2 **a** $(x+2)(x+5)$ **b** $(x-7)(x-2)$

c $(3x-4)(2x+1)$ **d** $(2x-1)(4x+3)$

e $(x+4)(5x-2)$ **f** $(2x-1)(6x-5)$

Now you try

Example 12

a $(2x+3)(3x+1)$ **b** $(4x-1)(2x+3)$

Example 13

1

Exercise 5E

1 **a** $(3x+1)(x+3)$ **b** $(2x+1)(x+1)$

c $(3x+2)(x+2)$ **d** $(3x-2)(x-1)$

2 **a** $(2x-1)(x-5)$ **b** $(5x-3)(x+1)$

c $(3x+1)(x-4)$ **d** $(3x+1)(x-1)$

e $(7x-5)(x+1)$ **f** $(2x-7)(x-1)$

g $(3x-4)(x+2)$ **h** $(2x-3)(x+4)$

i $(2x+1)(x-5)$ **j** $(13x+6)(x-1)$

k $(5x-2)(x-4)$ **l** $(4x-5)(2x-1)$

m $(3x-4)(2x+3)$ **n** $(5x-2)(2x+3)$

o $(3x+2)(2x+3)$ **p** $(4x-1)(x-1)$

q $(4x-5)(2x-1)$ **r** $(2x-5)(4x-3)$

s $(3x-2)(2x-3)$ **t** $(3x-2)(3x+5)$

3 **a** $(6x+5)(3x+2)$ **b** $(4x+3)(5x+6)$

c $(7x-2)(3x+4)$ **d** $(5x-2)(6x+5)$

e $(8x+3)(5x-2)$ **f** $(7x+2)(4x-3)$

g $(6x-5)(4x-3)$ **h** $(9x-2)(5x-4)$

i $(5x-2)(5x-8)$

4 **a** $2(3x+4)(x+5)$ **b** $3(2x+3)(x-4)$

c $3(8x+1)(2x-1)$ **d** $4(4x-5)(2x-3)$

e $8(2x-1)(x-1)$ **f** $10(3x-2)(3x+5)$

g $-5(5x+4)(2x+3)$ **h** $3(2x-3)^2$

i $5(4x-1)(x-1)$

5 **a** $(3x-4)(x-5)$

b −10 m; the cable is 10 m below the water.

c $x = \frac{4}{3}$ or $x = 5$

6 **a** $2x-5$ **b** $4x-1$ **c** $3x-2$

d $\frac{2}{3x+2}$ **e** $\frac{2}{7x-2}$ **f** $\frac{4}{2x-3}$

g $\frac{x+4}{3x+1}$ **h** $\frac{3x-1}{2x+3}$

7 **a** $\frac{3x+4}{x-3}$ **b** $\frac{3x+2}{4}$ **c** $\frac{1-x}{3}$

d $\frac{4x-3}{5x+1}$ **e** 125 **f** $\frac{x+2}{5}$

g 1 **h** $\frac{(4x-5)^2}{(x-3)^2}$

8 $-12x^2-5x+3$

$= -(12x^2+5x-3)$

$= -(3x-1)(4x+3)$

$= (1-3x)(4x+3)$

a $(3-2x)(4x+5)$ **b** $(5-2x)(3x+2)$

c $(4-3x)(4x+1)$ **d** $(3-4x)(2x-3)$

e $(2-7x)(2x-5)$ **f** $(3-5x)(3x+2)$

9 Answers will vary.

10 **a** $\frac{9x+2}{(2x-3)(4x+1)}$ **b** $\frac{5x+15}{(3x-1)(2x+5)}$

c $\frac{16x^2+5x}{(2x-5)(4x+1)}$ **d** $\frac{7x-12x^2}{(3x-2)(4x-1)}$

e $\frac{8x-5}{(2x+1)(2x-1)(3x-2)}$ **f** $\frac{11-3x}{(3x+5)(3x-5)(3x-2)}$

g $\frac{2}{(2x-5)(3x-2)}$ **h** $\frac{12x+3}{(5x-2)(2x-3)(2x+7)}$

Progress quiz

1 a $-24x + 10$ b $4a^2 - 7a$
c $m^2 + 7m + 10$ d $k^2 - 6k + 9$
e $9m^2 - 4$ f $8h^2 - 6h - 35$
g $5x^2 - 35x + 60$ h $19p + 4$

2 a $4(a - 5)$
b $-6m(2m - 3)$
c $(x + 5)(4 - x)$
d $(a - 9)(a + 9)$
e $(4a - 11b)(4a + 11b)$
f $5(m - 5)(m + 5)$
g $(k - 5)(k + 9)$
h $(x - 3)(x + 1)$
i $(x - \sqrt{15})(x + \sqrt{15})$
j $(h + 3 - \sqrt{7})(h + 3 + \sqrt{7})$
k $(x + 5)(x + a)$
l $(x - 2m)(4x - 5)$

3 a 4k b $a - 4$
c $\frac{9a}{2}$ d $\frac{5}{2}$

4 a $(x - 4)(x + 5)$ b $(a - 3)(a - 7)$
c $3(k - 9)(k + 2)$ d $(m - 6)^2$

5 a $x - 3$ b $\frac{x + 5}{2}$

6 a $(3a + 2)(2a + 5)$ b $(2m - 3)(4m + 3)$
c $(3x - 2)(5x - 4)$ d $(2k - 7)(3k + 5)$

7 $\frac{2x + 5}{2x - 3}$

5F

Building understanding

1 a 9 b 1 c 4
d 16 e $\frac{25}{4}$ f $\frac{81}{4}$

2 a $(x + 2)^2$ b $(x + 4)^2$ c $(x + 5)^2$
d $(x - 6)^2$ e $(x - 3)^2$ f $(x - 9)^2$

3 a $(x + 1 + \sqrt{5})(x + 1 - \sqrt{5})$
b $(x + 4 + \sqrt{10})(x + 4 - \sqrt{10})$
c $(x - 3 + \sqrt{11})(x - 3 - \sqrt{11})$

Now you try

Example 14

a $36, (x + 6)^2$ b $\frac{81}{4}, \left(x - \frac{9}{2}\right)^2$

Example 15

a $(x + 3 + \sqrt{10})(x + 3 - \sqrt{10})$
b $(x - 2)^2 + 3$ cannot be factorised.

Example 16

a $\left(x + \frac{5 + \sqrt{23}}{2}\right)\left(x + \frac{5 - \sqrt{23}}{2}\right)$
b $2\left(x - 1 + \sqrt{\frac{5}{2}}\right)\left(x - 1 - \sqrt{\frac{5}{2}}\right)$

Exercise 5F

1 a $9, (x + 3)^2$ b $36, (x + 6)^2$
c $4, (x + 2)^2$ d $16, (x + 4)^2$
e $25, (x - 5)^2$ f $1, (x - 1)^2$
g $16, (x - 4)^2$ h $36, (x - 6)^2$
i $\frac{25}{4}, \left(x + \frac{5}{2}\right)^2$ j $\frac{81}{4}, \left(x + \frac{9}{2}\right)^2$
k $\frac{49}{4}, \left(x + \frac{7}{2}\right)^2$ l $\frac{121}{4}, \left(x + \frac{11}{2}\right)^2$
m $\frac{9}{4}, \left(x - \frac{3}{2}\right)^2$ n $\frac{49}{4}, \left(x - \frac{7}{2}\right)^2$
o $\frac{1}{4}, \left(x - \frac{1}{2}\right)^2$ p $\frac{81}{4}, \left(x - \frac{9}{2}\right)^2$

2 a $(x + 2 + \sqrt{3})(x + 2 - \sqrt{3})$
b $(x + 3 + \sqrt{7})(x + 3 - \sqrt{7})$
c $(x + 1 + \sqrt{5})(x + 1 - \sqrt{5})$
d $(x + 5 + \sqrt{29})(x + 5 - \sqrt{29})$
e $(x - 4 + \sqrt{3})(x - 4 - \sqrt{3})$
f $(x - 6 + \sqrt{26})(x - 6 - \sqrt{26})$
g $(x - 2 + \sqrt{7})(x - 2 - \sqrt{7})$
h $(x - 4 + \sqrt{21})(x - 4 - \sqrt{21})$
i $(x + 7 + \sqrt{43})(x + 7 - \sqrt{43})$

3 a Not possible
b Not possible
c $(x + 4 + \sqrt{15})(x + 4 - \sqrt{15})$
d $(x + 2 + \sqrt{2})(x + 2 - \sqrt{2})$
e $(x + 5 + \sqrt{22})(x + 5 - \sqrt{22})$
f $(x + 2 + \sqrt{10})(x + 2 - \sqrt{10})$
g Not possible
h $(x - 3 + \sqrt{3})(x - 3 - \sqrt{3})$
i $(x - 6 + \sqrt{34})(x - 6 - \sqrt{34})$
j Not possible
k $(x - 4 + \sqrt{17})(x - 4 - \sqrt{17})$
l Not possible

4 a $\left(x + \frac{3 + \sqrt{5}}{2}\right)\left(x + \frac{3 - \sqrt{5}}{2}\right)$
b $\left(x + \frac{7 + \sqrt{41}}{2}\right)\left(x + \frac{7 - \sqrt{41}}{2}\right)$
c $\left(x + \frac{5 + \sqrt{33}}{2}\right)\left(x + \frac{5 - \sqrt{33}}{2}\right)$
d $\left(x + \frac{9 + \sqrt{93}}{2}\right)\left(x + \frac{9 - \sqrt{93}}{2}\right)$
e $\left(x - \frac{3 + \sqrt{7}}{2}\right)\left(x - \frac{3 - \sqrt{7}}{2}\right)$
f $\left(x - \frac{5 + \sqrt{23}}{2}\right)\left(x - \frac{5 - \sqrt{23}}{2}\right)$
g $\left(x - \frac{5 + \sqrt{31}}{2}\right)\left(x - \frac{5 - \sqrt{31}}{2}\right)$
h $\left(x - \frac{9 + \sqrt{91}}{2}\right)\left(x - \frac{9 - \sqrt{91}}{2}\right)$

5 a $2(x + 3 + \sqrt{5})(x + 3 - \sqrt{5})$
b $3(x + 2 + \sqrt{5})(x + 2 - \sqrt{5})$
c $4(x - 1 + \sqrt{5})(x - 1 - \sqrt{5})$
d $3(x - 4 + \sqrt{14})(x - 4 - \sqrt{14})$
e $-2(x + 1 + \sqrt{6})(x + 1 - \sqrt{6})$
f $-3(x + 5 + 2\sqrt{6})(x + 5 - 2\sqrt{6})$
g $-4(x + 2 + \sqrt{7})(x + 2 - \sqrt{7})$
h $-2(x - 4 + 3\sqrt{2})(x - 4 - 3\sqrt{2})$
i $-3(x - 4 + \sqrt{11})(x - 4 - \sqrt{11})$

6 a $3\left(x + \frac{3 + \sqrt{5}}{2}\right)\left(x + \frac{3 - \sqrt{5}}{2}\right)$
b $5\left(x + \frac{3 + \sqrt{37}}{2}\right)\left(x + \frac{3 - \sqrt{37}}{2}\right)$

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c $2\left(x-\frac{5+\sqrt{17}}{2}\right)\left(x-\frac{5-\sqrt{17}}{2}\right)$
d $4\left(x-\frac{7+\sqrt{37}}{2}\right)\left(x-\frac{7-\sqrt{37}}{2}\right)$
e $-3\left(x+\frac{7+\sqrt{57}}{2}\right)\left(x+\frac{7-\sqrt{57}}{2}\right)$
f $-2\left(x+\frac{7+\sqrt{65}}{2}\right)\left(x+\frac{7-\sqrt{65}}{2}\right)$
g $-4\left(x-\frac{3+\sqrt{29}}{2}\right)\left(x-\frac{3-\sqrt{29}}{2}\right)$
h $-3\left(x-\frac{3+\sqrt{17}}{2}\right)\left(x-\frac{3-\sqrt{17}}{2}\right)$
i $-2\left(x-\frac{5+\sqrt{41}}{2}\right)\left(x-\frac{5-\sqrt{41}}{2}\right)$

7 a $2\left(x+1+\sqrt{\frac{5}{2}}\right)\left(x+1-\sqrt{\frac{5}{2}}\right)$
b $2\left(x+2+\sqrt{\frac{7}{2}}\right)\left(x+2-\sqrt{\frac{7}{2}}\right)$
c $3\left(x-1-\sqrt{\frac{2}{3}}\right)\left(x-1+\sqrt{\frac{2}{3}}\right)$
d $3\left(x-2-\sqrt{\frac{14}{3}}\right)\left(x-2+\sqrt{\frac{14}{3}}\right)$
e $2\left(x+\frac{3-\sqrt{3}}{2}\right)\left(x+\frac{3+\sqrt{3}}{2}\right)$
f Not possible
g $-3\left(x+\frac{7+\sqrt{13}}{6}\right)\left(x+\frac{7-\sqrt{13}}{6}\right)$
h $-2\left(x-\frac{3+\sqrt{41}}{4}\right)\left(x-\frac{3-\sqrt{41}}{4}\right)$
i $-3\left(x+\frac{4}{3}\right)(x+1)$

8 a $x^2-2x-24$
$=x^2-2x+(-1)^2-(-1)^2-24$
$=(x-1)^2-25$
$=(x-1+5)(x-1-5)$
$=(x+4)(x-6)$

b Using a quadratic trinomial and finding two numbers that multiply to −24 and add to −2.

9 a If the difference of perfect squares is taken, it involves the square root of a negative number.
b i Yes ii Yes iii No iv No
v No vi Yes vii Yes viii No
c i $m \leqslant 4$ ii $m \leqslant 9$ iii $m \leqslant 25$

10 a i The square of any number will be greater than or equal to zero
ii $-2x$ when squared will always be greater than or equal to zero
iii $x-1$ when squared will always be greater than or equal to zero
iv Since $(x-3)^2 \geqslant 0$ then $-2(x-3)^2 \geqslant 0$
v Since $(x-1)^2 \geqslant 0$ then $(x-1)^2+1 \geqslant 1$
vi Since $(2-x)^2 \geqslant 0$ and $-(2-x)^2 \leqslant 0$ then $-(2-x)^2-3 \leqslant -3$
b i $(x+1)^2 \geqslant 0$
ii $(2x-1)^2 \geqslant 0$
iii $(x-3)^2-9+6=(x-3)^2-3 \geqslant -3$ because $(x-3)^2 \geqslant 0$
iv $\left(x+\frac{5}{2}\right)^2-\frac{25}{4}+1=\left(x+\frac{5}{2}\right)^2-\frac{21}{4} \geqslant -\frac{21}{4}$
because $\left(x+\frac{5}{2}\right)^2 \geqslant 0$
v $-x^2-4x+3=-(x^2+4x-3)=-((x+2)^2-7)$
$=-(x+2)^2+7 \leqslant 7$ because $-(x+2)^2 \leqslant 0$
vi $2x^2-8x+3=2\left(x^2-4x+\frac{3}{2}\right)=2\left((x-2)^2-\frac{5}{2}\right)$
$=2\ (x-2)^2-5 \geqslant -5$ because $2(x-2)^2 \geqslant 0$

5G

Building understanding

1 a 0, −1 b 0, 4 c 3, −2
d $\sqrt{3}, -\sqrt{3}$ e $\frac{1}{2}, -\frac{7}{3}$ f $-\frac{3}{8}, -\frac{3}{4}$
2 a $x^2+2x-3=0$ b $x^2-5x+6=0$
c $4x^2+4x-3=0$ d $2x^2-6x-5=0$
e $x^2-4x+12=0$ f $3x^2+2x+4=0$
3 a 2 b 1 c 2
d 2 e 1 f 1

Now you try

Example 17
a $x=0, x=3$ b $x=\pm\sqrt{11}$ c $x=\pm 3$

Example 18
a $x=-3$ or $x=4$
b $x=-3$
c $x=-\frac{2}{3}$ or $x=\frac{1}{2}$

Example 19
a $x=-6$ or $x=8$ b $x=-4$ or $x=5$

Exercise 5G

1 a $x=0, 4$ b $x=0, 3$
c $x=0, -2$ d $x=0, 4$
e $x=0, 5$ f $x=0, -2$
g $x=\sqrt{7}, -\sqrt{7}$ h $x=\sqrt{11}, -\sqrt{11}$
i $x=\sqrt{5}, -\sqrt{5}$
2 a $x=0, 2$ b $x=0, -5$
c $x=0, -\frac{1}{7}$ d $x=2, -2$
e $x=3, -3$ f $x=6, -6$
3 a $x=-2, -1$ b $x=-3, -2$
c $x=2, 4$ d $x=5, 2$
e $x=-6, 2$ f $x=-5, 3$
g $x=5, -4$ h $x=8, -3$
i $x=4, 8$ j $x=-2$
k $x=-5$ l $x=4$
m $x=7$ n $x=12$
o $x=-9$
4 a $x=-\frac{3}{2}, -4$ b $x=-\frac{1}{2}, -\frac{7}{2}$ c $x=5, \frac{7}{2}$
d $x=\frac{1}{2}, 11$ e $x=-\frac{5}{3}, 3$ f $x=-\frac{3}{5}, 2$
g $x=\frac{4}{3}, -\frac{5}{2}$ h $x=\frac{3}{7}, -4$ i $x=\frac{5}{4}, \frac{2}{5}$
5 a $x=-2, -6$ b $x=-1, 11$ c $x=3$
d $x=2$ e $x=\frac{3}{2}, -2$ f $x=\frac{2}{3}, \frac{5}{2}$
6 a $x=6, -4$ b $x=8, -4$ c $x=3$
d $x=-2, -5$ e $x=5, 3$ f $x=3, -3$

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g $x = 4, -4$ h $x = -1, -9$ i $x = 5, -1$
j $x = -5$ k $x = 8$ l $x = 8, -8$
m $x = 3, -1$ n $x = -\frac{2}{3}, -4$ o $x = -\frac{1}{4}, -\frac{3}{2}$

7 a $x = 12, -7$ b $x = -5, 14$ c $x = -9, 2$
d $x = \frac{5}{2}, -4$ e $x = -\frac{4}{5}, 2$ f $x = 2, -\frac{5}{6}$
g $x = -3, 1$ h $x = 1, \frac{1}{2}$ i $x = 3, -2$

8 a $x = -4, 7$ b $x = -10, -2$
c $x = -2$ d $x = 3$
e $x = -4, \frac{3}{2}$ f $x = -5, \frac{1}{3}$

9 a i $x = 1, -2$ ii $x = 1, -2$
b No difference
c $3x^2 - 15x - 18 = 3(x^2 - 5x - 6)$ and, as seen in part **a**, the coefficient of 3 makes no difference when solving.

10 This is a perfect square $(x + 8)^2$, which only has 1 solution; i.e. $x = -8$.

11 The student has applied the null factor law incorrectly; i.e. when the product does not equal zero. Correct solution is:
$x^2 - 2x - 8 = 7$
$x^2 - 2x - 15 = 0$
$(x - 5)(x + 3) = 0$
$x = 5$ or $x = -3$

12 a $x = -2, -1$ b $x = 1$ c $x = \frac{1}{2}, 5$
d $x = 8, -6$ e $x = -6, -2$ f $x = \frac{3}{2}, -4$
g $x = 8, -3$ h $x = 5, -3$ i $x = 2$
j $x = 4, -3$ k $x = 5, -2$ l $x = -5, 3$

5H

Building understanding

1 a Let x m be the width of the rectangle.
b $x + 5$
c $x(x + 5) = 24$
d $x^2 + 5x - 24 = 0, x = -8, 3$
e Width = 3 m, length = 8 m

2 a Width = 6 m, length = 10 m
b Width = 9 m, length = 7 m

Now you try

Example 20
Width = 6 m, length = 8 m

Exercise 5H

1 Width = 3 m, length = 4 m
2 Width = 6 m, length = 9 m
3 Height = 8 cm, base = 6 cm
4 Height = 2 m, base = 7 m
5 8 and 9 or −9 and −8
6 12 and 14
7 15 m
8 a 6 b 13 c 14
9 1 m
10 Father 64, son 8
11 5 cm
12 a 55
b i 7 ii 13 iii 23
13 a 3.75 m
b $t = 1$ second, 3 seconds
c The ball will reach this height both on the way up and on the way down.
d $t = 0$ seconds, 4 seconds
e $t = 2$ seconds
f The ball reaches a maximum height of 4 m .
g No, 4 metres is the maximum height. When $h = 5$, there is no solution.
14 a $x = 0, 100$
b The ball starts at the tee (i.e. at ground level) and hits the ground again 100 metres from the tee.
c $x = 2$ m or 98 m
15 5 m × 45 m
16 150 m × 200 m

5I

Building understanding

1 a 1 b 100 c 4 d $\frac{25}{4}$

2 a $(x + \sqrt{3})(x - \sqrt{3}) = 0$
b $(x + \sqrt{10})(x - \sqrt{10}) = 0$
c $(x + 1 + \sqrt{5})(x + 1 - \sqrt{5}) = 0$

3 a $x = \sqrt{2}, -\sqrt{2}$
b $x = \sqrt{7}, -\sqrt{7}$
c $x = 3 - \sqrt{5}, 3 + \sqrt{5}$
d $x = -5 - \sqrt{14}, -5 + \sqrt{14}$

Now you try

Example 21
a $x = 3 \pm \sqrt{7}$ b $x = -2 \pm 3\sqrt{2}$ c $x = \frac{5 \pm \sqrt{17}}{2}$

Exercise 5I

1 a $x = -3 - \sqrt{6}, -3 + \sqrt{6}$
b $x = -2 - \sqrt{2}, -2 + \sqrt{2}$
c $x = -5 - \sqrt{10}, -5 + \sqrt{10}$
d $x = -2 - \sqrt{6}, -2 + \sqrt{6}$
e $x = -4 - \sqrt{19}, -4 + \sqrt{19}$
f $x = -3 - \sqrt{14}, -3 + \sqrt{14}$
g $x = 4 - \sqrt{17}, 4 + \sqrt{17}$
h $x = 6 - \sqrt{39}, 6 + \sqrt{39}$
i $x = 1 - \sqrt{17}, 1 + \sqrt{17}$
j $x = 5 - \sqrt{7}, 5 + \sqrt{7}$
k $x = 3 - \sqrt{5}, 3 + \sqrt{5}$
l $x = 4 - \sqrt{7}, 4 + \sqrt{7}$
m $x = -3 - \sqrt{13}, -3 + \sqrt{13}$
n $x = -10 - \sqrt{87}, -10 + \sqrt{87}$
o $x = 7 - \sqrt{55}, 7 + \sqrt{55}$

2 a $x = -4 - 2\sqrt{3}, -4 + 2\sqrt{3}$
b $x = -3 - 2\sqrt{2}, -3 + 2\sqrt{2}$
c $x = 5 - 2\sqrt{5}, 5 + 2\sqrt{5}$
d $x = 2 - 3\sqrt{2}, 2 + 3\sqrt{2}$
e $x = 5 - 2\sqrt{7}, 5 + 2\sqrt{7}$

f $x = -4 - 2\sqrt{6}, -4 + 2\sqrt{6}$
g $x = 1 - 4\sqrt{2}, 1 + 4\sqrt{2}$
h $x = -6 - 3\sqrt{6}, -6 + 3\sqrt{6}$
i $x = -3 - 5\sqrt{2}, -3 + 5\sqrt{2}$

3 a $x = \frac{-5 + \sqrt{17}}{2}, \frac{-5 - \sqrt{17}}{2}$
b $x = \frac{-3 + \sqrt{5}}{2}, \frac{-3 - \sqrt{5}}{2}$
c $x = \frac{-7 + \sqrt{29}}{2}, \frac{-7 - \sqrt{29}}{2}$
d $x = \frac{3 + \sqrt{17}}{2}, \frac{3 - \sqrt{17}}{2}$
e $x = \frac{1 + \sqrt{13}}{2}, \frac{1 - \sqrt{13}}{2}$
f $x = \frac{-5 + \sqrt{33}}{2}, \frac{-5 - \sqrt{33}}{2}$
g $x = \frac{7 + \sqrt{41}}{2}, \frac{7 - \sqrt{41}}{2}$
h $x = \frac{9 + \sqrt{61}}{2}, \frac{9 - \sqrt{61}}{2}$
i $x = \frac{-1 + \sqrt{17}}{2}, \frac{-1 - \sqrt{17}}{2}$
j $x = \frac{-9 + 3\sqrt{5}}{2}, \frac{-9 - 3\sqrt{5}}{2}$
k $x = \frac{3}{2} + \sqrt{3}, \frac{3}{2} - \sqrt{3}$
l $x = -\frac{5}{2} + \sqrt{5}, -\frac{5}{2} - \sqrt{5}$

4 a 2 **b** 2 **c** 0 **d** 0
e 0 **f** 2 **g** 2 **h** 0
i 0 **j** 2 **k** 2 **l** 0

5 Width $= \frac{-3 + \sqrt{89}}{2}$ cm, length $= \frac{3 + \sqrt{89}}{2}$ cm

6 a No real solution. **b** $x = \frac{-5 \pm \sqrt{17}}{2}$
c $x = \frac{5 \pm \sqrt{17}}{2}$ **d** $x = \frac{-9 \pm \sqrt{69}}{2}$
e $x = \frac{-5 \pm \sqrt{21}}{2}$ **f** $x = 3 \pm \sqrt{5}$

7 a $x = \frac{-3 \pm \sqrt{29}}{2}$ **b** $x = \frac{-5 \pm \sqrt{61}}{2}$
c No real solutions. **d** $x = 4 \pm \sqrt{5}$
e $x = -5 \pm 2\sqrt{5}$ **f** No real solutions.

8 a i 1.5 km **ii** 1.5 km
b i 0 km or 400 km **ii** 200 km
c $200 \pm 100\sqrt{2}$ km

9 a $x^2 + 4x + 5 = 0$
$(x + 2)^2 + 1 = 0$, no real solutions
b $(x - 1)^2 + 2 = 0$, no real solutions

10 Factorise by quadratic trinomial; i.e. $(x + 6)(x - 5) = 0, 6 \times (-5) = -30$, and $6 + (-5) = -1$.
Therefore, $x = -6, 5$.

11 a Use the dimensions of rectangle *BCDE* and *ACDF* and the corresponding side lengths in similar rectangles.
b $a = \frac{1 + \sqrt{5}}{2}$

12 a i $30 - 2x$
ii $40 - 2x$
iii $(30 - 2x)(40 - 2x) = 4x^2 - 140x + 1200$
b 5
c $\frac{35 - 5\sqrt{29}}{2}$

5J

Building understanding

1 a $a = 3, b = 2, c = 1$ **b** $a = 5, b = 3, c = -2$
c $a = 2, b = -1, c = -5$ **d** $a = -3, b = 4, c = -5$

2 a -8 **b** 49 **c** 41 **d** -44

3 a 1 **b** 0 **c** 2

Now you try

Example 22
a 2 solutions **b** No solutions **c** 1 solution

Example 23
a $x = \frac{-3 \pm \sqrt{5}}{2}$ **b** $x = \frac{1 \pm \sqrt{13}}{4}$

Exercise 5J

1 a 2 **b** 0 **c** 1 **d** 2
e 2 **f** 2 **g** 0 **h** 0
i 2 **j** 1 **k** 0 **l** 2

2 a $x = \frac{-3 \pm \sqrt{17}}{2}$ **b** $x = \frac{-7 \pm \sqrt{65}}{2}$
c $x = \frac{7 \pm \sqrt{29}}{2}$ **d** $x = 4$
e $x = -1, -4$ **f** $x = -1, -7$
g $x = \frac{-7 \pm \sqrt{65}}{8}$ **h** $x = \frac{-5 \pm \sqrt{37}}{6}$
i $x = \frac{2 \pm \sqrt{22}}{3}$ **j** $x = \frac{5 \pm \sqrt{65}}{4}$
k $x = -\frac{4}{3}, 1$ **l** $x = \frac{-3 \pm \sqrt{19}}{5}$

3 a $x = -2 \pm \sqrt{3}$ **b** $x = 3 \pm \sqrt{5}$
c $x = -3 \pm \sqrt{11}$ **d** $x = \frac{-3 \pm 3\sqrt{5}}{2}$
e $x = 2 \pm 2\sqrt{2}$ **f** $x = \frac{4 \pm \sqrt{10}}{3}$
g $x = \frac{1 \pm \sqrt{7}}{2}$ **h** $x = \frac{3 \pm 2\sqrt{3}}{3}$
i $x = \frac{4 \pm \sqrt{31}}{5}$

4 $\frac{-5 + \sqrt{105}}{2}$

5 a $x = \frac{3 \pm 2\sqrt{3}}{3}$ **b** $x = \frac{-2 \pm \sqrt{10}}{2}$
c $x = \frac{-5 \pm \sqrt{57}}{8}$ **d** $x = \frac{5 \pm \sqrt{17}}{4}$
e $x = \frac{-2 \pm \sqrt{13}}{3}$ **f** $x = 1 \pm \sqrt{6}$
g $x = \frac{1 \pm \sqrt{11}}{5}$ **h** $x = \frac{3 \pm \sqrt{41}}{4}$
i $x = \frac{5 \pm \sqrt{19}}{6}$

6 $6\sqrt{2} + 10$ units

7 $\frac{3 + \sqrt{53}}{2}, \frac{-3 + \sqrt{53}}{2}$

8 63 cm

9 When $b^2 - 4ac = 0$, the solution reduces to $x = \frac{-b}{2a}$; i.e. a single solution.

10 Answers will vary.

11 $k = 6$ or -6

12 a i $k > 4$ ii $k = 4$ iii $k < 4$

b i $k > \frac{9}{8}$ ii $k = \frac{9}{8}$ iii $k < \frac{9}{8}$

c i $-2 < k < 2$

ii ± 2

iii $k > 2, k < -2$

d i No values

ii No values

iii All values of k

Problems and challenges

1 $b = -4, c = 1$

2 47

3 a $\pm 2, \pm 1$ b ± 3

4 a $x = 0, 1$ b $x = 1, -2$

5 144 cm^2

6 25 km/h

7 1.6 units

8 $x^2 - 2x + 2 = (x-1)^2 + 1$, as $(x-1)^2 \geqslant 0, (x-1)^2 + 1 > 0$

9 Square area − rectangle area $= \frac{(x-y)^2}{4} > 0$ for all x and y; hence, square area is greater than rectangle area.

10 $w : p = 1 : 3$; $t : q = 1 : 9$

Answers to success criteria example questions

1 $6x^2 - 19x + 15$

2 $2x^2 + 5x - 12$

3 $9x^2 - 4$

4 $x^2 + 10x + 25$

5 $6x(2x - 3)$

6 $(3x + 4)(3x - 4)$

7 $(x + \sqrt{7})(x - \sqrt{7})$

8 $(x + 2)(x - a)$

9 $2x - 1$

10 $\frac{3}{8}$

11 $(x - 10)(x + 2)$

12 $3(x - 3)(x - 5)$

13 3

14 $(5x - 2)(x + 3)$

15 $(x + 3 + \sqrt{7})(x + 3 - \sqrt{7})$

16 $2\left(x + \frac{3 - \sqrt{3}}{2}\right)\left(x + \frac{3 + \sqrt{3}}{2}\right)$

17 Cannot be factorised

18 $x = 0$, or $x = 3$

19 $x = 3$, or $x = -1$

20 Width = 6 m, length = 10 m

21 $x = -2 - \sqrt{26}$ or $x = -2 + \sqrt{26}$.

22 2 solutions

23 $x = \frac{-3 - \sqrt{41}}{4}$ or $x = \frac{-3 + \sqrt{41}}{4}$

Short-answer questions

1 a $-2x + 26$ b $3x^2 + 11x - 20$

c $25x^2 - 4$ d $x^2 - 12x + 36$

e $7x + 22$ f $12x^2 - 23x + 10$

2 a $x^2 + 4x + 4$ b $4x^2 + 18x$

c $x^2 + 3x + 21$

3 a $3x(x + 6)$ b $(x + 1)(4 - b)$

c $(x - a)(x + 2)$

4 a $(x + 7)(x - 7)$

b $(3x + 4)(3x - 4)$

c $3(x + 5)(x - 5)$

d $(x + \sqrt{11})(x - \sqrt{11})$

e $(x + 5)(x - 3)$

f $(x - 3 + \sqrt{10})(x - 3 - \sqrt{10})$

5 a $(x - 6)(x - 2)$ b $(x + 12)(x - 2)$

c $-3(x - 6)(x - 1)$

6 a $(3x + 2)(x + 5)$ b $(2x - 3)(2x + 5)$

c $(6x + 1)(2x - 3)$ d $(3x - 2)(4x - 5)$

7 a $\frac{4}{x}$ b $\frac{2}{x}$

c $\frac{2x}{x + 3}$ d $\frac{x - 4}{4}$

8 a $(x + 4 + \sqrt{6})(x + 4 - \sqrt{6})$

b $(x - 3 + \sqrt{13})(x - 3 - \sqrt{13})$

c $\left(x + \frac{3 + \sqrt{17}}{2}\right)\left(x + \frac{3 - \sqrt{17}}{2}\right)$

d $2\left(x - 1 - \sqrt{\frac{5}{2}}\right)\left(x - 1 + \sqrt{\frac{5}{2}}\right)$

9 a $x = 0, -4$ b $x = 0, 3$

c $x = 5, -5$ d $x = 3, 7$

e $x = 4$ f $x = -9, 4$

g $x = -2, \frac{1}{2}$ h $x = \frac{2}{3}, -\frac{5}{2}$

i $x = \frac{1}{9}, -\frac{3}{2}$

10 a $x = 3, -3$ b $x = 5, -1$

c $x = 4, -7$ d $x = -3, 6$

11 Length = 8 m, width = 6 m

12 a $x = -2 \pm \sqrt{7}$ b $x = 3 \pm \sqrt{7}$

c $x = \frac{3 \pm \sqrt{17}}{2}$ d $x = \frac{-5 \pm 3\sqrt{5}}{2}$

13 a 1 solution b 2 solutions

c 0 solutions d 2 solutions

14 a $x = \frac{-3 \pm \sqrt{33}}{2}$ b $x = 1 \pm \sqrt{5}$

c $x = \frac{2 \pm \sqrt{14}}{2}$ d $x = \frac{1 \pm \sqrt{37}}{6}$

Multiple-choice questions

1 D 2 B 3 C 4 A

5 B 6 D 7 C 8 C

9 E 10 C 11 A 12 B

Extended-response questions

1 a i $(15 + 2x)$ m

ii $(12 + 2x)$ m

b Overall area $= 4x^2 + 54x + 180 \text{ m}^2$

c Trench area $= 4x^2 + 54x \text{ m}^2$

d Minimum width is 1 m.

2 a $S = 63\pi \text{ m}^2$

b 0.46 m

c i $420 = 3\pi r^2 + 12\pi r$

ii $3\pi r^2 + 12\pi r - 420 = 0$

iii $r = 4.97$ m

Semester review 1

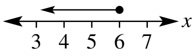
Algebra, equations and linear relationships

Short-answer questions

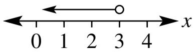
1 **a** **i** $x = -4$ **ii** $x = 2$

iii $x = 13$ **iv** $x = 2$

b **i** $x \leqslant 6$,

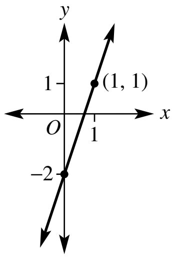


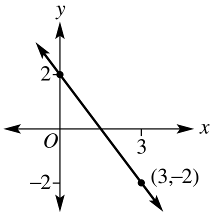
ii $x < 3$,



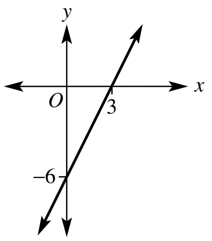
2 $\frac{x+3}{9}$

3 **a** **i** $m = 3, c = -2$ **ii** $m = -\frac{4}{3}, c = 2$





b **i**



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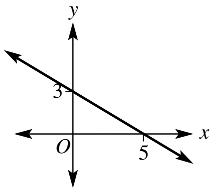

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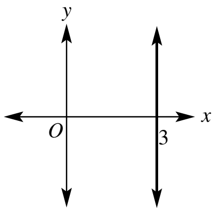
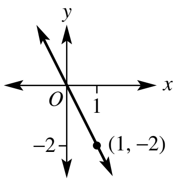


iv



4 **a** $y = -x + 3$ **b** $y = \frac{8}{5}x - \frac{9}{5}$

5 **a** $a = -3$ **b** $a = -4$

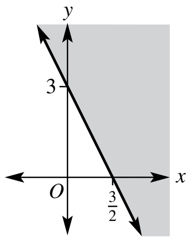
c $a = 1$ or $a = 7$ **d** $a = -4$

6 **a** $x = -3, y = -7$ **b** $x = -2, y = -4$

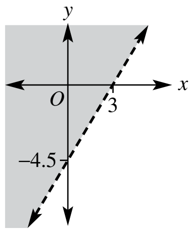
c $x = -1, y = 4$ **d** $x = 3, y = -5$

7 A hot dog costs \$3.50 a and soft drink \$2.

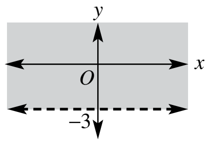
8 **a**



b



c



Multiple-choice questions

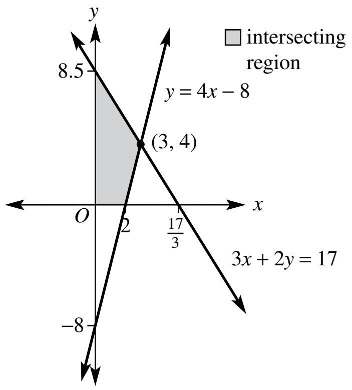
1 A **2** C **3** D

4 E **5** C

Extended-response question

a $x = 3, y = 4$

b, c



d 167 500 m^2

Geometry and networks

Short-answer questions

1 **a** $AB = DE$ (given) $AC = DF$ (given)

$\angle BAC = 60° = \angle EDF$ (given)

$\therefore \Delta ABC \equiv \Delta DEF$ (SAS)

$a = 35$ (corresponding angles in congruent triangles)

b $BC = DC$ (given)

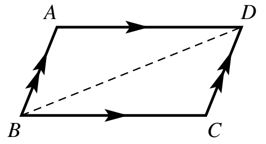
AC is common.

$\angle ABC = 90° = \angle ADC$ (given)

$\therefore \Delta ABC \equiv \Delta ADC$ (RHS)

$x = 3$ (corresponding sides in congruent triangles)

2



$\angle DBC = \angle BDA$ (alternate angles in parallel lines)
$\angle BDC = \angle DBA$ (alternate angles in parallel lines)
BD is common.
$\therefore \Delta BAD = \Delta DCB$ (AAS)
Using congruence, $BC = DA$ and $AB = CD$, corresponding sides in congruent triangles.

3 a $x = 6.75$ b $x = 2$

4 a $x = 8$ b $x = 5$
c $a = 32, b = 65$ d $x = 40$
e $a = 55$ f $a = 90, b = 60, c = 70$

5 a $x = 20$ b $x = 8$
c $a = 63, b = 55$

6 a $x = \frac{47}{5}$ b $x = \frac{29}{3}$ c $x = \frac{39}{5}$

7 a Zero
b $e = 8, v = 5, f = 5$ so $e + 2 = v + f$
c No, B is repeated
d Yes, zero odd vertices. All trails are circuits since there are no odd vertices.

8 a Yes b Yes

Multiple-choice questions

1 D 2 B 3 C 4 E 5 C

Extended-response question

a $CD = 6$ cm, chord theorem 2
b $OA = OD$ (radii of circle)
$OB = OC$ (radii of circle)
$AB = DC$ (chord theorem 2)
$\therefore \Delta OAB \equiv \Delta OCD$ (SSS).
c $OM = 4$ cm, area $= 12\text{cm}^2$
d 30.6%
e $\angle BOD = 106.2°$

Indices, exponentials and logarithms

Short-answer questions

1 a $6x^6y^2$ b $81x^5$ c $3a^2b^2$
d $\frac{3b^2}{a^5}$ e $4m^3$ f $\frac{2x^2}{5y^3}$

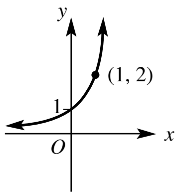
2 a i 37 200 ii 0.0000049
b i 7.30×10^{-5} ii 4.73×10^9

3 a i $10^{\frac{1}{2}}$ ii $7^{\frac{1}{2}}x^3$, when $x > 0$
iii $4x^{\frac{3}{5}}$ iv $15^{\frac{3}{2}}$
b i $\sqrt{6}$ ii $\sqrt[5]{20}$
iii $\sqrt[4]{7^3}$ or $\left(\sqrt[4]{7}\right)^3$ or $\sqrt[4]{343}$

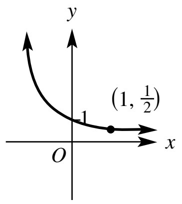
4 a $\frac{1}{5}$ b $\frac{1}{16}$ c 3 d $\frac{1}{2}$

5 a $x = 3$ b $x = 2$
c $x = \frac{3}{2}$ d $x = -\frac{1}{2}$

6 a



b



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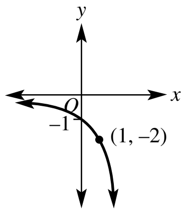
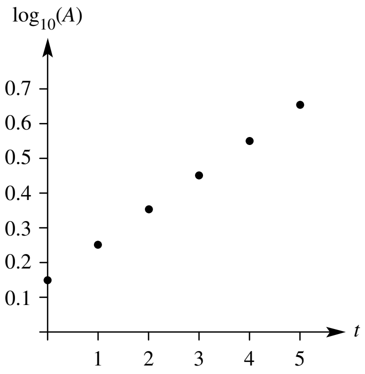


7 a \$2382.03 b \$7658.36

8 a 3 b −2 c 3 d 0
e 2 f 2 g 1 h 3

9 a



b Graph is linear which says that relationship between A and t is exponential.

10 a $x = 3$ b $x = 3$ c $x = 81$

11 a i $x = \log_3(30)$ ii $x = \log_{2.4}(4)$
b i $x = 2.460$ ii $x = 9.955$

Multiple-choice questions

1 D 2 E 3 C 4 D 5 A

Extended-response question

a $V = 80\,000\,(1.08)^n$
b i \$86 400 ii \$108 839
c 11.91 years
d 6% per year

Measurement and surds

Short-answer questions

1 a $3\sqrt{6}$ b $20\sqrt{3}$ c $3\sqrt{6}$
d $\sqrt{10}$ e 21 f $48\sqrt{3}$
g $\sqrt{3}$ h $\frac{\sqrt{5}}{3}$ i $\frac{10\sqrt{2}}{7}$

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2 a $7\sqrt{5}-\sqrt{7}$ b 0 c $-\sqrt{2}-4$
3 a $2\sqrt{15}-4\sqrt{3}$ b 45
4 a $\frac{3\sqrt{2}}{2}$ b $\frac{\sqrt{2}}{5}$ c $\frac{2\sqrt{5}-5}{5}$
5 a 36 cm, 52 cm^2 b 1.3 m, 0.1 m^2
c 220 mm, 2100 mm^2
6 a 188.5 m^2, 197.9 m^3 b 7.3 m^2, 1.3 m^3
c 50.3 cm^2, 23.7 cm^3
7 a 63.5 cm to 64.5 cm b 14.15 kg to 14.25 kg
8 a 1.8 cm b 58.8 cm^2
9 $\sqrt{\frac{27}{\pi}}$ cm

Multiple-choice questions

1 B 2 D 3 D 4 A 5 D

Extended-response question

a 753.98 cm^3 b 206.02 cm^3
c 17 cm d 1.79 cm

Quadratic expressions and equations

Short-answer questions

1 a $9x^2-1$ b $4x^2-20x+25$
c $-x^2+30x-5$
2 a $(2x-y)(2x+y)$ b $(x+2+\sqrt{7})(x+2-\sqrt{7})$
c $3(x-4)(x+4)$ d $(x-2)(x+7)$
e $(x-5)^2$ f $2(x-6)(x-2)$
3 a $3-2x$ b 20
c $\frac{3}{x+5}$
4 a $(3x+4)(x-2)$ b $(3x-1)(2x+3)$
c $(5x-4)(2x-3)$
5 a $x=0, 3$ b $x=-4, \frac{1}{2}$
c $x=0, -5$ d $x=4, -4$
e $x=\sqrt{7}, -\sqrt{7}$ f $x=2$
g $x=8, -3$ h $x=-2, \frac{1}{3}$
6 a $x=-8, 5$ b $x=3, 7$ c $x=-4, 5$
7 a i $(x-3+\sqrt{5})(x-3-\sqrt{5})$
ii $(x+2)^2+3$, does not factorise further.
iii $\left(x+\frac{3}{2}-\frac{\sqrt{5}}{2}\right)\left(x+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)$
b i $x=3\pm\sqrt{5}$
ii No solutions
iii $x=\frac{-3\pm\sqrt{5}}{2}$
8 a $x=\frac{-3\pm\sqrt{57}}{4}$
b $x=2\pm\sqrt{10}$

Multiple-choice questions

1 C 2 B 3 B 4 B 5 D

Extended-response question

a $4x^2+40x$ b 44 m^2
c $x=3$ d $x=2.2$

Chapter 6

6A

Building understanding

1 a 0.799 b 0.951 c 1.192 d 0.931
2 a $\sin\theta$ b $\cos\theta$ c $\tan\theta$
3 a 1.80 b 2.94 c 3.42 d 2.38
e 22.33 f 12.47

Now you try

Example 1
a 2.74 b 25.03
Example 2
a 11.65 b 26.27

Exercise 6A

1 a 1.15 b 3.86 c 13.74 d 5.07
e 2.25 f 2.79 g 1.97 h 13.52
i 37.02 j 9.30 k 10.17 l 13.11
2 a 8.55 b 4.26 c 13.06 d 10.04
e 5.55 f 1.52 g 22.38 h 6.28
i 0.06 j 12.12 k 9.80 l 15.20
3 a $x=2.5$ cm, $y=4.33$ cm
b $x=12.26$ cm, $y=6.11$ cm
c $x=0.20$ m, $y=0.11$ m
4 a 125 m b 327 m
5 22.3 m
6 7.54 m
7 28.5 m
8 26.4 cm
9 a 4.5 cm b 8.5 mm
10 The student rounded tan 65° too early.
11 a 3.7 b 6.5 c 7.7
12 a i $a=c\sin\theta$
ii $b=c\cos\theta$
iii $\tan\theta=\frac{a}{b}$
iv $\tan\theta=\frac{c\sin\theta}{c\cos\theta}=\frac{\sin\theta}{\cos\theta}$
v Answers may vary.
b i $a=c\sin\theta$
ii $b=c\cos\theta$
iii $c^2=a^2+b^2$
iv
$$\begin{aligned} c^2 &= (c\sin\theta)^2+(c\cos\theta)^2 \\ c^2 &= c^2(\sin\theta)^2+c^2(\cos\theta)^2 \\ \therefore 1 &= (\sin\theta)^2+(\cos\theta)^2 \end{aligned}$$

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6B

Building understanding

1 a 60° b $\frac{1}{2}$ c 0.75
2 a 23.58° b 60° c 11.31° d 5.74°
3 a Tangent b Cosine c Sine

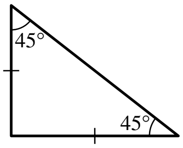
Now you try

Example 3
a $\theta = 30°$ b $\theta = 38.94°$

Example 4
38.7°

Exercise 6B

1 a $\theta = 60°$ b $\theta = 45°$ c $\theta = 48.59°$
d $\theta = 30°$ e $\theta = 52.12°$ f $\theta = 32.74°$
2 a $\alpha = 60°, \theta = 30°$ b $\alpha = 45°, \theta = 45°$
c $\alpha = 53.1°, \theta = 36.9°$ d $\alpha = 22.6°, \theta = 67.4°$
e $\alpha = 28.1°, \theta = 61.9°$ f $\alpha = 53.1°, \theta = 36.9°$
3 a 44.4°, 45.6° b 74.7°, 15.3°
c 58.3°, 31.7° d 23.9°, 66.1°
e 82.9°, 7.1° f 42.4°, 47.6°
4 70.02°
5 31.1°
6 47.1°
7 a 66.4° b 114.1° c 32.0°
8 a i 45° ii 33.7°
b 11.3°
9 a Once one angle is known, the other can be determined by subtracting the known angle from 90°.
b $\alpha = 63.4°, \beta = 26.6°$
10 a



b $\tan 45° = \frac{x}{x} = 1$
c $\sqrt{2}\,x$
d $\sin 45° = \frac{x}{\sqrt{2}\,x} = \frac{1}{\sqrt{2}}$, $\cos 45°$ also equals $\frac{1}{\sqrt{2}}$.
11 a $\theta = 30°$ b $\alpha = 60°$ c $\sqrt{3}$
d i $\frac{1}{2}$ ii $\frac{1}{2}$ iii $\frac{\sqrt{3}}{2}$
iv $\frac{\sqrt{3}}{2}$ v $\frac{\sqrt{3}}{3}$ vi $\sqrt{3}$
e $AB = \frac{1}{2}x + \frac{\sqrt{3}}{2}x = \left(\frac{\sqrt{3}+1}{2}\right)x$

6C

Building understanding

1 $\angle ABC$ and $\angle DCB$
2 a B b C c A

Now you try

Example 5
70.02 m

Example 6
27.98°

Exercise 6C

1 1866.03 m
2 52 m
3 37.6 m
4 1509.53 m
5 32°
6 a 1.17 m b 1.50 m
7 8.69 cm
8 299 m
9 a 1.45° b 3.44° c 1.99°
10 Yes
11 89.12 m
12 a i 8.7 cm ii 5 cm
b i 17.3 cm ii 20 cm
c Answers may vary.
13 321.1 km/h
14 a i 18° ii 72° iii 36° iv 54°
b i 0.77 m ii 2.38 m
iii 2.02 m iv 1.47 m
c 3.85 m
d 4.05 m
e Answers may vary

6D

Building understanding

1 a 0°T b 45°T c 90°T d 135°T
e 180°T f 225°T g 270°T h 315°T
2 a 050°T b 060°T c 139°T
3 a 200°T b 082°T c 335°T d 164°T

Now you try

Example 7
a A is 060°T
B is 140°T
C is 250°T
D is 310°T
b i 240°T ii 130°T

Example 8
a 4.10 km b 19.28 km

Exercise 6D

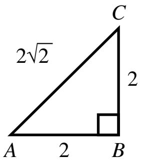
1 a A is 050°T
B is 150°T
C is 215°T
D is 320°T
b i 230°T ii 140°T

2 a 220°T b 305°T c 075°T d 150°T
3 a 1.7 km b 3.6 km
4 a 121°T b 301°T
5 a 3.83 km b 6.21 km
6 a 14.77 cm b 2.6 cm
7 a 211°T b 31°T
8 a 1.414 km b 1.414 km c 2.914 km
9 a 1.62 km b 5.92 km c 2.16 km
10 10.032 km
11 a i 045°T ii 236.3°T
iii 26.6°T iv 315°T
b i 296.6°T ii 116.6°T
iii 101.3°T iv 246.8°T
12 a i 2.5 km ii 2.82 km iii 5.32 km
b i 4.33 km ii 1.03 km iii 5.36 km
c i 45.2°T ii 7.6 km (or 7.5 km)
13 a 229.7°, 18.2 km b 55.1°, 12.3 km
14 a 212.98 m
b i 99.32 m ii 69.20 m
c 30.11 m
15 a 38.30 km b 57.86 km c 33.50°
16 a 4.34 km b 2.07 km c 4.81 km

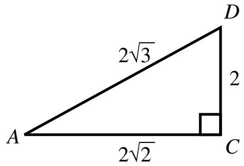
6E

Building understanding

1 3
2 a



b



c 35.3° d 45°

Now you try

Example 9
a 31.896 m b 47.72°

Exercise 6E

1 a 37.609 m b 45.47°
2 a 1.888 m b 43.35°
3 a 57.409 m b 57.91°
4 a i 26.57° ii 11.18 cm
b 10.14°
5 a 7.31 m b 6.87 m
6 138.6 m
7 a i 2.25 m ii 2.59 m
b 49.03° c 3.43 m
8 a i 1.331 km ii 1.677 km
b 0.346 km
9 a Camera C b 609.07 m
10 a 5.5 m b 34.5° c 34.7° d 0.2°
11 a 45° b 1.41 units
c 35.26° d 1.73 units
12 a i 1.55 ii 1.27 iii 2.82
b 34.34°
13 22°

Progress quiz

1 a 12.58 b 6.59
2 a 39° b 52°
3 63°
4 23.84 m
5 a A 060°
B 150°
C 288°
b 240°
6 a 13.65 km b 048.4°
7 a 36.77 m b 61°
8 a $\sqrt{8}$ cm b 35°

6F

Building understanding

1 $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
2 a 1.9 b 2.5
3 a 50.3° b 29.2°

Now you try

Example 10
7.3

Example 11
a 36.2° b 121.0°

Exercise 6F

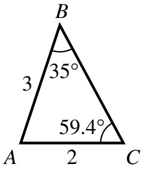
1 a 7.9 b 16.5 c 19.1
d 9.2 e 8.4 f 22.7
2 a 38.0° b 51.5° c 28.8°
d 44.3° e 47.5° f 48.1°
3 a 1.367 km b 74° c 2.089 km
4 27.0°
5 131.0 m
6 a $\angle ABC = 80^\circ$, $\angle ACB = 40^\circ$
b 122 km
7 a $\angle ABC = 80^\circ$ b 61.3 km c 53.9 km
8 a 147.5° b 102.8° c 126.1°
d 100.5° e 123.9° f 137.7°

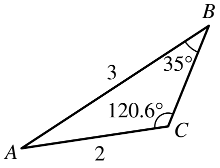
9 Impossible to find θ as such a triangle does not exist.

10 37.6° or 142.4°

11 a 59.4° or 120.6°

b

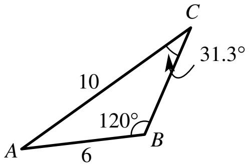




c 31.3°

d A triangle can only have one obtuse angle.

e



6G

Building understanding

1 a $c^2 = 3^2 + 4^2 - 2 \times 3 \times 4 \times \cos 105°$

b $7^2 = 5^2 + 9^2 - 2 \times 5 \times 9 \times \cos\theta$

2 a 9.6 b 1.5 c 100.3° d 36.2°

Now you try

Example 12
4.29 m

Example 13
26.53°

Exercise 6G

1 a 16.07 cm b 8.85 m c 14.78 cm
d 4.56 m e 2.86 km f 8.14 m

2 a 81.79° b 104.48° c 64.62°
d 61.20° e 92.20° f 46.83°

3 310 m

4 32.2°, 49.6°, 98.2°

5 a 145.9° b 208.2°

6 383 km

7 7.76 m

8 a Cosine rule b Sine rule c Sine rule
d Cosine rule e Sine rule f Cosine rule

9 a $\cos C = \frac{a^2+b^2-c^2}{2ab}$ b 121.9°

10 Obtuse, as cosine of an obtuse angle gives a negative result.

11 a $AP = b - x$

b $a^2 = x^2 + h^2$

c $c^2 = h^2 + (b-x)^2$

d $c^2 = a^2 - x^2 + (b-x)^2 = a^2 + b^2 - 2bx$

e $\cos C = \frac{x}{a}$

f $x = a\cos C$ substitute into part **d**.

6H

Building understanding

1 a 3.7 b 48.0

2 a α b θ c β

3 a 56.44° or 123.56° b 45.58° or 134.42°

Now you try

Example 14
30.1 cm^2

Example 15
18.64

Exercise 6H

1 a 4.4 cm^2 b 26.4 m^2 c 0.9 km^2
d 13.7 m^2 e 318.4 m^2 f 76.2 cm^2

2 a 11.9 cm^2 b 105.6 m^2 c 1.6 km^2

3 a $x = 5.7$ b $x = 7.9$ c $x = 9.1$
d $x = 18.2$ e $x = 10.6$ f $x = 1.3$

4 a 59.09 cm^2 b 1.56 mm^2 c 361.25 km^2

5 a 35.03 cm^2 b 51.68 m^2 c 6.37 km^2

6 a 965.88 m^2 b 214.66 m^2 c 0.72 km^2

7 a 17.3 m^2 b 47.2 cm^2 c 151.4 km^2

8 a Area $= ab\sin\theta$

b Area $= \frac{1}{2}a^2 \sin 60° = \frac{\sqrt{3}}{4}a^2$

c Area $= \frac{1}{2}a^2 \sin(180° - 2\theta) = \frac{1}{2}a^2 \sin 2\theta$

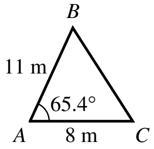
9 a i 129.9 cm^2 ii 129.9 cm^2

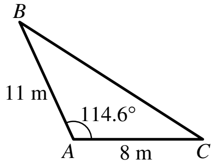
b They are equal because sin 60° and sin 120° are equal.

c Same side lengths with included angle 140°.

10 a 65.4°, 114.6°

b





11 a i 540° ii 108° iii 11.89 cm^2
iv 8.09 cm v 72°, 36° vi 19.24 cm^2
vii 43.0 cm^2

b 65.0 cm^2 c Answers may vary.

6I

Building understanding

1 a Quadrant 1 b Quadrant 3
c Quadrant 4 d Quadrant 2

2 a Quadrants 1 and 2 b Quadrants 2 and 4
c Quadrants 2 and 3 d Quadrants 1 and 4
e Quadrants 1 and 3 f Quadrants 3 and 4

3

θ	0°	90°	180°	270°	360°
sin θ	0	1	0	–1	0
cos θ	1	0	–1	0	1
tan θ	0	undefined	0	undefined	0

4 a 0.139 b –0.530 c –0.799
d –0.259 e 0.900 f –1.036

Now you try

Example 16
a 140° b 106° c 133°

Example 17
a Quadrant 4, $\sin\theta$ is negative
$\cos\theta$ is positive
$\tan\theta$ is negative
b Quadrant 3, $\sin\theta$ is negative
$\cos\theta$ is negative
$\tan\theta$ is positive
c Quadrant 2, $\sin\theta$ is positive
$\cos\theta$ is negative
$\tan\theta$ is negative

Example 18
a $-\sin 50°$ b $-\cos 54°$ c $\tan 80°$

Exercise 6I

1 a 140° b 115° c 155°
d 99° e 143° f 172°
2 a 30° b 86° c 70°
d 9° e 21° f 37°
3 a Quadrant 2, $\sin\theta$ positive, $\cos\theta$ negative, $\tan\theta$ negative
b Quadrant 4, $\sin\theta$ negative, $\cos\theta$ positive, $\tan\theta$ negative
c Quadrant 3, $\sin\theta$ negative, $\cos\theta$ negative, $\tan\theta$ positive
d Quadrant 1, $\sin\theta$ positive, $\cos\theta$ positive, $\tan\theta$ positive
e Quadrant 4, $\sin\theta$ negative, $\cos\theta$ positive, $\tan\theta$ negative
f Quadrant 2, $\sin\theta$ positive, $\cos\theta$ negative, $\tan\theta$ negative
g Quadrant 3, $\sin\theta$ negative, $\cos\theta$ negative, $\tan\theta$ positive
h Quadrant 3, $\sin\theta$ negative, $\cos\theta$ negative, $\tan\theta$ positive
i Quadrant 3, $\sin\theta$ negative, $\cos\theta$ negative, $\tan\theta$ positive
j Quadrant 1, $\sin\theta$ positive, $\cos\theta$ positive, $\tan\theta$ positive
k Quadrant 4, $\sin\theta$ negative, $\cos\theta$ positive, $\tan\theta$ negative
l Quadrant 2, $\sin\theta$ positive, $\cos\theta$ negative, $\tan\theta$ negative
m Quadrant 4, $\sin\theta$ negative, $\cos\theta$ positive, $\tan\theta$ negative
n Quadrant 1, $\sin\theta$ positive, $\cos\theta$ positive, $\tan\theta$ positive
o Quadrant 3, $\sin\theta$ negative, $\cos\theta$ negative, $\tan\theta$ positive
p Quadrant 2, $\sin\theta$ positive, $\cos\theta$ negative, $\tan\theta$ negative
4 a $-\sin 80°$ b $\cos 60°$
c $\tan 40°$ d $\sin 40°$
e $-\cos 55°$ f $-\tan 45°$
g $-\sin 15°$ h $-\cos 58°$
i $\tan 47°$ j $\sin 68°$
k $\cos 66°$ l $-\tan 57°$
5 a 30° b 60° c 24°
d 40° e 71° f 76°
g 50° h 25° i 82°
6 a $0 < \theta < 90°$ b $90° < \theta < 180°$
c $270° < \theta < 360°$ d $180° < \theta < 270°$
7 a 42° b 47° c 34° d 9°
e 33° f 62° g 14° h 58°
i 40° j 27° k 66° l 44°

8

θ_2	150°	315°	350°	195°	235°	140°	100°	35°	55°

9 a Quadrant 4 b Quadrant 1 c Quadrant 2
d Quadrant 2 e Quadrant 1 f Quadrant 3
10 As $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and both $\sin\theta$ and $\cos\theta$ are negative over this range, $\tan\theta$ is positive in the third quadrant.
11 As $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cos\theta = 0$ at 90° and 270°, the value of $\frac{\sin\theta}{\cos\theta}$ and, hence, $\tan\theta$ is undefined at these values.
12 a True b True c False d True
e True f False g True h False
i False j True k True l False
13 a i 0.17 ii 0.17 iii 0.59 iv 0.59
v 0.99 vi 0.99 vii 0.37 viii 0.37
b $\sin a = \cos b$ when $a + b = 90°$.
c i $90° - \theta$ ii $90° - \theta$
d i 70° ii 5° iii 19° iv 38°
e i $90° - \theta$ ii $\frac{b}{c}$ iii $\frac{b}{c}$
f $\frac{2\sqrt{5}}{5}$
14 a i Proof
ii True for these values.
b i $\sin 60° = \cos 30° = 0.866$,
$\sin 80° = \cos 10° = 0.985$,
$\sin 110° = \cos(-20°) = 0.940$,
$\sin 195° = \cos(-105°) = -0.259$
ii Their values are the same.
iii They add to 90°.
iv $\sin\theta = \cos(90° - \theta)$
v True for these values.
c Answers may vary.

6J

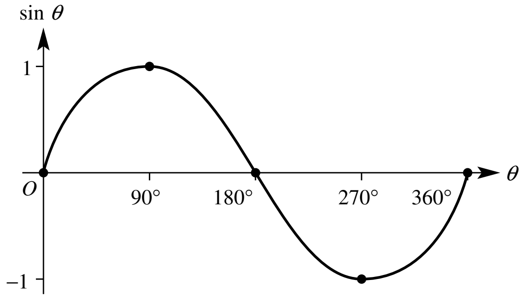
Building understanding

1 a

θ	0°	30°	60°	90°	120°	150°
sin θ	0	0.5	0.87	1	0.87	0.5

θ	180°	210°	240°	270°	300°	330°	360°
sin θ	0	–0.5	–0.87	–1	–0.87	–0.5	0

b

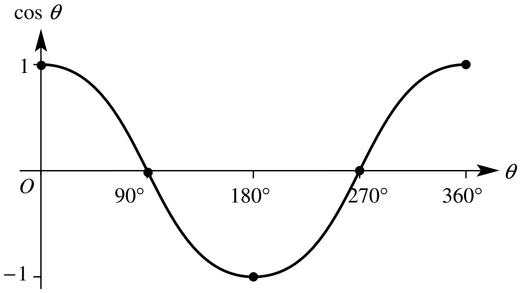


2 **a**

θ	0°	30°	60°	90°	120°	150°
cos θ	1	0.87	0.5	0	-0.5	-0.87

θ	180°	210°	240°	270°	300°	330°	360°
cos θ	-1	-0.87	-0.5	0	0.5	0.87	1

b



3 **a** **i** Maximum = 1, minimum = -1
ii 0°, 180°, 360°
b **i** Maximum = 1, minimum = -1
ii 90°, 270°
c **i** $90° < \theta < 270°$ **ii** $180° < \theta < 360°$

Now you try

Example 19
a ≈ -0.17 **b** $\approx$ 127° and 233°

Example 20
a False **b** True

Exercise 6J

1 **a** **i** 0.82 **ii** -0.98 **iii** 0.87
iv -0.77 **v** -0.17 **vi** 0.26
vii -0.42 **viii** 0.57
b **i** 37°, 323° **ii** 53°, 307° **iii** 73°, 287°
iv 84°, 276° **v** 114°, 246° **vi** 102°, 258°
vii 143°, 217° **viii** 127°, 233°

2 **a** **i** 0.42 **ii** 0.91 **iii** -0.64
iv -0.77 **v** 0.34 **vi** -0.82
vii -0.64 **viii** 0.94
b **i** 37°, 143° **ii** 12°, 168° **iii** 17°, 163°
iv 64°, 116° **v** 204°, 336° **vi** 233°, 307°
vii 224°, 316° **viii** 186°, 354°

3 **a** True **b** False **c** False **d** True
e False **f** True **g** True **h** True
i True **j** False **k** True **l** True

4 **a** 110° **b** 60° **c** 350° **d** 260°
e 27° **f** 326° **g** 233° **h** 357°

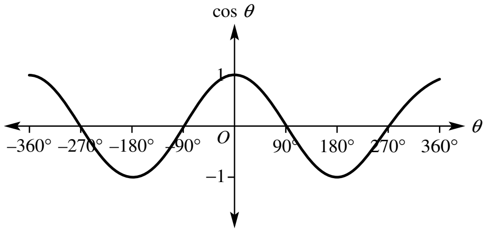
5 **a** 280° **b** 350° **c** 195° **d** 75°
e 136° **f** 213° **g** 24° **h** 161°

6 **a** **i** 110° **ii** $-70°$
b **i** 140° **ii** $-40°$
c **i** 70° **ii** $-110°$
d **i** 25° **ii** $-155°$
e **i** $-165°$ **ii** 15°
f **i** $-112°$ **ii** 68°
g **i** $-80°$ **ii** 100°
h **i** $-35°$ **ii** 145°

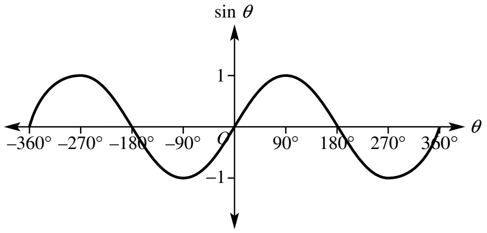
7 **a** 0, the maximum value of sin θ is 1.
b 0, the minimum value of cos θ is -1.

8 **a** 17.5°, 162.5° **b** 44.4°, 135.6°
c 53.1°, 306.9° **d** 36.9°, 323.1°
e 191.5°, 348.5° **f** 233.1°, 306.9°
g 113.6°, 246.4° **h** 49.5°, 310.5°
i 28.7°, 151.3°

9 **a**



b

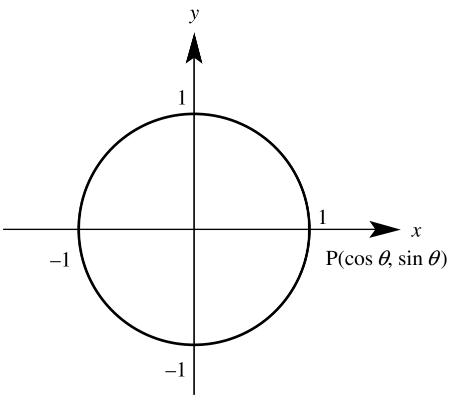


10 **a** Graph is reflected in the x-axis.
b Graph is reflected in the x-axis.
c Graph is dilated and constricted from the x-axis.
d Graph is dilated and constricted from the y-axis.
e Graph is translated up and down parallel to the x-axis.
f Graph is translated left and right parallel to the y-axis.

6K

Building understanding

1 On the unit circle, $(\cos\theta, \sin\theta) = (1, 0)$.



2 a 0 b 1

3

$\theta°$	0	30	45	60	90	180	270	360
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	–1	0	1
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	0	undefined	0

4 a 30° b 30° c 45°

Now you try

Example 21

a $\frac{\sqrt{3}}{2}$ b $\frac{\sqrt{2}}{2}$ c $-\sqrt{3}$

Example 22

a 60°, 300° b 225°, 315°

Exercise 6K

1 a $\frac{\sqrt{3}}{2}$ b $\frac{\sqrt{2}}{2}$ c $\sqrt{3}$
d $\frac{\sqrt{2}}{2}$ e $-\frac{\sqrt{3}}{2}$ f $-\sqrt{3}$
g $\frac{\sqrt{2}}{2}$ h $-\frac{\sqrt{2}}{2}$ i $\frac{\sqrt{3}}{2}$
j $-\frac{\sqrt{3}}{3}$ k $-\frac{1}{2}$ l $\frac{1}{2}$
m –1 n 1 o 0
p Undefined

2 a 45°
b i $-\frac{\sqrt{2}}{2}$ ii $-\frac{\sqrt{2}}{2}$ iii 1
c 30°
d i $-\frac{1}{2}$ ii $\frac{\sqrt{3}}{2}$ iii $-\frac{\sqrt{3}}{3}$

3 a $-\frac{1}{2}$ b $-\frac{1}{2}$
c $-\sqrt{3}$ d $-\frac{\sqrt{3}}{2}$
e –1 f $-\frac{\sqrt{3}}{2}$
g $-\frac{1}{2}$ h 0
i $-\frac{\sqrt{2}}{2}$ j –1
k $-\frac{\sqrt{3}}{3}$ l $\frac{1}{2}$
m 0 n Undefined
o 1 p –1

4 a 30°, 150° b 60°, 120°
c 60°, 300° d 45°, 315°
e 60°, 240° f 45°, 225°

5 a $3\sqrt{2}$ b $3\sqrt{2}$
c $\frac{20\sqrt{3}}{3}$ d 14
e $5\sqrt{3}$ f 3

6 a 45° b 30°
c 60°

7 a 120°, 240° b 225°, 315°
c 135°, 315° d 135°, 225°
e 240°, 300° f 150°, 330°

8 a 13
b i $\frac{5}{13}$ ii $\frac{12}{13}$ iii $\frac{5}{12}$

9 a 30°, 150°, 390°, 510°
b 45°, 315°, 405°, 675°
c 240°, 300°, 600°, 660°

10 a 30°, 150° b 45°, 315°
c 30°, 210° d 150°, 210°
e 210°, 330° f 135°, 315°

11 a –300°, –240°, 60°, 120°
b –135°, –45°, 225°, 315°
c –60°, 60°, 300°, 420°, 660°
d –135°, 135°

Problems and challenges

1 a 120°, 60° b 8.7 cm
2 225°
3 Use the cosine rule.
4 514 m
5 a 2 hours 9 minutes b 308°
6 17.93°
7 4.33 cm

Answers to success criteria example questions

1 $x = 9.23$
2 $x = 10.77$

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3 $\theta = 23.20°$
4 21 m
5 38.23°
6 220°T; 040°T
7 1.29 km
8 57.8°
9 $x = 8.2$
10 $\theta = 35.1°$
11 132.3°
12 8.64 m
13 $\theta = 56.25°$
14 $100.3\ \text{cm}^2$
15 $x = 6.22$
16 250°: Quadrant 3; $\sin\theta$ is negative, $\cos\theta$ is negative, $\tan\theta$ is positive.
$-100°$: Quadrant 3; $\sin\theta$ is negative, $\cos\theta$ is negative, $\tan\theta$ is positive.
17 $\cos 33° = -\cos 147°$
18 $\cos 300° = \cos 60°$; $\sin 195° = -\sin 15°$
19 $\cos 130° \approx -0.6$; $\theta = 66°$ and $\theta = 294°$
20 $\cos 50° = \cos 310°$
21 $\frac{\sqrt{3}}{2}; \frac{\sqrt{2}}{2}$
22 $\theta = 60°, 120°$

Short-answer questions

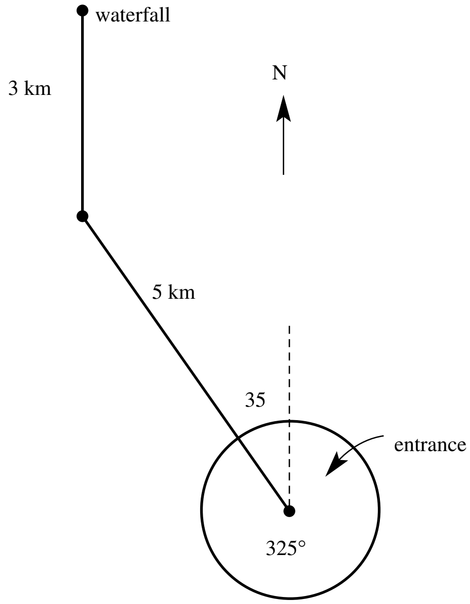
1 **a** $x = 14.74$ **b** $x = 13.17$
c $x = 11.55, y = 5.42$
2 **a** $\theta = 45.6°$ **b** $\theta = 64.8°$
3 6.1 m
4 **a** $A = 115°, B = 315°, C = 250°, D = 030°$
b **i** 295° **ii** 070°
5 **a** 98.3 km **b** 228.8 km **c** 336.8°
6 **a** 15.43 m
b 52°
7 **a** **i** 15.5 cm **ii** $135.0\ \text{cm}^2$
b **i** 14.9 cm **ii** $111.3\ \text{cm}^2$
8 28.1 m
9 **a** 52.6° **b** 105.4°
10 **a** 12.5 **b** 42.8°
11 **a** **i** Negative **ii** Positive
iii Negative **iv** Positive
b **i** $\sin 60°$ **ii** $-\cos 30°$
iii $-\tan 45°$ **iv** $-\sin 45°$
c **i** $\frac{\sqrt{3}}{2}$ **ii** $-\frac{\sqrt{3}}{2}$ **iii** -1 **iv** $-\frac{\sqrt{2}}{2}$
12 **a** $x = 4$ **b** $x = \frac{5\sqrt{3}}{3}$
13 **a** **i** 0.77 **ii** -0.97
b **i** 53°, 127° **ii** 197°, 343° **iii** No value
c **i** True **ii** True **iii** False
14 **a** 45°, 315° **b** 210°, 330°

Multiple-choice questions

1 D
2 B
3 E
4 D
5 A
6 C
7 A
8 D
9 C
10 C

Extended-response questions

1 **a**



b 2.9 km west
c 7.7 km
d **i** 21.9 m **ii** 38.0°
2 **a** 33.646° **b** $3177.54\ \text{m}^2$
c 41.00 m **d** 61.60 m
e **i** 65.66°, 114.34° **ii** 80.1 m, 43.1 m

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Chapter 7

7A

Building understanding

1 a maximum b $(-2, 4)$ c 2
d $-5, 1$ e $x = -2$
2 a minimum b $(1, -3)$ c -2
d $-1, 3$ e $x = 1$

Now you try

Example 1
a i Minimum at $(-1, -3)$ ii $x = -1$
iii $(-3, 0)$ and $(1, 0)$ iv $(0, -2)$
b i Maximum at $(2, 0)$ ii $x = 2$
iii $(2, 0)$ iv $(0, -2)$

Example 2

	Formula	Maximum or minimum	Reflecfed in the x-axis (yes/no)	Turning point	y-value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = \frac{1}{2}x^2$	minimum	no	$(0, 0)$	$\frac{1}{2}$	wider
b	$y = (x - 2)^2$	minimum	no	$(2, 0)$	1	same
c	$y = -x^2 - 1$	maximum	yes	$(0, -1)$	-2	same

Exercise 7A

1 a i $(2, -5)$, min ii $x = 2$
iii $(-1, 0)$, $(5, 0)$ iv $(0, -3)$
b i $(2, 0)$, max ii $x = 2$
iii $(2, 0)$ iv $(0, -1)$
c i $(2, 5)$, min ii $x = 2$
iii No x-intercept iv $(0, 7)$
d i $(-3, 0)$, min ii $x = -3$
iii $(-3, 0)$ iv $(0, 4)$
e i $(2, -2)$, min ii $x = 2$
iii $(1, 0)$, $(3, 0)$ iv $(0, 6)$
f i $(0, 3)$, max ii $x = 0$
iii $(-3, 0)$, $(3, 0)$ iv $(0, 3)$

2

	Formula	Max or min	Reflected in the x-axis (yes/no)	Turning point	y-value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 3x^2$	min	no	$(0, 0)$	$y = 3$	narrower
b	$y = \frac{1}{2}x^2$	min	no	$(0, 0)$	$y = \frac{1}{2}$	wider
c	$y = 2x^2$	min	no	$(0, 0)$	$y = 2$	narrower
d	$y = -4x^2$	max	yes	$(0, 0)$	$y = -4$	narrower
e	$y = -\frac{1}{3}x^2$	max	yes	$(0, 0)$	$y = -\frac{1}{3}$	wider
f	$y = -2x^2$	max	yes	$(0, 0)$	$y = -2$	narrower

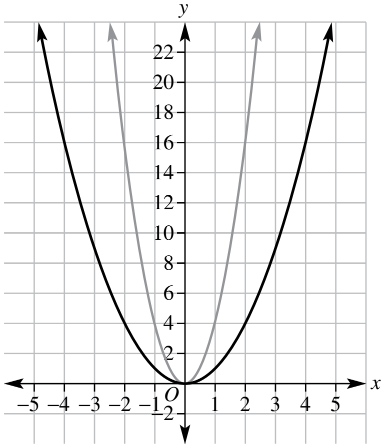
3

	Formula	Turning point	Axis of symmetry	y-intercept	x-intercept
a	$y = (x + 3)^2$	$(-3, 0)$	$x = -3$	$(0, 9)$	$(-3, 0)$
b	$y = (x - 1)^2$	$(1, 0)$	$x = 1$	$(0, 1)$	$(1, 0)$
c	$y = (x - 2)^2$	$(2, 0)$	$x = 2$	$(0, 4)$	$(2, 0)$
d	$y = (x + 4)^2$	$(-4, 0)$	$x = -4$	$(0, 16)$	$(-4, 0)$

4

	Formula	Turning point	y-intercept	y-value when $x = 1$
a	$y = x^2 + 3$	$(0, 3)$	$(0, 3)$	$y = 4$
b	$y = x^2 - 1$	$(0, -1)$	$(0, -1)$	$y = 0$
c	$y = x^2 + 2$	$(0, 2)$	$(0, 2)$	$y = 3$
d	$y = x^2 - 4$	$(0, -4)$	$(0, -4)$	$y = -3$

5 a $x = 0$ b $x = 0$ c $x = 0$
d $x = 0$ e $x = 0$ f $x = 2$
g $x = -1$ h $x = -3$ i $x = 0$
j $x = 0$ k $x = 0$ l $x = -4$
6 a $(0, 0)$ b $(0, 7)$ c $(0, 0)$
d $(0, 0)$ e $(0, -4)$ f $(2, 0)$
g $(-1, 0)$ h $(-3, 0)$ i $(0, -3)$
j $(0, 2)$ k $(0, -16)$ l $(-4, 0)$
7 a $(0, 0)$ b $(0, 7)$ c $(0, 0)$
d $(0, 0)$ e $(0, -4)$ f $(0, 4)$
g $(0, 1)$ h $(0, -9)$ i $(0, -3)$
j $(0, 2)$ k $(0, -16)$ l $(0, -16)$
8 a H b C c G
d D e A f E
g B h F
9 a i



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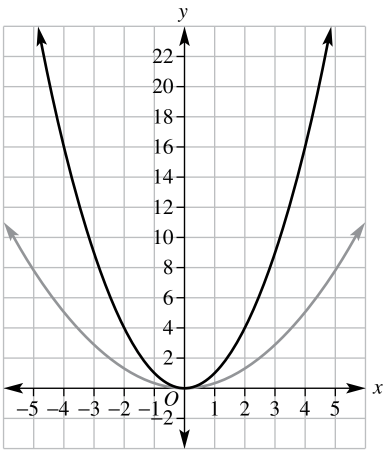

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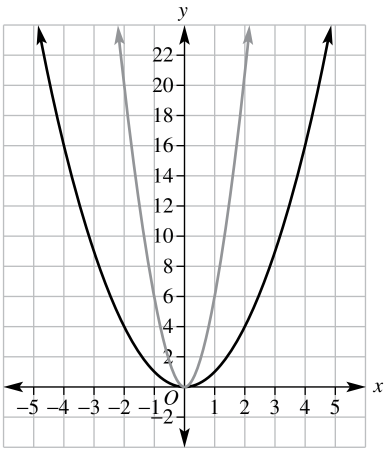

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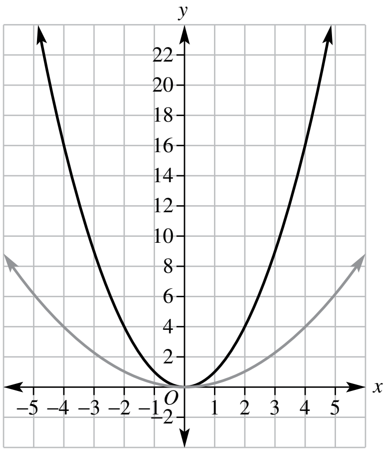

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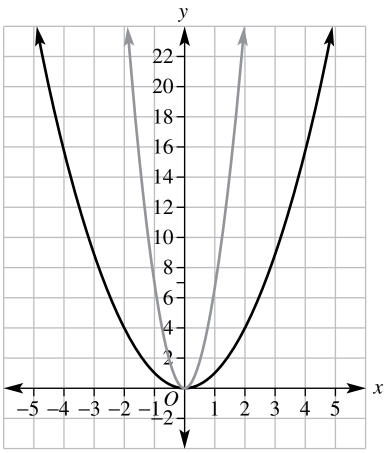
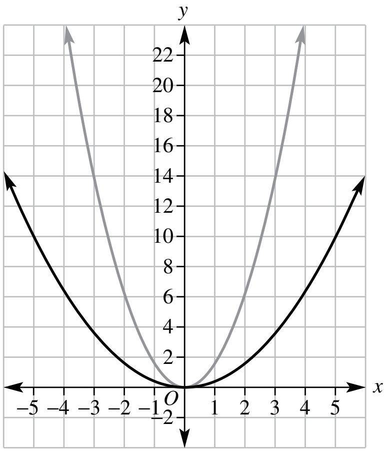
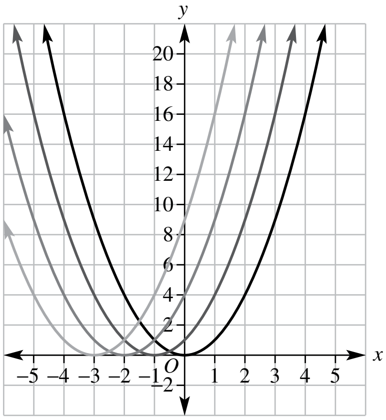


vi



b The constant a determines the narrowness of the graph.

10 a i



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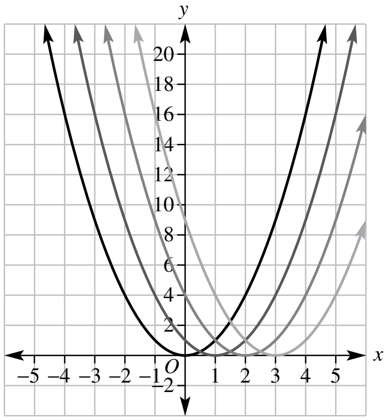
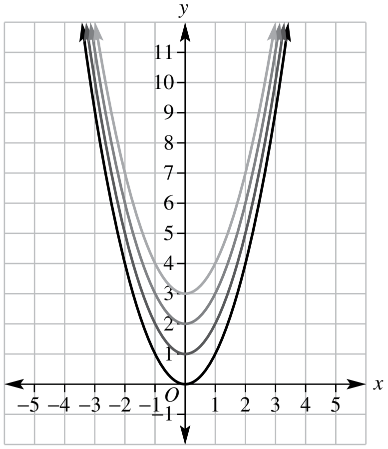


b The constant h determines whether the graph moves left or right from $y = x^2$.

11 a i



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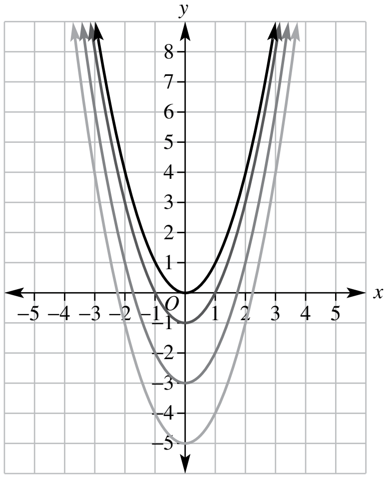


b The constant k determines whether the graph moves up or down from $y = x^2$.

12 Answers could be:

a $y = x^2 - 4$ **b** $y = (x - 5)^2$ **c** $y = x^2 + 3$

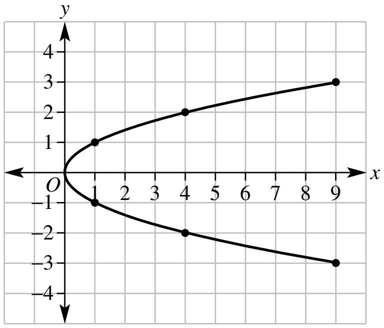
13 a $y = x^2 + 2$ **b** $y = -x^2 + 2$

c $y = (x + 1)^2$ **d** $y = (x - 2)^2$

e $y = 2x^2$ **f** $y = -3x^2$

g $y = (x + 1)^2 + 2$ **h** $y = \frac{1}{8}(x - 4)^2 - 2$

14 Parabola on its side.



7B

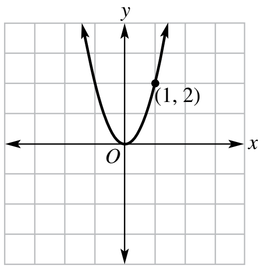
Building understanding

1	**a**	(0, 0)	**b**	(0, 3)	**c**	(0, −4)		
	d	(2, 0)	**e**	(−5, 0)	**f**	(0, 0)		
2	**a**	(0, 5)	**b**	(0, −3)	**c**	(0, 4)	**d**	(0, 2)
3	**a**	up	**b**	right	**c**	left	**d**	down
	e	down	**f**	left	**g**	right	**h**	up

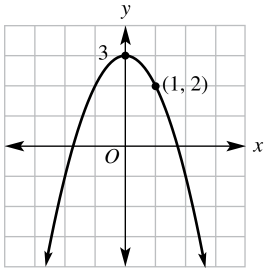
Now you try

Example 3

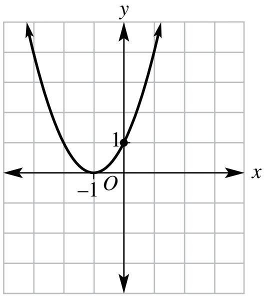
a



b

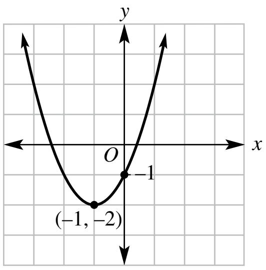


c



Example 4

a



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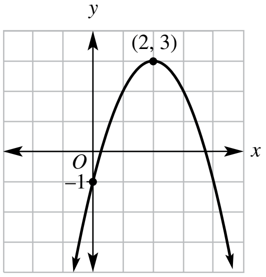
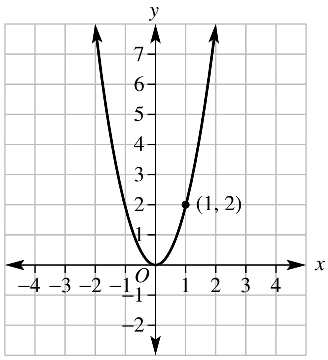


Example 5

$y = 2x^2 - 1$

Exercise 7B

1 a



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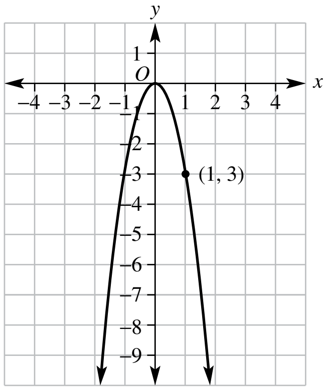

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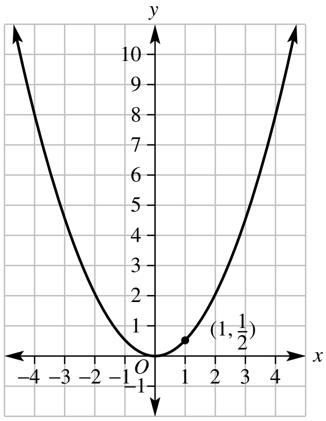

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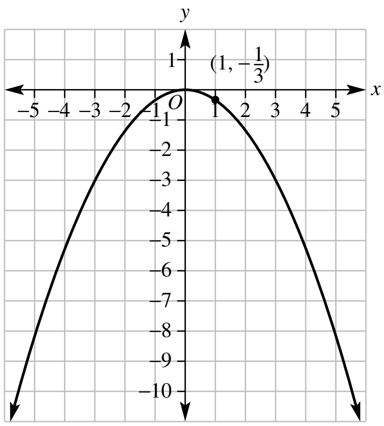
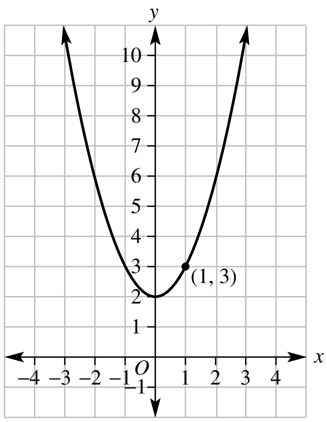


2 a



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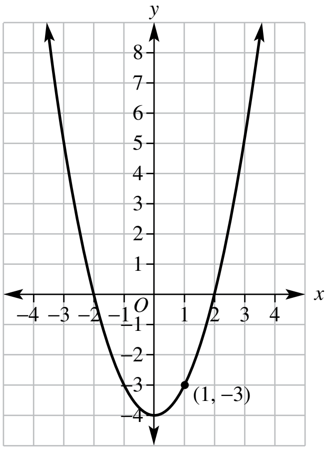
b



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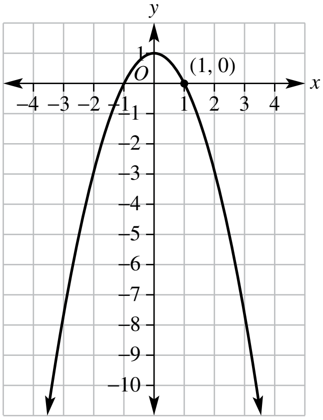

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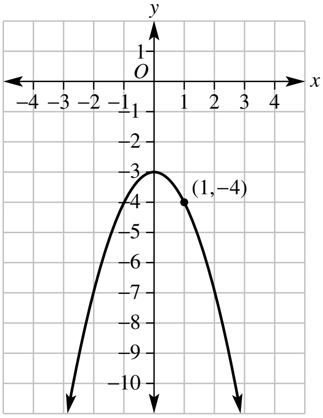

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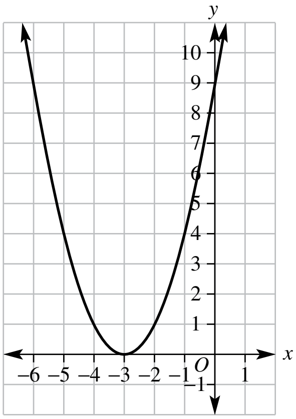

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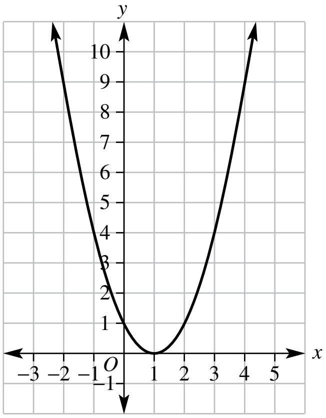

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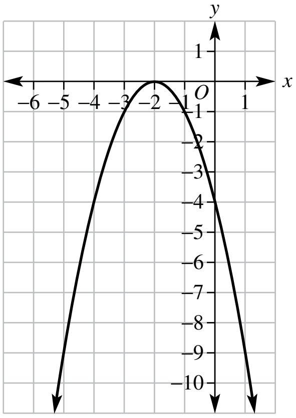

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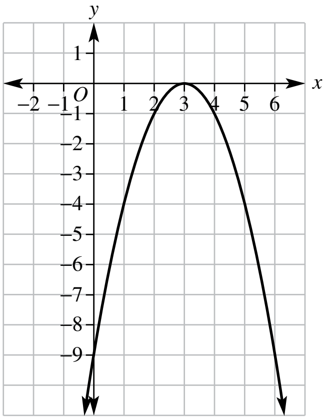
h



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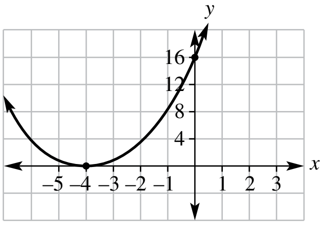
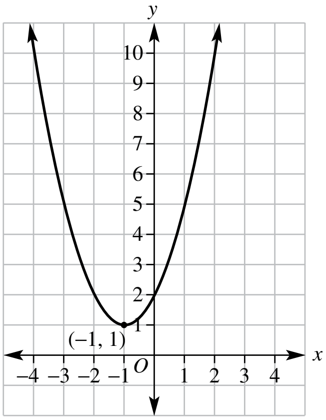


3 **a** (−3, 1) **b** (−2, −4) **c** (1, 3)
d (4, −2) **e** (3, −5) **f** (2, 2)
g (3, 3) **h** (2, 6) **i** (−1, 4)
j (2, −5) **k** (−1, −1) **l** (4, −10)

4 **a**



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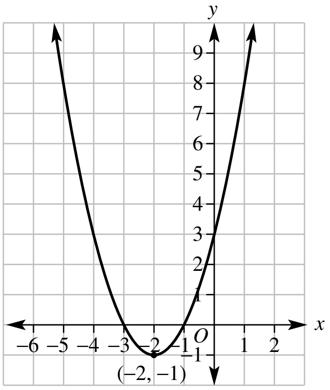

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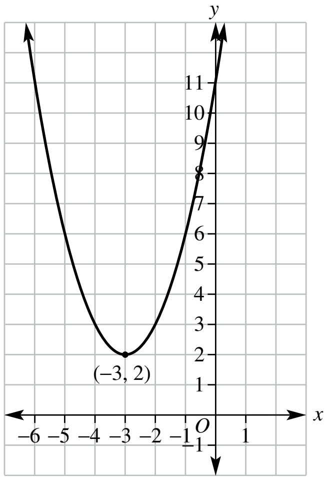

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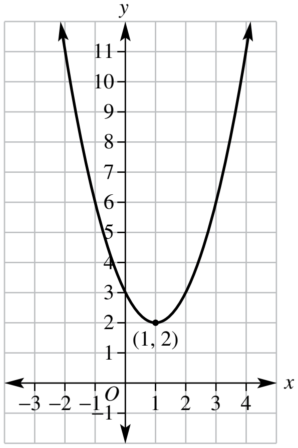

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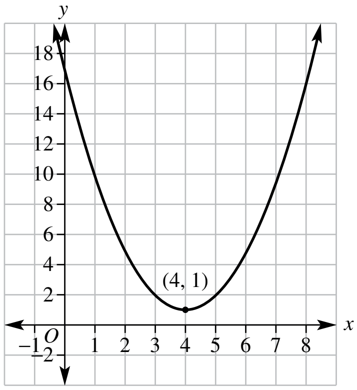

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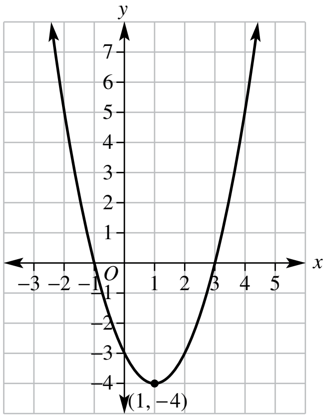

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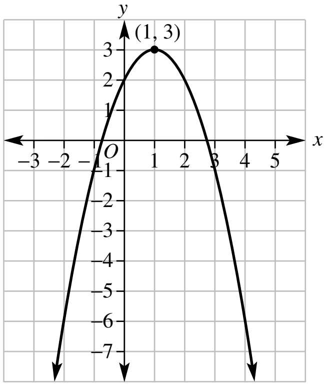

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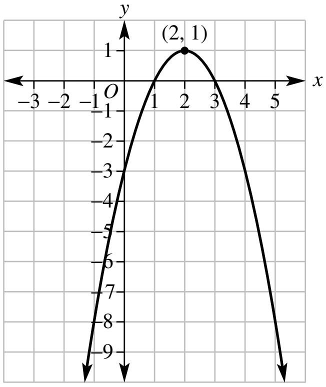

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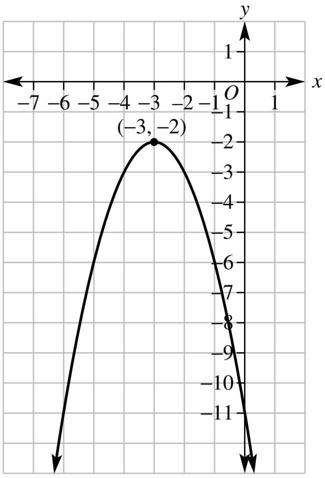

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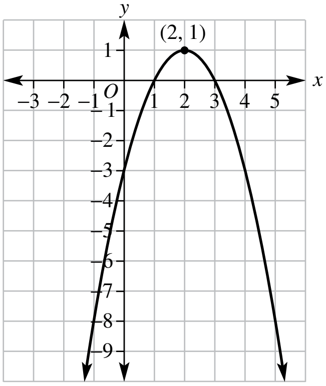

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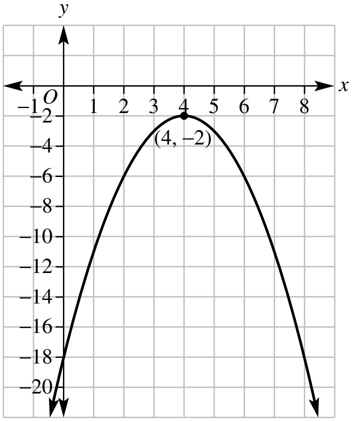

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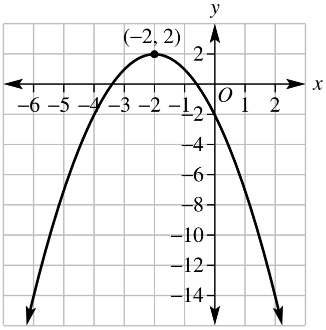


5 **a** $y = 6x^2$
b $y = x^2 + 4$
c $y = (x - 3)^2$
d $y = -(x + 2)^2$
e $y = \frac{1}{2}x^2$
f $y = -x^2 + 2$
g $y = x^2 - 1$
h $y = (x - 1)^2$
i $y = -7x^2$

6 **a** $y = -x^2$
b $y = (x + 2)^2$
c $y = x^2 - 5$
d $y = x^2 + 4$
e $y = (x - 1)^2$
f $y = -x^2 + 2$
g $y = -(x + 3)^2$
h $y = (x + 5)^2 - 3$
i $y = (x - 6)^2 + 1$

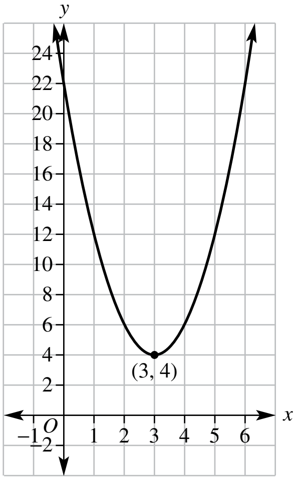
7 **a** Maximum **b** (5, 25)
c (0, 0) **d** 25 m
e **i** 21 m **ii** 21 m **iii** 0 m

8 **a** (1, 0) **b** (−2, 0) **c** (−3, 0)
d (0, −4) **e** (0, −2) **f** (0, 5)
g (−4, −1) **h** (−2, 3) **i** (5, 4)
j (−2, 3) **k** (−3, −5) **l** (3, −3)

9 **a** Translate 3 units right
b Translate 2 units left
c Translate 3 units down
d Translate 7 units up
e Reflect in x-axis
f Translate 2 units left and 4 units down
g Translate 5 units right and 8 units up
h Reflect in x-axis, translate 3 units left
i Reflect in x-axis, translate 6 units up

10 **a** (h, k) **b** $\left(0, ah^2 + k\right)$

11 **a**



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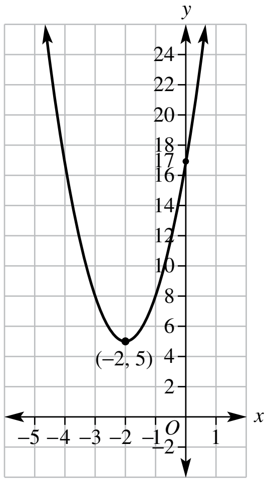

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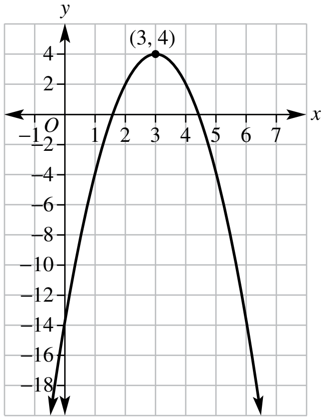

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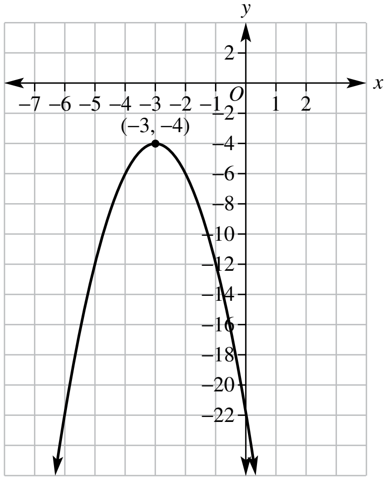

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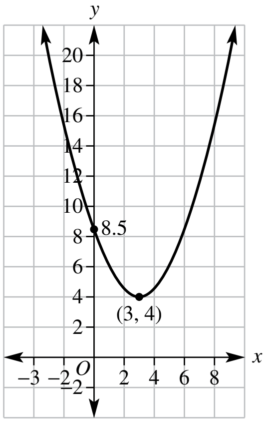
e



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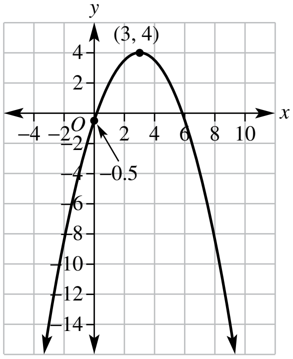

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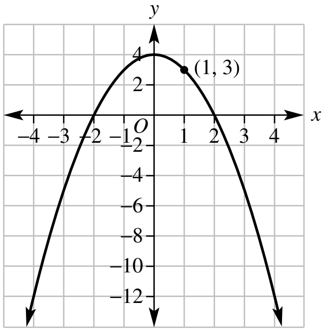

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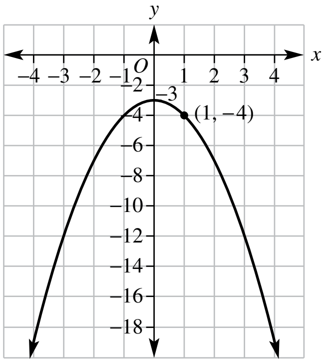

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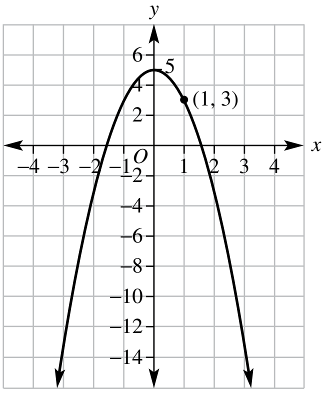

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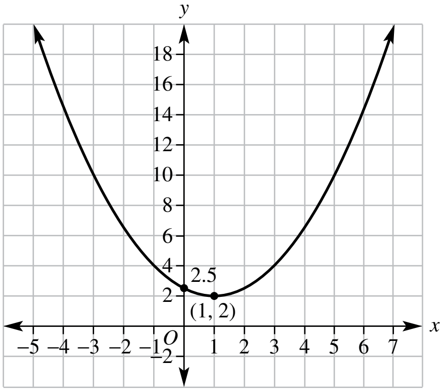

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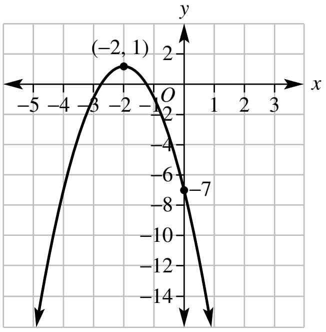

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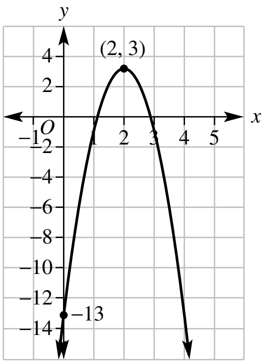

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7C

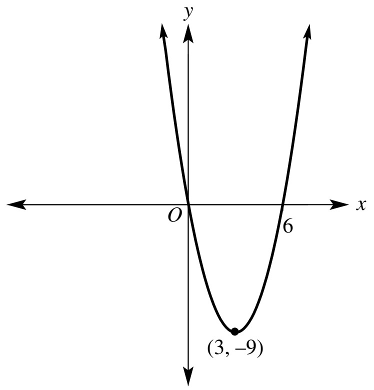
Building understanding

1 a $(-1, 0), (2, 0)$ b $(0, 0), (3, 0)$
c $(0, 0), (-2, 0)$
2 a $y = x(x - 4)$ b $y = (x + 4)(x - 2)$
c $y = (x - 4)^2$ d $y = (x + 5)(x - 5)$
3 a $(0, 0)$
b $(0, -8)$
c $(0, 16)$
d $(0, -25)$
4 a $(4, -4)$ b $(-1, 9)$

Now you try

Example 6

a



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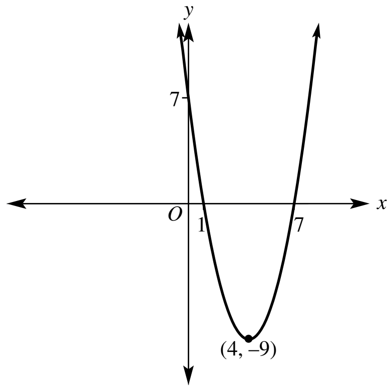
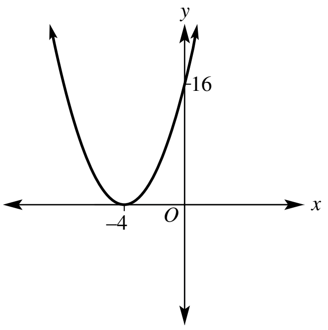


Example 7



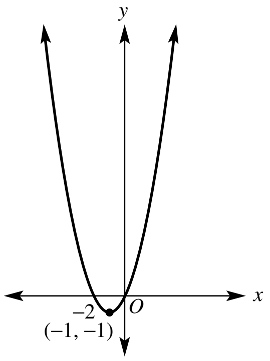
Example 8

$y = (x + 3)(x - 7)$

Turning point is $(2, -25)$

Exercise 7C

1 a



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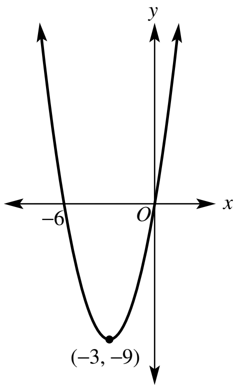

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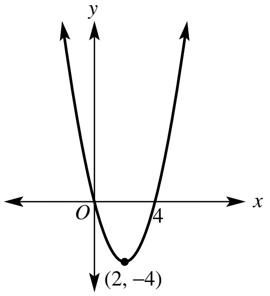

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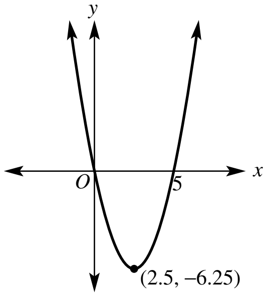
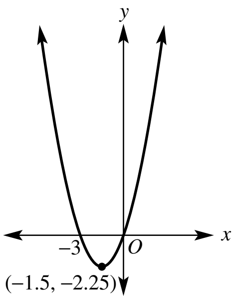
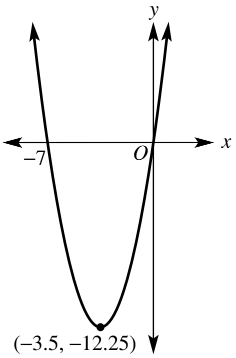


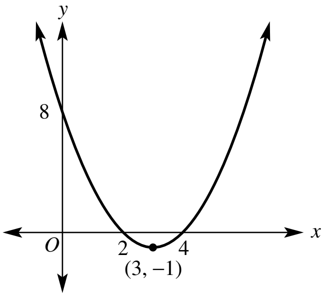
e


y
x
−3
O
(−1.5, −2.25)

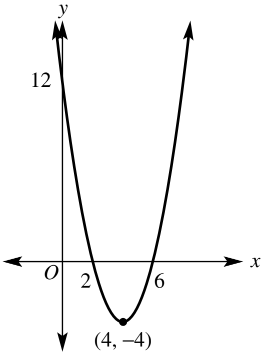
f


y
x
−7
O
(−3.5, −12.25)

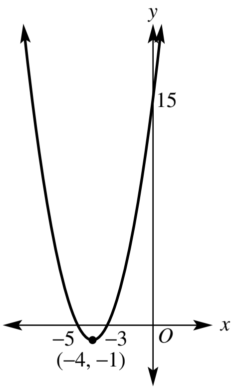
2 a


y
8
x
O
2
4
(3, −1)

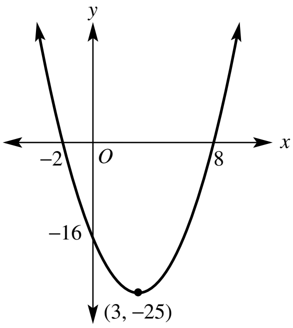
b


y
12
x
O
2
6
(4, −4)

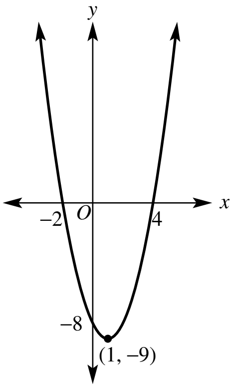
c


y
15
x
−5
−3
O
(−4, −1)

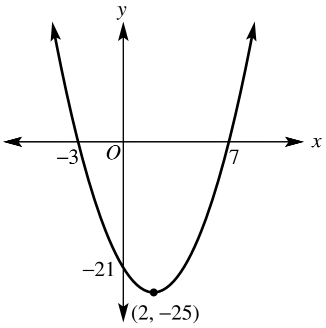
d


y
x
−2
O
8
−16
(3, −25)

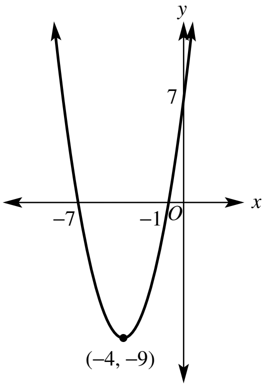
e


y
x
−2
O
4
−8
(1, −9)

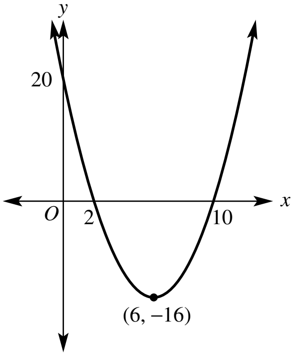
f


y
x
−3
O
7
−21
(2, −25)

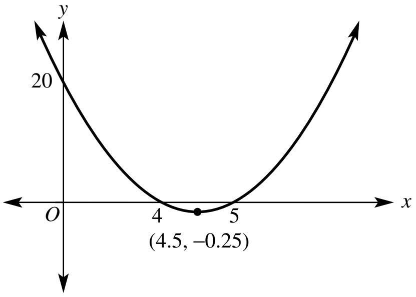
g


y
7
x
−7
−1
O
(−4, −9)

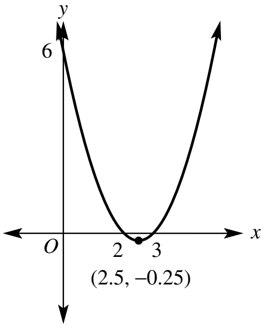
h


y
20
x
O
2
10
(6, −16)

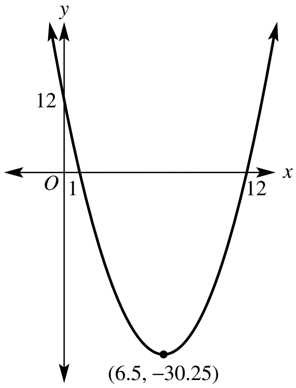
3 a


y
20
x
O
4
5
(4.5, −0.25)

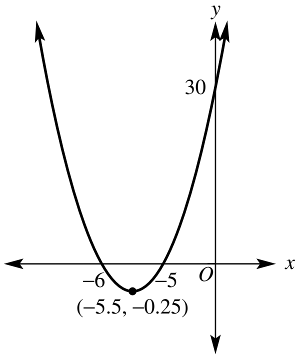
b


y
6
x
O
2
3
(2.5, −0.25)

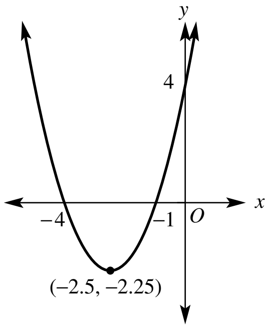
c


y
12
x
O
1
12
(6.5, −30.25)

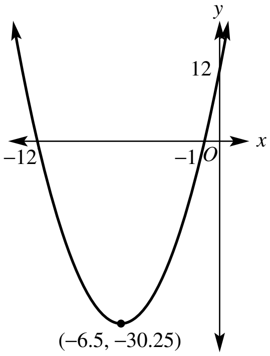
d


y
30
x
−6
−5
O
(−5.5, −0.25)

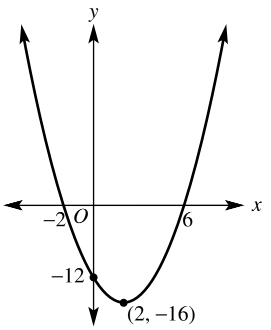
e


y
4
x
−4
−1
O
(−2.5, −2.25)

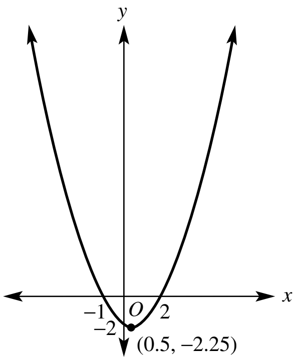
f


y
12
x
−12
−1
O
(−6.5, −30.25)

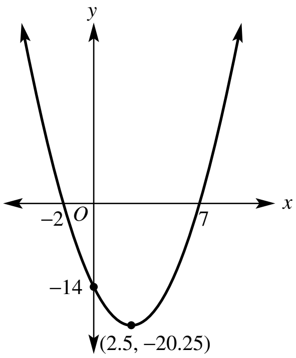
g


y
x
−2
O
6
−12
(2, −16)

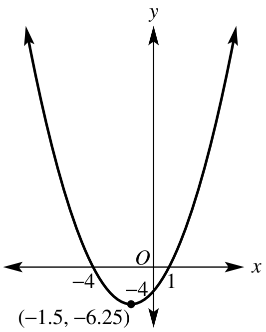
h


y
x
−1
O
2
−2
(0.5, −2.25)

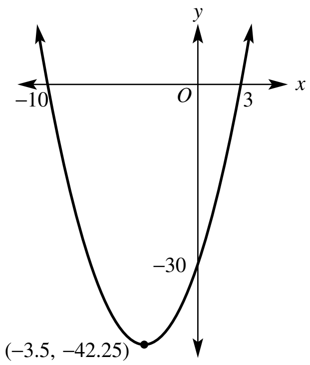
i


y
x
−2
O
7
−14
(2.5, −20.25)

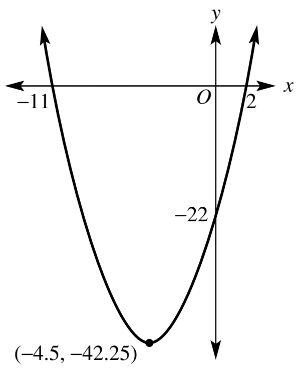
j


y
x
O
−4
−4
1
(−1.5, −6.25)

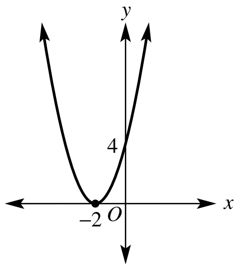
k


y
x
−10
O
3
−30
(−3.5, −42.25)

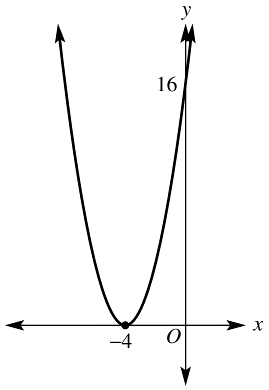
l


y
x
−11
O
2
−22
(−4.5, −42.25)

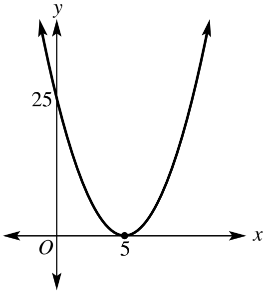
4 a


y
4
x
−2
O

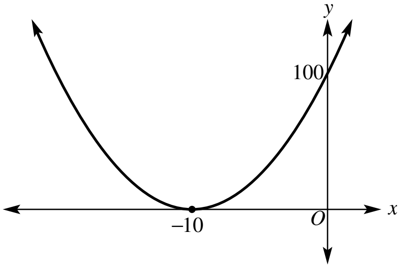
b


y
16
x
−4
O

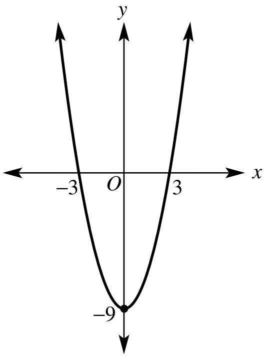
c



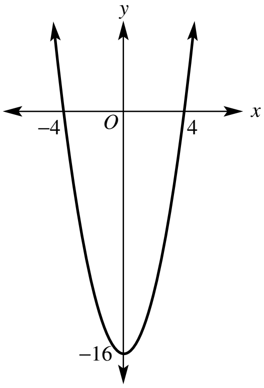
d



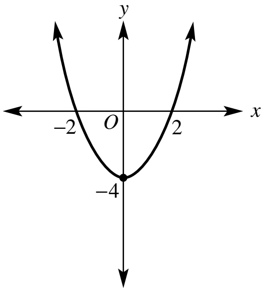
5 **a**



b



c

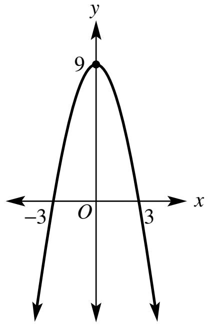


6 **a** (3.5, –4.5) **b** (3.5, –6.75) **c** (–3, –3)
d (–3, –4) **e** (0, –196) **f** (0, 196)
g (1, 0) **h** (1, 0) **i** (1, 8)

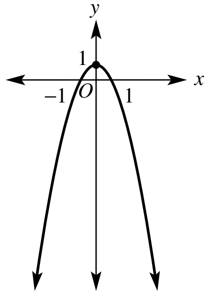
7 **a** $a = -1, b = -3$, TP(2, –1)
b $a = 5, b = -1$, TP(–2, –9)
c $a = 2, b = -6$, TP(2, –16)

8 **a** x-intercepts: $(\sqrt{2}, 0)(-\sqrt{2}, 0)$; TP(0, –2)
b x-intercepts: $(\sqrt{11}, 0)(-\sqrt{11}, 0)$; TP(0, –11)
c x-intercepts: $(\sqrt{5}, 0)(-\sqrt{5}, 0)$; TP(0, –10)

9 **a**



b



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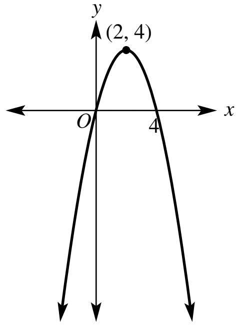

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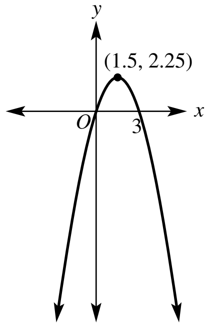
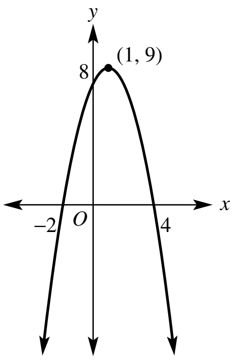
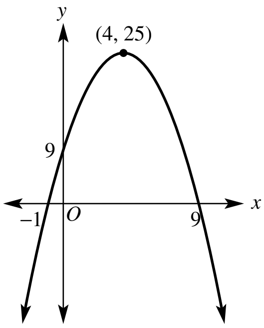


e



f



10 $a = -2$, TP (1, 18)

11 The coefficient does not change the x-intercepts.

12 **a** $y = x^2 - 2x + 1 = (x - 1)^2$. Only one x-intercept, which is the turning point.

b Graph has a minimum (0, 2), therefore its lowest point is 2 units above the x-axis.

13 **a** $x = 4,\ x = -2$

b (1, −9), (1, 9)

c Same x-coordinate, y-coordinate is reflected in the x-axis.

14 **a** (0, 0)

b (0, 0), $(-b, 0)$

c $\left(-\frac{b}{2}, -\frac{b^2}{4}\right)$

15 **a** $y = x(x - 4)$ **b** $y = x(x - 2)$

c $y = x(x + 6)$ **d** $y = (x + 3)(x - 3)$

e $y = (x + 2)(x - 2)$ **f** $y = (x + \sqrt{5})(x - \sqrt{5})$

g $y = 2(x + 4)(x - 2)$ **h** $y = 3(x - 1)(x - 5)$

i $y = 2(x + 1)(x - 3)$ **j** $y = -x(x - 4)$

k $y = \frac{-1}{2}(x + 2)(x - 6)$ **l** $y = -(x - \sqrt{10})(x + \sqrt{10})$

7D

Building understanding

1 **a**

$$\begin{aligned} y &= x^2 + 2x - 5 \\ &= x^2 + 2x + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 - 5 \\ &= (x + 1)^2 - 6 \\ \text{TP} &= (-1, -6) \end{aligned}$$

b

$$\begin{aligned} y &= x^2 - 6x + 10 \\ &= x^2 - 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 10 \\ &= (x - 3)^2 + 1 \\ \text{TP} &= (3, 1) \end{aligned}$$

2 **a** $x = \pm 3$ **b** $x = \pm\sqrt{3}$

c $x = 5, x = -3$ **d** $x = -4 \pm \sqrt{2}$

Now you try

Example 9

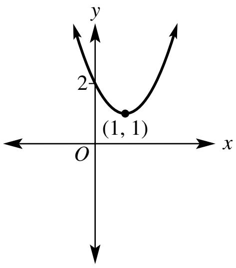
a Maximum at (−1, 18)

b (0, 16)

c (−4, 0) and (2, 0)

Example 10

a



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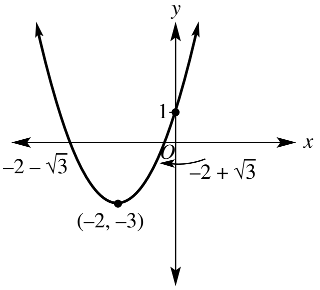

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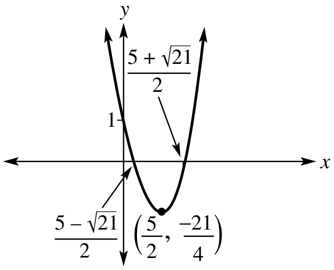


Exercise 7D

1 **a** Min (3, 5) **b** Max (1, 3)

c Max (−1, −2) **d** Min (−2, −5)

e Min (−5, 10) **f** Max (7, 2)

g Max (3, 8) **h** Min (3, −7)

2 **a** (0, 6) **b** (0, −2) **c** (0, 7) **d** (0, 9)

e (0, −16) **f** (0, −55) **g** (0, 3) **h** (0, 1)

i (0, −5) **j** (0, −8) **k** (0, 13) **l** (0, −5)

3 **a** (5, 0), (1, 0) **b** (−7, 0), (−1, 0)

c (9, 0), (−3, 0) **d** $(-2 + \sqrt{5}, 0), (-2 - \sqrt{5}, 0)$

e $(1 + \sqrt{10}, 0), (1 - \sqrt{10}, 0)$ **f** $(5 + \sqrt{3}, 0), (5 - \sqrt{3}, 0)$

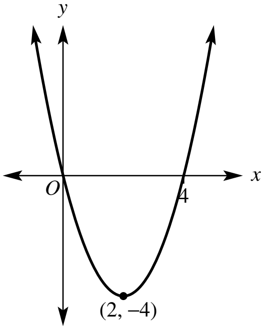
g (4, 0) **h** (−6, 0)

i No x-intercept **j** No x-intercept

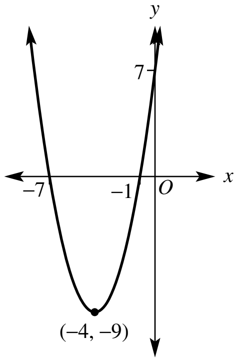
k $(2 + \sqrt{5}, 0), (2 - \sqrt{5}, 0)$ **l** $(3 + \sqrt{10}, 0), (3 - \sqrt{10}, 0)$

4 **a** $(-1, 0), (-5, 0)$
b $(-3 + \sqrt{7}), (-3 - \sqrt{7}, 0)$
c $(-4 + \sqrt{21}, 0), (-4 - \sqrt{21}, 0)$
d $(-1 + \sqrt{7}, 0), (-1 - \sqrt{7}, 0)$
e No x-intercept
f $(6 + \sqrt{41}, 0), (6 - \sqrt{41}, 0)$

5 **a**



b



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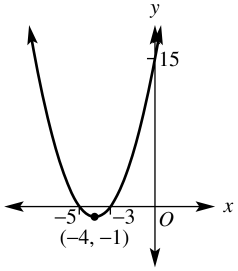

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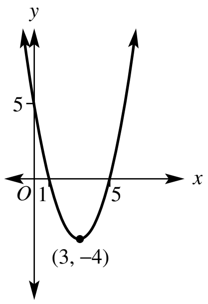

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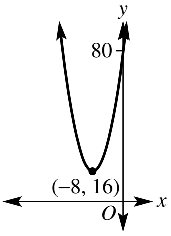
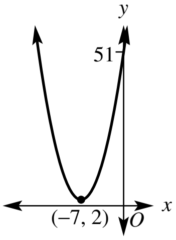


f



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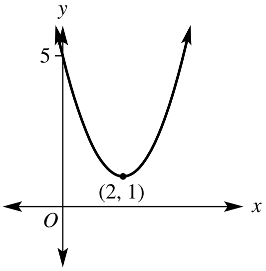

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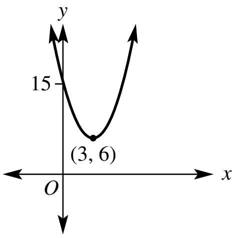

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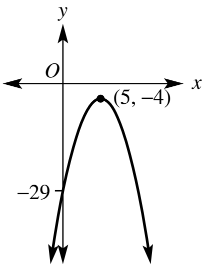

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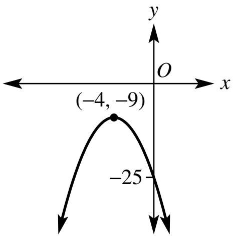

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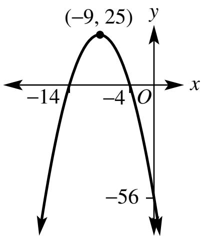
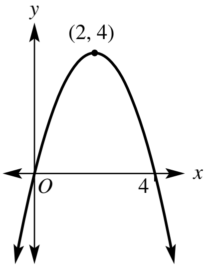
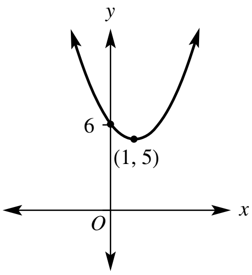
k



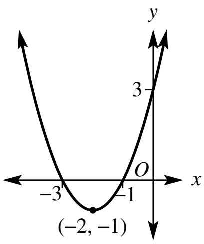
l



6 a



b



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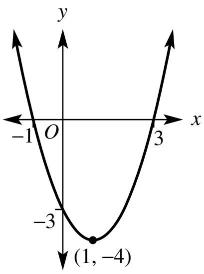

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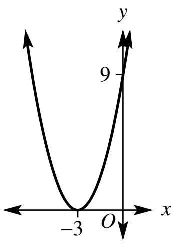

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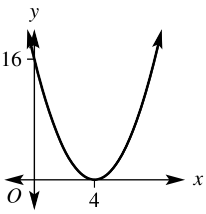

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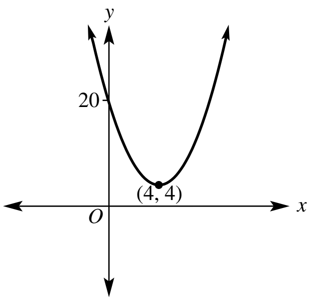

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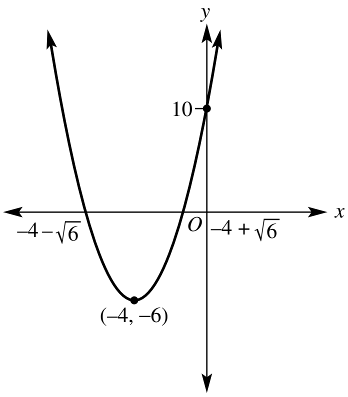

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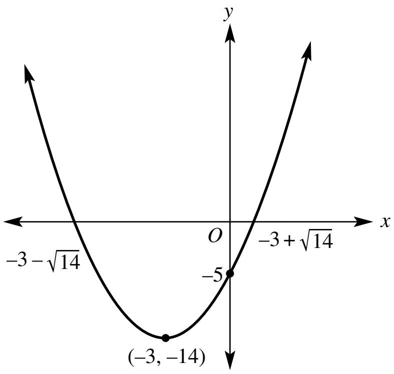

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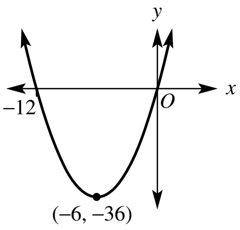

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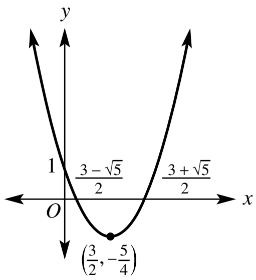
7 **a** 2 **b** 1 **c** 1
d 0 **e** 0 **f** 2

8 **a**



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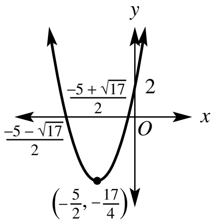
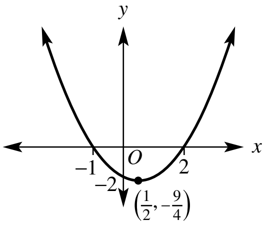


c



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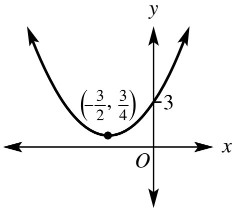
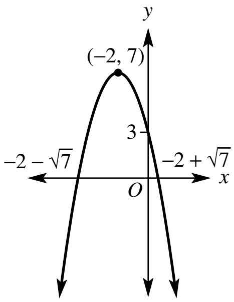
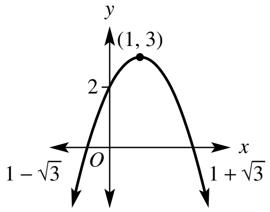


9 **a** $x = -1 \pm \sqrt{6}$ **b** $x = 3, x = 1$
c $x = 7, x = -1$ **d** $x = -2 \pm \sqrt{10}$
e $x = -2 \pm \sqrt{11}$ **f** $x = \frac{3 \pm \sqrt{5}}{2}$

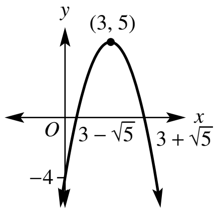
10 **a** $y = -(x - 12)^2 + 7$



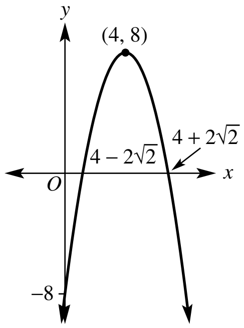
b $y = -(x - 1)^2 + 3$



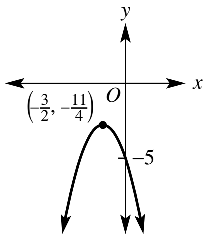
c $y = -(x - 3)^2 + 5$



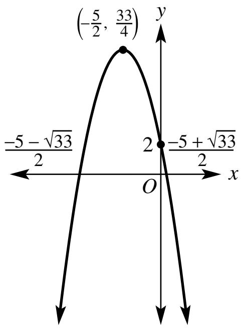
d $y = -(x - 4)^2 + 8$



e $y = -\left(x + \frac{3}{2}\right)^2 - \frac{11}{4}$



f $y = -\left(x + \frac{5}{2}\right)^2 + \frac{33}{4}$



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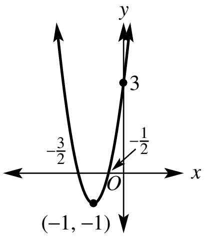
11 **a** $k > 0$

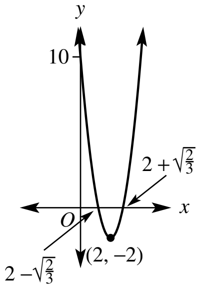
b $k = 0$

c $k < 0$

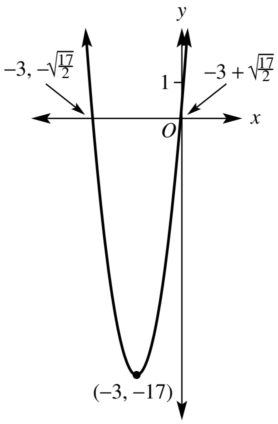
12 **a** $y = 4(x + 1)^2 - 1$



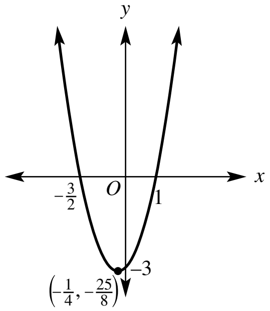
b $y = 3(x - 2)^2 - 2$



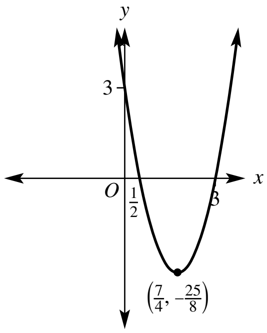
c $y = 2(x + 3)^2 - 17$



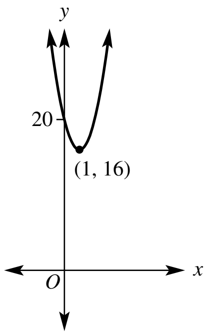
d $y = 2\left(x + \frac{1}{4}\right)^2 - \frac{25}{8}$



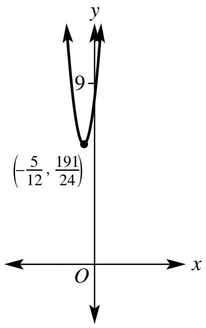
e $y = 2\left(x - \frac{7}{4}\right)^2 - \frac{25}{8}$



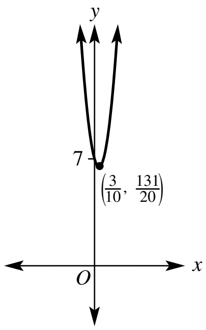
f $y = 4(x - 1)^2 + 16$



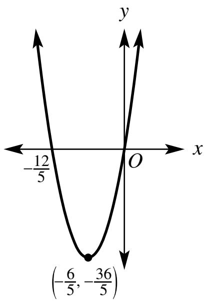
g $y = 6\left(x + \frac{5}{12}\right)^2 + \frac{191}{24}$



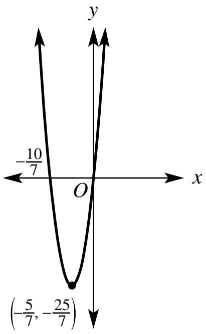
h $y = 5\left(x - \frac{3}{10}\right)^2 + \frac{131}{20}$



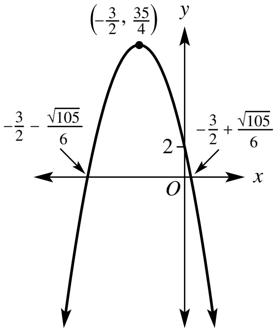
i $5\left(x+\frac{6}{5}\right)^2-\frac{36}{5}$



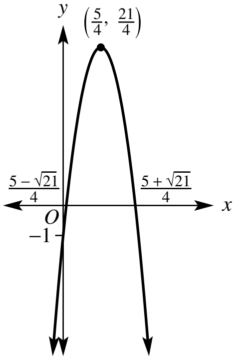
j $y=7\left(x+\frac{5}{7}\right)^2-\frac{25}{7}$



k $y=-3\left(x+\frac{3}{2}\right)^2+\frac{35}{4}$



l $y=-4\left(x-\frac{5}{4}\right)^2+\frac{21}{4}$



13 $$\begin{aligned} x^2+bx+c &= x^2+bx+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c \\ &= \left(x+\frac{b}{2}\right)^2-\frac{b^2}{4}+\frac{4c}{4} \\ &= \left(x+\frac{b}{2}\right)^2-\frac{b^2-4c}{4} \end{aligned}$$

14 a $y=2(x-3)^2+2$
b $y=\frac{1}{2}(x+2)^2+4$
c $y=-2(x+1)^2+2$
d $y=\frac{1}{3}(x-2)^2$
e $y=3(x-2)^2-12$
f $y=-(x+1)^2+5$

7E

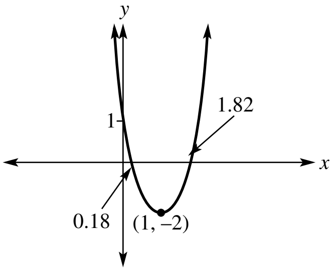
Building understanding

1 a 2 intercepts b 0 intercepts c 1 intercept
2 a $-1\pm\sqrt{2}$ b 2.5, −1 c $\frac{-1\pm\sqrt{17}}{4}$
3 a Zero b Positive c Negative

Now you try

Example 11
a 2 b (0, 1) c (1, −1)

Example 12



Exercise 7E

1 a 1 intercept b 0 intercepts c 2 intercepts
d 2 intercepts e 0 intercepts f 1 intercept
g 2 intercepts h 2 intercepts i 2 intercepts
j 0 intercepts k 2 intercepts l 2 intercepts
2 a (0, 3) b (0, 5) c (0, −2)
d (0, −4) e (0, 8) f (0, −10)
g (0, 0) h (0, 0) i (0, −7)
3 a (−1, 3) b (−2, −5) c (2, −1)
d (1, −5) e $\left(-\frac{3}{2}, 6\frac{1}{4}\right)$ f $\left(\frac{7}{2}, 5\frac{1}{4}\right)$
g $\left(-\frac{3}{4}, -5\frac{1}{8}\right)$ h $\left(\frac{3}{8}, -\frac{9}{16}\right)$ i (0, −9)
j $\left(\frac{1}{4}, -2\frac{3}{4}\right)$ k $\left(-\frac{1}{3}, \frac{1}{3}\right)$ l (0, 2)

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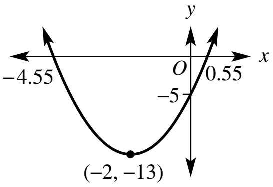
4 **a**



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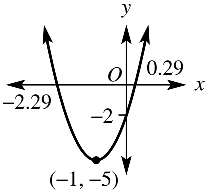

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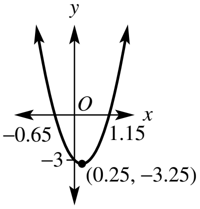

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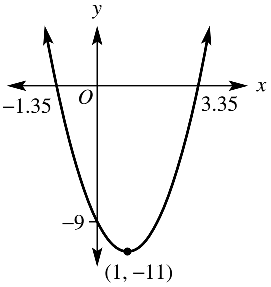

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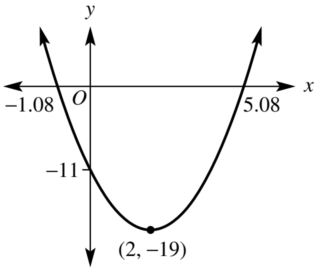

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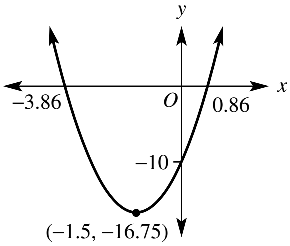

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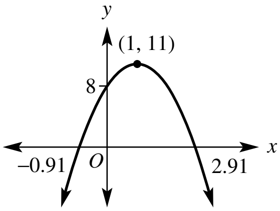

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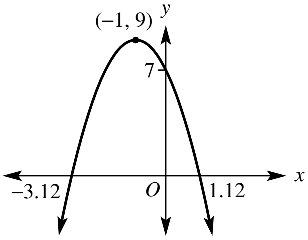

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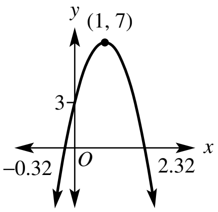

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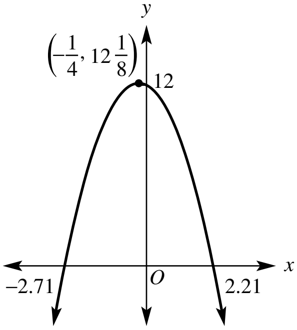

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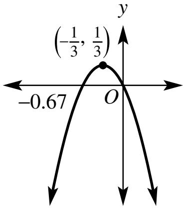

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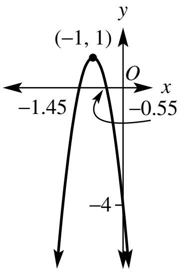

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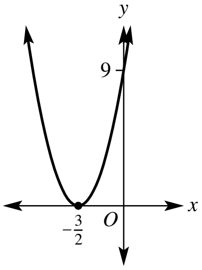
5 **a** $x = 1 \pm \frac{2\sqrt{3}}{3}$ **b** $x = -1 \pm \frac{\sqrt{10}}{2}$

c $x = 1 \pm \frac{\sqrt{10}}{2}$ **d** $x = \frac{-3 \pm \sqrt{15}}{2}$

e $x = 2 \pm \frac{\sqrt{6}}{2}$ **f** $x = 1 \pm \frac{\sqrt{30}}{5}$

6 **a**



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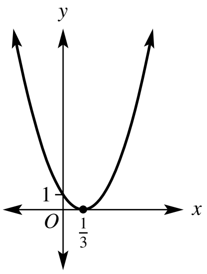

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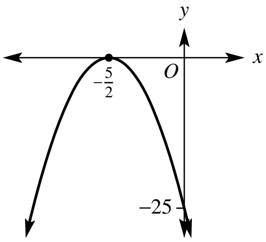

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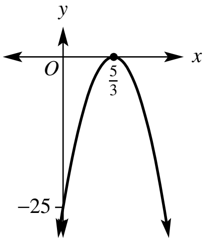

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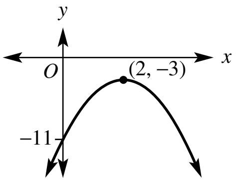

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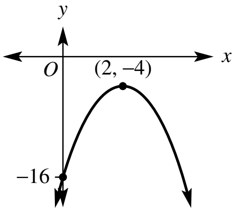

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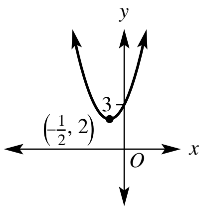

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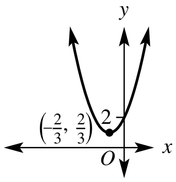


7 $y = (x + 1)^2 - 6 = x^2 + 2x - 5$

8 **a** Anything with $b^2 - 4ac > 0$

b Anything with $b^2 - 4ac = 0$

c Anything with $b^2 - 4ac < 0$

9 Number under square root = 0, therefore $x = \frac{-b}{2a}$ (one solution).

10 $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

11 $y = -\frac{b^2}{4a} + c$

12 $x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0$

$$x^2 + \left(\frac{b}{a}\right) + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ as required

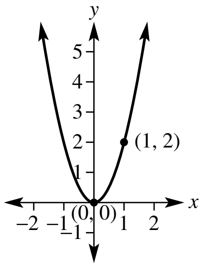
Progress quiz

1 **a** Maximum at (3, 4)

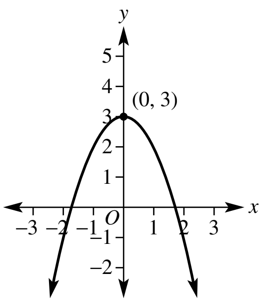
b $x = 3$

c (0, −5), (1, 0) and (5, 0)

2 **a**



b



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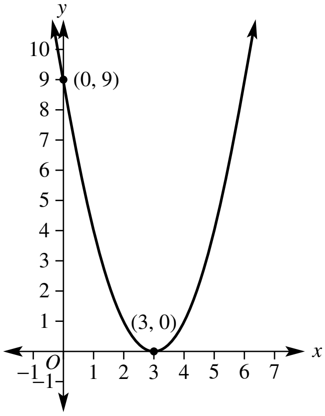
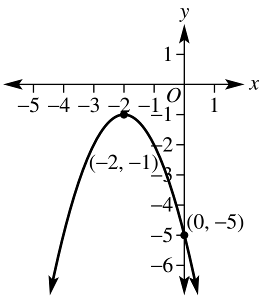
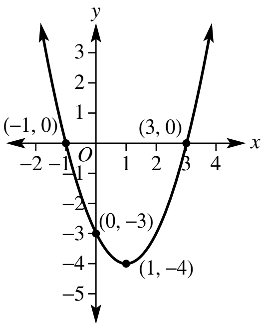


d



3 $y = 3x^2 + 2$

4 **a**



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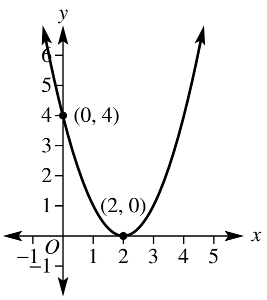


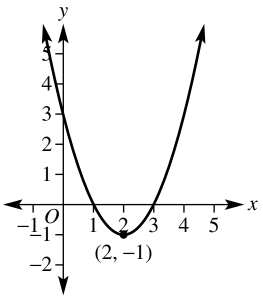
5 $a = 2,\ b = -4;\ y = (x + 2)(x - 4)$; Turning point is at (1, −9).

6 **a** Turning point is a maximum at (3, 8).

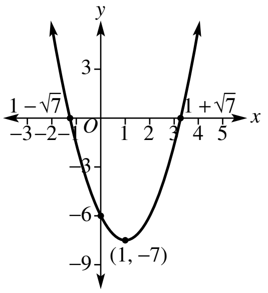
b y-intercept is at (0, −10).

c x-intercepts are at (5, 0) and (1, 0).

7 **a** $y = (x - 2)^2 - 1$



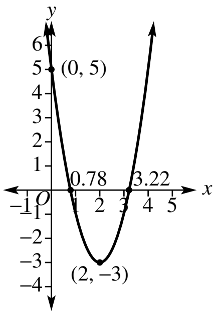
b $y = (x - 1)^2 - 7$



8

	i number of x-intercepts	ii y-intercept	iii Turning point
a	$\Delta < 0$; no x-intercepts	(0, 5)	(2, 1)
b	$\Delta > 0$; two x-intercepts	(0, −7)	(−3, −16)
c	$\Delta = 0$; one x-intercept	(0, −16)	(−4, 0)

9



7F

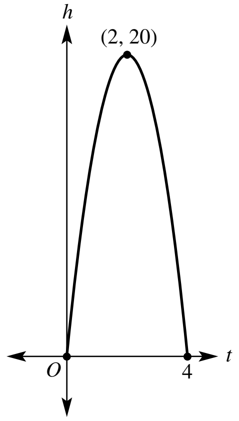
Building understanding

1 **a** 9 seconds **b** 12 metres **c** 20 metres
2 **a** $2x + 2y = 20$ **b** $x(40 - x)$

Now you try

Example 13

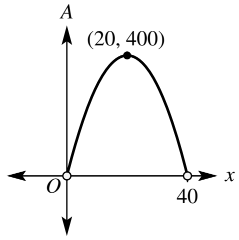
a



b 20 m
c 4 seconds

Example 14

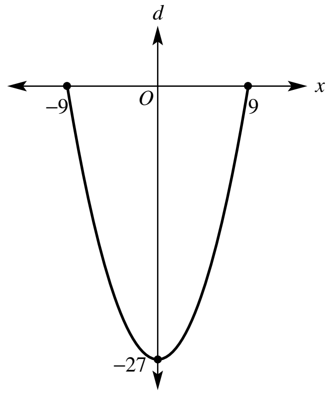
a $40 - x$ **b** $A = x(40 - x)$ **c** $0 < x < 40$
d



e 400 cm^2 **f** Square with side length 20 cm.

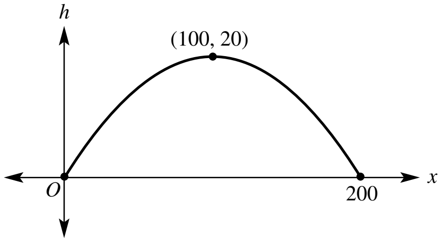
Exercise 7F

1 **a**



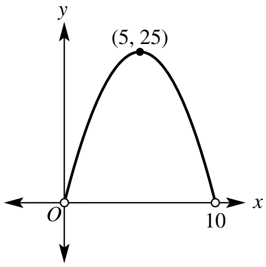
b 18 cm **c** 27 cm

2 **a** (100, 20) **b** 0 and 200
c



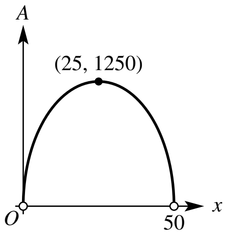
d 200 m **e** 20 m

3 **a** $10 - x$
b $A = x(10 - x)$
c $0 < x < 10$
d



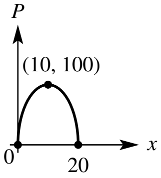
e 25 cm^2 **f** 5 cm by 5 cm

4 **a** $100 - 2x$ **b** $A = x(100 - 2x)$
c $0 < x < 50$
d



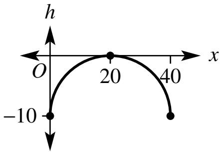
e 1250m^2
f Width = 25 m, length = 50 m

5 **a** $20 - x$
b $P = x(20 - x)$
c



d **i** $x = 0$ or 20 **ii** $x = 10$
e 100

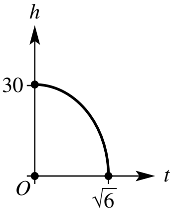
6 **a** (20, 0)
b



c 40 m
d 10 m

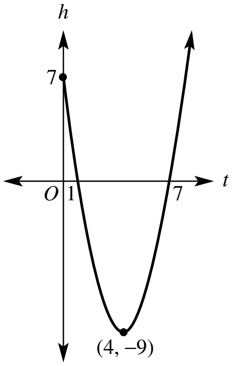
7 **a** $\sqrt{6}$ seconds

b



c $\sqrt{2}$ seconds

8 **a**



b **i** 1 second

ii 7 seconds

iii 4 seconds

c 9 m below sea level

d At 3 and 5 seconds

9 **a** 1 m

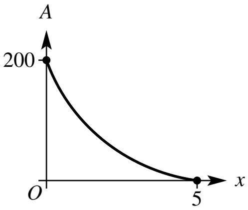
b No, 1 metre is the minimum height the kite falls to.

10 $P = x(64 - x)$ so maximum occurs at $x = 32$.
Maximum product $= 32(64 - 32) = 1024$

11 **a** $A = (20 - 2x)(10 - 2x)$

b Min $x = 0$, max $x = 5$

c



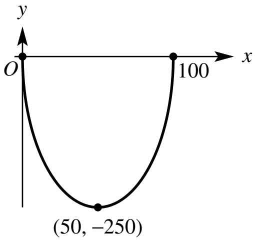
d Turning point occurs for an x-value greater than 5.

e 1 cm

12 **a** 6 m

b No, the maximum height reached is 4.5 m.

13 **a**



b **i** 2

ii None

c **i** (27.6, –200) and (72.4, –200)

ii (1.0, –10) and (99.0, –10)

d The highway meets the edge of the river (50 metres along).

14 $5\frac{1}{24}$m

7G

Building understanding

1 **a** (2, 12) **b** (–1, –3)

2 **a** $x^2 + 3x + 6 = 0$ **b** $x^2 - 5x + 3 = 0$

c $x^2 + 3x - 12 = 0$

3 **a** $b^2 - 4ac < 0$ **b** $b^2 - 4ac > 0$ **c** $b^2 - 4ac = 0$

Now you try

Example 15

a (–1, 2) and (2, 2) **b** No intersection points

Example 16

a (0, 0) and (4, 16) **b** (1, 12)

c (–2, 3) and $\left(\frac{1}{2}, -\frac{3}{4}\right)$

Example 17

(–0.62, –1.85) and (1.62, 4.85)

Example 18

a 2 solutions **b** 0 solutions

Exercise 7G

1 **a** (–3, 6) and (2, 6) **b** (–2, 12) and (6, 12)

c No solutions **d** (–3, –2) and $\left(-\frac{1}{2}, -2\right)$

e $\left(\frac{3}{2}, 0\right)$ **f** No solutions

2 **a** $x = 0, y = 0$ and $x = 3, y = 9$

b $x = 0, y = 0$ and $x = -2, y = 4$

c $x = -3, y = 9$ and $x = 6, y = 36$

d $x = 0, y = 5$ and $x = 3, y = 8$

e $x = -6, y = 34$ and $x = -2, y = 22$

f $x = -2, y = -3$ and $x = 3, y = 17$

g No solutions

h No solutions

i $x = -\frac{9}{2}, y = \frac{65}{2}$ and $x = -1, y = 8$

j $x = -\frac{5}{3}, y = -\frac{25}{3}$ and $x = 3, y = 1$

k $x = -3, y = 6$

l $x = -1, y = 2$

3 **a** $x = -4, y = 16$ and $x = 2, y = 4$

b $x = -1, y = 1$ and $x = 2, y = 4$

c $x = -1, y = 1$ and $x = \frac{1}{3}, y = \frac{1}{9}$

d $x = -2, y = 7$ and $x = -\frac{1}{2}, y = \frac{13}{4}$

e $x = -2, y = 0$ and $x = \frac{2}{3}, y = \frac{16}{9}$

f $x = -8, y = -55$ and $x = 2, y = 5$

4 **a** **i** No solutions

ii $x = -0.7, y = 1.5$ and $x = 2.7, y = 8.5$

   iii $x = -1.4, y = -2.1$ and $x = 0.4, y = 3.1$
   iv $x = -2.6, y = 8.2$ and $x = -0.4, y = 3.8$

 b i $x = \frac{-1 + \sqrt{21}}{2}, y = \frac{-1 + \sqrt{21}}{2}$ and $x = \frac{-1 - \sqrt{21}}{2}, y = \frac{-1 - \sqrt{21}}{2}$
   ii $x = \frac{3 + \sqrt{13}}{2}, y = 3 + \sqrt{13}$ and $x = \frac{3 - \sqrt{13}}{2}, y = 3 - \sqrt{13}$
   iii $x = \frac{-1 + \sqrt{13}}{2}, y = 1 - \sqrt{13}$ and $x = \frac{-1 - \sqrt{13}}{2}, y = 1 + \sqrt{13}$
   iv $x = \frac{-1 + \sqrt{17}}{2}, y = +\sqrt{17}$ and $x = \frac{-1 - \sqrt{17}}{2}, y = -\sqrt{17}$

5 a 2 b 0 c 2 d 0 e 1 f 2

6 Yes, the ball will hit the roof. This can be explained in a number of ways. Using the discriminant, we can see that the path of the ball intersects the equation of roof $y = 10.6$.

7 a $x = -1, y = -2$ and $x = -\frac{1}{2}, y = -\frac{7}{4}$
 b $x = \frac{5}{2}, y = -\frac{15}{4}$ and $x = 2, y = -4$
 c $x = 1, y = 8$ and $x = 2, y = 7$
 d $x = -6, y = -14$ and $x = 2, y = 2$

8 a $(-1, 4)$ and $\left(\frac{1}{2}, 5\frac{1}{2}\right)$ b 212 m

9 a $(3, -4)$
 b i $c > -4$ ii $c = -4$ iii $c < -4$

10 a $1 + 4k$
 b i $k > -\frac{1}{4}$ ii $k = -\frac{1}{4}$ iii $k < -\frac{1}{4}$

11 a Discriminant from resulting equation is less than 0.
 b $k \geq 2$

12 a $m = 2$ or $m = -6$
 b The tangents are on different sides of the parabola, where one has a positive gradient and the other has a negative gradient.
 c $m > 2$ or $m < -6$

7H

Building understanding

1 a Linear
 b Constant, the gradient does not change.
 c Constant

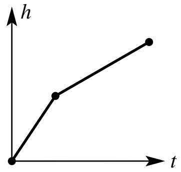
2 a B b A
 c Increasing d Decreasing

3 a positive b negative c zero

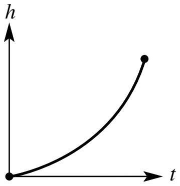
Now you try

Example 19

a



b



Example 20

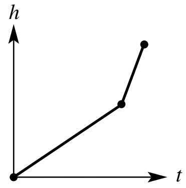
a i E to F ii B to C and E to F
 iii C to D iv A to B and D to E

b Yes, the gradient looks steeper at point D.

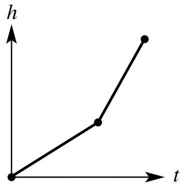
c i A ii D

Exercise 7H

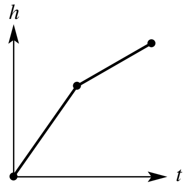
1 a



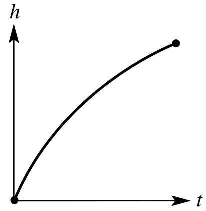
b



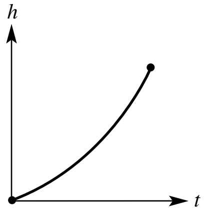
c



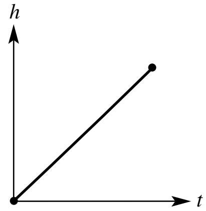
2 a



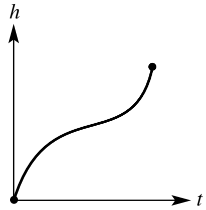
b



c



3 a



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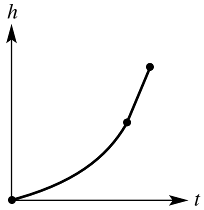
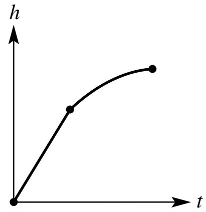


c



4 a i D to E ii B to C and D to E
 iii A to B and E to F iv C to D

b No, the graph looks steeper at point B.

c i B ii C

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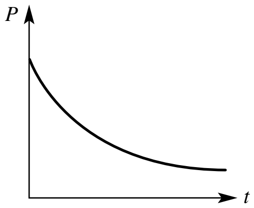
5 **a** **i** C to D
  ii A to B, C to D and E to F
  iii D to E
  iv B to C
b No, the graph looks steeper at B.
c **i** B **ii** E

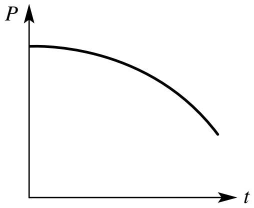
6 **a** Fixed distance from home, zero gradient, stationary.
b Decreasing distance from home, negative constant gradient, lower constant speed.
c Increasing distance from home, positive varying gradient, increasing speed, accelerating.
d Increasing distance from home, positive varying gradient, decreasing speed, decelerating.
e Decreasing distance from home, negative varying gradient, decreasing speed, decelerating.
f Decreasing distance from home, negative varying gradient, increasing speed, accelerating.

7 **A** Positive varying rate of change, increasing speed, accelerating.
B Positive constant rate of change, constant speed.
C Positive varying rate of change, decreasing speed, decelerating.
D Zero rate of change, stationary.
E Negative varying rate of change, increasing speed, accelerating.
F Negative constant rate of change, constant speed.
G Negative varying rate of change, decreasing speed, decelerating.

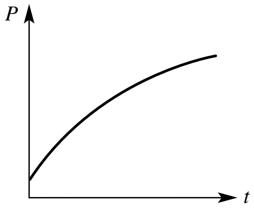
8 **a**



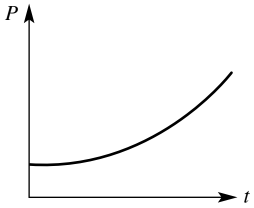
b



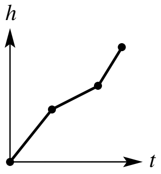
c



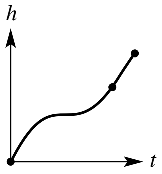
d



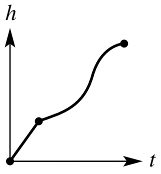
9 **a**



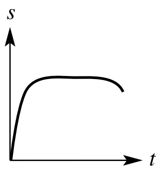
b



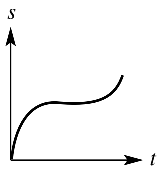
c



10 **a**

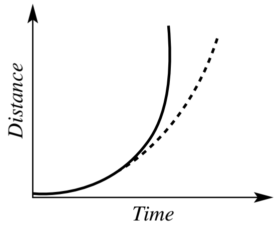


b



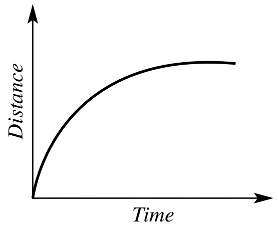
11 Corrected graphs are shown with a dashed line.

a



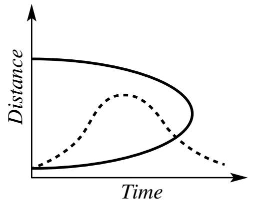
Vertical line incorrect. Can't change distance instantaneously.

b



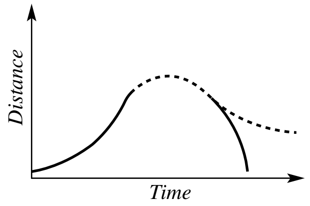
Graph correct.

c



Can't be in two places simultaneously. Curve must increase in gradient, turn, then decrease in gradient.

d



Continuous motion means that no breaks in the curve are possible.
Final deceleration segment needs a curve becoming flatter, showing a decreasing gradient.

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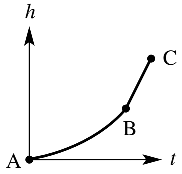
12 A & d: School bus; distance increases at an increasing rate (acceleration), then a constant rate (steady speed) and then a decreasing rate (deceleration) becoming a zero rate (stopped).
B & a: Soccer player; distance increases at a constant rate (steady speed), then a zero rate (stopped) and then at an increasing rate (acceleration).
C & c: Motor bike; distance increases at a constant rate (steady speed), then at an increasing rate (acceleration), and then at a decreasing rate (deceleration) becoming a zero rate (stopped).
D & b: Rocket booster; distance increases at an increasing rate (upward acceleration), then a decreasing rate (deceleration when detached) becoming zero (fleetingly stopped). Distance then decreases at an increasing rate (acceleration towards Earth) and finally distance decreases at a constant rate (steady fall to Earth with parachute).

13 **a** The join at B should be smooth.
b



14 Various solutions; check with your teacher.

7I

Building understanding

1 **a** 1 **b** 1
c AC, the gradient of AC is very close to the gradient of the tangent.
2 **a** 3 **b** 3 km/h
c AC, the gradient of AC is very close to the gradient of the tangent.

Now you try

Example 21
a -3 **b** 1

Example 22
a **i** $-\frac{4}{5}$ **ii** $-\frac{3}{2}$
b BC, the gradient of BC is closer to the gradient of the tangent.

Exercise 7I

1 **a** 4 **b** 3
2 **a** 0 **b** 3
3 **a** 1 **b** -2
4 **a** -1 **b** 9
5 **a** **i** $\frac{1}{3}$ **ii** 1
b *BC*, the gradient of *BC* is closer to the gradient of the tangent.
6 **a** **i** $-\frac{3}{2}$ **ii** -2
b *BC*, the gradient of *BC* is closer to the gradient of the tangent.
7 **a** **i** -1 **ii** $\frac{1}{2}$
b *BD*, the gradient of *BD* is closer to the gradient of the tangent.
8 40 km/h
9 3 km/h
10 8 m/s
11 **a** **i** 10 L **ii** 0 L
b -2 L/sec
c **i** Approx. -1.6 L/sec
ii Approx. -3.2 L/sec
12 **a** *AB*, the gradient of *AB* is closer to the gradient of the tangent.
b Approx. 0.5
13 **a** **i** 3 **ii** 2.5 **iii** 2.1
b 1 and 1.1, the gradient is closer to the gradient of the tangent at $x = 1$.
c Choose $x = 1$ and another x value even closer to $x = 1$ like $x = 1.01$.
14 **a** Approx. 1.4 **b** Approx. -0.25
c Approx. 12 **d** Approx. 0.35
15 **a** 6 **b** 4 **c** -4
d 8 **e** 9 **f** -32

7J

Building understanding

1 **a** Direct proportion **b** Inverse proportion
c Inverse proportion **d** Direct proportion
e Neither **f** Inverse proportion
2 **a** Straight line with y-intercept; neither direct nor inverse (indirect) proportion.
b Straight line starting at (0, 0); direct proportion.
c Upward sloping curve so as x increases, y increases; neither direct nor inverse (indirect) proportion.
d Hyperbola shape so as x increases, y decreases; inverse (indirect) proportion.

Now you try

Example 23
a $a = \frac{3}{2}b$ **b** 60 **c** 8

Example 24
a $k = 60, n = \frac{60}{m}$ **b** 5 **c** 2

Exercise 7J

1 **a** **i** $m = 5h$ **ii** 30 **iii** $\frac{8}{5}$
b **i** $p = 4q$ **ii** $p = 60$ **iii** $q = 25$
c **i** $p = 50q$ **ii** $p = 750$ **iii** $q = 4$

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2 **a** **i** $k = 24, y = \frac{24}{x}$
    ii 3
    iii 2
  b **i** $k = 72, y = \frac{72}{x}$
    ii $y = 2$
    iii $x = 24$
  c **i** $k = 50, y = \frac{50}{x}$
    ii $y = 0.5$
    iii $x = 0.5$

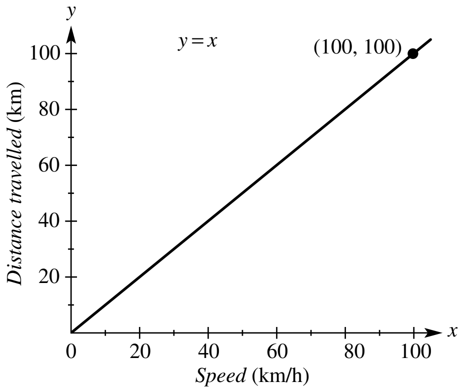
3 **a** $d = 80t$
  b Directly
  c **i** 320 km **ii** 2.5 hours

4 **a** $s = \frac{20}{t}$
  b Inversely
  c **i** 8 km/h **ii** 2 hours

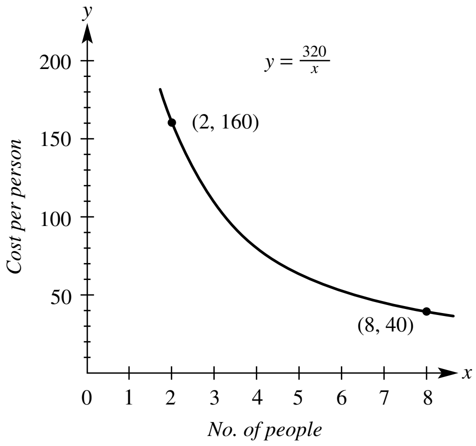
5 **a** $k =$ \$244/tonne
  b $P = 244n$
  c \$33 184
  d 1175 tonnes

6 **a** $c = \frac{74}{s}$ **b** \$4.93 **c** \$2.47

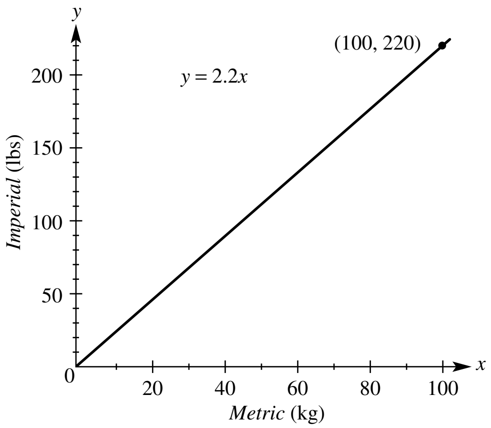
7 **a**



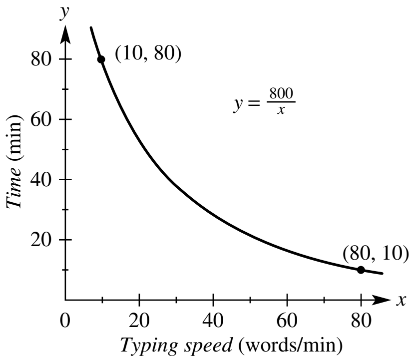
  b



  c



  d



8 **a** *D* **b** *D* **c** *I* **d** *I*
  e *D* **f** *D* **g** *D* **h** *I*

9 **a** $C = 200 + 0.5n$
  b \$230
  c 85

10 **a** $\frac{1}{2}$ **b** Increase **c** Decrease

11 **a** $t = \frac{kp}{n}$
  b $k = \frac{1}{200}$
  c **i** 5 hours **ii** 10 people

Problems and challenges

1 **a** $-\frac{2}{3} \leqslant x \leqslant \frac{1}{2}$
  b $x < -\frac{3}{4}$ or $x > \frac{1}{3}$
  c $\frac{7 - \sqrt{41}}{2} < x < \frac{7 + \sqrt{41}}{2}$

2 **a** $b^2 - 4ac < 0$ **b** $b^2 - 4ac = 0$ **c** $b^2 - 4ac > 0$

3 **a** $k = \frac{1}{3}$ **b** $k < \frac{1}{3}$ **c** $k > \frac{1}{3}$

4 **a** $k = \pm\sqrt{20} = \pm 2\sqrt{5}$
  b $k > 2\sqrt{5}$ or $k < -2\sqrt{5}$
  c $-2\sqrt{5} < k < 2\sqrt{5}$

5 **a** $y = -(x + 1)(x - 3)$
  b $y = \frac{3}{4}(x + 2)^2 - 3$
  c $y = x^2 - 2x - 3$

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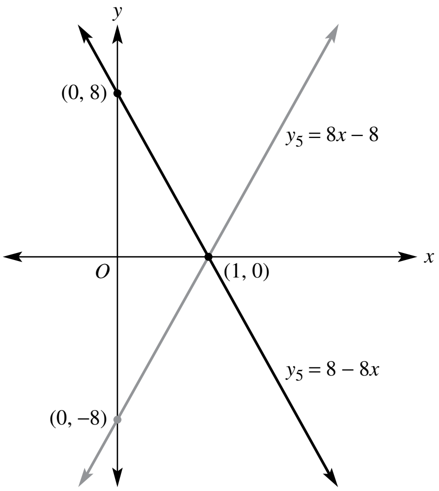
6 $a = 2, b = -3, c = -8$; TP $\left(\frac{3}{4}, -\frac{73}{8}\right)$

7 20

8 $4\sqrt{3}$

9



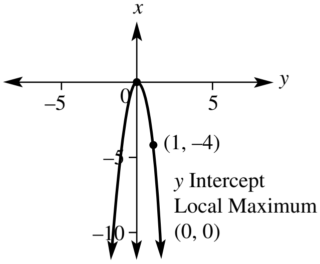
Answers to success criteria example questions

1 Turning point is a maximum at (2, 9)
Axis of symmetry is $x = 2$
x-intercepts are (−1, 0) and (5, 0)
y-intercept is (0, 5)

2

Formula	Max or min	Turning point	y-value when $x = 1$
$y = -x^2 + 2$	Max	(0, 2)	1

3



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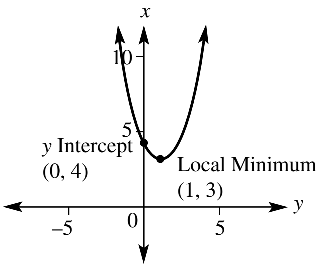
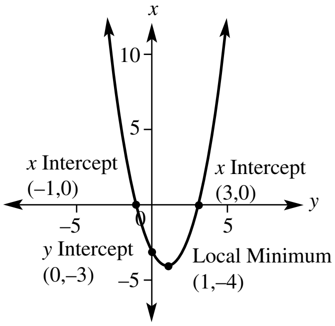
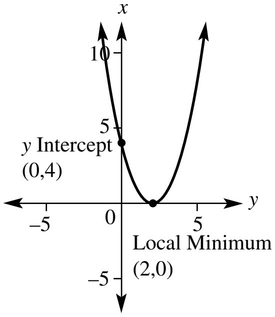


5 $y = 2x^2 - 1$

6

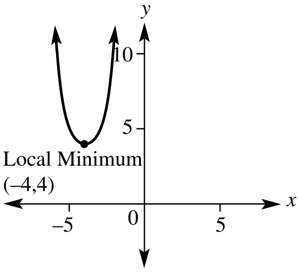


7



8 Turning point is at (1, −18)
x-intercepts are (−2, 0) and (4, 0)
y-intercept is (0, −16)

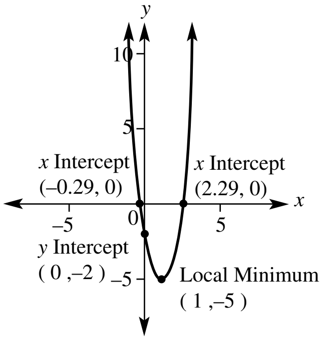
9



10 There are no x-intercepts

11 (−2, −13)

12



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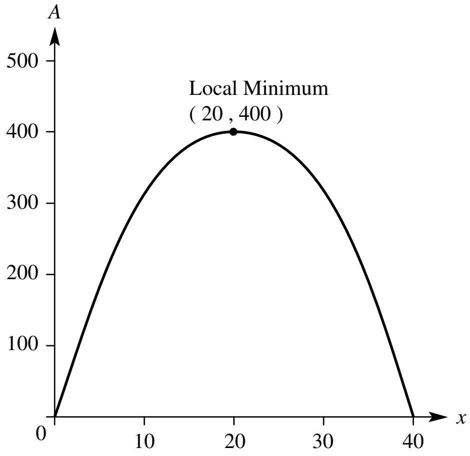
13 **i** $L = 40 - x$; **ii** $A = x(40 - x)$

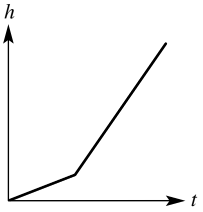


iii Maximum area is 400cm^2 when $x = 20$, width = 20 cm and length = 20 cm.

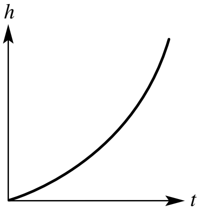
14 $x = 2$, $y = 3$ and $x = -1.5$, $y = -0.5$

15 There is one solution to the pair of equations.

16 **a**



b



17 **i** A to B and B to C **ii** *D* to *E*

iii *D*

18 Average rate of change = 4

19 $m_{AB} = 1$; $m_{AC} = \frac{5}{3}$; *AC* gives a better approximation as the gradient of *AC* is closer to the instantaneous rate of change at *B*.

20 72

21 $k = 20$, $y = \frac{20}{x}$

Short-answer questions

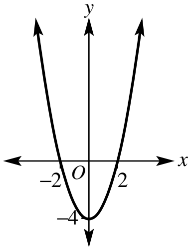
1 **a** Minimum at (1, −4) **b** $x = 1$

c (−1, 0) and (3, 0) **d** (0, −3)

2 **a** Minimum at (2, 0) **b** Maximum at (0, 5)

c Maximum at (−1, −2) **d** Minimum at (3, 4)

3 **a**



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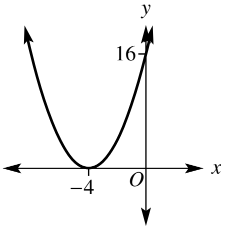

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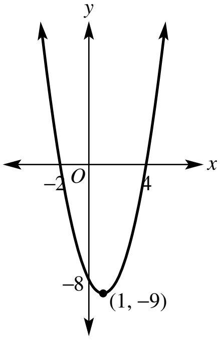
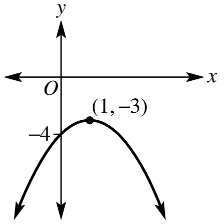


4 **a** **i** Maximum at (1, −3) **ii** (0, −4)

iii No x-intercepts

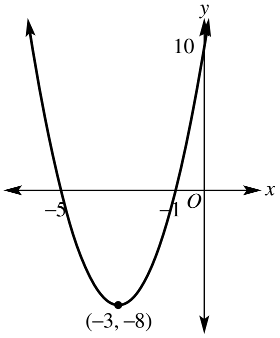
iv



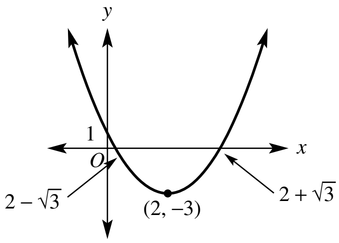
b **i** Minimum at (−3, −8) **ii** (0, 10)

iii (−1, 0) and (−5, 0)

iv



5 **a**



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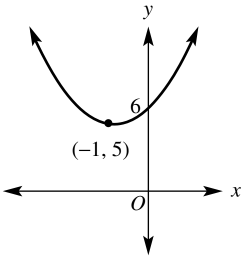

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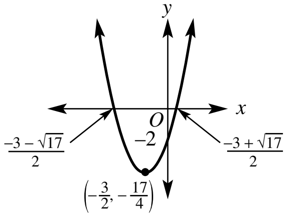
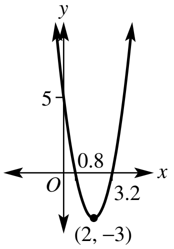
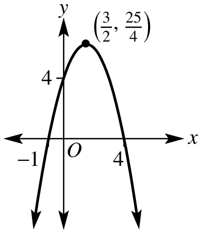


6 **a** 1 **b** 0 **c** 2 **d** 0

7 **a** **i** (0, 5)
ii (2, −3)
iii 0.8 and 3.2
iv



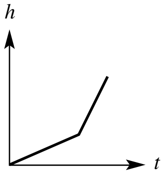
b **i** (0, 4) **ii** $\left(\frac{3}{2}, \frac{25}{4}\right)$ **iii** −1 and 4
iv



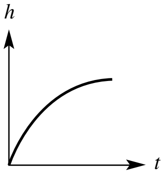
8 **a** $x = 2, y = 10$ and $x = -6, y = 10$
b No solutions
c $x = \frac{1}{3}, y = \frac{10}{9}$ and $x = -1, y = 2$

9 Show $b^2 - 4ac = 0$.

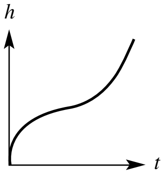
10 **a**



b



c



11 Increasing distance from home, decreasing positive gradient and decreasing speed.

12 **a** **i** 4 **ii** 5
b **i** −4 **ii** −5

13 AC, −1

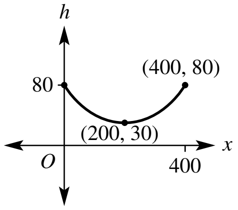
14 **a** $y = 5x$
b **i** $k = 72$ **ii** 18 **iii** $\frac{720}{7}$

Multiple-choice questions

1	B	**2**	D	**3**	E	**4**	D	**5**	A	**6**	A
7	C	**8**	D	**9**	A	**10**	C	**11**	B	**12**	E

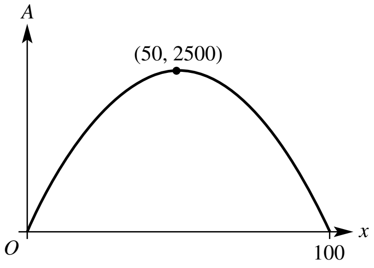
Extended-response questions

1 **a** (200, 30)
b $0 \leqslant x \leqslant 400$
c $30 \leqslant h \leqslant 80$
d



e 400 m
f 30 m
g 80 m

2 **a** $100 - x$
b $A = x(100 - x)$
c $0 < x < 100$
d



e 2500 m^2
f 50 m by 50 m

Chapter 8

8A

Building understanding

1 **a** 2 **b** {H, T} **c** Yes
d $\frac{1}{2}$ **e** $\frac{1}{2}$ **f** 1

2 **a** $\frac{1}{4}$ **b** $\frac{1}{6}$ **c** $\frac{3}{8}$ **d** 0

3 **a** 4 **b** 20 **c** 100

Now you try

Example 1

a $\frac{1}{11}$ **b** $\frac{2}{11}$ **c** $\frac{9}{11}$ **d** $\frac{3}{11}$

Example 2

a 0.1 **b** 0.09 **c** 0.87 **d** 0.9

Exercise 8A

1 **a** $\frac{1}{7}$ **b** $\frac{2}{7}$ **c** $\frac{5}{7}$ **d** $\frac{3}{7}$

2 **a** $\frac{3}{10}$ **b** $\frac{2}{5}$ **c** $\frac{3}{5}$ **d** $\frac{1}{2}$

3 **a** $\frac{1}{10}$ **b** $\frac{1}{2}$ **c** $\frac{1}{2}$ **d** $\frac{1}{2}$
e $\frac{2}{5}$ **f** $\frac{1}{5}$ **g** $\frac{3}{10}$ **h** $\frac{1}{10}$

4 **a** 0.09 **b** 0.43 **c** 0.47 **d** 0.91

5 **a** 0.62 **b** 0.03 **c** 0.97 **d** 0.38

6 **a** $\frac{1}{50}$ **b** $\frac{3}{10}$ **c** $\frac{49}{50}$

7 **a** $\frac{1}{2}$ **b** $\frac{3}{8}$ **c** $\frac{1}{4}$ **d** $\frac{5}{24}$ **e** 1 **f** 0

8 **a** $\frac{6}{25}$ **b** $\frac{1}{50}$ **c** $\frac{21}{25}$ **d** $\frac{2}{5}$ **e** $\frac{2}{25}$ **f** $\frac{4}{25}$

9 **a** **i** $\frac{7}{10}$ **ii** $\frac{1}{5}$ **iii** $\frac{1}{20}$
iv 0 **v** $\frac{1}{20}$
b $\frac{1}{10}$

10 **a** 59
b 4, as $\frac{41}{100}$ of 10 is closest to 4.
c 8, as $\frac{41}{100}$ of 20 is closest to 8.

11 **a** $\frac{1}{4}$ **b** $\frac{1}{13}$ **c** $\frac{1}{52}$ **d** $\frac{1}{2}$
e $\frac{2}{13}$ **f** $\frac{4}{13}$ **g** $\frac{12}{13}$ **h** $\frac{9}{13}$

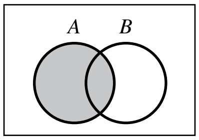
12 **a** $\frac{7}{15}$
b 15; any multiple of 15 is a possibility as 3 and 5 must be factors.

13 **a** 625π
b **i** 25π **ii** 200π **iii** 400π
c **i** $\frac{1}{25}$ **ii** $\frac{8}{25}$ **iii** $\frac{16}{25}$ **iv** $\frac{9}{25}$
v $\frac{24}{25}$ **vi** $\frac{17}{25}$ **vii** 1 **viii** $\frac{17}{25}$
d No, it doesn't.

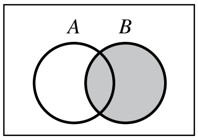
8B

Building understanding

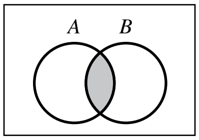
1 **a**



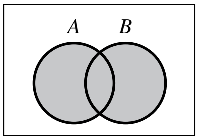
b



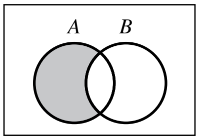
c



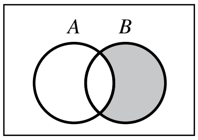
d



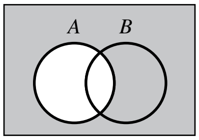
e



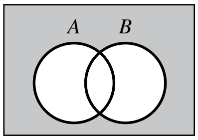
f



g



h



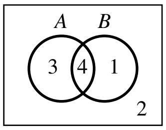
2 **a** $\varnothing$ **b** $\cap$ **c** $\cup$ **d** $\cup, \cup$

3 **a** No **b** Yes **c** No

Now you try

Example 3

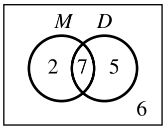
a



b **i** {2, 3, 5, 7}
ii {1, 2, 3, 4, 5, 6, 7, 8}
c **i** $\frac{7}{10}$ **ii** $\frac{2}{5}$ **iii** $\frac{4}{5}$
d No, $A \cap B \neq \varnothing$

Example 4

a



b **i** 2 **ii** 6
c **i** $\frac{9}{20}$ **ii** $\frac{1}{4}$ **iii** $\frac{7}{20}$

Example 5

a

	A	***A′***	
B	3	2	5
B′	4	1	5
	7	3	10

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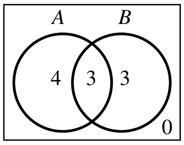
b **i** 3 **ii** 2 **iii** 4 **iv** 1
v 7 **vi** 5 **vii** 9

c **i** $\frac{3}{10}$ **ii** $\frac{3}{10}$ **iii** $\frac{2}{5}$

Exercise 8B

1 **a**

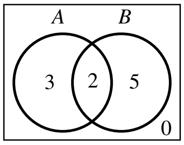


b **i** $A \cap B = \{2, 5, 8\}$
ii $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

c **i** $\frac{7}{10}$ **ii** $\frac{3}{10}$ **iii** 1

d No, since $A \cap B \neq \varnothing$

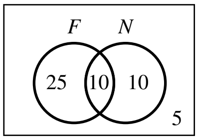
2 **a**



b **i** $A \cap B = \{2, 13\}$
ii $A \cup B = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$

c **i** $\frac{1}{2}$ **ii** $\frac{7}{10}$ **iii** $\frac{1}{5}$ **iv** 1

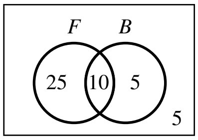
3 **a**



b **i** 25 **ii** 5

c **i** $\frac{2}{5}$ **ii** $\frac{1}{5}$ **iii** $\frac{1}{5}$

4 **a**



b **i** 25 **ii** 5

c **i** $\frac{7}{9}$ **ii** $\frac{2}{9}$ **iii** $\frac{8}{9}$ **iv** $\frac{2}{9}$ **v** $\frac{1}{9}$

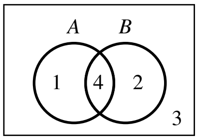
5 **a**

	A	**A'**	
B	2	6	8
B'	5	3	8
	7	9	16

b **i** 2 **ii** 6 **iii** 5 **iv** 3
v 7 **vi** 8 **vii** 13 **viii** 16

c **i** $\frac{1}{8}$ **ii** $\frac{9}{16}$ **iii** $\frac{5}{16}$

6 **a**



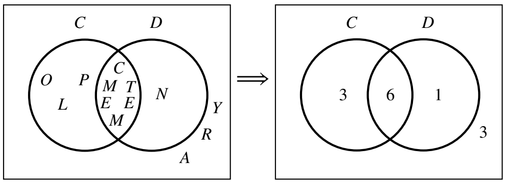
b

	A	**A'**	
B	4	2	6
B'	1	3	4
	5	5	10

c **i** 2 **ii** 3 **iii** $\frac{2}{5}$ **iv** $\frac{7}{10}$

7 **a** 4 **b** 10, 12 **c** a, c, e **d** Nothing

8 **a**



b **i** $\frac{9}{13}$ **ii** $\frac{6}{13}$ **iii** $\frac{10}{13}$
iv $\frac{4}{13}$ **v** $\frac{3}{13}$

9 **a**

	A	**A'**	
B	3	3	6
B'	4	1	5
	7	4	11

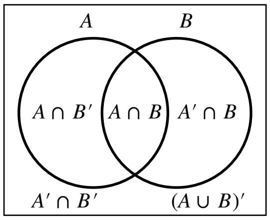
b

	A	**A'**	
B	2	7	9
B'	2	1	3
	4	8	12

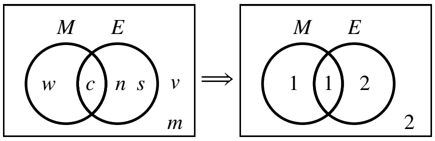
10 3

11 **a** $1 - a$ **b** $a + b$ **c** 0

12

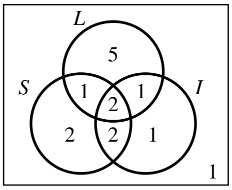


13 **a**



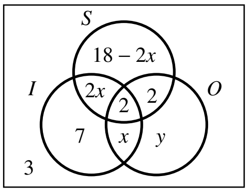
b **i** $\frac{1}{3}$ **ii** $\frac{2}{3}$ **iii** $\frac{1}{6}$ **iv** $\frac{2}{3}$ **v** $\frac{1}{3}$

14 **a**



b 1

c **i** $\frac{3}{5}$ **ii** $\frac{1}{3}$ **iii** $\frac{13}{15}$ **iv** $\frac{1}{15}$

15 a


S = Own state
I = Interstate
O = Overseas

b i 4 ii 10

c i $\frac{5}{19}$ ii $\frac{1}{19}$ iii $\frac{7}{38}$

iv $\frac{35}{38}$ v $\frac{25}{38}$

8C

Building understanding

1 a i {4, 5, 6} ii {2, 4, 6}

iii {2, 4, 5, 6} iv {4, 6}

b No, $A \cap B \neq \varnothing$

c $\frac{2}{3}$

2 a 0.8 b 0.7

3 0.05

Now you try

Example 6

a i 13 ii 4 iii 1

b i $\frac{1}{4}$ ii $\frac{3}{4}$ iii $\frac{1}{52}$

c $\frac{4}{13}$

d $\frac{10}{13}$

Example 7

a 0.35 b 0.25

Exercise 8C

1 a i 13 ii 4 iii 1

b i $\frac{1}{4}$ ii $\frac{3}{4}$ iii $\frac{1}{52}$

c $\frac{4}{13}$

d $\frac{10}{13}$

2 a i {3, 6, 9, 12, 15, 18}

ii {2, 3, 5, 7, 11, 13, 17, 19}

b i $\frac{1}{20}$ ii $\frac{13}{20}$

c $\frac{7}{20}$

3 $\frac{7}{10}$

4 a 0.1 b 0.2

5 a 0.3 b 0.1

6 $\frac{7}{12}$

7 a $\frac{3}{8}$ b $\frac{5}{32}$

8 a $\frac{4}{13}$ b $\frac{4}{13}$ c $\frac{7}{13}$

d $\frac{49}{52}$ e $\frac{10}{13}$ f $\frac{10}{13}$

9 a 0.4 b 0.45

10 Because $\Pr(A \cap B) = 0$ for mutually exclusive events.

11 a $\Pr(A) < \Pr(A \cap B)$ b $\Pr(A) + \Pr(B) < \Pr(A \cup B)$

12 $\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C)$

13 a $\frac{3}{10}$ b $\frac{1}{4}$ c $\frac{3}{20}$

d $\frac{13}{20}$ e $\frac{9}{20}$ f $\frac{3}{5}$

14 a $\frac{1}{4}$ b $\frac{71}{500}$ c $\frac{33}{500}$

d $\frac{7}{100}$ e $\frac{1}{25}$ f $\frac{7}{500}$

8D

Building understanding

1 a $\frac{1}{3}$ b $\frac{1}{2}$

2 a i 2 ii 9

b $\frac{2}{9}$

3 a i 7 ii 10

b $\frac{7}{10}$

c $\frac{7}{12}$

Now you try

Example 8

a $\frac{7}{13}$ b $\frac{3}{13}$ c $\frac{3}{8}$ d $\frac{3}{7}$

Example 9

a

	A	**A'**	
B	9	6	15
B'	4	4	8
	13	10	23

b $\frac{4}{23}$ c $\frac{3}{5}$ d $\frac{9}{13}$

Exercise 8D

1 a i $\frac{9}{13}$ ii $\frac{3}{13}$ iii $\frac{3}{7}$ iv $\frac{1}{3}$

b i $\frac{14}{17}$ ii $\frac{4}{17}$ iii $\frac{4}{7}$ iv $\frac{2}{7}$

c i $\frac{3}{4}$ ii $\frac{5}{8}$ iii $\frac{5}{7}$ iv $\frac{5}{6}$

d i $\frac{7}{16}$ ii $\frac{1}{8}$ iii $\frac{1}{4}$ iv $\frac{2}{7}$

2 a i $\frac{7}{18}$ ii $\frac{1}{9}$ iii $\frac{1}{5}$ iv $\frac{2}{7}$

b i $\frac{4}{9}$ ii $\frac{1}{9}$ iii $\frac{1}{5}$ iv $\frac{1}{4}$

c i $\frac{8}{17}$ ii $\frac{7}{17}$ iii $\frac{7}{10}$ iv $\frac{7}{8}$

d i $\frac{3}{4}$ ii $\frac{1}{4}$ iii $\frac{2}{3}$ iv $\frac{1}{3}$

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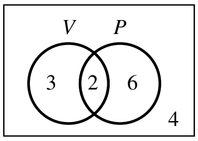
3 a

	A	A'	
B	9	6	15
B'	4	1	5
	13	7	20

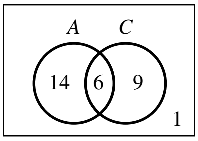
b $\frac{1}{5}$ **c** $\frac{3}{5}$ **d** $\frac{9}{13}$

4 a



b 4 **c** $\frac{2}{5}$ **d** $\frac{1}{4}$

5 a



	A	A'	
C	6	9	15
C'	14	1	15
	20	10	30

b i $\frac{3}{10}$ **ii** $\frac{7}{15}$

c $\frac{2}{5}$

d $\frac{3}{10}$

6 a

	A	A'	
B	2	2	4
B'	3	1	4
	5	3	8

i 1 **ii** $\frac{2}{5}$ **iii** $\frac{1}{2}$

b

	A	A'	
B	3	13	16
B'	5	6	11
	8	19	27

i 6 **ii** $\frac{3}{8}$ **iii** $\frac{3}{16}$

7 a $\frac{1}{13}$ **b** $\frac{1}{13}$ **c** $\frac{1}{4}$ **d** $\frac{1}{2}$

8 a $\frac{1}{3}$ **b** $\frac{1}{2}$

9 $\Pr(A \mid B) = \Pr(B \mid A) = 0$ as $\Pr(A \cap B) = 0$

10 a 1 **b** $\frac{1}{5}$

11 a $\Pr(A \cap B) = \Pr(A) \times \Pr(B \mid A)$

b 0.18

12 a 329 **b** $\frac{174}{329}$ **c** $\frac{81}{329}$

d $\frac{24}{155}$ **e** $\frac{31}{231}$ **f** $\frac{18}{31}$

8E

Building understanding

1 a i 9 **ii** 6

b i $\frac{1}{3}$ **ii** $\frac{5}{9}$ **iii** $\frac{4}{9}$

c i 0 **ii** $\frac{2}{3}$ **iii** $\frac{1}{3}$

2 a 9 **b** 6

Now you try

Example 10

a $\frac{1}{36}$

b i $\frac{1}{18}$ **ii** $\frac{5}{18}$ **iii** $\frac{1}{12}$

c $\frac{1}{3}$

Example 11

a

		1st			
		R	R	B	W
2nd	R	X	(R, R)	(B, R)	(W, R)
	R	(R, R)	X	(B, R)	(W, R)
	B	(R, B)	(R, B)	X	(W, B)
	W	(R, W)	(R, W)	(B, W)	X

b i $\frac{1}{6}$ **ii** $\frac{1}{6}$ **iii** $\frac{1}{3}$ **iv** $\frac{1}{5}$

Exercise 8E

1 a

		1st roll			
		1	2	3	4
2nd roll	1	(1, 1)	(2, 1)	(3, 1)	(4, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)
	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)

b 16 **c** $\frac{1}{16}$

d i $\frac{1}{4}$ **ii** $\frac{5}{8}$ **iii** $\frac{13}{16}$

2 a

		1st toss	
		H	T
2nd toss	H	(H, H)	(T, H)
	T	(H, T)	(T, T)

b 4 **c** $\frac{1}{4}$

d i $\frac{1}{2}$ **ii** $\frac{3}{4}$

e 250

3 a

		1st		
		S	E	T
2nd	S	X	(E, S)	(T, S)
	E	(S, E)	X	(T, E)
	T	(S, T)	(E, T)	X

b i $\frac{1}{6}$ **ii** $\frac{2}{3}$ **iii** $\frac{2}{3}$ **iv** $\frac{1}{3}$ **v** 1

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4 **a**

		1st				
		L	E	V	E	L
2nd	L	X	(E, L)	(V, L)	(E, L)	(L, L)
	E	(L, E)	X	(V, E)	(E, E)	(L, E)
	V	(L, V)	(E, V)	X	(E, V)	(L, V)
	E	(L, E)	(E, E)	(V, E)	X	(L, E)
	L	(L, L)	(E, L)	(V, L)	(E, L)	X

b 20

c **i** 8 **ii** 12 **iii** 12

d **i** $\frac{2}{5}$ **ii** $\frac{3}{5}$ **iii** $\frac{3}{5}$

e $\frac{1}{5}$

5 **a**

		Die 1					
		1	2	3	4	5	6
Die 2	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

b 36

c **i** 2 **ii** 6 **iii** 15

d **i** $\frac{1}{6}$ **ii** $\frac{1}{6}$ **iii** $\frac{35}{36}$ **iv** $\frac{1}{12}$

e $\frac{1}{6}$. Min's guess is wrong.

6 **a**

		1st		
		O	L	D
2nd	C	(O, C)	(L, C)	(D, C)
	O	(O, O)	(L, O)	(D, O)
	L	(O, L)	(L, L)	(D, L)
	L	(O, L)	(L, L)	(D, L)
	E	(O, E)	(L, E)	(D, E)
	G	(O, G)	(L, G)	(D, G)
	E	(O, G)	(L, G)	(D, G)

b 21

c $\frac{1}{7}$

7 **a** **i** 100 **ii** 90

b **i** $\frac{1}{10}$ **ii** $\frac{1}{10}$ **iii** $\frac{4}{5}$

c $\frac{19}{100}$

8 **a** **i** $\frac{1}{4}$ **ii** $\frac{5}{8}$

b **i** $\frac{2}{5}$ **ii** $\frac{1}{10}$ **iii** $\frac{2}{3}$

9 **a** Without **b** With **c** With **d** Without

10 **a** 30

b **i** $\frac{1}{15}$ **ii** $\frac{1}{15}$ **iii** $\frac{2}{15}$ **iv** $\frac{4}{15}$

c $\frac{1}{18}$

11 **a**

		1st			
		2.5	5	10	20
2nd	2.5	5	7.5	12.5	22.5
	5	7.5	10	15	25
	10	12.5	15	20	30
	20	22.5	25	30	40

b 16

c **i** 1 **ii** 8 **iii** 8

d **i** $\frac{1}{16}$ **ii** $\frac{1}{8}$ **iii** $\frac{1}{4}$ **iv** $\frac{3}{16}$

e $\frac{7}{16}$

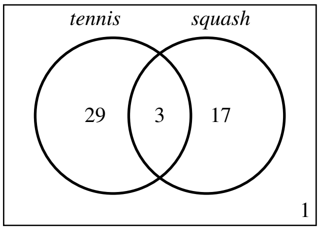
Progress quiz

1 **a** $\frac{1}{10}$ **b** $\frac{1}{10}$ **c** $\frac{1}{20}$ **d** $\frac{2}{5}$ **e** $\frac{3}{5}$

2 **a** 0.17 **b** 0.29 **c** 0.33 **d** 0.62

3 **a**

	Like tennis	**Dislike tennis**	**Total**
Like squash	3	17	20
Dislike squash	29	1	30
Total	32	18	50



b 29

c $\frac{1}{50}$

4 **a** 7 **b** 20 **c** 5

d $\frac{7}{33}$ **e** $\frac{13}{33}$ **f** $\frac{28}{33}$

5 0.91

6 **a** 0.83 **b** 0.17

7 **a** $\frac{1}{2}$ **b** $\frac{1}{6}$ **c** $\frac{1}{4}$ **d** $\frac{1}{3}$

8 **a**

	Like soft drink	**Dislike soft drink**	**Total**
Like water	15	5	20
Dislike water	20	0	20
Total	35	5	40

b $\frac{1}{4}$

9 **a** $\frac{5}{9}$

b $\frac{1}{6}$

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8F

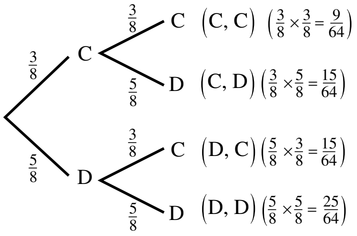
Building understanding

1 **a** **i** $\frac{2}{5}$ **ii** $\frac{3}{5}$

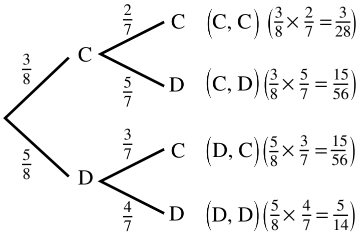
b **i** $\frac{2}{5}$ **ii** $\frac{3}{5}$

c **i** $\frac{1}{4}$ **ii** $\frac{3}{4}$

2 **a**



b



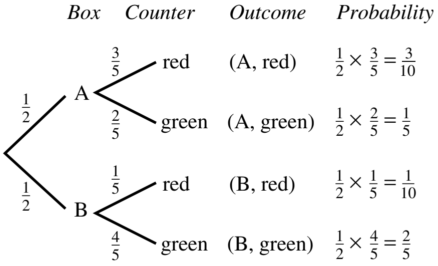
Now you try

Example 12

a $\frac{3}{5}$

b $\frac{1}{5}$

c

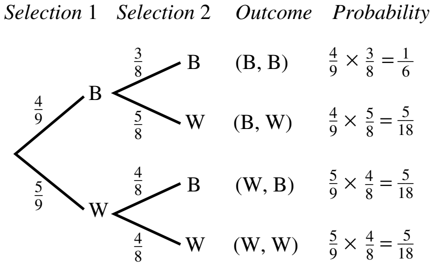


d $\frac{1}{10}$

e $\frac{2}{5}$

Example 13

a



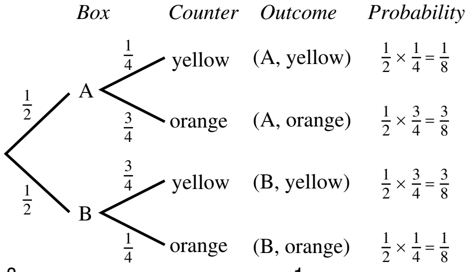
b **i** $\frac{5}{18}$ **ii** $\frac{1}{6}$ **iii** $\frac{5}{9}$

c **i** $\frac{20}{81}$ **ii** $\frac{16}{81}$ **iii** $\frac{40}{81}$

Exercise 8F

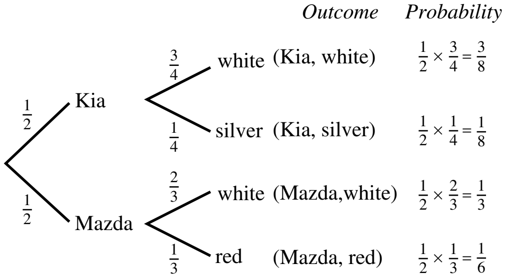
1 **a** $\frac{1}{4}$ **b** $\frac{3}{4}$

c



d $\frac{3}{8}$ **e** $\frac{1}{2}$

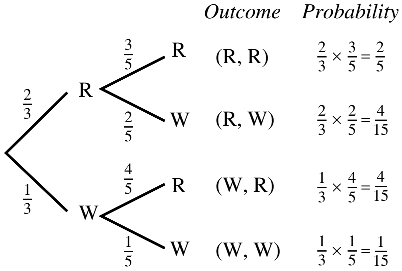
2 **a**



b **i** $\frac{3}{8}$ **ii** $\frac{1}{6}$ **iii** $\frac{17}{24}$

iv $\frac{7}{24}$ **v** $\frac{5}{6}$ **vi** $\frac{1}{3}$

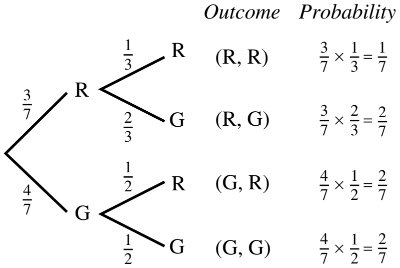
3 **a**



b **i** $\frac{4}{15}$ **ii** $\frac{2}{5}$ **iii** $\frac{8}{15}$

c **i** $\frac{2}{9}$ **ii** $\frac{4}{9}$ **iii** $\frac{4}{9}$

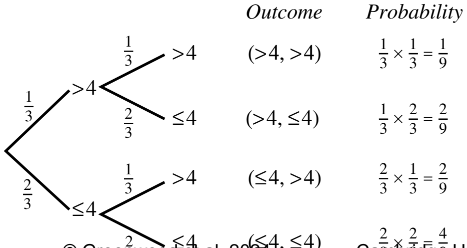
4 **a**



i $\frac{1}{7}$ **ii** $\frac{2}{7}$ **iii** $\frac{4}{7}$ **iv** $\frac{3}{7}$

b **i** $\frac{9}{49}$ **ii** $\frac{16}{49}$ **iii** $\frac{24}{49}$ **iv** $\frac{25}{49}$

5 **a**



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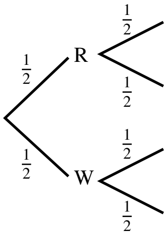
b **i** $\frac{1}{9}$ **ii** $\frac{4}{9}$ **iii** $\frac{5}{9}$ **iv** $\frac{4}{9}$

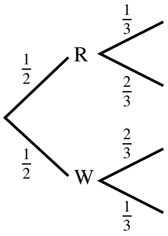
6 **a**



Outcome	*Probability*
(R, R)	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
(R, W)	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
(W, R)	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
(W, W)	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

i $\frac{1}{4}$ **ii** $\frac{1}{2}$ **iii** $\frac{3}{4}$ **iv** $\frac{3}{4}$

b

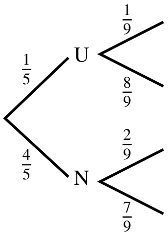


Outcome	*Probability*
(R, R)	$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$
(R, W)	$\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$
(W, R)	$\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$
(W, W)	$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

i $\frac{1}{6}$ **ii** $\frac{2}{3}$ **iii** $\frac{5}{6}$ **iv** $\frac{5}{6}$

7 **a** **i** $\frac{1}{5}$ **ii** $\frac{4}{5}$

b



Outcome	*Probability*
(U, U)	$\frac{1}{5} \times \frac{1}{9} = \frac{1}{45}$
(U, N)	$\frac{1}{5} \times \frac{8}{9} = \frac{8}{45}$
(N, U)	$\frac{4}{5} \times \frac{2}{9} = \frac{8}{45}$
(N, N)	$\frac{4}{5} \times \frac{7}{9} = \frac{28}{45}$

i $\frac{1}{45}$ **ii** $\frac{16}{45}$ **iii** $\frac{44}{45}$

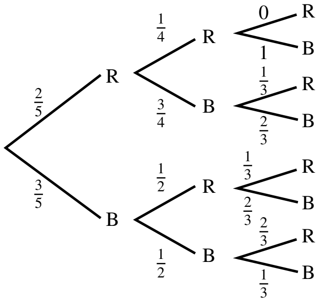
c 62.2%

8 **a** **i** 0.17 **ii** 0.11 **iii** 0.83
b **i** 0.1445 **ii** 0.0965 **iii** 0.8555

9 **a** $\frac{3}{7}$ **b** $\frac{4}{7}$

10 **a** **i** 0.21 **ii** 0.30 **iii** 0.50
b **i** 0.21 **ii** 0.30 **iii** 0.50
c Answer are the same correct to two decimal places. This is because there is a large supply of each flavour so choosing one can without replacement does not have a big effect on the probability of the flavour of the second can.

11 **a**

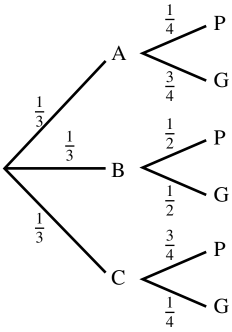


Outcome	*Probability*
(R, R, R)	0
(R, R, B)	$\frac{1}{10}$
(R, B, R)	$\frac{1}{10}$
(R, B, B)	$\frac{1}{5}$
(B, R, R)	$\frac{1}{10}$
(B, R, B)	$\frac{1}{5}$
(B, B, R)	$\frac{1}{5}$
(B, B, B)	$\frac{1}{10}$

i $\frac{1}{10}$ **ii** $\frac{3}{10}$ **iii** 0
iv $\frac{9}{10}$ **v** $\frac{9}{10}$

b **i** 1 **ii** $\frac{2}{5}$

12 **a**



Outcome	*Probability*
(A, P)	$\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$
(A, G)	$\frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$
(B, P)	$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$
(B, G)	$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$
(C, P)	$\frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$
(C, G)	$\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$

b 6
c **i** $\frac{1}{12}$ **ii** $\frac{1}{6}$ **iii** $\frac{1}{4}$
d $\frac{1}{2}$

13 **a** **i** $\frac{7}{8}$ **ii** $\frac{1}{8}$
b \$87.50 to player A, \$12.50 to player B.
c **i** A\$68.75, B\$31.25 **ii** A\$50, B\$50
iii A\$81.25, B\$18.75 **iv** A\$34.38, B\$65.62
v Answers may vary.

8G

Building understanding

1 **a** **i** $\frac{1}{2}$ **ii** $\frac{1}{2}$
b Yes
c $\frac{1}{2}$

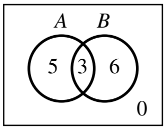
2 **a** **i** $\frac{3}{10}$ **ii** $\frac{1}{3}$
b No
c No

3 **a** with **b** without

Now you try

Example 14

a



b **i** $\frac{4}{7}$ **ii** $\frac{1}{3}$
c No, $\Pr(A \mid B) \neq \Pr(A)$

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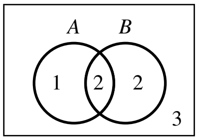
Exercise 8G

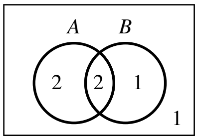
1 a



b i $\frac{3}{8}$ ii $\frac{1}{2}$

c Not independent

2 a



b i $\frac{2}{3}$ ii $\frac{2}{3}$

c Independent

3 a i $\Pr(A) = \frac{3}{4}, \Pr(A \mid B) = \frac{1}{2}$ ii Not independent

b i $\Pr(A) = \frac{1}{4}, \Pr(A \mid B) = \frac{1}{4}$ ii Independent

c i $\Pr(A) = \frac{1}{3}, \Pr(A \mid B) = \frac{1}{3}$ ii Independent

d i $\Pr(A) = \frac{2}{7}, \Pr(A \mid B) = 0$ ii Not independent

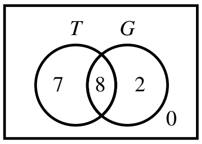
4 a $\Pr(A) = \frac{1}{2}, \Pr(A \mid B) = \frac{1}{2}$, independent

b $\Pr(A) = \frac{3}{10}, \Pr(A \mid B) = \frac{1}{4}$, not independent

c $\Pr(A) = \frac{5}{12}, \Pr(A \mid B) = \frac{3}{20}$, not independent

d $\Pr(A) = \frac{1}{9}, \Pr(A \mid B) = \frac{1}{9}$, independent

5 a



	T	T'	
G	8	2	10
G'	7	0	7
	15	2	17

i $\frac{15}{17}$ ii $\frac{7}{17}$ iii $\frac{4}{5}$

b No

6 a $\frac{1}{32}$ b $\frac{31}{32}$ c $\frac{31}{32}$

7 a $\frac{1}{216}$ b $\frac{1}{216}$ c $\frac{1}{72}$ d $\frac{1}{36}$

8 False; $\Pr(A \mid B) = 0$ but $\Pr(A) = \frac{2}{9}$.

9 a 6 b 22 c 2

10 a 0.24 b 0.76

11 $\frac{5}{6}$

8H

Building understanding

1 a i 11 ii Addition

b i 28 ii Multiplication

2 a 6 b 24 c 120 d 20

Now you try

Example 15

a Multiplication, 18 b Addition, 13

Example 16

a 6 b 20 c 24

Exercise 8H

1 a 9 b 6 c 16 d 24

2 a 6 b 18 c 50 d 120

3 a Addition, 34 b Multiplication, 6

4 a 6 b 120 c 2 d 720

e 4 f 210 g 360 h 120

i 2 j 24 k 1 l $\frac{1}{6}$

5 12

6 60

7 60

8 a 10 000 b 17 576 c 15 625

9 No, the multiplication principle should be used.

10 No, the addition principle should be used.

11 a 20 b 10 c 30 d 9900

12
$$\begin{aligned}\frac{100!}{2!98!}\times\frac{197!}{198!} &= \frac{100\times 99\times 98!}{2\times 98!}\times\frac{197!}{198\times 197!}\\ &= \frac{100\times 99}{2}\times\frac{1}{198}\\ &= \frac{100\times 99}{2}\times\frac{1}{99\times 2}\\ &= 25\end{aligned}$$

13
$$\begin{aligned}\frac{8!(x+3)!}{7!(x+2)!} &= \frac{8\times 7!\times (x+3)\times (x+2)!}{7!\times (x+2)!}\\ &= 8\times (x+3)\\ &= 8x+24\end{aligned}$$

14
$$\begin{aligned}\frac{a!}{(a-3)!}\times\frac{1}{a^2-a} &= \frac{a(a-1)(a-2)(a-3)!}{(a-3)!}\times\frac{1}{a(a-1)}\\ &= a-2\end{aligned}$$

8I

Building understanding

1 120 2 12 3 12 4 120

Now you try

Example 17

a 720 b 20

Example 18

a 36 b 1680

Exercise 8I

1 a 6 b 5040

c 362 880 d 479 001 600

2 a 120 b 120

c 840 d 151 200

3 a 40 320 b 336

4 a 720 b 360

5 720
6 a 12 b 4320
7 10 080
8 a 20 b 3024
9 665 280
10 a 720 b 1000
11 50 400
12 a 17 280 b 5760
13 a 60 b 125
14 a 12 b 24 c 42
15 276

8J

Building understanding

1 a AB, AC, AD, BC, BD, CD
BA, CA, DA, CB, DB, DC
b AB, AC, AD, BC, BD, CD
c 12
d 6
e 6
2 a 24
b ABC, ABD, ACD, BCD
c 4
d 4

Now you try

Example 19
a 10 b 35 c 1 d 45

Example 20
35

Example 21
525

Exercise 8J

1 a 10 b 28 c 12 d 1
2 a 15 b 330 c 1 d 15
3 21
4 1365
5 84
6 a 30 b 60 c 200
7 66 528
8 a 50 b 19 900 c 120
9 a The number of ways of choosing one object from n available is n. Also $^nC_1 = \frac{n!}{1!(n-1)!} = \frac{n \times (n-1)!}{(n-1)!} = n$
b The number of ways of choosing zero objects from n available is 1, that is, to choose none of them.
Also $^nC_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1$
c The number of ways of choosing n objects from n available is 1, that is, to choose all of them.
Also $^nC_n = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = 1$
10 a 56 b 30
11 a 60 b 76
12 a 32 b 2^n
13 63
14 a 1820 b 675 c 1605

8K

Building understanding

1 a 30 b $\frac{1}{30}$
2 a 24 b $\frac{1}{24}$
3 a 35 b $\frac{1}{35}$

Now you try

Example 22
a $\frac{1}{120}$ b $\frac{1}{20}$

Example 23
a $\frac{1}{165}$ b $\frac{1}{75}$

Exercise 8K

1 a $\frac{1}{5040}$ b $\frac{1}{210}$
2 a $\frac{1}{120}$ b $\frac{1}{60}$
3 $\frac{1}{336}$
4 a $\frac{1}{792}$ b $\frac{1}{336}$
5 a $\frac{1}{84}$ b $\frac{1}{40}$
6 $\frac{1}{504}$
7 $\frac{1}{48}$
8 $\frac{1}{840}$
9 a $\frac{1}{8008}$ b $\frac{1}{3300}$ c $\frac{1}{462}$
10 a $\frac{1}{5040}$ b $\frac{1}{10\,000}$
11 a $\frac{1}{30}$ b $\frac{1}{15}$
12 a $\frac{1}{6}$ b $\frac{1}{36}$ c $\frac{5}{18}$ d $\frac{7}{12}$
13 a $\frac{1}{143}$ b $\frac{56}{143}$ c $\frac{17}{143}$ d $\frac{17}{143}$
14 a $\frac{1}{120}$ b $\frac{1}{360}$ c $\frac{2}{3}$ d $\frac{13}{18}$

Problems and challenges

1 a 0.16 b 0.192 c 0.144
2 0.593 75
3 a $\frac{7}{8}$ b 1 c $\frac{4}{7}$
4 a $\frac{1}{12}$ b $\frac{1}{2}$ c $\frac{3}{4}$ d $\frac{2}{3}$
5 $\frac{63}{64}$
6 $\frac{1}{13\,983\,816}$
7 $\frac{1}{12}$

8 $\frac{8}{9}$

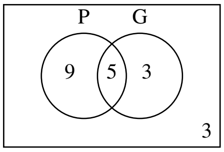
9 5

Answers to success criteria example questions

1 $\frac{2}{9}$

2 $\frac{7}{50}$

3



; $\frac{3}{20}$

4

	A	A'	
B	2	3	5
B'	4	1	5
	7	3	10

$n(A' \cap B) = 3$; $\Pr(A \cup B) = \frac{9}{10}$

5 $\frac{7}{13}$

6 0.1

7 $\Pr(A\,|\,B) = \frac{1}{2}$; $\Pr(B\,|\,A) = \frac{3}{7}$

8

	A	A'	
B	2	3	5
B'	4	2	6
	6	5	11

$\Pr(A\,|\,B) = \frac{2}{5}$

9

	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

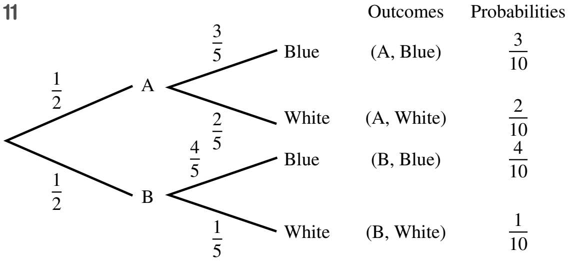
Pr(sum of at least 5) $= \frac{5}{8}$

10

	B	B	R	G
B	X	(B, B)	(B, R)	(B, G)
B	(B, B)	X	(B, R)	(B, G)
R	(R, B)	(R, B)	X	(R, G)
G	(G, B)	(G, B)	(G, R)	X

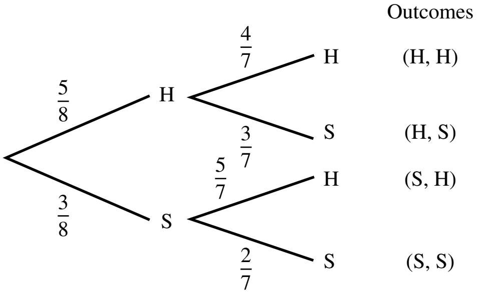
; $\frac{1}{3}$

11



Pr(Blue) $= \frac{7}{10}$

12



; $\frac{15}{28}$

13 The events A and B are not independent.

14 8

15 30

16 120 ; 30

17 1680

18 i 120

ii 144

19 21; 6

20 35

21 60

22 $\frac{1}{45}$

Short-answer questions

1 a $\frac{1}{8}$ b $\frac{1}{4}$ c $\frac{3}{8}$

d $\frac{5}{8}$ e $\frac{1}{2}$

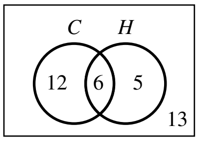
2 a $\frac{5}{8}$ b $\frac{1}{2}$ c $\frac{5}{8}$

3 a i $\frac{2}{5}$ ii $\frac{1}{4}$ iii $\frac{1}{5}$

iv $\frac{1}{10}$ v $\frac{1}{20}$

b i $\frac{3}{5}$ ii $\frac{17}{20}$

4 a



b

	C	C'	
H	6	5	11
H'	12	13	25
	18	18	36

c 13

d i $\frac{1}{6}$ ii $\frac{5}{36}$ iii $\frac{1}{2}$

5 a 6 b $\frac{6}{13}$

6 a i 13 ii 4 iii 1

b i $\frac{3}{4}$ ii $\frac{1}{52}$

c $\frac{4}{13}$

d $\frac{10}{13}$

7 a 0.1 b 0.5

8 a $\frac{2}{5}$ b $\frac{1}{5}$

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9 a i $\frac{4}{11}$ ii $\frac{5}{11}$ iii $\frac{1}{5}$

b No, $\Pr(A \mid B) \neq \Pr(A)$

c i $\frac{1}{2}$ ii $\frac{1}{4}$ iii $\frac{1}{2}$

d Yes, $\Pr(A \mid B) = \Pr(A)$

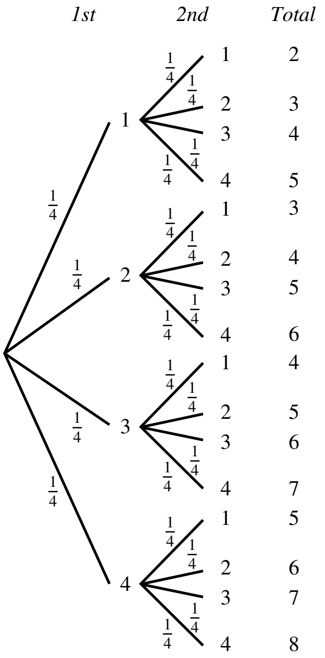
10 a

		1st				
		H	A	P	P	Y
2nd	H	(H, H)	(A, H)	(P, H)	(P, H)	(Y, H)
	E	(H, E)	(A, E)	(P, E)	(P, E)	(Y, E)
	Y	(H, Y)	(A, Y)	(P, Y)	(P, Y)	(Y, Y)

b 15

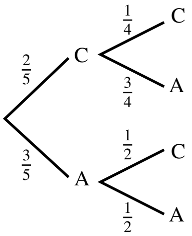
c i $\frac{1}{15}$ ii $\frac{2}{15}$ iii $\frac{13}{15}$

11 a



b i $\frac{1}{16}$ ii $\frac{1}{4}$ iii 0 iv 1

12



a $\frac{2}{5}$ **b** $\frac{3}{4}$ **c** $\frac{3}{10}$

d $\frac{3}{5}$ **e** $\frac{7}{10}$

13 a 0.12

b 0.58

14 a 720 **b** 60 **c** 56 **d** 9

15 a 720 **b** 120 **c** 240

16 $\frac{1}{20}$

17 $\frac{1}{5}$

18 a 792

b $\frac{1}{336}$

Multiple-choice questions

1 A **2** B **3** D **4** C **5** B
6 A **7** C **8** A **9** E **10** E
11 D **12** C **13** A **14** B

Extended-response questions

1 a 3

b i $\frac{7}{15}$ ii $\frac{1}{15}$

c

	R	R'	
S	3	1	4
S'	3	8	11
	6	9	15

d i $\frac{1}{2}$ ii $\frac{3}{4}$

2 a

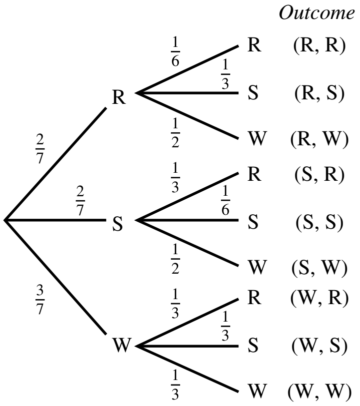
		1st		
		R	S	W
2nd	R	(R, R)	(S, R)	(W, R)
	S	(R, S)	(S, S)	(W, S)
	W	(R, W)	(S, W)	(W, W)

b i $\frac{1}{9}$ ii $\frac{1}{3}$ iii $\frac{5}{9}$ iv $\frac{4}{9}$

c 4

d $\frac{5}{9}$

e



i $\frac{1}{21}$ ii $\frac{10}{21}$ iii $\frac{6}{7}$ iv $\frac{16}{21}$

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Chapter 9

9A

Building understanding

1 **a** C **b** D **c** A **d** B **e** E
2 **a** B **b** E **c** C
d D **e** F **f** A
3 **a** Numerical **b** Categorical **c** Categorical
4 D

Now you try

Example 1
a Numerical and continuous
b Categorical and ordinal

Example 2
Students who study in the library may not reflect a typical group of Year 12 students.

Exercise 9A

1 **a** Numerical and discrete
  b Numerical and continuous
  c Categorical and nominal
  d Categorical and ordinal
2 **a** Numerical and discrete
  b Numerical and discrete
  c Categorical and nominal
  d Numerical and continuous
  e Categorical and ordinal
3 **a** No **b** No
  c Yes **d** Yes
4 Answers will vary and should be discussed in class.
5 **a** D
  b D is the most representative sample. A may pick out the keen students; B probably are good maths students who like maths; and C will have different-sized classes.
6 **a** For example, likely to be train passengers.
  b For example, email will pick up computer users only.
  c For example, electoral roll will list only people aged 18 years and over.
7 Check with your teacher.
8 **a** A small survey, misinterpreted their data.
  b Survey more companies and make it Australia-wide.
  c No, data suggest that profits had reduced, not necessarily that they were not making a profit. Also, sample size is too small.
9 **a** Graph A
  b Graph B
  c The scale on graph A starts at 23, whereas on graph B it starts at 5.
  d Graph A because the scale expands the difference in column heights.
10 For example, showing only part of the scale, using different column widths, including erroneous data values.
11–13 Research required.

9B

Building understanding

1 **a** 10 **b** 1.4 **c** 1 **d** 1 **e** 90%
2

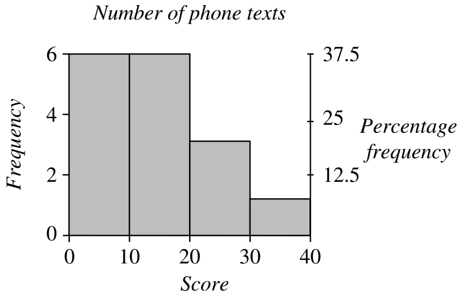
Class interval	Frequency	Percentage frequency
0–	2	20
10–	1	10
20–	5	50
30–40	2	20
Total	**10**	**100**

Now you try

Example 3
a

Class interval	Frequency	Percentage frequency
0–	6	37.5
10–	6	37.5
20–	3	18.75
30–40	1	6.25
Total	**16**	**100**

b



c Positively skewed

Example 4
a

Stem	Leaf
3	6 8 9
4	4 5 8
5	2 6 6 8
6	0 2

5|2 means $52

b **i** $56 **ii** $50 **iii** $49.50

Exercise 9B

1 **a**

Class interval	Frequency	Percentage frequency
0–	3	15
10–	7	35
20–	9	45
30–40	1	5
Total	**20**	**100**

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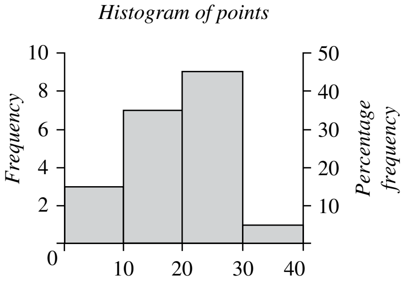
b

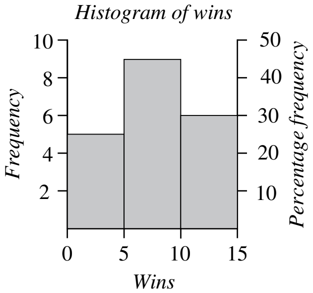


c Negatively skewed

2 **a**

Class interval	Frequency	Percentage frequency
0–	5	25
5–	9	45
10–15	6	30
Total	**20**	**100**

b



c Symmetrical

3 **a**

Type of transport	Frequency	Percentage frequency
Car	16	40
Train	6	15
Tram	8	20
Walking	5	12.5
Bicycle	2	5
Bus	3	7.5
Total	**40**	**100**

b **i** 6 **ii** Car **iii** 40%
iv 17.5% **v** 42.5%

4 **a** Symmetrical **b** Negatively skewed
c Positively skewed **d** Symmetrical

5 **a**

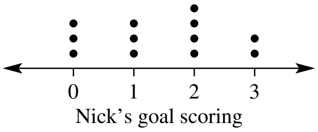
Stem	Leaf
1	6 8 8 8 9
2	3 5 6 7
3	0

2|6 means 26°C

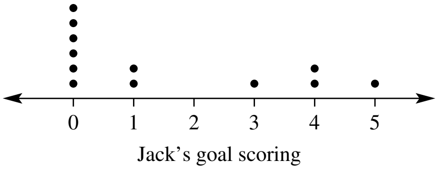
b **i** 18°C **ii** 21°C **iii** 22°C

6 **a** **i** 34.3 **ii** 38 **iii** 39
b **i** 19.4 **ii** 20 **iii** No mode

7 **a**



b



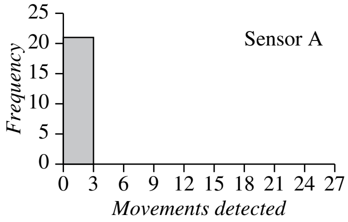
c Well spread performance

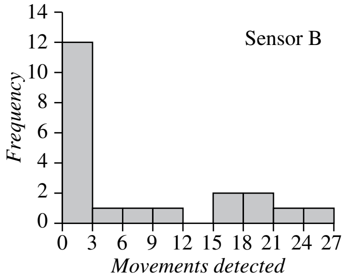
d Irregular performance, positively skewed

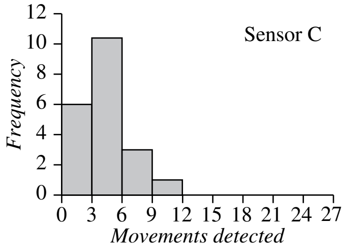
8 **a**

	Sensor A frequency	Sensor B frequency	Sensor C frequency
0–	21	12	6
3–	0	1	11
6–	0	1	3
9–	0	1	1
12–	0	0	0
15–	0	2	0
18–	0	2	0
21–	0	1	0
24–26	0	1	0
Total	**21**	**21**	**21**

b







c **i** Low sensitivity **ii** Very sensitive
iii Moderately sensitive

9 **a**

Mass	Frequency	Percentage frequency
10–	3	6
15–	6	12
20–	16	32
25–	21	42
30–35	4	8
Total	**50**	**100**

b 50
c 32%
d At least 25 g but less than 30 g.
e 42%
f 94%

10 **a**

Section	Frequency	Percentage frequency
Strings	21	52.5
Woodwind	8	20
Brass	7	17.5
Percussion	4	10
Total	**40**	**100**

b 40 **c** 52.5% **d** 47.5%
e 9.3% **f** 65.6%

11 8 students scored between 20 and 30 and there are 32 students all together, so this class interval makes up 25% of the class.

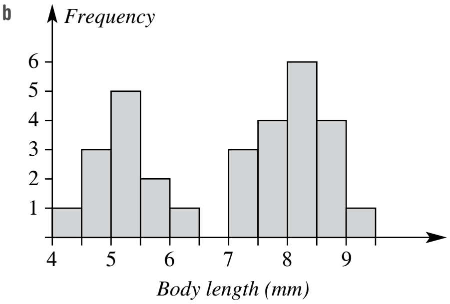
12 No discrete information, only intervals are given and not individual values.

13 $3 \leqslant a \leqslant 7$, $0 \leqslant b \leqslant 4$, $c = 9$

14 **a**

Body length (mm)	Frequency	Percentage frequency (1 dp)
4.0–	1	3.3
4.5–	3	10.0
5.0–	5	16.7
5.5–	2	6.7
6.0–	1	3.3
6.5–	0	0.0
7.0–	3	10.0
7.5–	4	13.3
8.0–	6	20.0
8.5–	4	13.3
9.0–	1	3.3
Total	30	100

b



c The distribution is bimodal with body lengths ranging from 4 to 6.5 mm and 7 to 9.5 mm.
d **i** 5.0–5.5 mm **ii** 8.0–8.5 mm
e Answers will vary.

9C

Building understanding

1 **a** $a = 18$, $b = 20$ and $c = 36$
b 36
c **i** 18 **ii** 12
d Yes

2 **a** **i** $\frac{2}{5}$ **ii** $\frac{1}{10}$ **iii** $\frac{3}{20}$ **iv** $\frac{1}{5}$
b **i** 5% **ii** 5%
c Yes

Now you try

Example 5

1 **a** **i** 10 **ii** 4 **iii** 2 **iv** 1
b **i** 50% **ii** 50% **iii** 30% **iv** 10%
c No. There is not a significant difference in the way teachers from the different types of schools responded to the question.

Example 6

1 **a**

	Pure	Mixed	Total
Country	18	12	30
City	7	13	20
Total	25	25	50

b **i** $\frac{2}{5}$

ii $\frac{7}{50}$

c No. The proportion of mixed breed dogs is higher for the city.

Exercise 9C

1 a i 26 ii 7 iii 1 iv 10
b i 48% ii 32% iii 2% iv 0%
c Yes. There is a greater proportion of older people agreeing with the idea compared to younger people.

2 a i 52 ii 25 iii 19 iv 14
b i 48% ii 20% iii 14% iv 2%
c Yes. A larger proportion of younger people agree with the idea compared to older people.

3 a

	Smoked	Do not smoke	Total
Over 55	5	25	30
Under 55	8	12	20
Total	13	37	50

b i $\frac{3}{5}$ ii $\frac{6}{25}$
c Yes. A larger proportion of the younger people smoked compared to the older people.

4 a

	Happy with waiter	Not happy with waiter	Total
Satisfied with coffee	26	6	32
Dissatisfied with coffee	3	5	8
Total	29	11	40

b i $\frac{4}{5}$ ii $\frac{3}{20}$
c Yes. A greater proportion of people not satisfied with their coffee were not happy with the waiter.

5 a i 46% ii 12% iii 22%
b $\frac{2}{11}$
c $\frac{6}{7}$
d Yes. A greater proportion of the people who prefer Horror and Drama were adults.

6 $a = 6, b = 13, c = 15, d = 17, e = 32$

7 a i 16 ii 8
b i 88.9% ii 46.7%
c Yes. A greater proportion of the weekend customers were dissatisfied compared to the weekday customers.

8 a $\frac{11}{20}$ or 0.55 b $\frac{1}{5}$ or 0.2

9 To give a neutral option and two opposing sides of equal size.

10 a $\frac{1}{4}$ b $\frac{1}{4}$
c No. The proportions for both local residents and tourists are equal.

11 So that percentages or probabilities could be easily read straight from the table, particularly for large surveys.

12 a Accept H_0. The proportion of A that are also B is very close to the proportion of A' that are also B. So A and B appear not to be connected.
b Reject H_0. There is a significant difference in the proportion of A that are also B and the proportion of A' that are also B. So A and B appear to be connected.

9D

Building understanding

1 a Min, lower quartile (Q_1), median (Q_2), upper quartile (Q_3), max
b Range is max – min; IQR is $Q_3 - Q_1$. Range is the spread of all the data, IQR is the spread of the middle 50% of data.
c An outlier is a data point (element) significantly outside the vicinity of the rest of the data.
d If the data point is greater than $Q_3 + 1.5 \times \text{IQR}$ or less than $Q_1 - 1.5 \times \text{IQR}$.

2 a 2
b i 1 ii 3
c 2

3 a i 10.5 ii 7.5 iii 12
b 4.5 c 0.75, 18.75 d No

Now you try

Example 7
a Range = 9, IQR = 4.5
b Range = 0.9, IQR = 0.6

Example 8
a i 0 and 36 ii 24 iii 19 and 26
iv 7 v Yes; 0
b Perhaps the bus was not taking passengers.

Exercise 9D

1 a Min = 3, $Q_1 = 4$, median = 8, $Q_3 = 10$, max = 13; range = 10, IQR = 6
b Min = 10, $Q_1 = 10.5$, median = 14, $Q_3 = 15.5$, max = 18; range = 8, IQR = 5
c Min = 1.2, $Q_1 = 1.85$, median = 2.4, $Q_3 = 3.05$, max = 3.4; range = 2.2, IQR = 1.2
d Min = 41, $Q_1 = 53$, median = 60.5, $Q_3 = 65$, max = 68; range = 27, IQR = 12

2 a Min = 0, max = 17 b Median = 13
c $Q_1 = 10, Q_3 = 15$ d IQR = 5
e 0
f Road may have been closed that day.

3 a i Min = 4, max = 14 ii 7.5
iii $Q_1 = 5, Q_3 = 9$ iv IQR = 4
v No outliers
b i Min = 16, max = 31 ii 25
iii $Q_1 = 21, Q_3 = 27$ iv IQR = 6
v No outliers

4 a i $5.\dot{3}$ ii 2.4
b i 2.5 ii 2
c i $2.9\dot{3}$ ii 0.5

5 a i Min = 25, max = 128 ii 47
iii $Q_1 = 38, Q_3 = 52.5$ iv IQR = 14.5
v Yes; 128 vi 51.25
b Median as it is not affected dramatically by the outlier.
c A more advanced calculator was used.

6 **a** No outliers **b** Outlier is 2.
c Outliers are 103, 182. **d** Outliers are 2, 8.
7 **a** IQR = 12 **b** No outliers.
c 24 **d** 22
8 1, 2, 3
9 **a** Increases by 5. **b** It is doubled.
c It is divided by 10.
10 **a** It stays the same. **b** It doubles.
c It is reduced by a scale factor of 10.
11 Answers may vary. Examples:
a 3, 4, 5, 6, 7 **b** 2, 4, 6, 6, 6 **c** 7, 7, 7, 10, 10
12 It is not greatly affected by outliers.
13 Answers will vary.

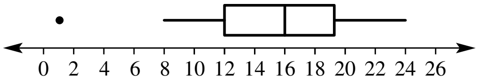
9E

Building understanding

1 **a** 15 **b** 5 **c** 25 **d** 20
e 10 **f** 20 **g** 10
2 **a** 4 **b** 2 **c** 18 **d** 20
e It is.

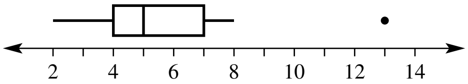
Now you try

Example 9
a Yes, 1 is an outlier.
b

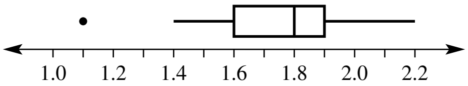


Exercise 9E

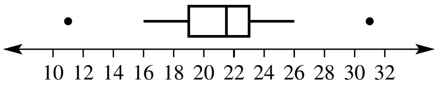
1 **a** **i** $Q_1 = 4, Q_3 = 7$; outlier is 13
ii



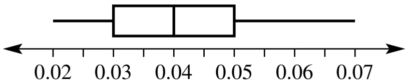
b **i** $Q_1 = 1.6, Q_3 = 1.9$; outlier is 1.1
ii



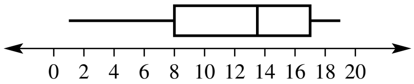
c **i** $Q_1 = 19, Q_3 = 23$; outliers are 11 and 31
ii



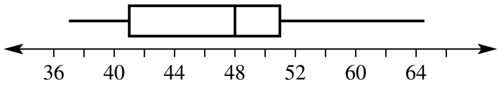
d **i** $Q_1 = 0.03, Q_3 = 0.05$; no outliers
ii



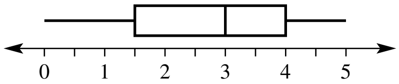
2 **a**



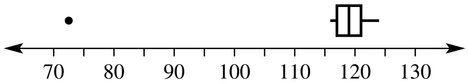
b



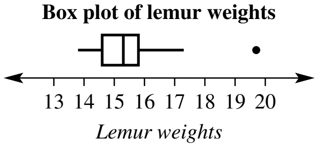
c



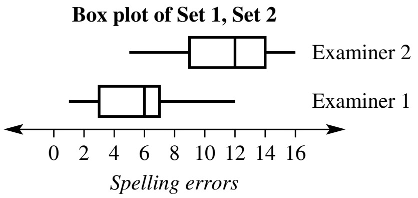
d



3 **a** $Q_1 = 14.6, Q_2 = 15.3, Q_3 = 15.8$
b 19.7 kg
c



4 **a** Same minimum of 1.
b B
c **i** 5 **ii** 10
d Data points for B are more evenly spread than those for A.
5 **a** They have the same median and upper quartile.
b B
c **i** 4 **ii** 5
d Set B is more spread out.
6 **a** A **b** B **c** B
7 **a**



b Yes, examiner 2 found more errors.
8 Answers may vary. Examples:
a **i,ii** Class results had a smaller spread in the top 25% and bottom 25% performed better.
iii State results have a larger IQR.
b The class did not have other results close to 0 but the school did.
9 Answers will vary.

Progress quiz

1 **a** Numerical and discrete
b Categorical and nominal

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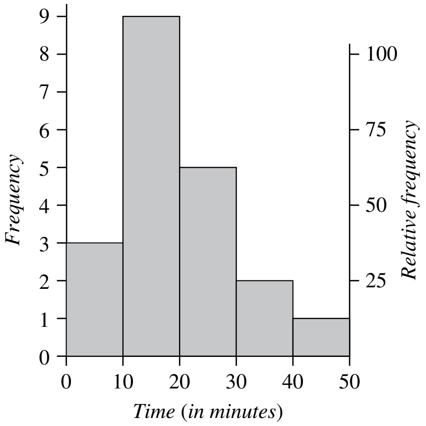
2 a

Class interval	Frequency	Percentage frequency
0–	3	15
10–	9	45
20–	5	25
30–	2	10
40–50	1	5

b


Travel time from home to school

c Positively skewed

3 a

Stem	Leaf
1	4 6 8 9
2	1 4 8
3	2 4 4 6 7
4	0 2

3|2 means

b **i** 34 km/h **ii** 30 km/h **iii** 28.2 km/h

4 a 10 **b** 12%

c Yes, a greater proportion of the people under 40 agree with the idea.

5 a Range $= 32 - 4 = 28$; $Q_2 = 17$, $Q_1 = 12$, $Q_3 = 23$, IQR $= 11$

b Range $= 6.6 - 4.2 = 2.4$; $Q_2 = 5.2$, $Q_1 = 4.5$, $Q_3 = 6.1$, IQR $= 1.6$

6 a **i** Min = 30, max = 69

ii Median = 39

iii Lower quartile = 34, upper quartile = 46

iv IQR = 12

v $Q_1 - 1.5 \times \text{IQR} = 16$; $Q_3 + 1.5 \times \text{IQR} = 64$

The outlier is 69.

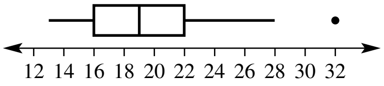
b For example, school open day or grandparents day.

7 a $Q_1 = 16$, $Q_3 = 22$, IQR $= 6$

$Q_3 + 1.5 \times \text{IQR} = 31$

32 is an outlier.

b



9F

Building understanding

1 a larger **b** smaller

2 a B **b** A

3 A. The data values in A are spread farther from the mean than the data values in B.

4 a Gum Heights **b** Oak Valley

Now you try

Example 10
Mean $= 2.8$, $s = 1.6$

Example 11

a More data values are centred around the 10s and 20s for Year 12 compared to 20s and 30s for Year 7.

b The Year 7 data are more spread-out.

c Given their studies, Year 12s are more likely to watch less television.

Exercise 9F

1 a Mean $= 6, s = 2.2$ **b** Mean $= 3.6, s = 2.6$

c Mean $= 8, s = 3.8$ **d** Mean $= 32.5, s = 3.6$

2 a Mean $= 2.7$, $s = 0.9$ **b** Mean $= 14.5$, $s = 6.6$

3 a The outer-suburb school has more data values in the higher range.

b There is less spread. Data values are closer to the mean.

c Students at outer-suburb schools may live some distance from the school. Answers will vary.

4 a False **b** True **c** True

5 a Mean $= 2$, $s = 1.0$ **b** Mean $= 5.25$, $s = 0.7$

6 a No **b** No **c** Yes

d Yes, one of the deviations would be calculated using the outlier.

7 a No, standard deviation reflects the spread of the data values from the mean not the size of the data values.

b No. As for part **a**.

8 The IQRs would be the same, making the data more comparable. The standard deviation would be affected by the outlier.

9 a 1.8

b 3.6

c Standard deviation is larger.

d Mean dev ≈ 5.3
Std dev ≈ 32.6
A much larger effect on standard deviation.

10 a **i** 85.16 **ii** 53.16 **iii** 101.16

iv 37.16 **v** 117.16 **vi** 21.16

b **i** 66% **ii** 96% **iii** 100%

c **i** Research required

ii One SD from the mean = 68%
Two SDs from the mean = 95%
Three SDs from the mean = 99.7%
Close to answers found.

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9G

Building understanding

1 a i 6 ii 1 iii 10 iv 18
b i 90% ii 100%
iii 40% iv 5%
2 a 8, 10, 12
b i 7 ii 10 iii 11 iv 13

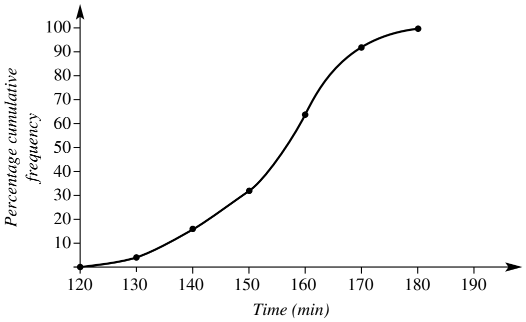
Now you try

Example 12
1 a i 8 ii 4 iii 1 iv 24
b i 30% ii 20%
iii 26.7% iv 50%
c i 7°C ii 5°C iii 9°C
d Approx. 3.6

Example 13
a

Time (min)	Frequency	Cumulative frequency	Percentage cumulative frequency
120–	1	1	4
130–	3	4	16
140–	4	8	32
150–	8	16	64
160–	7	23	92
170–180	2	25	100

b



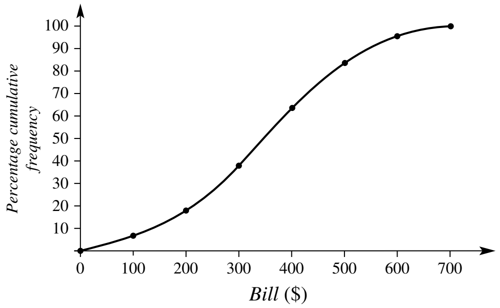
c i 166 mins ii 156 mins iii 146 mins
d Approx. 82%
e Approximately 50% of the marathon runners complete the race under 156 minutes.

Exercise 9G

1 a i 2 ii 15 iii 24 iv 27
b i 7.1% ii 67.9%
iii 25% iv 3.6%
c i 204 ii 198 iii 219
d 16
2 a i 4 ii 8
iii 29 iv 4
b i 56.3% ii 18.7%
iii 31.3% iv 6.3%
c i 19 ii 13 iii 22
d 8
3 a

Bill ($)	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	3	3	6
100–	6	9	18
200–	10	19	38
300–	13	32	64
400–	10	42	84
500–	6	48	96
600–700	2	50	100

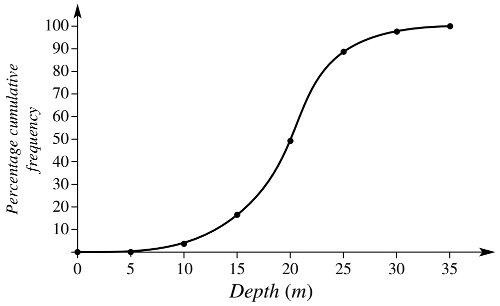
b



c i Approx. 545
ii Approx. 346
iii Approx. 260
d Approx. 74%
e Approximately 50% of the gas bills were below $346
4 a

Depth (m)	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	0	0	0.0
5–	2	2	3.6
10–	7	9	16.4
15–	18	27	49.1
20–	22	49	89.1
25–	5	54	98.2
30–35	1	55	100.0

b



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c **i** Approx. 22.6
  ii Approx. 20.1
  iii Approx. 12.5

d Approx. 81

e Approximately 50% of the back yards have a depth of less than 20.1 m.

5 **a** $a = 9, b = 28, c = 12.5, d = 53.1$

b $a = 20, b = 29, c = 48.3, d = 86.2$

c $a = 6, b = 42, c = 43.2, d = 90.9$

d $a = 26$, $b = 38$, $c = 13.0$, $d = 95.7$

6 **a** **i** 19 **ii** 36

b **i** 14.3% **ii** 95.2%

c **i** 45.2% **ii** 54.8%

7 **a** 170.5 cm **b** 14 cm

8 **a** 20–30 **b** 2–4

9 **a** Rising steeply especially for the 5–10 category then levelling off through 10–30.

b Rising slowly initially then rising steeply especially for the 300–500 category then levelling off.

10 Less class intervals would provide less data points for the plotting of the percentage cumulative frequency curve and hence provide less accuracy when reading off percentiles.

11 **a** **i** 13.3 **ii** 26.7 **iii** 43.3 **iv** 55.0

b Approx. 18.3

9H

Building understanding

1 **a** Linear **b** No trend

c Non-linear **d** Linear

2 **a** **i** 28°C **ii** 33°C **iii** 33°C **iv** 35°C

b 36°C

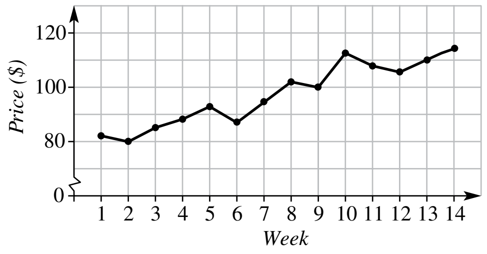
c **i** 12 p.m. to 1 p.m. **ii** 3 p.m to 4 p.m.

d Temperature is increasing from 8 a.m. to 3 p.m. in a generally linear way. At 3 p.m. the temperature starts to drop.

Now you try

Example 14

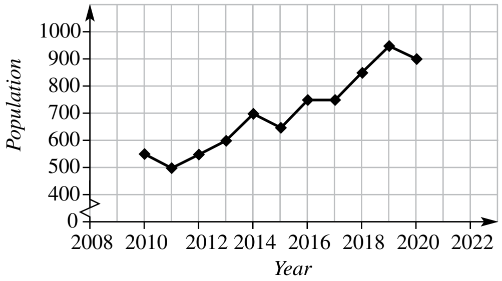
a



b General linear upward trend.

Exercise 9H

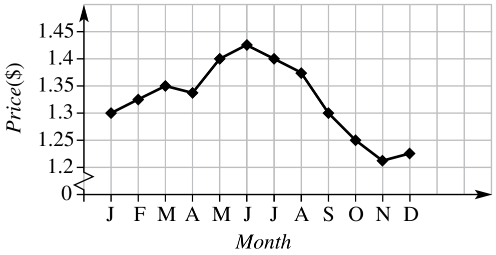
1 **a**



b Generally linear in a positive direction.

c **i** 500 **ii** 950

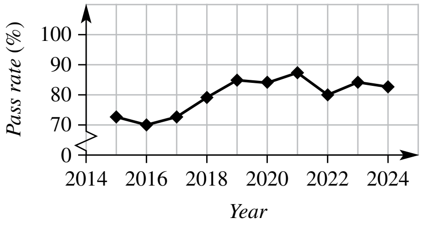
2 **a**



b The share price generally increased until it peaked in June and then continually decreased to a yearly low in November before trending upwards again in the final month.

c $0.21

3 **a**



b The pass rate for the examination has increased marginally over the 10 years, with a peak in 2021.

c 2021 **d** 11%

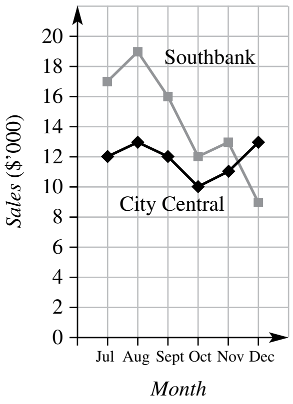
4 **a** Linear

b **i** $650 000 **ii** $750 000

5 **a** **i** $6000 **ii** $4000

b 1

c



d **i** The sales trend for City Central for the 6 months is fairly constant.
  ii Sales for Southbank peaked in August before taking a downturn.
e About $5000

6 **a** **i** 5.8 km **ii** 1.7 km
  b **i** Blue Crest slowly gets closer to the machine.
    ii Green Tail starts near the machine and gets further from it.
  c 8:30 p.m.

7 **a** The yearly temperature is cyclical and January is the next month after December and both are in the same season.
  b No
  c Northern hemisphere, as the seasons are opposite, June is summer.

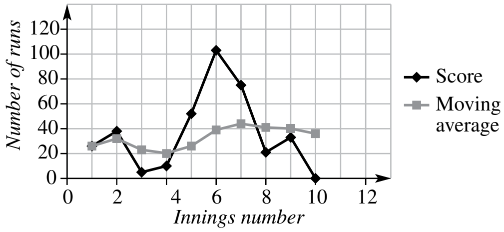
8 **a** Increases continually, rising more rapidly as the years progress.
  b Compound interest—exponential growth.

9 **a** Graphs may vary, but it should decrease from room temperature to the temperature of the fridge.
  b No. Drink cannot cool to a temperature *lower* than that of the internal environment of the fridge.

10 **a**

Innings	1	2	3	4	5	6	7	8	9	10
Score	26	38	5	10	52	103	75	21	33	0
Moving average	26	32	23	20	26	39	44	41	40	36

b



c **i** The score fluctuates wildly.
  ii The graph is fairly constant with small increases and decreases.
d The moving average graph follows the trend of the score graph but the fluctuations are much less significant.

9I

Building understanding

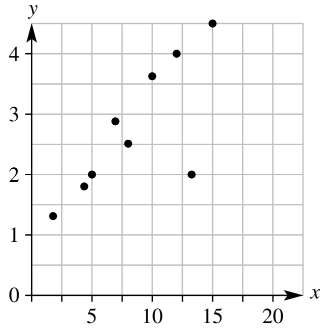
1 **a** Unlikely **b** Likely **c** Unlikely
  d Likely **e** Likely **f** Likely

2 **a** y generally increases as x increases.
  b y generally decreases as x increases.

Now you try

Example 15

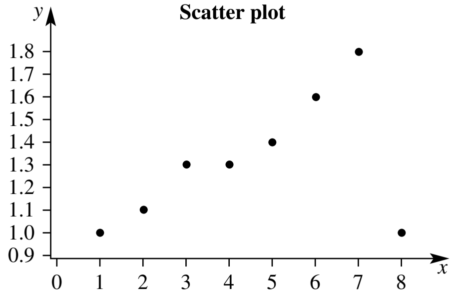
a



b Positive **c** Strong **d** (13, 2.0)

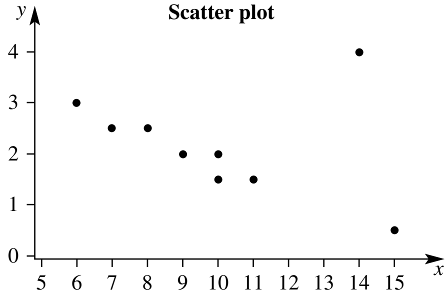
Exercise 9I

1 **a**



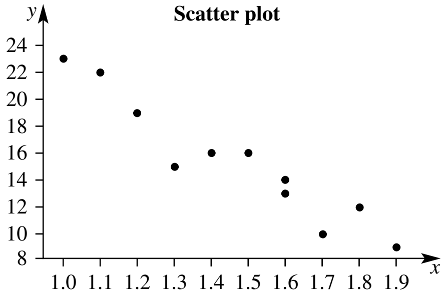
b Positive **c** Strong **d** (8, 1.0)

2 **a**



b Negative **c** Strong **d** (14, 4)

3 **a** Negative



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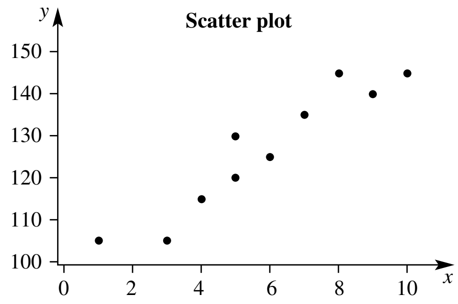
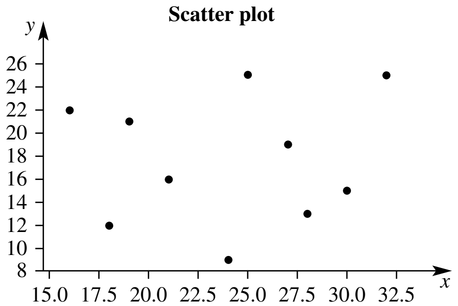
b Positive



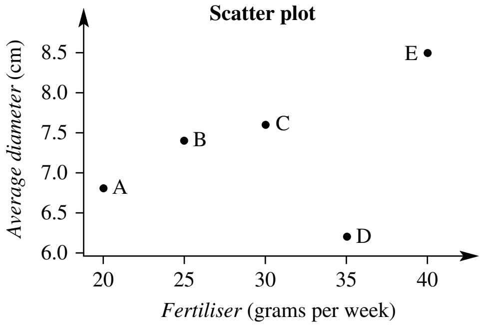
c None



4 **a** None **b** Weak negative
c Positive **d** Strong positive

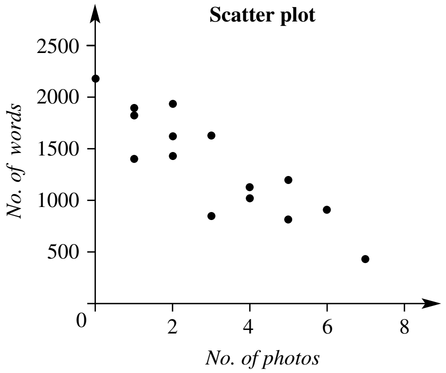
5 **a** Yes
b Decrease
c **i** Yes **ii** Car H

6 **a**



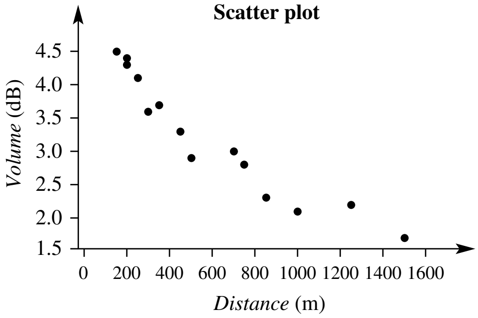
b D
c Seems likely but small sample size does lead to doubt.

7 **a**



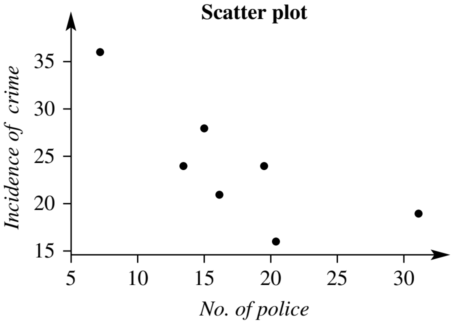
b Negative, weak correlation.

8 **a**

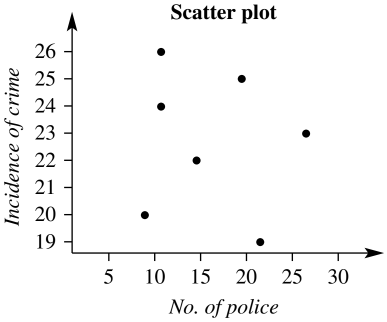


b Negative
c As distance increases, volume decreases.

9 **a** **i** Weak, negative correlation



ii No correlation



b Survey 1, as this shows an increase in the number of police has seen a decrease in the incidence of crime.

10 The positive correlation shows that as height increases it is predicted that ability to play tennis increases.

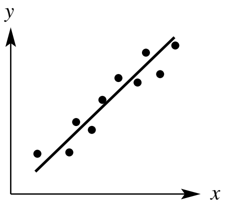
11 Each axis needs a better scale. All data are between 6 and 8 hours sleep and show only a minimal change in exam marks. Also, very small sample size.

12 **a** **i** Students I, T **ii** Students G, S
b **i** Students H, C **ii** Students B, N
c Students C, G, H, S,
d Students B, I, N, T
e No

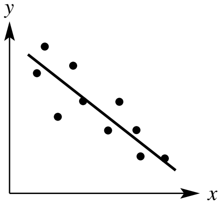
9J

Building understanding

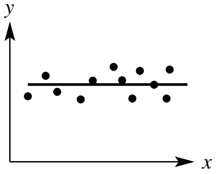
1 **a**



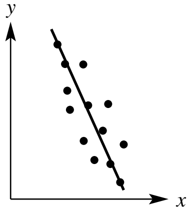
b



c



d



2 **a** $y = \frac{1}{2}x + \frac{7}{2}$ **b** $y = -\frac{2}{3}x + \frac{17}{3}$

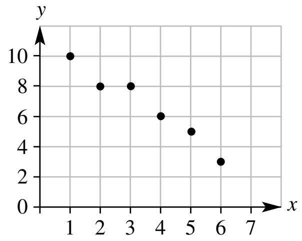
3 **a** **i** 17 **ii** $\frac{23}{4}$

b **i** $\frac{28}{5}$ **ii** $\frac{14}{5}$

Now you try

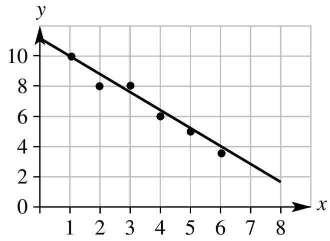
Example 16

a



b Negative correlation

c



d **i** ≈ 7 **ii** ≈ 11

iii ≈ 8 **iv** ≈ 4.5

Example 17

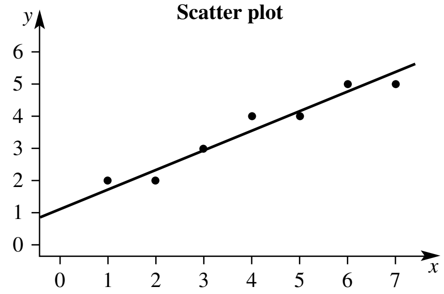
a $y = \frac{5}{3}x + \frac{10}{3}$

b **i** 45 cm **ii** 90 cm

c **i** 34 kg **ii** 46 kg

Exercise 9J

1 **a**



b Positive correlation

c As above.

d All answers are approximate.

i 3.2 **ii** 0.9 **iii** 1.8 **iv** 7.4

2 **a** ≈ 4.5 **b** ≈ 6

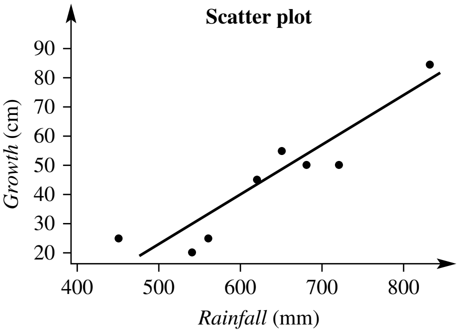
c ≈ 0.5 **d** ≈ 50

3 **a** $y = \frac{3}{5}x + 18$

b **i** 42 **ii** 72

c **i** 30 **ii** 100

4 **a, b**



c **i** ≈ 25 cm **ii** ≈ 85 cm

d **i** ≈ 520 mm **ii** ≈ 720 mm

5 **a** $y = 5x - 5$

b 85 cm

c 21 kg

6 **a** Data do not appear to have any correlation.

b Too few data points to determine a correlation.

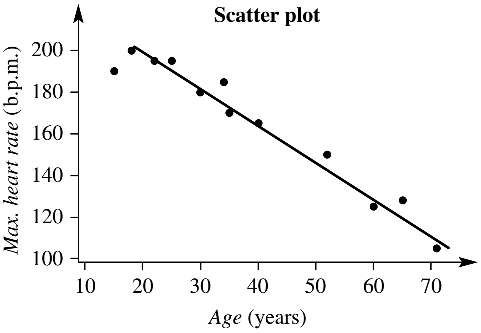
7 **a** Too few data points to determine a correlation.

b The data points suggest that the trend is not linear.

8 **a** **i** 50 **ii** 110

b It is possible to obtain scores of greater than 100%.

9 **a** Experiment 1



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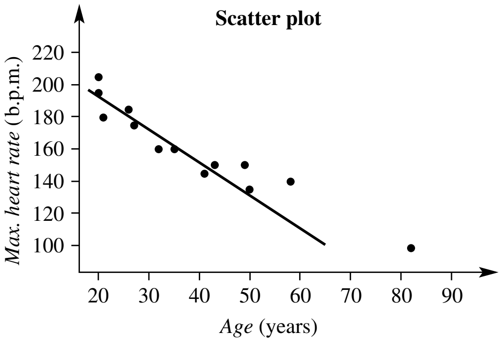
Experiment 2



b **i** ≈ 140 **ii** ≈ 125

c **i** ≈ 25 **ii** ≈ 22

d Experiment 2

e Research required.

Problems and challenges

1 66 kg

2 88%

3 19

4 1.1

5 **a** Larger by 3 **b** Larger by 3 **c** No change
d No change **e** No change

6 $y = x^2 - 3x + 5$

7 $5.8 \leqslant a < 6.2$

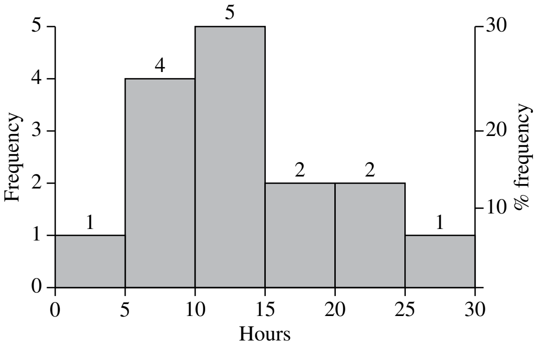
8 Physics, Biology. The number of standard deviations from the mean shows the relative position of Emily's mark within the spread of all results from each class. This number gives a meaningful comparison of results. Physics 1.67, Maths 1.11, Biology 0.

Answers to success criteria example questions

1 Numerical and discrete.

2 Surveying people listed in the phone book is restricted to people who are listed in the phone, ruling out some sections of the community from participating in the survey.

3



Positively skewed.

4 Mean = 17.1; median = 15; mode = 11

5 **i** 4

ii Yes, a greater proportion of people from overseas agree with the idea.

6

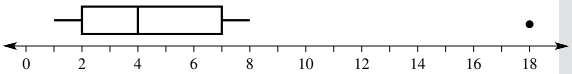
	Mortgage	No mortgage	Total
50 or over	5	5	10
Under 50	11	4	15
Total	16	9	25

$; \frac{1}{5}$

7 Minimum = 7; maximum = 30; median = 16; upper quartile = 25; lower quartile = 12; range = 23; IQR = 13

8 Upper quartile = 18; lower quartile = 12; 2 is an outlier

9



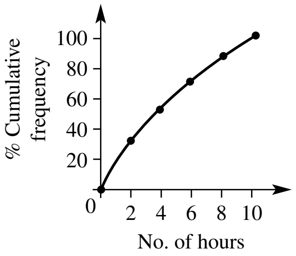
10 Mean = 5.6; standard deviation = 3.0

11 The minimum temperature for New York is 4°C and for Melbourne is 13°C. New York's mean is affected by several values smaller than 13°C. The temperatures for New York are more spread out from the mean. Melbourne's temperatures are more closely clustered near its mean.

12 55 kg; IQR = 18 kg

13

No. of hours	Frequency	Cumulative Frequency	Percentage Cumulative Frequency
0–	15	15	37.5
2–	6	21	52.5
4–	10	31	77.5
6–	6	37	92.5
8–10	3	40	100



50^{th} percentile: Approx. 3.7 hours

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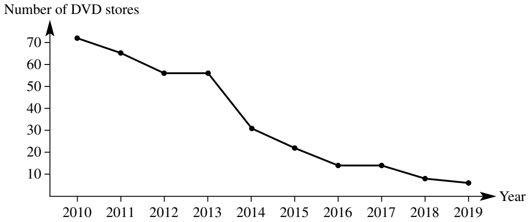
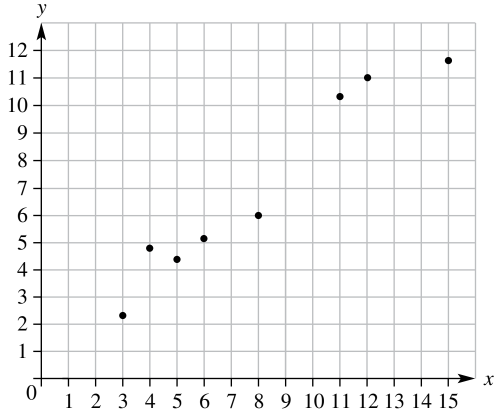
14



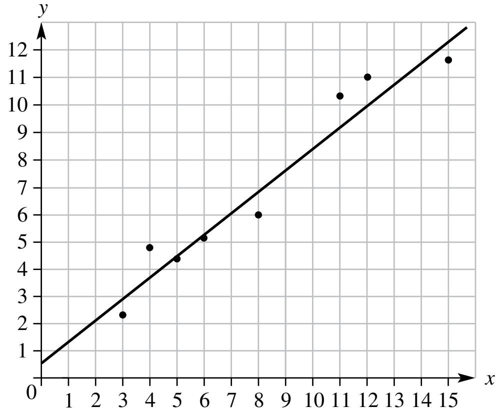
General linear downward trend.

15



Strong positive correlation.

16



When $x = 10$, $y \approx 8.5$; when $y = 8$, $x \approx 9.3$

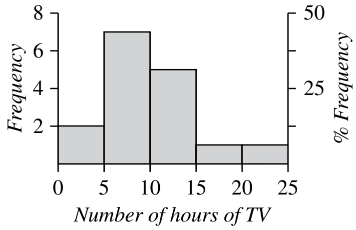
17 $y = \frac{1}{2}x + 5$; $x = 90$

Short-answer questions

1 a

Class interval	Frequency	Percentage frequency
0–	2	12.5
5–	7	43.75
10–	5	31.25
15–	1	6.25
20–25	1	6.25
Total	**16**	100

b



c It is positively skewed.

d

Stem	Leaf
0	1 3 5 6 6 7 8 8 9
1	0 1 2 3 4 6
2	4

1 | 3 means 13

e 8.5 hours

2 a 8

b $6\frac{2}{3}\%$

c Yes, a much of higher proportion of non-pet owners agreed with the idea of cats being inside at night.

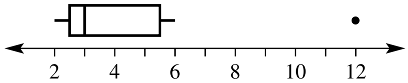
3 a i 10

ii $Q_1 = 2.5, Q_3 = 5.5$

iii 3

iv 12

v



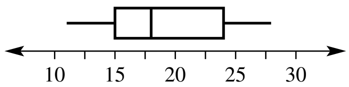
b i 17

ii $Q_1 = 15, Q_3 = 24$

iii 9

iv None

v



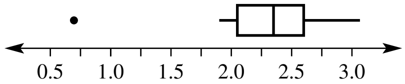
c i 2.4

ii $Q_1 = 2.1, Q_3 = 2.6$

iii 0.50

iv 0.7

v



4 a False **b** True **c** True **d** True

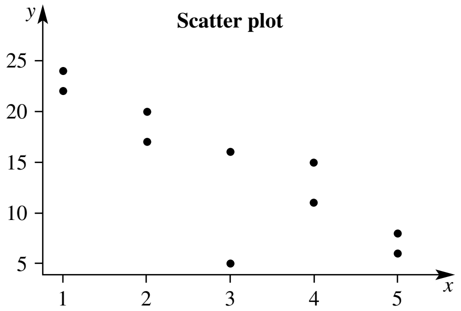
5 a

Time mins	Frequency	Cumulative frequency	Percentage cumulative frequency
30–	3	3	15
35–	4	7	35
40–	8	15	75
45–	3	18	90
50–55	2	20	100

b 35%

c 60%

6 **a**



b Negative **c** Weak **d** $(3, 5)$

7 **a** $y = \frac{3}{5}x + \frac{7}{5}$

b **i** 3.8 **ii** 7.4

c **i** $2\frac{2}{3}$ **ii** $17\frac{2}{3}$

8 **a** Mean $= 7, s = 2.5$

b Mean $= 4, s = 3.0$

9 **a** The JackJumpers **b** The JackJumpers

c The JackJumpers **d** The JackJumpers

10 **a** Non-linear

b Linear

Multiple-choice questions

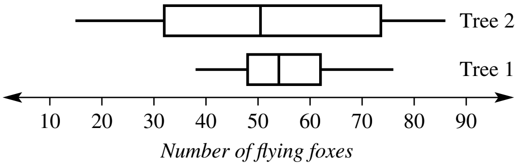
1 D	**2** B	**3** D	**4** C	**5** C
6 A	**7** C	**8** B	**9** D	**10** E

Extended-response questions

1 **a** **i** 14 **ii** 41

b **i** No outliers **ii** No outliers

c

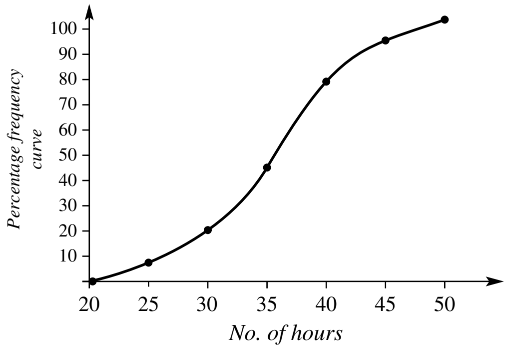


d More flying foxes regularly take refuge in tree 1 than in tree 2, for which the spread is much greater.

2 **a**

No. of hours	Frequency	Cumulative frequency	Percentage cumulative frequency
20–	3	3	7.5
25–	5	8	20
30–	10	18	45
35–	14	32	80
40–	6	38	95
45–50	2	40	100

b



c **i** 38 hours **ii** 36 hours

d 23%

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Chapter 10

10A

Building understanding

1 a $f(x) = 8x$ b $f(x) = 9 - x^2$ c $f(x) = 2^x$
2 a True b True c False d False e True
3 a $y \geqslant 0$ b $y > 0$ c $y > 9$
d $0 \leqslant y \leqslant 1$ e $y \geqslant 0$

Now you try

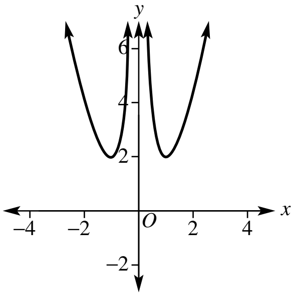
Example 1
a 2 b 16 c $k^2 - 5k + 2$

Example 2
a Function b Function c Not a function

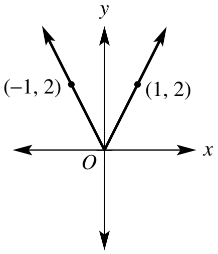
Example 3
a Domain: all real x
Range: all real y
b Domain: all real x
Range: $y \leqslant 2$

Exercise 10A

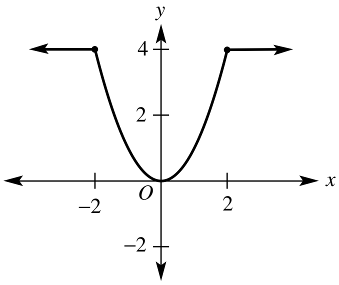
1 a 4 b 10 c 19
d 124 e 7 f 14
g 4 h $2a^2 - a + 4$ i $2k^2 - k + 4$
2 a Function b Function c Not a function
d Function e Function f Function
3 a Function b Function c Function
d Function e Not a function f Function
g Not a function h Function i Not a function
4 a $f(0) = 0, f(2) = 8, f(-4) = -16,$
$f(a) = 4a, f(a+1) = 4a + 4$
b $f(0) = 1, f(2) = -3, f(-4) = -15,$
$f(a) = 1 - a^2, f(a+1) = -a^2 - 2a$
c $f(0) = -12, f(2) = 0, f(-4) = -12,$
$f(a) = a^2 + 4a - 12, f(a+1) = a^2 + 6a - 7$
d $f(0) = 9, f(2) = 25, f(-4) = 73,$
$f(a) = 4a^2 - 9, f(a+1) = 4a^2 + 8a + 13$
5 a All real x b All real x c All real x d All real x
6 a All real y b All real y c $y \geqslant 0$ d $y \leqslant 2$
7 a i 5 ii -2 iii 3
iv -15 v 5 vi -4
b $a = \frac{5}{3}$ represents the x-value of the point where the line graphs intersect.
8 a i False ii False
b i False ii True
c i False ii False
9 $4x + 2h - 3$
10 a They all pass the vertical line test, as each x-value has only one y-value.
b Vertical lines in the form $x = a$, since a single x-value has multiple y-values.
c The y-value of the vertex is the maximum or minimum value of the parabola and therefore is essential when finding the range.
d i $y \geqslant -4$ ii $y \geqslant -12\frac{1}{4}$
iii $y \leqslant 1\frac{1}{8}$ iv $y \geqslant 1$
11 a $x \neq 1$ b $x \neq -\frac{1}{2}$ c $x \neq 1$
12 a $x \geqslant 0$ b $x \geqslant 2$ c $x \geqslant -2$ d $x \leqslant 2$
13 a $f(a) = f(-a) = a^2 + \frac{1}{a^2}$
b



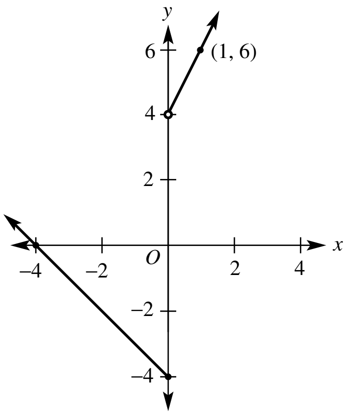
c The y-axis is the axis of symmetry for the function.
14 a i



ii



iii



b iii
c i $y \geqslant 0$
ii $0 \leqslant y \leqslant 4$
iii $y \geqslant -4$
d i 8, 8, 2
ii 34, 18, -2

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10B

Building understanding

1 **a** 5
b **i** 3 **ii** −2 **iii** 1 **iv** −1
c 2
2 **a** Linear **b** Quadratic **c** Quartic
d Quadratic **e** Constant **f** Cubic
3 **a** 3 **b** 4 **c** 6

Now you try

Example 4
a Yes and degree is 3 **b** No

Example 5
a 14 **b** 8

Exercise 10B

1 **a**, **b**, **f** are polynomials.
a Degree = 3 **b** Degree = 4 **c** Degree = 8
2 **a** −5 **b** 11 **c** 1 **d** −45
3 **a** 0 **b** 92 **c** −4 **d** 42
4 **a** 14 **b** 92 **c** 8 **d** 4
5 **a** −2 **b** 25 **c** −22 **d** −17
e 17 **f** −351
6 **a** **i** 30 m **ii** 24 m **iii** 0 m
b Yes, when $5 < x < 7$.
7 **a** $-\frac{1}{2}$ **b** −1 **c** $\frac{1}{2}$
8 **a** 0 **b** 4 **c** −108
9 **a** 8 **b** $n + 1$ **c** 1 **d** 1
10 **a** $-\frac{9}{8}$ **b** $-\frac{20}{27}$ **c** $\frac{5}{8}$ **d** $\frac{27}{64}$
e $\frac{16}{27}$ **f** $-\frac{216}{125}$ **g** $-\frac{1}{2}$ **h** $-\frac{9}{8}$
11 **a** $2k^3 - k^2 - 5k - 1$
b $2b^3 - b^2 - 5b - 1$
c $16a^3 - 4a^2 - 10a - 1$
d $-2a^3 - a^2 + 5a - 1$
e $-16a^3 - 4a^2 + 10a - 1$
f $-54k^3 - 9k^2 + 15k - 1$
g $2a^3b^3 - a^2b^2 - 5ab - 1$
h $-2a^3b^3 - a^2b^2 + 5ab - 1$
12 **a** **i** 10 **ii** 2 **iii** 1
iv −13 **v** −9 **vi** −18
b **i** 3 **ii** −11 **iii** −22
c $a = 2$ and $b = -1$

10C

Building understanding

1 **a** $x^2 + 2x$
b $x^2 + 6x - 55$
c $8x^2 - 26x + 15$
2 **a** $x^4 - 5x^3 + 4x^2 - 3$
b $-x^6 - 3x^4 + x^3 - x^2 + 13$
3 **a**, **b**, **c** are true.

Now you try

Example 6
a $3x^4 - x^3$
b $x^4 - 2x^3 + 4x^2 - 11x + 6$

Example 7
a $x^5 + x^4 - 5x^3 + 2x^2 + 7x - 12$
b $x^6 - 2x^4 + 6x^3 + x^2 - 6x + 9$

Exercise 10C

1 **a** $x^3 - 3x^2$ **b** $x^4 - x^2$ **c** $2x^2 + 6x^3$
d $x^3 - x^4$ **e** $x^5 + 3x^4$ **f** $-3x^6 + 3x^3$
g $-2x^5 - 2x^4$ **h** $-x^7 + x^4$ **i** $-4x^7 + 8x^{10}$
2 **a** $x^5 + x^3 + 2x^2 + 2$
b $x^5 - x$
c $x^5 - x^4 - 3x^3 + 3x^2$
d $x^5 - x^3 - 2x^2 - 2x + 4$
e $x^5 + 2x^4 + 2x^3 - 2x^2 - 3x$
f $x^5 - 2x^4 + 5x^3 - 4x^2$
g $x^6 - x^5 + x^4 - 4x^3 + 2x^2 - x + 2$
h $x^6 - 5x^5 - x^4 + 8x^3 - 5x^2 - 2x + 2$
i $x^8 - x^6 + x^5 - 2x^4 - x^3 + 3x^2 + x - 3$
3 **a** $x^5 - 2x^4 + 2x^3 - 3x^2 + 3x - 1$
b $x^6 + 2x^4 - 2x^3 + x^2 - 2x + 1$
c $x^4 - 4x^3 + 6x^2 - 4x + 1$
4 **a** $x^5 + 3x^4 - x^3 - 9x^2 - 2x + 8$
b $x^4 + 2x^3 - 3x^2 - 4x + 4$
c $x^6 + 4x^5 + 2x^4 - 12x^3 - 15x^2 + 8x + 16$
5 **a** $x^3 + x^2 - 4x + 1$
b $x^3 - x^2 + 6x - 1$
c $2x^3 + 5x^2 - 23x + 5$
d $-x^5 + 5x^4 - 2x^3 + 5x^2 - x + 1$
e $-x^6 - 2x^4 - x^2 + 4$
f $-x^6 - x^4 - 10x^3 + 26x^2 - 10x + 1$
6 $\left(x^2 + x - 1\right)^4 = x^8 + 4x^7 + 2x^6 - 8x^5 - 5x^4 + 8x^3 + 2x^2 - 4x + 1$
7 $(x^2 - x - 1)^2 - (x^2 - x + 1)^2 = x^4 - 2x^3 - x^2 + 2x + 1 - (x^4 - 2x^3 + 3x^2 - 2x + 1) = 4x - 4x^2$ as required (or could use DOPS)
8 Yes. Multiplicative axiom $ab = ba$.
9 **a** 3 **b** 5 **c** 7 **d** 12
10 **a** m **b** m **c** $m + n$
d $2m$ **e** $2m$ **f** $3n$
11 **a** $x^4 - x^3 + x^2 - x$
b $x^5 + 2x^4 - 3x^3$
c $x^3 + 4x^2 + x - 6$
d $6x^3 + 23x^2 - 5x - 4$
e $15x^3 - 11x^2 - 48x + 20$
f $x^5 + 3x^4 - x^3 - 3x^2 - 2x - 6$

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10D

Building understanding

1 **a** 1 **b** 3 **c** 0

2 **a** If $182 \div 3 = 60$ remainder 2 then $182 = 3 \times 60 + 2$.
b If $2184 \div 5 = 436$ remainder 4 then $2184 = 5 \times 436 + 4$.
c If $617 \div 7 = 88$ remainder 1 then $617 = 7 \times 88 + 1$.

3 **a** x^2 **b** $2x^2$ **c** $4x$ **d** -7

Now you try

Example 8
a $(x-1)(x^2+2x-2)+1$
b $(x+2)(3x^2-8x+21)-44$

Exercise 10D

1 $P(x) = (x-1)(x^2+2x-1)+1$

2 $P(x) = (x-2)(x^2+3x+4)+11$

3 $P(x) = (x+2)(2x^2-5x+14)-30$

4 $P(x) = (x+1)(3x^2-4x+5)-3$

5 **a** $2x^3 - x^2 + 3x - 2 = (x-2)(2x^2+3x+9)+16$
b $2x^3 + 2x^2 - x - 3 = (x+2)(2x^2-2x+3)-9$
c $5x^3 - 2x^2 + 7x - 1 = (x+3)(5x^2-17x+58)-175$
d $-x^3 + x^2 - 10x + 4 = (x-4)(-x^2-3x-22)-84$
e $-2x^3 - 2x^2 - 5x + 7 = (x+4)(-2x^2+6x-29)+123$
f $-5x^3 + 11x^2 - 2x - 20 = (x-3)(-5x^2-4x-14)-62$

6 **a** $6x^4 - x^3 + 2x^2 - x + 2$
$= (x-3)\left(6x^3+17x^2+53x+158\right)+476$
b $8x^5 - 2x^4 + 3x^3 - x^2 - 4x - 6$
$= (x+1)\left(8x^4-10x^3+13x^2-14x+10\right)-16$

7 **a** $x^2 - 2x + 3 - \dfrac{5}{x+2}$
b $x^2 + 2x + 2 - \dfrac{1}{x-1}$
c $x^3 - 3x^2 + 9x - 27 + \dfrac{79}{x+3}$
d $x^3 + 4x^2 + 15x + 60 + \dfrac{240}{x-4}$

8 $-1, 1, 2$

9
$$
\begin{array}{r}
6x^2 - 7x - 3 \\
x-5\overline{)6x^3 - 37x^2 + 32x + 15} \\
\underline{6x^3 - 30x^2} \qquad\qquad \\
-7x^2 + 32x \qquad \\
\underline{-7x^2 + 35x} \qquad \\
-3x + 15 \\
\underline{-3x + 15} \\
0
\end{array}
$$
Remainder of 0, as required.

10 **a** 4 **b** $\dfrac{13}{8}$

11 **a** -8 **b** $\dfrac{-253}{16}$ **c** $-\dfrac{41}{27}$

12 **a** $x^3 - x^2 + 3x + 2 = (x^2-1)(x-1)+4x+1$
b $2x^3 + x^2 - 5x - 1 = (x^2+3)(2x+1)-11x-4$
c $5x^4 - x^2 + 2 = 5x(x^3-2) - x^2 + 10x + 2$

10E

Building understanding

1 **a** -1 **b** 41 **c** -19 **d** -141

2 **a** 3 **b** -2

3 0

Now you try

Example 9
a 2 **b** -37

Example 10
a No **b** Yes

Example 11
$k = -2$

Exercise 10E

1 **a** 3 **b** 11 **c** 27 **d** 57
e -127 **f** -33 **g** -13 **h** -69

2 **a** 3 **b** 20 **c** 36 **d** 5

3 **b**, **c** and **e** are factors of $P(x)$.

4 **b**, **d**, **f**, **g** are factors of $P(x)$.

5 **a** $x+1$ **b** $x-1$, $x+1$ or $x+2$
c $x+2$ **d** $x-2$

6 **a** $x-2$, $x-1$ and $x+1$ **b** $x-3$, $x-1$ and $x+2$
c $x-3$, $x-2$ and $x+1$ **d** $x-5$, $x-1$ and $x+4$

7 **a** -4 **b** -2 **c** -14 **d** 96

8 -38

9 **a** 5 **b** 1 **c** 5 **d** -3

10 **a** -2 **b** 23

11 **a** $a = -1$ and $b = 2$ **b** $a = 3$ and $b = -4$

Progress Quiz

1 **a** 4 **b** 13 **c** $a^2 - 2a + 5$

2 **a** and **b**

3 **a** Domain: all real x-values, range: all real y-values
b Domain: all real x-values, range: $y \geqslant -3$

4 The term involving $\sqrt{x}$ has a fractional index when written in index notation.

5 **a** 8
b 58
c 7

6 **a** 3 **b** 1 **c** 3 **d** $4x^3$

7 **a** $x^7 - 2x^5 + x^4$
b $x^4 + 2x^3 + 5x^2 - 2x - 6$

8 **a** $x^4 + x^3 + 3x + 2$
b $-x^4 + x^3 - x + 2$
c $x^8 + 4x^5 + 4x^2$
d $x^7 + x^5 + 4x^4 + 2x^2 + 4x$

9 **a** $P(x) = (x-2)(x^2-x+2)+11$
b $P(x) = (x-2)(2x^2+9x+15)+26$

10 **a** 1 **b** -23

11 $a = -3$

10F

Building understanding

1 $P(-1) = 0$

2 a $x = -3$ or 1 b $x = -3$ or 4

Now you try

Example 12

a $x = 2, -1, -6$ b $x = \frac{1}{2}, -3, -\frac{2}{5}$

Example 13

$x = -1, -4$ or 3

Exercise 10F

1 a $x = -3$, 1 or 2 b $x = -7, -2$ or 1
c $x = -4$, 3 or 4 d $x = -\frac{1}{2}, -\frac{1}{3}$ or 3
e $x = -\frac{2}{3}, -\frac{1}{2}$ or 3 f $x = -\frac{2}{7}, \frac{1}{4}$ or $\frac{2}{5}$
g $x = -\frac{12}{11}, -\frac{1}{2}$ or $-\frac{11}{3}$ h $x = -\frac{3}{5}, -\frac{2}{19}$ or $\frac{1}{2}$

2 a $(x-3)(x-2)(x+1)$; $x = -1$, 2 or 3
b $(x+1)(x+2)(x+3)$; $x = -3, -2$ or -1
c $(x-3)(x-2)(x-1)$; $x = 1$, 2 or 3
d $(x-4)(x-3)(x-1)$; $x = 1$, 3 or 4
e $(x-6)(x+1)(x+2)$; $x = -2, -1$ or 6
f $(x-2)(x+3)(x+5)$; $x = -5, -3$ or 2

3 a $x = 1$ or $1 + \sqrt{5}$ or $1 - \sqrt{5}$
b $x = -2$

4 a $x = -1$, 3 or 5 b $x = -3, -2$ or 1

5 a $x = -4$, 1 or 3 b $-2, -1$ or 3

6 a 3 b 4 c n

7 a $x^2(x-1)$; $x = 0$ or 1
b $x^2(x+1)$; $x = -1$ or 0
c $x(x-4)(x+3)$; $x = -3$, 0 or 4
d $2x^3(x+1)^2$; $x = -1$ or 0

8 $0 = x^4 + x^2 = x^2(x^2 + 1)$
No solution to $x^2 + 1 = 0$.
Thus, $x = 0$ is the only solution.

9 The discriminant of the quadratic is negative, implying solutions from the quadratic factor are not real. $x = 2$ is the only solution.

10 a $x = -4, -3, -2$ or 1 b $x = -2$ or 3
c $x = -3, -2$, 1 or 3 d $x = -2, \frac{1}{2}$, 1 or 2

10G

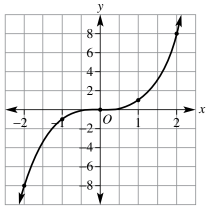
Building understanding

1 a 27 b 125
c -64 d -1000
e -216 f 2
g 1 h -3
i -5

2 a 0 b 54
c -2 d -128

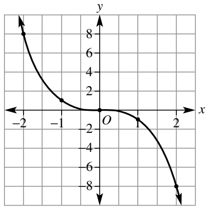
3 a

x	-2	-1	0	1	2
y	-8	-1	0	1	8



b

x	-2	-1	0	1	2
y	8	1	0	-1	-8



4 a (0, 2)
b D

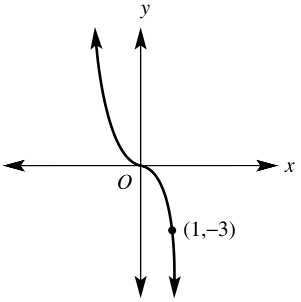
Now you try

Example 14

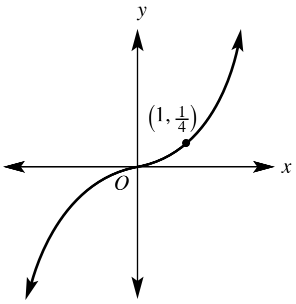
a $x = 2$
b $x = -3$
c $x = 4$

Example 15

a

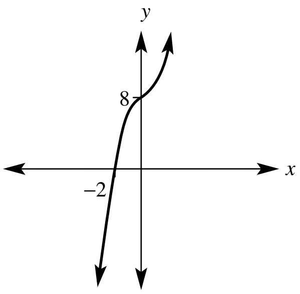


b

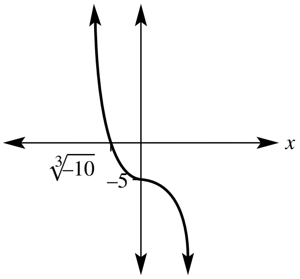


Example 16

a



b



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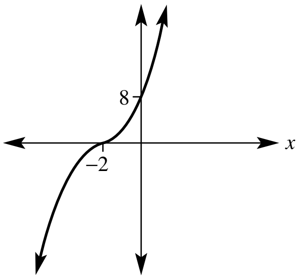


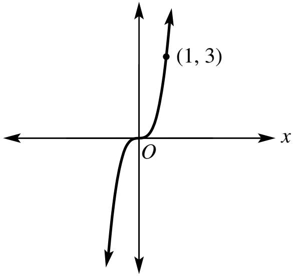
Exercise 10G

1 **a** $x = 4$ **b** $x = 5$ **c** $x = -3$
d $x = -2$ **e** $x = 10$ **f** $x = -3$
g $x = 6$ **h** $x = -3$ **i** $x = \frac{1}{2}$
j $x = \frac{1}{3}$ **k** $x = \frac{1}{5}$ **l** $x = \frac{1}{2}$

2 **a** $x = 1$ **b** $x = 3$ **c** $x = -1.4$
d $x = 3$ **e** $x = 4$ **f** $x = 6.1$
g $x = -2.9$ **h** $x = -4.3$ **i** $x = -3$

3 **a** $x = \sqrt[3]{21}$ **b** $x = \sqrt[3]{6}$
c $x = \sqrt[3]{-2}$ **d** $x = \sqrt[3]{18}$

4 **a**



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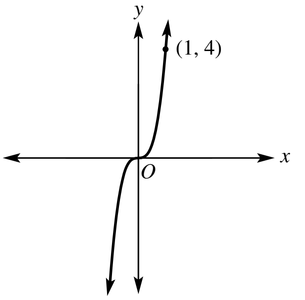

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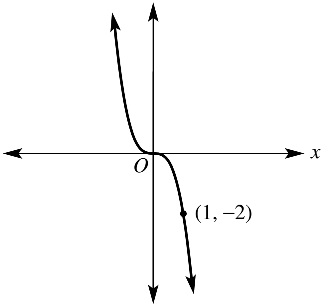

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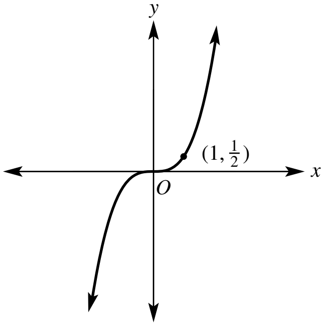

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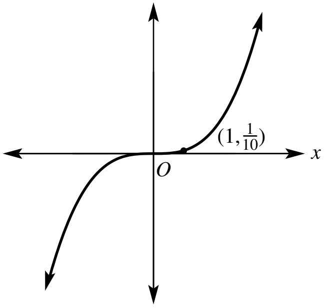

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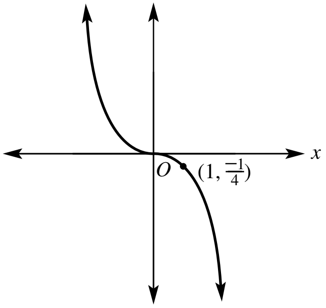
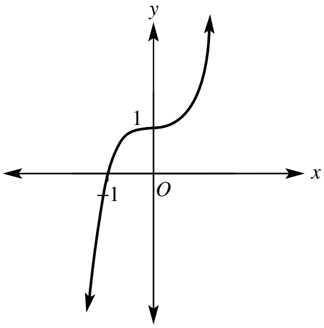
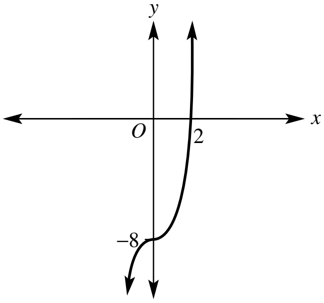


5 **a**



b



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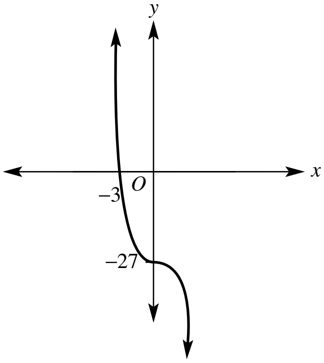

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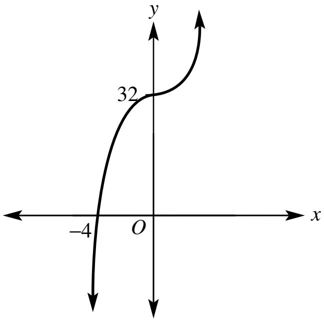

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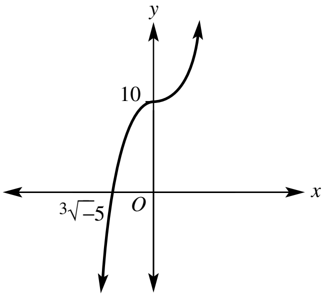

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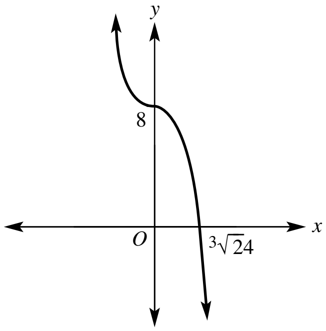
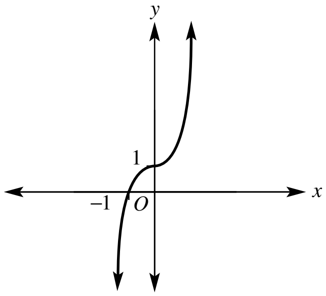


6 **a**



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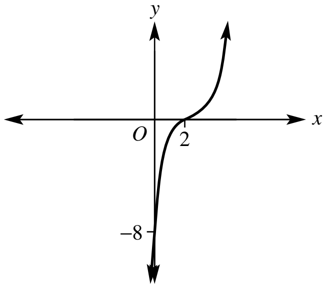

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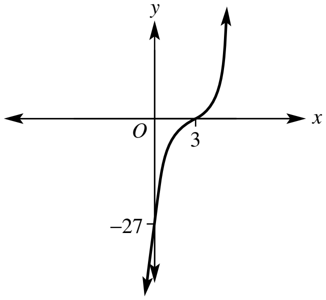

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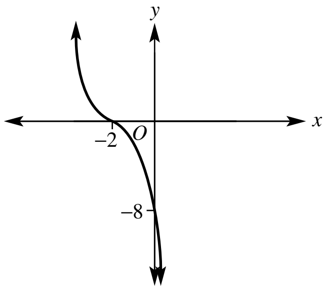

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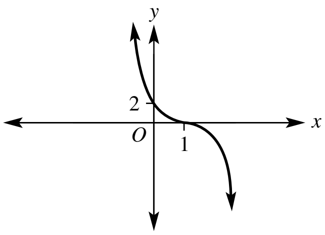
e



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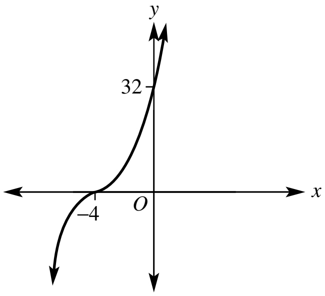


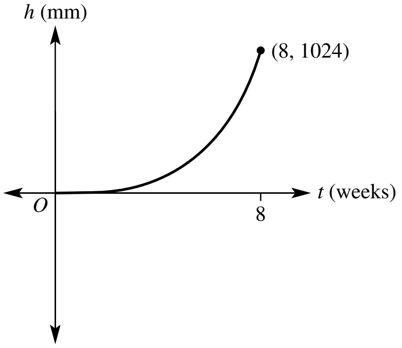
7 384 cm^2

8 **a** 54 mm

b 5 weeks

c After 8 weeks

d



9 **a** $x = 1$ **b** $x = 5$ **c** $x = -7$

d $x = 3$ **e** $x = -2$ **f** $x = -14$

10 6371 km

11 **a** $(-x)^2 = (-x) \times (-x) = x^2$

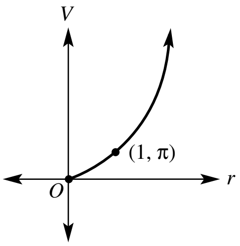
b $(-x)^3 = (-x) \times (-x) \times (-x) = -x^3$

c If n is odd, then $(-x)^n = -x^n$ for all values of x.

12 **a** $V = \pi r^3$

b $\sqrt{\frac{8000}{\pi}} = \frac{20}{\sqrt[3]{\pi}}$ units

c Volume increases by a factor of 8; i.e. $(2)^3$.



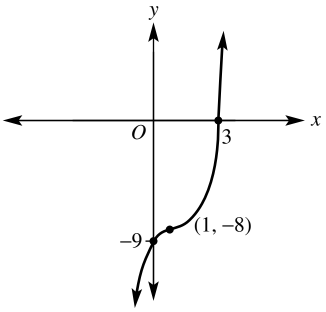
13 **a** $a = -\frac{1}{2}, d = 0$

b $a = 2, d = 3$

c $a = 1, d = -2$

d $a = 1, d = 2$

14 **a**



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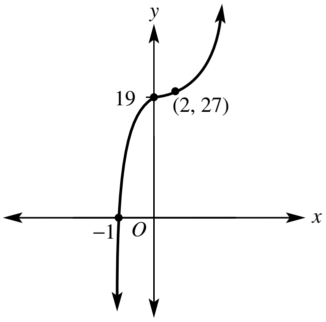

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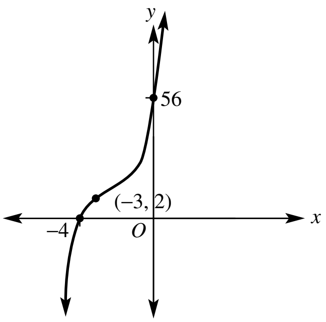

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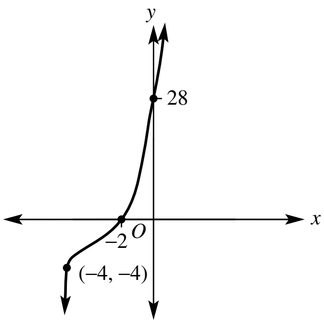

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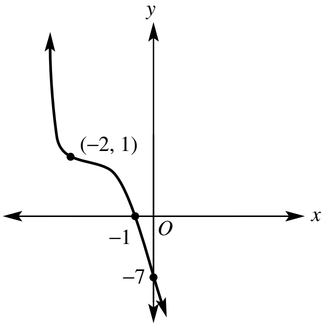

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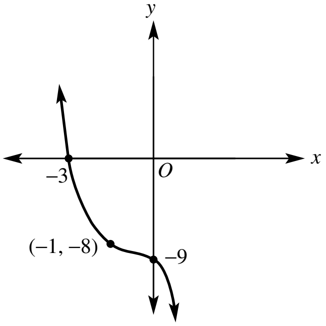
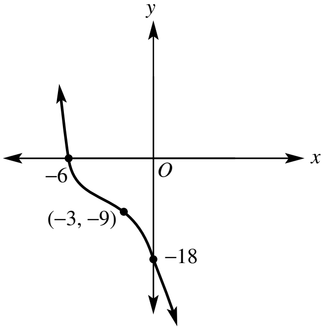
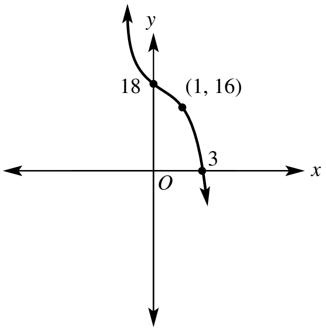


g



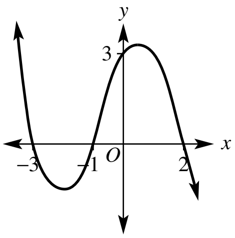
h



10H

Building understanding

1 a



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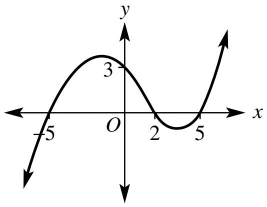


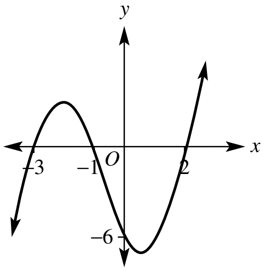
2 a y-intercept is $(0, 12)$
x-intercepts are $(-1, 0)$, $(3, 0)$, $(4, 0)$

b y-intercept is $(0, 0)$
x-intercept are $(-7, 0)$, $(0, 0)$, $(5, 0)$

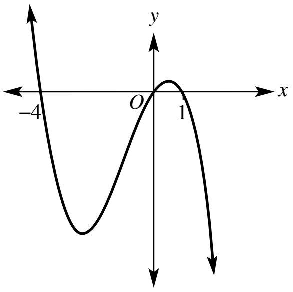
Now you try

Example 17

a

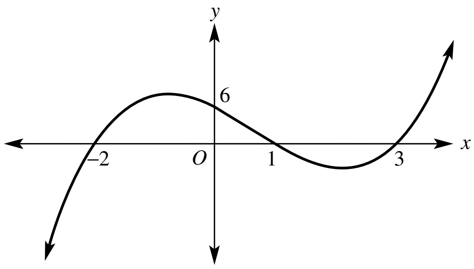


b

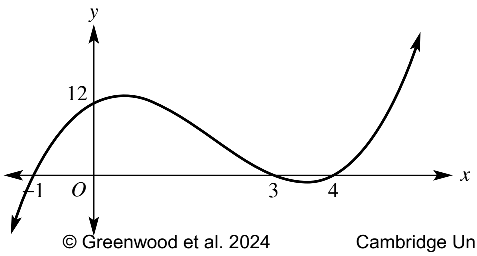


Exercise 10H

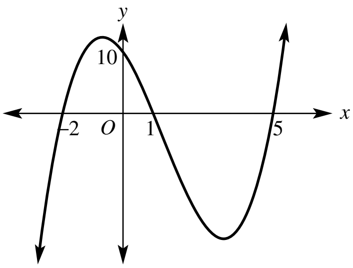
1 a y-intercept: 6
x-intercepts: $-2, 1, 3$



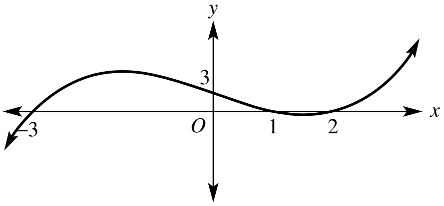
b y-intercept: 12
x-intercepts: $-1, 3, 4$



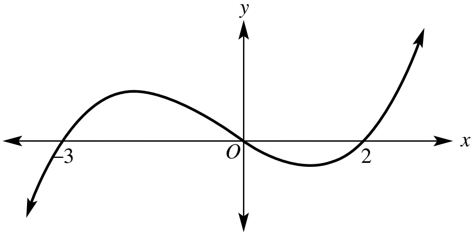
c y-intercept: 10
x-intercepts: $-2, 1, 5$



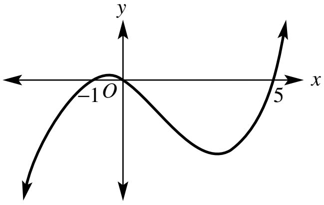
d y-intercept: 3
x-intercepts: $-3, 1, 2$



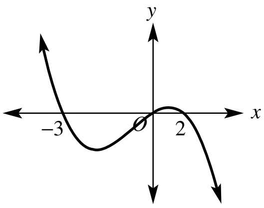
e y-intercept: 0
x-intercepts: $-3, 0, 2$



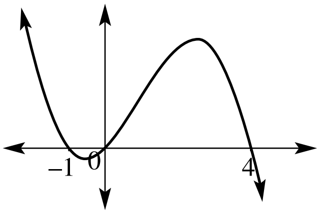
f y-intercept: 0
x-intercepts: $-1, 0, 5$



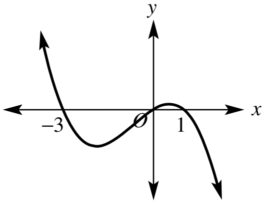
2 **a** y-intercept: 0
x-intercepts: $-3, 0, 2$



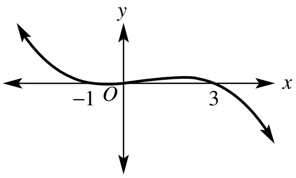
b y-intercept: 0
x-intercepts: $-1, 0, 4$



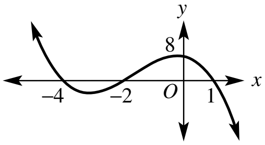
c y-intercept: 0
x-intercepts: $-3, 0, 1$



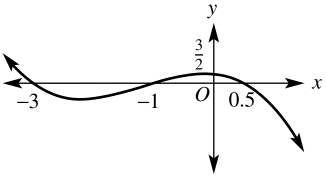
d y-intercept: 0
x-intercepts: $-1, 0, 3$



e y-intercept: 8
x-intercepts: $-4, -2, 1$



f y-intercept: $\frac{3}{2}$
x-intercepts: $-3, -1, \frac{1}{2}$



3 **a** P **b** P **c** N
d N **e** N **f** P

4 C

5 B

6 **a** $y = (x - 2)(x + 1)(x + 4)$
b $y = (x + 3)(x - 1)(x - 3)$
c $y = \frac{1}{2}x(x - 2)(x + 3)$
d $y = -\frac{1}{2}(x + 3)(x + 1)(x - 2)$

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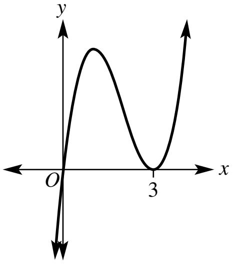
7 a



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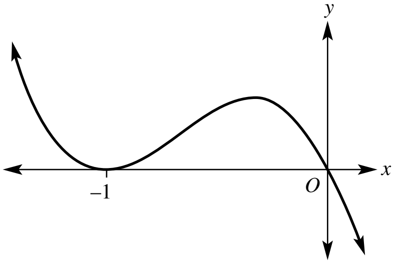

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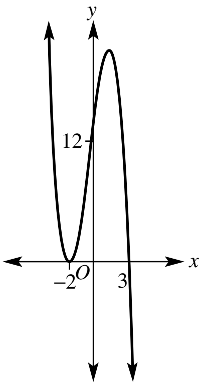

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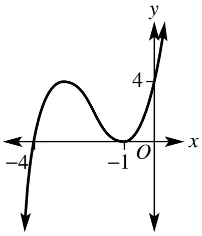

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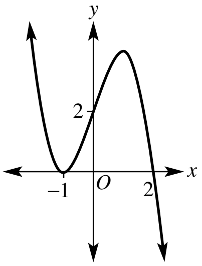
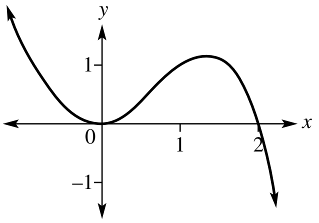
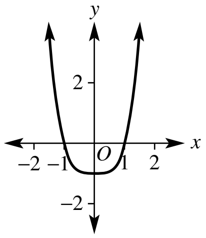


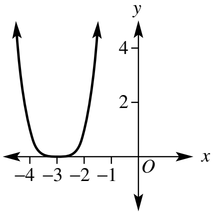
f



8 a



b



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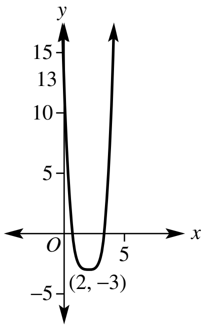
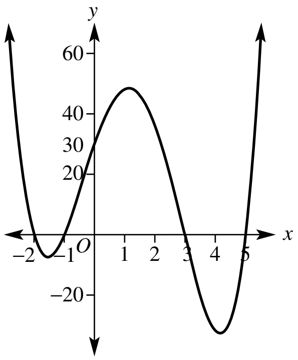


9 a



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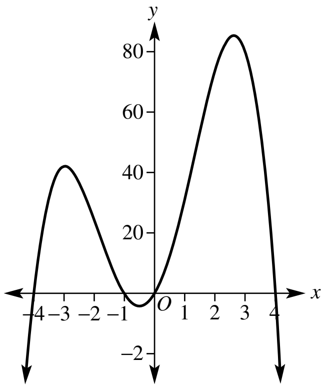

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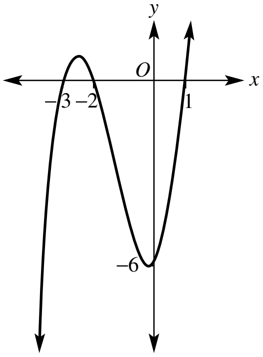
10 **a** **i** y-intercept $= (0, -6)$

ii $y = (x - 1)(x + 2)(x + 3)$

iii x-intercepts: $(-3, 0), (-2, 0), (-1, 0)$

iv

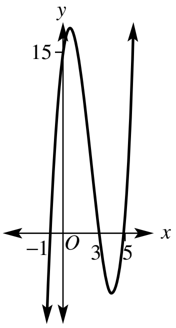


b **i** y-intercept $= (0, 15)$

ii $y = (x - 5)(x - 3)(x + 1)$

iii x-intercepts: $(-1, 0), (3, 0), (5, 0)$

iv

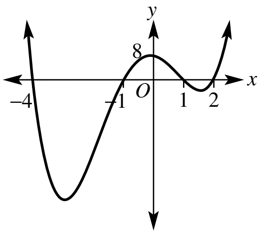


c **i** y-intercept $= (0, 8)$

ii $y = (x - 2)(x - 1)(x + 1)(x + 4)$

iii x-intercepts: $(-4, 0), (-1, 0), (1, 0), (2, 0)$

iv

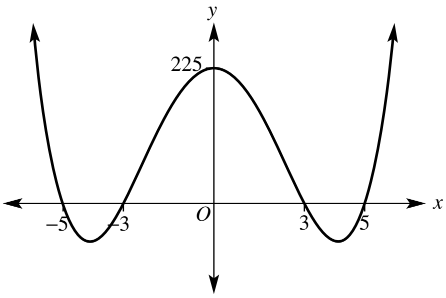


d **i** y-intercept $= (0, 225)$

ii $y = (x - 5)(x - 3)(x + 3)(x + 5)$

iii x-intercepts: $(-5, 0), (-3, 0), (3, 0), (5, 0)$

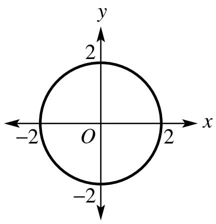
iv



10I

Building understanding

1



2 **a** $x = \pm\sqrt{5}$ **b** $x = \pm 4$ **c** $y = \pm\sqrt{11}$

3 **a** $(0, 0)$

b r

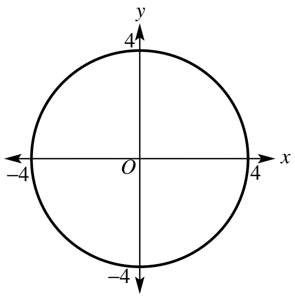
Now you try

Example 18

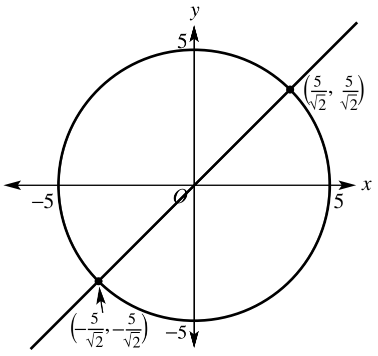
a $(0, 0)$ **b** $r = 4$

c $y = \pm\sqrt{15}$ **d** $x = \pm\frac{\sqrt{63}}{2} = \pm\frac{3\sqrt{7}}{2}$

e



Example 19



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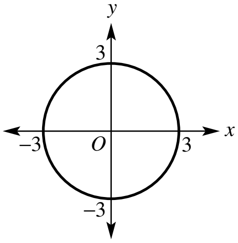
Exercise 10I

1 **a** (0, 0) **b** $r = 3$ **c** $y = \pm\sqrt{5}$

d $x = \pm\frac{\sqrt{27}}{2} = \frac{\pm 3\sqrt{3}}{2}$

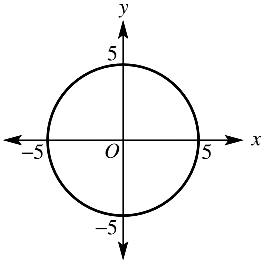
e



2 **a** (0, 0) **b** $r = 5$

c $y = \pm\frac{\sqrt{19}}{2}$ **d** $x = \pm 3$

e



3 **a** $r = 6$ **b** $r = 9$

c $r = 12$ **d** $r = \sqrt{5}$

e $r = \sqrt{14}$ **f** $r = \sqrt{20} = 2\sqrt{5}$

4 **a** $x^2 + y^2 = 4$ **b** $x^2 + y^2 = 49$

c $x^2 + y^2 = 10\,000$ **d** $x^2 + y^2 = 2601$

e $x^2 + y^2 = 6$ **f** $x^2 + y^2 = 10$

g $x^2 + y^2 = 1.21$ **h** $x^2 + y^2 = 0.25$

5 **a** $(1, \sqrt{3}), (1, -\sqrt{3})$

b $(-1, \sqrt{3}), (-1, -\sqrt{3})$

c $\left(\frac{1}{2}, \frac{\sqrt{15}}{2}\right), \left(\frac{1}{2}, -\frac{\sqrt{15}}{2}\right)$

d $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right), \left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$

e $(0, -2)$

f $(2, 0), (-2, 0)$

6 **a** x-intercepts: ± 1, y-intercepts: ± 1

b x-intercepts: ± 4, y-intercepts: ± 4

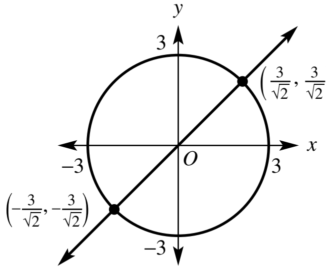
c x-intercepts: $\pm\sqrt{3}$, y-intercepts: $\pm\sqrt{3}$

d x-intercepts: $\pm\sqrt{11}$, y-intercepts: $\pm\sqrt{11}$

7 **a** $r = 2\sqrt{2}$ **b** $r = 2$ **c** $r = 3$

d $r = \sqrt{10}$ **e** $r = 2\sqrt{3}$ **f** $r = 2\sqrt{5}$

8



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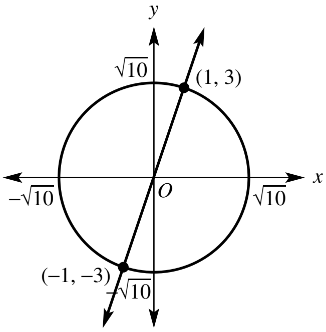

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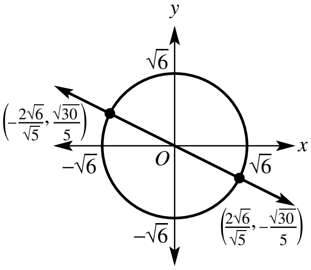

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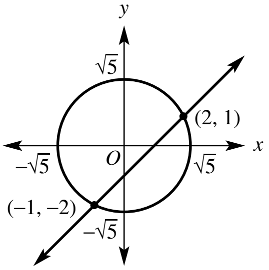


Chord length $= 3\sqrt{2}$ units

12 **a** $m = \pm\sqrt{3}$

b $m > \sqrt{3}$ or $m < -\sqrt{3}$

c $-\sqrt{3} < m < \sqrt{3}$

13 **a** D **b** A **c** E **d** C **e** F **f** B

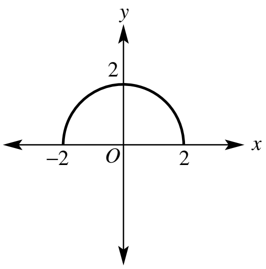
14 **a** $y = \pm\sqrt{16 - x^2} = \pm\sqrt{4^2 - x^2}$

b $x = \pm\sqrt{3 - y^2} = \pm\sqrt{\left(\sqrt{3}\right)^2 - y^2}$

15 **a** Radius of graph is 2, so points are 2 units from (0, 0); i.e.< 2.

b Radius of graph is 1, so points are 1 unit from (0, 0); i.e. −1 is the leftmost point, which is not as far as −2.

16 **a**



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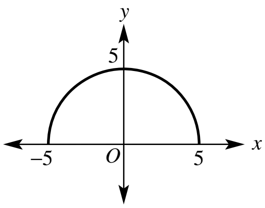
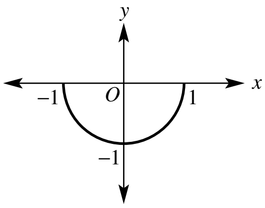
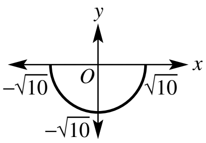
b



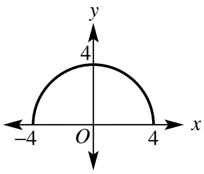
c



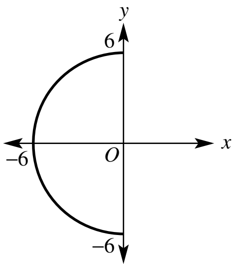
d



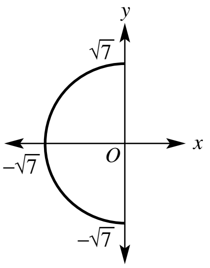
e



f



g



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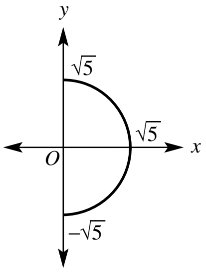
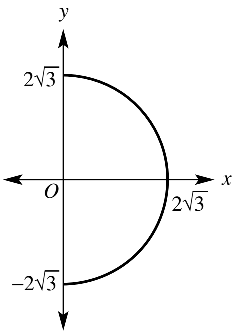


i



17 **a** $y = \sqrt{25 - x^2}$ **b** $y = -\sqrt{16 - x^2}$
c $x = \sqrt{4 - y^2}$ **d** $x = -\sqrt{1 - y^2}$
e $y = \sqrt{3 - x^2}$ **f** $y = -\sqrt{5 - x^2}$
g $x = \sqrt{10 - y^2}$ **h** $x = -\sqrt{8 - y^2}$
i $y = -\sqrt{18 - x^2}$

10J

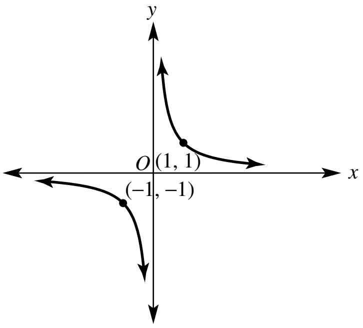
Building understanding

1 **a** C **b** A **c** B
2 **a** 1 ÷ 0.1, 1 ÷ 0.01, 1 ÷ 0.001, 1 ÷ 0.00001
b $x = \frac{1}{100}$
c 0.099
d 998

Now you try

Example 20

a



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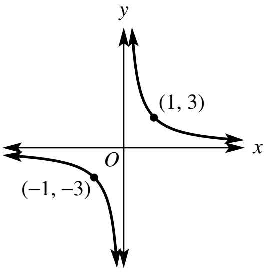

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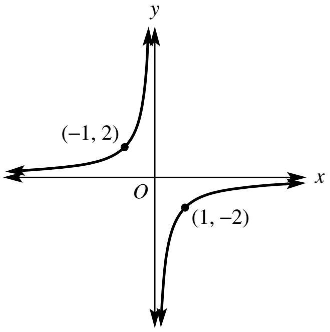


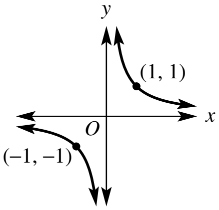
Example 21

a $\left(\frac{1}{4}, 4\right)$

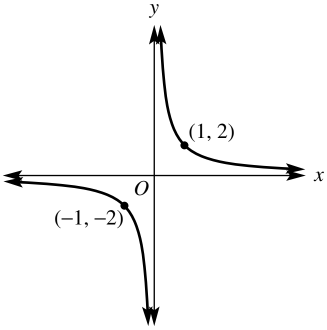
b $\left(-\frac{1}{\sqrt{3}}, -\sqrt{3}\right), \left(\frac{1}{\sqrt{3}}, \sqrt{3}\right)$

Exercise 10J

1 a



b



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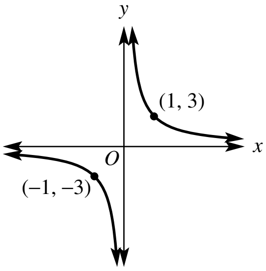
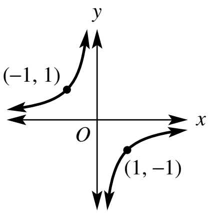
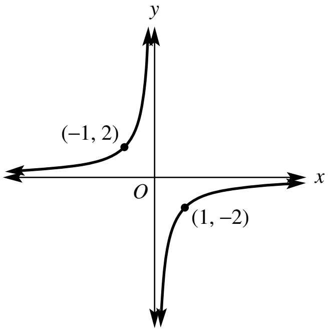


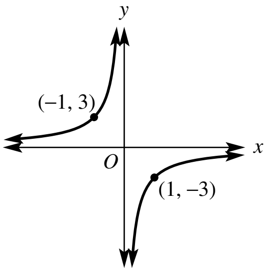
d



e



f



2 a (2, 1)
b $\left(4, \frac{1}{2}\right)$
c (−1, −2)
d $\left(-6, -\frac{1}{3}\right)$

3 a $\left(10, -\frac{1}{2}\right)$
b $\left(-4, \frac{5}{4}\right)$
c $\left(-7, \frac{5}{7}\right)$
d $\left(9, -\frac{5}{9}\right)$

4 a (1, 3)
b (3, 1)
c $\left(-\frac{3}{2}, -2\right)$
d $\left(-\frac{1}{2}, -6\right)$

5 a Yes
b Yes
c No
d No

6 a $\left(\frac{1}{2}, 2\right)$
b $\left(\frac{1}{6}, 6\right)$
c (−1, −1)
d $\left(-\frac{1}{10}, -10\right)$
e (1, 1), (−1, −1)
f $\left(-\frac{1}{2}, -2\right), \left(\frac{1}{2}, 2\right)$
g $\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right), \left(-\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$
h $\left(\frac{1}{\sqrt{5}}, \sqrt{5}\right), \left(-\frac{1}{\sqrt{5}}, -\sqrt{5}\right)$

7 a $\left(\frac{2}{3}, -3\right)$
b $\left(-\frac{1}{2}, 4\right)$
c $\left(4, -\frac{1}{2}\right)$
d $\left(-6, \frac{1}{3}\right)$
e (1, −2), (−1, 2)
f $\left(\frac{1}{2}, -4\right), \left(-\frac{1}{2}, 4\right)$
g (2, −1), (−2, 1)
h $(\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

8 a E
b C
c D
d B
e A
f F

9 Yes, $x = 0$ or $y = 0$.

10 a zero
b zero
c infinity
d negative infinity

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11 Greater the coefficient, the closer the graph is to the asymptote.

12 **a** **i** $x = \frac{1+\sqrt{5}}{2}, y = \frac{-1+\sqrt{5}}{2}$ or $x = \frac{1-\sqrt{5}}{2}, y = \frac{-1-\sqrt{5}}{2}$

ii $x = 1+\sqrt{2}, y = -1+\sqrt{2}$ or $x = 1-\sqrt{2}, y = -1-\sqrt{2}$

iii $x = -1+\sqrt{2}, y = 1+\sqrt{2}$ or $x = -1-\sqrt{2}, y = 1-\sqrt{2}$

b No intersection, $\Delta < 0$.

c $y = -x + 2, y = -x - 2$

10K

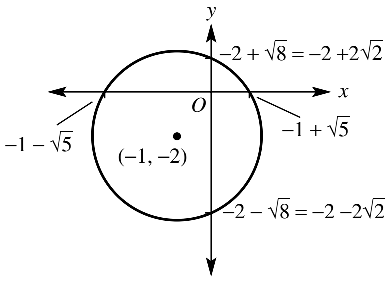
Building understanding

1 **a** down **b** right **c** left
d down **e** left **f** up
2 **a** $k = 3$ **b** $k = -1$ **c** $k = 4$
3 **a** $h = 0, k = 2$
b $h = 3, k = 0$
c $h = -2, k = -1$

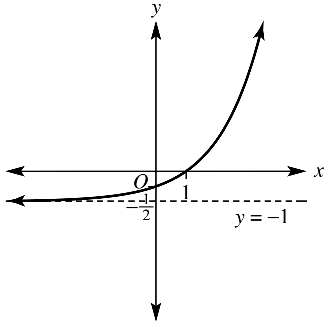
Now you try

Example 22

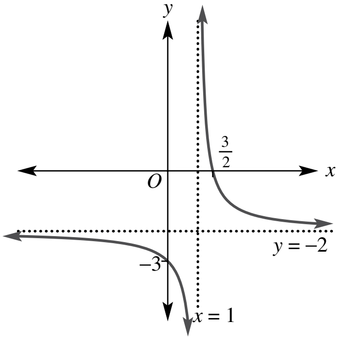
a



b

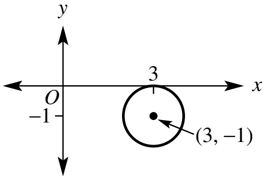


c

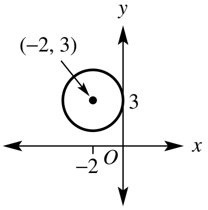


Exercise 10K

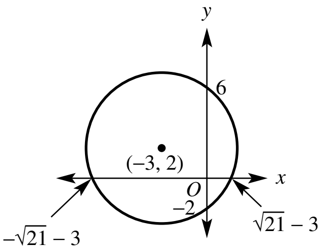
1 **a**



b



c



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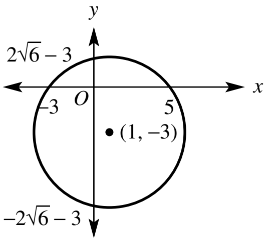

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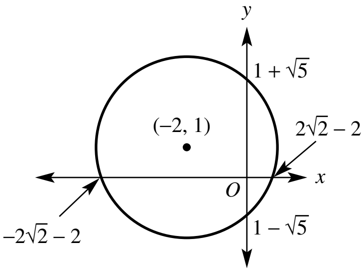

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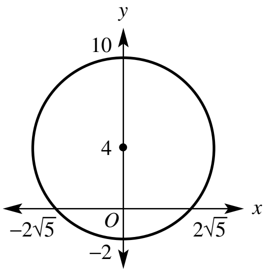

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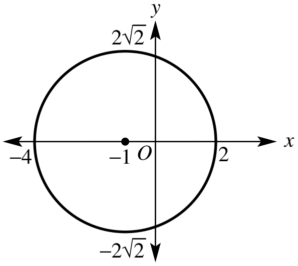

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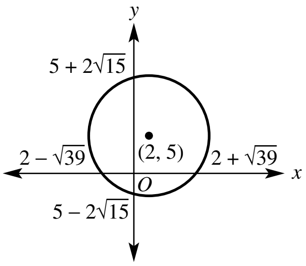

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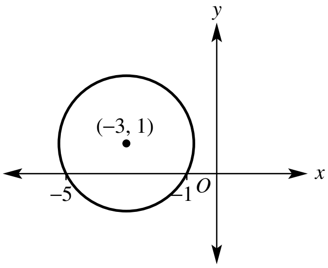

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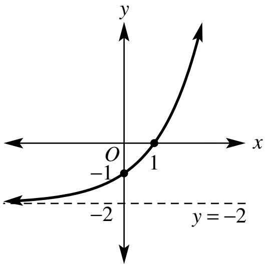

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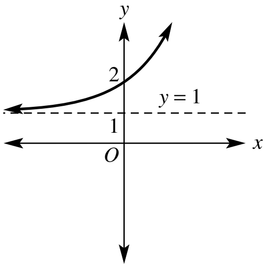

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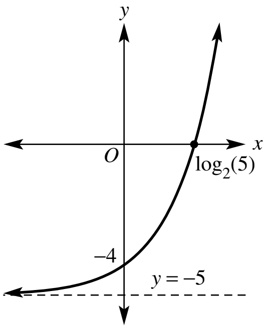

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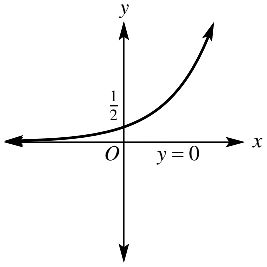

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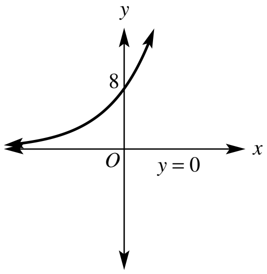

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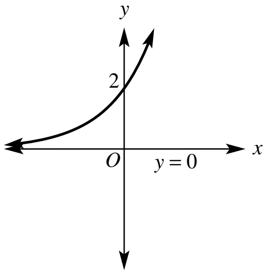

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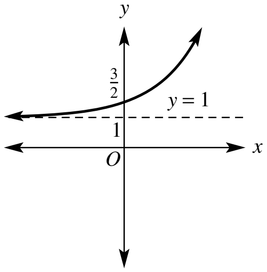

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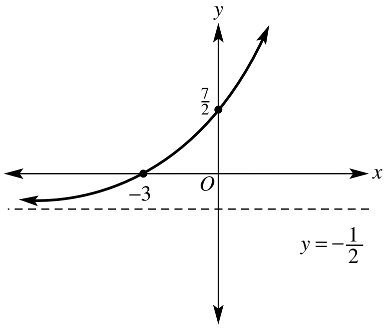

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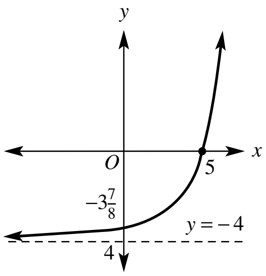
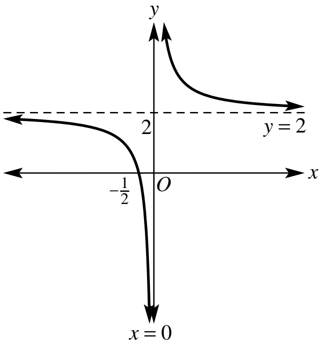
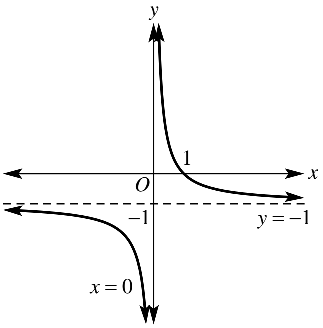

y
x
O
5
$-3\frac{7}{8}$
$y = -4$
4

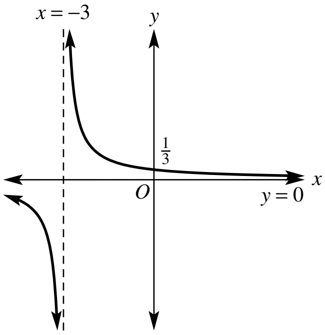
3 a


y
2
$y = 2$
x
$-\frac{1}{2}$
O
$x = 0$

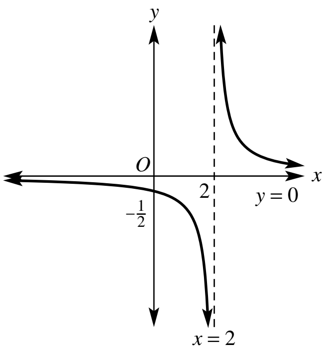
b


y
1
x
O
-1
$y = -1$
$x = 0$

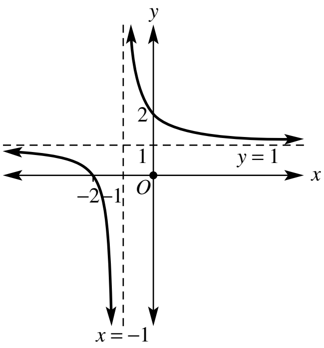
c


$x = -3$
y
$\frac{1}{3}$
x
O
$y = 0$

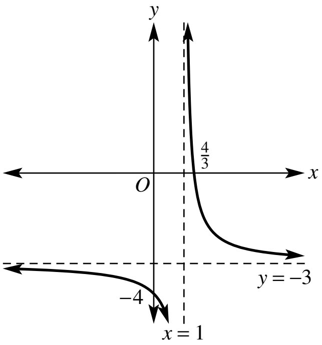
d


y
O
x
2
$y = 0$
$-\frac{1}{2}$
$x = 2$

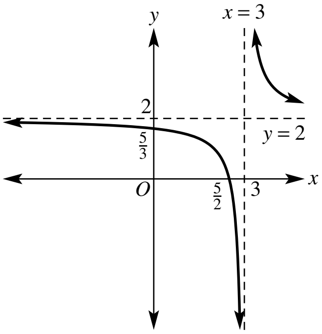
e


y
2
1
$y = 1$
x
-2
-1
O
$x = -1$

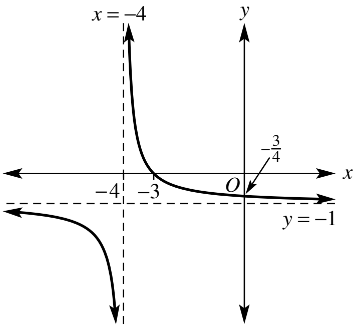
f


y
$\frac{4}{3}$
x
O
$y = -3$
-4
$x = 1$

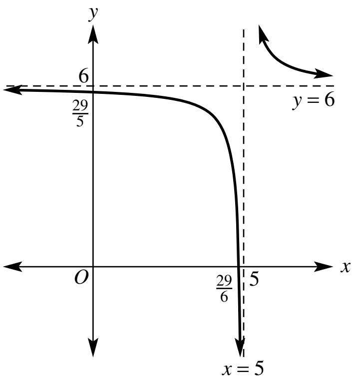
g


y
$x = 3$
2
$y = 2$
$\frac{5}{3}$
x
O
$\frac{5}{2}$
3

h


$x = -4$
y
$-\frac{3}{4}$
x
-4
-3
O
$y = -1$

i



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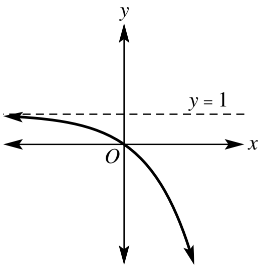

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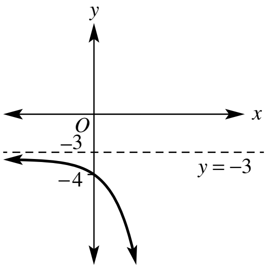

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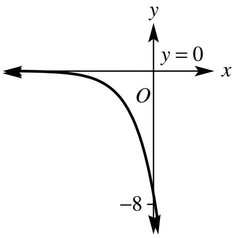

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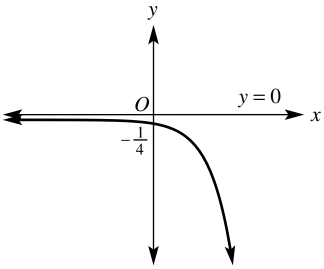

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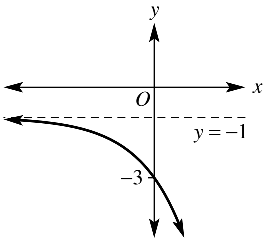

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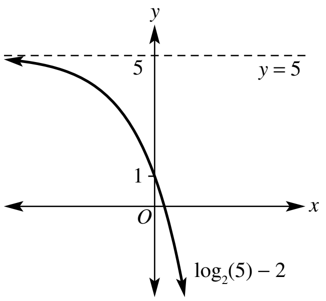
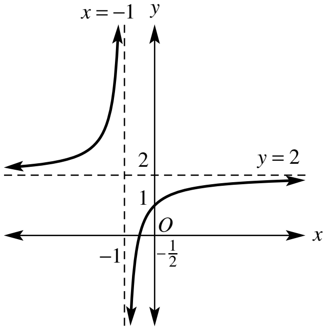


5 a



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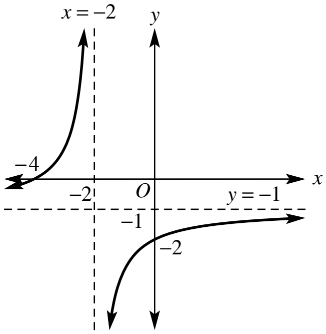

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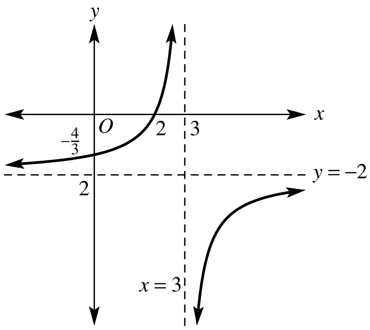

6 a $y = \dfrac{1}{x-2} - 1$ **b** $y = \dfrac{1}{x+1} + 3$

c $y = \dfrac{1}{x-1} + \dfrac{3}{2}$

7 a $\left(\dfrac{-3-\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}\right), \left(\dfrac{-3+\sqrt{5}}{2}, \dfrac{1+\sqrt{5}}{2}\right)$

b $(\sqrt{5}, 3+\sqrt{5}), (-\sqrt{5}, 3-\sqrt{5})$

c $\left(\dfrac{-1-\sqrt{11}}{2}, \sqrt{11}\right), \left(\dfrac{-1+\sqrt{11}}{2}, -\sqrt{11}\right)$

d $(1, 2), \left(-\dfrac{3}{5}, -\dfrac{6}{5}\right)$

e $(-6, 3), (-2, -1)$

f $(3, 0), (-3, -2)$

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8 a Max $x = 5$, min $x = 1$
  b Max $y = 0$, min $y = -4$

9 a $(x-2)^2 + (y-1)^2 = 8$ b $(x+2)^2 + y^2 = 25$
  c $(x+5)^2 + (y+3)^2 = 18$ d $y = \frac{1}{x-1} + 1$
  e $y = \frac{1}{x+2} - 1$ f $y = \frac{-1}{x+3}$

10 a Solving $\frac{1}{x} = -x$ would require $x^2 = -1$, which is not possible.
  b Circle has centre (1, −2) and radius 2, so maximum y-value on the circle is 0, which is less than 1.
  c Exponential graph rises more quickly than the straight line and this line sits below the curve.
  d Solving $\frac{2}{x+3} - 1 = \frac{1}{3x}$ gives a quadaratic with $\Delta < 0$, thus no points of intersection.

11 a $(x+2)^2 + (y-1)^2 = 4$, $C(-2, 1)$, $r = 2$
  b $(x+4)^2 + (y+5)^2 = 36$, $C(-4, -5)$, $r = 6$
  c $(x-3)^2 + (y-2)^2 = 16$, $C(3, 2)$, $r = 4$
  d $(x-1)^2 + (y+3)^2 = 15$, $C(1, -3)$, $r = \sqrt{15}$
  e $(x+5)^2 + (y+4)^2 = 24$, $C(-5, -4)$, $r = 2\sqrt{6}$
  f $(x+3)^2 + (y+3)^2 = 18$, $C(-3, -3)$, $r = 3\sqrt{2}$
  g $\left(x + \frac{3}{2}\right)^2 + (y-3)^2 = \frac{29}{4}$, $C\left(-\frac{3}{2}, 3\right)$, $r = \frac{\sqrt{29}}{2}$
  h $\left(x + \frac{5}{2}\right)^2 + (y-2)^2 = \frac{49}{4}$, $C\left(-\frac{5}{2}, 2\right)$, $r = \frac{7}{2}$
  i $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{3}{2}$, $C\left(\frac{1}{2}, -\frac{3}{2}\right)$, $r = \sqrt{\frac{3}{2}}$
  j $\left(x - \frac{3}{2}\right)^2 + \left(\mathrm{y} - \frac{5}{2}\right)^2 = \frac{25}{2}$, $C\left(\frac{3}{2}, \frac{5}{2}\right)$, $r = \frac{5}{\sqrt{2}}$

12 $(x+2)^2 + (y-3)^2 = -2$; radius can't be negative.

Problems and challenges

1 −2

2 $a = 5$, $b = -2$

3 Proof using long division required.
  a $(x^3 - a^3) \div (x - a) = x^2 + ax + a^2$
  b $(x^3 + a^3) \div (x + a) = x^2 - ax + a^2$

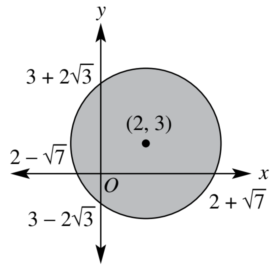
4 a $2 \leqslant x \leqslant 5$ or $x \leqslant -1$
  b $-4 < x < 1$ or $x > 4$

5 $y = \frac{1}{9}(x-3)^2(x+2)$

6 16

7 $y = -\frac{1}{10}x^2(x-3)(x+3)$

8 $(x-2)^2 + (y-3)^2 \leqslant 16$



9 $(x-2)^2 + (y+3)^2 = -15 + 9 + 4 = -2$, which is impossible

Answers to success criteria example questions

1 7

2 i A function
  ii Not a function
  iii Not a function

3 Domain is the set of all real x-values
  Range is the set of y-values, where $y \geq 6$

4 $3x^3 - 2x + 7$ is a polynomial of degree 3

5 −17

6 $2x^5 - x^4 + 10x^3 - 7x^2 + 14x - 8$

7 $x^2 - 3x + 6$

8 14

9 Yes, $(x + 2)$ is a factor

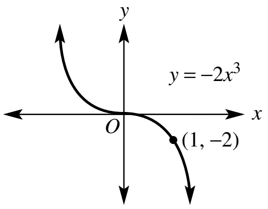
10 $k = -1$

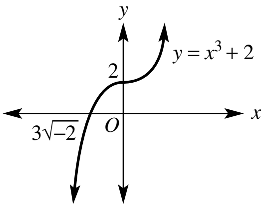
11 $x = -\frac{5}{2}$, $x = 3$, $x = -2$

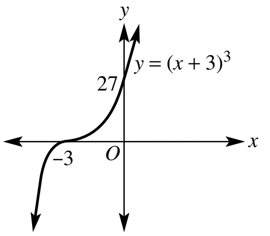
12 $x = -1$, $x = 2$, $x = 4$

13 $x = -5$; $x = 4$

14







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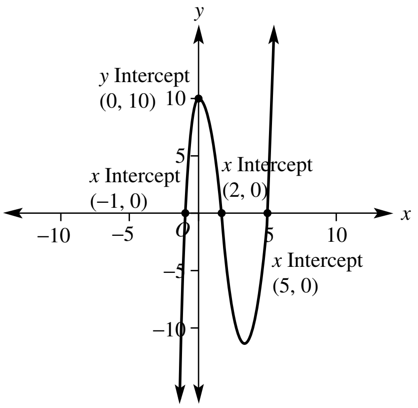

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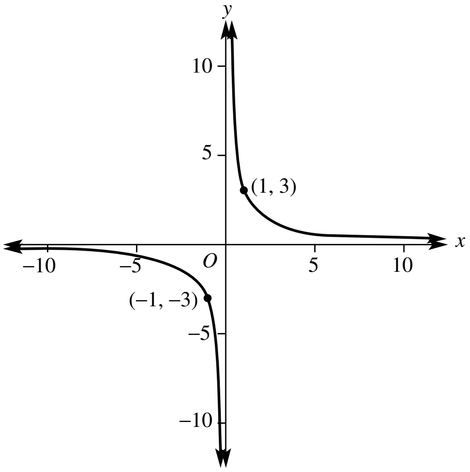
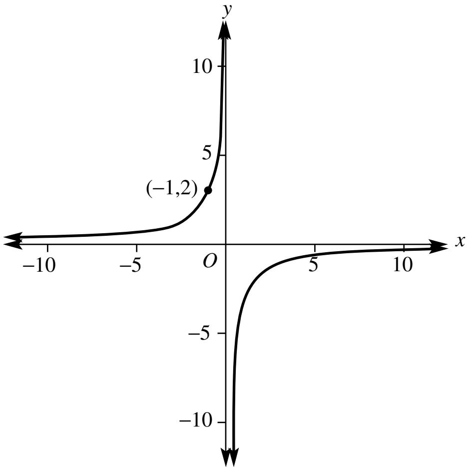
16 (0, 0); $r = 4$

17 $x = \frac{-3\sqrt{5}}{5}, y = \frac{-6\sqrt{5}}{5}$ and $x = \frac{3\sqrt{5}}{5}, y = \frac{6\sqrt{5}}{5}$

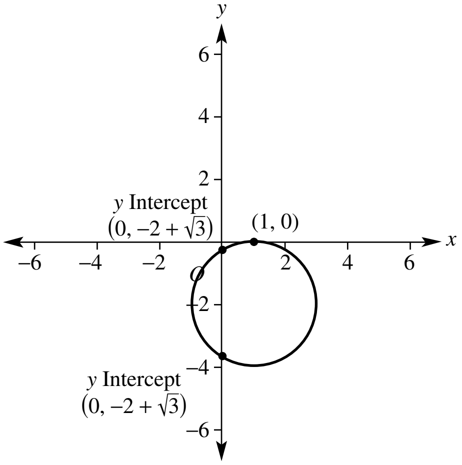
18

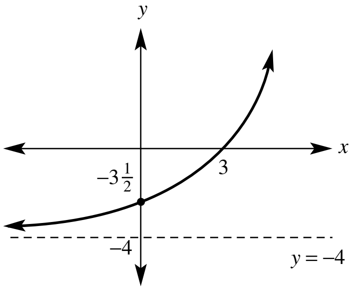


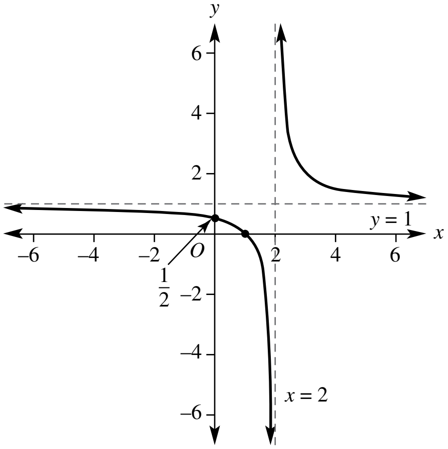


19 $x = -\frac{1}{3}, y = -3$ and $x = \frac{1}{3}, y = 3$

20







Short-answer questions

1 **a** 9
b 24
c 3
d $2k^2 - k + 3$

2 **a** All real x, all real y.
b All real x, only $y = 4$.
c Only $x = 1$, all real y
d All real x except $x = 0$, all real y except $y = 0$.
e All real x, $y \geqslant -3$.
f All real x, $y \geqslant -\frac{1}{4}$.

3 **a** −1 **b** 1
c −2 **d** −34

4 **a** $x^4 + 3x^2 + 2$
b $x^5 - x^4 - 3x^3$
c $x^5 + x^4 - 3x^3 - x^2 - x + 3$
d $x^6 + 2x^4 - 4x^3 + x^2 - 4x + 3$

5 **a** $x^3 + x^2 + 2x + 3 = (x - 1)(x^2 + 2x + 4) + 7$
b $x^3 - 3x^2 - x + 1 = (x + 1)(x^2 - 4x + 3) - 2$
c $2x^3 - x^2 + 4x - 7 = (x + 2)(2x^2 - 5x + 14) - 35$
d $-2x^3 - x^2 - 3x - 4 = -(x - 3)(2x^2 + 7x + 24) - 76$

6 **a** −3 **b** −39
c −91 **d** 41

7 **b**, **c** and **d** are factors.

8 **a** $x = -2$, 1 or 3 **b** $x = -\frac{1}{3}, \frac{3}{2}$ or 5

9 **a** $(x - 1)(x + 2)(x + 3) = 0$ $x = -3, -2$ or 1
b $(x + 2)(x - 5)(x - 6) = 0$ $x = -2, 5$ or 6

10 **a** $x = -3$ **b** $x = 4$

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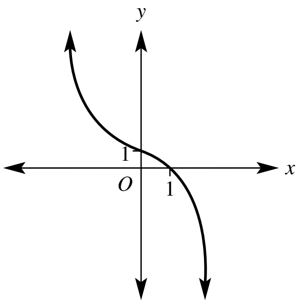
11 a



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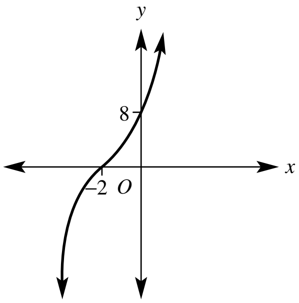

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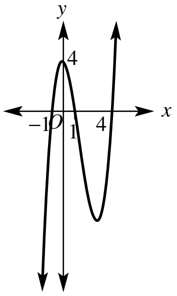

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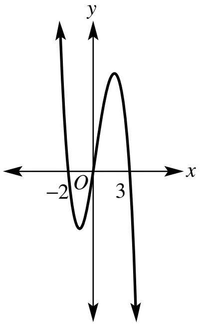

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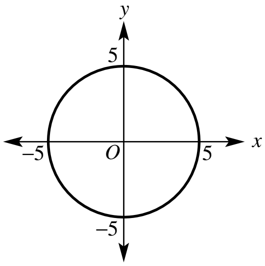

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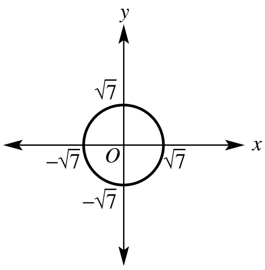

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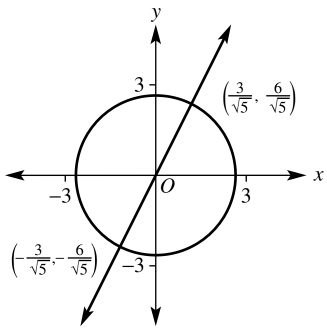
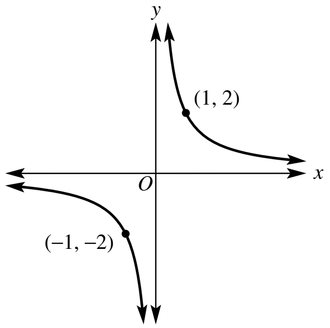


15 a



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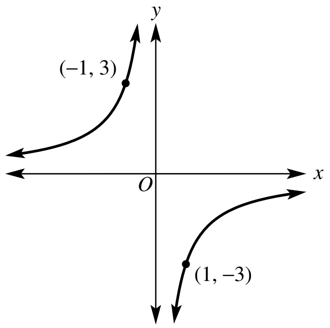
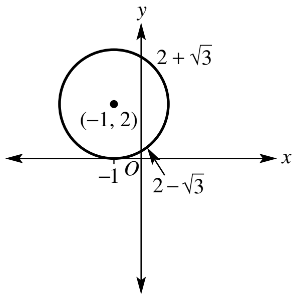


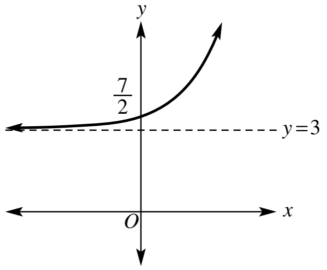
16 a $\left(\frac{4}{3}, 3\right)$

b $(\sqrt{2}, 2\sqrt{2})$ and $(-\sqrt{2}, -2\sqrt{2})$

17 a



b



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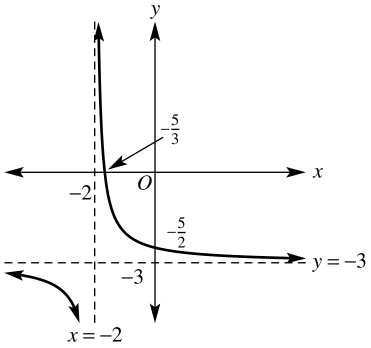
c



Multiple-choice questions

1	D	**2**	D	**3**	D	**4**	A	**5**	B
6	E	**7**	E	**8**	E	**9**	C	**10**	A

Extended-response questions

1 **a** **i** 32

ii 0

b There is no remainder; i.e. $P(1) = 0$.

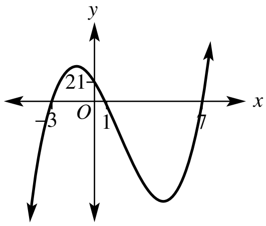
c $x^2 - 4x - 21$

d $(x-7)(x-1)(x+3)$

e $x = 7, 1$ or -3

f $P(0) = 21$

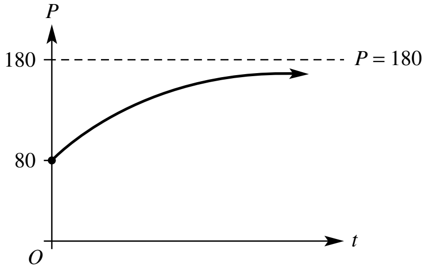
g



2 **a** 80

b 3 years

c



d It continues to increase with a limiting population of 180.

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Semester review 2

Trigonometry

Short-answer questions

1 **a** $x = 19.5$ **b** $\theta = 43.8°$, $y = 9.4$

2 **a** **i** 150°T **ii** 330°T

b **i** 310°T **ii** 130°T

3 **a** 32.174 m **b** 52.2°

4 **a** $x = 9.8$ **b** $\theta = 125.3°$

5 95.1°

6 **a** $\tan\theta$

b **i** $\theta = 155°$ **ii** $\theta = 35°$ **iii** $\theta = 42°$

c **i** $\frac{1}{2}$ **ii** $\frac{\sqrt{2}}{2}$ **iii** $-\frac{\sqrt{3}}{3}$ **iv** $\frac{\sqrt{3}}{2}$

7 **a** ≈ 0.34 **b** $\theta \approx 233°, 307°$

c Yes

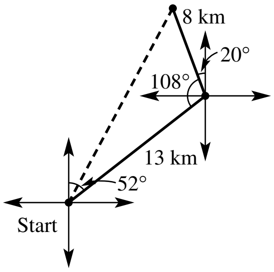
Multiple-choice questions

1 E **2** B **3** A **4** D **5** C

Extended-response question

a 104.3 m

b



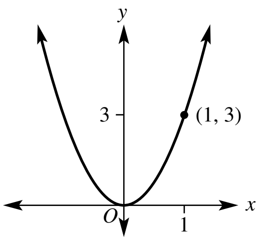
c 17.242 km

d 206°T

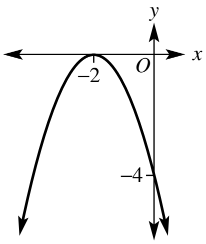
Parabolas and rates of change

Short-answer questions

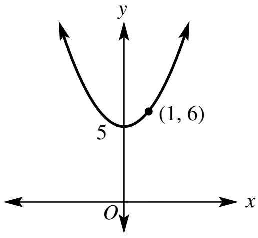
1 **a** Dilated by a factor of 3 from the x-axis.



b Reflected in x-axis and translated 2 units left.



c Translated 5 units up.

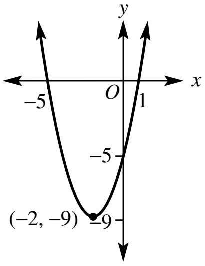


2 **a** $(0, -5)$

b $(-5, 0), (1, 0)$

c $(-2, -9)$

d

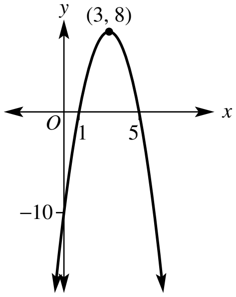


3 **a** Maximum at (3, 8).

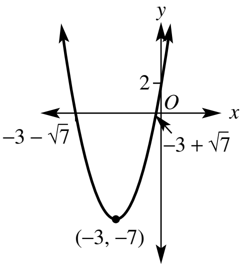
b $(0, -10)$

c $(1, 0), (5, 0)$

d



4 **a** $y = (x + 3)^2 - 7$



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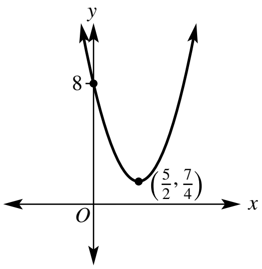
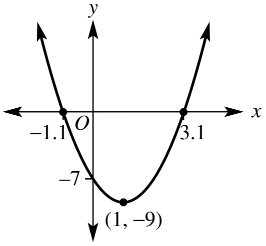
b $y = \left(x - \frac{5}{2}\right)^2 + \frac{7}{4}$



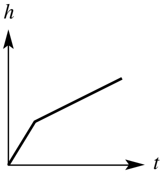
5 **a** Discriminant = 72, thus two x-intercepts.

b

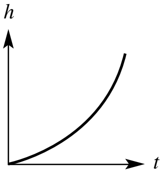


c (1, −9) and (−2, 9)

6 **a**



b



7 **a** 6

b $m_{AC} = -\frac{4}{5}$, $m_{BC} = -\frac{3}{2}$

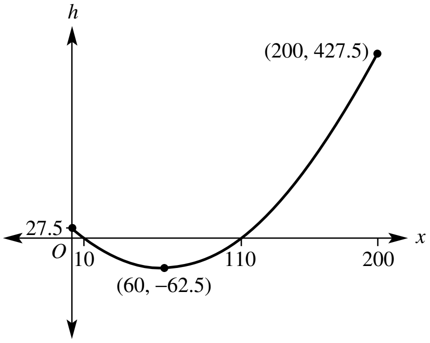
BC is a better approximation as it is closer to the gradient at C.

Multiple-choice questions

1 D **2** A **3** D **4** B **5** D

Extended-response question

a



b 27.5 m

c 10 m and 110 m from start

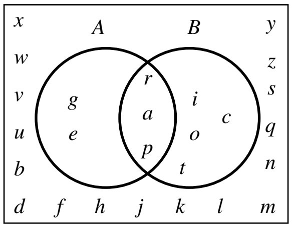
d 427.5 m

e 62.5 m

Probability and counting techniques

Short-answer questions

1 **a**



b **i** $\frac{5}{26}$ **ii** $\frac{3}{26}$ **iii** $\frac{9}{26}$ **iv** $\frac{19}{26}$

c No, $A \cap B \neq \varnothing$

2 **a**

	B	**B′**	
A	3	1	4
A′	4	4	8
	7	5	12

b **i** 3 **ii** 4 **iii** 5 **iv** 8

c **i** $\frac{1}{4}$ **ii** $\frac{1}{12}$ **iii** $\frac{7}{12}$ **iv** $\frac{3}{4}$

3 **a** 0.18 **b** 0.37

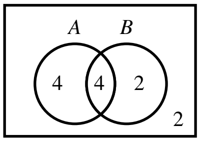
4 **a**

		Die 1			
		1	2	3	4
Die 2	1	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

b 16

c **i** $\frac{3}{16}$ **ii** $\frac{5}{8}$ **iii** $\frac{1}{5}$

5 **a**



b **i** $\frac{2}{3}$ **ii** $\frac{2}{3}$

c Yes they are, since $\Pr(A \mid B) = \Pr(A)$

6 **a** 24 **b** 210 **c** 15 **d** 1

7 **a** 120 **b** 240

8 **a** 120 **b** 36 **c** $\frac{1}{36}$

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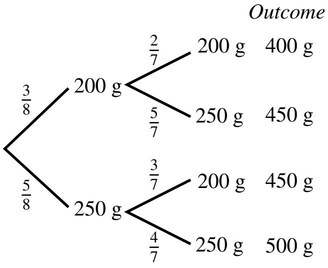
Multiple-choice questions

1 C **2** E **3** B **4** D **5** B

Extended-response question



a **i** $\frac{3}{28}$ **ii** $\frac{15}{28}$ **iii** $\frac{5}{14}$

b $\frac{9}{14}$

c $\frac{3}{5}$

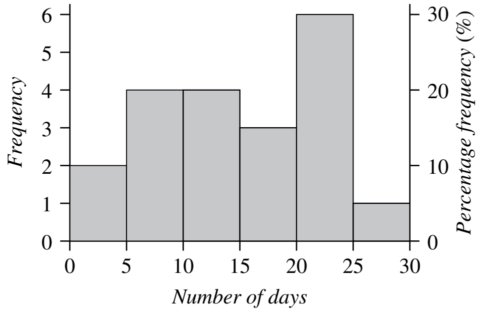
Statistics

Short-answer questions

1 **a**

Class interval	Frequency	Percentage frequency
0–	2	10%
5–	4	20%
10–	4	20%
15–	3	15%
20–	6	30%
25–30	1	5%
Total	**20**	**100%**

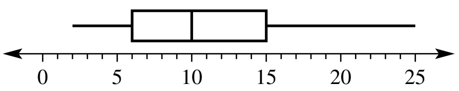
b



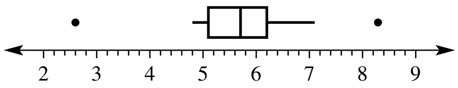
c **i** 14 **ii** 50%

iii 20–24 days, those that maybe catch public transport to work or school each week day.

2 **a**



b

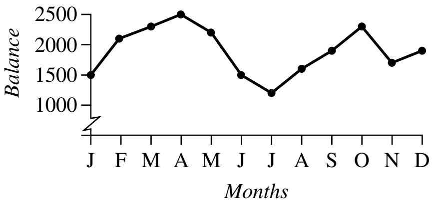


3 **a** 10

b 12%

c Yes, a much larger proportion of adults disagreed that watching their team was better in person.

4 **a**

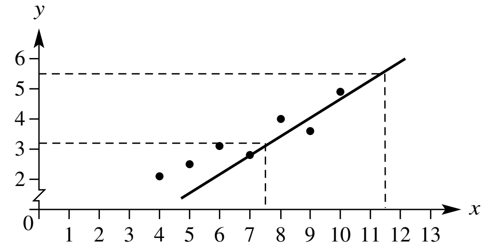


b Balance fluctuated throughout the year but ended up with more money after 12 months.

c May and June

d Increase of $500

5 **a, c**



b Positive

d **i** ≈ 3.2 **ii** ≈ 11.5

6 **a** **i** Under 40 **ii** Over 40

b Over 40: mean = 11, standard deviation = 7.3; under 40: mean = 24.1, standard deviation = 12.6

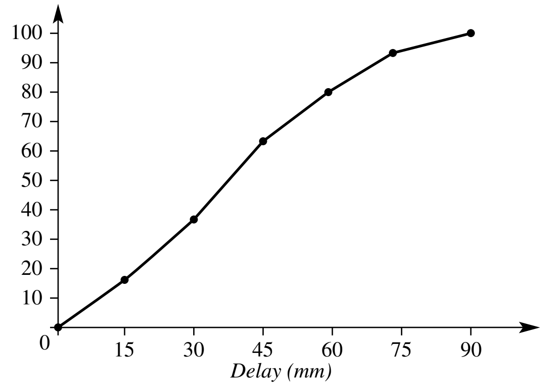
Multiple-choice questions

1 E **2** B **3** C **4** B **5** A

Extended-response question

a

Delay mins	Frequency	Cumulative frequency	Percentage cumulative frequency (1dp)
0–	5	5	16.7
15–	6	11	36.7
30–	8	19	63.3
45–	5	24	80
60–	4	28	93.3
75–90	2	30	100



b **i** 51 mins **ii** 37 mins, the median

c 54%

Polynomials, functions and graphs

Short-answer questions

1 **a** Each x-value produces a unique y-value (any vertical line will cut the graph at most once).

b **i** 3

ii 12

iii $(a-2)^2+3$

c All real x, $y \geqslant 3$

2 **a** **i** 6 **ii** 0 **iii** −49 **iv** −5

b **i** $2x^6+6x^5-11x^4-25x^3+34x+24$

ii $4x^6-12x^4-16x^3+9x^2+24x+16$

3 $P(x)=(x-3)(x^2-x-1)+4$

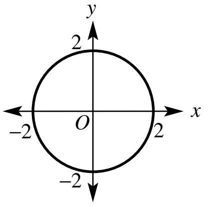
4 **a** −24, not a factor **b** 0, a factor

c −40, not a factor

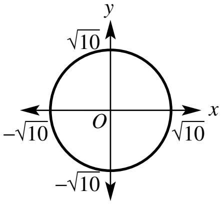
5 **a** $x=-1, 3$ or -6 **b** $x=0, \frac{5}{2}$ or $-\frac{2}{3}$

c $x=-4, -2$ or 1 **d** $x=-1, \frac{1}{2}$ or 2

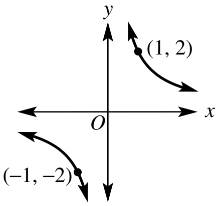
6 **a**



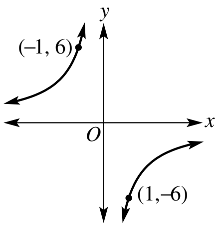
b



c

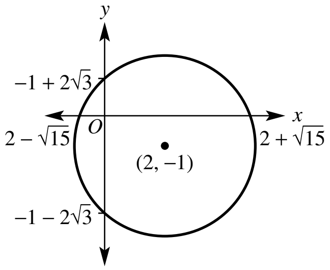


d



7 **a** $(\sqrt{3}, 2\sqrt{3}),\ (-\sqrt{3}, -2\sqrt{3})$ **b** $\left(\frac{1}{2}, 4\right), \left(-\frac{1}{2}, -4\right)$

8 **a**



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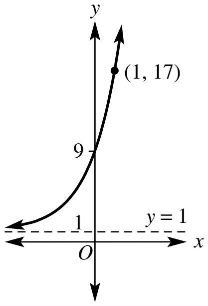

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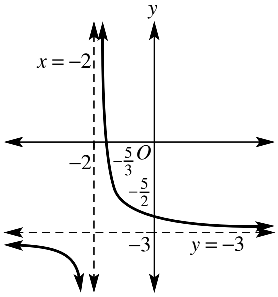

Multiple-choice questions

1 D **2** C **3** D **4** C **5** B

Extended-response question

a $P(-3)=0$

b $P(x)=-(x+3)(2x-1)(x-4)$

c