The Mathematical Association of Victoria

2006 FURTHER MATHEMATICS

VCAA Sample Examination 1 and 2

Suggested Answers and Solutions

Examination 1

The sample examination questions for Further Mathematics Examination 1 address new content areas from the core and each of the six modules. A full set of questions has been provided for the new Application Module 6: Matrices. The sample exam is found at:
www.vcaa.vic.edu.au/vce/studies/mathematics/further/pastexams/sample/fm1sample06.pdf

Examination 2

The sample examination questions for Further Mathematics Examination 2 provide a full set of extended-response questions for the new Application Module 6: Matrices. The sample exam is found at:
www.vcaa.vic.edu.au/vce/studies/mathematics/further/pastexams/sample/fm2sample06.pdf

These answers and solutions have been written and published to assist students in their preparations for the 2006 Further Mathematics Examinations. The answers and solutions do not necessarily reflect the views of the Victorian Curriculum and Assessment Authority.

Published by The Mathematical Association of Victoria
"Cliveden", 61 Blyth Street, Brunswick, 3056
Phone: (03) 9380 2399 Fax: (03) 9389 0399
E-mail: office@mav.vic.edu.au website: http://www.mav.vic.edu.au

Further Mathematics Examination 1
Suggested answers and solutions

Core Data analysis

1A z = -2.1 is in the range where z < -2
95% of the data values lie within two standard deviations (z=2) from the mean.
Hence 2.5% of the data values have z scores less than -2 and 2.5% have z scores greater than +2
Since the student's standard score is less than -2, his score is within the bottom 2.5%

2A The mean is 55 and standard deviation is 2.85
For this student, z = -2.1 means his score is (2.1 lots of 2.85) below the mean
His score is $55 - 2.1\times 2.85 = 49.015$ which rounds to 49
This is really a simple interpretation of the formula $z = \dfrac{x - \bar{x}}{s}$ that some may choose to use.
Substituting into this formula gives $s = \dfrac{20+30+18}{2} = 34$
Multiplying both sides of this equation produces $2.85 \times (^-2.1) = x - 55$
Adding 55 to both sides now produces $55 + 2.85 \times (^-2.1) = x$
This is the same as $x = 55 - 2.1\times 2.85 = 49.015$

3B IQR for this data set is $Q_3 - Q_1 = 32 - 22 = 10$
$1.5 \times IQR = 1.5 \times 10 = 15$
Any outliers would be less than $Q_1 - 15 = 22 - 15 = 7$ or greater than $Q_3 + 15 = 32 + 15 = 47$
Data values 2 and 5 are both less than $Q_1 - 1.5 \times IQR$ and so are outliers

4B The slope in the equation $y = a + bx$ is the value of b
Substitute into the formula $b = r\dfrac{s_y}{s_x}$

$$b = 0.5675 \times \frac{67.98}{8.67} = 4.4520$$

This rounds to give $b = 4.45$

5C Winter 2007 is Quarter number 7
The seasonalised sales figure for Winter 2007 comes from the formula given
$\mathit{Deseasonalised\ sales} = 230 + 45.6 \times 7 = 540.2(\$000) = \$549\,200$

The deseasonalised sales for winter 2007 is found from:
Seaonalised sales = (Deseasonalised sales) x (Seasonal index) = 549 200 x 0.85 = $466 820

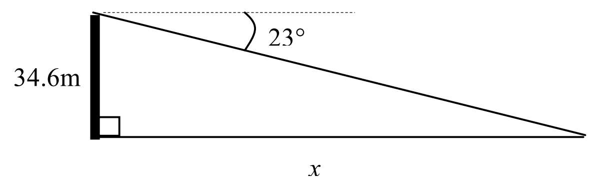
Module 1 Number patterns and their applications

1D In a Fibonnaci sequence, $t_{26} + t_{27} = t_{28}$
Then, $t_{27} = t_{28} - t_{26}$
Therefore, $t_{27} = 317811 - 121393 = 196418$

2D Successive values of t_n can be found up to t_6 by using the given rule
$t_3 = t_1 + t_2 = 1 + 4 = 5$
$t_4 = t_2 + t_3 = 4 + 5 = 9$
$t_5 = t_3 + t_4 = 5 + 9 = 14$
$t_6 = t_4 + t_5 = 9 + 14 = 23$

Module 2

1D A diagram may assist to see x is required.



$$\tan 23^0 = \frac{34.6}{x}$$

$$so\ x = \frac{34.6}{\tan 23^0}$$ which rounds to 81.5m

2C Use Heron's rule $A = \sqrt{s(s-a)(s-b)(s-c)}$ where s is half the perimeter

ie $s = \dfrac{a+b+c}{2}$

$$s = \frac{20+30+18}{2} = 34$$

Therefore $A = \sqrt{34(34-20)(34-30)(34-18)}$ which rounds to 175cm²

3C Volume of cylinder = $\pi r^2 h = \pi \times 2.9^2 \times 3.3 = 87.189$

Volume of cone $= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi \times 2.9^2 \times 1.8 = 15.852$

Total volume = 87.189 + 15.852 = 103.041 which rounds to 103m^3

Note:
This could be simplified if you know some algebra as follows:
Total volume = volume of cylinder + volume of cone

$= \pi r^2 h_1 + \frac{1}{3}\pi r^2 h_2$

$= \pi r^2 (h_1 + \frac{1}{3} h_2)$ by taking out common factors

$= \pi \times 2.9^2 \times (3.3 + \frac{1}{3} \times 1.8) = \pi \times 2.9^2 \times (3.3 + 0.6)$

$= \pi \times 2.9^2 \times 3.9 = 103.041$

Module 4 Business and related mathematics

1D Convert 23.5% to a decimal to get 0.235.
Then multiply 9650 x 0.235 = 2267.75

2C Finding the price in successive years involves multiplying
previous year's price × increase factor for this year

We do not need to work out the actual price for any year other than 2008.
Hence, we can eventually do only one multiple calculation for the year 2008.
2005 price = $12.50

2006 price = \$12.50 × 1.023
2007 price = 2006 price × 1.031 = (\$12.50 × 1.023) × 1.031
2008 price = 2007 price × 1.035 = {(\$12.50 × 1.023) × 1.031} × 1.035
= \$12.50 × 1.023 × 1.031 × 1.035 = 13.6453 which rounds to \$13.65

3B In a perpetuity, each regular payment is exactly equal to the interest earned by the investment in that compounding period. The principal value of the investment remains constant throughout.
All regular payments are equal. There is no compounding effect and interest earned may be calculated on a *simple interest* basis

The question states that interest is calculated monthly. Since the investment is defined as a perpetuity, it follows that payments are also made monthly. Therefore, we must work with a monthly interest rate and not an annual one.

An annual interest rate of 3.5% becomes $\frac{3.5}{12}$% per month. This is the same as $\frac{0.035}{12}$

Therefore, the interest earned in one month is $100\,000 \times \frac{0.035}{12}$ = \$291.67 after rounding to two decimal places. This is the value of the regular payment from the perpetuity.

This is the same as using the *simple interest* formula $I = \frac{Prt}{100}$ where P = 100 000, $r = \frac{3.5}{12}$ and t = 1.

Alternatively, we could also use the TVM calculator function.
We will use the convention that money paid out by you is **negative** and money you receive is **positive**.

Use TVM function with

N=1	(The value of the first payment is the same as all others – so let N=1)
I% = 3.5	(This is always the annual rate)
PV = -100 000	(Jodie paid this sum into the investment; therefore it is negative)
PMT =	**value required**
FV = 100 000	(Positive since it would be paid back to Jodie if the perpetuity was ended)
P/Y = 12	(Number of compounding periods per year)

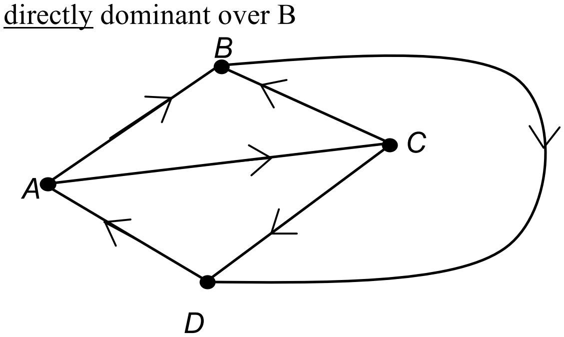
This gives PMT = 291.66667 which rounds to \$291.67pm paid to Jodie (that's why it is positive)

Module 5 Networks and decision mathematics

1B Matrix arithmetic is not required for this module. The answer here can be found by separately counting the one-step and two-step dominance values for each team. The team with the highest total is ranked highest in dominance.

One-step dominance values can be listed as follows.
We will let AB to mean that A beat B and is, therefore, directly dominant over B

The results of the six games are
AB, AC
BD
CB, CD
DA



These results can be summarised in a dominance matrix.

The convention is that the *Winners* and *Losers* are represented as shown. The figures indicate wins against other teams and could be counted directly off the diagram given without the need to list the wins.

$$\begin{array}{cc} & \textit{Loser} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \textit{Winner} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Two step dominance values can also be counted from the diagram and listed.
We will let **A**B**D** mean that A is <u>indirectly</u> dominant over D through its one-step dominance over B.

The two-step dominance list is:
AB**D**, **A**C**B**, **A**C**D**
BD**A**
CB**D**, **C**D**A**
DA**B**, **D**A**C**

These results can be put into a two-step dominance matrix $\begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

By adding up the individual one-step and two-step dominance values, we arrive at
$\begin{bmatrix} 0 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ which is answer B

Alternatively, matrix arithmetic can be used to find the two-step dominance matrix without additional counting. You still need to count to form the one-step dominance matrix which we will call *W*.

Then, the two-step dominance matrix is $W^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

To show both one-step and two-step dominances, you must add

$W + W^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ which, again, is answer B

2A Ranking is determined by adding up all the one-step and two-step dominance values for each team as found in answer B for Question 1.

Team A – 0+2+1+2 = 5 …Team A is directly or indirectly dominant over others in five different ways
Team B – 1+0+0+1 = 2 …Team B is directly or indirectly dominant over others in two different ways
Team C – 1+1+0+2 = 4 …Team C is directly or indirectly dominant over others in four different ways
Team D – 1+1+1+0 = 3 …Team D is directly or indirectly dominant over others in three different ways

Therefore, team A is most dominant

Module 6 Matrices

1A The element $z_{3,4}$ refers to the number that is in the 3rd row, 4th column.
This number is –3

2B $$\begin{bmatrix}1 & -2\\2 & 0\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix} - 3\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}2\\8\end{bmatrix} - \begin{bmatrix}6\\9\end{bmatrix} = \begin{bmatrix}-4\\-1\end{bmatrix}$$

3C The order of the product AB is given as $(r \times c)$ where
r is the same as the number of rows in A
c is the same as the number of columns in B

Therefore, $r = 3$ and $c = 2$
So the order of AB is (3×2)

4A AB will be a (1×1) matrix.
The single row of A will multiply with the single column of B
Therefore $AB = [\ 1 \times 2 + 4 \times (-1) + 0 \times 3 + (-1) \times (-2)\ ] = [\ 0\ ]$

5E The product BA is undefined as the number of columns in B does not equal the number of rows of A.
(Note - Matrix multiplication of AB would be possible; but that is not the same as BA).

6C The matrix that displays the number of business passengers carried on each of the three flights can be extracted from this table. As we only need to show three numbers, we can use either a column or a row matrix. The possible answers only offer the column matrix version of $\begin{bmatrix}22\\2\\27\end{bmatrix}$

7C $\begin{bmatrix} 2 & 3 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -4 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 9 \end{bmatrix}$

You could do the multiplication of these two matrices on your calculator.
Just remember that one of them is the inverse of the coefficient matrix.
If done by hand, the following would complete the calculation:

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 5 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 9 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{22}\begin{bmatrix} -22 \\ 22 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

8E If the determinant of a coefficient matrix equals zero, then there can be no unique solution to a system of equations. Such a matrix is said to be *singular* and cannot have an inverse.
If simultaneous equations are to have a unique solution, then the matrix form of the simultaneous equations must involve a square matrix for the coefficients for which an inverse can be found.

This question, therefore, expects you to find if any of the relevant determinants are zero.
Looking at the four given systems of equations:

I $\begin{aligned} 4x + 2y &= 9 \\ x - 5y &= 4 \end{aligned}$ The coefficient matrix is $\begin{bmatrix} 4 & 2 \\ 1 & -5 \end{bmatrix}$ whose determinant = 4 × 5 – 1 × 2 = 18

II $\begin{aligned} 2y &= 9 \\ x - 2y &= 5 \end{aligned}$ The coefficient matrix is $\begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix}$ whose determinant = 0 × (-2) – 1 × 2 = -2

III $\begin{aligned} 3x - 12y &= 3 \\ x - 3y &= 4 \end{aligned}$ The coefficient matrix is $\begin{bmatrix} 3 & -12 \\ 1 & -3 \end{bmatrix}$ whose determinant = 3 ×(-3) – 1 × (-12) = 3

IV $\begin{aligned} 2x \quad &= 3 \\ x - 2y &= 1 \end{aligned}$ The coefficient matrix is $\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$ whose determinant = 2 × (-2) – 1 × 0 = -4

Since none of the determinants is zero, all coefficient matrices have an inverse.
Hence, all four systems of equations given here have unique solutions.

9D This is an application of scalar multiplication. That is, we are scaling elements of a matrix by a scale factor of 380.

Each of the elements in the matrix $P = \begin{bmatrix} 0.78 \\ 0.18 \\ 0.04 \end{bmatrix}$ represent proportions of the plane's passenger capacity.

They must all be multiplied by 380.

ie $N = 380 \begin{bmatrix} 0.78 \\ 0.18 \\ 0.04 \end{bmatrix}$

Hence our answer is $N = 380P$

Note that the proportion numbers must all add up to 1.0 . This means that 100% of the passengers are seated in one of these three sections of the plane.

Note also that, if the product was then to be calculated (which is beyond the requirements of this question), we would get

$$N = 380 \begin{bmatrix} 0.78 \\ 0.18 \\ 0.04 \end{bmatrix} = \begin{bmatrix} 296.4 \\ 68.4 \\ 15.2 \end{bmatrix}$$

Rounding down these figures, to avoid decimal numbers of people, would give a total of only 379 which obviously is not exactly 100% of 380.

This is an issue which can arise when rounding. If one accepts the rounded answers are only approximates to the actual figures, then the issue is overcome.

Alternatively, we could say that, "on average, these are the number of passengers". In that case, we could leave the decimals unrounded since averages permit decimals of numbers that must otherwise be whole numbers.

Further Mathematics Examination 2
Suggested answers and solutions

Module 6 - MATRICES

1

a

$$\begin{bmatrix} 2 & 1 & 7 \\ 3 & 1 & 4 \\ 5 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9556 \\ 5899 \\ 3155 \end{bmatrix}$$

1 mark

b From the calculator, rounded to one decimal place:

If $A = \begin{bmatrix} 2 & 1 & 7 \\ 3 & 1 & 4 \\ 5 & 2 & 1 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} -0.7 & 1.3 & -0.3 \\ 1.7 & -3.3 & 1.3 \\ 0.1 & 0.1 & -0.1 \end{bmatrix}$

1 mark

c

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -0.7 & 1.3 & -0.3 \\ 1.7 & -3.3 & 1.3 \\ 0.1 & 0.1 & -0.1 \end{bmatrix}\begin{bmatrix} 9556 \\ 5899 \\ 3155 \end{bmatrix} = \begin{bmatrix} 33 \\ 880 \\ 1230 \end{bmatrix}$$

Therefore, it is estimated that there are 33 cats, 880 rats and 1230 lizards on the island.

2 marks

2

a $T = \begin{bmatrix} 0.85 & 0.15 & 0.05 \\ 0.05 & 0.80 & 0.05 \\ 0.10 & 0.05 & 0.90 \end{bmatrix}$

2 marks

b $N_{2006} = \begin{bmatrix} 4000 \\ 2500 \\ 3300 \end{bmatrix}$

1 mark

c **i** $N_{2007} = T \times N_{2006} = \begin{bmatrix} 0.85 & 0.15 & 0.05 \\ 0.05 & 0.80 & 0.05 \\ 0.10 & 0.05 & 0.90 \end{bmatrix}\begin{bmatrix} 4000 \\ 2500 \\ 3300 \end{bmatrix}$

ii $N_{2007} = \begin{bmatrix} 3940 \\ 2365 \\ 3495 \end{bmatrix}$

2 marks

d $$N_{2010} = T^4 \times N_{2006} = \begin{bmatrix} 0.85 & 0.15 & 0.05 \\ 0.05 & 0.80 & 0.05 \\ 0.10 & 0.05 & 0.90 \end{bmatrix}^4 \begin{bmatrix} 4000 \\ 2500 \\ 3300 \end{bmatrix} = \begin{bmatrix} 3764 \\ 2131 \\ 3905 \end{bmatrix}$$

Analysis (not required by this question) - In 2010, we expect 3764 birds to nest at site A, 2131 at site B and 3905 at site C.

1 mark

e This question requires you to show values of N for **several** larger values of n.
If you were to only do one calculation for a chosen value of, say, $n = 80$, then you will only have demonstrated the result is true for that particular value of n. You need to 'show' the matrix N is constant (within a reasonable limit) for at least two increasingly large values of n.

$N_{\text{long term}} = T^n N_{2006}$ where n is large

For $n = 30$, $N = T^{30} N_{2006} = \begin{bmatrix} 3431.8 \\ 1960.1 \\ 4408.1 \end{bmatrix}$

For $n = 40$, $N = T^{40} N_{2006} = \begin{bmatrix} 3430.2 \\ 1960.0 \\ 4409.8 \end{bmatrix}$

For $n = 60$, $N = T^{60} N_{2006} = \begin{bmatrix} 3430.0 \\ 1960.0 \\ 4410.0 \end{bmatrix}$ = *about* the same result as two prior

For $n = 80$, $N = T^{80} N_{2006} = \begin{bmatrix} 3430.0 \\ 1960.0 \\ 4410.0 \end{bmatrix}$ = same result as previous

Thus in the long term, the number of birds nesting at each of the sites is given by the matrix

$$N = \begin{bmatrix} 3430.0 \\ 1960.0 \\ 4410.0 \end{bmatrix}$$

2 marks

3

a The product CP **is** defined since the number of columns in C (=1) equals the number of rows in P (=1)

The product PC is not defined as the number of columns in P (=4) does not equal the number of rows in $C6$ (=3)

1 mark

b

i $R = \begin{bmatrix} 10000 \\ 6500 \\ 9750 \end{bmatrix} \begin{bmatrix} 0.015 & 0.01 & 0.005 & 0.018 \end{bmatrix}$

$$\therefore R = \begin{bmatrix} 150 & 100 & 50 & 180 \\ 97.5 & 65 & 32.5 & 117 \\ 146.25 & 97.5 & 48.75 & 175.5 \end{bmatrix}$$

ii Interpret - $R = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{cccc} \textit{cats} & \textit{rats} & \textit{lizards} & \textit{gulls} \\ \end{array}$

$$R = \begin{array}{c} A \\ B \\ C \end{array}\begin{bmatrix} 150 & 100 & 50 & 180 \\ 97.5 & 65 & 32.5 & 117 \\ 146.25 & 97.5 & 48.75 & 175.5 \end{bmatrix}$$

The matrix *R* lists the number of chicks killed by each type of predator at each site.

2 marks

Note: If the numbers in the matrix for *R* represented 'average' numbers, then rounding is not needed as 'averages' may have decimals.
If exact numbers were required, rounding would be needed but this may cause a "rounding error" in the **total** killed at each site.

Total: 15 marks