INSIGHT

Trial Exam Paper

2010

FURTHER MATHEMATICS

Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions.
- tips and guidelines.

SECTION A

Core: Data Analysis

Question 1

2006 FIFA World Cup performances of 18 of the 32 qualified countries are given in the table below. The table shows the total goals scored, the total goals conceded and the total games won by these 18 countries.

Countries	Total goals scored	Total goals conceded	Total games won
Germany	18	8	6
England	7	5	3
Paraguay	2	2	1
Argentina	13	7	3
Netherlands	3	2	2
Cote d’Ivoire	5	6	1
Portugal	9	3	5
Mexico	5	5	1
Italy	17	5	6
Ghana	4	6	2
United States	2	6	0
Brazil	10	2	4
Australia	5	6	1
Japan	2	7	0
Switzerland	4	3	2
France	12	8	4
Korea Republic	3	4	1
Spain	9	4	3

Table 1

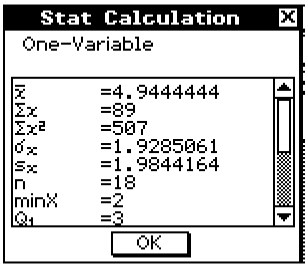
a. List the names of two countries whose standardised scores of total goals conceded are closest to 1.

Country 1:

Country 2:

Worked solution

We first determine the mean and the standard deviation of the values in the second column by calculator.



Mean= 4.9444 Standard deviation= 1.9844164

A standard score of +1 means the value that's one standard deviation above the mean.

So, $4.94444 + 1.9844164 = 6.9288564 \cong 7$

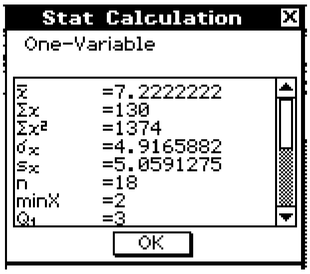
Argentina and Japan are the two countries with a total of 7 goals conceded.

1 mark

b. Germany, the host of 2006 FIFA World Cup, scored a total of 18 goals. Find the standardised score for the total goals scored by Germany and interpret its meaning.

Worked solution

We need to determine the mean and the standard deviation of the values in the first column by calculator.



Mean= 7.22222222 (for total goals scored)

Standard deviation=5.0591275

$$z - score = \frac{18 - 7.22222222}{5.0591275} \cong 2.13$$

This score means that Germany's total goals are more than two standard deviations above the mean.

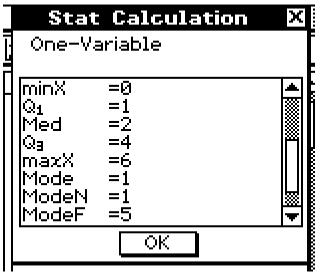
Tip

- *It also means that Germany is in the top 2.5% of these 18 countries in terms of the total goals scored.*

1 mark

c. Germany and Italy both won 6 games. Are these values outliers? Justify your answer by showing appropriate calculations.

Worked solution



$Q_1 = 1 \; and \; Q_3 = 4$
$IQR = 4 - 1 = 3$
Now let's find the upper fence to check if 6 is an outlier or not.
$Upper\ fence = Q_3 + 1.5IQR = 4 + 1.5 \times 3 = 8.5$
Since 6 is less than 8.5, the upper limit, the number of games won by Germany and Italy are not outliers.

1 mark

d. For all 32 countries qualified to play in 2006 World Cup, the mean number of total goals conceded was 5.125. For the sample of 18 countries in table 1, determine the percentage of countries that conceded more goals than the World Cup mean of 5.125.Write your answer correct to two decimal places.

Percentage:

Worked solution

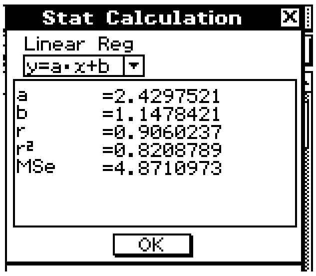
8 countries conceded more than 5.125 goals.
$Percentage = \frac{8}{18} \times 100 = 44.44\%$

1 mark

e. Use the data in table 1 to determine the equation of the least squares regression line that will enable the total number of goals scored to be predicted from the total number of games won. Write the coefficients correct to three decimal places.

Worked solution

Independent variable: the total number of games won
Dependent variable: the total number of goals scored
We determine the equation of the least squares regression line by calculator.



$$the\ total\ number\ of\ goals\ scored = 1.148 + 2.430 \times the\ total\ number\ of\ games\ won$$

Mark allocation

- 1 mark if the correct values are used in wrong places.

Tip

- *Please allocate 2 full marks if the coefficients are given in more than 3 decimal places that can be rounded off to the correct values. Do not allocate any marks if they are given in less than 3 decimal places.*

2 marks

f. Complete the following sentences by filling in the spaces.

i. $\boxed{}$ % of the variation in the total number of goals scored can be explained by the variation in the total number of games won.

ii. On average, a country's total scored goals increase by $\boxed{}$ goals per game won.

Worked solution

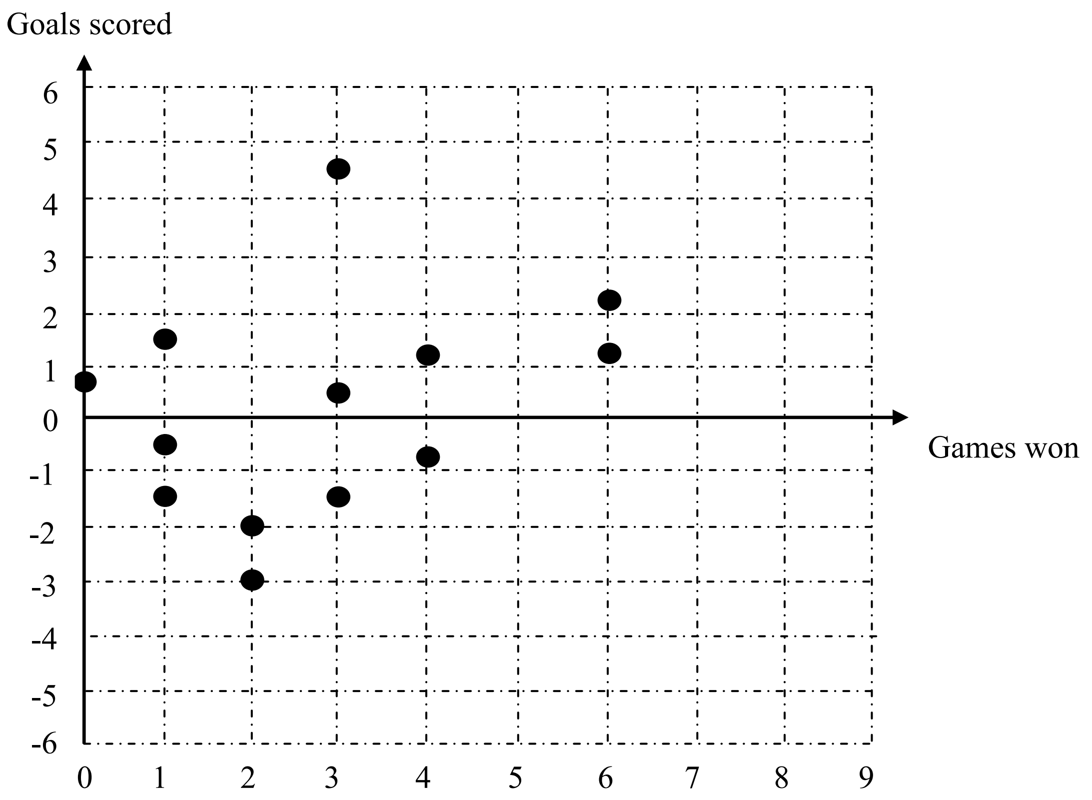
i) $Coefficient\ of\ determination = r^2 = 82\%$

ii) $Gradient = 2.430$

1+1=2 marks

Question 2

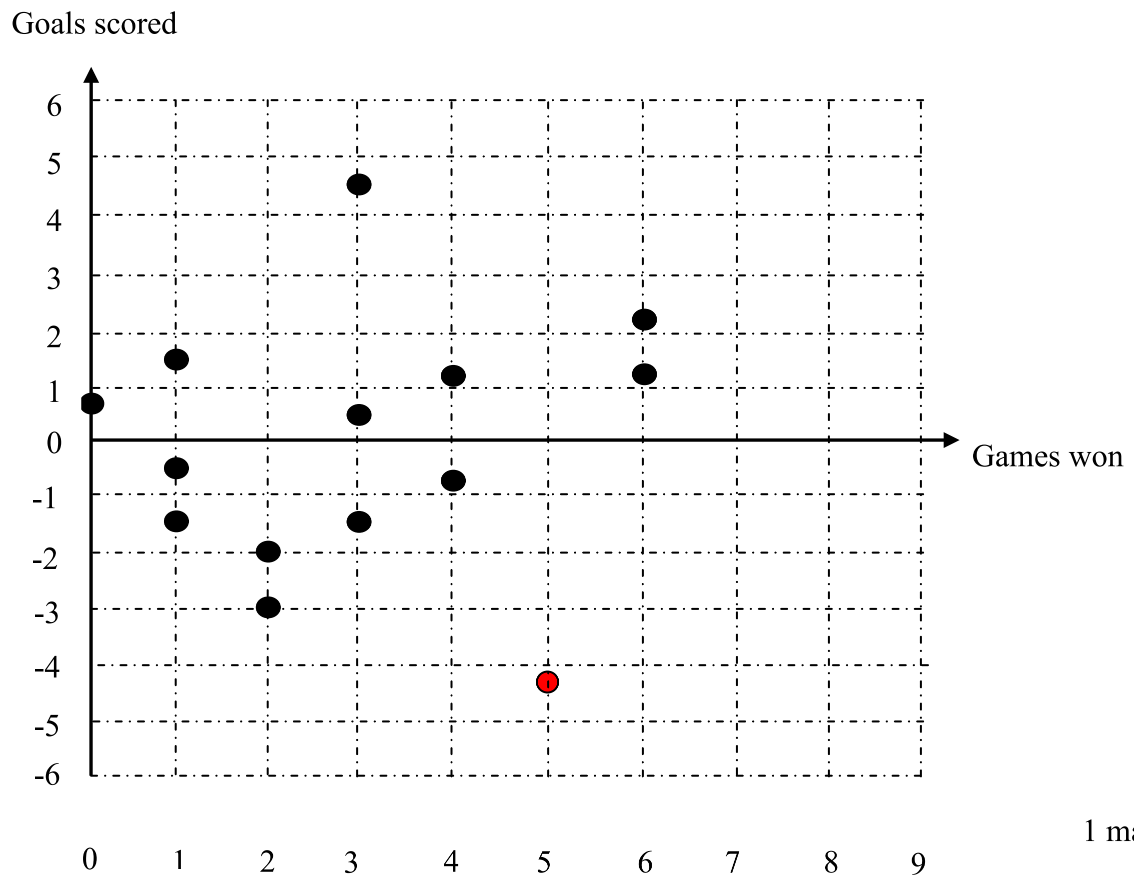
To investigate the form of the relationship between the total number of games won and the total number of goals scored, the following residual plot is constructed. It is incomplete.



a. Complete the residual plot above by marking in the missing residual value that belongs to the country that won 5 games.

Worked solution

$$residual\ value = y_{actual} - y_{predicted}$$
$$y_{predicted} = 1.148 + 2.430 \times 5 = 13.298$$
$$residual\ value = 9 - 13.298 = -4.298$$



1 mark

b. When complete does the residual plot suggest that the original data probably has a linear relationship? Justify your response.

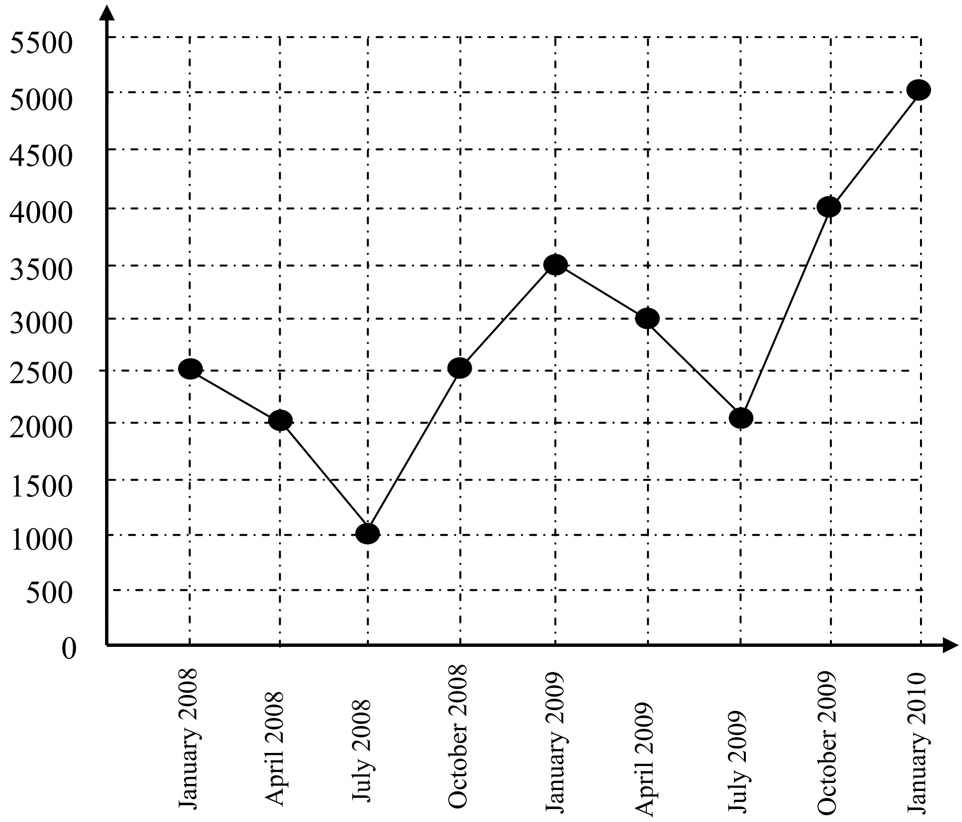
Worked Solution

Yes, the residual plot suggests that the original data has a linear relationship because the residual plot has no clear pattern and it consists of random distribution of points on both positive and negative sides of the horizontal axis.

1 mark

Question 3

Charlie has been a casual employee in a roof insulation company since 2008. His quarterly net pay is shown on the time series plot below.

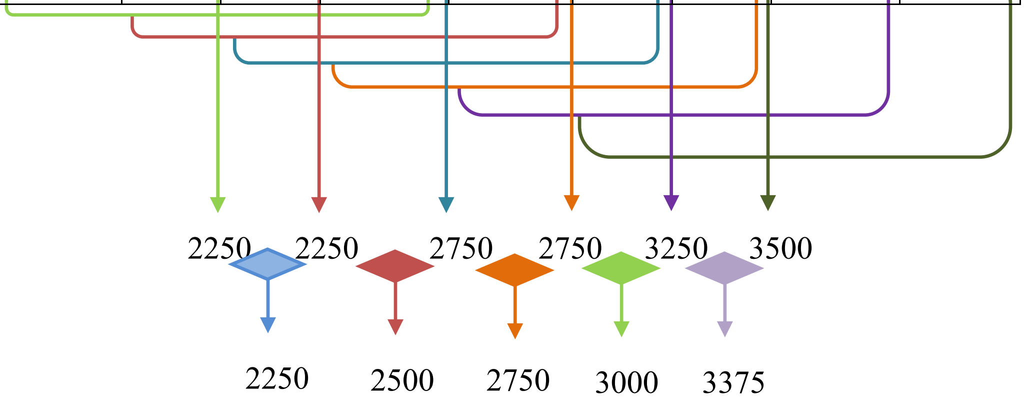


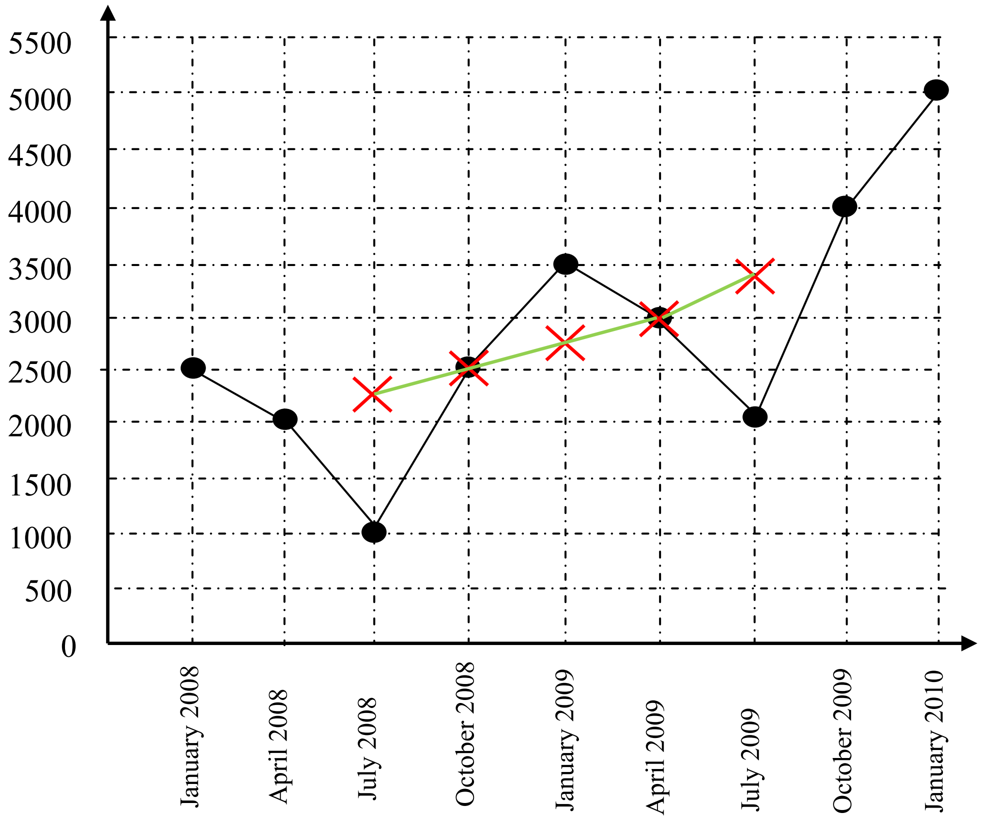
a. Use four-median smoothing with centring to smooth the time series plot. Plot the smoothed series on the same graph above**.** Mark each smoothed data point with a cross (×).

Worked Solution

The table of values given in the graph is as follows:

Quarter	January 2008	April 2008	July 2008	October 2008	January 2009	April 2009	July 2009	October 2009	January 2010
Quarter number	1	2	3	4	5	6	7	8	9
Pay	2500	2000	1000	2500	3500	3000	2000	4000	5000





Mark allocation

- You can allocate 1 mark for finding the three of the five points correct with no value shown for January 2008, April 2008, October 2009 and January 2010.

2 marks

b. Describe the general pattern in the quarterly pay that is revealed by the smoothed time series plot.

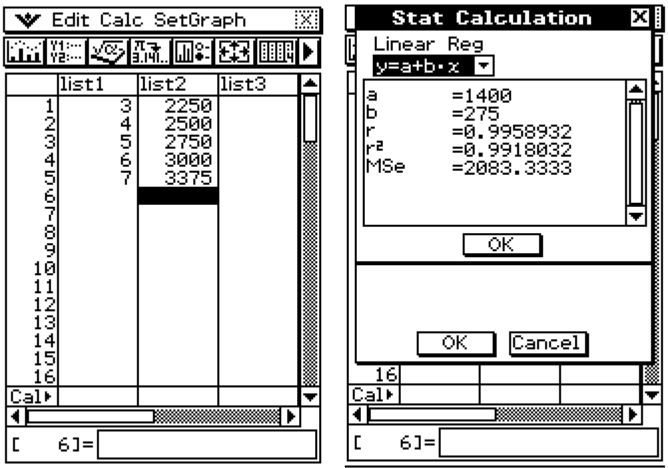
Worked Solution

There is an increase in the quarterly pay throughout the smoothed time series graph.

1 mark

c. Find the equation of the regression line that is fitted to the smoothed time series graph above.

Worked solution



$$Quarterly\ pay = 1400 + 275 \times quarter\ number$$

1 mark

d. Use the regression equation to predict Charlie's quarterly net pay for October quarter in 2010.
Give your answer correct to the nearest thousand dollars.

Worked solution

Quarter 1: January 2008
Quarter 2: April 2008

.
.
.

Quarter 12: October 2010

$$Quarterly\ pay = 1400 + 275 \times 12 = 4700 \cong \$5000$$

1 mark

Total 15 Marks

END OF CORE

END OF SECTION A

SECTION B

Module 1: Number Patterns

Question 1

A group of students from Bright Starts Kindergarten is taken for monthly excursions to Melbourne Luna Park. Melbourne Luna Park has got some height regulations for most of the rides. The children need to be at least 110 cm tall to ride twin dragon and at least 120 cm tall to ride sky rider which both happen to be the favourite rides of the students. The teachers measure the height of the children and find out that 35% of the group are eligible to ride twin dragon. Tony, one of the teachers in the group, believes that each month a further 6% of the children will exceed the 110 cm height limit.

a. If this occurs, what type of sequence will be formed by the monthly percentage of students who are eligible to ride twin dragon?

Worked Solution

This is clearly an arithmetic sequence with a common difference of 6%.

1 mark

b. If in their *n*th visit, the percentage of students who are eligible to ride twin dragon is given by t_n, write down a formula for t_n in terms of *n*.

Worked solution

$t_n = a + (n-1)d$
$t_n = 35 + (n-1)(6)$
$t_n = 6n + 29$

1 mark

c. If their first visit to Luna Park is in February 2010, determine the percentage of students who will be eligible to ride twin dragon in July 2010.

Worked solution

We can solve the question by iteration:

$t_1 = 35 \quad (on\ February\ 2010)$
$t_2 = 41 \quad (on\ March\ 2010)$
$t_3 = 47 \quad (on\ April\ 2010)$
$t_4 = 53 \quad (on\ May\ 2010)$
$t_5 = 59 \quad (on\ June\ 2010)$
$t_6 = 65 \quad (on\ July\ 2010)$

An alternative way to solve this question is to use the formula we found in question 1b.Since February is the first month, July should be the 6^{th} month. We'll substitute 6 for *n* in the equation.

$t_6 = 6 \times 6 + 29 = 65$

1 mark

d. Under this scenario, in which month will all the students in this group be eligible to ride twin dragon?

Worked solution

First way: Using algebra

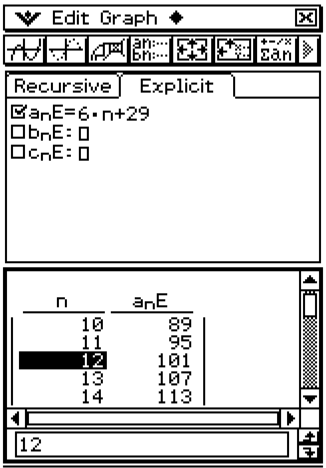
We can solve the general rule for an arithmetic sequence.

$$6n + 29 \geq 100$$
$$n \geq \frac{100 - 29}{6}$$
$$n \geq 11.833$$

$\therefore n = 12$, so in the 12^{th} month, in January 2011, all the students will be eligible to ride twin dragon.

Second way: Using calculator

Enter the arithmetic sequence rule $t_n = 6n + 29$ in the calculator to generate the percentage sequence and scroll where the percentage first becomes more than 100.



12^{th} value is 101 which is more than 100, so in the 12^{th} month, in January 2011, all the students will be eligible to ride twin dragon.

1 mark

Question 2

Another group of students from the same kindergarten also visits Luna Park once a month, starting from June 2010. Jessie, a student in this group, is 105cm tall initially. She grows 6 cm in the first month and each consecutive month she grows 45% less than the previous month.

a. How tall is Jessie going to be in her fifth visit to Luna Park? Give your answer correct to two decimal places.

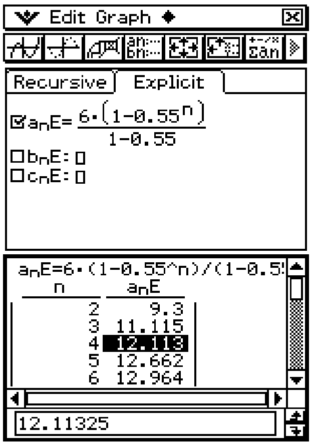
Worked solution

First way: Generating the sequence.

She was 105 cm in her first visit to the Luna Park.
She grew 6 centimetres in the next month and reached 111 cm in the second visit.
She grew 6×0.55=3.3 centimetres in the next month and reached 114.3 cm in the third visit.
She grew 3.3×0.55=1.815 centimetres in the next month and reached 116.115 cm in the fourth visit.
She grew 1.815×0.55=0.99825 centimetres in the next month and reached 117.11325 cm in the fifth visit.

Therefore, her height will be 117.11325 cm.≈117.11 cm in her fifth visit to the Luna Park.

Second way: Using calculator
Enter the geometric sequence rule $S_n = 6 \times \frac{0.55^n - 1}{0.55 - 1}$ in the calculator to generate the sequence for the height growth.



Her height will be 105+12.11=117.11 cm in her fifth visit to the Luna Park.

1 mark

b. Write an expression that will determine the total growth in Jessie's height in her n^{th} visit to the Luna Park?

Worked solution

Her total height growth can be found by using the geometric sum formula

So her total growth per Luna Park visit forms this geometric sequence: 6, 3.3, 0.815,...

$$\therefore S_n = \frac{6 \times (0.55^{n-1} - 1)}{0.55 - 1}$$

Tip

- *Since her first growth will be observed in her second visit, the geometric sum formula,* $S_n = \frac{6\times(0.55^n - 1)}{0.55-1}$ *had to be changed to* $S_n = \frac{6\times(0.55^{n-1} - 1)}{0.55-1}$.

1 mark

c. Totally how many centimetres will she grow between her eight and eleventh visits to the Luna Park? (The eighth and eleventh visits are inclusive.) Write your answer correct to three decimal places.

Worked solution

$$S_{11} - S_7 = \frac{6\times(0.55^{11-1}-1)}{0.55-1} - \frac{6\times(0.55^{7-1}-1)}{0.55-1} = 13.29956 - 12.964258 = 0.3353 \cong 0.335\ cm$$

1 mark

d. If she continues growing with this pattern, in which month will she be eligible to go on sky rider? Explain your answer with doing appropriate calculations.

Worked solution

$$\frac{6 \times (0.55^{n-1} - 1)}{0.55 - 1} \geq 120 - 105$$

This inequation has no solutions showing that she will never be eligible to go on sky rider.

We can also prove this by finding $S_\infty = \frac{6}{1-0.55} = 13.333.$ so maximum growth is 13.333.
So the maximum height that she can ever get to is 105+13.333=118.333 cm.

Mark allocation

- You can allocate 1 mark for doing the necessary calculations. The correct explanation should be made as well as the necessary calculations in order to get 2 full marks.

2 marks

Question 3

The kindergarten started operating at the start of 2006 with only 50 children. The number of students enrolled at the kindergarten at the start of its n^{th} year of operation is given by a difference equation

$A_{n+1} = 0.8 \times A_n + 80 \quad , where\ A_1 = 50$

a. How many students are enrolled at this kindergarten at the start of 2008?

Worked solution

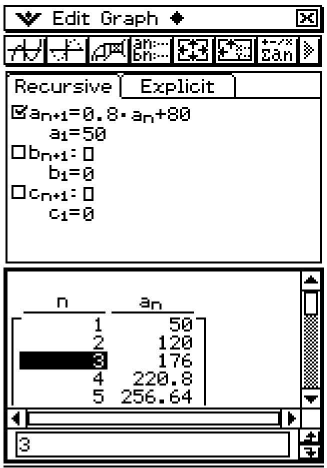
At the start of 2007:

$A_2 = 0.8 \times 50 + 80 = 120\ students\ are\ enrolled\ at\ the\ kindergarten.$

At the start of 2008:

$A_3 = 0.8 \times 120 + 80 = 176\ students\ are\ enrolled\ at\ the\ kindergarten.$

Alternatively, we can use the sequence application of the calculator.



1 mark

b. Show that the number of students enrolled at the kindergarten at the start of its n^{th} year of operation does not follow an arithmetic or a geometric sequence.

Worked solution

$A_2 - A_1 = 120 - 50 = 70 \quad \text{and} \quad A_3 - A_2 = 176 - 120 = 56$

$A_2 - A_1 \neq A_3 - A_2 \quad$ therefore the sequence is not arithmetic.

$\frac{A_2}{A_1} = \frac{120}{50} = 2.4 \quad \text{and} \quad \frac{A_3}{A_2} = \frac{176}{120} = 1.47$

$\frac{A_2}{A_1} \neq \frac{A_3}{A_2} \quad$ therefore the sequence is not geometric.

1 mark

c. Show that the solution to the difference equation
$A_{n+1} = 0.8 \times A_n + 80 \quad , where\ A_1 = 50$

is given by $A_n = -350 \times 0.8^{n-1} + 400$.

Worked solution

For the general first order difference equation given by $t_n = at_{n-1} + b$ a solution is given by $t_n = a^{n-1}t_1 + b\frac{(a^{n-1}-1)}{a-1}$
Let's substitute $a = 0.8\ and\ b = 80$ into the equation

$$A_n = 0.8^{n-1} \times 50 + 80\frac{(0.8^{n-1} - 1)}{0.8 - 1}$$

$$A_n = 0.8^{n-1} \times 50 + 400 - 400 \times 0.8^{n-1}$$

$$A_n = -350 \times 0.8^{n-1} + 400$$

1 mark

d. Explain why the total number of students in this kindergarten can never exceed 400?

Worked solution:

Let's assume $A_n = 400,$
Now let's evaluate A_{n+1}
$A_{n+1} = 0.8 \times 400 + 80 = 400$
This shows that the number of students in the kindergarten can never exceed 400.

1 mark

Question 4

The yearly fees paid by the kindergarten students at the start of its n^{th} year of operation is specified with the second order difference equation

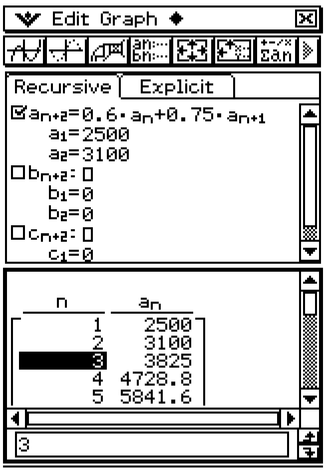
$$F_{n+2} = 0.6F_n + 0.75F_{n+1}\ ,\ where\ F_1 = 2\,500, \quad and \quad F_2 = 3\,100$$

a. What is the yearly fee of the kindergarten at the start of its 3rd year of operation?

Worked solution

$F_3 = 0.6F_1 + 0.75F_2\ = 0.6 \times 2\,500 + 0.75 \times 3\,100 = 3\,825$

Alternatively, we can use the sequence application of the calculator.



1 mark

b. Find the total amount of fees collected by the kindergarten owners in its first three years of operation. Give your answer correct to the nearest thousand dollars.

Worked solution

$Total\ fees = 50 \times \$2\,500 + 120 \times \$3\,100 + 176 \times \$3\,825 = \$1\,170\,200 \cong \$1\,170\,000$

1 mark

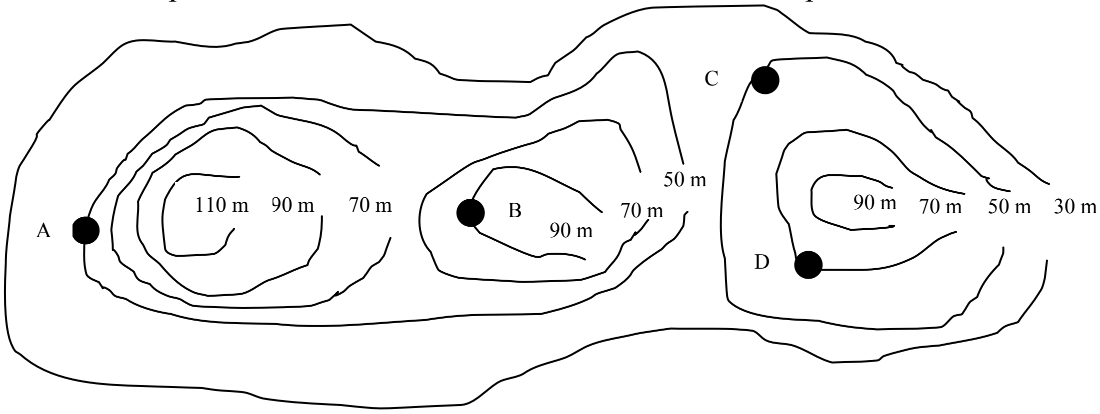
Total 15 marks

END OF MODULE 1

Module 2: Geometry and trigonometry

Question 1

A contour map of the amusement park is shown below. It has contours drawn at intervals of 20 metres. The map shows four observation decks in four different spots.



Scale 1:2500

a. The horizontal distance between observation deck A and observation deck B on the contour map is 36 mm.

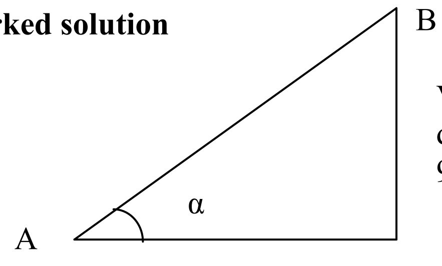
i. Determine the horizontal distance between observation deck A and observation deck B to the nearest metre.

Worked solution

Since the scale is 1:2500, horizontal distance is $36 \times 2500\ mm = 90\ m$

ii. Hence determine the angle of depression of a person looking at observation deck A from observation deck B. Give your answer correct to the nearest degree.

Worked solution



Vertical distance= 90-50=40 m

$$tan \propto = \frac{40}{90}$$
$$\propto = 23.96° \cong 24°$$

Horizontal distance= 90 m

1+1=2 marks

b. What is the difference in altitudes of observation deck C and observation deck A?

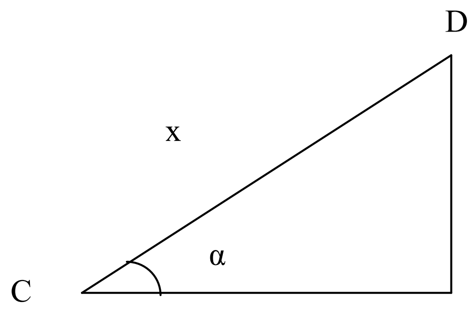
Worked solution

Vertical distance= 50 – 50 = 0 m

1 mark

c. The horizontal distance between observation deck C and observation deck D is 44 metres. The owners of the botanical garden want to make a top tree walk between these observation decks. What is the shortest possible length of this top tree walk? Write your answer correct to two decimal places.

Worked solution



Horizontal distance= 44m

Vertical distance= 70-50=20 m

Shortest distance= x

$$x^2 = 20^2 + 44^2$$

$$x = 48.33218389$$

$$\cong 48.33\ m$$

1 mark

d. Find the slope (gradient) of the top tree walk. Write your answer correct to three decimal places.

Worked solution

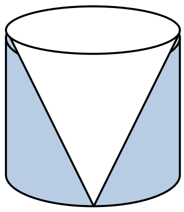
$$Slope = \frac{vertical\ distance}{horizontal\ distance} = \frac{20}{44} = 0.455$$

1 mark

Question 2

Martha uses two measuring cups to cook her mother's favourite cupcakes from her secret recipe. One of the measuring cups is in cylindrical shape and the other one is a cone. They both have the same radii and equal heights.

Her mother told her to use a secret ingredient in a specific amount to give the cupcakes their unique taste. She measured it in a very strange way. She first put some secret ingredient into the cylindrical cup and then she put the cone in the cylinder. The part between the cylinder and the cone is meant to be full of secret ingredient as shown in the figure below.



If the volume of the cylinder is V cm^3 what fraction of V is the volume of the secret ingredient?

Worked solution

$$Volume\ of\ the\ cylinder = \pi r^2 h = V$$
$$Volume\ of\ the\ cone = \frac{1}{3}\pi r^2 h = \frac{1}{3}V$$
$$Volume\ of\ the\ secret\ ingredient = V - \frac{1}{3}V = \frac{2}{3}V$$

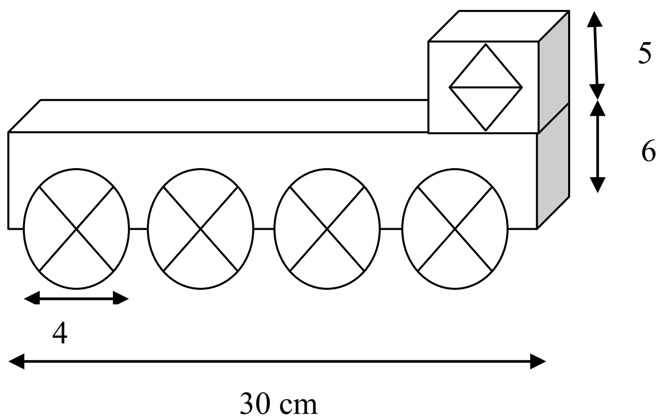
2 marks

Mark allocation

- 1 mark if the ratio of the cone to the cylinder is found.

Question 3

Josh made a little train with his wooden blocks. The train consists of a rectangular prism shaped body, a cube shaped conductor compartment, eight circular wheels (four on each side), two triangular windows. The wheels are stuck to the train from their centres and half of each wheel is below the train. He wants to paint the train including the windows but excluding the wheels.



Find the total surface area of the train that needs to be painted. Give your answer correct to three decimal places.

Worked solution

$$Total\ surface\ area = 2(5 \times 6 + 5 \times 30 + 6 \times 30) + 4 \times 5^2 - 4 \times \pi \times 2^2$$
$$= 769.735\ cm^2$$

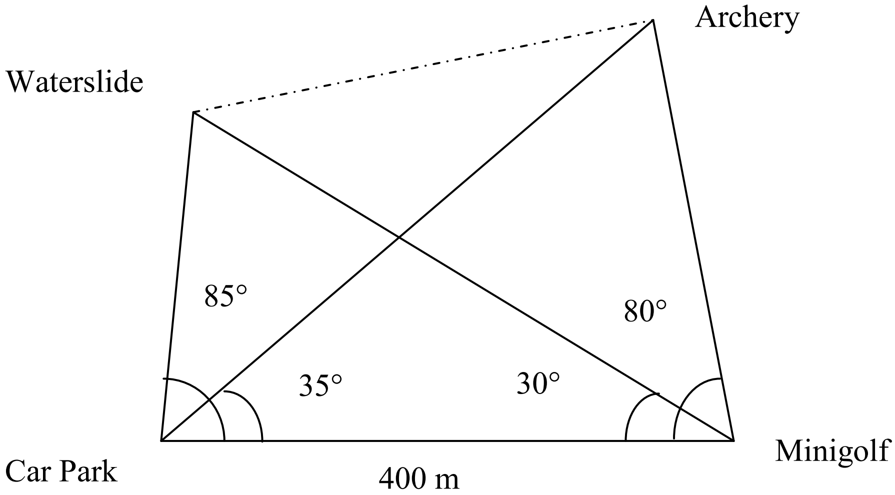
Mark Allocation

- 1 mark if the total surface area of the rectangular and square surfaces is found. (820 cm^2)

2 marks

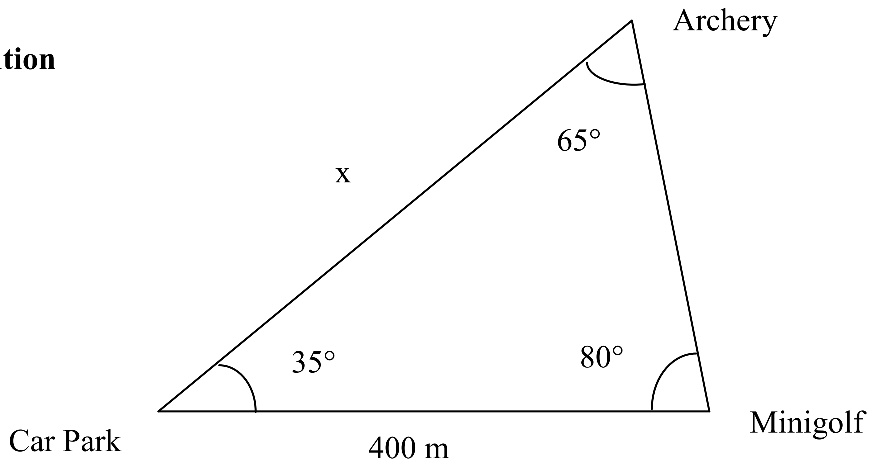
Question 4

An amusement park in Melbourne is visited by so many tourists every year. Jack and Kate came from Mildura to visit the amusement park with their three year old daughter Jenny. They all started to walk from the car park. Kate and Jenny first went on the waterslides while Jack went to the archery. Then they all joined in the minigolf course. After playing minigolf they went back to the car park together. The diagram below shows the angles and distances between the car park, waterslide, archery and minigolf course.



a. How far did Jack walk on his way from the car park to the archery? Write your answer correct to three decimal places.

Worked solution



Let's apply the sine rule to the triangle to find x.

$$\frac{x}{\sin 80°} = \frac{400}{\sin 65°}$$

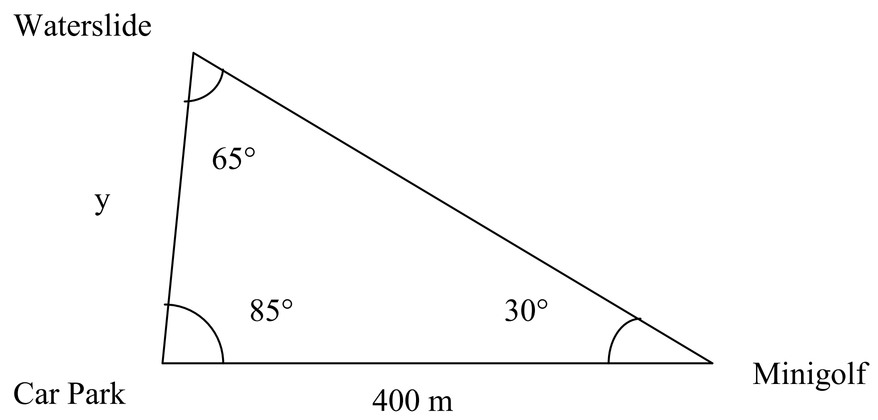
$$x = \frac{400 \times \sin 80°}{\sin 65°}$$

$$x = 434.6460516\ m \cong 434.646\ m$$

1 mark

b. How far did Kate and Jenny walk from the car park to the waterslides? Give your answer correct to the nearest metre.

Worked solution



Let's apply the sine rule to the triangle to find y.

$$\frac{y}{\sin 30°} = \frac{400}{\sin 65°}$$

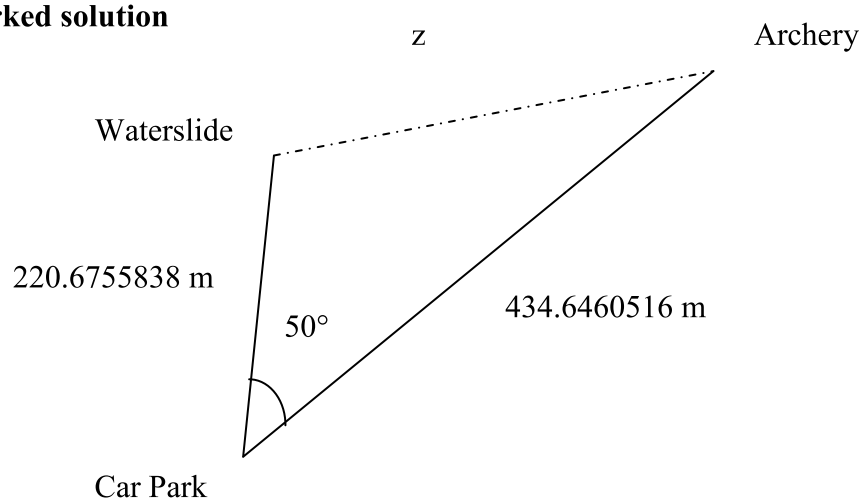
$$y = \frac{400 \times \sin 30°}{\sin 65°}$$

$$y = 220.6755838\ m \cong 221\ m$$

1 mark

c. Find the shortest distance between waterslide and the archery. Write your answer correct the nearest metre.

Worked solution



Let's apply the cosine rule to the triangle to find z.

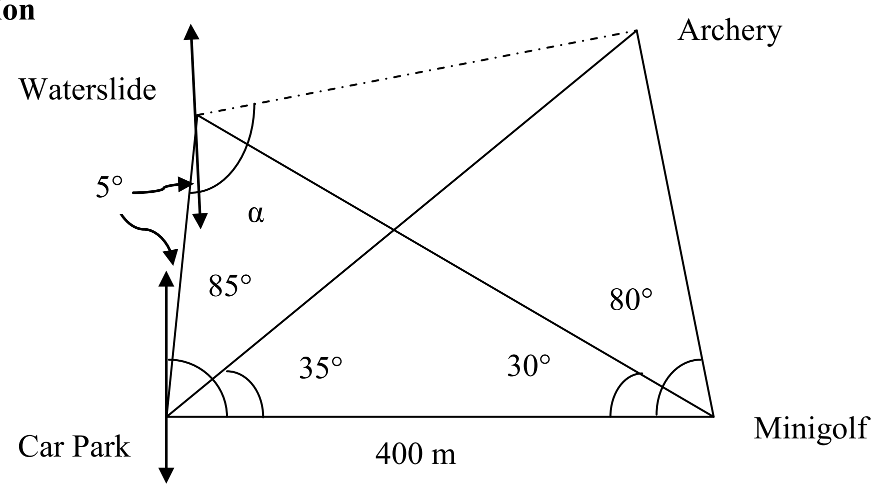
$$z^2 = 220.6755838^2 + 434.6460516^2 - 2 \times 220.6755838 \times 434.6460516 \times \cos 50°$$

$$z = 338.0946094\ m \cong 338\ m$$

1 mark

d. The minigolf course is due east of the car park. By using the answers obtained from questions 4a, 4b and 4c, find the bearing of the archery from the waterslides. Write your answer correct to the nearest degrees and minutes.

Worked solution



$$\cos \alpha = \frac{338^2 + 221^2 - 434.646^2}{2 \times 338 \times 221}$$

$\alpha = 99°57'$
The bearing of the archery from waterslide $= 180° - (99°57' - 5°) = 85°3'$T

2 marks

e. What is the area of the quadrilateral that is formed by the car park, waterslides, archery and minigolf course? Give your answer, in km², correct to three decimal places.

Worked solution

$$Area = \frac{1}{2} \times 220.6755838 \times 434.6460516 \times sin50° + \frac{1}{2} \times 434.6460516 \times 400 \times \sin 35°$$

$$Area = 86598.41843\ m^2 \cong 86598.4\ m^2 = 0.087\ km^2$$

Note: We obtain the same answer by using the rounded off values obtained from question 4a and 4b.

1 mark

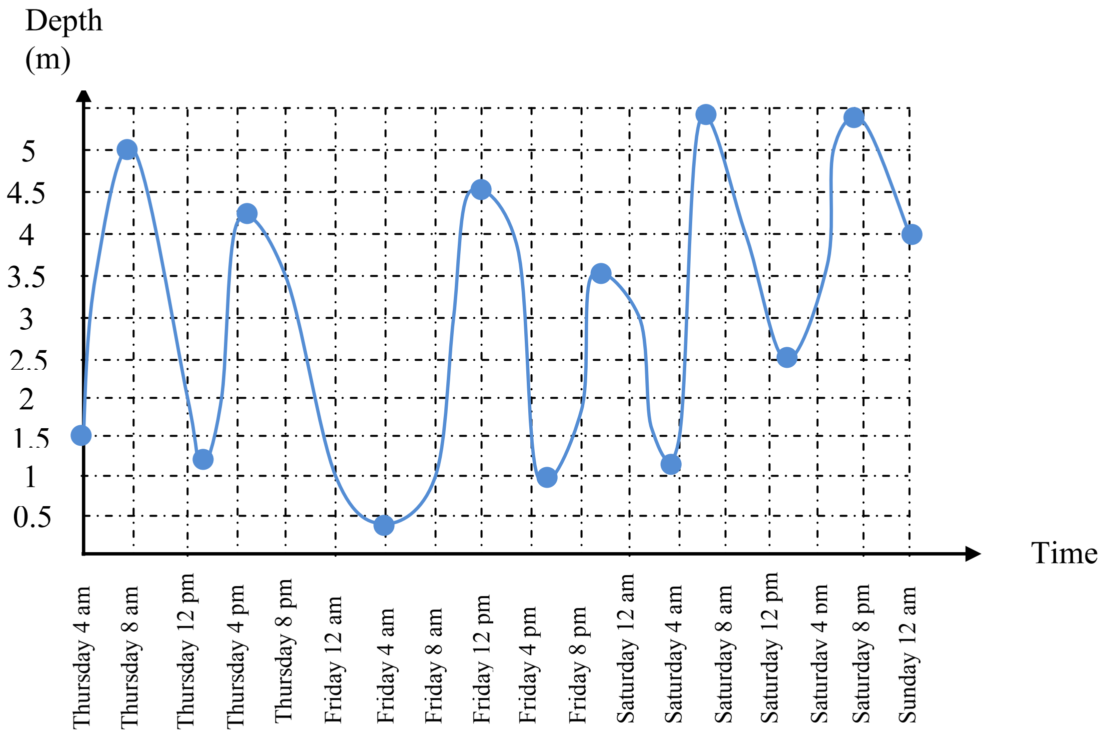
Total 15 marks

END OF MODULE 2

Module 3: Graphs and Relations

Question 1

Jennifer, a professional swimmer does her swimming practice in the natural environment. The depth of water across the sand bar that she prefers swimming in varies depending on the tide as shown in the graph below.



a. When does the third high tide occur?

Worked Solution

Third high tide occurs on Friday at 12 pm.

1 mark

b. The water levels should be on or above 1metre level for Jennifer to swim comfortably. Approximately between what times and on which day can Jennifer not swim comfortably?

Worked Solution

Jennifer cannot swim comfortably on Friday between 12 am and 8 am.

1 mark

c. At what time on Friday morning will there be the least movement of water in the sand bar?

Worked Solution

There will be the least movement of water in the sandbar at 4 am on Friday.

1 mark

d. What is the difference in heights of the first high tide and the fourth low tide?

Worked Solution

$5\ metres - 1\ metre = 4\ metres$

1 mark

e. How many high tides and low tides occur between Thursday 8 pm and Saturday 4 pm?

Worked Solution

3 high tides and 4 low tides occur between 8 pm on Thursday and 4 pm on Saturday.

1 mark

Question 2

Jennifer is a member of a swimming club which has many other male and female members. The club has a swimming coach training each swimmer for two different swimming styles, butterfly and freestyle. On average, it takes 10 minutes for the swimming coach to train each female swimmer and 25 minutes to train each male swimmer for butterfly. He has a total of 250 minutes a day to train for the butterfly. He also spends an average of 45 minutes with each male swimmer and 40 minutes with each female swimmer to train them for freestyle and he has a total of 720 minutes a day for training freestyle. Let x be the number of female swimmers and y be the number of male swimmers in the club in one day.

a. Two of the constraints are $x \geq 0\ and\ y \geq 0$.

i. Write down the constraint that is formed by the daily available times of the coach to train female and male swimmers for butterfly.

Worked solution

$$10x + 25y \leq 250$$

ii. The following inequation represents the constraint of the daily available times of the coach to train female and male swimmers for freestyle. Complete the following constraint by filling in the spaces.

$$\frac{x}{\square} + \frac{y}{\square} \leq 1$$

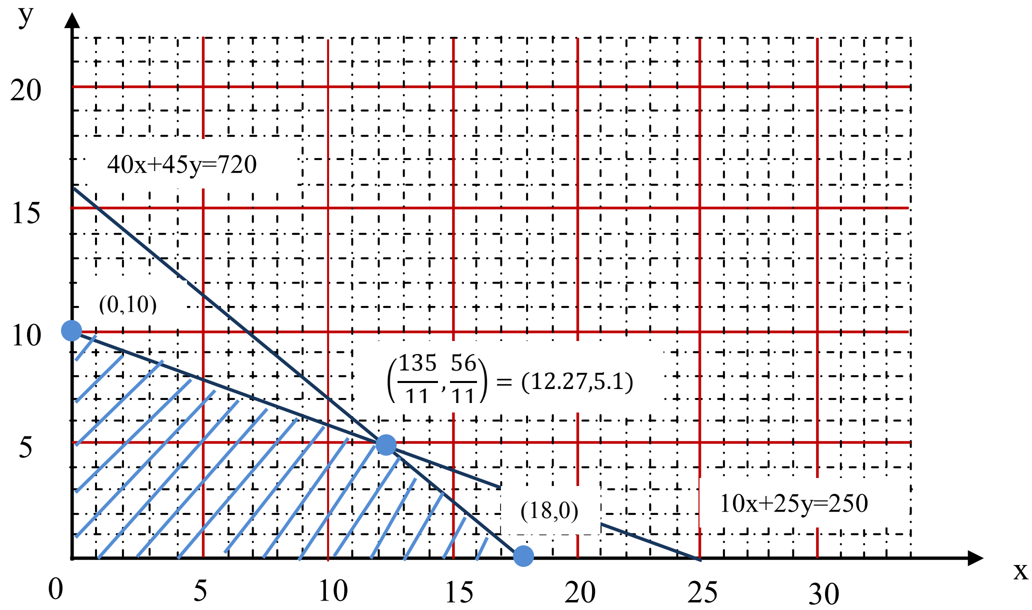
Worked solution

$$40x + 45y \leq 720$$
$$\frac{40x}{720} + \frac{45y}{720} \leq \frac{720}{720}$$
$$\frac{x}{18} + \frac{y}{16} \leq 1$$

1+1=2 marks

c. Sketch and shade the feasible region described by the four constraints on the graph below. Label each line and clearly identify the coordinates of all vertices of the feasible region.

Worked solution



3 marks

Mark allocation

- 1 mark if the lines are drawn correctly. 2 marks if the lines are drawn and intersection points are clearly shown.

d. The profit made by the owners of the club on each male swimmer is \$50 and on each female swimmer is 60% less than a male swimmer.

i. Write down an equation for the profit P.

Worked solution

$$P = 20x + 50y$$

ii. How many male swimmers and female swimmers are needed in a day in order to maximise the profit.

Worked solution

For the vertex (0.10) $P = 20 \times 0 + 50 \times 10 = 500$
For the vertex (18.0) $P = 20 \times 18 + 50 \times 0 = 360$
For the vertex $\left(\frac{135}{11}, \frac{56}{11}\right)$ $P = 20 \times \frac{135}{11} + 50 \times \frac{56}{11} = 500$

Since we have two values giving the same maximum profit of \$500, all discrete valued points between the point (0,10) and $\left(\frac{135}{11}, \frac{56}{11}\right)$ are included in the solution set.
So solution points are (0,10), (5,8) and (10,6).

In order to get a maximum daily profit of \$5000, the club has three alternatives.

1) 0 female and 10 male swimmers would give a maximum profit of \$5000.
2) 5 female and 8 male swimmers would give a maximum profit of \$5000.
3) 10 female and 6 male swimmers also would give a maximum profit of \$5000.

1+1=2 marks

d. One month before the swimming competition the club decided to hire another swimming coach to train the swimmers for breaststroke. The new coach trains a female swimmer in 20 minutes and a male swimmer in 10 minutes and he is available in the club for a total of 200 minutes a day.

i. 5 male and 10 female swimmers want to train for breaststroke in the club on an average day during the busy competition preparations. Will the club be able to meet the needs of these swimmers?

Worked solution

(10,5) is within the feasible region of our previous graph. Let's check if this point satisfies the new constraint.

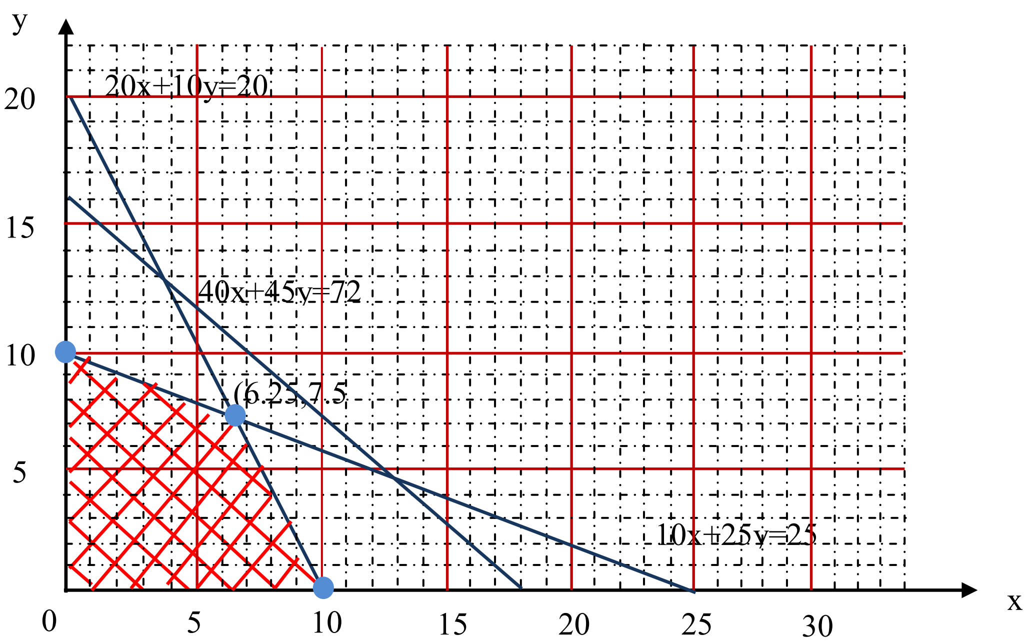
New constraint: $20x + 10y \leq 200$

$20 \times 10 + 10 \times 5 = 250$

Since 250 is greater than 200, this point is not in the new feasible region. This shows that the club will not be able to meet the needs of these swimmers.

ii. What is the maximum number of female and male swimmers that the club can accept on a busy day?

Worked solution:



The objective function for the total number of male and female swimmers is $N = x + y$. Let's substitute the vertices of the feasible region into our objective function.

For the vertex (10.0) N=10+0=10
For the vertex (0.10) N=0+10=10
For the vertex (6.25.7.5) $N = 6.25 + 7.5 = 13.75$

Since our maximum value is not a discrete number, we look for the discrete points in the feasible region that gives the maximum value to the objective function. (5,8), (7,6) and (6,7) all give a maximum value of 13. So the maximum number of swimmers who can visit the club a day is 13 people with 5 females and 8 males or 6 females and 7 males or 7 females and 6 males.

1+2=3 marks
Total 15 marks

END OF MODULE 3

Module 4: Business-related mathematics

Question 1

a. Adam needs \$45 000 to renovate his first class restaurant. He wants to borrow money from the Bestrates Bank at $9\frac{4}{9}$ % p.a. simple interest for four years. How much interest will Bestrates Bank charge Adam in these four years?

Worked solution

$I = \frac{PrT}{100}$, where $P = \$45\ 000,\quad r = 9\frac{4}{9}\ \%\ \text{p.a},\quad T = 4\ years$

$$I = \frac{45\ 000 \times 9\frac{4}{9} \times 4}{100}$$

$$I = \$17\ 000$$

1 mark

b. Adam's friend, Sheila recommended him the Cheaper Building Society. Adam finds that the Cheaper Building Society will lend the \$45 000 to him at 0.8% per month simple interest for four years. Find the total interest that Adam will have to pay to the Cheaper Building Society if he decides to borrow the loan from them?

Worked solution

$I = \frac{PrT}{100}$, where $P = \$45\ 000,\quad r = 0.8\%\ per\ month,\quad T = 48\ months$

$$I = \frac{45\ 000 \times 0.8 \times 48}{100}$$

$$I = \$17\ 280$$

1 mark

c. Hence or otherwise, which institution would you recommend Adam to borrow the loan from?

Worked solution

Adam should borrow the loan from the Bestrates Bank.

1 mark

Question 2

Sheila has \$63 000 invested in an account which pays interest at the rate of 4.2% per annum compounding monthly.

a. Show that the interest rate per month is 0.35%.

Worked solution

$Monthly\ interest\ rate = \frac{4.2}{12} = 0.35$

1 mark

SECTION B – continued

b. Determine the value of the \$63 000 investment after five years. Write your answer in dollars correct to the nearest cent.

Worked solution

$A = P\left(1 + \frac{r}{100}\right)^n$ $,where\ P = \$63\ 000$ $,\ r = 0.35\ ,\ n = 5 \times 12 = 60$

$$A = \$63\ 000\left(1 + \frac{0.35}{100}\right)^{60}$$

$$A = \$77\ 693.23$$

1 mark

c. If Sheila had decided to invest her money into another account find the interest rate per annum that would enable her \$63 000 investment to grow to \$80 000 over five years if interest is compounded quarterly. Write your answer correct to two decimal places.

Worked solution

$A = P\left(1 + \frac{r}{100}\right)^n$ $,where\ P = \$63\ 000$ $,\ A = \$80\ 000$ $,\ n = 5 \times 4 = 20$

$$\$80\ 000 = \$63\ 000\left(1 + \frac{r}{100}\right)^{20}$$

$$\left(1 + \frac{r}{100}\right)^{20} = \frac{\$80\ 000}{\$63\ 000}$$

$$\left(1 + \frac{r}{100}\right) = \sqrt[20]{\frac{\$80\ 000}{\$63\ 000}}$$

$$\frac{r}{100} = \sqrt[20]{\frac{\$80\ 000}{\$63\ 000}} - 1$$

$$r = \left(\sqrt[20]{\frac{\$80\ 000}{\$63\ 000}} - 1\right) \times 100$$

$$r = 1.2016\%\ per\ quarter$$

$$Annual\ rate = 1.2016\% \times 4 = 4.81\%$$

2 marks

Question 3

A school bought 20 new computers with each of them valuing at $1 950 and having a scrap value of $220.

a. If they depreciate by 11.1% per annum by flat rate depreciation method, how long will it be before the scrap value is reached?

Worked solution

$$Number\ of\ years = \frac{cost\ price - scrap\ value}{rate\ of\ depreciation}$$

$$Number\ of\ years = \frac{\$1950 - \$220}{\$1950 \times \frac{11.1}{100}} = 7.99 \cong 8\ years$$

Mark allocation

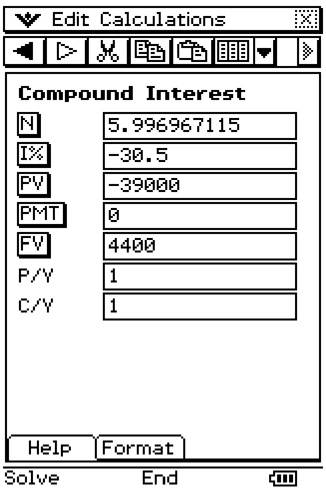
- 2 marks for correct answer

2 marks

b. Use the reducing balance depreciation method with a depreciation rate of 30.5% to calculate the total amount of time that they can be used before they will need to be replaced.

Worked solution

We'll use *TVM* solver to find the total amount of time that the computers can be used before they will need to be replaced.



So, the school can use the computers for a total of 6 years before they reach their scrap value.

1 mark

c. Which method, flat rate depreciation or reducing balance depreciation is better to use for school to utilise the computers for a longer period of time? Write down the difference in the lifetime of the school computers calculated by two different methods.

Worked solution

Flat rate depreciation method is better to use. The difference in the lifetime of the school computers calculated by two different methods is 8-6=2 years.

1 mark

Question 4

a. Joseph's monthly rent payments are $1 300. Assuming an average inflation rate of 5.6% per annum, calculate the amount of monthly rent he will be paying in five years' time. Write your answer in dollars correct to the nearest cent.

Worked solution

$$A = P\left(1+\frac{r}{100}\right)^n \text{ , } where\ P = \$1\,300 \quad , \ r = 5.6\%\ p.a.\ , \ n = 5\ years$$

$$A = \$1\,300\left(1+\frac{5.6}{100}\right)^5$$
$$A = \$1707.115648 \cong \$1707.12$$

1 mark

b. His annual gross salary is $45 000. His employer decided to raise his salary by 3% every year. Calculate his annual salary in five years' time. Write your answer in dollars correct to the nearest cent.

Worked solution

$$A = P\left(1+\frac{r}{100}\right)^n \text{ , } where\ P = \$45\,000 \quad , \ r = 3\%\ p.a.\ , \ n = 5\ years$$
$$A = \$45\,000\left(1+\frac{3}{100}\right)^5$$
$$A = \$52\,167.33$$

1 mark

c. John is 42 years old and is planning to retire at 61. His employer contributions to his superannuation fund are 8% of his gross monthly income. John also contributes a further $450 a month as a salary sacrifice. The superfund has been returning an interest rate of 4.2% p.a. compounded monthly and his current balance in the superfund is $52 000.

i. Calculate John's total monthly contributions to the superannuation account made by him and his employer.

Worked solution

The employer contribution is 8% of his gross monthly income.

$$8\% \text{ of } \$\frac{\$45\,000}{12} = 0.08 \times \$3750$$
$$= \$300$$
$$total\ monthly\ contributions = \$300 + \$450 = \$750$$

SECTION B – continued

ii. Calculate the lump sum that he can receive for his planned retirement at age 61 assuming that his salary remains constant every year. Write your answer in dollars correct to the nearest cent.

Worked solution

$P = \$52\ 000\,, \qquad Q = \$750\,, \qquad n = 228\,, \quad R = 1.0035$

$A = PR^n + \dfrac{Q(R^n - 1)}{R - 1}$

$= \$52\ 000 \times 1.0035^{228} + \dfrac{\$750(1.0035^{228} - 1)}{1.0035 - 1}$

$= \$376\ 336.5086 \cong \$376\ 336.51$

1+1=2 marks

Total 15 marks

END OF MODULE 4

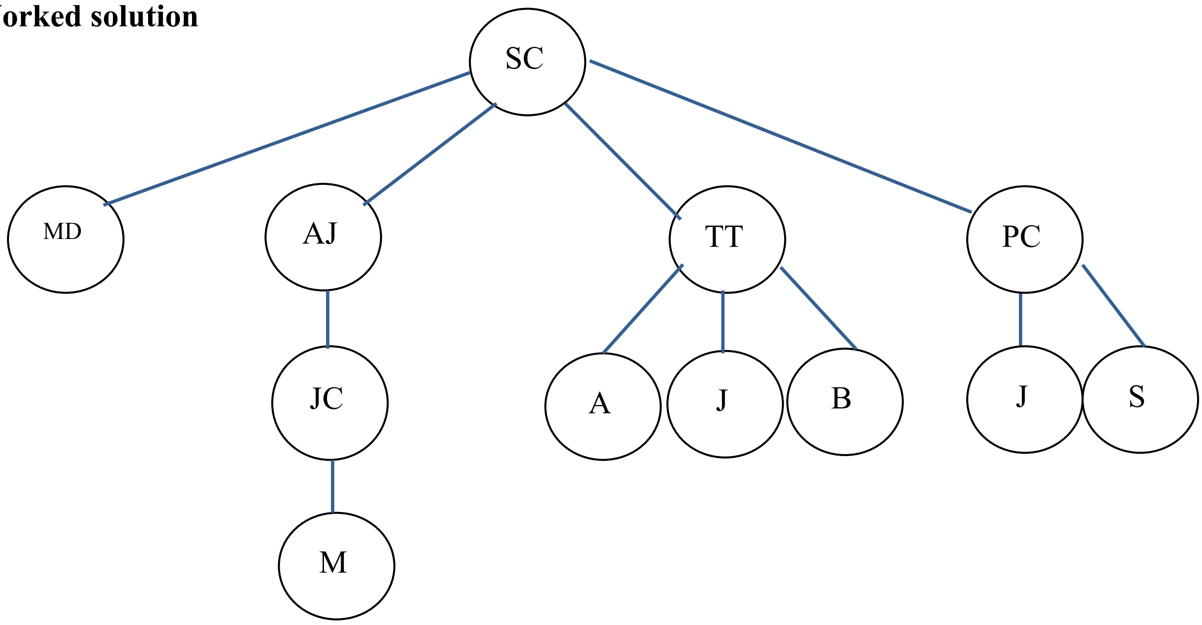
Module 5: Networks and decision mathematics

Question 1

Santos and Carla are married with four children, Maria, Arlene, Tommy and Peter.
Arlene married Jim and had one child, Jenny.
Tommy married Tina and had three children, Adele, John and Bony.
Peter married Cindy and had two children, Juliet and Sammy.
Jenny married Charlie and had one child, Morgan.
Maria married David and they don't have any children.

a. Construct a network representing the given family tree. Use a single node to represent each married couple and each child.

Worked solution



1 mark

b. How many vertices and edges does the network have?

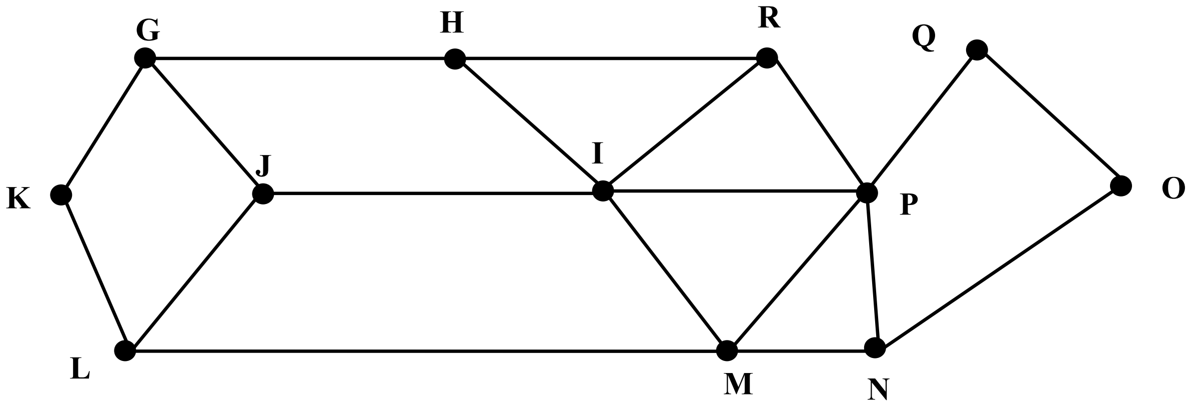
Worked solution

The network has 12 vertices and 11 edges.

1 mark

Question 2

Santos and Carla visit an art gallery during their trip to Spain. The art gallery contains 12 drawings denoted as vertices G to R on the network diagram below. The edges on this network represent the pathways that link the 12 drawings.



a. Write down the degree of vertex P.

Solution

5

1 mark

b. Carla wishes to examine each drawing only once. She wants to begin her tour with drawing H and end it at drawing G.

i. Write down the order Carla will examine the drawings.

Solution

The Hamiltonian path is H-R-P-Q-O-N-M-I-J-L-K-G.

ii. How many pathways doesn't she need to use at all during her tour? List their names.

Solution

She doesn't need to use 8 pathways, GH, H-I, I-R, I-P, P-M, P-N, M-L and JG.

iii. Suppose she starts and finishes her tour with drawing L. List two different ways that she can examine the drawings.

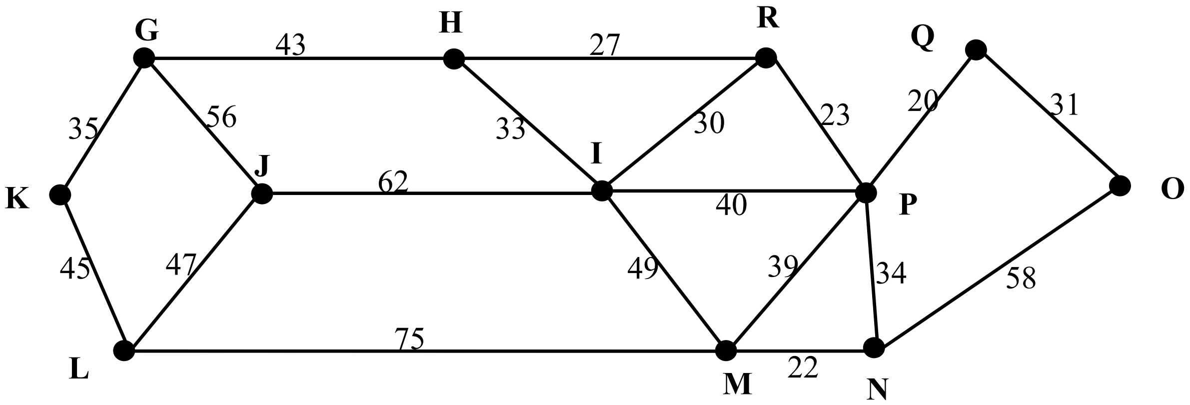
Solution

Fist Hamiltonian circuit: L-K-G-H-R-P-Q-O-N-M-I-J-L
Second Hamiltonian circuit: L-J-I-M-N-O-Q-P-R-H-G-K-L

1+1+1=3 marks

Question 3

The following network represents the distances (in metres) of pathways connecting 12 drawings.



a. Explain why an Euler circuit does not exist for this graph.

Solution

An Euler circuit does not exist for this graph because not all vertices are of even degree.

1 mark

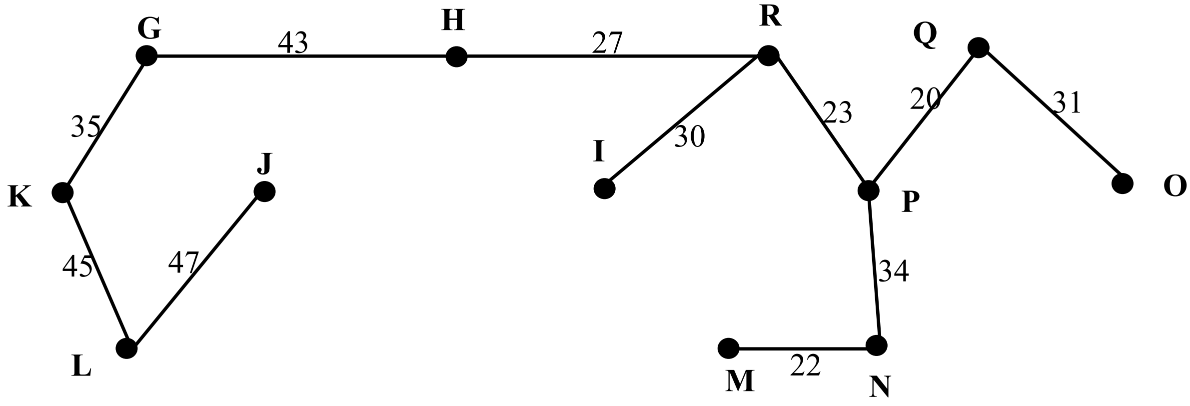
b. i. Explain why L-K-G-H-R-P-Q-O-N-P-M-I-J doesn't form a tree.

Solution

A tree is a connected graph with no circuits. Since L-K-G-H-R-P-Q-O-N-P-M-I-J contains the circuit P-Q-O-N-P it cannot be a tree.

ii. Draw the minimum spanning tree for this network on the above graph.

Worked Solution



iii. Find the total distance of the minimum spanning tree.

Worked solution

$$Total\ distance = 47 + 45 + 35 + 43 + 30 + 27 + 23 + 20 + 31 + 34 + 22 = 357\ m$$

1+1+1=3 marks

Question 4

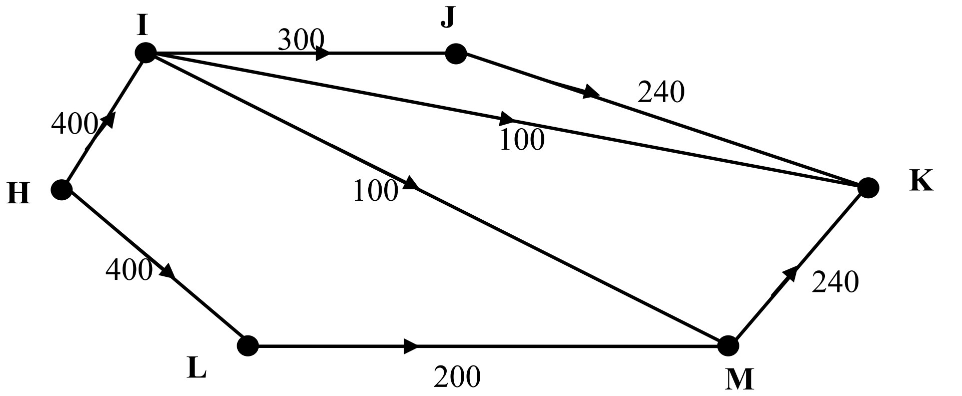
The following table represents the flow capacity of the water pipes between some of the main streets in a town. The letters represent the street names and flow capacities are given in litres per minute.

From	To	Flow capacity
H	I	400
H	L	400
I	J	300
I	M	100
L	M	200
J	K	240
M	K	240
I	K	100

SECTION B – continued

a. Convert the information presented in the following table to a network diagram, clearly indicating the direction and quantity of the flow.

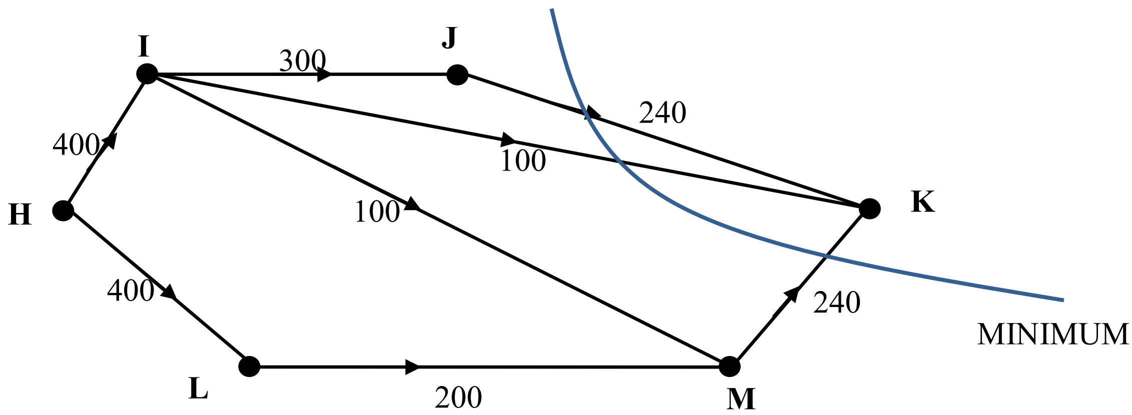
Worked solution



1 mark

b. Show the minimum cut and hence, determine the maximum flow capacity of the network.

Worked solution:



$Maximum\ flow\ capacity\ of\ the\ network = 240 + 100 + 240 = 580\ litre\ per\ minute$

2 marks

Question 5

In the “Brightest Brains” chess tournament, each player competes against each of the other players in a chess game after which one player is judged the winner and the other is the loser. The results of these games between five competitors; Susan, Keith, Cathy, Malcolm and Terry, are shown in the one-step dominance matrix below.

$$A = \begin{array}{c} \\ S \\ K \\ C \\ M \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} S & K & C & M & T \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

a. Find the two step dominance matrix.

Worked solution

$$A^2 = \begin{array}{c} \\ S \\ K \\ C \\ M \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} S & K & C & M & T \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

1 mark

b. Using the information on one-step and two-step dominances, find the winner of the chess competition.

Worked solution

$$A^2 + A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 3 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 2 & 0 \end{bmatrix}$$

The sum of the first row is 7, second row is 3, third row is 8, fourth row is 2 and fifth row is 6. Since the third row has the highest sum, the winner of the chess competition is Cathy.

1 mark

Total 15 marks

END OF MODULE 5

Module 6: Matrices

Question 1

The table below displays the goods to be delivered from a technological equipment company's warehouse to its four different shops.

	To shop A	To shop B	To shop C	To shop D
Mobile phones	24	15	35	22
Televisions	32	23	31	20
Cameras	45	76	63	55
Laptops	12	7	9	11

a. Write down a 2×4 matrix that displays the number of mobile phones, televisions, cameras and laptops to be delivered to shop B and shop D.

$$A = \begin{array}{c} \\ To\ shop\ B \\ To\ shop\ D \end{array} \begin{array}{c} \begin{array}{cccc} \text{M} & \text{T} & \text{C} & \text{L} \end{array} \\ \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \end{array}$$

Worked solution

$$A = \begin{array}{c} \\ To\ shop\ B \\ To\ shop\ D \end{array} \begin{array}{c} \begin{array}{cccc} \text{M} & \text{T} & \text{C} & \text{L} \end{array} \\ \begin{bmatrix} 15 & 23 & 76 & 7 \\ 22 & 20 & 55 & 11 \end{bmatrix} \end{array}$$

1 mark

b. Let A be the matrix found in the previous question and B be a 4×1 matrix representing the unit cost prices (in dollars) of mobile phones, televisions, cameras and laptops produced by the warehouse. Matrix B is defined as follows.

$$B = \begin{bmatrix} 560 \\ 1250 \\ 360 \\ 1850 \end{bmatrix}$$

i. Evaluate the matrix product AB.

Worked solution

$$A \times B = \begin{bmatrix} 15 & 23 & 76 & 7 \\ 22 & 20 & 55 & 11 \end{bmatrix} \times \begin{bmatrix} 560 \\ 1250 \\ 360 \\ 1850 \end{bmatrix} = \begin{bmatrix} 77\,460 \\ 77\,470 \end{bmatrix}$$

ii. What information does the matrix product AB represent? Explain its meaning.

Worked solution

Matrix product AB represents the total cost of all equipments that shop B and shop D received from the warehouse.

iii. Explain why the matrix product BA does not exist.

Worked solution

The number of columns of matrix B is 1 and the number of rows of matrix A is 4. Since the number of columns of matrix B is not equal to the number of rows of matrix A, the matrix product BA does not exist.

iv. If K is a 1×6 matrix, what is the order of the matrix product BK?

Worked solution

The matrix product of a 4×1 matrix and a 1×6 matrix would yield a 4×6 matrix.

1+1+1+1=4 marks

c. All of the mobile phones, televisions, cameras and laptops in the shops were sold within a month. Selling price of each item was constant in every store. The revenues that shop A, shop B, shop C and shop D received from selling all these technological items are \$119 560, \$106 035, \$129 725 and \$103 105, respectively.

Let the selling price, in dollars; of each of the items be represented by ***m*** for the mobile phones, ***t*** for the televisions, ***c*** for the cameras and ***l*** for the laptops.

This situation is represented in the matrix equation below.

$$\begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \times \begin{bmatrix} \boldsymbol{m} \\ \boldsymbol{t} \\ \boldsymbol{c} \\ \boldsymbol{l} \end{bmatrix} = \begin{bmatrix} 119\,560 \\ 106\,035 \\ 129\,725 \\ 103\,105 \end{bmatrix}$$

i. On the above equation, find the 4×4 matrix that displays the number of mobile phones, televisions, cameras and laptops that were delivered to shops A, B, C and D.

Worked solution

$$\boldsymbol{A} = \begin{array}{c} \begin{array}{cccc} \boldsymbol{m} & \boldsymbol{t} & \boldsymbol{c} & \boldsymbol{l} \end{array} \\ \begin{bmatrix} 24 & 32 & 45 & 12 \\ 15 & 23 & 76 & 7 \\ 35 & 31 & 63 & 9 \\ 22 & 20 & 55 & 11 \end{bmatrix} \end{array}$$

ii. Solve the matrix equation and hence, find the values of **m**, **t,** **c** and **l**.

Worked solution

$$\begin{bmatrix} \boldsymbol{m} \\ \boldsymbol{t} \\ \boldsymbol{c} \\ \boldsymbol{l} \end{bmatrix} = \begin{bmatrix} 24 & 32 & 45 & 12 \\ 15 & 23 & 76 & 7 \\ 35 & 31 & 63 & 9 \\ 22 & 20 & 55 & 11 \end{bmatrix}^{-1} \times \begin{bmatrix} 119\,560 \\ 106\,035 \\ 129\,725 \\ 103\,105 \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{m} \\ \boldsymbol{t} \\ \boldsymbol{c} \\ \boldsymbol{l} \end{bmatrix} = \begin{bmatrix} 720 \\ 1550 \\ 580 \\ 2215 \end{bmatrix}$$

1+2=3marks

SECTION B – continued

Question 2

The technological equipment company did a questionnaire on the customer behaviour in order to attract more customers. The feedback of the questionnaire showed that:

20% of the people who went to shop A this month will go to shop C next month.
15% of the people who went to shop A this month will go to shop B next month.
Only 5% of the people who went to shop A this month will go to shop D next month.

One fifth of the people who went to shop B this month will go to the same shop next month.
None of the people who went to shop B this month will go to shop C next month.
8% of the people who went to shop B this month will go to shop D next month.

16% of the people who went to shop C this month will go to shop B next month.
Three quarters of the people who went to shop C this month will go to shop A next month.
4% of the people who went to shop C this month will go to shop D next month.

Half of the people who went to shop D this month will go to the same shop next month.
Half of the remaining people will go to shop B and the other half will go to shop C next month.

a. Enter this data into the transition matrix T below.

$$T = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{this month} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \end{array} \text{next month}$$

Worked solution

$$T = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{this month} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0.6 & 0.72 & 0.75 & 0 \\ 0.15 & 0.2 & 0.16 & 0.25 \\ 0.2 & 0 & 0.05 & 0.25 \\ 0.05 & 0.08 & 0.04 & 0.5 \end{bmatrix} \end{array} \text{next month}$$

2 marks

b. This questionnaire was conducted in May, 2010. At that time shop A, B, C and D had 315, 426, 790 and 1200 customers respectively. Write down the initial state matrix, S_0, for the number of customers that each shop had when the questionnaire was conducted.

Worked solution

$$S_0 = \begin{bmatrix} 315 \\ 426 \\ 790 \\ 1200 \end{bmatrix}$$

1 mark

c. How many of the people involved in the questionnaire moved from shop C to shop B one month later?

Worked solution

16% of the 790 people moved from shop C to shop B in the first month.

$$So, \quad \frac{16}{100} \times 790 = 126.4$$
$$\cong 126 \; people\ moved\ from\ shop\ C\ to\ shop\ B\ one\ month\ later.$$

1 mark

d. How many customers will shop C have in September?

Worked solution

$$S_4 = \begin{bmatrix} 0.6 & 0.72 & 0.75 & 0 \\ 0.15 & 0.2 & 0.16 & 0.25 \\ 0.2 & 0 & 0.05 & 0.25 \\ 0.05 & 0.08 & 0.04 & 0.5 \end{bmatrix}^4 \begin{bmatrix} 315 \\ 426 \\ 790 \\ 1200 \end{bmatrix} = \begin{bmatrix} 1545.89 \\ 473.24 \\ 406.01 \\ 305.86 \end{bmatrix}$$

So, there will be 406 customers in shop C in September.

1 mark

e. The technological equipment company decided to close down the shops if they have less than 300 customers a month.

i. Write down the number of customers that each shop is going to have in the long run.

Worked solution

$$S_{50} = \begin{bmatrix} 0.6 & 0.72 & 0.75 & 0 \\ 0.15 & 0.2 & 0.16 & 0.25 \\ 0.2 & 0 & 0.05 & 0.25 \\ 0.05 & 0.08 & 0.04 & 0.5 \end{bmatrix}^{50} \begin{bmatrix} 315 \\ 426 \\ 790 \\ 1200 \end{bmatrix} = \begin{bmatrix} 1595.40 \\ 463.5 \\ 405.92 \\ 266.17 \end{bmatrix}$$

The steady state matrix is $\begin{bmatrix} 1595.40 \\ 463.5 \\ 405.92 \\ 266.17 \end{bmatrix}$.

So, in the long run shop A, B, C and D will have 1595, 464, 406 and 266 customers respectively.

ii. Which one of the four shops is going to have the highest number of customers and which one will have to be closed down?

Worked solution

In the long run, shop A is going to have the highest number of customers and shop D will have to be closed down.

1+1=2 marks

Total 15 marks

END OF MODULE 6

END OF SOLUTIONS BOOK