THE SCHOOL FOR EXCELLENCE (TSFX)

UNIT 3 & 4 FURTHER MATHEMATICS 2020

WRITTEN EXAMINATION 1 – SOLUTIONS

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SECTION A – CORE

DATA ANALYSIS

QUESTION 1   Answer D

25 people were surveyed and ten of these people answered "No". The remaining 15 people answered "Yes" with one of three different outcomes.

Therefore $\frac{15}{25} \times 100 = 60\%$ lied on their resumes.

QUESTION 2   Answer A

The type of data recorded for the response to "Have you lied on a resume?" is nominal categorical data. It is categorical because the data response is worded, even though numbers were allocated to each response. It is nominal because there is no specific order required.

QUESTION 3   Answer B

Examining the data, the five figure summary can be determined as:

Minimum	Q_1	Median	Q_3	Maximum
15.3º	16.9º	19.2º	21.5º	23.5º

The only box plot that shows all these five statistics correctly is option B.

QUESTION 4   Answer C

$100 is represented by two on a log scale as $\log_{10} 100 = 2$ and $1000 is represented by three on a log scale as $\log_{10} 1000 = 3$. The frequency value between two and three on the histogram is four.

QUESTION 5   Answer A

The value *a* must be between 57 and 62 as it is between these two values in the list.

If *a* = 57 then the interquartile range is 57 – 52 = 5. This places the upper and lower fences at 52 – 1.5 × 5 = 44.5 and 57 + 1.5 × 5 = 64.5 respectively. This would result in one outlier because 44 < 44.5.

If *a* = 58 then the interquartile range is 58 – 52 = 6. This places the upper and lower fences at 52 – 1.5 × 6 = 43 and 58 + 1.5 × 6 = 67 respectively. This would result in no outliers as there are no values larger than 67 or less than 43.

Any larger value would also exclude the presence of outliers, so the answer must be A.

QUESTION 6   Answer E

All values for weight have 6 added to them. This means that the mean will increase by six units. As all values move upward by the same amount the standard deviation, which is a measure of spread, will stay the same.

QUESTION 7   Answer E

This is a normal distribution with a mean of 500.

$\dfrac{40}{250} \times 100 = 16\%$ of the packets weigh more than 512 grams.

Using the 68 – 95 – 99.7% rule, 16% of the sample will be more than one standard deviation above the mean, therefore one standard deviation is 12 grams.

QUESTION 8   Answer A

The standard score is calculated using $\dfrac{data\ value - mean}{standard\ deviation}$, but this question is better

approached using understanding. The highest standard score will be the result of the highest time. The highest time is 12.1 seconds for William.

QUESTION 9   Answer E

The statement "53% of the variation in a student's exam scores can be explained by the variation in the number of days absent from class", tells us that the value of the coefficient of determination is $r^2 = 0.53$.

The correlation coefficient is therefore $r = \pm\sqrt{0.53} = \pm 0.72801...$

It would be expected that as the days absent increased, the exam score would decrease, so the value of the correlation coefficient will be negative, $r = -0.72801... \approx -0.73$.

QUESTION 10   Answer C

To justify an association between the sale prices of houses and their location, a difference in medians, as stated in Option C, would be an appropriate comparison. The median is appropriate because it is indicative of the whole distribution.

Option A is not appropriate because there is no difference observed.

Option B is not appropriate because it compares the maxima and the maximum in each location is only a single value.

Option D is not appropriate as outliers are not indicative of the whole distribution.

Option E is not appropriate because the range may be affected by outliers.

QUESTION 11   Answer C

The mean score for Further Mathematics is (47+25+36+32+39+31+22+15+29+27)/10 = 30.3.

The least squares regression equation that can be used to predict the *FM score* from the *Eng score* is *FM score* $= 12.84 + 0.58 \times Eng\ score$.

Using the formula $y = a + bx$, the value of *a* is given by $a = \bar{y} - b\bar{x}$. From the formula given, $a = 12.84$ and $b = 0.58$ so:

$12.84 = 30.3 - 0.58 \times \bar{x}$

$\bar{x} = \dfrac{12.84 - 30.3}{-0.58} = 30.103...$

The mean of English scores is $\bar{x} = 30.1$.

QUESTION 12   Answer B

Exploring each option:

Option A is true as it reflects the definition of a least squares line.

Option B is false as while the number of positive and negative residuals are often equal, there may be some difference between them.

Option C is true as the sign of the gradient is an indication of the direction of the relationship, as is the sign of the correlation coefficient.

Option D is true as the statement is an appropriate interpretation of the gradient.

Option E is true as the equation of a least square line is given by $y = a + bx$ where $a = \bar{y} - b\bar{x}$.
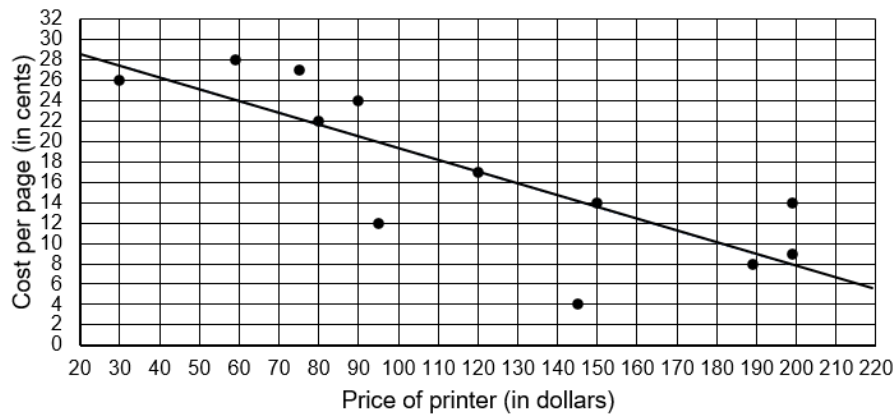
If $\bar{x}$ is substituted into the equation:

$y = \bar{y} - b\bar{x} + b\bar{x}$

$y = \bar{y}$

Therefore, Option B is not true.

QUESTION 13   Answer D

The line must be added to the scatterplot as shown below:

The line can be placed by determining values as shown:

$$\text{When Price = 20, } Cost\ per\ page = 30.83 - 0.12 \times 20 = 28.43$$
$$\text{When Price = 220, } Cost\ per\ page = 30.83 - 0.12 \times 220 = 4.43$$

By observation it can be seen that the first few residuals will be approximately (30, -1), (60, 4), (75, 5), (80, 0.7) and (90, 4).

The only residual plot that has points similar to these is Option D.

QUESTION 14   Answer D

The graph curve indicates that any of the following transformations would be suitable:
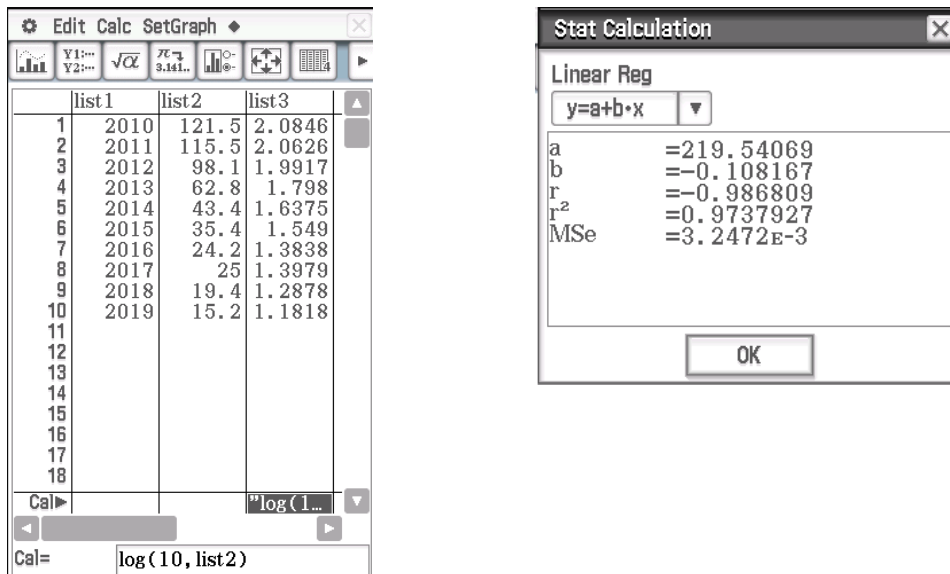
$quarter^2$

$\log(cumulative\ sales)$

$\dfrac{1}{cumulative\ sales}$

The only option that is present is a reciprocal transformation of the *cumulative sales*.

QUESTION 15   Answer B

The relationship is best determined using a CAS calculator as shown:

It can be seen that the vertical intercept of the relationship is 219.54069 which is 220, correct to two significant figures and the gradient of the relationship is -0.108167, which is -0.11 correct to two significant figures, so the relationship is

$$\log_{10}(sales(in\,millions)) = 220 - 0.11 \times year \, .$$

QUESTION 16   Answer C

Exploring each option:

Option A is true: Quarter 3, 2017 is t = 3, so:

$$\frac{1}{Revenue(in\,\$millions)} = 0.17 - 0.016 \times 3$$

$$Revenue(in\,\$millions) = \frac{1}{0.17 - 0.016 \times 3} = 8.196$$

This is $8.2 million correct to two significant figures.

Option B is true: The gradient of the reciprocal relationship is negative which means that the linear relationship would be positive as a reciprocal transformation compresses the axis and twists it to reverse the direction.

Option C is false: The reciprocal relationship has a gradient of -0.016, but the gradient of the linear relationship will be positive with a different gradient.

Option D is true: In Quarter 1, 2018 the actual *Revenue (in $millions)* was $12.8 million.

The predicted value is:

$$\frac{1}{Revenue(in\ \$millions)} = 0.17 - 0.016 \times 5$$

$$Revenue(in\ \$millions) = \frac{1}{0.17 - 0.016 \times 5} = 11.1...$$

The residual is therefore 12.8 – 11.1 = $1.7 million.

Option E is true: Once the value of t is 11 or more the revenue is predicted to be negative, so Quarter 3, 2019 and onwards are unreliable.

QUESTION 17   Answer A

The time series graph displays a clear seasonal pattern with repeating patterns every seven values. There is, however, a clear change in the pattern starting from t = 22, where although a seven-point seasonal pattern continues the values are significantly different to previous values. This is an indication of structural change.

QUESTION 18   Answer C

The two-point moving mean with centring for Thursday is calculated as follows:

$$\frac{32 + 31}{2} = 31.5, \frac{31 + 22}{2} = 26.5$$

$$\frac{31.5 + 26.5}{2} = 29°$$

The four-point moving mean with centring for Thursday is calculated as follows:

$$\frac{35 + 32 + 31 + 22}{4} = 30, \frac{32 + 31 + 22 + 26}{4} = 27.75$$

$$\frac{30 + 27.75}{2} = 28.875°$$

Therefore, the two-point moving mean is less than one degree more than the four-point moving median.

QUESTION 19   Answer A

The seasonal index for Quarter 3 is 0.71. The seasonal index is the average of the seasonal proportions for that point in the cycle, so it will be similar to 0.71.

The seasonal proportion is the actual value ÷ the average for the cycle. If x is the income in Quarter 3, the following equation can be solved:

$$\frac{x}{(98\,700 + 78\,600 + x + 86\,400)/4} = 0.71$$

Using CAS, the value of x is $56 907.90 which is closest to $56 900.

QUESTION 20   Answer E

The actual sales in Quarter 3, 2018 were 3964, so the deseasonalised sales are

$$\frac{actual\ value}{seasonal\ index} = \frac{3964}{0.5} = 7928 .$$

The actual sales in Quarter 4, 2019 were 12 837, so the deseasonalised sales are

$$\frac{actual\ value}{seasonal\ index} = \frac{12837}{1.5} = 8558 .$$

Both of these points have zero residuals, so the line passes through the points (3, 7928) and (8, 8558).

The gradient of this line is $\frac{8558 - 7928}{8 - 3} = 126$.

The y – intercept can be calculated as follows:

$y = mx + c$

$8558 = 126 \times 8 + c$

$c = 8558 - 126 \times 8 = 7550$

Therefore, in the equation *deseasonalised sales* $= a + b \times t$, the value of *a* = 7550 and *b* = 126.

RECURSION AND FINANCIAL MODELLING

QUESTION 21   Answer C

The sequence is an example of geometric growth. It represents growth because the numbers are increasing and the growth is not arithmetic because there is a different increase each time.

The increase is geometric with a multiple of 1.08 as shown below:

$$\frac{t_2}{t_1} = \frac{10800}{10000} = 1.08$$

$$\frac{t_3}{t_2} = \frac{11664}{10800} = 1.08$$

$$\frac{t_4}{t_3} = \frac{12597.12}{11664} = 1.08$$

QUESTION 22   Answer B

The recurrence relation has no multiplier of the previous term, so it cannot represent a compound interest account. Therefore options C, D and E are not correct.

The interest rate of the simple interest account is $\frac{888}{12000} \times 100 = 7.4\%$, so option B is correct.

QUESTION 23   Answer C

For an interest rate to be equivalent to 4.8% per annum, the interest rate multiplied by the times per year must be 4.8. Each option is shown below:

2.4% per half year	$2.4 \times 2 = 4.8\%$
1.2% per quarter	$1.2 \times 4 = 4.8\%$
0.2% per month	$0.2 \times 12 = 2.4\%$
$\frac{6}{65}\%$ per week	$\frac{6}{65} \times 52 = 4.8\%$
$\frac{24}{1825}\%$ per day	$\frac{24}{1825} \times 365 = 4.8\%$

Therefore, Option C is the rate not equivalent to 4.8%.

QUESTION 24   Answer D

The value decreases by $30 000 – $12 000 = $18 000 over a three year period, so $6000 per year.

As he travels 40 000 km per year, each kilometre the utility travels represents a depreciation of $\dfrac{6000}{40000} = \$0.15$ per kilometre.

The question requires the recurrence relation that would give the book value, in dollars, after *n* kilometres, so it could be option D as this recurrence relation has a starting value of $30 000 and a unit decrease of $0.15.

QUESTION 25   Answer E

The interest rate per month can be calculated using $\dfrac{12.54}{2280} \times 100 = 0.55\%$.

The amount of interest during month 4 is calculated as 0.55% of $574.27, so it is $\dfrac{0.55}{100} \times 574.27 \approx \$3.16$. This is value *b*.

In order to have a zero balance the principal reduction must be the same as the previous balance, so *c* is $574.27.

The payment, *a*, must include the principal reduction and the interest so it is $574.27 + 3.16 = \$577.43$.

The answer that matches these three values is E.

QUESTION 26   Answer D

The nominal rate of interest per month is 0.325%, because the multiplier is $(1 + \dfrac{r}{100}) = 1.00325$. This is the same as a nominal rate of $0.325 \times 12 = 3.9\%$ per annum.

The effective rate, equivalent to a nominal rate of 3.9% per annum, can be calculated using the interest conversion function on the CAS calculator as shown below:

Interest Conversion

N	12
EFF	3.97047327
APR	3.9

This is an effective rate of 3.97%.

QUESTION 27   Answer D

Daniel's initial balance of his perpetuity would be $82 000:

$$348.50 \times 12 = \$4182$$
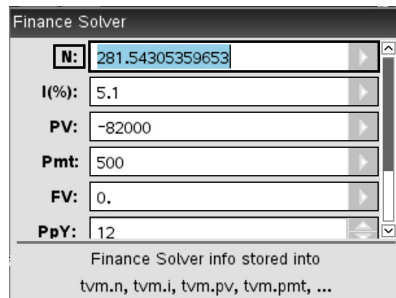
$$4182 = \frac{5.1}{100} \times balance$$

$$balance = \frac{4182 \times 100}{5.1} = \$82\,000$$

Using CAS the annuity can be explored from a starting balance of $82 000:

Compound Interest

N	281.5430536
I%	5.1
PV	−82000
PMT	500
FV	0
P/Y	12
C/Y	12

Finance Solver

N:	281.54305359653
I(%):	5.1
PV:	−82000
Pmt:	500
FV:	0.
PpY:	12

Finance Solver info stored into
tvm.n, tvm.i, tvm.pv, tvm.pmt, ...

After 281 months there will be a smaller balance which will not reduce to zero until the end of the next month, so the time will be 282 months.

QUESTION 28   Answer B

This question must be approached using the CAS Financial function:

Compound Interest

N	12
I%	8.5
PV	−468749.9976
PMT	4000
FV	460268.37
P/Y	12
C/Y	12

The initial amount in the annuity can be calculated as shown to be $468 750, correct to the nearest cent.

Compound Interest

N	36
I%	8.5
PV	−468750
PMT	4000
FV	440989.7552
P/Y	12
C/Y	12

The balance after three years can then be calculated as $440989.76.

In three years of the annuity, the balance has fallen by $468750 - 440989.76 = \$27\,760.24$.

Manual has taken $4000 \times 36 = \$144\,000$ from the annuity in the three years. Therefore, the interest earned is $144000 - 27760.24 = \$116\,239.76$ which is closest to $116 240.

QUESTION 29   Answer C

Option A is represented by crosses. The crosses form a straight line, so they represent flat rate depreciation. The rate of depreciation can be determined by the annual reduction of $4000, using $\dfrac{4000}{40000} \times 100 = 10\%$.

Option B is represented by dots. The crosses form a curve, so they represent reducing balance depreciation. The rate of depreciation can be calculated using the reduction of $8000 in the first year, using $\dfrac{8000}{40000} \times 100 = 20\%$.

Exploring each option:

A is incorrect as shown above because the rate of depreciation is 20% not 15%.

B is incorrect as shown above because the rate of depreciation is 10% not 20%.

C is true because after three years, the book value is lower for Option B. This means there has been a greater amount of depreciation.

D is incorrect because after eight years, the book value is higher for Option A. This means there has been less depreciation compared to Option B.

E is incorrect because, for Option B, the depreciation is represented by a curve as the annual amount of depreciation reduces.

QUESTION 30   Answer B

This question requires the use of the CAS Financial function:

Compound Interest

N	360
I%	3.85
PV	480000
PMT	-2250.279515
FV	0
P/Y	12
C/Y	12

If the loan is paid with both principal and interest from the beginning, the monthly payment will be $2250.28.

Over the 30 years, the total paid for the loan will be $2250.28 \times 360 = \$810\,100.80$

Compound Interest

N	336
I%	3.85
PV	480000
PMT	-2336.388633
FV	0
P/Y	12
C/Y	12

If there is an interest only period at the start of the loan, they will pay $\dfrac{3.85}{100 \times 12} \times 480000 \times 24 = \$36\,960$ in the first two years with no reduction in the loan balance.

As shown they then will then pay $2336.39 per month for 28 years at a cost of $2336.39 \times 336 = \$785\,027.04$.

This means there is a total cost of $36960 + 785027.04 = \$821\,987.04$.

The amount saved is $821987.04 - 810100.80 = \$11\,886.24$ which is closest to $12 000.

SECTION B – MODULES

MODULE 1: MATRICES

QUESTION 1   Answer B

$$3\begin{bmatrix} 5 & 4 \\ -6 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 3 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} 3\times 5-2\times 1 & 3\times 4-2\times 3 \\ 3\times -6-2\times -4 & 3\times 3-2\times -1 \end{bmatrix} = \begin{bmatrix} 13 & 6 \\ -10 & 11 \end{bmatrix}$$

QUESTION 2   Answer C

The determinant of the matrix $\begin{bmatrix} 5 & -4 \\ 6 & -7 \end{bmatrix}$ is given by $(5\times -7)-(-4\times 6)=-11$.

QUESTION 3   Answer D

There are five different letters in THE EYES and in THEY SEE, but seven different positions because E is repeated three times.

Three of the rows of the permutation matrix are interchangeable because of the three E's, row 3, row 6 and row 7, each aligned with an E in the matrix $\begin{bmatrix} T \\ H \\ E \\ Y \\ S \\ E \\ E \end{bmatrix}$.

All six options are shown below:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

QUESTION 4   Answer D

The following rules will determine whether each matrix is defined:

- A matrix multiplication will be defined if the number of columns in the first matrix is equal to the number of rows in the next matrix.

- The matrix product will have the number of rows from the first matrix and columns from the last matrix.

- Matrix powers, including inverses, can only be defined with square matrices.

- Addition and subtraction of matrices is defined when the order of two matrices is the same.

Each option is explored below:

Option A ACB Using order of matrices: $3 \times \underline{4 \times 4} \times \underline{2 \times 2} \times 3$ so the product is defined.

Option B $(BD)^{-1}$ Using order of matrices: $2 \times \underline{3 \times 3} \times 2$ so the product is defined and

will have order 2×2. As the product is square, an inverse may exist.

Option C $(DB)^3$ Using order of matrices: $3 \times \underline{2 \times 2} \times 3$ so the product is defined and

will have order 3×3. As the product is square, a power exists.

Option D $(AC)^2$ Using order of matrices: $3 \times \underline{4 \times 4} \times 2$ so the product is defined and

will have order 3×2. As the product is not square, a power will not exist.

Option E $AC - D$ Using order of matrices for AC: $3 \times \underline{4 \times 4} \times 2$ so the product is defined

and will have order 3×2. This is the same order as D, so the subtraction is defined.

QUESTION 5   Answer E

The elements of the matrix are defined by the rule $3i - j$, this means that the following values exist for a matrix up to 4×4.

$$A = \begin{array}{c c} & \begin{array}{cccc} j=1 & j=2 & j=3 & j=4 \end{array} \\ \begin{array}{c} i=1 \\ i=2 \\ i=3 \\ i=4 \end{array} & \begin{bmatrix} 3\times1-1=2 & 3\times1-2=1 & 3\times1-3=0 & 3\times1-4=-1 \\ 3\times2-1=5 & 3\times2-2=4 & 3\times2-3=3 & 3\times2-4=2 \\ 3\times3-1=8 & 3\times3-2=7 & 3\times3-3=6 & 3\times3-4=5 \\ 3\times4-1=11 & 3\times4-2=10 & 3\times4-3=9 & 3\times4-4=8 \end{bmatrix} \end{array}$$

Exploring each option:

Option A   If $m = n$, the elements on the leading diagonal could also be calculated using $i + j$ is true. It can be seen that the elements on the leading diagonal are 1+1 = 2, 2+2 = 4, 3+3 = 6, 4+4 = 8 etc.

Option B   If $m < n$, there could be negative elements in Matrix A is true. The position where i = 1 and j = 4 has a value of -1. This would be where $m < n$.

Option C   If $m > n$, there will be no negative elements in Matrix A is true. Wherever the number of rows exceeds the number of columns the values are positive.

Option D   The transpose of Matrix A would have the rule $3j - i$ is true. In the transpose the positions of i and j are reversed as they are in $3j - i$.

Option E   If $n = 4$ there could be two zero elements in Matrix A is not true. As can be seen above there would only be one zero element in column 3.

QUESTION 6   Answer C

Using the information given the following matrix equation can be written:

$$\begin{bmatrix} 4 & 12 \\ 5 & 15 \end{bmatrix} \begin{bmatrix} P \\ M \end{bmatrix} = \begin{bmatrix} 33.80 \\ 42.25 \end{bmatrix}$$

This matrix equation cannot be solved because the determinant of $\begin{bmatrix} 4 & 12 \\ 5 & 15 \end{bmatrix}$ is $(4 \times 15) - (12 \times 5) = 0$, so the matrix is singular.

The singular matrix indicates that the simultaneous equations are either dependent (the same line) or inconsistent (parallel lines).

Because each value in the second row is 1.25 times the value above it, these simultaneous equations represent the same line, so they are dependent.

QUESTION 7   Answer B

from

The transition matrix from the diagram is
$$\begin{array}{ccc} A & B & C \end{array}$$
$$\begin{bmatrix} 0.20 & 0.50 & 0.25 \\ 0.35 & 0.40 & 0.60 \\ 0.45 & 0.10 & 0.15 \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array} \text{ to}$$

There are 1000 people in the system, so there needs to be 1000 people in the state matrix used. It is unimportant where in this matrix they are, so long as the steady state is reached.

$$\begin{bmatrix} 0.20 & 0.50 & 0.25 \\ 0.35 & 0.40 & 0.60 \\ 0.45 & 0.10 & 0.15 \end{bmatrix}^{30} \times \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.20 & 0.50 & 0.25 \\ 0.35 & 0.40 & 0.60 \\ 0.45 & 0.10 & 0.15 \end{bmatrix}^{31} \times \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 340.26... \\ 429.11... \\ 230.62.... \end{bmatrix}$$

Therefore, the number at B is closest to 429 people.

QUESTION 8   Answer D

In order to obtain the total amount paid, the number of adults must be summed as must the number of children. The total number of adults must be multiplied by 50 and the total number of children must be multiplied by 30, and then these two values added. Finally, 25% of this amount must be determined. The final matrix will need to be a 1×1 matrix with the value $\begin{bmatrix} 7532.50 \end{bmatrix}$.

All of the options A, B, C and E result in this matrix, but option D is not defined for multiplication as the orders of the matrices are $2 \times 2 \times 2 \times 1 \times 2 \times 7 \times 7 \times 1$. The red highlighted numbers show that the number of columns in the second matrix is not equal to the number of rows in the third matrix.

QUESTION 9   Answer B

Using the supplied information, a matrix calculation can be set up as follows:

$$\begin{bmatrix} 0.8 & Q & R \\ P & 0.8 & R \\ P & Q & 0.8 \end{bmatrix} \begin{bmatrix} 820 \\ 110 \\ 235 \end{bmatrix} = \begin{bmatrix} 747 \\ 381 \\ 478 \end{bmatrix}$$

Using matrix multiplication, the following set of simultaneous equations can be determined:

$0.8 \times 820 + 110Q + 235R = 747$
$820P + 0.8 \times 110 + 235R = 381$
$820P + 110Q + 0.8 \times 235 = 478$

This is best solved using the simultaneous equation solver on CAS:

$\begin{bmatrix} 0.8 \times 820 + 110Q + 235R = 747 \\ 820P + 0.8 \times 110 + 235R = 381 \\ 820P + 110Q + 0.8 \times 235 = 478 \end{bmatrix} P, Q, R$

$\{P = 0.3, Q = 0.4, R = 0.2\}$

However, these are decimal values and *P, Q* and *R* are percentages so *P* = 30, *Q* = 40 and *R* = 20.

QUESTION 10   Answer C

The best approach to this question is to draw a dominance diagram. The diagram can be started by noting that Tim beats three people who beat Wahid. As there are **only** five people in tournament, the first three paths must be T – U – W, T – V – W and T – X – W.

Tim also beat two people who beat Xavier, so these must be paths T – U – X and T – V – X. From there the rest of the diagram falls into place. A completed diagram is shown below:
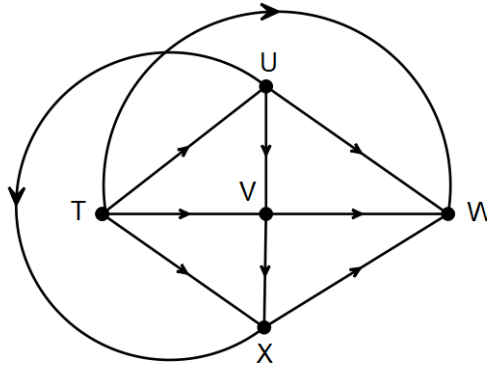
From the diagram it can be seen that the only correct statement is that Uriah defeated Vincenzo.

MODULE 2: NETWORKS AND DECISION MATHEMATICS

QUESTION 1   Answer C

There are eight edges and as each edge has two ends, the degree of the vertices is $8 \times 2 = 16$.

Alternatively, the degrees at each vertex can be counted: deg (A) = 3, deg (B) = 4, deg (C) = 5 and deg (D) = 4. 3 + 4 + 5 + 4 = 16.

QUESTION 2   Answer B

A number of possible Hamiltonian paths and cycles are available in this network. One Hamiltonian path is ABDC and one Hamiltonian is ABDCA.

The network has two vertices of odd degree (A = 3 and C = 5) and two vertices of even degree (B = 4 and D = 4). As there are two vertices of odd degree an Eulerian trail exists with start/finish points at A and C.

QUESTION 3   Answer E

Using Euler's rule for planar graphs, $V + F = E + 2$. As V and E are equal, that leaves that F = 2.

QUESTION 4   Answer E

Using the adjacency matrix, the following network or an isometric equivalent can be drawn:
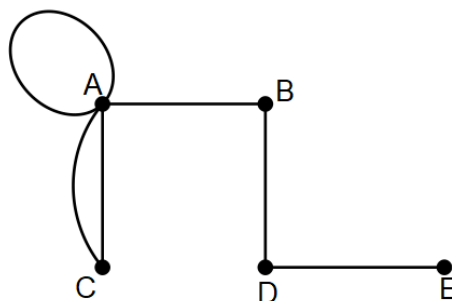
It can be seen that the network is, in fact, undirected, it has a loop at A and there are two bridges AB and BD. As there are six edges including the loop, the sum of the degrees of the vertices is twelve, so options A, B, C and D are true.

Option E is not true because the graph is not simple as it has both a loop and multiple edges from A to C.

QUESTION 5   Answer C

The minimum spanning tree determined using Prim's algorithm is shown below:
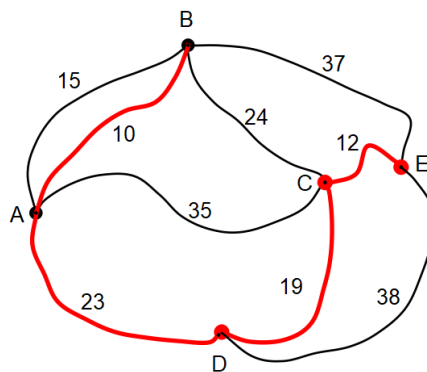
The shortest path from A to E using Dijkstra's algorithm is shown below:
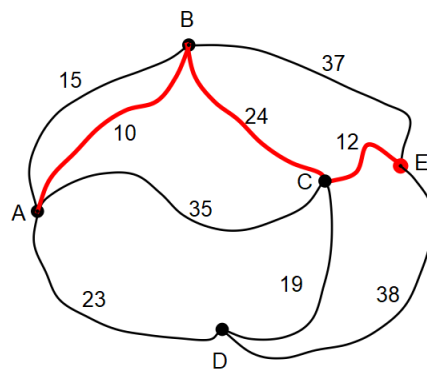
It can be seen that the number of edges the minimum spanning tree and the shortest path from A to E have in common is two, AB and CE.
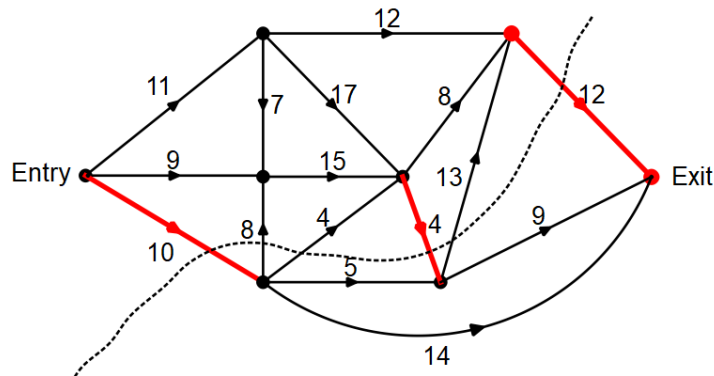

QUESTION 6   Answer B

The capacity of Cut A is 55 as it is 10 + 9 + 7 + 17 + 12 = 55. The edges that are cut that weigh 8 and 13 are not counted as they will have no flow if the edges closer to the source are cut already, thereby restricted their flow to zero.
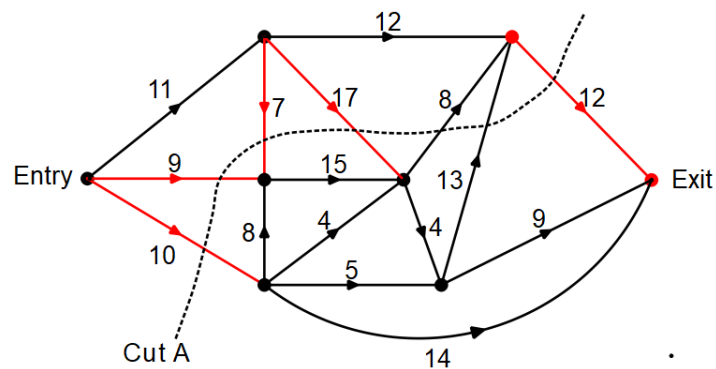
QUESTION 7   Answer A

The maximum number of people per hour who could visit the museum is 26 as given by the cut shown below:

As in the previous question the edges of weight 8, 4 and 13 are not counted as cuts closer to the source restrict flow through these edges.

QUESTION 8   Answer C

The first steps in the Hungarian Algorithm showing row reduction followed by column reduction are below:

	F	R	M	K
P	15	10	21	13
Q	16	9	22	13
R	18	11	24	14
S	17	10	23	13

	F	R	M	K
P	5	0	11	3
Q	7	0	13	4
R	7	0	13	3
S	7	0	13	3

	F	R	M	K
P	0	0	0	0
Q	2	0	2	1
R	2	0	2	0
S	2	0	2	0

The zeros in the reduced matrix can be covered with three lines, so the matrix requires further reduction as shown in the steps below:

	F	R	M	K
P	0	0	0	0
Q	2	0	2	1
R	2	0	2	0
S	2	0	2	0

	F	R	M	K
P	0	2	0	2
Q	0	0	0	1
R	0	0	0	0
S	0	0	0	0

Producing a bipartite graph where allocations are allowed:
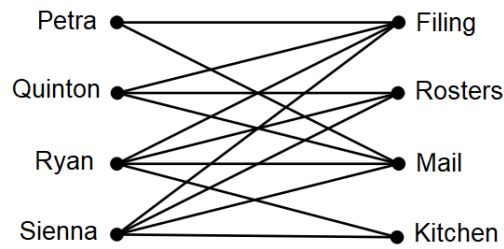
It can be seen that Quinton cannot be allocated the kitchen, so option C is not appropriate.

QUESTION 9   Answer D

The table of predecessors for each activity is shown below:

Activity	Predecessor
A	None
B	A
C	A
D	A
E	B
F	B
G	D
H	D
I	C E F G
J	C E G
K	H I J

The largest number of predecessors is four. This is for activity I that has F as a direct predecessor and C, E and G are predecessors via the dummy.

QUESTION 10   Answer D
The earliest start time (EST), latest start time (LST) and float time for each activity is listed below:

Activity	EST	LST	Float
A	0	0	0
B	23	23	0
C	23	36	13
D	23	35	12
E	37	40	3
F	37	37	0
G	31	43	12
H	31	49	18
I	52	52	0
J	44	47	3
K	58	58	0

Exploring each option:

Option A: The critical path in this network is ABFIK is true as all of these activities have zero float time.

Option B: The smallest float time for a non-critical activity is 3 minutes is true as both E and J have a float time of three.

Option C: The latest start time for activity C is 36 minutes is true as it is followed by J with a latest start time of 47 and the duration of C is 11, so 47 – 11 = 36.

Option D: A reduction in time of activity F by 2 minutes would result in a second critical path is not true. If F was reduced by 2 minutes then the ABFIK would still be the longest path now at 59 minutes duration. The next longest path is ABEJK at 58 minutes, still 1 minute less than the reduced time for ABFIK.

Option E: The largest activity float time is 18 minutes is true as H has a float time of 18 hours.

MODULE 3: GEOMETRY AND MEASUREMENT

QUESTION 1   Answer E

The major sector has an angle of 360º – 32º = 328º. The area of the sector is calculated as follows:

$$Area = \frac{328°}{360°} \times \pi \times 8.5^2 \approx 206.8 \ cm^2$$

QUESTION 2   Answer C

The size of the angle can be found using the sine rule:

$$\frac{\sin(x°)}{181} = \frac{\sin(39°)}{122}$$
$$\sin(x) = \frac{\sin(39°)}{122} \times 181 = 0.93366...$$
$$x = \sin^{-1}(0.93366...) \approx 69°$$

The angle is obtuse and this is a case of the ambiguous case of the sine rule. The answer is therefore 180º – 69º = 111º.

QUESTION 3   Answer C

The wooden cube has a surface area of $20 \times 20 \times 6 = 2400 \ cm^2$.

The hemispherical bowl will have a surface area of $\frac{1}{2} \times 4 \times \pi \times 8^2 = 402.1238... \ cm^2$, but it will replace a circle of area of $\pi \times 8^2 = 201.0619... \ cm^2$.

Therefore the area of the feeding bowl is 2400 – 201.0619 + 402.1238 ≈ 2601.0619 … cm² which is closest to 2601 cm².

QUESTION 4   Answer B

A diagram of the six points A at (0ºN, 150ºW), B at (0ºN, 120ºW), C at (30ºS, 150ºW), D at (30ºS, 120ºW), E at (60ºS, 150ºW) and F at (60ºS, 120ºW) is shown below:
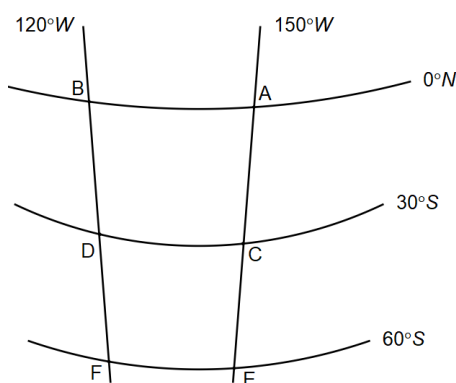
The shortest direct distance would be the arc EF as the radius of the small circle at 60ºS is the smallest. The lengths AB, BD, DF, AC and CE are all equally the longest distance as they are all on great circles.

QUESTION 5   Answer D

Both Hokkaido, Japan, (45ºN, 141ºE) and Minneapolis, USA, (45ºN, 124ºW) are on the 45º N parallel of latitude. The radius of this circle is $6400 \times \cos(45°)$.

The two points are 141 + 124 = 265º apart, but the shortest distance around the circle is 360 – 265 = 95º.

Therefore the distance would be given by $6400 \times \cos(45) \times \dfrac{\pi}{180} \times 95$.

QUESTION 6   Answer A

The original pyramid would have a perpendicular height of the slanted face of 50 cm as it would be formed from the triangle below:
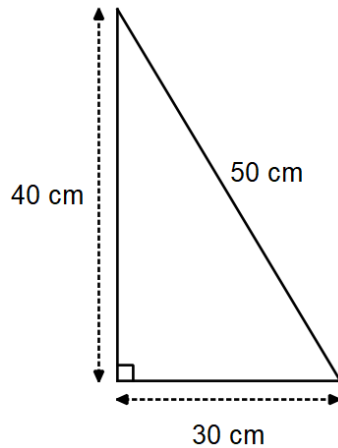
The length of 50 cm can be seen as this is a Pythagorean triple where $50 = \sqrt{30^2 + 40^2}$.

Using similarity, the removed section has a perpendicular height of the slanted face of 50 – 20 = 30 cm, so the removed height will be $\dfrac{30}{50} \times 40 = 24$ cm. This means that the remaining height will be 40 – 24 = 16 cm.
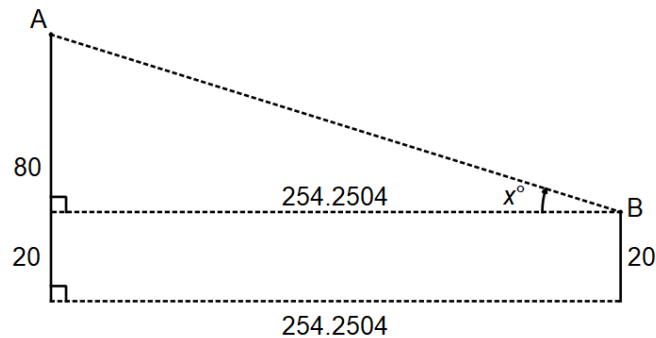
QUESTION 7   Answer B

The location on horizontal ground of Tom and Towers A and B is shown in the diagram below:
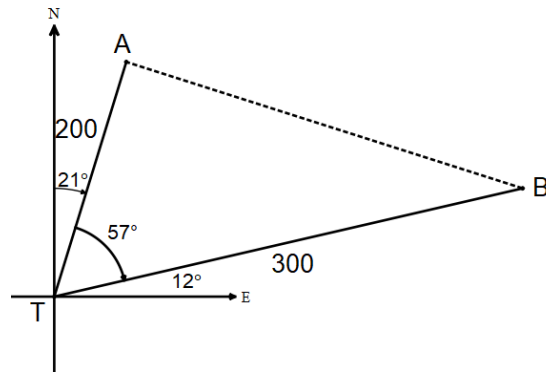
From this diagram is can be seen that the distance between A and B can be calculated as follows: $AB = \sqrt{200^2 + 300^2 - 2 \times 200 \times 300 \times \cos(57°)} = 254.2504...m$

The vertical angle of elevation can be determined using the diagram and calculation that follows:

$$\tan(x°) = \frac{80}{254.2504}$$

$$x° = \tan^{-1}(\frac{80}{254.2504}) = 17.46... \approx 17.5°$$

QUESTION 8   Answer A

The shape is formed from one segment of a 15 m radius circle less another segment formed from a 12 m circle.

The angle at the centre of the larger segment is given by $2 \times \tan^{-1}(\frac{13}{4}) = 145.7945...°$

The angle at the centre of the smaller segment is given by $2 \times \tan^{-1}(\frac{10}{4}) = 136.3971...°$

The area of the larger segment is $\frac{145.79...}{360} \times \pi \times 15^2 - \frac{1}{2} \times 15^2 \times \sin(145.79...) = 232.023...$
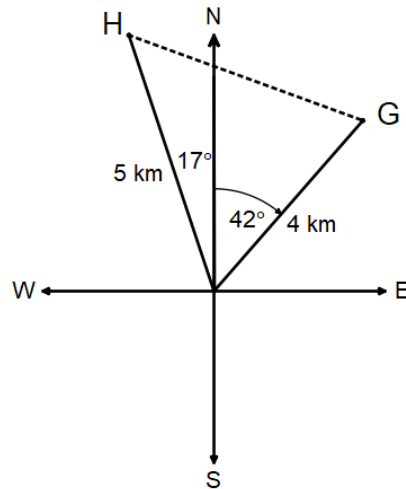
The area of the smaller segment is $\frac{136.39...}{360} \times \pi \times 12^2 - \frac{1}{2} \times 12^2 \times \sin(136.39...) = 121.7465...$

The area of the arch is therefore 232.023… − 121.7465… = 101.27.. ≈ 101 cm².

QUESTION 9   Answer C

The diagram showing the positions of Geraldine and Hugo after their walks is shown below:

The angle between the two walks is 59⁰ and the length between them is given by the cosine rule, so it would be $\sqrt{4^2 + 5^2 - 2 \times 4 \times 5 \times \cos(59°)}$.

QUESTION 10   Answer A

The lengths CD : BE : AF have the ratio 3 : 5 : 7.

Therefore the areas of the triangles CID : BHE : AGF is 9 : 25 : 49.

As the triangles overlap, the area of grey is 25 – 9 = 16 and the area of white is 9 + 49 – 25 = 33.

Therefore the ratio of grey to white is 16 : 33.

MODULE 4: GRAPHS AND RELATIONS

QUESTION 1   Answer D

A horizontal line has the equation $y = constant$ and because the line passes through the point (3, 5), it has the equation $y = 5$.

QUESTION 2   Answer B

Using m = -3 and $x_1 = -4$ and $y_1 = -5$:

$$y - y_1 = m(x - x_1)$$
$$y - -5 = -3(x - -4)$$
$$y + 5 = -3x - 12$$
$$y = -3x - 17$$
$$3x + y = -17$$

QUESTION 3   Answer D

A coffee van proprietor makes at total of 800 coffees and hot chocolates, so if h is the number of hot chocolates, then the number of coffees is $800 - h$.

The total revenue for hot chocolates is $4.8h$ and the revenue for coffees is $4.1(800 - h)$, so the equation is $4.1(800 - h) + 4.8h = 3476$.

QUESTION 4   Answer A

The average decrease in depth from the 8th of April to the 28th of April is given by the gradient from the point (8, 260) to (28, 110) given by $\dfrac{260 - 110}{8 - 28} = -7.5$ cm/day.

This is a decrease of 7.5 cm/day.

QUESTION 5   Answer C

The equation of the line passing through the points (1, 3000) and (8, 2419) can be calculated as follows:

$$m = \frac{3000 - 2419}{1 - 8} = -83$$

Using the point (1,3000) and the gradient of -83:

$$y = mx + c$$
$$3000 = -83 \times 1 + c$$
$$c = 3000 + 83 = 3083$$

So, the equation is *Number of customers* = 3083 – 83 × *n*.

February, 2019 would be equivalent to n = 14:

Number of customers $= 3083 - 83 \times 14 = 1921$

QUESTION 6   Answer C

The graph shows that the relationship is a relationship of the form $b = \dfrac{k}{a}$ where k is the gradient of the straight line. The gradient is given by $\dfrac{4-0}{5-0} = \dfrac{4}{5}$ so the relationship is $b = \dfrac{4}{5} \times \dfrac{1}{a} = \dfrac{4}{5a}$.

QUESTION 7   Answer B

The revenue relationship passes through the points (0, 0) and (10, 5500), so it has a gradient of $\dfrac{5500-0}{10-0} = 550$ and a vertical intercept of 0. The equation is $R = 550n$.

The cost relationship passes through (0, 3000) and (10, 6000), so it has a gradient of $\dfrac{6000-3000}{10-0} = 300$ and a vertical intercept of 3000. The equation is $C = 300n + 3000$.

The profit equation can be calculated as follows:

$Profit = R - C$
$P = 550n - (300n + 3000)$
$P = 250n - 3000$

QUESTION 8   Answer D

If H is the hire fee and t is the number of hours, the equation for Bloo Roo is $H = 2t$.

The line segment graph for Pink Galah will pass through the following points:

$t = 0,\ H = \$6$
$t = 12,\ H = 6 + 1 \times 12 = \$18$
$t = 20,\ H = 6 + 1 \times 12 + 8 \times 4 = \$50$

The first segment has the equation $H = t + 6$.

The second segment has a gradient of 4 (\$4 per hour) and passes through (12, 18) so the equation can be determined as follows:
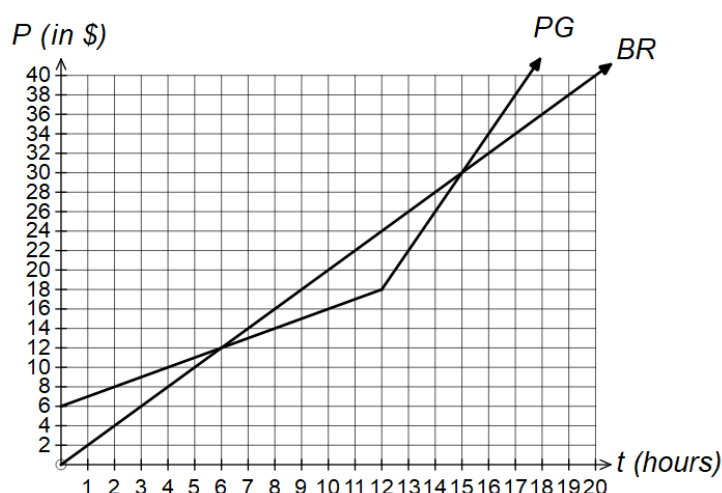
$H = 4t + c$
$18 = 4 \times 12 + c$
$c = 18 - 48 = -30$

So, the equation for Pink Galah is $H = \begin{cases} t+6, & 0 < t \le 12 \\ 4t - 30, & t > 12 \end{cases}$.

The graph that represents these two relationships is below:

It can be seen from the graphs that Pink Galah would be cheapest for more than six hours and less than 15 hours of use.

Exploring each option:

Option A: Incorrect. Bloo Roo is cheapest for up to six hours of use.

Option B: Incorrect. Bloo Roo is cheapest for more than 15 hours of use but is also cheapest for up to six hours of use.

Option C: Incorrect. Pink Galah is cheapest for more than six hours and less than 15 hours of use.

Option D: Correct. As can be seen from the graph above, Pink Galah is cheapest for more than six hours and less than 15 hours of use.

Option E: Incorrect. As can be seen from the graph above, Pink Galah, not Bloo Roo is cheapest for more than six hours and less than 15 hours of use.

QUESTION 9   Answer B

The shaded region is between x values of 5 and 35 so $5 \leq x \leq 35$.

The shaded region is above y values of 10 so $y \geq 10$.

The line $x + y = 50$ passes from (50, 0) on the x axis to (0, 50) on the y axis and the shaded region is below this so $x + y \leq 50$.

The last line passes through (0, 0), (5, 15) and (10,30), so each y value is three times the x value, so the line is $y = 3x$. As the shaded region is below $y = 3x$ it will be $y \leq 3x$.

Therefore, the region is $5 \leq x \leq 35, y \geq 10, x + y \leq 50, y \leq 3x$.

QUESTION 10   Answer D

The feasible region formed by these inequalities is shown shaded below:
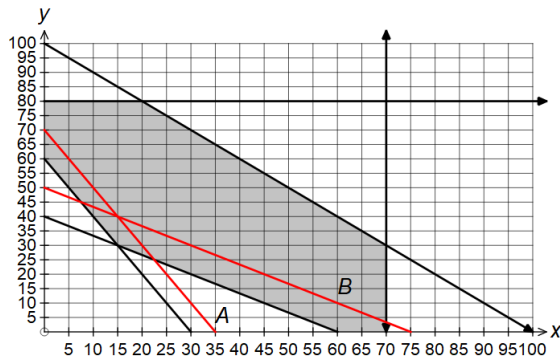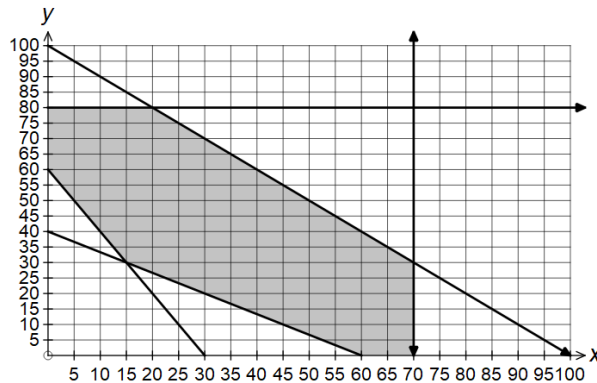

The objective functions $Z = 4x + 2y$ (Option A) and $Z = 2x + 3y$ (Option B) are shown.

Both of these lines will slide down and, as can be seen, A minimises along the line $6x + 3y = 180$ while B minimises along the line $4x + 6y = 240$. This means that while (15, 30) is a minimum, it is not the only minimum for these objective functions.
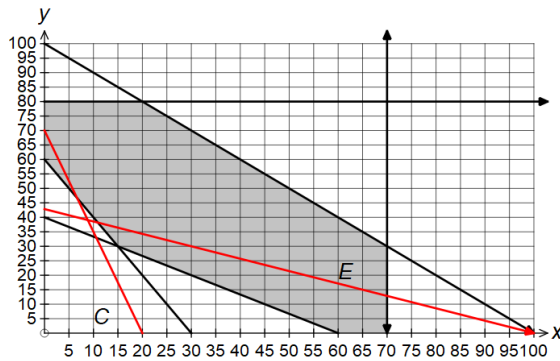
The objective functions $Z = 7x + 2y$ (Option C) and $Z = 3x + 7y$ (Option E) are shown.

Both of these lines slide down to minimise and C will minimise at (0, 60) and E will minimise at (60, 0).
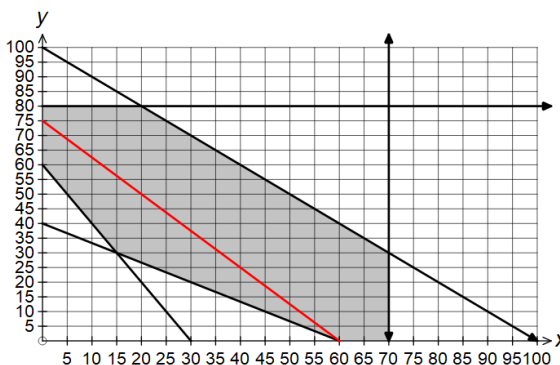
The line $Z = 5x + 4y$ (Option D) is shown.

It can be seen that the objective function will minimise at (15, 30) only.