Suggested solutions to 1995 Mathematical Methods CAT 2 - part I

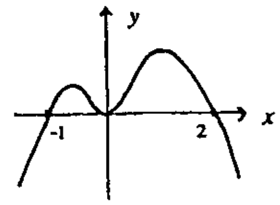
Question 1 E

x intercept : let $y = 0$

$\therefore 0 = x^2(2-x)(x+1)$

$\therefore x = 0,\ 2,\ -1$

$x = 0$ is a turning point



y intercept : let $x = 0$

$\therefore y = 0 \times 2 \times 1 = 0$

General shape is a negative quartic

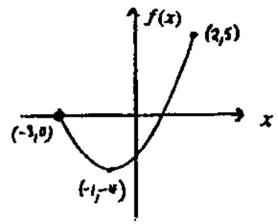
Question 2 B

$f(x) = (x+1)^2 - 4$

From translations, turning point at $(-1,4)$

$f(-3) = (-2)^2 - 4 = 0$

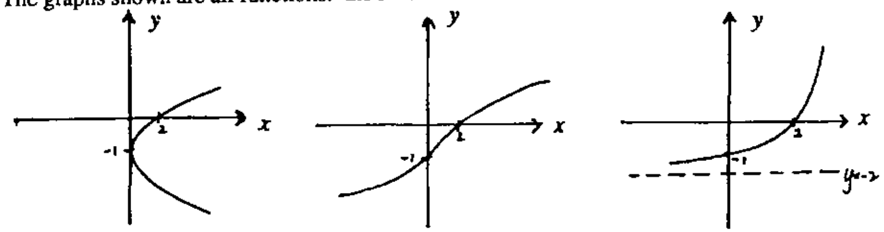
$f(2) = 3^2 - 4 = 5$



The minimum y value is -4 and the maximum y value is 5, therefore the range is $[-4,5]$

Question 3 D

The graphs shown are all functions. the inverse of each function is as follows.



The inverse of (i) is not a function, but the inverses of (ii) and (iii) are both functions.

Question 4 C

Using the asymptotes given, the equation is of the form: $f(x) = \frac{A}{x-1} + 2$

Substitute the point (0,0) : $\therefore 0 = \frac{A}{-1} + 2$

$\therefore A = 2$

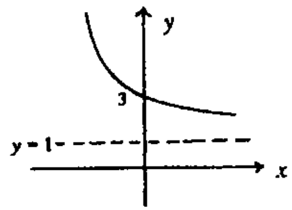
$\therefore f(x) = \frac{2}{x-1} + 2$

$= \frac{2+2(x-1)}{x-1}$

$= \frac{2x}{x-1}$

Question 5 **A**

$y = 1 + 2e^{-x}$



Horizontal asymptote: $y = 1$
Basic shape is reflected in the y axis.
y intercept : $y = 1 + 2e^0 = 3$

Question 6 **C**

Let the model be of the form $y = A\cos n(x + b)$

amplitude = 1, $\therefore A = 1$ period $= \frac{2\pi}{3}$, $\therefore n = 3$ $\therefore y = \cos 3(x + b)$

The cosine curve is translated $\frac{\pi}{6}$ units to the right, $\therefore b = -\frac{\pi}{6}$

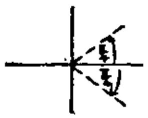
The equation of the curve is $y = \cos 3(x - \frac{\pi}{6}) = \cos(3x - \frac{\pi}{2})$

Question 7. **D**

$\sqrt{2}\cos 3x = 1$

$\therefore \cos 3x = \frac{1}{\sqrt{2}}$

Cosine is positive, angles in 1st & 4th quadrants
Basic angle is $\frac{\pi}{4}$ as $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$



$\therefore 3x = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}$

$\therefore\ x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}$

Question 8. **E**

The gradient of $f(x)$ is negative over the domain $(-\infty, 0)$, therefore statement E is incorrect.

Question 9. **A**

Let $f(x) = \frac{3x^2+2}{x^2} = 3 + 2x^{-2}$

$\therefore\ f'(x) = -4x^{-3} = -\frac{4}{x^3}$

Question 10. **D**

Using the Product rule:

$\frac{dy}{dx} = x(2e^{2x}) + e^{2x}(1)$

$= 2xe^{2x} + e^{2x}$

$= (2x+1)e^{2x}$

Question 11. **C**

Using the Chain rule:

Let $u(x) = x^2 - 4$ $\therefore f(u) = u^{\frac{1}{2}}$

$\therefore\ u'(x) = 2x$ $f'(u) = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{x^2-4}}$

$f'(x) = \frac{2x}{2\sqrt{x^2-4}} = \frac{x}{\sqrt{x^2-4}}$

Question 12. **A**

Using the Quotient rule:

Let $f(t) = \frac{2t-1}{t+4}$

then $f'(t) = \frac{(t+4)(2) - (2t+1)(1)}{(t+4)^2} = \frac{2t+8-2t+1}{(t+4)^2} = \frac{9}{(t+4)^2}$

Question 13. **D**

Let $f(x) = 4x^2 - 2x + 3$
For local maximum or minimum solve $f'(x) = 0$
$\therefore 8x - 2 = 0$
$\therefore\ x = \frac{1}{4}$

Test for local minimum:

$f'(0) = -2 < 0$
$f'(1) = 6 > 0$

x	$< \frac{1}{4}$	$\frac{1}{4}$	$> \frac{1}{4}$
$f'(x)$	< 0	0	> 0
	\	_	/

$x = \frac{1}{4}$ gives minimum value

minimum value $= 4(\frac{1}{4})^2 - 2(\frac{1}{4}) + 3 = 2\frac{3}{4}$

Question 14. **D**

Gradient of tangent $f'(x) = -2e^{-2x}$
At $x = \frac{1}{2}$ gradient of tangent $= -2e^{-1} = -\frac{2}{e}$
At $x = \frac{1}{2}$ gradient of normal $= -1 \div -\frac{2}{e} = \frac{e}{2}$

Question 15. **B**

$V(t)=\frac{2}{5}(15t^2-\frac{1}{4}t^3)$

Rate of change $=\frac{2}{5}(30t-\frac{3}{4}t^2)$

For maximum rate of change let $V''(t)=0$

$\therefore \frac{2}{5}(30-\frac{3}{2}t)=0$

$\therefore \quad 30-\frac{3}{2}t=0$

$\therefore \quad t=20$

Test for maximum:

$V'(19)=\frac{2}{5}\left(30-\frac{3}{2}(19)\right)>0$

$V'(21)=\frac{2}{5}\left(30-\frac{3}{2}(21)\right)<0$

t	<20	20	>21
$V'(t)$	>0	0	<0
	/	–	\

Volume is changing at the greatest rate after 20 minutes.

Question 16. **C**

$(e^x-1)(e^{2x}-4)=0$

either $e^x=1$ or $e^{2x}=4$

$\therefore \quad x=0$ or $2x=\log_e 4$

$x=\frac{1}{2}\log_e 4=\log_e 4^{\frac{1}{2}}=\log_e 2$

Question 17. **B**

$(3-2x)^5=(3)^5-5(3)^4(2x)+10(3)^3(2x)^2-10(3)^2(2x)^3+5(3)(2x)^4-(2x)^5$

coefficient of $x^3=-10\times3^2\times2^3=-720$

Question 18. **C**

Let $y=(x-1)^2-4$

Interchanging x and y gives

$x=(y-1)^2-4$

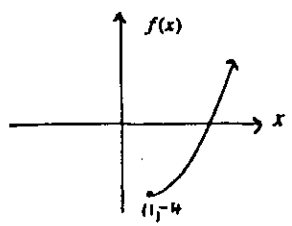
$\therefore \quad x+4=(y-1)^2$

$\therefore \sqrt{x+4}=y-1$

$\therefore \quad y=1+\sqrt{x+4}$

$\therefore f^{-1}(x)=1+\sqrt{x+4}$

Inverse of $f=[-4,\infty)\to R,\ f^{-1}(x)=1+\sqrt{x+4}$



Domain of f^{-1} = range of $f=[-4,\infty)$

Question 19. **E**

If $x=0.5,\ y=4(0.5)^3=0.5$

If $x=1,\ y=4(1)^3=4$

If $x=1.5,\ y=4(1.5)^3=13.5$

Approximate area $=0.5\times0.5+0.5\times4+0.5\times13.5=9$ square units

Question 20. **A**

$\int_3^0 g(x)\,dx=\int_3^0 2f(x)-1\,dx$

$=2\int_3^0 f(x)\,dx-\int_3^0 1\,dx$

$=-2\int_0^3 f(x)\,dx-[x]_3^0$

$=-2(4)-(0-3)$

$=-5$

Question 21. **E**

$\int_0^{\frac{\pi}{2}} 4\sin 2x\,dx=[-2\cos 2x]_0^{\frac{\pi}{2}}$

$=-2\cos\pi+2\cos 0$

$=2+2$

$=4$

Question 22. **B**

$f'(x)=\frac{6}{\sqrt{3x-1}}=6(3x-1)^{-\frac{1}{2}}$

$f(x)=\frac{6}{\frac{1}{2}\times3}(3x-1)^{-\frac{1}{2}}+c$

$=4\sqrt{3x-1}+c$

Question 23. **A**

Area $=\int_{\frac{1}{2}}^{2}\frac{3}{7-2x}\,dx=-\frac{3}{2}\int_{\frac{1}{2}}^{2}\frac{-2}{7-2x}\,dx$

$=-\frac{3}{2}\left[\log_e(7-2x)\right]_{\frac{1}{2}}^{2}$

$=-\frac{3}{2}(\log_e 3-\log_e 6)$

$=\frac{3}{2}(\log_e 6-\log_e 3)$

$=\frac{3}{2}\log_e 2$

Question 24. **C**

$\sum \Pr(X=x)=1$

$\therefore 3c^2+8c^2+c^2+4c^2=1$

$\therefore \quad 16c^2=1$

$\therefore \quad c^2=\frac{1}{16}$

$\Pr(X>2)=\Pr(X=3)+\Pr(X=4)$

$=\frac{1}{16}+\frac{4}{16}$

$=\frac{5}{16}$

Question 25. **D**
Let B denote the bonus paid

b	$\Pr(B=b)$	$b\Pr(B=b)$
10	0.4	4
2	0.3	0.6
0	0.3	0
		4.6

The expected weekly bonus is $4.60

Question 26. **B**
$\mu = 6.2, \quad \sigma = \sqrt{2.89} = 1.7$

$\mu + 2\sigma = 6.2 + 2(1.7) = 9.6 \qquad \mu - 2\sigma = 6.2 - 2(1.7) = 2.8$

Since X is a discrete random variable, the 95% confidence interval is 3 to 9.

Question 27. **E**
Let X denote the number of globes which need to be replaced in the year.
$n = 4, \quad p = 0.4$

$\Pr(X \le 1) = \Pr(X = 0) + \Pr(X = 1)$

$= \binom{4}{0}(0.4)^0(0.6)^4 + \binom{4}{1}(0.4)^1(0.6)^3$

$= 0.475$

Question 28. **C**
$E(X) = np = 4 \times 0.4 = 1.6$

On average 1.6 globes per year would need to be replaced. Therefore it would be expected that 16 globes would need to be replaced over a ten year period.

Question 29. **D**
$\Pr(X \ge 1) = 0.7599$

$\therefore \Pr(X = 0) = 1 - 0.7599 = 0.2401$

$\therefore \quad \binom{n}{0}(0.3)^0(0.7)^n = 0.2401$

$\therefore \quad (0.7)^n = 0.2401$

$\therefore \quad \log_{10}(0.7)^n = \log_{10} 0.2401$

$\therefore \quad n = \dfrac{\log_{10} 0.2401}{\log_{10} 0.7}$

$= 4$

$n = 4, \quad p = 0.3, \quad \sigma^2 = np(1-p) = 4 \times 0.3 \times 0.7 = 0.84$

Question 30. **A**
$\mu = 375, \qquad \sigma = \sqrt{4} = 2$

$\Pr(X < 372) = \Pr(Z < \frac{372-375}{2})$

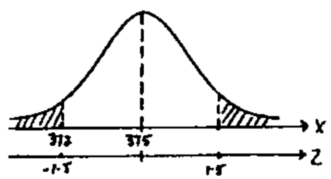
$= \Pr(Z < -1.5)$

$= \Pr(Z > 1.5)$

$= 1 - \Pr(Z < 1.5)$

$= 1 - 0.9332$

$= 0.0668$



Question 31. **E**
Both normal distributions are centred about the same value, $\therefore \mu_A = \mu_B$
Distribution A has a smaller spread than B, $\therefore \sigma_A < \sigma_B$

Question 32. **A**
$\mu = 20$

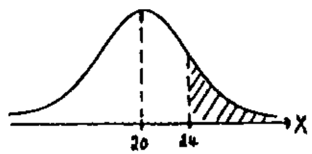
$\Pr(X > 24) = 0.4$

$\therefore \Pr(X < 24) = 0.6$

$\therefore \frac{24-20}{\sigma} = 0.253$

$\therefore \quad \sigma = 15.8$

$\therefore \quad \sigma^2 = 250$



Question 33. B

$\hat{p} = \frac{10}{25} = 0.4$

$se(\hat{p}) = \sqrt{\frac{0.4 \times 0.6}{25}} = 0.098$

lower limit = 0.4 - 2(0.098) = 0.204

Upper limit = 0.4 + 2(0.098) = 0.596

95% confidence interval is 0.204 to 0.596

Suggested solutions to 1995 Mathematical Methods CAT 2 - part II

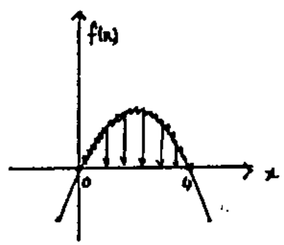
Question 1

$f(x)$ is defined when $4x - x^2 \geq 0$

$\therefore\ x(4-x) \geq 0$

$\therefore\ 0 \leq x \leq 4$

The largest possible domain is [0,4]



Question 2

Interchanging x and y gives:

$\therefore\quad x = 4e^{y-1} + 2$

$\therefore\ \frac{x-2}{4} = e^{y-1}$

$\therefore\ y - 1 = \log_e\left(\frac{x-2}{4}\right)$

$\therefore\quad y = 1 + \log_e\left(\frac{x-2}{4}\right)$

The inverse of the function is $y = 1 + \log_e\left(\frac{x-2}{4}\right)$

Question 3

x intercepts: let $y = 0$

$\therefore 0 = 4x^2 - x^4$

$\therefore 0 = x^2(4 - x^2)$

$\therefore 0 = x^2(2-x)(2+x)$

$\therefore x = 0,\ 2,\ -2$

stationary points: let $\frac{dy}{dx} = 0$

$\therefore 0 = 8x - 4x^3$

$\therefore 0 = 4x(2 - x^2)$

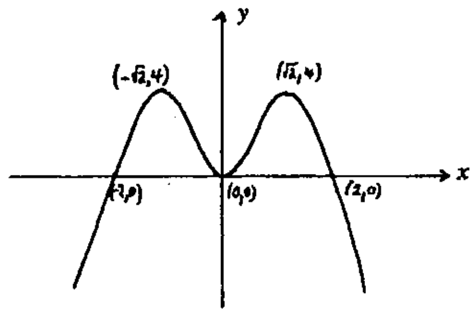
$\therefore 0 = 4x(\sqrt{2} - x)(\sqrt{2} + x)$

$\therefore x = 0,\ \sqrt{2},\ -\sqrt{2}$

When $x = 0,\ y = 0$

When $x = \sqrt{2},\ y = 4(\sqrt{2})^2 - (\sqrt{2})^4 = 4$

When $x = -\sqrt{2},\ y = 4(-\sqrt{2})^2 - (-\sqrt{2})^4 = 4$



Question 4

$f(x) = 3(3 + 2x - x^2) = 9 + 6x - 3x^2$

$$\text{Area} = \int_1^3 (9 + 6x - 3x^2)\,dx + \left|\int_3^4 (9 + 6x - 3x^2)\,dx\right|$$

$$= \left[9x + 3x^2 - x^3\right]_1^3 + \left|\left[9x + 3x^2 - x^3\right]_3^4\right|$$

$$= (27 + 27 - 27) - (9 + 3 - 1) + |(36 + 48 - 64) - (27 + 27 - 27)|$$

$$= 27 - 11 + |20 - 27|$$

$$= 27 - 11 + 7$$

$$= 23 \text{ square units}$$

Question 5

a. Using the Product Rule:

$f'(x) = 4x^2\left(\frac{1}{x}\right) + 8x\log_e x = 4x + 8x\log_e x = 4x(1 + 2\log_e x)$

b. From part a

$\int (4x + 8x\log_e x)\, dx = 4x^2 \log_e x + C$

$\therefore \quad \int 8x\log_e x\, dx = 4x^2\log_e x - \int 4x\, dx + C$

$\therefore \quad 2\int 4x\log_e x\, dx = 4x^2\log_e x - 2x^2 + C$

$\therefore \quad \int 4x\log_e x\, dx = 2x^2\log_e x - x^2 + C$

Question 6

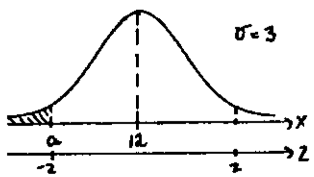
$\Pr(X < a) = 0.05$

$\therefore \Pr(Z < -z) = 0.05$

$\therefore \Pr(Z < z) = 0.95$

$\therefore \quad \dfrac{a-12}{3} = -1.645$

$\therefore \quad a = 7.065$



END SUGGESTED SOLUTIONS
1995 MATHEMATICAL METHODS CAT 2.
FACTS, SKILLS AND APPLICATIONS.