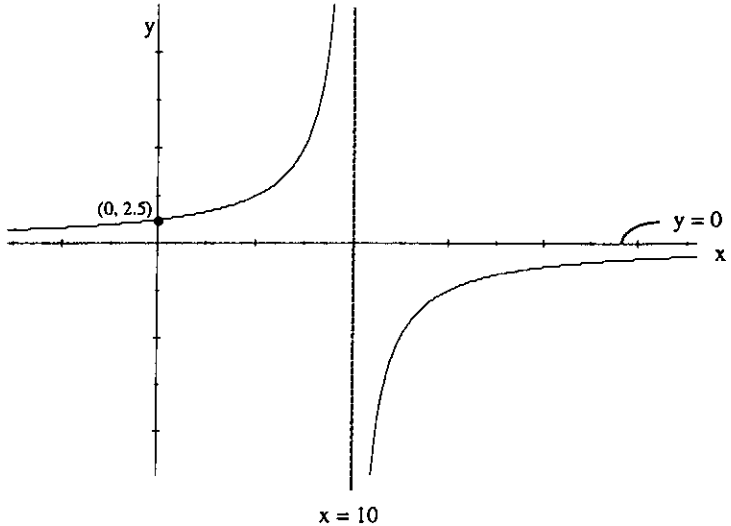
1995 CAT 3

SOLUTIONS.

Question 1

a.



Shape A1
Intercept A1
Asymptotes A1

b. **i.**

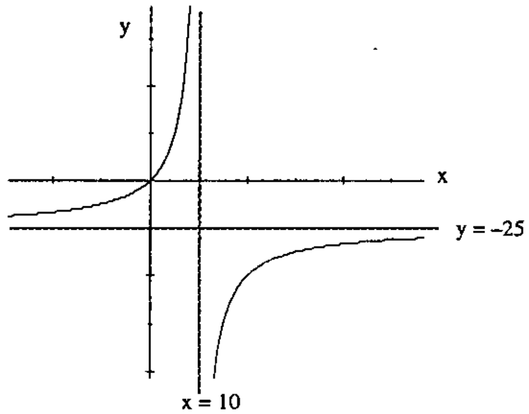
Using long division we have:

$$\begin{array}{r|l} & -25 \\ \hline x + 10 & 25x \\ & 25x - 250 \\ \hline & 250 \end{array}$$

Therefore, $g(x) = -25 + \dfrac{250}{10 - x}$ as required. M1 A1

ii. Now, $g(x) = -25 + \dfrac{250}{10-x} = -25 + 10\left(\dfrac{25}{10-x}\right) = -25 + 10f(x)$ M1

c.



Shape A1
Asymptotes A1

d. When x = 60, $C(60) = \frac{25 \times 60}{100 - 60} = 37.5$

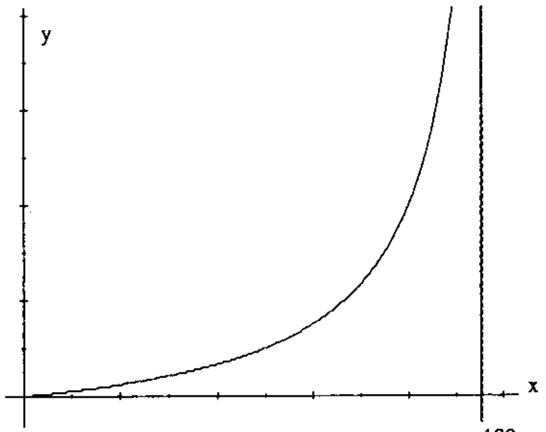
Therefore it would cost $37.5 million. A1

e. When C = 10, $10 = \frac{25x}{100 - x} \Leftrightarrow 1000 - 10x = 25x \Leftrightarrow 35x = 1000$ M1

$x = 28.57$

That is, 28.57% of pollutants can be removed. That is, 71.43% remains. A1

f.



Shape A1
Asymptotes A1

g. Let $y = C(x)$, interchanging x and y, we have:

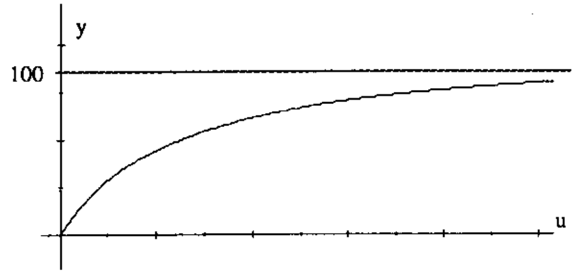
$$x = \frac{25y}{100 - y} \Leftrightarrow x(100 - y) = 25y$$ M1

$$\Leftrightarrow 100x - xy = 25y$$
$$\Leftrightarrow 100x = 25y + xy$$
$$\Leftrightarrow 100x = y(25 + x)$$
$$\Leftrightarrow y = \frac{100x}{25 + x}$$ A1

Therefore, $C^{-1}(x) = \frac{100x}{25 + x}$.

h. $N(u)$ represents the percentage of pollutants removed from the river system when $u million dollars are spent on the cleaning process. A1

i.



Shape A1
Asymptotes A1

j. It is impossible to remove all pollutants from the river system (asymptote exists at y = 100). A1

This means that in order to get close to removing 100% of the pollutants, you would need to continually fund the cleaning process.

A1

Question 2

a. Set up a table of values:

x	0.5	1	1.5	2	3.5
R(x) $(0 \le x \le 3)$	1.736	0.889	0.375	0.111	
R(x) $(3 \le x \le 4)$					0.0417

Therefore Area = 0.5(1.736 + 0.889 + 0.375 + 0.111 + 0.0417) M1

A_1 = 1.576 sq units A1

b. As above, we have:

x	0	0.5	1	1.5	2	4
R(x) $(0 \le x \le 3)$	3	1.736	0.889	0.375	0.111	
R(x) $(3 \le x \le 4)$						0.333

Therefore Area = 0.5(3 + 1.736 + 0.889 0.375 + 0.111 + 0.333) M1

A_2 = 3.2221 sq units A1

c. Exact area = A = $\int_0^3 \frac{1}{9}(3-x)^3\,dx + \int_3^4 \frac{1}{3}(x-3)^3\,dx$ M1

$= \frac{1}{9}\left[-\frac{1}{4}(3-x)^4\right]_0^3 + \frac{1}{3}\left[\frac{1}{4}(x-3)^3\right]_3^4$ M1 A1

$= \frac{1}{36}\left[0-(-81)\right] + \frac{1}{12}(1-0) = \frac{84}{36} = 2.3333$ A1

Therefore, $A_1 < A < A_2$ as required.

d. Using R(x) for $3 \le x < 4$, $\frac{dy}{dx} = (x-3)^2$. M1 A1

When x = 4, $\frac{dy}{dx} = (4-3)^2 = 1$. A1

e. **i.** Because the transition from x < 3 to x > 3 must be smooth, then m = 1 (from **d.**).
A1

ii. Now, when x = 4, $y = \frac{1}{3}(1)^3 = \frac{1}{3}$. M1

Therefore, $y - \frac{1}{3} = 1(x-4)$, so that $y = x - \frac{11}{3}$ for $4 \le x \le a$, as required. A1

f. When y = 2.5 we have $2.5 = a - \frac{11}{3}$. So that $a = \frac{37}{6}$. M1

g. **i.** $A = \int_0^3 \frac{1}{9}(3-x)^3\,dx + \int_3^4 \frac{1}{3}(x-3)^3\,dx + \int_4^{\frac{37}{6}} \left(x - \frac{11}{3}\right)dx$ M1

$= \frac{84}{36} + \left[\frac{1}{2}x^2 - \frac{11}{3}x\right]_4^{\frac{37}{6}}$ A1

$= \frac{84}{36} + \left[\frac{1}{2}\left(\frac{37}{6}\right)^2 - \frac{11}{3}\left(\frac{37}{6}\right)\right] - \left[\frac{1}{2}(4)^2 - \frac{11}{3}(4)\right]$

= 5.403 sq units A1

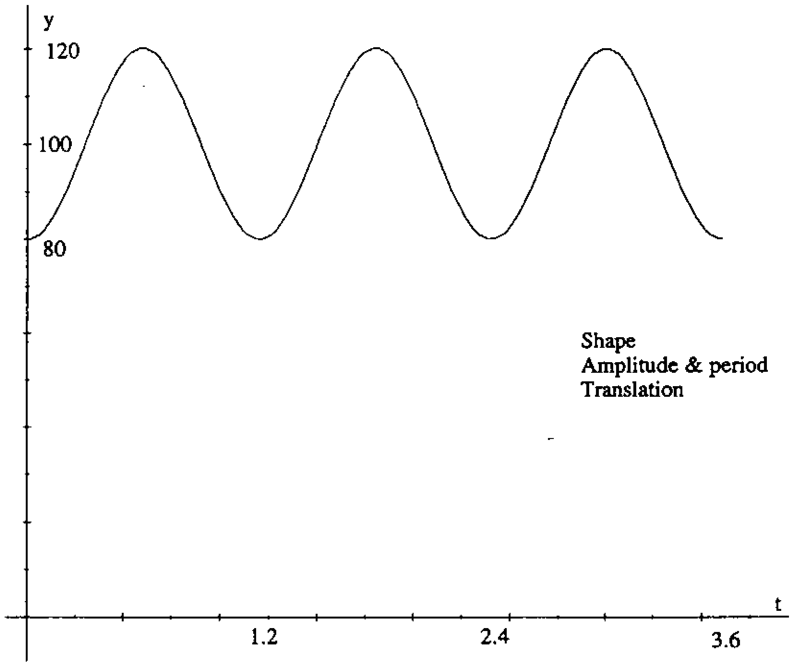
ii. Volume = 8(5.403) = 43.2 ≈ 43 units cubed. A1

Question 3

a. **i.** period $= \frac{2\pi}{\frac{5\pi}{3}} = \frac{6}{5} = 1.2$ seconds A1

ii. Amplitude = 20 A1

b.



Shape A1
Amplitude & period A1
Translation A1

c. Need to solve for $f(t) = 110 \Rightarrow 100 - 20\cos\left(\frac{5\pi}{3}t\right) = 110.$ M1

Therefore,
$$20\cos\left(\frac{5\pi}{3}t\right) = -10 \Leftrightarrow \cos\left(\frac{5\pi}{3}t\right) = -\frac{1}{2}$$
$$\frac{5\pi}{3}t = \frac{2\pi}{3}, \frac{4\pi}{3}$$
$$t = 0.4, 0.8, \ldots,$$ A1

Therefore P lies above 110 millimeters for 0.4 sec every 1.2 seconds, and so the percentage is $\left(\frac{0.4}{1.2} = \right)33.33\%$. A1

Question 4

a. **i.** 40 A1

ii. 6 A1

b. Let the r.v X denote the time taken to complete set homework per subject.

$P(X > 50) = P\left(Z > \frac{50-40}{6}\right) = P(Z > 1.6667) = 1 - P(Z < 1.6667)$ M1 M1

$= 1 - 0.9521 = 0.0479$ A1

c. $P(X < 45) = P\left(Z < \frac{45-40}{6}\right) = P(Z < 0.8333) = 0.7975$ M1 A1

Let N denote the number of subjects in which the homework is completed within 45 minutes.

d. Therefore $N \overset{d}{=} Bin(3, 0.7975) \Rightarrow P(N = 3) = (0.7975)^3 = 0.5072$ M1 A1

e. This time, $N \overset{d}{=} Bin(5, 0.7975) \Rightarrow P(N = 3) = {}^5C_3(0.7975)^3(0.2025)^2 = 0.2080$ M1 A1

f. We need to find x such that $P(X < x) = 0.90$

That is $\frac{x-40}{6} = 1.2816.$ M1 A1

Therefore x = 47.6896, so that dinner should be served at seven forty eight. A1

g. Let $T = X_1 + X_2 + X_3$, where each $X_i \overset{d}{=} N(40, 36)$

Therefore E(T) = 120 and Var(T) = 108* so that $T \overset{d}{=} N(120, 108)$. M1 M1

So, $P(T < 150) = P\left(Z < \frac{150-120}{\sqrt{108}}\right) = P(Z < 2.8867) = 0.9980$ M1 A1

*NB: Do not use the expresssion $Var(kT) = k^2 Var(T)$.
In this instance, as each Xi is an i.i.d.r.v, then Var(Sum) = Sum(Var).