The Mathematical Association of Victoria

1997

MATHEMATICAL METHODS

Trial Examination 2

(Practice papers based on MAV 1997 trial CATs)

Reading time: 15 minutes
Writing time: 1 hour 30 minutes

Student's Name: ____

Directions to students

This examination consists of four questions.

Answer all questions.

All working and answers should be written in the spaces provided.

The marks allotted to each part of each question appear at the end of each part.

There are **60 marks** available for this task.

A formula sheet is attached.

These questions have been written and published to assist students in their preparations for the Mathematical Methods Examination 2. The questions and associated answers and solutions do not necessarily reflect the views of the Board of Studies Assessing Panels. The Association gratefully acknowledges the permission of the Board to reproduce the formula sheet.

Published by The Mathematical Association of Victoria
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Mathematical Methods

Practice papers based on MAV 1997 trial CATs

Examination 2 (Analysis task)

Question 1

The dosage, $D(t)$ mg, of a particular type of medicine for children aged 1 to 14 years, where t years is the child's age, has been modelled by the equation

$$D(t) = \frac{at}{t+14,},\ 1 \leq\ t \leq 14,$$

where a mg is the adult dosage.

a. **i.** The dosage for an adult for this drug has been set at 600 mg. How much should a child who is exactly 4 years old be given? [1 mark]

ii. Based on this model, does a child become an 'adult' straight after its 14^{th} birthday? [2 marks]

b. What is the average rate of change of the dosage of this drug given to a child from when the child is one to when the child is 14 years old? [2 marks]

c. What is the rate of change of the dosage of this drug at the time when the child turns 10 years old. [2 marks]

d. Show that $D(t)$ can be expressed in the form $a + \dfrac{k}{t+14}$ and hence show that the value of k is –8400 for this drug. [2 marks]

e. **i.** Sketch the graph of $\mathrm{D}(t) = \dfrac{at}{t+14}, 1 \leq t \leq 14$ for **this particular drug** [3 marks]

ii. Clearly state the range of $D(t)$, $1 \leq t \leq 14$. [2 marks]

f. Based on a dosage d mg, the age of a child, $T(d)$ years, can also be determined. Find an expression for the child's age, $T(d)$, for a dosage d mg of this drug. [3 marks]

g. Using part **e.**, sketch the graph of $T(d)$, stating both range and domain. [3 marks]

Total 20 marks

Question 2

The concentration $C(t)$ of a drug, in milligrams per litre, in the bloodstream of a patient t hours after the drug has been administered as tablets, is given by the formula

$$C(t) = \frac{1}{58}\, e^{(-t^2+5t-1)}, \qquad t \geq 0$$

a. Find the concentration of the drug in the patient's bloodstream one hour after the tablets have been administered (answer in milligrams per litre correct to 2 decimal places). [2 marks]

b. Find a formula for the rate at which the concentration of the drug is changing with time. State the units of all the main elements of your formula. [3 marks]

c. Find the rate at which the concentration of the drug is changing 3 hours after the tablets have been administered (to 2 decimal places). Explain the significance of your answer to this 'real' situation. [2 marks]

d. Find the time(s) at which the concentration of the drug in the patient's bloodstream is at a maximum and find the maximum concentration correct to 2 decimal places. [6 marks]

[Total 13 marks]

Question 3

The resistances of heating elements produced by an electrical firm are normally distributed with mean 50 ohms and standard deviation 4 ohms.

a. Find the probability that a randomly selected element has a resistance less than 40 ohms. [3 marks]

b. If the specifications require that acceptable elements shall have a resistance between 45 and 55 ohms, find the probability that a randomly selected element has these specifications. [3 marks]

c. The profit on an acceptable element, that is, one whose resistance is within the specified limits, is $2.00, while unacceptable elements result in a loss of $0.50. If P dollars is the profit on a randomly selected element produced by the firm, find the mean and variance of P. [4 marks]

[Total 10 marks]

Question 4

a. **i.** Find the derivative (with respect to θ) of $y = a(1 - \tan\theta)$ [1 mark]

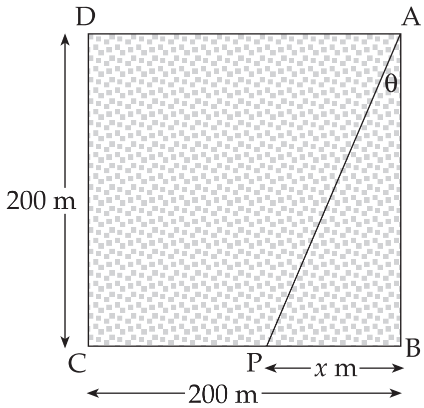
ii. Find the derivative of $y = \dfrac{b}{\cos\theta}$ [2 marks]

iii. Hence, show that $\dfrac{d}{d\theta}\left(a(1-\tan\theta)+\dfrac{b}{\cos\theta}\right) = -a\sec^2\theta + b\sec^2\theta\sin\theta$ [2 marks]

b. **i.** Show that if $-a\sec^2\theta + b\sec^2\theta\sin\theta = 0$ then $a = b\sin\theta$. [2 marks]

ii. Hence, show that the function $T(\theta) = a(1 - \tan\theta) + \dfrac{b}{\cos\theta}$ will have a stationary point when $\sin\theta = \dfrac{a}{b}$. [2 marks]

Simi the athelete is training for the Olympics. Part of her training program is to run "Square laps". On a particularly wet afternoon she decides to "cheat". Rather than running the last 400m along the perimeter of the track from A to B to C, she decides to run from the vertex A onto the muddy field and cut across to a point P somewhere on the last 200m stretch.



Simi can run through the muddy field at a constant speed of 5 m/s and on the track at a constant speed of 8 m/s.

c. Given that $\angle PAB = \theta$ where $0 \le \theta \le \dfrac{\pi}{4}$ and that $PB = x$,

i. Show that $x = 200\tan\theta$. [1 mark]

ii. Find an expression in terms of θ for the time it takes Simi to run in a straight line from A to P. [1 mark]

iii. Find an expression in terms of θ for the time it takes Simi to run in a straight line from P to C. [1 mark]

iv. Hence show that the total time taken for Simi to run from A to C via P can be expressed in the form $T(\theta) = 200\left[a(1-\tan\theta)+\dfrac{b}{\cos\theta}\right]$ [2 marks]

v. Write down the values of a and b. [1 mark]

vi. Hence, using part **b**, find the angle, θ, for which Simi's running time will be a minimum. [You do not need to justify that the stationary value is a minimum.] [2 marks]

[Total 17 marks]

Examination 2 Total: 60 marks

Mathematical Methods

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Examination 2 (Analysis task)

Solutions

Question 1

a. **i.** $a = 600 \therefore D(4) = \dfrac{600 \times 4}{4 + 14}$

$= 133\dfrac{1}{3}$ **[A]**

ii. $D(14) = 300 \ (\neq 600)$ **[M]**

$\therefore$ Child does not become an adult. **[A]**

b. $\dfrac{D(14) - D(1)}{14 - 1} = \dfrac{300 - 40}{13}$ **[M]**

$= 20$ **[A]**

i.e., 20 mg/year

c. $D'(t) = \dfrac{600(t + 14) - 600t}{(t + 14)^2}$

$= \dfrac{8400}{(t + 14)^2}$ **[A]**

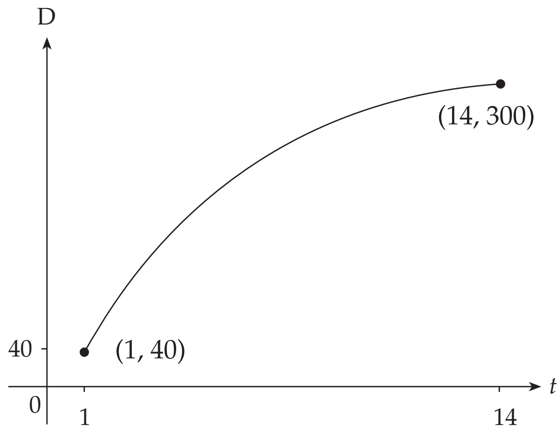
$\therefore \ D'(10) = 14.58$ **[A]**

d. $D(t) = \dfrac{600(t + 14) - 8400}{(t + 14)}$ **[M]**

$= 600 + \dfrac{-8400}{(t + 14)}$

$\therefore k = -8400$ **[A]**

e. **i.**



shape **[A]**
endpoints **[A][A]**

ii. Use of graph **[M]**

[40, 300] **[A]**

f. $d = \dfrac{at}{t+14} \Leftrightarrow dt + 14d = at$ **[M]**

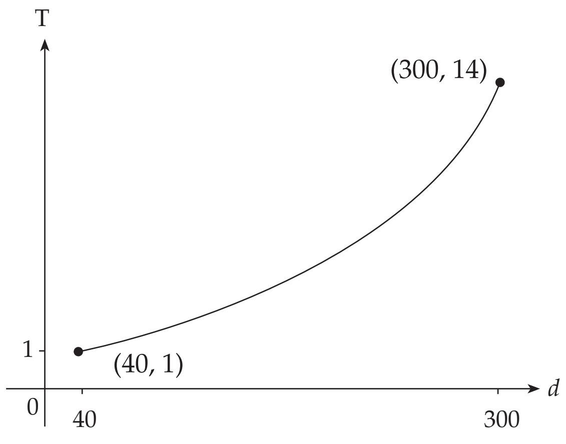
$\therefore\ t = \dfrac{14d}{a-d}$ **[A]**

with $a = 600$,

$T(d) = \dfrac{14d}{600-d},\quad 40 \le d \le 300$ **[A]**

g. Domain = [40, 300] **[A]**

Range = [1, 14] **[A]**



shape **[M]**

Question 2

a. $C(1) = \frac{1}{58}e^{(-1^2+5-1)} \approx 0.35$ milligram per litre **[M][A]**

b. Rate of change = $C'(t) = \frac{-2t+5}{58}e^{(-t^2+5t-1)}$ [chain rule] **[M][A]**

Units: t is in hours and $C'(t)$ is in milligrams per litre per hour. **[A]**

c. $C'(3) = -2.56$ (mg per litre per hour). **[A]**

The negative answer means that the drug is being eliminated from the bloodstream of the patient. **[A]**

d. $C'(t) = 0$ when, $0 = \frac{-2t+5}{58}e^{(-t^2+5t-1)}$ **[M]**

Therfore, either $\frac{-2t+5}{58} = 0 \Leftrightarrow t = 2.5$ **[M][A]**

or $e^{(-t^2+5t-1)} = 0,$ for which there are no real solutions. **[M]**

That is, there is only one value of t that gives a stationary point. Using the sign of the first derivative, we find that this is in fact a local maximum. **[M]**

Therefore, maximum concentration = $C(2.5) \approx 3.29$ milligrams per litre. **[A]**

Question 3

a. $X \sim N(\mu = 50, \sigma^2 = 16)$ and $Z = \frac{X-\mu}{\sigma}$, **[M]**

where X ohms = the resistance of the element.

$\Pr(X < 40) = \Pr(Z < -2.5) = 0.0062$ **[M][A]**

b. $\Pr(45 < X < 55) = \Pr(-1.25 < Z < 1.25)$ **[M][A]**

$= 0.7887$ **[A]**

c. **[A]**

p	2.00	–0.50
$\Pr(P = p)$	0.7887	0.2113

$E(P) = 1.472$ and $E(P^2) = 3.208$. **[M][A]**

Therefore, $Var(P) = E(P^2) - [E(P)]^2 = 3.208 - [1.472]^2 = 1.041$ **[A]**

Question 4

a. **i.** $\frac{d}{d\theta}(a - a\tan\theta) = -a\sec^2\theta$ **[A]**

ii. $y = b(\cos\theta)^{-1} \quad \therefore \frac{dy}{d\theta} = -b(-\sin\theta)(\cos\theta)^{-2}$ **[M]**

$= \frac{b\sin\theta}{\cos^2\theta}$ **[A]**

iii. $\frac{d}{d\theta}\left(a - a\tan\theta + \frac{b}{\cos\theta}\right)$

$= -a\sec^2\theta + \frac{b\sin\theta}{\cos^2\theta}$ **[M]**

$= -a\sec^2\theta + b\frac{1}{\cos^2\theta}.\sin\theta$

$= -a\sec^2\theta + b\sec^2\theta.\sin\theta$ **[A]**

b. **i.** $\sec^2\theta(-a + b\sin\theta) = 0$

As $\sec\theta \neq 0, \quad -a + b\sin\theta = 0$ **[M]**

$\therefore a = b\sin\theta$ **[A]**

ii. $T'(\theta) = 0 \Rightarrow b\sin\theta = a$ **[M]**

$\therefore \ \sin\theta = \frac{a}{b}$ **[A]**

c. **i.** $\tan\theta = \frac{PB}{200} = \frac{x}{200}$ **[M]**

$\therefore x = 200\tan\theta$

ii. $t_{AP} = \frac{AP}{5} = \frac{200}{5\cos\theta}$ **[M]**

iii. $t_{PC} = \frac{200 - x}{8} = \frac{200(1 - \tan\theta)}{8}$ **[M]**

iv. $T(\theta) = \frac{200}{5\cos\theta} + \frac{200(1 - \tan\theta)}{8}$ **[M]**

$= 200\left[\frac{1}{5}.\frac{1}{\cos\theta} + \frac{1}{8}(1 - \tan\theta)\right]$ **[A]**

v. $a = \frac{1}{8}, \quad b = \frac{1}{5}$ **[A]**

vi. $\sin\theta = \frac{1/8}{1/5} = \frac{5}{8}$ **[M]**

angle = 0.6751 radians = 38°41′ **[A]**

