Question 1

The dosage, $D(t)$ gm, of a particular type of medicine for children aged 1 to 14 years, where t is the child's age, has been modelled by the equation

$$D(t) = \frac{at}{t+14},\ 1 \le t \le 14,$$

where a is the adult dosage.

a. i. The dosage for an adult for this drug has been set at 600gm. How much should a four year old child be given?

$a = 600 \quad \therefore D(4) = \dfrac{600 \times 4}{4+14}$

$= 133\frac{1}{3}$ A1

[1 mark]

ii. Based on this model, does a child become an 'adult' straight after its 14th birthday?

$D(14) = 300\ (\neq 600)$ M1

$\therefore$ Child does not become an 'adult' A1

[2 marks]

b. What is the average rate of change of dosage given to a child from when the child is one to when the child is 14 years old?

$\dfrac{D(14) - D(1)}{14 - 1} = \dfrac{300 - 40}{13}$ M1

$= 20$ A1

i.e, 20 gm/year

[2 marks]

c. What is the rate of change at which the dosage can be administered at the time when the child turns 10 years old?

$D'(t) = \dfrac{600(t+14) - 600t}{(t+14)^2}$

$= \dfrac{8400}{(t+14)^2}$ A1

$\therefore D'(10) = 14.58$ A1

[2 marks]

d. Show that $D(t)$ can be expressed in the form $a + \frac{k}{t+14}$ and hence show that the value of k is -8400.

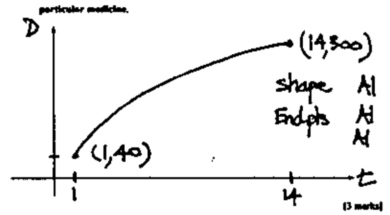
$D(t) = \dfrac{600(t+14) - 8400}{(t+14)}$ M1

$= 600 + \dfrac{-8400}{(t+14)}$

$\therefore k = -8400$ A1

[2 marks]

e. i. On the set of axes below, sketch the graph of $D(t) = \frac{at}{t+14}$, $1 \le t \le 14$ for this particular medicine.



[3 marks]

ii. Clearly state the range of $D(t)$, $1 \le t \le 14$.

Use of graph M1

$[40, 300]$ A1

[2 marks]

f. Based on a dosage d mg, the age of a child, $T(d)$ years, can also be determined. Find an expression for the child's age, $T(d)$, for a dosage d mg.

$d = \dfrac{at}{t+14} \Leftrightarrow dt + 14t = at$ M1

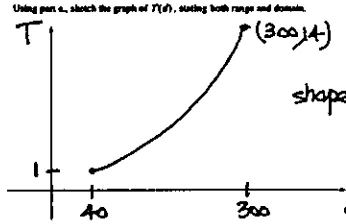
$\therefore t = \dfrac{14d}{a-d}$ A1

with $a = 600$,

$T(d) = \dfrac{14d}{600-d}$, $40 \le d \le 300$ A1

[3 marks]

g. Using part e., sketch the graph of $T(d)$, stating both range and domain.



Domain $= [40, 300]$ A1

Range $= [1, 14]$ A1

[3 marks]

Total 21 marks

Question 2

a. i. Given that $\frac{d}{d\theta}(\tan\theta) = \sec^2\theta$, find the derivative (with respect to θ) of $y = a(1 - \tan\theta)$.

$\dfrac{d}{d\theta}(a - a\tan\theta) = -a\sec^2\theta$ A1

[1 mark]

ii. Find the derivative of $y = \frac{b}{\cos\theta}$

$y = b(\cos\theta)^{-1} \therefore \dfrac{dy}{d\theta} = -b \cdot -\sin\theta \cdot (\cos\theta)^{-2}$ M1

$= \dfrac{b\sin\theta}{\cos^2\theta}$ A1

[2 marks]

iii. Hence, show that $\frac{d}{d\theta}\left(a(1 - \tan\theta) + \frac{b}{\cos\theta}\right) = -a\sec^2\theta + b\sec\theta\tan\theta$, where $\sec\theta = \frac{1}{\cos\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

$\dfrac{d}{d\theta}\left(a - a\tan\theta + \dfrac{b}{\cos\theta}\right)$

$= -a\sec^2\theta + \dfrac{b\sin\theta}{\cos^2\theta}$

$= -a\sec^2\theta + \dfrac{b\sin\theta}{\cos\theta} \cdot \dfrac{1}{\cos\theta}$ M1

$= -a\sec^2\theta + b\tan\theta \cdot \sec\theta$ A1

[2 marks]

b. i. Show that if $-a\sec^2\theta + b\sec\theta\tan\theta = 0$ then $a\sec\theta = b\tan\theta$.

$\left.\begin{array}{l}\sec\theta(-a\sec\theta + b\tan\theta) = 0 \\ \text{As } \sec\theta \neq 0,\ -a\sec\theta + b\tan\theta = 0\end{array}\right\}$ M1

$\therefore b\tan\theta = a\sec\theta$ A1

[2 marks]

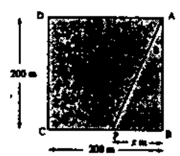
ii. Hence, show that the function $T(\theta) = a(1 - \tan\theta) + \frac{b}{\cos\theta}$ will have a stationary point when $\sin\theta = \frac{a}{b}$.

$T'(\theta) = 0 \Rightarrow b\tan\theta = a\sec\theta$ M1

$\left.\begin{array}{l}b \cdot \dfrac{\sin\theta}{\cos\theta} = \dfrac{a}{\cos\theta} \\ \sin\theta = \dfrac{a}{b}\end{array}\right\}$ A1

[2 marks]

Sind the athlete is training for the Sydney 2000 Olympics. Part of her training programme is to run "Square laps". On a particularly wet afternoon she decides to "cheat". Rather than running the last 400m along the perimeter of the track, she decides to run from the vertex A onto the muddy field and cut across to a point P somewhere on the last 200m stretch.



Sind can run through the muddy field at a constant speed of 5 m/s and on the track at a constant speed of 8 m/s.

c. Given that $\angle PAB = \theta$ where $0 \le \theta \le \frac{\pi}{4}$ and that $PB = x$.

i. Show that $x = 200\tan\theta$.

$\tan\theta = \dfrac{PB}{200} = \dfrac{x}{200}$ M1

$\therefore x = 200\tan\theta$

[1 mark]

ii. Find an expression in terms of θ for the time it takes Sind to run in a straight line from A to P.

$t_{AP} = \dfrac{AP}{5} = \dfrac{\frac{200}{\cos\theta}}{5} = \dfrac{200}{5\cos\theta}$ M1

[1 mark]

iii. Find an expression in terms of θ for the time it takes Sind to run in a straight line from P to C.

$t_{PC} = \dfrac{200 - x}{8} = \dfrac{200(1 - \tan\theta)}{8}$ M1

[1 mark]

iv. Hence show that the total time taken for Sind to run from A to C via P can be expressed in the form $T(\theta) = 200\left[a(1 - \tan\theta) + \frac{b}{\cos\theta}\right]$

$T(\theta) = \dfrac{200}{5\cos\theta} + \dfrac{200(1 - \tan\theta)}{8}$ M1

$= 200\left[\dfrac{1}{5} \cdot \dfrac{1}{\cos\theta} + \dfrac{1}{8}(1 - \tan\theta)\right]$ A1

[2 marks]

v. Write down the values of a and b.

$a = 1/8,\ b = 1/5$ A1

[1 mark]

vi. Hence, using part b., find the angle, θ, for which Sind's running time will be a minimum.

$\sin\theta = \dfrac{1/8}{1/5} = 5/8$ M1

$\therefore \theta = \sin^{-1}\left(\frac{5}{8}\right) = 38°41'$ A1

[2 marks]

Total 18 marks

Question 3

The rate, $\frac{dL}{dt}$, at which inexperienced students at the Pakenham Cooking School are able to learn to produce 'A Grade' quality dishes at the end of their course has been shown to follow an exponential function;

$$\frac{dL}{dt} = -Ake^{-kt}, 0 \le t \le 20$$

where t is measured in the number of weeks from when an inexperienced student starts his first cooking class and L is a measure of the number of dishes he has produced in the last t weeks.

a. How long does the course run for?

20 wks A1

[1 mark]

b. Show that $L(t) = Ae^{-kt} + C, 0 \le t \le 20$

$\int \frac{dL}{dt}\,dt = \int -Ake^{-kt}\,dt$

$\therefore L(t) = Ae^{-kt} + C, 0 \le t \le 20$ A1

[1 mark]

All student must have had 15 attempts at cooking prior to attending the course.

c. i. Show that $A + C = 15$

$L(0) = 15, \therefore 15 = Ae^{0} + C$ M1

$\therefore 15 = A + C$ A1

[2 marks]

ii. During the five weeks of the course, the student will have produced 45 dishes. Show that $Ae^{-5k} + C = 45$.

$L(5) = 45 \therefore 45 = e^{-5k} + C$ A1

[1 mark]

iii. Given that $k = \frac{1}{5}$, find, to the nearest integer, the values of A and C.

$15 = A + C$ —①

$45 = Ae^{-1} + C$ —②

②−①: $30 = A(e^{-1} - 1) \therefore A = -47.45$ M

$= -47$ (2 dp) A

Sub into ①: $C = 62$ A

[3 marks]

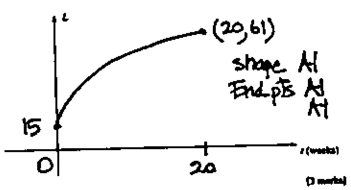
iv. How many dishes will an inexperienced student make by the end of the course?

$L = 62 - 47e^{-1/5 t} \quad 0 \le t \le 20$

$\therefore L(20) = 62 - 47e^{-4} \approx 61$ A1

[1 mark]

d. On the set of axes shown below, sketch the graph of $L(t)$.



[3 marks]

e. During which week will the student have learned the most?

All working leading to your answer must be shown.

t	L(t)	L(t+1) − L(t)	
0	15	8.52	
1	23.52	6.97	M1
2	30.49	5.72	M1
3	36.21	⋮	
⋮	⋮	⋮	

[3 marks]

Total 15 marks

$\therefore$ learns the most in the 1st week. A1

Question 4

The discrete random variable X denotes the number of faults (called seeds) that occur in glass sheets. The probability distribution for the number of faults in sheets measuring 'A' square metres is given by

$$Pr(X = x) = \frac{\mu^x}{x!}e^{-\mu}, x = 0, 1, 2, \ldots$$

The parameter μ measures the expected number of faults in a glass sheet of area A square metres and is given by $\mu = k \times A$, where k is the rate at which faults occur per square metre of glass.

A manufacturer finds that glass sheets from her plant have faults which occur at random at a rate of 0.05 per square metre.

a. For glass sheets that measure 5 metres by 4 metres, show that $\mu = 1$

$\mu = 5 \times 4 \times 0.05 = 1$ A1

[1 mark]

b. For glass sheets that measure 5 metres by 4 metres, find the probability that there will be

i. no seeds

$Pr(X=0) = \frac{1^0 \times e^{-1}}{0!} = 0.3679$ A1

[1 mark]

ii. one seed

$Pr(X=1) = \frac{1^1 \times e^{-1}}{1!} = 0.3679$ A1

[1 mark]

iii. at least two seeds

$Pr(X \ge 2) = 1 - Pr(X=0) - Pr(X=1)$
$= 1 - 2 \times 0.3679$ } M1

$= 0.2642$ A1

[2 marks]

c. Find the probability that if a glass sheet that measures 5 metres by 4 metres has at least one seed, it will in fact have 2 seeds.

$Pr(X=2 \mid X \ge 1) = \frac{Pr(X=2)}{Pr(X \ge 1)} = \frac{1^2 \times e^{-2}/2!}{1 - 0.3679}$ M1 M1

$= 0.2909$ A1

[3 marks]

d. If sheets with two or more seeds are rejected, find the probability that in a batch of 10 glass sheets that measure 5 metres by 4 metres, there will be

i. none that are rejected

$p = 0.2642, n = 10 \therefore Y \stackrel{d}{=} Bin(10, 0.2642)$ M1

$Pr(Y=0) = {}^{10}C_0 (0.2642)^0 (0.7358)^{10} = 0.0465$ A1

[2 marks]

ii. exactly one that is rejected

$Pr(Y=1) = {}^{10}C_1 (0.2642)^1 (0.7358)^9$ M1

$= 0.1670$ A1

[2 marks]

iii. at least two that are rejected.

$Pr(Y \ge 2) = 1 - Pr(Y \le 1)$ M1

$= 0.7865$ A1

[2 marks]

e. What assumption have you made in answering part c. ?

Each sheet is identical and independent. A1

[1 mark]

Each sheet that measures 5 metres by 4 metres costs the manufacturer \$8.00 to produce. Sheets that contain no seeds are sold at \$15.00, sheets that contain one seed are sold at \$10.00 and those that have at least two seeds are destroyed.

f. What is the expected profit on 100 (5m by 4m) glass sheets?

$P = S - 8$ P = profit, S = selling

$E(P) = E(S - 8)$

$= E(S) - 8$ M1

$= 0.3679 \times 15 + 0.3679 \times 10 - 8$ M1

$= 1.1975$

$\therefore$ For each 100 sheets

Expt. profit = \$119.75 A1

[3 marks]

Total 15 marks