THE
HEFFERNAN
GROUP
[illegible]

MATHEMATICAL METHODS
TRIAL CAT 3
1998
SOLUTIONS

Question 1

a. From the graph, we see that one complete cycle takes 20 hours. The period is therefore 20 hours.
Alternatively, period $= \frac{2\pi}{n}$ where $n = \frac{\pi}{10}$. So, the period $= 2\pi \times \frac{10}{\pi} = 20$ hours. ....(1m)

b. $t = 48$ represents midnight on Tuesday and so 12 hours after that, that is, $t = 60$, represents midday Wednesday. ....(1m)

c. We see from the graph that the maximum height of the water above the river bed is 14 metres. Therefore, the height of the jetty above the river bed is 15 metres. ....(1m)

d. The passengers last chance to access the motor boat on Monday morning is when the river is at

$15 - 2 = 13$metres. So, we require that $2\cos\frac{\pi t}{10} + 12 = 13$ ....(1m)

$$\cos\frac{\pi t}{10} = \frac{1}{2}$$
$$\frac{\pi t}{10} = \frac{\pi}{3}$$
$$t = \frac{10}{3} = 3\frac{1}{3}$$ ....(1m)

The latest time for passengers to access the motor boats on Monday morning is 3.20am.......(1m)

e. From part d., we have $\cos\frac{\pi t}{10} = \frac{1}{2}$

So,
$$\frac{\pi t}{10} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}$$
$$t = \frac{10}{3}, \frac{50}{3}, \frac{70}{3}, \frac{110}{3}, \frac{130}{3}$$
$$= 3\frac{1}{3}, 16\frac{2}{3}, 23\frac{1}{3}, 36\frac{2}{3}, 43\frac{1}{3}$$

We require those time intervals when the water level is 13 m or higher. By looking at the graph we see that the passengers can access the motor boats on

i) Monday - between 12am and 3.20am. ....(1m)
ii) Monday - between 4.40pm and 11.20pm ....(1m)
iii) Tuesday - between 12.40pm and 7.20pm ....(1m)

f. On Monday between 4.40pm and 6pm and on Tuesday between 12.40pm and 6pm, the water level is 13 m or above and the motor boats are running. There is therefore 1 hour 20 minutes plus 5 hours 20 minutes which gives a total of 6 hours 40 minutes when the motor boats can be accessed by passengers. ....(1m)

g. **i.** The jetty is fixed at 15 metres above the river bed and the lowest height above the river bed of the river is 10 metres (this can be read from the graph). Therefore a ladder of length 5 metres would be required to enable passengers to access the motor boats at all times. ....(1m)

ii. At $t = 5$, $h(5) = 2\cos\frac{\pi \times 5}{10} + 12$
$$= 2\cos\frac{\pi}{2} + 12$$
$$= 2 \times 0 + 12$$
$$= 12$$

By symmetry, at $t = 15$, $h(15) = 12$. ....(1m)

For $t \in [0,5] \cup [15,20]$, that is for 10 hours out of the first 20, or 50% of the time, the height of the water is 12 metres or above. Therefore, since the jetty is fixed at 15 metres, a ladder of 3 metres length is required to enable passengers to access the motor boats 50% of the time. ....(1m)

Total 13 marks

Question 2

a. **i.** $\Pr(X = 5) = 0.2$ ....(1m)

ii. $\Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)$ ....(1m)
$= 0.2 + 0.3 + 0.2$
$= 0.7$ ....(1m)

b. Now, $a + b + 2b + a + a + a = 1$
Since $b = 2a$, $4a + 2a + 4a = 1$
So, $a = 0.1$ and therefore $b = 0.2$. ....(1m)
So, $\Pr(X = 2) = 2b = 0.4$ ....(1m)

c. Mrs. Pearson and her team are more successful in correctly answering geography questions. (1m)
They are more likely to obtain 3 or more correct answers in geography than in sport. They are more likely to obtain 0, 1 or 2 correct answers in sport than in geography. ....(1m)

d. We require 4 correct answers in geography **and** 5 correct answers in sport **or** 5 correct answers in geography **and** 4 correct answers in sport **or** 5 correct answers in geography **and** 5 correct answers in sport. ....(1m)
The probability is therefore given by $0.3 \times 0.1 + 0.2 \times 0.1 + 0.2 \times 0.1$ ....(1m)
$= 0.03 + 0.02 + 0.02$
$= 0.07$ ....(1m)

e. To obtain a perfect score of 30, we need a score of 5 in each of the 6 subjects.
The probability of the team obtaining a score of 5 in each of geography, history and science is 0.2.
The probability of the team obtaining a score of 5 in each of sport, current affairs and politics is 0.1 ....(1m)
The probability of a perfect score of 30 is $0.2 \times 0.2 \times 0.2 \times 0.1 \times 0.1 \times 0.1$
$= 0.000008$ ....(1m)

f. Let Y equal the number of times Mrs. Pearson's regular team plays. Now Y has a binomial distribution where $n = 5, x = 5$ and $p = 0.8$.

So, $\Pr(Y = 5) = {}^5C_5(0.8)^5(0.2)^0$
$= (0.8)^5$
$= 0.32768$ ....(1m)

g. The probability of Mrs. Pearson's regular team competing at the next

- 1 game is 0.8
- 2 games is $0.8 \times 0.8 = 0.64$
- 3 games is $0.8 \times 0.8 \times 0.8 = 0.512$
- 4 games is $0.8 \times 0.8 \times 0.8 \times 0.8 = 0.4096$ ....(1m)

So, Mrs. Pearson has a probability greater than 0.5 of having her regular team playing at the next 3 trivia nights. ....(1m)

Alternatively, we have $0.8^n \geq 0.5$

$$\log_e 0.8^n \geq \log_e 0.5$$

$$n\log_e 0.8 \geq \log_e 0.5$$

$$n \leq \frac{\log_e 0.5}{\log_e 0.8}$$

(Note that we divide both sides by a negative number, that is, $\log_e 0.8 \approx -0.2231$, and so we must reverse the inequality sign.) ....(1m)

$$n \leq 3.1063$$

So, Mrs. Pearson has a probability greater than 0.5 of having her regular team playing at the next 3 trivia nights. ....(1m)

Total 15 marks

Question 3

a. **i.** Since one of the x-intercepts of the curve is at $x = -2.5$, then a factor of the expression is $(x+2.5)$. ....(1m)

ii. $x^3 + 2.5x^2 - 100x - 250 = x^2(x+2.5) - 100(x+2.5)$ ....(1m)

$$= (x+2.5)(x^2-100)$$

$$= (x+2.5)(x-10)(x+10) \quad ....(1m)$$

Alternatively,

$$\begin{array}{r|l} & x^2 - 100 \\ x+2.5 & x^3 + 2.5x^2 - 100x - 250 \\ & x^3 + 2.5x^2 \\ & \qquad -100x - 250 \\ & \qquad -100x - 250 \end{array}$$

So, $x^3 + 2.5x^2 - 100x - 250 = (x+2.5)(x-10)(x+10)$

iii. From part ii., the other 2 x-intercepts are at $x = -10$ and $x = 10$. So, the width of the puzzle, including the border, is $10 + 10 + 2 + 2 = 24$cm. ....(1m)

b. Find the local minimum value for the function $y = \frac{1}{100}(x^3 + 2.5x^2 - 100x - 250)$

Now $\frac{dy}{dx} = \frac{1}{100}(3x^2 + 5x - 100)$

A maximum or minimum occurs when $\frac{dy}{dx} = 0$.

So, we need to solve $3x^2 + 5x - 100 = 0$ ....(1m)

So, $(3x+20)(x-5) = 0$

$$x = -\frac{20}{3} \text{ or } 5 \quad ....(1m)$$

From figure 3, we see that the minimum occurs at $x = 5$.

At $x = 5$, $y = \frac{1}{100}(125 + 2.5 \times 25 - 500 - 250)$

$$= -5.625 \quad ....(1m)$$

So, L is the point $(5, -5.625)$.

So, the length of the puzzle is $2 \times (5.625 + 8.375) = 28$cm. ....(1m)

c. Width of puzzle minus the borders = 20cm

So, width of piece 1 = 20cm.

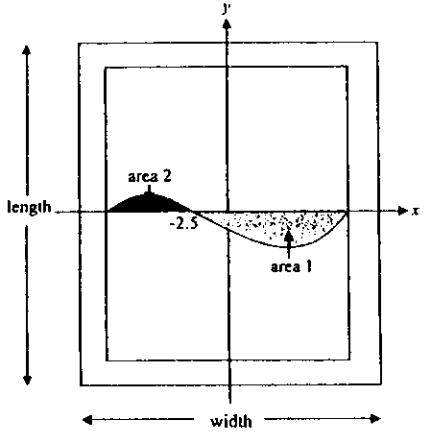
Length of puzzle minus the borders = 24cm.

So, the "height" of piece 1 (that is, the vertical distance from the x-axis to the top of piece 1) = 12cm

Area of piece 1 $= 20 \times 12 +$ area 1 - area 2 ....(1m)

where area 1 and area 2 are indicated in the diagram below.

Since area 1 falls below the x-axis, the value $\int_{-2.5}^{10} \frac{1}{100}(x^3 + 2.5x^2 - 100x - 250)dx$ will be negative and we must take this into account when writing down an expression for the area of piece 1.



Area of piece 1

$$= 20 \times 12 + \text{area 1 - area 2}$$

$$= 20 \times 12 + \left|\int_{-2.5}^{10} \frac{1}{100}(x^3 + 2.5x^2 - 100x - 250)dx\right| - \int_{-10}^{-2.5} \frac{1}{100}(x^3 + 2.5x^2 - 100x - 250)dx \quad .... (1m)$$

$$= 240 + \left|\frac{1}{100}\left[\frac{x^4}{4} + \frac{2.5x^3}{3} - \frac{100x^2}{2} - 250x\right]_{-2.5}^{10}\right| - \frac{1}{100}\left[\frac{x^4}{4} + \frac{2.5x^3}{3} - \frac{100x^2}{2} - 250x\right]_{-10}^{-2.5} \quad (1m)$$

$$= 240 + \left|\frac{1}{100}\left\{\left(\frac{10000}{4} + \frac{2500}{3} - \frac{10000}{2} - 2500\right) - \left(\frac{39.0625}{4} - \frac{39.0625}{3} - \frac{625}{2} + 625\right)\right\}\right|$$

$$- \frac{1}{100}\left\{\left(\frac{39.0625}{4} - \frac{39.0625}{3} - \frac{625}{2} + 625\right) - \left(\frac{10000}{4} - \frac{2500}{3} - \frac{10000}{2} + 2500\right)\right\}$$

..........(1m)

$$= 240 + \left|\frac{1}{100}(-4166.\dot{6} - 309.2448)\right| - \frac{1}{100}(309.2448 + 833.\dot{3})$$

$= 240 + |-44.7591| - 11.4258$ ....(1m)

$= 240 + 44.7591 - 11.4258$

$= 273.3$ square units (correct to 1 decimal place) ....(1m)

d. domain $= [-8, -2) \cup (-2, 8]$ ....(1m)

range $= [-8, 3) \cup (3, 8]$ ....(1m)

e. <u>To find the horizontal straight edge.</u>

When $y = 8$, $y = \frac{1}{x+2} + 3$

becomes
$$8 = \frac{1}{x+2} + 3$$
$$5 = \frac{1}{x+2}$$
$$5(x+2) = 1$$
$$5x + 10 = 1$$
$$5x = -9$$
$$x = -\frac{9}{5} = -1.8$$ ………(1m)

The horizontal straight edge extends from $x = -1.8$ to $x = 8$, that is, a length of 9.8 cm

<u>To find the vertical straight edge.</u>

When $x = 8$, $y = \frac{1}{8+2} + 3$
$$= \frac{1}{10} + 3$$
$$= 3\frac{1}{10} = 3.1$$ ………(1m)

The vertical straight edge extends from $y = 3.1$ to $y = 8$, that is, a length of 4.9cm.

The total length of the straight edges of piece 1 of the puzzle is $9.8 + 4.9 = 14.7$ cm. ………(1m)

f. We are looking for the rule of the inverse function of $y = \frac{1}{x+2} + 3$

So, swap x and y. $x = \frac{1}{y+2} + 3$ ………(1m)
$$(x-3)(y+2) = 1$$
$$y + 2 = \frac{1}{x-3}$$
$$y = \frac{1}{x-3} - 2$$ ………(1m)

Total 21 marks

Question 4

a. a is the x-intercept of the graph.

Let $x\log_e(x) - x = 0$
$$\log_e(x) = \frac{x}{x}$$
$$\log_e(x) = 1$$

So, $e^1 = x$

The x-intercept is $x = e$, so $a = e$. ………(1m)

b. $f(x) = x\log_e(x) - x$
$$f'(x) = x.\frac{1}{x} + 1 \times \log_e(x) - 1$$ ………(1m)
$$= 1 + \log_e(x) - 1$$
$$= \log_e x$$ ………(1m)

c. A minimum occurs when $f'(x) = 0$

So, $\log_e x = 0$
$$e^0 = x$$

So, $x = 1$ ………(1m)

When $x = 1$, $y = 1 \times \log_e 1 - 1$
$$= 1 \times 0 - 1$$
$$= -1$$

Minimum point is $(1, -1)$ ………(1m)

d. When $f'(x) = 1$, we have $\log_e x = 1$, so, $x = e$.

So the point of tangency is $(e, 0)$. ………(1m)

The equation of the tangent is $y - 0 = 1(x - e)$
$$y = x - e$$ ………(1m)

The y-intercept occurs when $x = 0$, that is at $y = -e$.

So, the point A is $(0, -e)$. ………(1m)

The equation of the normal is $y - 0 = -1(x - e)$
$$y = -x + e$$ ………(1m)

The y-intercept occurs when $x = 0$, that is at $y = e$.

So the point B is $(0, e)$. ………(1m)

The distance between point A and point B is $2e$ units. ………(1m)

Total 11 marks

END OF SOLUTIONS