The Mathematical Association of Victoria

1998

MATHEMATICAL METHODS

Trial Examination 2

(Practice papers based on MAV 1998 trial CATs)

Reading time: 15 minutes
Writing time: 1 hour 30 minutes

Student's Name: ______________________________________

Directions to students

This examination consists of four questions.

Answer all questions.

All working and answers should be written in the spaces provided.

The marks allotted to each part of each question appear at the end of each part.

There are **60 marks** available for this task.

A formula sheet is attached.

These questions have been written and published to assist students in their preparations for the Mathematical Methods Examination 2. The questions and associated answers and solutions do not necessarily reflect the views of the Board of Studies Assessing Panels. The Association gratefully acknowledges the permission of the Board to reproduce the formula sheet.

Published by The Mathematical Association of Victoria
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Mathematical Methods

Practice papers based on MAV 1998 trial CATs

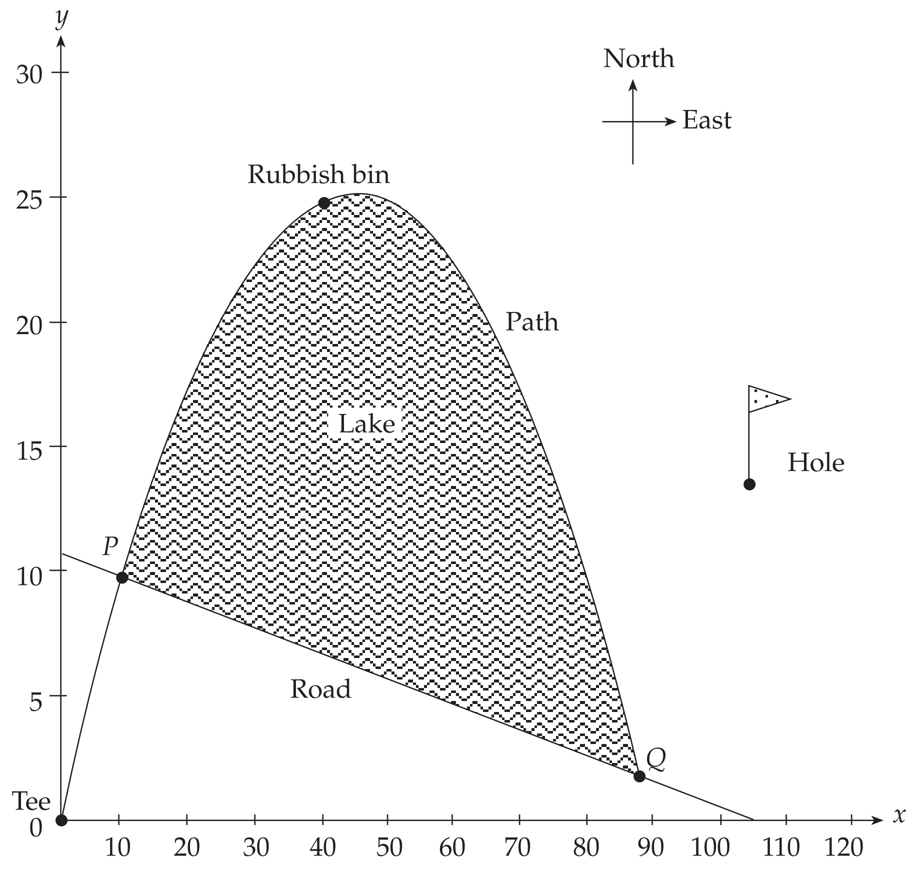
Examination 2 (Analysis task)

Question 1

Betty Irons is playing golf at Sanctuary Bay Golf Club. She is standing on the first tee and notices that there is a large lake beween her and the hole. On one edge of the lake is a curved path for the golfers to walk to the hole. On the other edge is a straight road for golf carts to travel on.

The straight road and curved path intersect at each end of the lake. The first intersection point, P, is 10 metres north and 10 metres east of the first tee. Part way along the path is a rubbish bin. The rubbish bin is a further 30 metres east and 15 metres north of this intersection.

If the tee is taken to be the origin (0, 0), then the diagram below represents Sanctuary Bay's first hole.



a. Give the coordinates of the first intersection point P, of the path and the road. [1 mark]

b. Give the coordinates of the rubbish bin. [1 mark]

c. The curved path follows a general quadratic shape given by the rule $y = ax^2 + bx + c$, where a, b, and c are real constants.

Find the exact values for a, b and c and thus the equation of the path. [4 marks]

d. Given that the road is 11 metres directly north of the tee, find a linear equation that describes the road. [2 marks]

e. Show that the coordinates of the second intersection point between the road and path, Q, are $(88, \frac{11}{5})$. [2 marks]

Betty works for The Southern Water Company and is interested in the amount of water in the lake. She notices that the lake is full to each edge and that there is a sign next to the lake which says, **"No diving! Lake is only 1.2 metres deep!"**

f. Find the surface area of the lake using antidifferentiation. Give your answer in square metres correct to one decimal place. [4 marks]

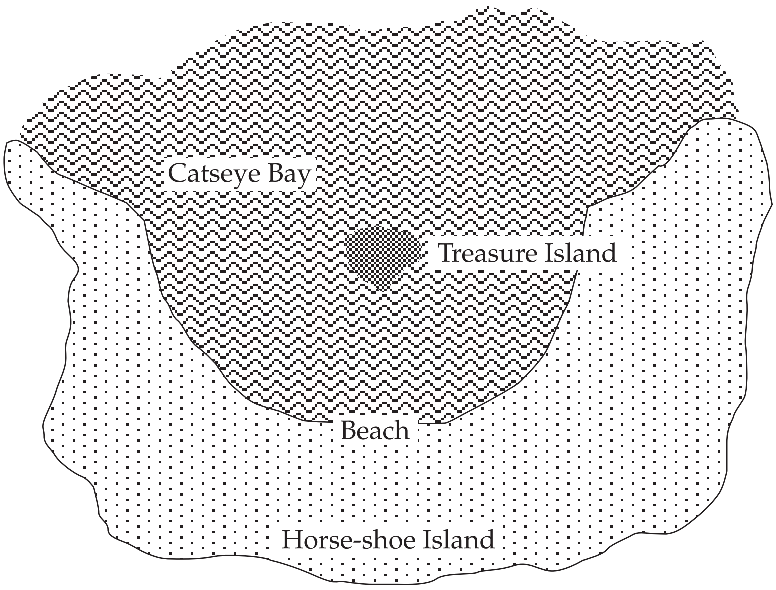
g. Assuming that the lake has a constant depth, what is the volume of water in the lake? Give your answer to the nearest cubic metre. [1 mark]

Total 15 marks

Question 2

Mark and Joanne are holidaying in a resort on Horse-shoe Island, which overlooks a small bay known as Catseye Bay. In Catseye Bay there is a small, off-shore island called Treasure Island. While lying on the beach, Mark and Joanne notice the dramatic tidal changes in Catseye Bay. At high tide, Treasure Island is almost covered with water. At low tide Mark and Joanne can easily wade out to Treasure Island.

A map of Horse-shoe Island and Catseye Bay is shown below.



The tidal movements in Catseye Bay can be approximated by the following equation

$$D(t) = 3.6 + 3.5\cos\frac{\pi}{6}(t-8)$$

where D(t) metres is the maximum depth of the water in Catseye Bay at time t hours. Time t = 0 is taken to be midnight, Saturday the 18th of April.

a. **i.** How many hours are there between consecutive high tides? [1 mark]

ii. What is the maximum depth of the water at high tide? [1 mark]

b. Draw the graph of $D(t) = 3.6 + 3.5\cos\frac{\pi}{6}(t-8),\ \ 0 \le t \le 24$.

Make sure you show an appropriate scale on both the t and D axes. [3 marks]

From this graph, or by some other method, determine the answers to parts **c.**, **d.** and **e.**

c. At what two times on Sunday 19th of April is it high tide? [3 marks]

d. How many hours are there between a low tide and the next high tide? [1 mark]

e. At what time on Sunday does the first low tide occur? [1 mark]

f. Between what times during the daylight hours on Sunday would Mark and Joanne be able to wade out to Treasure Island through water less than 0.5m deep? Give you answer(s) to the nearest minute. [5 marks]

[Total 15 marks]

Question 3

Professor Digger Jones, the world-renowned archaeologist, decides to study an ancient document unearthed at a Melbourne construction site. Suspecting that it is the mysterious "Paper Number 6", he prepares to have it dated. He does this using carbon dating, a process involving the decay of the radioactive isotope carbon-14. Carbon dating assumes that the percentage of carbon-14 in a sample of paper when it was buried was the same as in a present-day sample of paper.

A sample of paper T years after it was buried has P% of the carbon-14 of a present day sample of paper. That is, the unearthed sample has P% of its initial carbon-14 remaining, where

$$P = 100e^{-0.000121t} \text{ and } t \ge 0.$$

a. Find the initial value of P. [1 mark]

Professor Jones finds that the unearthed document has 90% of the carbon-14 of a present-day sample of paper.

b. Find the age of the document to the nearest year. [2 marks]

c. What is the instantaneous rate of change of the percentage of its initial carbon-14 remaining in a sample after 5730 years, accurate to four decimal places? [2 marks]

Professor Jones' rival, Doctor Kelvin, discovers a new secret radioisotope, dawnitium-28. The amount, D grams, of dawnitium-28 per 1000 grams remaining in a sample of paper after t years can be modelled by

$$D = D_0e^{-kt}, \text{ where } t \ge 0 \text{ and } D_0 \text{ and } k \text{ are real constants.}$$

Doctor Kelvin finds that a 1000-year-old document contains 111.57 grams of dawnotium-28 per 1000 grams and that a 2000-year-old document contains only 24.90 grams of dawnitium-28 per 1000 grams.

d. Use this data to find k, accurate to four decimal places, and D_0, accurate to the nearest whole number. [5 marks]

Unlike Professor Jones, Doctor Kelvin is a fairly straight fellow and so dislikes curved graphs. He rearranges the above expression for D in terms of t to develop a **linear** model for t in terms of $\log_e D$. That is,

$$t = a\log_e D + b, \text{ where } a \text{ and } b \text{ are constants.}$$

e. Find the values of a and b accurate to one decimal place. [3 marks]

f. Hence, or otherwise, find, correct to the nearest 10 years, the age of a sample of paper that contains 5.55 grams of dawnitium-28 per 1000 grams. [2 marks]

[Total 15 marks]

Question 4

Uncle Albert Jones keeps hens in his backyard. He regularly records the weights of the eggs that they lay and finds that the weights are normally distributed with a mean of 61 grams and a standard deviation of 8 grams.

a. One afternoon Uncle Albert checks to find a freshly laid egg in the hen coop.

Calculate the probability, correct to four decimal places, that the egg weighs

i. more than 67 grams. [2 marks]

ii. less than 59 grams [2 marks]

iii. more than 67 grams, given that he knows it weighs more than 61 grams. [2 marks]

b. The next morning, Albert finds 6 freshly laid eggs in the coop. Find the probability that at least two of these eggs weigh more than 67 grams. [3 marks]

Uncle Albert 's neighbour, Victor Cab, also keeps hens, and they lay eggs whose weights are normally distributed with a standard deviation of only 2 grams.

c. Victor brags that 98% of his eggs weigh more than 67 grams. What would the mean weight of Victor's eggs be? Give your answer correct to one decimal place. [3 marks]

d. Victor packs his eggs for market in containers holding 12 eggs each.

Out of 50 such containers, how many could be expected to contain at least one egg that weighs less than 67 grams. [3 marks]

[Total 15 marks]

Examination 2 Total: 60 marks

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Examination 2 (Analysis task)

Solutions

Question 1

a. (10, 10) **[A]**

b. (40, 25) **[A]**

c. when $x = 0$, $y = 0$

$\therefore c = 0$ **[A]**

when $x = 10$, $y = 10$

$\therefore 10 = 100a + 10b$ ①

when $x = 40$, $y = 25$

$\therefore 25 = 1600a + 40b$ ② **[M]**

$$\begin{array}{rcrcr} 25 & = & 1600a & + & 40b \\ 40 & = & 400a & + & 40b \\ \hline -15 & = & 1200a & & \end{array}$$

$$\therefore a = \frac{-15}{1200}$$

$$\therefore a = \frac{-1}{80}$$ **[A]**

Substitute a into ①:

$10 = 100a + 10b$

$$10 = 100\left(\frac{-1}{80}\right) + 10b$$

$$10 = \frac{-10}{8} + 10b$$

$$\frac{90}{8} = 10b$$

$$\frac{9}{8} = b$$

$$\therefore y = \frac{-1}{80}x^2 + \frac{9}{8}x$$ ③ **[A]**

d. when $x = 0, y = 11$

when $x = 10, y = 10$

Gradient $= \dfrac{10-11}{10-0} = \dfrac{-1}{10}$ **[A]**

$\therefore y = \dfrac{-1}{10}x + 11$ (4) **[A]**

e. Substitute $x = 88$ into both (3) and (4):

In (3) $y = \dfrac{-1}{80}(88)^2 + \dfrac{9}{8}(88)$

$y = \dfrac{-7744}{80} + 99$

$y = \dfrac{11}{5}$

In 4 $y = \dfrac{-1}{10}(88) + 11$ **[M]**

$y = \dfrac{-88}{10} + 11$

$y = \dfrac{11}{5}$

[OR solve (3) and (4) simultaneously]

Hence Q is $\left(88, \dfrac{11}{5}\right)$. **[M]**

f. $\displaystyle\int_{10}^{88}\left(-\frac{1}{80}x^2 + \frac{9}{8}x - \left[-\frac{1}{10}x + 11\right]\right)dx$ **[M]**

$= \displaystyle\int_{10}^{88}\left(-\frac{1}{80}x^2 + \frac{9}{8}x + \frac{1}{10}x - 11\right)dx$

$= \displaystyle\int_{10}^{88}\left(-\frac{1}{80}x^2 + \frac{98}{80}x - 11\right)dx$

$= \left[-\dfrac{1}{240}x^3 + \dfrac{98}{160}x^2 - 11x\right]_{10}^{88}$ **[M]**

$= [-2839.466 + 4743.2 - 968] - [-4.166 + 61.25 - 110]$

$= [935.73] + 52.92$ **[M]**

$= 988.65$

Area = 988.7 square metres **[A]**

g. Volume $= \text{Area} \times \text{Depth}$

$= 988.7 \times 1.2$

$= 1186.44 \text{ m}^3$

$= 1186 \text{ m}^3$ **[A]**

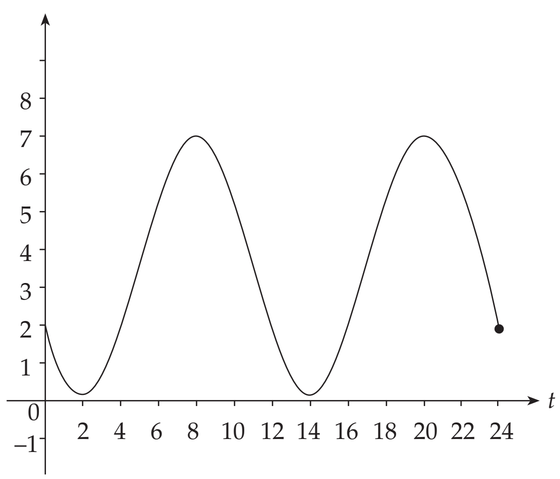
Question 2

a. **i.** Period = $\dfrac{2\pi}{\frac{\pi}{6}} = 12$ hours **[A]**

ii. Amplitude = 3.5 m

Depth = 3.6 + 3.5 = 7.1m **[A]**

b.



1 mark for the scales **[A]**

1 mark for the shape **[A]**

1 mark for correct position of graph **[A]**

c. $\cos\frac{\pi}{6}(t-8) = 1$

$\cos(0) = 1$ and $\cos(2\pi) = 1$ **[M]**

$\therefore t - 8 = 0$ or $t - 8 = 12$

$\therefore t = 8$ or $t = 20$ **[M]**

$\therefore$ times are 8 am and 8 pm **[A]**

Alternatively, use the graph ....

d. Half the period = $\dfrac{12}{2} = 6$ hours **[A]**

e. High tide is at 8 am and there is a six hour time difference between high and low tides.

$8 - 6 = 2$, $\therefore$ 2 am is the first time. **[A]**

f. Depth = 0.5 m when

$3.6 + 3.5\cos\frac{\pi}{6}(t-8) = 0.5$ **[A]**

$\cos\frac{\pi}{6}(t-8) = \frac{-3.1}{3.5}$

$\frac{\pi}{6}(t-8) = 2.6588$ or... **[M]**

$t \approx 13.0780$

Time = 13 hours and 5 mins **[A]**

By symmetry the other time is 14 hours and 55 mins. **[M]**

Time is between 1.05 pm and 2.55 pm **[A]**

Question 3

a. When $t = 0$, $P = 100e^0 = 100$ **[A]**

b. $90 = 100\mathrm{e}^{-0.000121t}$ **[M]**

$0.9 = e^{-0.000121t}$

$\log_e(0.9) = -0.000121t$

$t = 870.748 \approx 871$ years **[A]**

c. $\frac{dP}{dt} = -0.0121\, e^{-0.000121t}$ **[M]**

when $t = 5730$.

$\frac{dP}{dt} = -0.0121\, e^{-0.000121\times 5730} = -0.0060$ **[A]**

d. $t = 1000$, $D = 111.57 \Rightarrow 111.57 = D_0e^{-1000k}$ (1)

$t = 2000$, $D = 24.90 \Rightarrow 24.90 = D_0e^{-2000k}$ (2) **[M]**

$$(1) \div (2) \Rightarrow 4.48 = \frac{e^{-1000k}}{e^{-2000k}}$$ **[M]**

$$= e^{k(-1000+2000)}$$

$$= e^{1000k}$$

$$1000k = \log_e 4.48$$

$$k \approx 0.0015$$ **[A]**

Substitute k into (1):

$$111.57 = D_0e^{-1000\times 0.0015}$$ **[M]**

$$D_0 \approx 500$$ **[A]**

e. $D = 500e^{-0.0015t}$

$\log_e D = \log_e 500e^{-0.0015t}$ **[M]**

$\log_e D = \log_e 500 - 0.0015t$

$0.0015t = -\log_e D + \log_e 500$ **[M]**

$t = \dfrac{-1}{0.0015}\log_e D + \dfrac{\log_e 500}{0.0015}$

$a = \dfrac{-1}{0.0015} \approx -666.7$

$b = \dfrac{\log_e 500}{0.0015} \approx 4143.1$ **[A]**

f. $t = -666.7 \log_e D + 4143.1$

When $D = 5.55$, $t = -666.7 \log_e 5.55 + 4143.1$ **[M]**

≈ 3000.5

Age is 3000 years (to nearest 10 years) **[A]**

Question 4

a. **i.** $\mu = 61$, $\sigma = 8$

Let E gm be the weight of an egg.

$\Pr(E > 67) = \Pr\left(Z > \dfrac{67-61}{8}\right)$ **[M]**

$= \Pr(Z > 0.75)$

$= 1 - \Pr(Z \le 0.75)$

$= 1 - 0.7734$

$= 0.2266$ **[A]**

ii. $\Pr(E < 59) = \Pr\left(Z < \dfrac{59-61}{8}\right)$ **[M]**

$= \Pr(Z < -0.25)$

$= 1 - \Pr(Z \le 0.25)$

$= 1 - 0.5987$

$= 0.4013$ **[A]**

iii. $\Pr(E > 67 \mid E > 61) = \dfrac{\Pr(E > 67)}{\Pr(E > 61)}$ **[M]**

$= \dfrac{0.2266}{\Pr(Z > 0)}$

$= \dfrac{0.2260}{0.5}$

$= 0.4532$ **[A]**

b. $\Pr(E > 67) = 0.2266$ (from part **a. i.**)

Let X be the number of eggs that weigh more than 67 gm.

Then $X \sim Bi\,(6, 0.2266)$ **[M]**

$\Pr\,(X \geq 2) = 1 - [Pr(X = 0) + \Pr(X = 1)]$

$= 1 - [(0.7734)^6 + 6(0.2266)(0.7734)^5]$ **[M]**

≈ 0.4098 **[A]**

c. Let Y be the number of Victor's eggs which weigh more than 67 gms.

Then $Y \sim N(\mu, 2^2)$

$\Pr(Y > 67) = 0.98$

$\Pr\left(Z > \dfrac{67 - \mu}{2}\right) = 0.98$ **[M]**

$Pr(Z > c) = 0.98$

$c = -2.054$

$\dfrac{67 - \mu}{2} = -2.054$ **[M]**

$-\mu = -4.108 - 67$

$\mu = 71.108$

$\mu \approx 71.1$ grams **[A]**

d. Probability that at least one egg in container weighs < 67 gm.

$= 1 - 0.98^{12} \approx 0.2153$ **[M][A]**

Out of 50 containers, expected number containing at least one egg weighing < 67 gm. $= 50 \times 0.2153$

≈ 11 **[A]**