Question 1

a. $A = (0, 50)$ [A]

$C = (70, 8)$ [A]

b. when $y = 0$, $0.02x^2 - 2x + 50 = 0$ [M]

$0.02(x - 50)^2 = 0$

$\therefore x = 50 \quad B = (50, 0)$ [A]

c. Gradient $= \dfrac{dy}{dx} = \dfrac{1}{25}x - 2$ [A]

d. i. When $x = 10$, $\dfrac{dy}{dx} = \dfrac{10}{25} - 2 = -1\frac{3}{5}$ (OR $x = -1.6$) [A]

ii. When $x = 70$, $\dfrac{dy}{dx} = \dfrac{70}{25} - 2 = \dfrac{4}{5}$ (OR $x = 0.8$) [A]

e. If θ = angle from x-axis,

$\tan\theta$ = gradient

$\therefore \tan\theta = 0.8$ [M]

$\therefore \theta = 0.6747^c$ (OR $38.66° \approx 39°$) [A]

f. i. Area of rectangle $= 50 \times 5 = 250\ \text{m}^2$ [A]

ii. $A = \displaystyle\int_0^{70} \left(\frac{1}{50}x^2 - 2x + 50\right) dx$ [M]

$= \left[\dfrac{x^3}{150} - x^2 + 50x\right]_0^{70}$ [A]

$= 886\frac{2}{3}\ \text{m}^2$ [A]

g. The point on the new equation is $(68, 10)$. [A]

$\therefore 10 = A(68 - 50)^3$

$\therefore A = \dfrac{10}{18^3} = \dfrac{5}{2916}$ [A]

h. $A = \displaystyle\int_{50}^{68} \frac{5}{2916}(x - 50)^3 dx$ [M]

$= \dfrac{5}{2916}\left[\dfrac{(x-50)^4}{4}\right]_{50}^{68}$ [A]

$= \dfrac{5}{2916}\dfrac{(68-50)^4}{4} - 0$

$= 45\ \text{m}^2$ [A]

Question 2

a. When $t = 0$, $N = \dfrac{2000}{25} = 80$ [A]

b. As $t \to \infty$, $N \to 2000$ [A]

c. When $t = 10$, $N = \dfrac{2000}{1 + 24e^{-1}} \approx 203$ [A]

d. $N = 2000(1 + 24e^{-0.1t})^{-1}$ [M]

$\dfrac{dN}{dt} = 2000(1 + 24e^{-0.1t})^{-2} \cdot -2.4e^{-0.1t}$ [M] [M]

$= \dfrac{4800e^{-0.1t}}{(1 + 24e^{-0.1t})^{2}}$ [A]

e. When $t = 10$, $\dfrac{dN}{dt} = \dfrac{4800e^{-1}}{(1 + 24e^{-1})^2}$ [M]

$= 18.28$ foxes/month [A]

f. At the minimum, $\dfrac{dN}{dt} = 0$

$\therefore 11.6t + B = 0$ [M]

When $t = 64$, $11.6 \times 64 + b = 0$

$\therefore B = -742.4$ [A]

g. When $t = 64$, substitute into $N = 5.8t^2 - 742.4t + 24200$ [A]

$\therefore N = 443$ [M]

Question 3

a. **i.** $\Pr(\text{received}) = \Pr(S_1 \cap S_2) = \Pr(S_1) \times \Pr(S_2)$

$= p \times p = p^2$ [M]

ii. $\Pr(\text{received}) = \Pr(S_1 \cup S_2) = \Pr(S_1) + \Pr(S_2) - \Pr(S_1 \cap S_2)$

$= p + p - p^2 = 2p - p^2$ [M]

b. **i.** $p = 0.7$, $\Pr(\text{received}) = 2 \times 0.7 - 0.7^2 = 0.91$ [A]

ii. $\Pr(X|\text{received}) = \dfrac{\Pr(X \cap \text{received})}{\Pr(\text{received})}$

$= \dfrac{\frac{1}{2}p^2}{\frac{1}{2}p^2 + \frac{1}{2}(2p - p^2)}$ [M]

$= \dfrac{\frac{1}{2}p^2}{p}$

$= \frac{1}{2}p$

$= 0.35$ [A]

c. **i.** $N \stackrel{d}{=} \text{Bin}(10, 0.91)$

$\therefore \Pr(N = 10) = {}^{10}C_{10}(0.91)^{10}(0.09)^0 = 0.3894$ [A]

ii. $\Pr(N \geq 9) = \Pr(N = 9) + \Pr(N = 10)$

$= {}^{10}C_9(0.91)^9(0.09) + {}^{10}C_{10}(0.91)^{10}(0.09)^0$

$= 0.3851 + 0.3894 = 0.7746$ [A]

iii. $\Pr(N \geq 2) = 1 - \Pr(N < 2)$

$= 1 - [\Pr(N = 0) + \Pr(N = 1)]$ [M]

$= 1 - [(0.09)^{10} + {}^{10}C_1(0.91)^1(0.09)^9]$

$= 0.9999$ [A]

d. **i.** $\Pr(\text{Y receives signal}) = 2p - p^2$

Let the random variable R denote the number of Y components that receive the signal in a batch of 10, so that $R \stackrel{d}{=} \text{Bin}(10, 2p - p^2)$.

We require that $\Pr(R \geq 1) \geq \frac{1}{2}$

i.e., $1 - \Pr(R = 0) \geq \frac{1}{2}$ [M]

$1 - {}^{10}C_0(2p - p^2)^0(1 - (2p - p^2))^{10} \geq \frac{1}{2}$ [M]

$\Leftrightarrow \quad (1 - 2p + p^2)^{10} \leq \frac{1}{2}$

$\Leftrightarrow \quad 2(1 - 2p + p^2)^{10} \leq 1$ [M]

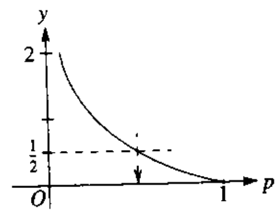
ii. Now let $y = 2(1 - 2p + p^2)^{10}$ [M]

$= 2(1 - p)^{20}$

If $2(1 - 2p + p^2)^{10} \leq 1$, $(1 - p)^{20} \leq \frac{1}{2}$

$\therefore 1 - p \leq \sqrt[20]{\frac{1}{2}}$

$p \geq 1 - \sqrt[20]{\frac{1}{2}}$



$\therefore$ smallest value of p is $1 - \sqrt[20]{\frac{1}{2}} \approx 0.0341$ [A]

e. Let the life span for component Y be denoted by the random variable T, so that $T \stackrel{d}{=} \text{Bin}(110, 16)$

i. $\Pr(T > 115) = \Pr\left(z < \frac{115 - 110}{4}\right)$ [M]

$= \Pr(z > 1.25)$

$= 0.1056$ [A]

ii. $\Pr(108 < T < 114) = \Pr(-0.5 < z < 1)$ [M]

$= 0.5328$ [A]

iii. $\Pr(T > 115 | T > 111) = \dfrac{\Pr(T > 115)}{\Pr(T > 111)}$ [M]

$= \dfrac{0.1056}{0.4013} = 0.2631$ [A]

f. $E(T) = 110$, $\sigma = 4$

$\therefore$ 95% confidence interval for μ_T is given by $110 - 2 \times 4 < \mu_T < 110 + 2 \times 4$

i.e. $102 < \mu_T < 118$ [A]

Question 4

a. $e^{-\frac{x}{2}}\cos x = 0 \Leftrightarrow \cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ [M]

$\therefore A\left(\frac{\pi}{2}, 0\right)$ and $B\left(\frac{3\pi}{2}, 0\right)$ [A]

b. **i.** $f'(x) = -\frac{1}{2}e^{-\frac{x}{2}}\cos x - e^{-\frac{x}{2}}\sin x = -e^{-\frac{x}{2}}\left(\frac{1}{2}\cos x + \sin x\right)$ [M] [A]

For stationary points: $f'(x) = 0$

$\Rightarrow \frac{1}{2}\cos x + \sin x = 0$ [M]

$\Leftrightarrow \sin x = -\frac{1}{2}\cos x$

i.e. $\tan x = -\frac{1}{2}$ [M]

ii. $x = \tan^{-1}\left(-\frac{1}{2}\right)$

$= 2.67794$

≈ 2.678 (3 d. p.) (0.234) [A]

c. **i.** $\frac{d}{dx}\left[e^{-\frac{x}{2}}(a\cos x + b\sin x)\right] = -\frac{1}{2}e^{-\frac{x}{2}}(a\cos x + b\sin x) + e^{-\frac{x}{2}}(-a\sin x + b\cos x)$ [M]

$\therefore \left(-\frac{1}{2}a + b\right)e^{-\frac{x}{2}}\cos x + \left(-\frac{1}{2}b - a\right)e^{-\frac{x}{2}}\sin x = e^{-\frac{x}{2}}\cos x$

$\therefore b - \frac{1}{2}a = 1$ (1)

$a + \frac{1}{2}b = 0$ (2) [A]

Substitute (2) into (1): $b + \frac{1}{4}b = 1$ $\therefore b = \frac{4}{5}$

Substitute b into (1): $a = -\frac{1}{2} \times \frac{4}{5} = -\frac{2}{5}$ [M]

ii. Required area $= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-\frac{x}{2}}\cos x \, dx$ [M]

$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{d}{dx}\left(e^{-\frac{x}{2}}(a\cos x + b\sin x)\right) dx$

$= \left[(a\cos x + b\sin x)e^{-\frac{x}{2}}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$ [A]

$= -\left(-be^{-\frac{3\pi}{4}} - be^{-\frac{\pi}{4}}\right)$

$= \frac{4}{5}e^{-\frac{\pi}{4}}\left(1 + e^{-\frac{\pi}{2}}\right)$ square units. [A]