The Mathematical Association of Victoria

Mathematical Methods 1998 CATs 2 and 3 Solutions

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"Cliveden", 61 Blyth Street, Brunswick, 3056.
Telephone: (03) 9380 2399 Fax: (03) 9380 8323
e-mail: office@mav.vic.edu.au website: http://www.mav.vic.edu.au

1998 Mathematical Methods CAT 2

Suggested Answers & Solutions

Part I (Multiple-choice) Answers

1.	B	2.	D	3.	C	4.	E	5.	E
6.	D	7.	E	8.	D	9.	B	10.	D
11.	E	12.	A	13.	C	14.	C	15.	A
16.	C	17.	A	18.	B	19.	C	20.	B
21.	B	22.	B	23.	E	24.	D	25.	C
26.	A	27.	E	28.	A	29.	C	30.	B
31.	D	32.	D	33.	B				

Part I (Multiple-choice) Solutions

Question 1 [B]

$$y = 5 - 2\cos\left(\frac{\pi t}{8}\right)$$

$$\text{period} = \frac{2\pi}{\frac{\pi}{8}} = 2\pi \times \frac{8}{\pi} = 16$$

high tide occurs half a period after low tide
9 am + 8 hours = 5 pm

Question 2 [D]
Let t = 12 hours (9 am to 9 pm)

$$y = 5 - 2\cos\left(\frac{12\pi}{8}\right)$$

$$= 5 - 2\cos\left(\frac{3\pi}{2}\right)$$

$$= 5 \text{ metres}$$

Question 3 [C]
amplitude = 2
vertical shift is up
period = 4

$$\frac{2\pi}{n} = 4$$

$$n = \frac{\pi}{2}$$

$$y = 1 + 2\cos\left(\frac{\pi x}{2}\right)$$

Question 4 [E]

$$\sin 2x - a\cos 2x = 0$$

$$\sin 2.\frac{\pi}{6} - a\cos 2.\frac{\pi}{6} = 0$$

$$\sin\frac{\pi}{3} - a\cos\frac{\pi}{3} = 0$$

$$\sin\frac{\pi}{3} = a\cos\frac{\pi}{3}$$

$$\tan\frac{\pi}{3} = a$$

$$a = \sqrt{3}$$

Question 5 [E]

$$2\sin x + \sqrt{3} = 0$$

$$\sin x = \frac{-\sqrt{3}}{2}$$

$$x = \pi + \frac{\pi}{3},\ 2\pi - \frac{\pi}{3}$$

$$x = \frac{4\pi}{3},\ \frac{5\pi}{3}$$

$$\frac{4\pi}{3} + \frac{5\pi}{3} = \frac{9\pi}{3} = 3\pi$$

Question 6 [D]

$$3\sin 2x = -\sqrt{3}\cos 2x$$

$$\tan 2x = \frac{-\sqrt{3}}{3}$$

$$= \frac{-\sqrt{3}}{3} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{-1}{\sqrt{3}}$$

Question 7 [E]

Line numbers	Gradients
L_1	$\frac{-3}{4}$
L_2	$\frac{-3}{4}$
L_3	$\frac{-5}{8}$
L_4	$\frac{8}{5}$

The gradients of L_3 and L_4 are negative reciprocals of one another. Hence, L_3 is perpendicular to L_4.

Question 8 [D]

A is not a fourth degree polynomial
$P(2) = 0$ satisfies **B, C, D, E**
$P(1.9) < 0$ satisfies **C** and **D**
$P(2.1) <$) satisfies only **D**.

Alternatively, the curve touching the x-axis at $x = 2$ implies a repeated root, $(x-2)^2$ as part of the quartic polynomial. The shape of the given graph also suggests a negative quartic. **D** is the only option with both a negative quartic and a repeated root.

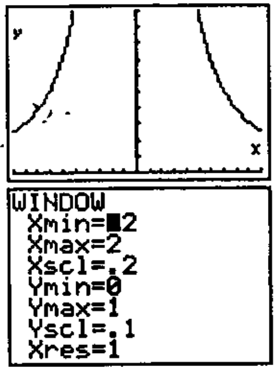
Question 9 [B]

- For functions to be multiples of one another, their x-intercepts must be equal. Hence, **C** and **E** are false.
- **D** describes a graph of the same shape simply shifted to the left. Since $g(x)$ and $h(x)$ have a different shape altogether, **D** is also false.
- In the domain $[0, 1]$, the value of $g(x) + h(x)$ is clearly greater than $f(x)$. Hence, **A** is also false.
- Upon inspection, the value of $f(x) + g(x)$, at the intersections of f and g and f and h, is equal to $h(x)$. Hence **B** is true.

Question 10 [D]

$f(-2) = f(2) = \frac{1}{4}$ and $f(-1) = f(1) = 1$

Upon inspection of the following graph generated by a graphing calculator, the range of **D** is incorrect and should be $[1, \infty)$.



Question 11 [E]

Only the graphs $y = ax^2$ and $y = ax^{\frac{1}{2}}$ pass through the origin. The data represented graphically has a similar basic shape to that of $y = ax^{\frac{1}{2}}$

Question 12 [A]

f^{-1} is a reflection of f about the line $y = x$. The only option that accurately reflects this is **A**.

Question 13 [C]

$$y = x^2 + 1$$
$$\frac{dy}{dx} = 2x$$
$$2x = -1 \quad \text{(gradient of tangent = -1)}$$
$$x = \frac{-1}{2}$$
$$\text{At } x = \frac{-1}{2},\ y = \left(\frac{-1}{2}\right)^2 + 1 = \frac{5}{4}$$

$$y = -x + c$$
$$c = y + x$$
$$c = \frac{5}{4} - \frac{1}{2} = \frac{3}{4}$$

Question 14 [C]

$$y = e^{(2\cos(3x))}$$
Let $u = 2\cos(3x)$

then $y = e^u$
$$\frac{du}{dx} = -6\sin(3x)$$
$$\frac{dy}{du} = e^u = e^{(2\cos(3x))}$$
$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$
$$= -6\sin(3x)e^{(2\cos(3x))}$$

Question 15 [A]

$$y = e^{(ax)}\cos(bx)$$
$$\frac{dy}{dx} = ae^{(ax)}\cos(bx) + -b\sin(bx).e^{(ax)}$$
$$= ae^{(ax)}\cos(bx) - be^{(ax)}\sin(bx)$$

Question 16 [C]

Let $u = 2x + 3$
$$\frac{du}{dx} = 2$$
Using $\int\left(\frac{1}{ax+b}\right)dx = \frac{1}{a}\log_e(ax+b)$,
$$\int\frac{1}{2x+3} - \cos(2x+3)dx = \int\left(\frac{1}{2x+3}\right)dx + \int(\cos(2x+3))dx$$
$$= \frac{1}{2}\log_e(2x+3) - \frac{1}{2}\sin(2x+3)$$

Question 17 [A]

- **A** is true because the derived function is negative at $x > 2$ and increases to zero at the turning point of the curve, and then is positive and increasing.
- **B** cannot be true because the derived function is positive for $x : x \geq 3.75$ (approximately)
- **C** is not true because the derived function does exist, but is equal to zero.
- **D** is not true because the function is not differentiable at $x = 0$ and $x = 2$.
- **E** is not true because the function is discontinuous at $x = 0$.

Question 18 [B]

$$f(x) = \frac{x^3}{3} - \log_e(x) + c$$
$$f(1) = 0 = \frac{1}{3} - 0 + c$$
$$c = \frac{-1}{3}$$
$$f(x) = \frac{x^3}{3} - \log_e(x) - \frac{1}{3}$$

Question 19 [C]

Since the area defined by $\int_0^1 f(x)dx$ will result in a negative value, the *total* area of the shaded area is given by
$$\int_{-2}^{0} f(x)dx - \int_0^1 f(x)dx$$

Question 20 [B]

$$\int_0^3 k(x^2+1)dx = 1$$
$$k\left[\frac{x^3}{3} + x\right]_0^3 = 1$$
$$k\left[\frac{3^3}{3} + 3 - \left(\frac{0^3}{3} + 0\right)\right] = 1$$
$$k[9 + 3 - 0] = 1$$
$$12k = 1$$
$$k = \frac{1}{12}$$

Question 21 [B]

$$\frac{d}{dx}x\log_e x = 1 + \log_e x$$
$$x\log_e x = \int(1 + \log_e x)dx$$
$$x\log_e x = \int 1dx + \int\log_e x dx$$
$$x\log_e x = x + \int\log_e x dx + c$$
$$\int\log_e x dx = x\log_e x - x - c$$
$$= x\log_e x - x + c$$
since c is a constant, $+c$ are $-c$ are equivalent

Question 22 [B]

$$(r+1)^{th}\text{ term} = {}^nC_r x^r a^{n-r}$$
$$4^{th}\text{ term} = {}^6C_3(2x)^3a^3$$
$$4320 = 160a^3$$
$$a = 3$$

Question 23 [E]

$$ax^4 - bx^2 = x^2(ax^2 - b)$$
$$= x.x[(\sqrt{a}x)^2 - (\sqrt{b})^2]$$
$$= x.x(\sqrt{a}x + \sqrt{b})(\sqrt{a}x - \sqrt{b})$$

Question 24 [D]

$$3\log_{10}x - \log_{10}(x^2) = 1 + \log_{10}y$$
$$\log_{10}x = \log_{10}10y$$
$$x = 10y$$

Question 25 [C]

As the graph below shows, f is only one-to-one to the left or right of the turning point $(1, 7)$. Possible domains are subsets of $(\infty, 1]$ or $[1, \infty)$. Only **C** satisfies this criteria.

Question 26 [A]

Let $x = e^{2y} + 1$
$$e^{2y} = x - 1$$
$$2y = \log_e(x-1)$$
$$y = \frac{1}{2}\log_e(x-1)$$

Question 27 [E]

$$\mu = \sum x.p(x)$$
$$= 2.9$$
$$\sigma = \sqrt{\sum x^2.p(x) - \mu^2}$$
$$= \sqrt{11.1 - 8.41}$$
$$= 1.64$$

Question 28 [A]

$$\Pr(X = x) = {}^nC_x p^x q^{n-x}$$
$$n = 10,\ p = 40\% = 0.4,\ q = 0.6$$
$$\Pr(X = 9) = {}^{10}C_9 0.4^9 0.6$$

Question 29 [C]

$$\Pr(X > 15.2) = \Pr(Z > \frac{X-\mu}{\sigma})$$
$$= \Pr(Z > \frac{15.2 - 10.2}{2.5})$$
$$= \Pr(Z > 2)$$

Question 30 [B]

$$\Pr(X > 1) = \Pr(Z > \frac{1 - 1.05}{0.03})$$
$$= \Pr(Z > -1.67)$$
$$= \Pr(Z < 1.67) \quad \text{by symmetry}$$
$$= 0.9525$$
$$\approx 0.95$$

Question 31 [D]

$$\Pr(X > 35) = 0.1$$
$$\Pr(X < 35) = 0.9$$
$$\Pr(Z < \frac{35-28}{\sigma}) = 0.9$$
$$\frac{35-28}{\sigma} = 1.281$$
$$\sigma = \frac{35-28}{1.281} \approx 5.46$$

Question 32 [D]

$N = 120 + 80 = 200$

$\hat{p} = \frac{120}{200} = 0.6$

$se(\hat{p}) = \sqrt{\frac{0.6 \times 0.4}{200}} = 0.0346$

Interval is

$[0.6 - 2 \times 0.0346,\ 0.6 + 2 \times 0.0346]$

$= [0.5308,\ 0.6692]$

$\approx [0.53,\ 0.67]$

Question 33 [B]

$se(\hat{p}) = \sqrt{\frac{0.15 \times 0.85}{100}} = 0.0357$

Part II (Short Answer Questions) Solutions

Question 1

a. From the graph, (x − 2) is one factor. By evaluating coefficients from expansion,

$x^2(x-2) - 3x(x-2) - 5(x-2)$

$= x^3 - 2x^2 - 3x^2 + 6x - 5x + 10$

$= (x-2)(x^2 - 3x - 5)$

(Note that long division of polynomials can also be used.)

b. Using the quadratic formula,

if $ax^2 + bx + c = 0$, then

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

let $x^2 - 3x - 5 = 0$, then

$x = \frac{3 \pm \sqrt{9 + 20}}{2}$

$= \frac{3 + \sqrt{29}}{2},\ \frac{3 - \sqrt{29}}{2}$

Graph cuts the x-axis at $x = 2,\ \frac{3 + \sqrt{29}}{2},\ \frac{3 - \sqrt{29}}{2}$

Question 2

$2\log_e x = 2\log_e a + \log_e 9$

$\log_e x^2 = \log_e a^2 + \log_e 9$

$\log_e x^2 = \log_e 9a^2$

$x^2 = 9a^2$

$x = 3a$

Question 3

a. f is a one-to-one function

b. i. The domain of f equals the range of f^{-1}.

From the graph, the dom f = ran $f^{-1} = R\backslash\{2\}$

ii. $b = -1$ (the vertical asymptote)

$B = 2$ (the horizontal asymptote)

Let $x = \frac{1}{y-1} + 2$

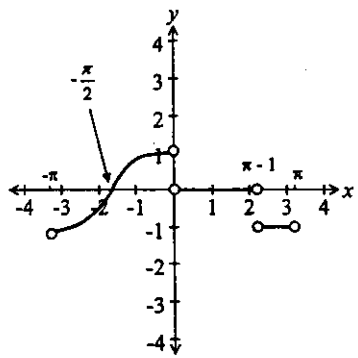
$x - 2 = \frac{1}{y-1}$

$y - 1 = \frac{1}{x-2}$

$y = \frac{1}{x-2} + 1$

$f^{-1}(x) = \frac{1}{x-2} + 1$

Question 4



Note that the function, f, is not necessarily a parabola in the domain (π, 0) and so its derived function will not be a straight line over that domain.

Question 5

a. $\sum p(x) = 1$

$\frac{2k^2 - 1 + 4k + 3k + k}{9} = 1$

$2k^2 + 8k - 1 = 9$

$2k^2 + 8k - 10 = 0$

$k^2 + 4k - 5 = 0$

$(k-1)(k+5) = 0$

$k = 1$

b. $\sum x.p(x) = \frac{4}{9} + \frac{6}{9} + \frac{3}{9} = 1\frac{4}{9} \approx 1.44$

c. $\Pr(X \le 2) = 1 - \Pr(X = 3)$

$= 1 - \frac{1}{9}$

$= \frac{8}{9}$

≈ 0.89

Question 6

Using symmetry,

$A = 2\int_0^{\frac{2\pi}{3}} \cos x - \cos 2x\,dx + \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \cos 2x - \cos x\,dx$

$= 2\left[\sin x - \frac{1}{2}\sin 2x\right]_0^{\frac{2\pi}{3}} + \left[\frac{1}{2}\sin 2x - \sin x\right]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$

$= 2\left(\sin\frac{2\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3}\right)$

$+\left[\left(\frac{1}{2}\sin\frac{8\pi}{3} - \sin\frac{4\pi}{3}\right) - \left(\frac{1}{2}\sin\frac{4\pi}{3} - \sin\frac{2\pi}{3}\right)\right]$

$= 2\left[\frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{\sqrt{3}}{2}\right] + \left[\left(\frac{1}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2} \times \frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right]$

$= \sqrt{3} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}$

$= 3\sqrt{3}$