1999 Mathematical Methods CAT 2

Suggested Answers and Solutions

Part I (Multiple-choice) Answers

1. B	2. E	3. A	4. E	5. C
6. E	7. C	8. D	9. A	10. B
11. D	12. B	13. D	14. A	15. D
16. A	17. E	18. D	19. A	20. C
21. D	22. B	23. E	24. A	25. B
26. D	27. C	28. C or E	29. B	30. C
31. E	32. A	33. B		

Question 1 [B]

$\sum \Pr(X = x) = 1$

So $k + 4k + 9k + 16k = 1$

$30k = 1$

$k = \frac{1}{30}$

Question 2 [E]

Binomial Variable $n = 6$, $p = 0.9$

Let X be the number of "cures" amongst patients.

$X \sim Bi(n, p)$ $X \sim Bi(6, 0.9)$ $\Pr(X = x) = {}^6C_3\, 0.9 0$

$\Pr(X \geq 3)$

$= 1 - [\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)]$

$= 1 - [0.1^6 + {}^6C_1 0.1^5 0.9 + {}^6C_2 0.1^4 0.9^2]$

Question 3 [A]

Note Binomial Variable $p = 0.2$ $n = 10$ $q = 0.8$

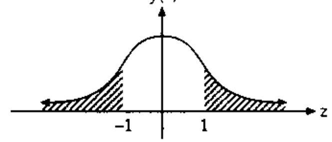
μ = Mean $= np$ Variance $= npq$

$\mu = 10 \times 0.2$ Var $= 2 \times 0.8$

$= 2$ $= 1.6$

Question 4 [E] from diagrams given.

Question 5 [C]



From diagram [C] is incorrect.

Note that $\Pr(Z < -1) = \Pr(Z > 1)$. Other options A, B, D and E correctly describe a standard normal distribution.

Question 6 [E]

$\mu = 100$ $\sigma = 8$

$\Pr(X > 110) = \Pr\left(Z > \frac{110 - 100}{8}\right)$

$= \Pr\left(Z > \frac{10}{8}\right)$

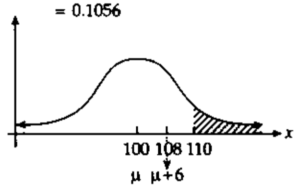
$= \Pr\left(Z > \frac{5}{4}\right)$

$= \Pr(Z > 1.25)$

$= 1 - \Pr(Z < 1.25)$

$= 1 - 0.8944$

$= 0.1056$



Note that from above diagram shaded area could not be answers A, B or C

Question 7 [C]

$p = 0.6$ = population proportion

Sample of 100 = n

We seek $\sigma_{\hat{p}}^2$ = variance of sample proportion

$= \frac{p(1-p)}{n}$

$= \frac{0.6 \times 0.4}{100}$

$= \frac{0.24}{100}$

$= 0.0024$

Omit

Question 8 [D]

Note $a > 0$

Note as graph has y-intercept 0, not a, we eliminate C ($Y = ae^x$)

Question 9 [A]

$y = \frac{A}{x + B}$

Vertical Asymptote at $x = -1 \Rightarrow B = 1$

So graph is of form $y = \frac{A}{x + 1}$

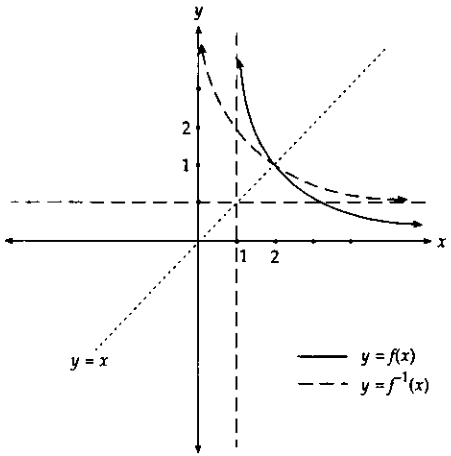
Substitute $(0, -1)$ $-1 = \frac{A}{1}$

$A = -1$

$\therefore$ Rule is $y = \frac{-1}{x + 1}$

Question 10 [B]

Note same scale on BOTH axes



Question 11 [D]

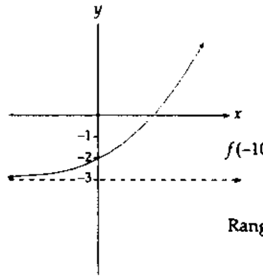
Note: derived function does not exist at both $x = 1$ and $x = -1$. The derivative will not exist when the function is not "smooth" and continuous. At both these points we are unable to draw a tangent to the graph.

Question 12 [B]

$f(x) = 2^{(0.1x + 0.1)} - 3$

$= 2^{0.1x} \times 2^{0.1} - 3$

$= \sqrt[10]{2} \times 2^{0.1x} - 3$



$f(0) = \sqrt[10]{2} - 3$

$\approx 1.0718 - 3$

$= -1.9282$

$f(-100) = \left(\sqrt[10]{2} \times 2^{-10}\right) - 3$

$= -2.9989$

Range from graph is $(-3, \infty)$

Alternatively,

the constant -3 translates the asymptote down

the $(0.1x + 0.1)$ power will dilate and translate the graph parallel to the x-axis (doesn't affect the range)

Question 13 [D]

Dilation from y-axis by scale factor of 2 causes

$y = \sin x \rightarrow y = \sin \frac{x}{2}$

A translation of $\frac{\pi}{2}$ to left in x direction causes

$y = \sin \frac{\pi}{2} \rightarrow y = \sin \frac{1}{2}\left(x + \frac{\pi}{2}\right)$

A translation of 2 units downwards in the y-direction causes

$y = \sin \frac{1}{2}\left(x + \frac{\pi}{2}\right) \rightarrow y = \left(\sin \frac{1}{2}\left(x + \frac{\pi}{2}\right)\right) - 2$

or $y = \sin\left[0.5\left(x + \frac{\pi}{2}\right)\right] - 2$

Question 14 [A]

We cannot draw a tangent to f at $x = 0$ or $x = 2$. So at these points there will be no derivative. Also if $0 < x < 2$, $f'(x) = 0$

Question 15 [D]

$f(x) = x(x^2 - 3x - 9)$ of form $f(x) = uv$

where $u = x$ and $v = x^2 - 3x - 9$

$\frac{du}{dx} = 1$ $\frac{dv}{dx} = 2x - 3$

$f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$ PRODUCT RULE

$= x(2x - 3) + (x^2 - 3x - 9) \times 1$

Question 16 [A]

$y = \frac{\cos 3t}{t^2}$ of form $y = \frac{u}{v}$

where $u = \cos 3t$

$\frac{du}{dt} = -3\sin 3t$

and $v = t^2$, $\frac{dv}{dt} = 2t$

$\frac{dy}{dt} = \frac{v\frac{du}{dt} - u\frac{du}{dt}}{v^2}$ QUOTIENT RULE

$\frac{dy}{dt} = \frac{t^2\left(-3t^2 \sin 3t\right) - (\cos 3t)2t}{t^4}$

$\frac{dy}{dt} = \frac{-3t^2 \sin 3t - 2t\cos 3t}{t^4}$

Question 17 [E]

$y = \log_e\left(\frac{1}{x}\right)$ $y = \log_e u$ where $u = \frac{1}{x}$

$\frac{du}{dx} = \frac{-1}{x^2}$

Chain Rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{d \log_e u}{du} \times \left(-\frac{1}{x^2}\right)$

$= \frac{1}{u} \times \left(-\frac{1}{x^2}\right)$

$= x \times \left(-\frac{1}{x^2}\right)$

$= -\frac{1}{x}$

Check:

NOT REQUIRED BY EXAMINERS

$y = \log_e\left(\frac{1}{x}\right)$

$= \log_e 1 - \log_e x$

$= 0 - \log_e x$

$\frac{dy}{dx} = -\frac{1}{x}$

Question 18 [D]

$y = x - e^{-x}$

$\frac{dy}{dx} = 1 - (-1) \times e^{-x}$

$= e^{-x} + 1$

If $x = 0$, $\frac{dy}{dx} = e^{-0} = 1 + 1 = 2$

Question 19 [A]

$y = \frac{x^3}{3} - x^2 - 15x$

$\frac{1}{3} > 0 \therefore$ "Positive" cubic shape

$\frac{dy}{dx} = x^2 - 2x - 15$

Local maximum if $x^2 - 2x - 15 = 0$

$(x+3)(x-5) = 0$

$x = -3$ or 5

From shape $x = -3$ for local maximum

Note that a quick alternative would be to substitute $x = -3, 0, 1, 3, 5$ into $y = \frac{x^3}{3} - x^2 - 15x$ to see which gives the greatest y-value or to check the sign of the derivative. A graphing calculator could also be used.

x	$y = \frac{x^3}{3} - x^2 - 15x$ $y = x\left(\frac{x^2}{3} - x - 15\right)$
-3	$-3\left(\frac{9}{3} - 3 - 15\right) = -3(-9) = 27$ maximum y-value
0	0
1	$1\left(\frac{1}{3} - 1 - 15\right) < 0$
3	$3(3 - 3 - 15) < 0$
5	$5\left(\frac{25}{3} - 5 - 15\right) < 0$

Question 20 [C]

$f'(x) = 2\sin 2x - 4e^{-2x}$ $f(o) = 2$

$f(x) = \int\left(2\sin 2x - 4e^{-2x}\right)dx$

$= -\cos 2x + 2e^{-2x} + c$

But $f(0) = 2$, so $2 = -1 + 2 + c$

$\therefore$ $2 = 1 + c$ $c = 1$

So $f(x) = 2e^{-2x} - \cos 2x + 1$

Question 21 [D]

Shaded area $= \int_0^6 \left(x(6-x) - 2x(x-6)\right)dx$

$= \int_0^6 \left(6x - x^2 - 2x^2 + 12x\right)dx$

$= \int_0^6 \left(18x - 3x^2\right)dx$

$= \left[9x^2 - x^3\right]_0^6$

$= 9 \times 36 - 6^3$

$= 108$

Question 22 [B]

$f(x) = x^4 - 4x$

$f'(x) = 4x^3 - 4$

Stationary point $f'(x) = 0$

$4x^3 - 4 = 0$

$x^3 - 1 = 0$

$x^3 = 1$

$x = 1$

As coefficient of $x^4 > 0$ we have an "arms up" quartic function:

Thus the stationary point must be a minimum.

Alternatively, use a graphics calculator to graph $y = x^4 - 4x$.

Question 23 [E]

We must sum two separate areas namely for $0 \le x \le a$ and for $a \le x \le b$

Shaded area $= \int_0^a \left(f(x) - g(x)\right)dx + \int_a^b \left(g(x) - f(x)\right)dx$

Question 24 [A]

as in $(bx + c)$

$= a \sin b(x + \frac{c}{b})$

period $= \frac{2\pi}{b}$

Period $= \pi$ $\therefore b = 2$

Amplitude = 2 $\therefore a = 2$ or -2

From shape $a = -2$

So $y = -2\sin(2x + c)$

$y = -2\sin 2\left(x + \frac{c}{2}\right)$

Extending the diagram the graph would "start" at about $\frac{-\pi}{12}$

$\therefore \frac{c}{2} = \frac{\pi}{12}$ $c = \frac{\pi}{6}$

So $y = -2\sin\left(2x + \frac{\pi}{6}\right)$

Question 25 [B]

$f(x) = 2\cos(3x + \pi) - 1$

Amplitude = 2 Period $= \frac{2\pi}{3}$

Range $-2 - 1 \le y \le 2 - 1$

$-3 \le y \le 1$

or $[-3, 1]$

Question 26 [D]

$2\cos(2x) = \sqrt{2}$

$\cos(2x) = \frac{\sqrt{2}}{2}$

$2x = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$

$x = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$

Sum of solutions $= \frac{1}{8}(\pi + 7\pi + 9\pi + 15\pi)$

$= \frac{32\pi}{8}$

$= 4\pi$

Question 27 [C]

$b \sin ax = \sqrt{2}\cos(ax)$

$\tan ax = \frac{\sqrt{2}}{b}$

Substitute $x = \frac{\pi}{4}$, $\tan\frac{\pi a}{4} = \frac{\sqrt{2}}{b}$

But $\tan\frac{\pi}{4} = 1 = \frac{\sqrt{2}}{\sqrt{2}}$

$\therefore a = 1,\ b = \sqrt{2}$ is a solution

Note we cannot independently solve for a and b, $\tan\frac{\pi a}{4} = \frac{\sqrt{2}}{b}$ as we have one equation in two unknowns. We *must* use elimination testing the given values in the multiple choice options.

Question 28 [C] or [E]

$f(x) = p\sin(2x) + q$ where $p > 0$.
Range $-p + q \le f(x) \le p + q$

If $f(x) > 0$ for all the values of x

$-p + q > 0$

$q > p$ or $p < q$

$\therefore q > 2p$ also correct

Question 29 [B]

$a, b, c > 0$

$P(x) = (x^2 + a)(x - b)(x - c)^2 = 0$

Null Factor Law

$x^2 + a = 0$ No real solution
$x - b = 0$ One solution $x = b$
$(x - c)^2 = 0$ One solution $x = c$

So $P(x) = 0$ has two distinct real solutions

Question 30 [C]

$(px + 4)^5 = (px)^5 + {}^5C_1(px)^4(4) + {}^5C_2(px)^3(4)^2 + \ldots$

Coefficient of $x^3 = {}^5C_2 \times p^3 \times 16$
$= 160p^3$

So $160p^3 = 4320$
$p^3 = 27$
$p = 3$

Question 31 [E]

$f(x) = \frac{1}{x+2} - 1$

Interchange the roles of x and y. For the inverse

$x = \frac{1}{y+2} - 1$

$x + 1 = \frac{1}{y+2}$

$y + 2 = \frac{1}{x+1}$

$y = -2 + \frac{1}{x+1}$

$f^{-1}(x) = \frac{1}{x+1} - 2$ Dom $R\backslash\{-1\}$

Question 32 [A]

$a > 0$ and $x > 0$

$\log_a x^2 - 2 = 2\log_a 5$

$\log_a x^2 - 2\log_a 5 = 2$

$\log_a\left(\frac{x^2}{5^2}\right) = 2$

$a^2 = \frac{x^2}{25}$

$x^2 = 25a^2$

$x = 5a$

Check: LHS $= \log_a x^2 - 2$
$= \log_a(25a^2) - 2$
$= \log_a 25 + \log_a a^2 - 2$
$= \log_a 5^2 = 2\log_a a - 2$
$= 2\log_a 5 =$ RHS check

Question 33 [B]

$4 \times 10^{2x} = 9$

$10^{2x} = \frac{9}{4}$

$\log_{10}\frac{9}{4} = 2x$

$x = \frac{1}{2}\log_{10}\frac{9}{4}$

$x = \log_{10}\frac{9^{\frac{1}{2}}}{4^{\frac{1}{2}}}$

$x = \log_{10}\left(\frac{3}{2}\right)$

Check: LHS $= 4 \times 10^{2x}$
$= 4 \times 10^{2\log_{10}(3/2)}$
$= 4 \times 10^{\log_{10}(9/4)}$
$= 4 \times \frac{9}{4}$
$= 9$
$=$ RHS check

Part II (Short answer questions) Solutions

Question 1

(a) Period = 8 (b) Amplitude = 1

Question 2

Let X be the distance Antonio can throw the ball.
$\mu = 80$m $\sigma = 3$m

$X \sim N(\mu, \sigma^2)$

$X \sim N(80, 3^2)$

$\Pr(X > d) = 0.25$ | $z = \frac{x - \mu}{\sigma}$

$\therefore \Pr(X < d) = 0.75$

From inverse normal tables corresponding z value is 0.6745

$0.6745 = \frac{d - 80}{3}$

$d - 80 = 2.0235$

$d = 82.0235$

$d \approx 82$

Question 3

p represents population proportion and $\hat{p}$ represents the sample proportion.
n = number in sample = 300

$\hat{p} = \frac{225}{300} = 0.75$

The 95% confidence limits for p are

$\hat{p} \pm 2\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.75 \pm 2\sqrt{\frac{0.75 \times 0.25}{300}}$
$= 0.75 \pm 2 \times 0.025$
$= 0.75 \pm 0.05$

So $0.7 \le p \le 0.8$

OMIT

Question 4

(a) Expand $3x(2x - 5)^3$

$= 3x\left((2x)^3 - 3(2x)^2(5) + 3(2x)(5)^2 - 5^3\right)$

$= 3x\left(8x^3 - 60x^2 + 150x - 125\right)$

$= 24x^4 - 180x^3 + 450x^2 - 375x$

Question 4 (cont)

(b) $P(x) = x^3 + 2x^2 + ax - 2$

$x - 2$ is a factor $\therefore P(2) = 0$

$2^2 + 2(2)^2 + 2a - 2 = 0$

$8 + 8 + 2a - 2 = 0$

$14 + 2a = 0$

$a = -7$

Check: (NOT REQUIRED BY EXAMINERS)

$P(x) = x^3 + 2x^2 + 7x - 2$

$(x^3 + 2x^2 - 7x - 2) \div (x - 2) = x^2 + 4x + 1$

$x^3 - 2x^2$

$4x^2 - 7x$

$4x^2 - 8x$

$x - 2$

$x - 2$

0 = Remainder check

Question 5

$V(t) = 1000 - 25t - \frac{t^2}{100}$, $t \in (0, 35)$

(a) $V(0) = 1000$

$V(10) = 1000 - 250 - 1$
$= 749$

Average rate of change of volume over first ten minutes $= \frac{V(10) - V(0)}{10}$

$= \frac{749 - 1000}{10} = -25.1$ cm^3/min

(b) $V(t) = 1000 - 25t - \frac{t^2}{100}$, $0 < t < 35$

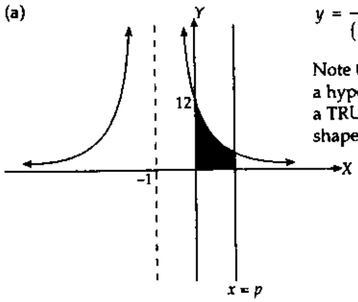
$V'(t) = -25 - \frac{t}{50}$

$V'(10)$ = instantaneous rate of change of volume when $t = 10$

$V'(10) = -25 - \frac{10}{50}$
$= -25.2$ cm^3/min

Question 6

(a)



$y = \dfrac{12}{(x+1)^2}$

Note this is not a hyperbola but a TRUNCUS shape.

$$\begin{aligned}
\text{Area} &= \int_0^p \frac{12}{(x+1)^2}\,dx \\
&= \left[-\frac{12}{x+1}\right]_0^p \\
&= -\frac{12}{(p+1)} - \left(\frac{-12}{1}\right) \\
&= \frac{-12}{(p+1)} + 12 \\
\text{or } &= \frac{12p}{(p+1)}
\end{aligned}$$

(b) Area = 6 ∴

$$\begin{aligned}
6 &= \frac{-12}{(p+1)} + 12 \\
\frac{12}{p+1} &= 6 \\
p+1 &= 2 \\
p &= 1
\end{aligned}$$