1999 Mathematical Methods CAT 3

Suggested Solutions

Question 1

$L = \frac{A}{x} + B$

L in mg/kg is lead concentration.
x m is distance of grass from roadside.

(a) As $x \to \infty$, $L \to 2$ mg/kg $\quad \therefore B = 2$

So $L = \frac{A}{x} + 2$

(b) Substitute $x = 10$m, $L = 50$ mg/kg

$50 = \frac{A}{10} + 2$

$48 = \frac{A}{10}$

$A = 480 \qquad$ So $L = \frac{480}{x} + 2$

(c) Substitute $L = 10$ mg/kg

$10 = \frac{480}{x} + 2$

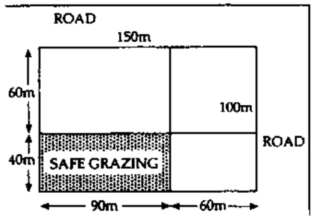
$8 = \frac{480}{x}$

$x = \frac{480}{8}$

$= 60$m

Least distance from roadside to meet requirement is 60 m.

(d)



Safe grazing area $= \frac{40 \times 90}{150 \times 100} \times \frac{100}{1}\%$

$= 24\%$

$G = e^{0.01x}$
x is distance of grass from roadside
G kg/m^2 is the density of grass
T mg/m^2 is overall concentration of lead in grass
$T = L \times G$

(e) $T = \left(\frac{480}{x} + 2\right)e^{0.001x}$

(f) $x = 50$m, $\quad T = \left(\frac{480}{x} + 2\right) e^{0.001x}$

$T = 11.6 \qquad e^{0.5}$

$T = 19.125167$

$T \approx 19.1$ mg/m^2

(g) $T = \left(\frac{480}{x} + 2\right) e^{0.01x}$ of form $T = uv$

where $u = \frac{480}{x} + 2$

$\frac{du}{dx} = -\frac{480}{x^2}$

and $v = e^{0.01x}$

$\frac{dv}{dx} = 0.01e^{0.01x}$

Using the product rule,

$\frac{dT}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= \left(\frac{480}{x} + 2\right)0.01e^{0.01x} + e^{0.01x}\left(-\frac{480}{x^2}\right)$

$= e^{0.01x}\left[\left(\frac{4.8}{x} + \frac{1}{50}\right) - \frac{480}{x^2}\right]$

$= \frac{e^{0.01x}}{50x^2}\left(240x + x^2 - 24000\right)$

$\frac{dT}{dx} = \frac{e^{0.01x}}{50x^2}\left(240x + x^2 - 24000\right)$

If $x = 50$m, $\frac{dT}{dx} = \frac{e^{0.5}}{125000}(2500 + 12000 - 24000)$

$= \frac{e^{0.5}}{125000}(-9500)$

$= -\frac{19e^{0.5}}{250}$

$= -0.1253028 \frac{\text{mg}}{\text{m}^2} \div \text{m}$

$\approx -0.125 \frac{\text{mg}}{\text{m}^3}$

Note carefully the unit of $\frac{dT}{dx}$.

Question 2

$f: D \to R$ where $f(x) = 2 - \log_e(x+1)$

(a) Domain $\quad x + 1 > 0$

$x > -1$

Domain $= \{x : x > -1\}$ or $(-1, \infty)$

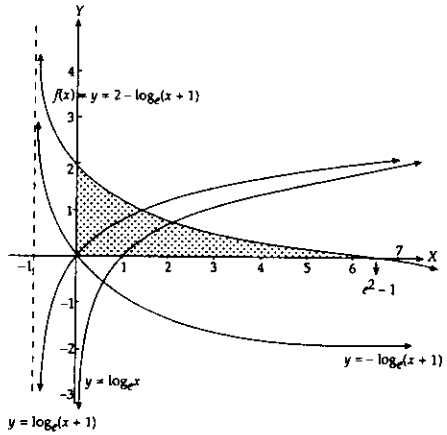
(b) $y = \log_e x$

(1) Translate above graph 1 unit to left parallel to the X-axis. Thus

$y = \log_e x \to y = \log_e(x+1)$

(2) Now reflect the graph of $y = \log_e(x+1)$ in the X-axis to obtain $y = -\log_e(x+1)$

(3) Now translate the graph of $y = -\log_e(x+1)$ 2 units upwards parallel to the Y-axis to obtain the graph of $f(x) = 2 - \log_e(x+1)$



Note that the sketch graphs above were not required by the examiners. They illustrate the transformation process, however.

(c) $f(x) = 2 - \log_e(x+1)$

Y-intercept $\quad f(0) = 2 - \log_e 1$

$= 2$

$\therefore (0, 2)$

X-intercept $\quad$ Put $\quad f(x) = 2 - \log_e 1$

$2 - \log_e(x+1) = 0$

$\log_e(x+1) = 2$

$x + 1 = e^2$

$\therefore x = e^2 - 1$

$\therefore (e^2 - 1, 0)$

See above sketch

(d) $f(x) = 2 - \log_e(x+1)$

Of form $f(x) = 2 - \log_e u \quad$ where $u = x + 1$

$\frac{du}{dx} = 1$

$f'(x) = \frac{dy}{du} \times \frac{du}{dx}$

$= -\frac{1}{u} \times 1$

$= -\frac{1}{x+1}$

Gradient of graph when $x = 4$ is $f'(4) = -\frac{1}{4+1}$

$= -\frac{1}{5}$

(This is reasonable from previous sketch graph).

(e) (i) $f(x) = 2 - \log_e(x+1)$

$y = 2 - \log_e(x+1)$

To find the inverse function interchange x and y

$x = 2 - \log_e(y+1)$

$x - 2 = -\log_e(y+1)$

$\log_e(y+1) = 2 - x$

$y + 1 = e^{2-x}$

$y = -1 + e^{2-x}$

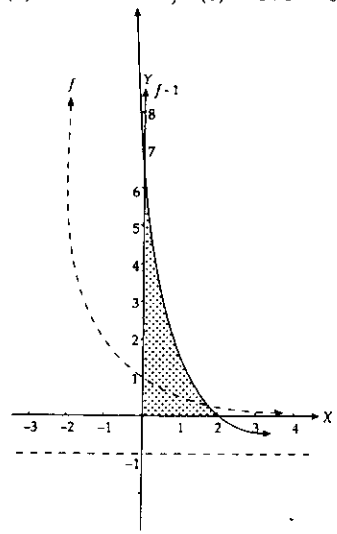
OR $y = -1 + \frac{e^2}{e^x}$

Rule for inverse function is $y = -1 + e^{2-x}$

(ii) Domain of $f^{-1}(x)$ is unrestricted $\therefore R$

(f) Sketch of $f^{-1}: R \to R$ where

$f^{-1}(x) = -1 + e^{2-x} \qquad f^{-1}(0) = -1 + e^2 \approx 64$



(g) Area of region bounded by the graph of $f(x)$ and coordinate axis $= \int_0^{e^2-1}\left(-1+e^{2-x}\right)d$

P6 $= \text{Area} = \int_0^2 \left(-1+e^{2-x}\right)dx$

$= \int_0^2 \left(-1+e^2e^{-x}\right)dx$

$= \left[-x+e^2\left(-e^{-x}\right)\right]^2$

$= \left(-2-e^2\left(e^{-2}\right)\right)-\left(-e^2\right)$

$= -2-1+e^2 = e^2-3$ sq. units

Question 3

(a) Let X be the amount of storage space on computer.

Then $X \sim N(\mu, \sigma^2)$ $\mu = 1.2$ megabytes, $\sigma = 0.3$ megabytes

(i) We seek $\Pr(X < 1.44)$

$z = \dfrac{x-\mu}{\sigma}$

$z = \dfrac{1.44-1.2}{0.3} = 0.8$

$\Pr(X < 1.44) = \Pr(z < 0.8)$

$= 0.7881$ from tables

≈ 0.788 to 3 dec. places

(ii) $\Pr(Z < z) = 0.85$

$z = 1.036$

$1.036 = \dfrac{x-1.2}{0.3}$

$x = 1.5108$

So minimum amount of storage spaces required on disc is 1.51 megabyte (correct to 2 dec. places).

(iii) $\Pr\begin{pmatrix}\text{one computer requires}\\ \text{more than 1.44 megabytes}\\ \text{of storage space}\end{pmatrix} = 1-0.7881$

$= 0.2119$

$\Pr\begin{pmatrix}\text{all 5 computers}\\ \text{need more than}\\ \text{1.44 megabytes}\\ \text{of storage space}\end{pmatrix} = 0.2119^5$

$= 0.000427\,22$

≈ 0.0004 to 4 dec. places

(b) Let Y be the number of computers affected by bug amongst 5 computers in Company Q

$p = 0.2$ = probability that computer is affected by bug.

$Y \sim Bi(n, p)$

$Y \sim Bi(5, 0.2)$

$\Pr(Y=4) = {}^5C_4\, 0.2^4\, 0.8$

$= 5 \times 0.2^4 \times 0.8$

$= 0.0064$

(c) (i)

Number of computers encountering bug = Y	0	1	2	3	4	5
Nett loss made by software company ($)	-5×20	1×120 $-(5\times 20)$	140	3×120 $-(5\times 20)$	4×120 $-(5\times 20)$	5×120 $-(5\times 20)$
	$= -100$	20		260	380	500

(ii) $\Pr\begin{pmatrix}\text{software company will make loss of more}\\ \text{than \$400 given that at least 4 of company}\\ \text{Q's computers are affected by bug.}\end{pmatrix}$

$= \Pr(y=5 \mid y \geq 4)$

Now $\Pr(B \mid A) = \dfrac{\Pr(A \cap B)}{\Pr(A)}$ Always true.

$\Pr(Y=5 \mid Y \geq 4) = \left(\dfrac{\Pr(Y=5 \text{ AND } Y \geq 4)}{\Pr(Y \geq 4)}\right)$

$= \left(\dfrac{\Pr(Y=5)}{\Pr(Y \geq 4)}\right) = \left(\dfrac{0.2^5}{0.0064+0.2^5}\right)$

$= 0.0476$ correct to 4 dec. places

(d) p = probability of failure in population = 0.2

n = sample size

$\sqrt{\dfrac{p(1-p)}{n}} = 0.05$

$\sqrt{\dfrac{0.2\times 0.8}{n}} = 0.05$

$\sqrt{\dfrac{0.16}{n}} = 0.025$

$\dfrac{0.16}{n} = 0.025^2$

$\dfrac{0.16}{n} = 0.000625$

$n = \dfrac{0.16}{0.000625}$

$n = 256$

So a sample size of 256 would be required to estimate the value of p to an accuracy of $\pm$ 0.05 with 95% confidence.

Question 4

(a) The equation $y = \sin\left(\dfrac{\pi x}{10}\right), 0 \leq x \leq W$ represents the rock surface ABCDE.

Now ABCDE is one complete cycle. So in $0 \leq x \leq W$ we see one complete cycle is shown.

Period $= \dfrac{2\pi}{n}$

$= 2\pi \div \dfrac{\pi}{10}$

$= 20$

$\therefore W = 20\text{m}$

(b) Vertical distance from surface to top of gold seam $= \sin\left(\dfrac{\pi x}{10}\right)-\left(\cos\left(\dfrac{\pi x}{10}\right)-3\right)$

$= 3+\sin\left(\dfrac{\pi x}{10}\right)-\cos\left(\dfrac{\pi x}{10}\right)$

(c) Let $d(x)$ be vertical distance

$d(x) = 3+\sin\left(\dfrac{\pi x}{10}\right)-\cos\left(\dfrac{\pi x}{10}\right)$

$d'(x) = \dfrac{\pi}{10}\cos\dfrac{\pi x}{10}+\dfrac{\pi}{10}\sin\dfrac{\pi x}{10}$

If $d(x)$ is a mimimum, $d'(x) = 0$

So $\dfrac{\pi}{10}\cos\left(\dfrac{\pi x}{10}\right)+\dfrac{\pi}{10}\sin\left(\dfrac{\pi x}{10}\right) = 0$

$\cos\left(\dfrac{\pi x}{10}\right)+\sin\left(\dfrac{\pi x}{10}\right) = 0$

$\sin\left(\dfrac{\pi x}{10}\right) = -\cos\left(\dfrac{\pi x}{10}\right)$

$\dfrac{\sin\left(\frac{\pi x}{10}\right)}{\cos\left(\frac{\pi x}{10}\right)} = -1$ provided $\cos\dfrac{\pi x}{10} \neq 0$

$\tan\dfrac{\pi x}{10} = -1$

$\dfrac{\pi x}{10} = \pi-\dfrac{\pi}{4}, 2\pi-\dfrac{\pi}{4}$

$\dfrac{\pi x}{10} = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

$x = \dfrac{30}{4}, \dfrac{70}{4}$

$x = 7.5, 17.5$

From the diagram given in question $d(x)$ will be a minimum at $x = 17.5$m

$\therefore$ minimum vertical distance is

$d(17.5) = 3+\sin\left(\dfrac{17.5\pi}{10}\right)-\cos\left(\dfrac{17.5\pi}{10}\right)$

$= 3-\sqrt{2}$

(d)

Cross-sectional area of granite he will remove $= \int_0^{20}\left(\sin\left(\dfrac{\pi x}{10}\right)-\left(\cos\left(\dfrac{\pi x}{10}\right)-3\right)\right)dx$

$= \int_0^{20}\left(3+\sin\left(\dfrac{\pi x}{10}\right)-\cos\left(\dfrac{\pi x}{10}\right)\right)dx$

$= \left[3x-\dfrac{10}{\pi}\cos\dfrac{\pi x}{10}-\dfrac{10}{\pi}\sin\dfrac{\pi x}{10}\right]_0^{20}$

$= \left(3\times 20-\dfrac{10}{\pi}\cos 2\pi-\dfrac{10}{\pi}\sin 2\pi\right)-\left(-\dfrac{10}{\pi}\cos 0-\dfrac{10}{\pi}\sin 0\right)$

$= 60-\dfrac{10}{\pi}-\left(-\dfrac{10}{\pi}-\dfrac{10}{\pi}\times 0\right)$

$= 60-\dfrac{10}{\pi}+\dfrac{10}{\pi}$

$= 60\text{m}^2$

(e) A = Cross-sectional area of goldseam $= \int_0^{20} T\,dx$

$= \int_0^{20}\left(0.2-0.002(20-x)^{1.5}\right)dx$

Recall $\int (ax+b)^n\,dx = \dfrac{1}{a(n+1)}(ax+b)^{n+1}+c, n \neq -1$

$A = \left[0.2x-\dfrac{0.002}{2.5\times(-1)}(20-x)^{2.5}\right]_0^{20}$

$A = \left(0.2\times 20+\dfrac{0.002}{2.5}(0)^{2.5}\right)-\left(0+\dfrac{0.002}{2.5}(20)^{2.5}\right)$

$\approx 4-1.43108$

$\approx 2.5689 \text{ m}^2$

Volume of seam $= V = 2.5689 \times 40 \text{ m}^3 \approx 102.7567 \text{ m}^3$

Volume of gold $= \dfrac{0.2}{100} \times$ Vol. of seam

$\approx 0.206 \text{ m}^3$

(f)

The reduced cross-sectional area of gold seam $= R = \int_{15}^{20} T\,dx$

$$= 0.002\left[100x + \frac{1}{2.5}(20 - x)^{2.5}\right]_{15}^{26}$$

$$R = 0.002\left[100 \times 20 - \left(100 \times 15 + \frac{1}{2.5} \times 5^{2.5}\right)\right]$$

$$= 0.002\left(2000 - 1500 - \frac{1}{2.5} \times 5^{2.5}\right)$$

$$= 0.9552786405 \text{ m}^2$$

Percentage of total amount of gold he can now mine $= \frac{0.9552786}{2.5689165} \times \frac{100}{1}\%$

$= 37.1861\ \%$

$\approx$ 37% to nearest per cent