THE HEFFERNAN GROUP

P.O. Box 1180
Surrey Hills North VIC 3127
ABN 20 607 374 020
Phone 9836 5021
Fax 9836 5025

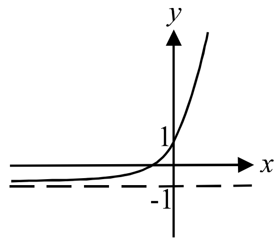
MATHEMATICAL METHODS TRIAL EXAM 2 2000 SOLUTIONS

Question 1

a. Given $f(0) = 1$, we have $ae^0 - 1 = 1$ So, $a = 2$ since $e^0 = 1$ **(1 mark)**

b. Do a quick sketch of the function *f* on your graphics calculator.

From this we see that $r_f = (-1, \infty)$ **(1 mark)**



c. The *x* intercept occurs when $y = 0$.

We have, $2e^x - 1 = 0$ **(1 mark)**

$$e^x = \frac{1}{2}, \text{ so } x = \log_e \frac{1}{2} \text{ and so } b = \frac{1}{2} \quad \textbf{(1 mark)}$$

d. **(1 mark)** for graph of $f^{-1}(x)$, **(1 mark)** for *x* intercept of 1, **(1 mark)** for *y* intercept of $\log_e \frac{1}{2}$ or -0.693

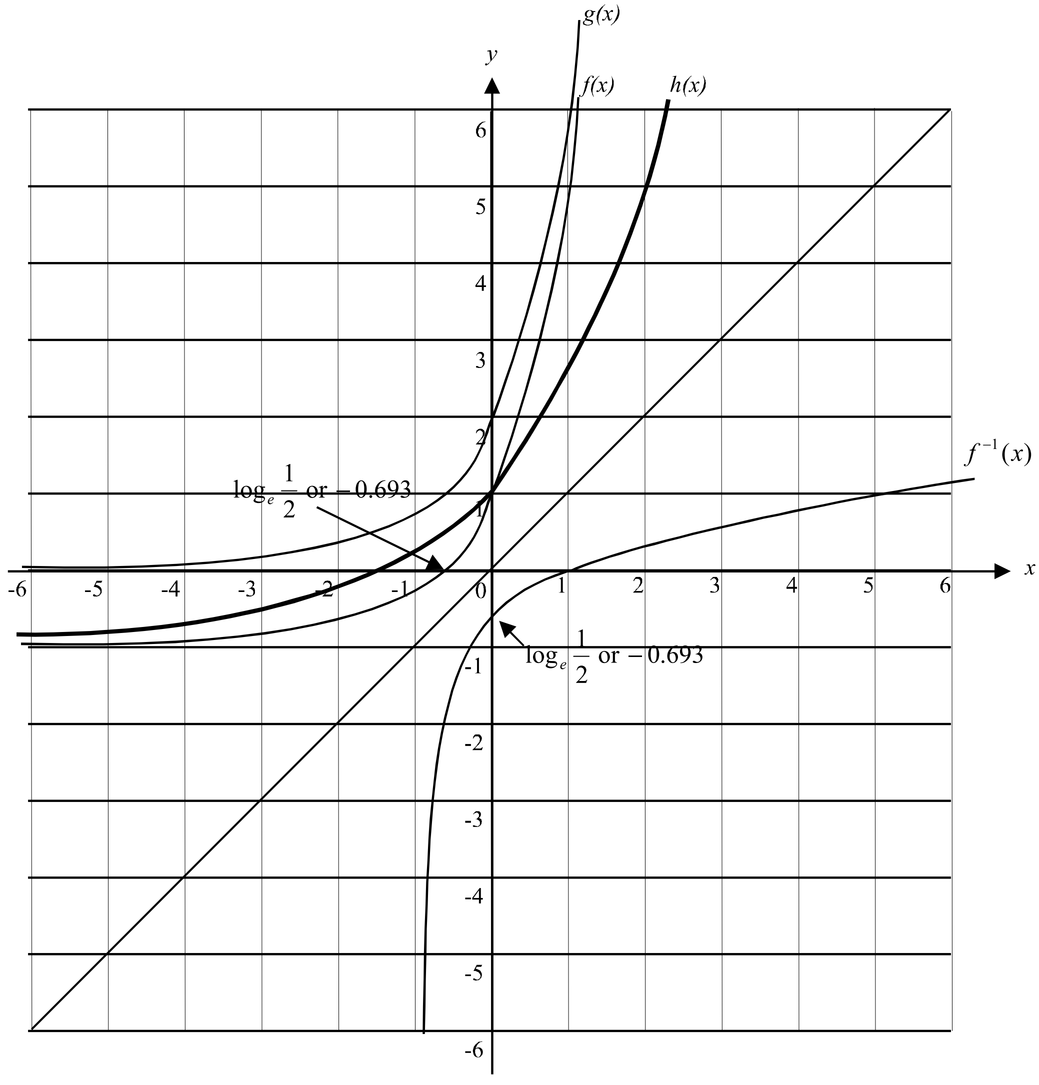


Figure A

e. Consider $f(x) = 2e^x - 1$

Let $y = 2e^x - 1$

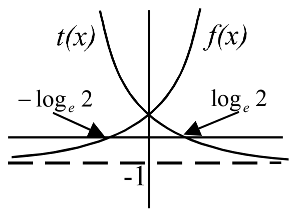
Swap *x* and *y*: $x = 2e^y - 1$

So, $\frac{x+1}{2} = e^y$

$y = \log_e\left(\frac{x+1}{2}\right)$

So, $f^{-1}(x) = \log_e\left(\frac{x+1}{2}\right)$ **(1 mark)** $d_{f^{-1}} = (-1, \infty)$ **(1 mark)**

f. i. and **ii.** See Figure A **(2 marks)**



g. i. The *x* intercept of *f(x)* is $\log_e \frac{1}{2}$

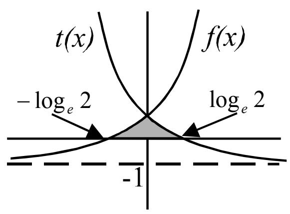
Now, $\log_e \frac{1}{2} = \log_e(2^{-1})$

$= -\log_e 2$

So the *x* intercept of *t(x)* is given by $\log_e 2$. So $m = 2$ **(1 mark)**

ii. We are looking for the shaded area. Since *f(x)* and *t(x)* are symmetrical about the *y* axis, we have, area required = $2\int_{\log_e \frac{1}{2}}^{0}(2e^x - 1)dx$ **(1 mark)** for integrand, **(1 mark)** for terminals

$= 2\left[2e^x - x\right]_{\log_e \frac{1}{2}}^{0}$ **(1 mark)**



$= 2\left\{(2e^0 - 0) - (2e^{\log_e \frac{1}{2}} - \log_e \frac{1}{2})\right\}$

$= 2(2 - 2 \times \frac{1}{2} + \log_e \frac{1}{2})$

$= 2(1 + \log_e \frac{1}{2})$ **(1 mark)**

OR $= 2(1 - \log_e 2)$

Total 16 marks

Question 2

a. i. Let X = the number of yellow balls occurring in a set.
X is a variable which has a binomial distribution.
So, $E(x) = np$

$= 3 \times 0.14$

$= 0.42$ **(1 mark)**

ii. $\Pr(X = 1) = {}^3C_1(0.14)^1(0.86)^2$ **(1 mark)**

$= 0.3106$ (correct to 4 decimal places) **(1 mark)**

iii. To win a point they need to draw out 3 balls, of which none are yellow.

So, $\Pr(X = 0) = {}^3C_0(0.14)^0(0.86)^3$ **(1 mark)**

$= 0.6361$ (correct to 4 decimal places) **(1 mark)**

iv. The probability of earning 5 points means that no yellow balls appear in 5 successive sets. Using **part iii.**, we have $0.6361^5 = 0.1041$ (to 4 places) **(1 mark)**

b. The probability of winning a point is the probability of there being no yellow balls in a draw. That is, ${}^3C_0 = (p)^0(1-p)^3 = 0.719$ where p is the probability of drawing a yellow ball from the container. **(1 mark)**

So, $1 \times 1(1-p)^3 = 0.719$

So, $(1-p)^3 = 0.719$ **(1 mark)**

Use trial and error.

If Mario's father removed 1 yellow ball, then the probability of a yellow ball being drawn would be $\frac{6}{49}$. So, $(1-p)^3 = (1-\frac{6}{49})^3$

$= 0.676$ to 3 places

$\neq 0.719$

If Mario's father removed 2 yellow balls then the probability of a yellow ball being drawn would be $\frac{5}{48}$. So, $(1-p)^3 = (1-\frac{5}{48})^3$

$= 0.719$ to 3 places

So, Mario's father took out 2 yellow balls. **(1 mark)**

c. i. Since the balls are no longer being replaced, the number of yellow balls in a set now has a hypergeometric distribution. So, $E(x) = n\frac{D}{N} = 3 \times \frac{7}{50} = 0.42$ **(1 mark)**

ii. To earn a point we require that there is no yellow balls in a set, that is, $\Pr(X = 0)$.

Now, $\Pr(X = x) = \frac{{}^DC_x \, {}^{N-D}C_{n-x}}{{}^NC_n}$ where $D = 7$, $x = 0$, $N = 50$ and $n = 3$ **(1 mark)**

So, $\Pr(X = 0) = \frac{{}^7C_0 \, {}^{43}C_3}{{}^{50}C_3} = 0.6296$ **(1 mark)**

iii. With these rules, the probability of scoring a point is 0.6296. With the other rules, the probability of scoring a point is 0.6361 from **part a. iii.**. In other words, it is harder to score a point under the new rules. **(1 mark)**

iv. $\Pr(X \geq 2) = \Pr(X = 2) + \Pr(X = 3)$ **(1 mark)**

$= \frac{{}^7C_2 \, {}^{43}C_1}{{}^{50}C_3} + \frac{{}^7C_3 \, {}^{43}C_0}{{}^{50}C_3}$

$= 0.046071 + 0.0017857$

$= 0.0479$ to 4 places **(1 mark)**

Total 15 marks

Question 3

a. a =60 (a is the amplitude) and b=60 **(2 marks)**

b. $y_B = -60\sin\frac{\pi x}{20} + 60$ **(1 mark)**

c. period $= \frac{2\pi}{n} = \frac{2\pi}{\pi/20}$ Alternatively, just read the period off the graph.

$= 40$ **(1 mark)**

d. Between the two fence posts there are 8 complete cycles of the graph of $y_A = 60\sin\frac{\pi x}{20} + 60$

(1 mark)

Since the period of this function is 40, then the distance between the two fence posts is $8 \times 40 \text{ cm} = 320 \text{ cm}$ or 3.2metres. **(1 mark)**

e. We want to find the first two points of intersection of the graphs with equations

$y_A = 60\sin\frac{\pi x}{20} + 60$ and $y = 90$ for $x \geq 0$

So we need to solve the equation:

$$60\sin\frac{\pi x}{20} + 60 = 90 \qquad \textbf{(1 mark)}$$

So,
$$60\sin\frac{\pi x}{20} = 30$$
$$\sin\frac{\pi x}{20} = \frac{1}{2}$$
$$\frac{\pi x}{20} = \frac{\pi}{6}, \frac{5\pi}{6}... \qquad \textbf{(1 mark)}$$
$$x = \frac{20}{6}, \frac{100}{6}...$$
$$x = 3\frac{1}{3}, 16\frac{2}{3}...$$

The first point of intersection is $3\frac{1}{3}$ cm from the left-hand post and the second point of intersection is $16\frac{2}{3}$ cm from the left-hand post. Look at Figure 2 to see if this seems reasonable. It does. **(1 mark)**

f. i. From Figure 2, we see that the "peak" of the curved wire closest to the left-hand fence post is located at the point (10, 120). The stick is 10 cm long and its ends must intersect with the wire at x = 5 and x = 15. **(1 mark)**

Now, at $x = 5$, $y_A = 60 \sin\frac{\pi \times 5}{20} + 60$

$= 102.4264$ **(1 mark)**

So the stick is wedged a vertical distance of 17.57 cm below the top of the fence. **(1 mark)**

Check your answer by substituting $x = 15$ into $y_A = 60\sin\frac{\pi \times 15}{20} + 60 = 102.4264$

ii. The shape of the graph with equation A between x = 0 and x = 20 is identical to the shape of the graph with equation B between x = 20 and x = 40. We know from Figure 2 that the horizontal distance between the peaks of these two graphs is 20 cm, as is the horizontal distance between all the corresponding points on the two graphs. The length of the little boy's stick is therefore 20 cm.
(1 mark) **Total 13 marks**

Question 4

a. i. At point A, $x = 50$, from the graph and therefore, $y = \frac{50+1}{50}\log_e(50+1)$

$= 4.01$ to 2 decimal places

(1 mark)

ii. The highest point in the park above the levelled flat rubbish tip is therefore 4.01 m **(1 mark)**

b. Substitute $x = 0$ into $y = \frac{x+1}{50}\log_e(x+1)$

We have $y = \frac{0+1}{50}\log_e(0+1)$

$= \frac{1}{50}\log_e 1$

$= 0$ since $\log_e 1 = 0$

So the graph with equation $y = \frac{x+1}{50}\log_e(x+1)$ passes through the origin. **(1 mark)**

c. Now, $y = \frac{x+1}{50}\log_e(x+1)$

So, $\frac{dy}{dx} = \frac{x+1}{50}.\frac{1}{x+1} + \frac{1}{50}\log_e(x+1)$

$= \frac{1}{50} + \frac{1}{50}\log_e(x+1)$

$= \frac{1}{50}(1 + \log_e(x+1))$ **(1 mark)**

At $x = 25$, $\frac{dy}{dx} = \frac{1}{50}(1 + \log_e 26)$

$= 0.085$ to 3 decimal places **(1 mark)**

d. Graph the gradient function $y = \frac{1}{50}(1 + \log_e(x+1))$. By looking at the graph and using the "TRACE" function or by looking at the table of values of the function, we see that the maximum value is 0.0986 (to 4 places) over the domain $x \in [0,\ 50]$ and therefore the gradient of the path never exceeds 0.1 **(2 marks)**

e. i. Now, $y = \frac{(x+1)^2}{100}\log_e(x+1)$

So, $\frac{dy}{dx} = \frac{2(x+1)^1 \times 1}{100}\log_e(x+1) + \frac{(x+1)^2}{100}.\frac{1}{x+1}$ **(1 mark)**

$= \frac{x+1}{50}\log_e(x+1) + \frac{x+1}{100}$ **(1 mark)**

ii. From part **e. i.** we know that

$$\frac{d\left\{\frac{(x+1)^2}{100}\log_e(x+1)\right\}}{dx}=\frac{x+1}{50}\log_e(x+1)+\frac{x+1}{100}$$

So, $$\int\frac{d\left\{\frac{(x+1)^2}{100}\log_e(x+1)\right\}}{dx}dx=\int\frac{x+1}{50}\log_e(x+1)dx+\int\frac{x+1}{100}dx$$ **(1 mark)**

So, $$\frac{(x+1)^2}{100}\log_e(x+1)=\int\frac{x+1}{50}\log_e(x+1)dx+\frac{x^2}{200}+\frac{x}{100}+c,$$ c is constant

(1 mark)

Rearranging, $$\int\frac{x+1}{50}\log_e(x+1)dx=\frac{(x+1)^2}{100}\log_e(x+1)-\frac{x^2}{200}-\frac{x}{100}-c$$ **(1 mark)**

iii. Area under graph is given by

$$\int_0^{50}\frac{x+1}{50}\log_e(x+1)dx=\left[\frac{(x+1)^2}{100}\log_e(x+1)-\frac{x^2}{200}-\frac{x}{100}\right]_0^{50}$$

(1 mark) for terminals
(1 mark) for integrand

$$=\left\{\left(\frac{51^2}{100}\log_e 51-\frac{2500}{200}-\frac{1}{2}\right)-\left(\frac{1}{100}\log_e 1-0-0\right)\right\}$$

$=89.2668$ to 4 decimal places **(1 mark)**

Since the path is 2 metres wide and does not slope from side to side, then the volume of clean fill under the path is $89.2668\times 2\text{ metre}^3=178.53\text{ metre}^3$ correct to 2 decimal places.

(1 mark)
Total 16 marks