Prepared for and published by the Incorporated Association of Registered Teachers of Victoria

YEAR 12
IARTV TEST — OCTOBER 2000
MATHEMATICAL METHODS Units 3 and 4
EXAMINATION 1 — ANSWERS & SOLUTIONS

SECTION A : MULTIPLE CHOICE QUESTIONS

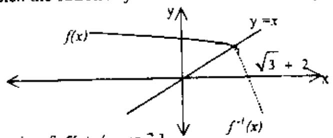
1. B	12. A	23. E
2. D	13. D	24. A
3. A	14. E	25. B
4. C	15. D	26. B
5. E	16. B	27. C
6. D	17. A	28. E
7. E	18. D	29. E
8. D	19. C	30. A
9. E	20. D	31. C
10. C	21. D ?	32. A
11. B	22. B	33. A

SECTION B: SHORT ANSWER QUESTIONS

QUESTION 1

$f:(-\infty,2] \to R$ where $f(x) = 2 + \sqrt{3-x}$

a) Sketch the **function** f and the **inverse function** f^{-1}



b) Domain of $f(x)$: $(-\infty,3]$
Range of $f(x)$: $[2,\infty)$

c) Domain of $f^{-1}(x)$: $[2,\infty)$
Rule of $f^{-1}(x)$:

$f^{-1}\,[2,\infty) \to R$ where $f^{-1}(x) = -(x-2)^2+3$

QUESTION 2

Find $\{x : \sqrt{3} + 2\cos 2x = 0, -\pi \le x \le \pi\}$

$\cos 2x = \dfrac{-\sqrt{3}}{2}$, where $\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$

4 solutions $\{-2\pi \le 2x \le 2\pi\}$ $2x = \dfrac{-7\pi}{6}, \dfrac{-5\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$

$x = \dfrac{-7\pi}{12}, \dfrac{-5\pi}{12}, \dfrac{5\pi}{12}, \dfrac{7\pi}{12}$

QUESTION 3

Derivative of $\log_e(x^2-2) = \dfrac{2x}{x^2-2}$

Therefore $\int \dfrac{6x}{x^2-2}\,dx = 3\log_e(x^2-2)$

QUESTION 4

$P(n) = -n(2n^2 - 900) = -2n^3 + 900n$

a) For maximum no. of employees $P'(n) = 0$
$P'(n) = -6n^2 + 900$
$-6n^2 + 900 = 0$

$n = \sqrt{\dfrac{900}{6}} = 12.25$ i.e 12 employees.

b) Maximum profit $= P(12)$
$= -2(12)^3 + 900(12)$
$= \$\,7{,}344$

QUESTION 5

Mean value $\bar{x} = 7.15$ am
Standard deviation $\sigma = 5$ minutes

Probability that Poppy's tram will arrive at least 10 minutes earlier:

$Z = \dfrac{x-\mu}{\sigma} = \dfrac{-10}{5} = -2$

$\Pr(Z < -2) = 1 - \Pr(Z<2)$
$= 1 - 0.9772 = 0.0228$

QUESTION 6

X~ Bi (n = 5 , p = 0.60)
q = 1-p = 1- 0.60 = 0.40

Probability that out of 5 people chosen at random at least 1 will be in favour of the GST:

$\Pr(X \ge 1) = 1 - \Pr(X = 0)$

$\Pr(X = 0) = {}^5C_0 q^5 p^0 = 0.40^5 = \dfrac{32}{3125} \approx 0.01$

$\Pr(X \ge 1) = 1 - \dfrac{32}{3125} = \dfrac{3093}{3125} \approx 0.99$

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YEAR 12
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MATHEMATICAL METHODS — EXAMINATION 2 (ANALYSIS TASK)
ANSWERS & SOLUTIONS

Question 1.

a) $c'(t) = \cos t - \sqrt{3}\sin t$

b) Turning points when $c'(t) = 0$,

$\cos t = \sqrt{3}\sin t$, $\tan t = \dfrac{1}{\sqrt{3}}$, $t = \dfrac{\pi}{6}, \dfrac{7\pi}{6}$

Coordinates $(\dfrac{\pi}{6}, 2)$, $(\dfrac{7\pi}{6}, -2)$

c) $(0, \sqrt{3})$

d) t – intercepts occur when $c(t) = 0$,

$\tan t = -\sqrt{3}$, $t = \dfrac{2\pi}{3}, \dfrac{5\pi}{3}$.

e) see ~~over page~~. below

f) A= amplitude = 2
B= 1
C= π/6

g) *solving* $1 = 2\cos(t - \dfrac{\pi}{6})$,

$t = \dfrac{\pi}{2}, \dfrac{11\pi}{6}$.

h) $\dfrac{2}{3}$

i) $\dfrac{4}{\pi}$

Question 2

a) $t = 2$, $P = 4.84$

b) $t = 10^{\frac{1}{15}(12-P)} - 1$

c) $t = 10^{0.6} - 1 = 2.98 \approx 3$ months

d) $10^y = x$, $\log_e(10^y) = \log_e x$

$y\log_e 10 = \log_e x \Rightarrow$ *result*

$\dfrac{dy}{dx} = \dfrac{1}{x\log_e 10}$

e) $\dfrac{dP}{dt} = \dfrac{-15}{(1+t)\log_e 10}$

f) when $t = 2$, $\dfrac{dP}{dt} = -2.17$

P is reducing by \$2170 per month after 2 months.

g) $t = 1.17$

h) $t = 5.31$ *months*

i) *Total bprofit* $= \int_0^7 P\,dt = 21.23$, ie. \$21,230.

Question 3

a) i) \$533.33
ii) \$300

b) $C = \begin{cases} \dfrac{8000}{n}, & n \le 20 \\ 400, & 20 \le n \le 30 \\ 700 - 10n, & 30 \le n \le 50 \end{cases}$ $n \in N$

c) $30 + x$

d) $R = n(700 - 10n) = (x + 30)(400 - 10x)$

$R = 12000 + 100x - 10x^2$

e) $x \in [0,20]$ and $x \in N$

f) $\dfrac{dR}{dx} = 0 = 100 - 20x \Rightarrow x = 5$

Thus 35 passengers maximises R, the receipts.

g) R contains points on a negative quadratic function and so R achieves a maximum.

Question 4

a) $\dfrac{28}{55}$

b) $\dfrac{63}{64}$

c)

i) 0.067

ii) 0.061

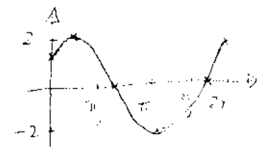
iii) 0.988

d) 108.42

e)

i) 0.871

ii) 0.129



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Q1 normal distribution

$\mu = 43 \quad \sigma = 6$

Use invNorm $(0.025, 43, 6)$
this finds the value for which 2.5% falls below.

$\Rightarrow 31.24.$

Use invNorm $(0.975, 43, 6)$
this finds the value for which 97.5% fall below (ie 2.5% is outside)

$\Rightarrow 54.76$

$\therefore \quad 31 \leq x \leq 55$

B.

Q2 $f(x) = \log_e x - \log_e(x^2 - x)$

$\dfrac{d \log_e x}{dx} = \dfrac{1}{x}.$

$\dfrac{d \log_e g(x)}{dx} = \dfrac{g'(x)}{g(x)} = \dfrac{2x-1}{x^2-x}.$

$\therefore f'(x) = \dfrac{1}{x} - \dfrac{2x-1}{x^2-x}.$

$= \dfrac{1}{x} - \dfrac{2x-1}{x(x-1)}$

$= \dfrac{x-1 - (2x-1)}{x(x-1)}$

$= \dfrac{x-1-2x+1}{x(x-1)}$

$= \dfrac{-\cancel{x}}{\cancel{x}(x-1)}$

$= \dfrac{-1}{x-1} \quad \therefore$ D

Q3/ Maximal domain is the maximum possible domain in the set of real numbers.

$y = \dfrac{2}{\sqrt{-6+x}}.$

Only has real values when

$\sqrt{-6+x} > 0$ it can't be $= 0$ $\because \frac{2}{0}$ is undefined

$\therefore \quad -6+x > 0$

$x > 6$

$\therefore (6, \infty)$

$\therefore$ A

Q4/ $r = 7$cm.

$r_{new} = 7 + h$

Surface Area $= 4\pi r^2$

$f(x+h) = f(x) + hf'(x)$ $\boxed{\text{page 242 of text}}$

The increase in the surface area will be $f(x+h) - f(x) = hf'(x)$

$= h \times 8\pi r$

When $r = 7$

$= 56\pi h.$

C.

Q5/ $y = 3 + \dfrac{2}{2x-1}$

will have a vertical asymptote when $2x-1=0$

ie $x = \frac{1}{2}.$

When $x \to \infty$

$\dfrac{2}{2x-1} \to 0$

$\therefore y \to 3$

$\therefore$ horizontal asymptote is $y=3.$

$\therefore \quad x = \frac{1}{2}, y = 3$

E

2000 MATH 1

Q6 If $y = (5x^3 - 3x)^5$

then $\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$

if $u = (5x^3 - 3x)$

then $y = u^5$.

$\therefore \frac{dy}{dx} = 5u^4 \times (15x^2 - 3)$

$= 5(5x^3 - 3x)^4 \, 3(5x^2 - 1)$

$= 15(5x^3 - 3x)^4(5x^2 - 1)$

$= (75x^2 - 15)(5x^3 - 3x)^4$

$= D$

Q7 $x^4 - 3x^3 + 9x^2 - 27x + 81$

$\Rightarrow$ 3^0 $(-3)^1$ $(-3)^2$ $(-3)^3$ $(-3)^4$

x^4 x^3 x^2 x x^0

This can be expressed as the

$\sum_{i=0}^{4} (-3)^i x^{4-i}$

$\therefore$ E

Q8 $y = b^2x^2 - x^3$

$= x^2(b^2 - x)$ (factor squared)

This graph touches at $x = 0$.

and intersects at $x = b^2$ (linear factor)

$\therefore$ D

Q9 $y = e^{a(x-b)}$

When $x = 0$ $y = e^2$

$\therefore e^2 = e^{-ab}$

$\therefore -ab = 2$.

When $x = 2$ $y = 1$.

$1 = e^{a(2-b)}$

Q9 (cont.) $1 = e^0 = e^{a(2-b)}$

$\therefore a(2-b) = 0$.

either $a = 0$ or $b = 2$

impossible since $-ab = 2$.

If $-ab = 2$ $a = -1$.

$\therefore y = e^{-(x-2)}$

$\therefore$ E

Q10 $f'(x) = 3x^2 - \frac{2}{x+2}$

$\therefore f(x) = \int 3x^2 - \frac{2}{x+2}$

$= \frac{3x^3}{3} - 2\log_e(x+2) + c$.

$f(-1) = -1 = (-1)^3 - 2\log_e 1 + c$

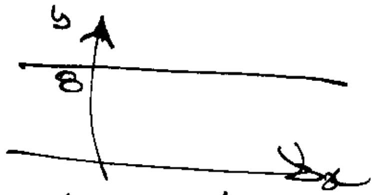
$-1 = -1 - 0 + c$

$\therefore c = 0$

$\therefore f(x) = x^3 - 2\log_e(x+2)$

C

Q4 $f(x) = 8$ is a straight line



This is not one-to-one it is many-to-one.

$\therefore$ B

Q12 $f(x) = -3\sin(2x + \pi)$

$= -3\sin 2(x + \frac{\pi}{2})$

period $= \frac{2\pi}{2} = \pi$.

the $\frac{\pi}{2}$ moves to the right

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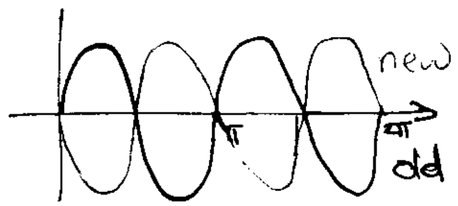
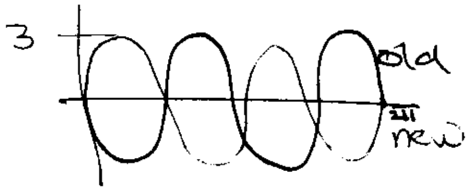
Q12 (cont.)



The (−3) reflects the graph in the x axis



∴ A

Q13 $f(x) = \cos x + 2$

The domain of the inverse f^{-1} is the range of f.

∴ range of $f = [1, 3]$

∴ domain $f^{-1} = [1, 3]$

= D

Q14 $e^{2x} - 6e^{x} = -5$.

$(e^{x})^2 - 6(e^{x}) + 5 = 0$.

Let $a = e^{x}$

$a^2 - 6a + 5 = 0$.

$(a-5)(a-1) = 0$

$a = 5$ or $a = 1$.

If $a = 5 = e^{x}$

$\log_e 5 = x$.

If $a = 1 = e^{x}$

$\log_e 1 = x$

$x = 0$.

∴ The solution set is

$\{0, \log_e 5\}$.

∴ E

Q15. $f(x) = 2x^2 + 1$

$f(2) = 2(2)^2 + 1 = 9$

$f(2+h) = 2(2+h)^2 + 1$

$= 2(4 + 4h + h^2) + 1$

$= 8 + 8h + 2h^2 + 1$

$= 8h + 2h^2 + 9$

$$\therefore \frac{f(2+h) - f(2)}{h} = \frac{2h^2 + 8h + 9 - 9}{h}$$

$$= \frac{h(2h+8)}{h}$$

$= 2h + 8 \quad (h \neq 0)$

∴ D

Q16 The period is the length along the horizontal axis, until the graph repeats itself.

In this case the period $= \pi$

$$\text{The amplitude} = \frac{\max - \min}{2} = \frac{1 - -5}{2}$$

$= 3$

∴ B

Q17 $\int e + (3+4x)^2 \, dx$

$= \int e \, dx + \int (3+4x)^2 \, dx$.

$$= ex + \frac{(3+4x)^3}{3 \times 4} + c$$

$$= ex + \frac{(3+4x)^3}{12} + c.$$

∴ A.

Q18 $y = e^{2\cos\frac{x}{2}}$

Let $u = 2\cos\frac{x}{2}$. ∴ $y = e^{u}$.

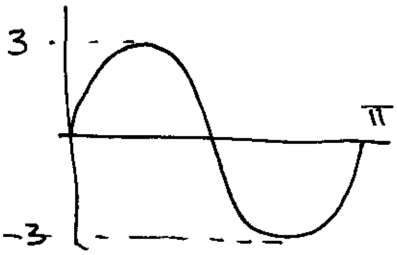
$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$= e^{u} \times 2 \times \frac{1}{2}\left(-\sin\left(\frac{x}{2}\right)\right)$

$= -e^{2\cos\frac{x}{2}} \sin\frac{x}{2}$ D

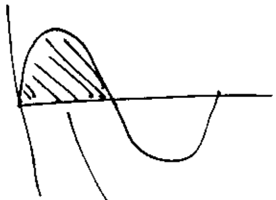
2000 HARTV 1

Q19 The graph of $y = 3\sin 2x$, $0 \le x \le \pi$ is



Plot the graph on your calculator, and then use $\boxed{\text{2nd}}$ $\boxed{\text{CALC}}$ $\boxed{7}$ to find the area under the graph.

Use "0" as your lower bound and $\frac{\pi}{2}$ as your upper bound.



area = 3 units^2

$\therefore$ the total area bounded between $x = 0$ and $x = \pi$ and the equation is $3 \times 2 = 6$ sq units.

$\therefore$ C

If you try to use an upper bound of π, with "0" lower bound, the calculator will show the area to be zero.

Q20 $f(x) = 1 + 2e^{-x}$

as $x \to \infty$

$$e^{-x} = \frac{1}{e^x} \to \frac{1}{\infty} \to 0$$

$\therefore$ as $x \to \infty$ $f(x) \to 1$

as $x = 0$; $1 + 2e^0 = 3$

$\therefore f(x) = 3$.

as $x \to -\infty$ $1 + 2e^{\infty} \to \infty$

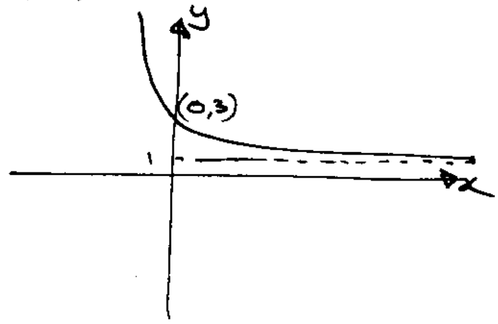
$f(x) \to \infty$

Q20 (cont.)

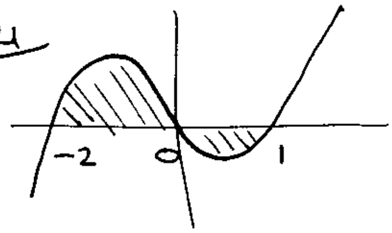
$\therefore$ The range of the function is $(1, \infty)$

$\therefore$ D

The graph looks like



Q21



Use $\boxed{\text{2nd}}$ $\boxed{\text{CALC}}$ $\boxed{7}$ to find the area between −2 or 0, and then 0, and 1.

<u>DO NOT</u> find the area using the domain $[-2, 1]$

The area for $[-2, 0] = 2\frac{2}{3}$

The area for $[0, 1] = 0.4167$

$= \frac{5}{12}$

$$\therefore 2\tfrac{2}{3} + \tfrac{5}{12}$$

$$= 2\tfrac{8}{12} + \tfrac{5}{12}$$

$$= 3\tfrac{1}{12}$$

$$= 3.08 \text{ units}^2$$

$\therefore$ D

1ARTV 2000 (1)

Q22 $y = \frac{5-x^2}{3x}$.

gradient function $= \frac{dy}{dx}$

$y = \frac{5}{3x} - \frac{x^2}{3x}$

$= \frac{5}{3x} - \frac{x}{3}$

$= \frac{5}{3}x^{-1} - \frac{x}{3}$

$\frac{dy}{dx} = -\frac{5}{3}x^{-2} - \frac{1}{3}$

$y'(1) = -\frac{5}{3}(1)^{-2} - \frac{1}{3}$

$= -\frac{5}{3} - \frac{1}{3}$

$= -2$.

$\therefore$ The gradient of the normal

$= \frac{-1}{(-2)} = \frac{1}{2}$.

(since $m_1 m_2 = -1$)

$\therefore$ equation of normal is

$(y - y_1) = m(x - x_1)$

$(y - \frac{4}{3}) = \frac{1}{2}(x - 1)$

$y - \frac{4}{3} = \frac{1}{2}x - \frac{1}{2}$

$y - \frac{4}{3} + \frac{1}{2} = \frac{1}{2}x$

$y - \frac{5}{6} = \frac{1}{2}x$

$\therefore$ B

Q23 $2\sin x - 1 = 0$ $\quad \frac{\pi}{2} \le x \le 2\pi$

$\therefore \sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$ ← too large. not in domain

not in domain

$\therefore \frac{5\pi}{6}$

$\therefore$ E

Q24 $y = \frac{5x^2}{e^{2x}}$

at maximum $\frac{dy}{dx} = 0$.

Let $y = 5x^2 e^{(-2x)}$
$\quad u \quad v$.

$\therefore \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$= e^{-2x}\,10x + (5x^2)(-2e^{-2x})$

$= e^{-2x}(10x - 10x^2)$

for $\frac{dy}{dx} = 0 \Rightarrow 10x - 10x^2 = 0$

$10x(1 - x) = 0$

$\therefore x = 1$.

$\therefore$ A

Q25 $P(t) = 0.3t^3 + 0.2t^2$.

$P(3) = 0.3(3)^3 + 0.2(3)^2$

$= 9.9$

$P(1) = 0.3(1)^3 + 0.2(1)^2$

$= 0.5$

$\therefore \frac{P(3) - P(1)}{3 - 1} = \frac{9.4}{2} = 4.7$ m/s.

$\therefore$ B

Q26 $\mu = 7.7$ $\quad$ ~~σ~~ VAR $= 0.25$

$\therefore \sigma = \sqrt{0.25}$

$= \frac{1}{2}$

$Pr(X \ge 6.6)$.

Use the calculator:

normal cdf (lower, upper, μ, σ)

becomes (6.6, 1000, 7.7, 0.5)

$\therefore Pr = 0.9861$

$\therefore$ B

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Q27 $\sum_0^5 Pr(Z=z) = 1$. ①

and $\sum_0^5 z\, Pr(Z=z) = 2.38$. ②

Use this to create two simultaneous equations to find k+m.

①∴ $0.05+0.25+k+0.3+m+0.03=1$

$0.63+k+m=1$

$k+m=0.37$. —③

② $1\times0.25+2k+3\times0.3+4m+5\times0.03=2.38$

$0.25+2k+0.9+4m+0.15=2.38$

$2k+4m=1.08$ —④.

④−③×2

$2k+2m=0.74$

$2m=0.34$

$m=0.17$.

∴ $k=0.20$

∴ C

Q28 This is a binomial distribution because the student either gets a coupon or doesn't.

$n=7$ $p=\frac{1}{5}$ $q=\frac{4}{5}$

∴ ${}^7C_3(0.2)^3(0.8)^4$

Since the number of successes is 3,

∴ E

Q29 This requires a failure, fail, fail, success

$= Pr(\text{failure})^3\, Pr(\text{success})$

$=(0.65)^3\times0.35$.

$=0.9612$

= E

Q30 $E(X)=5$.

then $E(2X-4)=2E(X)-4$

$=10-6$

$=6$.

∴ A

Q31 $n=15$. $x=6$.

$p=0.85$ $q=0.15$

∴ ${}^{15}C_6(0.85)^6(0.15)^9$

∴ C.

Q32 Since there is no replacement this is a hypergeometric distribution

Pop = N = 12 Sample n = 3

8 caramel 1 caramel +2p

4 peppermint. or 2c + 1p.

$$\therefore \frac{\binom{8}{1}\binom{4}{2}+\binom{8}{2}\binom{4}{1}}{\binom{12}{3}}$$

∴ A.

Q33 $Pr(\text{Success})=0.74$

$n=15$ $x\geq1$.

∴ $1-Pr(X=0)$

$Pr(X=0)=\binom{15}{0}(0.74)^0(0.26)^{15}$.

∴ $Pr(x\geq1)=1-\binom{15}{0}(0.74)^0(0.26)^{15}$

= A.