Resource material to assist in the implementation of VCE Mathematical Methods Units 3 and 4

May 2000

Introduction

Mathematical Methods Units 3 and 4 is a course consisting of the following areas of study: Coordinate geometry, Circular (trigonometric) functions, Calculus, Algebra, and Statistics and probability. These must be covered in progression from Unit 3 to Unit 4, with an appropriate selection of content for each of Unit 3 and Unit 4. The selection of content from the Coordinate geometry, Circular (trigonometric) functions, Calculus and Algebra areas of study should be constructed so that there is a development in the complexity and sophistication of problem types and mathematical processes used (modelling, transformations, graph sketching and equation solving) in application to contexts related to these areas of study. There should be a clear progression of skills and knowledge from Unit 3 to Unit 4 in each area of study. This structure provides teachers with considerable flexibility in devising their intended course implementation. The following material describes one such possible implementation, and some sample assessment tasks.

Sample course plan

This course plan includes a possible sequencing of content for Mathematical Methods Units 3 and 4, and includes a suggested schedule for school-assessed coursework. The scheduling of coursework assessment should allow time for teachers to assess student work and provide feedback as part of the school reporting process. Students are expected to demonstrate achievement of the three outcomes through the completion of the coursework tasks. The course implementation plan shown is an example of how this might be achieved. Teachers might expect students to complete other assessment activities as part of the teaching and learning process; however, only the tasks which are to be reported for school-assessed coursework have been incorporated in this sample plan.

Term 1			Term 2		
Week	**Content**	**Assessment**	**Week**		**Assessment**
1	Coordinate geometry		1	Circular functions	
2	Coordinate geometry		2	Circular functions	
3	Coordinate geometry		3	Circular functions	
4	Coordinate geometry		4	Circular functions	Test
5	Calculus		5	Application task	
6	Calculus		6		
7	Calculus		7	Exponential and logarithmic functions	
8	Calculus	Test	8	Exponential and logarithmic functions	
9	Exponential and logarithmic functions		9	Applications of differentiation	
10	Exponential and logarithmic functions		10	Applications of differentiation	

Term 1			Term 2		
Week	**Content**	**Assessment**	**Week**		**Assessment**
1	Applications of differentiation		1	Examination preparation	
2	Integration and Areas under Curves		2		
3	Integration and areas under curves		3		
4	Integration and areas under curves				
5	Analysis task 1				
6	Statistics and probability				
7	Statistics and probability				
8	Statistics and probability				
9	Statistics and probability				
10	Analysis task 2				

In this course plan, all coursework assessment for Unit 3 would be completed by week 6 of Term 2. The first test would be on the Coordinate geometry and Calculus areas of study. This would take place during week 8 of Term 1, after the completion of the Coordinate Geometry and Calculus work. The completion of the first test during week 8 gives schools with multiple classes the opportunity to moderate work; for example, by sample moderation of work within selected score ranges, and to provide students with feedback on their progress prior to the end of Term 1. The second test, on exponential, logarithmic and circular functions, would be scheduled for week 4 of Term 2, just prior to the application task.

The application task has been scheduled within weeks 5 and 6 of Term 2. In completing this task by the end of week 6, schools should be able to provide feedback on student achievement by the end of Unit 3. In this course implementation plan, both tests have been scheduled before the application task, so that students might have the opportunity to review work on calculus and functions material prior to commencing the application task. Alternatively, the application task could be scheduled for early in Term 2, with the second test later in the term.

In Unit 4, the analysis tasks have been scheduled at the completion of two major areas of work: anti-differentiation and integration, and statistics and probability. Term 4 has been designated as examination preparation time.

Student access to an approved graphics calculator will be assumed by the setting panel for each examination.

Students will need to be able to use analytical, numerical or graphical approaches to tackle questions, consistent with the mathematics study design. Where a numerical answer to a question, or part of a question, is required, this may be obtained using any of these approaches is appropriate and applicable. Students will need sufficient opportunity for practise with each of these approaches to identify which one is likely to be more effective or efficient in a particular context. There may well be questions where no readily accessible analytical approach or no analytical approach, is available, in other instances it may be the case that only an analytical approach will be suitable. Where a numerical answer is required, students should provide the answer to the specified degree of accuracy.

Sample application task

The following sample task has been devised using the suggested theme 'Fitting function to data' and the starting point 'Determining curves'.

Component 1

Consider the following set of information:

Two points on a curve are (0,2) and (3,–1), and the curve has gradient $-\frac{1}{2}$ at the point where $x = 1$.

a. **i** Could the curve be a straight line? Try to find the equation of a straight line which satisfied the information.

ii How many pieces of the given information are needed to 'determine' a straight line?

b. **i** Find the equation of a parabola which satisfied the given information.

ii Do the three pieces of given information determine a parabola?

Component 2

a. Consider a cubic function with equation $f(x) = ax^3 + bx^3 + cx + d$. Use the given information from **Component 1** to:

i find the value of d

ii express b and c in terms of a

iii find the equation of the cubic function, with coefficients in terms of a

iv Does the given information determine a cubic function? Explain your answer.

b. Find the condition on a for which there are

i no stationary points

ii one stationary point

iii two stationary points.

c. Illustrate each of the 4 cases of b for suitably chosen values of a, and also consider when a is positive and when a is negative for **iii**.

Component 3

Consider a parabola. What properties must the two points and a gradient at a third x value have if a parabola can be found to satisfy that information? When is a parabola determined by these three pieces of information?

Mapping criteria and outcomes with the sample task

The following solution notes and mapping illustrate one way in which the criteria and outcomes can be linked to developing a marking scheme for the sample task.

Component 1

1. **a.** Using $(x_1, y_1) = (0,2)$ and $(x_1,y_2) = (3,–1)$, gradient $= \frac{-1-2}{3-0} = -1$

$y–y_1 = m\,(x–x_1)$ $\therefore$ $y = 2–x$, $\therefore$ the gradient is not $-\frac{1}{2}$ when $x = 1$

Using $(x_1, y_1) = (0,2)$ and $\frac{dy}{dx} = \frac{1}{2}$, $y–2 = -\frac{1}{2}(x–0)$ $\therefore$ $y = 2-\frac{1}{2}x$.

When $x = 3$, $y = \frac{1}{2}$, $\therefore$ the line does not go through (3, –1)

Using $(x_2, y_2) = (3, –1)$ and $\frac{dy}{dx} = -\frac{1}{2}$, $y + 1 = -\frac{1}{2}(x–3)$ $\therefore$ $y = \frac{1}{2} - \frac{1}{2}x$

When $x = 0$, $y = \frac{1}{2}$, $\therefore$ the line does not go through (0,2)

The curve could not be a straight line.

ii Any two pieces of the information are needed to determine a straight line, but the third is incompatible.

b. **i** $y = ax^2 + bx + c$, $\frac{dy}{dx} = 2ax + b$

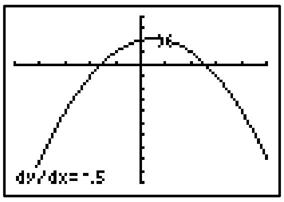
When $x = 0$, $y = 2$ $\therefore$ $c = 2$ $\therefore$ $y = ax^2 + bx + 2$

When $x = 3$, $y = –1$ $\therefore$ $–1 = 3a + b$ [1]

When $x = 1$, $\frac{dy}{dx} = -\frac{1}{2}$, $\therefore$ $-\frac{1}{2} = 2a + b$ [2]

Subtract [2] from [1] $\therefore$ $a = -\frac{1}{2}$

Substitute in [1] $\therefore$ $–1 = 3\left(-\frac{1}{2}\right) + b$ $\therefore$ $b = \frac{1}{2}$ $\therefore$ $y = -\frac{1}{2}x^2 + \frac{1}{2}x + 2$



ii Yes, the three pieces of information do determine a parabola.

Component 2

2. **a.** **i** $d = 2$

ii $y = ax^3 + bx^2 + cx + 2 \quad \frac{dy}{dx} = 3ax^2 + 2bx + c$

At (3,–1), $-1 = 9a + 3b + c$ [1]

When $x = 1 \quad \frac{dy}{dx} = -\frac{1}{2} \therefore -\frac{1}{2} = 3a + 2b + c$ [2]

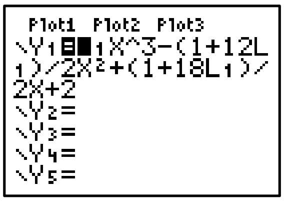
Subtract [2] from [1] $\therefore -\frac{1}{2} = 6a + b \therefore b = -\frac{1 + 12a}{2}$

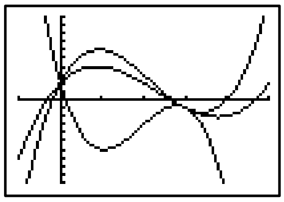
Substitute in [1] $\therefore \; -1 = 9a + 3\left(-\frac{1 + 12a}{2}\right) + c \therefore c = \frac{1 + 18a}{2}$

iii $y = ax^3 - \frac{1 + 12a}{2} x^2 + \frac{1 + 18a}{2} x + 2$

iv No, the given information does not determine a cubic function, because the cubic function is not uniquely defined. The constant *a* can take real value except 0. In the case $a = 0$, the equation to the parabola is found.

The cubic function can be entered as shown in L1 as a parameter. Values for L1 entered in the Home Screen and the graphs shown.





b. For y = $ax^3 + bx^2 + cx + 2$

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

Stationary points occur where $\frac{dy}{dx} = 0$

Consider the discriminant

$\Delta = 4b^2$

$= 4\left(\frac{1 + 12a}{2}\right) - 12a\left(\frac{1 + 18a}{2}\right)$

$= 1 + 24a + 144a^2 - 6a - 108a^2$

$= 36a^2 + 18a + 1$

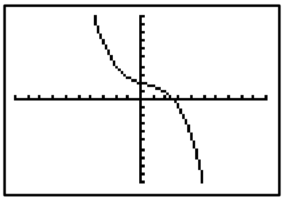
$\Delta = 0$ implies $\quad a = \frac{-18 \pm \sqrt{180}}{72}$

$= \frac{-18 \pm 6\sqrt{5}}{72}$

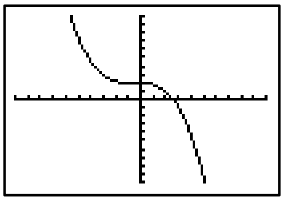
$= \frac{-3 \pm \sqrt{5}}{12}$

i no stationary points when $\frac{-3-\sqrt{5}}{12} < a < \frac{-3+\sqrt{5}}{12}$

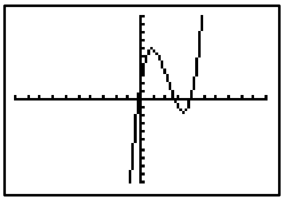
note: $\frac{-3-\sqrt{5}}{12} \approx -0.4363$ and $\frac{-3+\sqrt{5}}{12} \approx -0.0637$

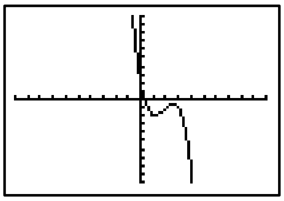


ii one stationary point when $a = \frac{-3-\sqrt{5}}{12}$ or $a = \frac{-3+\sqrt{5}}{12}$



iii two stationary points when $a > \frac{-3-\sqrt{5}}{12}$ or $a < \frac{-3-\sqrt{5}}{12}$





Component 3

3. Let the two points be (x_1, y_1), (x_2, y_2) where $x_1 \neq x_2$, and $\frac{dy}{dx} = m$ when $x = x_3$

and the equation of the parabola be given by $y = ax^2 + bx + c$, $a \neq 0$.

At (x_1, y_1), $y_1 = ax_1^2 + bx_1 + c$ $\boxed{1}$

At (x_2, y_2), $y_2 = ax_2^2 + bx_2 + c$ $\boxed{2}$

Now $\frac{dy}{dx} = 2ax + b$

At $x = x_3$, let $\frac{dy}{dx} = m \therefore 2ax_3 + \mathrm{b} = m$ $\boxed{3}$

Rearrange $\boxed{3}$ to make b the subject.

$$b = m–2ax_3 \boxed{4}$$

Subtracting $\boxed{2}$ from $\boxed{1}$ gives

$y_1 – y_2 = a(x_1^2 – x_2^2) + b(x_1 – x_2)$

$\therefore b = \dfrac{y_1 - y_2 - a\left(x_1^2 - x_2^2\right)}{x_1 - x_2}$

$\therefore b = \dfrac{y_1 - y_2}{x_1 - x_2} - a\,(x_1 + x_2)$ $\boxed{5}$

Equating $\boxed{4}$ and $\boxed{5}$

$$m–2ax_3 = \frac{y_1 - y_2}{x_1 - x_2} – a\,(x_1 + x_2)$$

$$\therefore m = \frac{y_1 - y_2}{x_1 - x_2} + 2ax_3 – a\,(x_1 + x_2)$$

$$\therefore a = \frac{m - \dfrac{y_1 - y_2}{x_1 - x_2}}{2x_3 - x_1 - x_2}$$

* If $m = \dfrac{y_1 - y_2}{x_1 - x_2}$, then $a = 0$

This is the case when the given gradient is the gradient of the chord between $(x_1–y_1)$ and $(x_2–y_2)$, and a straight line can be found, not the parabola.

If $x_2 + x_1 = 2x_3$

then $x_3 = \dfrac{x_1 - x_2}{2}$. Consider this case more fully.

If x_3 is average of x_1 and x_2. a value of a cannot be found using equation $\boxed{6}$

If $x_3 = \dfrac{x_1 - x_2}{2}$

then from $\boxed{4}$ $b = m–2a\dfrac{x_1 - x_2}{2}$ $\boxed{7}$

$= m–a\,(x_1 + x_2)$

and from $\boxed{5}$ $b = \dfrac{y_1 - y_2}{x_1 - x_2} – a\,(x_1 + x_2)$ $\boxed{8}$

Equating $\boxed{7}$ and $\boxed{8}$

$m–a = a\,(x_1 + x_2)\dfrac{y_1 - y_2}{x_1 - x_2} – a\,(x_1 + x_2)$

$\therefore m = \dfrac{y_1 - y_2}{x_1 - x_2}$

If $y_1 = y_2$, then $m = 0$ and $b = -a(x_1 + x_2)$ and x_3 is the x coordinate of the vertex of the parabola.

If $y_1 \neq y_2$, then $m \neq 0$ and a straight line can be found.

If the points are (1.5, 6) (2, 8) and $\frac{dy}{dx} = 3$ when $x = 4$

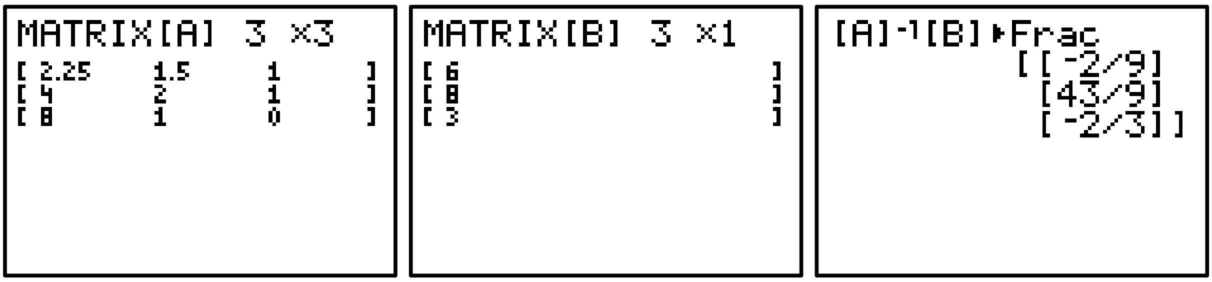
The equations are

$6 = 2.25a + 1.5b + c$

$8 = 4a + 2b + c$

$3 = 8a + b$

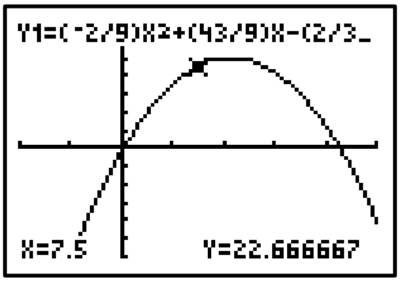
A graphics calculator can be used to solve these using matrices as shown in the screens below.



The equation of the parabola is

$$\frac{-2}{9}x^2 + \frac{43}{9}x - \frac{2}{3}$$

The graph is as shown.



Outcome 1

Criterion 1 (3 marks) **Appropriate use of mathematical conventions, symbols and terminology.**

Correct use of algebra, representation of graphs correct.
Applied in all three components

3 correct notation throughout
2 a graduation
1 a graduation
0 incorrect notation used throughout, poor graphs and definition of parameters

Criterion 2 (6 marks) **Definition and explanation of key concepts.**

Component 1 (2 marks) quadratic and linear requirements
Component 2 (2 marks) cubic
Component 3 (2 marks)

Criterion 3 (6 marks) **Accurate application of mathematical skills and techniques.**

Component 1 (2 marks)
Component 2 (2 marks)
Component 3 (2 marks)

Outcome 2

Criterion 1 (4 marks) **Identification of important information, variables and constraints.**

Applied in all three components
Component 1 (2 marks)
Component 2 (2 marks)
Component 3 (2 marks)

Criterion 2 (8 marks) **Application of mathematical ideas and contents from the specified areas of study.**

Component 1 (2 Marks) straight line and quadratics, concepts applied correctly
Component 2 (4 marks) use of differentiation, discriminant, graphic cubics
Component 3 (2 marks) use of differentiation, simultaneous equations

Criterion 3 (8 marks) **Analysis and interpretation of results.**

Component 1 (2 marks) 3 pieces if necessary and sufficient for parabola
Component 2 (3 marks) conditions for defining a parabola fully explained
Component 3 (3marks) conditions for defining a parabola fully explained

Outcome 3

Criterion 1 (2 marks) **Appropriate selection and effective use of technology.**

Choice of graphs to illustrate cases and examples
Possible use of algebra packages
Correct windows chosen to show relevant features
Exact results rather than approximations.
All of this applied throughout

Criteria (3 marks) **Application of technology.**

Use of technology to help explore the family of cubics in Component 2
Correct choices of values of *a*
Systematic use of graphs throughout

Sample test 1 Coordinate geometry

Multiple-choice section (10 marks)

1. For $f(x) = \dfrac{2}{x-1} + 3$ which of the following statements is not true:

A. The asymptotes are $x = 1$ and $y = 3$

B. The domain of the function is $R \setminus \{1\}$

C. Axis intercepts occur when $x = \dfrac{1}{3}$ and $y = 1$

D. There is only one asymptote at $x = 1$

E. As $x \to \infty$, $f(x) \to 3$

2. The graph of $f : R \rightarrow R$, where $f(x) = ax^2(b - x)^2$ where $a \in R^-$ and $b \in R^+$ has:

A. x intercepts at $x = a$ and $x = b$

B. x intercepts at at $x = 0$ and $x = b$

C. local minimum turning points at $x = a$ and $x = b$

D. local minimum turning points at $x = 0$ and $x = b$

E. local maximum turning points at $x = a$ and $x = b$

3. If $f : \mathrm{R}^+ \setminus \{0\} \rightarrow \mathrm{R}$ where $f(x) = \frac{1}{x} + 2$, then the rule and domain for f^{-1} are, respectively:

A. $f^{-1}(x) = \frac{1}{x-2}$ and $\mathrm{R} \setminus \{2\}$

B. $f^{-1}(x) = \frac{1}{x} - 2$ and $\mathrm{R} \setminus \{0\}$

C. $f^{-1}(x) = \frac{1}{x-2}$ and $\mathrm{R} \setminus \{0\}$

D. $f^{-1}(x) = \frac{1}{x} - 2$ and $\mathrm{R} \setminus \{2\}$

E. $f^{-1}(x) = \frac{1}{y} + 2$ and $\mathrm{R} \setminus \{0\}$

4. The relationship between two variables is shown in the table below.

x	1.1	2.3	2.7	3.9
y	13.012	33.932	48.639	152.210

If a is a positive constant, then the equation relating x and y is most likely to be of the form:

A. $y = \frac{a}{x^2}$

B. $y = ax^2$

C. $y = ax^{0.5}$

D. $y = \frac{a}{x}$

E. $y = ae^x$

5. The complete set of linear factors for $f(x) = x^4 - 16x^2$ is:

A. $\{0, 4, -4\}$

B. $\{x^2, (x^2 - 16)\}$

C. $\{x^2, (x - 4), (x + 4)\}$

D. $\{0, 0, 4, -4\}$

E. $\{x, x, (x - 4), (x + 4)\}$

Short-answer section

6. Consider the function $f(x) = 2\,log_e\,x - 3$

a. State why f^{-1} exists.

b. Write down the domain of f^{-1}.

c. Write down the rule for f^{-1}.

4 marks

7. State a set of transformations, in order, which will produce the graph of $y = \dfrac{-2}{(x+3)^2} + 4$ from the graph of $y = \dfrac{1}{x^2}$

2 marks

8. Consider the function f, with the rule $f(x) = x^4 - 3x^3 + 5x$

a. Sketch the graph of the gradient function.

b. Write down the interval (correct to two decimal places) within which the gradient function is positive.

4 marks

9. Consider the function, f with the rule $f(x) = 2x^{0.5} - 1$

a. Write down the domain and range of f

b. Find $f^{-1}(x)$.

c. Sketch the graph of the inverse function.

6 marks

10. Consider the function f, with the rule $f(x) = \dfrac{-2}{x+1} - 1$

a. Write down the domain and range of the function.

b. Find the exact points where the graph cuts the axes.

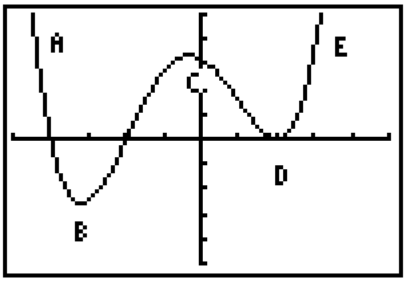
c. The graph passes through the point (m, 2.505). Find the value of m to two decimal places.

5 marks

Extended-response section

11. A car race is taking place on a track, the graph of which is shown below.

9 marks



The race starts at A and finishes at E. The track follows the curve with equation $y = a(x - b)(x - c)(x - d)(x - e)$ where $a, b, c, d, e > 0$

a. Let $a = 0.11$. Given that $(x - 2)$ and $(x - 4)$ are factors of y, find the other two linear factors.

b. Hence, state the exact points where the race track cuts the axes.

c. Find the average gradient of the race track between B and D, and between C and D.

d. The finish of the race is at the point E (k, 7.038). Find the value of k to two decimal places.

e. Hence, find the exact gradient of the race track at the finish point of the race.

f. The race officials decide to change the track because this track could be dangerous under very wet conditions. Vary the value of a and discuss the effect that this has on the track. Use your results to indicate what might be an appropriate value of a would be for the race track.

Mapping criteria and outcomes to components of the test

For a test, Outcomes 1 and 3 and the associated criteria should be considered. The criteria give an indication of the type of knowledge, skills and processes that students are likely to demonstrate through completion of the test. The test should include questions which require the use of technology in relation to Outcome 3. This test has a total of 40 marks (Outcome 1: Outcome 2 in a 30:10 ratio) and the final result would need to be re-scaled accordingly, to a coursework score out of 10, or the combined score on the two tests re-scaled to a coursework score out of 20.

The table below shows the relationship between questions, criteria and outcomes, in terms of anticipated likely response. It should be noted that this mapping is applied when devising the task, student responses would be marked according to the marks allocated as a consequence of this process. For this test, the multiple- choice questions have been linked to Criterion 2 of Outcome 1.

		Outcome 1			**Outcome 3**	
Question	**Available marks**	**Criterion 1**	**Criterion 2**	**Criterion 3**	**Criterion 1**	**Criterion 2**
1–5	10		**/10**			
6a	1		**/1**			
6b	1	**/1**				
6c	2			**/2**		
7	2	**/2**				
8a	2			**/1**	**/1**	
8b	2			**/1**		**/1**
9a	2			**/2**		
9b	2			**/2**		
9c	2					**/2**
10a	2	**/2**				
10b	2			**/2**		
10c	1				**/1**	
11a	2			**/2**		
11b	1		**/1**			
11c	1	**/1**				
11d	1				**/1**	
11e	1				**/1**	
11f	3					**/3**
	Total					
	Mark /40			**Coursework**	**Score /10**	

Sample calculus analysis task 1 a set of application-type questions

1. Consider the curve *y= 0.5* sin *(2x)* + cos *(x) 0* £ *x* £ *2*p

The area between the curve and the x-axis is going to be used for garden beds in a park. The three garden beds are going to be planted with different types of plants. The largest garden bed is to be planted with native plants, the smallest with roses and the other with small flowers. A gardener has to determine the area of each region to decide on the number of plants required.

a. Sketch the graph in the required domain.
2 marks

b. Write down the range for this function
1 mark

c. Find the axes intercepts for the curve.
1 mark

d. Find $\frac{dy}{dx}$
1 mark

e. Find the coordinates of the stationary points, giving values to three decimal places.
2 marks

f. Use calculus to determine the exact area of the garden bed closest to the y-axis.
4 marks

g. Determine the area of each of the other two garden beds.
2 marks

h. The cost of seeds is given in the following table:

Plant	Cost ($ / m^2)
Native	15
Roses	35
Flowers	18

Determine the cost of planting the garden beds.
1 mark

i. Another gardener suggests that a less-expensive design would be of the form:
y= 0.5 sin *(2x)* + cos *(x)* *0* £ *x* £ *2*p

Consider this new design and compare the costs of the two designs.
5 marks
Total marks 19

2. Lead contamination of grass is often attributed to leaded petrol used in motor vehicles. Lee's farm is next to a country road. The grass near the roadside is found to have a lead content which can be modelled by the formula

$$L = \frac{A}{x} + B$$

where *L* milligram per kilogram is the lead concentration in the grass,

x metres is the distance of the grass from the roadside

and *A* and *B* are constants.

a. The further away from the roadside the lead content of the grass is measured, the closer its value gets to 2 milligrams per kilogram. State the value of B.
1 mark

b. At a distance of 10 metres from the roadside, the lead content in the grass is measured at 50 milligrams per kilogram. Calculate the value of A.
1 mark

c. State the domain and range of L.
1 mark

d. Use your values for A and B to sketch a graph of L against x over a suitable domain. Explain why you chose this domain.
2 marks

e. Describe the key features of your graph.
1 mark

f. The maximum allowable content of lead in grass used for grazing sheep is 10 milligrams per kilogram. What is the least distance from the roadside that Lee can graze sheep in order to meet this requirement?
1 mark

g. A paddock on Lee's farm is a rectangle measuring 100 m by 150 m, with the road going along two adjacent sides. What percentage of the paddock is suitable for grazing sheep while meeting the requirement in part **f?** Give your answer to the nearest per cent.
2 marks

The pollution from the traffic also affects the rate at which the grass is growing. Lee finds that the density of the grass can be modelled by the formula

$$G = e^{0.01x}$$

where x metres is the distance of the grass from the roadside
and G kilogram per square metre is the density of the grass.
Lee decides that the overall concentration of lead in the grass, T milligrams per square metre, can be modelled by $T = L \cdot G$

h. Write down an expression for T in terms of x.
1 mark

i. Sketch a graph of T against x.
2 marks

j. What is the overall concentration of lead (T milligrams per square metre) in grass which is 50 metres from the roadside? Give your answer correct to one decimal place.
1 mark

k. Find an expression for the instantaneous rate of change of T with respect to x.
3 marks

l. At what rate is the overall concentration of lead in the grass changing, with respect to distance from the roadside, in grass which is 50 metres from the roadside? Give your answer correct to three decimal places.
1 mark

m. Compare your answer to **i.** with grass that is 80 metres from the roadside.
1 mark
(1999 Analysis task examination Question 1 modified)

Total marks 18

3. In a country town, it is decided that a new road should be built. It is decided that the road should follow the path whose equation is $y=(2x^2 - 3x)e^{ax}$ where $a > 0$.
The town can be represented on a grid where the railway line is along the x-axis and the post office is at the coordinate (2, 3). In each direction on the axes, let 1 unit represent 1 kilometre.

a. Find the value of a for which the road will pass through the post office. Give your answer correct to three decimal places.
2 marks

b. Sketch a graph of the road for this case, and describe the key features of the graph.
3 marks

Now consider the case where $a = 1$.

c. Sketch a graph for this value of *a*.
2 marks

d. Find the *x* coordinates where the road crosses the railway line.
2 marks

e. Use calculus to find the coordinates of the turning point for the graph. Give your answers correct to three decimal places.
4 marks

f. The town council wishes to develop the area bounded by the road and the railway line between the x coordinates where the road crosses the railway line. This area is to be made into a lake for native water birds. Find the values of *m* and *n* for which

$$\frac{d}{dx}\left\{\left(2x^2 + mx + n\right)e^x\right\} = \left(2x^2 - 3x\right)e^x$$

and hence find the exact area of the lake which is to be developed.
6 marks

g. Consider the original curve $y=(2x^2 - 3x)e^{ax}$ where $a > 0$. Choose suitable values for *a* and produce graphs to show the effect of varying *a*. Describe the effect that *a* has on the road.
4 marks

(1997 Examination analysis task Question 3 modified)

Total marks 23

Mapping of criteria and outcomes for analysis task 1

Outcome and criteria	**Available**				**Score**
Outcome 1	**24**	**Question 1**	**Question 2**	**Question 3**	
Criterion 1: Appropriate use of mathematical conventions, symbols and terminology	6	/2	/2	/2	
Criterion 2: Definition and explanation of key concepts	9	/3	/4	/2	
Criterion 3: Accurate application of mathematical skills and techniques	9	/3	/3	/3	
Outcome 2	**21**				
Criterion 1: Identification of important information, variables and constraints	6	/1	/1	/4	
Criterion 2: Application of mathematical ideas and content from the specified areas of study	6	/1	/2	/3	
Criterion 3: Analysis and interpretation of results	9	/3	/3	/3	
Outcome 3	**15**				
Criterion 1: Appropriate selection and effective use of technology	6	/2	/2	/2	
Criterion 2: Application of technology	9	/2	/2	/5	
Total	**60**	**/17**	**/19**	**/23**	

Sample calculus analysis task 2 (probability) an item response analysis for a set of multiple-choice questions

For each of the questions shown, students are required to explain the mathematical reasoning as to why particular multiple-choice options are correct or incorrect. Each multiple-choice option for each question must be discussed separately. Students should include any diagrams or working which helps to show their mathematical thinking when undertaking these problems. Any graphs should be drawn carefully, clearly labelled and annotated to demonstrate key features. It is anticipated that each question will require approximately half a page of explanation.

1. If X has a standard normal distribution, which one of the following is not true?

   A The mean, median and mode of Z are all equal.
   B The mean of Z equals 0 and the standard deviation of Z equals 1.
   C $\Pr(Z < -1) = 1 - \Pr(Z>1)$
   D $\Pr(Z < 0) = 0.5$
   E $\Pr(-2 < Z < 2) \approx 0.95$

2. The random variable X has a normal distribution with mean 11.3 and variance 4. The probability that X is less than 9.3 is equal to:

   A $1 - \Pr(Z < 1)$
   B $1 - \Pr(Z > 1)$
   C $\Pr(Z > 1)$
   D $\Pr(Z > -1)$
   E $1 - \Pr(Z < -1)$

3. A person has a 30% probability that they arrive at work late due to traffic. In a ten-day period, the probability they get to work late exactly eight times is:

   A ${}^{10}C_8\,(0.7)(0.3)^8$
   B ${}^{10}C_8\,(0.3)(0.7)^8$
   C ${}^{10}C_8\,(0.7)(0.3)^8 + (0.3)^{10}$
   D ${}^{10}C_8\,(0.7)(0.3)^8 + (0.7)^{10}$
   E $1 - (0.3)^{10}$

4. A survey of the students at a school found that 40% of students had a part-time job. The probability that at least 2 students out of a group of six have a part-time job is given by:

   A $1 - [\,{}^6C_0(0.4)^6 + {}^6C_1(0.6)(0.4)^5\,]$
   B $1 - {}^6C_0(0.6)^6$
   C ${}^6C_2(0.4)^2(0.6)^4$
   D $1 - [\,{}^6C_0(0.6)^6 + {}^6C_1(0.4)(0.6)^5\,]$
   E ${}^6C_2(0.6)^2(0.4)^4$

5. Consider the following discrete probability distribution:

y	1	2	4	5
$\Pr(Y = y)$	0.4	0.3	0.1	0.2

   The expectation, variance and 95% confidence interval for Y, correct to 2 decimal places, is:

   A 2.4, 0.58, $1.64 \le Y \le 3.16$
   B 2.4, .0.76, $1.24 \le Y \le 3.56$
   C 2.4, .0.76, $0.88 \le Y \le 3.92$
   D 2.4, .0.58, $1.24 \le Y \le 3.56$
   E 2.4, .0.58, $0.88 \le Y \le 3.92$

6. Which one of the following situations is most accurately characterised by a hypergeometric distribution model?

   A The number of kings obtained if a card is drawn from a deck of cards, replaced and then another card is drawn.
   B The number of times you hit a target in 4 shots if there is a 30% probability of hitting the target for any given shot.

C The number of tails when a fair coin is tossed 4 times.
D The number of females on a committee of 7 people from a collection of males and females.
E The number of rolls required on a die before a 6 is obtained.

7. From a committee of 10 people there are 4 people who support a particular proposal. What is the probability that a subcommittee of 3 will contain 2 people who support the proposal?

A 0.4×0.4
B $\dfrac{{}^4C_2\,{}^6C_1}{{}^{10}C_3}$
C ${}^3C_2(0.4)^2(0.6)^1$
D $10\,[\,{}^3C_2(0.4)^2(0.6)^1\,]$
E 3×0.4

8. For a particular binomial distribution with n independent trials, each with probability of success p, the mean and variance are 5 and $3\frac{3}{4}$ respectively. Which one of the following gives the correct values for n and p?

A n=20, $p=\frac{1}{5}$
B n=25, $p=\frac{1}{5}$
C n=25, $p=\frac{4}{5}$
D n=20, $p=\frac{1}{4}$
E n=20, $p=\frac{3}{4}$

(Facts, skills and applications task, 1996 question 28)

9. The random variable Z has a standard normal distribution with mean 0 and a standard deviation of 1. If $\Pr(Z > a) = 0.1977$, then the value of a is, correct to 2 decimal places.

A –0.85
B –0.15
C 0.15
D 0.85
E 1.70

10. Callum carefully measures the quantity of milk contained in one-litre cartons of a particular brand, and he finds that the actual quantity of milk is approximately normally distributed with a mean of 1.05 litres and a standard deviation of 0.03 litres.

The proportion of cartons that contain more than one litre of milk is closest to:

A 0.98
B 0.95
C 0.90
D 0.10
E 0.05

(Facts, skills and applications task, 1998 question 30)

11. Anastasia takes the bus to and from school each day, making a total of 10 trips per week. The probability that the bus is running late on exactly 3 occasions is given by

$$^{10}C_3(0.2)^3(0.87)^7$$

The mean and variance of the number of occasions that Anastasia finds that the bus is running late is:

	mean	**variance**
A	2	1.6
B	2	0.4
C	8	1.6
D	8	0.4
E	0.2	0.8

(Facts, skills and applications task, 1999 question 3)

12. A group of children's heights are normally distributed with a mean of 150 cm and a variance of 36 cm. One child is selected at random. The probability that the child's height is more than 160 cm is:

A $\Pr\left(Z < \frac{-10}{6}\right)$

B $\Pr\left(Z > \frac{10}{6}\right)$

C $\Pr\left(Z > \frac{-10}{6}\right)$

D $\Pr\left(Z > \frac{-10}{36}\right)$

E $1-\Pr\left(Z > \frac{10}{6}\right)$

Mapping of outcomes and criteria to item response task

For students' responses to contribute towards demonstration of achievement of the outcomes for this task, students should clearly indicate the key mathematical ideas and processes which are relevant to specific questions. They should demonstrate an understanding of the mathematical concepts through explanations about why specific multiple-choice responses are incorrect. Technology may be used to produce graphs, tables and numerical values, and to calculate probabilities, in order to show that given responses are correct or incorrect. Any results produced from an application of technology should be clearly related to the context of the question.

In relation to Outcome 1, students should be able to use correct mathematical conventions, symbols and terminology, define and explain key concepts, and should be able to apply mathematical skills and techniques correctly. Criteria 1 and 2 for Outcome 1 may be demonstrated through the explanation of why specific responses are incorrect. Criterion 3, which requires correct application of mathematical skills, could be demonstrated by illustrating the correct multiple-choice response. Linking student responses to Outcome 2 will be achieved largely through consideration of their systematic and thorough analysis of correct and incorrect responses. This analysis and interpretation is essential if students are to highlight their understandings of the mathematical ideas and content from the areas of study. Responses can be related to Outcome 3 through the production of graphs and diagrams, excerpts from tables of values, calculations and explanations provided by students. Students should clearly indicate how they have applied technology to the given problem, and how the application of technology can be interpreted in the context of the given task. Teachers can make a judgement about awarding marks to the criteria for each outcome, based on the students' overall responses to the item analyses.

Sample analysis task (probability) a short and focused investigative, challenging problem or modelling task

Teachers may wish to construct a map of elements of this tasks against the criteria and outcomes for Unit 4.

Analysis question Probability

1. An experiment consists of 250 trials. The probability of success for a trial is 0.48. Each trial results in a success or failure, and the trials are independent.

a. **i)** Use a graphics calculator to plot the distribution for this Binomial distribution ; for example, using commands such as seq (a,x,x,0,250) ® L1 and binom pdf (250,0.48) ® L2, or similar. Use zoom stat from the STAT menu to plot.

ii) Find the number of successes which has the maximum probability of being achieved.

iii) If the probability of a success in a trial is 0.5, what is the number of successes which has the maximum probability of being achieved?

iv) Plot the graphs of other Binomial distributions with different numbers of trials and probabilities of success.

2. For the Binomial distribution with 250 trials and probability of success of 0.48:

a. State the expected value μ and standard deviation s of the distribution.

b. Consider the normal distribution with random variable x, mean μ and standard deviation s.

i) Pr $(124.5 < X < 125.5)$

ii) Plot the probability density function of this normal distribution on the same screen as the plot of the binomial distribution with 250 trials and probability of success of a trial being 0.48.

3. Investigate other binomial distributions and the normal distribution formed in this way, and comment on the relationship between these distributions.