Mathematical Methods Examination 1
Answers & Solutions

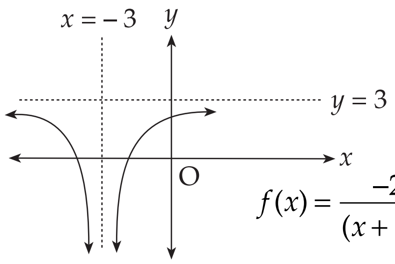
Part I (Multiple-choice) Answers

1. **E**	2. **D**	3. **D**	4. **B**	5. **C**
6. **B**	7. **B**	8. **A**	9. **A**	10. **D**
11. **E**	12. **A**	13. **D**	14. **C**	15. **D**
16. **C**	17. **D**	18. **E**	19. **E**	20. **A**
21. **B**	22. **B**	23. **E**	24. **A**	25. **E**
26. **B**	27. **B**			

Solutions

Multiple Choice

1. **[E]**



$$f(x) = \frac{-2}{(x+3)^2} + 3$$

range $(-\infty, 3)$

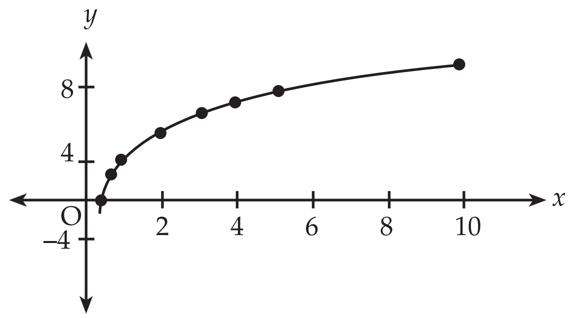
2. **[D]**

Repeated root $(x-a)^2$

Negative cubic $-(x-a)^2(x-c)$

Therefore $y = (x-a)^2(c-x)$

3. **[D]**



The graph is in the form of $y = a\log_e(x) + b$, where a and b are constants

When $\quad x = 1, y = 4$

$$4 = a\log_e(1) + b$$
$$b = 4$$

When $\quad x = 10, y = 8.6$

$$8.6 = a\log_e(10) + 4$$
$$a \approx 2$$

Hence $\quad y \approx 2\log_e(x) + 4$

4. **[B]**

Horizontal asymptote $y = b$

Vertical asymptote $x = a$

Reflected in 1 axis

$$y = \frac{-1}{(x-a)} + b$$
$$y = \frac{1}{(a-x)} + b$$

5. **[C]**

In C, f is a many-to-one function. Therefore, the inverse function f^{-1} is not defined.

6. **[B]**

$$y = e^{(x-1)} + 5$$
$$f^{-1}: x = e^{(y-1)} + 5$$
$$x - 5 = e^{(y-1)}$$
$$\log_e(x-5) = y - 1$$
$$y = \log_e(x-5) + 1$$

7. **[B]**

$$(x^2+a)(x^3+b)(x+c)^2 = 0$$
$$a > 0, (x^2+a) = 0,\ 0\ \textit{real solutions}$$
$$b > 0, (x^3+b) = 0,\ 1\ \textit{real solution}$$
$$c > 0, (x+c)^2 = 0,\ 1\ \textit{real, distinct solution}$$

Therefore, 2 real distinct solutions

8. **[A]**

The term independent of a is

$$^{16}C_{12}\left(a^3b\right)^4\left(-\frac{1}{ab^2}\right)^{12}$$
$$= {}^{16}C_{12}\frac{b^4}{b^{24}}$$
$$= {}^{16}C_{12}\frac{1}{b^{20}}$$

or

This should be found by trial and error. For example by looking at the a.

$\left(a^3\right)^6\left(\frac{1}{a}\right)^{10} = \frac{a^{18}}{a^{10}} = a^8$ – a term still exists

$\left(a^3\right)^5\left(\frac{1}{a}\right)^{11} = \frac{a^{15}}{a^{11}} = a^4$ – a term still exists

$\left(a^3\right)^4\left(\frac{1}{a}\right)^{12} = \frac{a^{12}}{a^{12}} = 1$ – no 'a' term

9. **[A]**

$$25^x = 5^x + 2$$
$$5^{2x} - 5^x - 2 = 0$$
$$\textit{let } a = 5^x$$
$$a^2 - a - 2 = 0$$
$$(a-2)(a+1) = 0$$

$a = 2$ $\qquad a = -1$

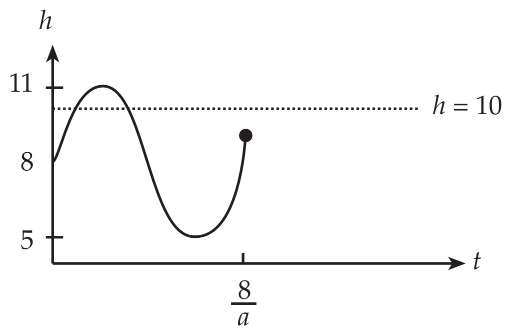
$5^x = 2$ $\qquad 5^x \neq -1$

$$x\log 5 = \log 2$$
$$x = \frac{\log 2}{\log 5}$$

10. **[D]**

$$\log_a(3r)^6 - 3\log_a(9r) - \log_a(r^4) = 2$$
$$\log_a(3r)^6 - \log_a(9r)^3 - \log_a(r^4) = 2$$
$$\log_a\frac{3^6r^6}{9^3r^3r^4} = 2$$
$$\log_a\frac{3^6r^6}{3^6r^7} = 2$$
$$\log_a\frac{1}{r} = 2$$
$$\frac{1}{r} = a^2$$
$$r = \frac{1}{a^2}$$

11. **[E]**



$$\text{Period} = \frac{2\pi}{\frac{\pi a}{4}}$$
$$= \frac{8}{a}\text{ hours}$$

Hence $\frac{24}{\frac{8}{a}}$ = 3a cycles in a day.

Thus $2 \times 3a = 6a$ times the height is 10m in the first day.

12. **[A]**

$$\cos 2x - \sqrt{3}\sin 2x = 0$$
$$\cos 2x = \sqrt{3}\sin 2x$$
$$1 = \frac{\sqrt{3}\sin 2x}{\cos 2x}$$
$$1 = \sqrt{3}\tan 2x$$
$$\tan 2x = \frac{1}{\sqrt{3}}$$
$$2x = -\frac{11\pi}{6}, \ -\frac{5\pi}{6}, \frac{\pi}{6}, \frac{7\pi}{6}$$
$$x = -\frac{11\pi}{12}, \ -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}$$
$$m = -\frac{11\pi}{12}, \ n = \frac{7\pi}{12}$$
$$m + n = -\frac{11\pi}{12} + \frac{7\pi}{12}$$
$$= -\frac{4\pi}{12}$$
$$= -\frac{\pi}{3}$$

13. **[D]**

$$\frac{dy}{dx} = 0.5\sec^2(0)$$
$$= 0.5 \neq 0$$

14. **[C]**

$f(1) = 3, \quad f(3) = 13$

Average rate of change $= \frac{\Delta y}{\Delta x} = \frac{10}{2} = 5$

15. **[D]**

$$\frac{dy}{dx} = -ae^{-x} = \text{gradient of tangent}$$
$At \ x = 0, \quad m = -a$
$$y - a = -a(x - 0)$$
$$y = -ax + a$$
$$y = a - ax$$

16. **[C]**

$$f(x) = \frac{x}{\sqrt{36 - x^2}}$$
$$= \frac{x}{\left(36 - x^2\right)^{\frac{1}{2}}}$$

Using the quotient rule

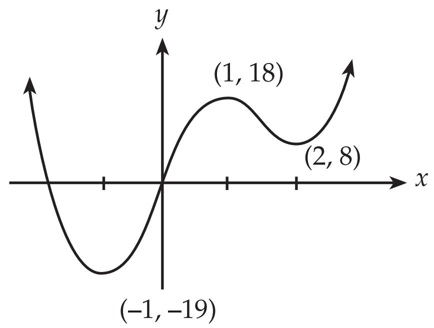
$$f'(x) = \frac{1.\left(36 - x^2\right)^{\frac{1}{2}} - x.\left(\frac{1}{2}\left(36 - x^2\right)^{\frac{-1}{2}}.(-2x)\right)}{36 - x^2}$$
$$= \frac{\left(36 - x^2\right)^{\frac{1}{2}} + \frac{x^2}{\left(36 - x^2\right)^{\frac{1}{2}}}}{36 - x^2}$$
$$= \frac{36 - x^2 + x^2}{\left(36 - x^2\right)\left(36 - x^2\right)^{\frac{1}{2}}}$$
$$= \frac{36}{\left(36 - x^2\right)^{\frac{3}{2}}}$$

17. **[D]**

$$y = \tan(x^3 + 3)$$

Using Chain Rule: $\frac{dy}{dx} = 2x\sec^2(x^2 + 3)$

18. **[E]**



Using the graphics calculator and find turning point.

$x > 0, \ (-1,1) \cup (2,\infty)$

N.B. at the turning point: $f'(x) = 0, \not> 0$

19. **[E]**

$$\begin{aligned} f'(x) &= k(x-a)(x-b) \\ &= k\left[x^2-(a+b)x+ab\right] \\ f(x) &= k\left[\frac{x^3}{3}-\frac{1}{2}(a+b)x^2+abx+c\right] \end{aligned}$$

20. **[A]**

The area of the rectangles

$= 0.5\,f(1.5) + 0.5\,f(2) + 0.5\,f(2.5) + 0.5\,f(3.0)$

$= 0.5(\sqrt{0.5}+\sqrt{1}+\sqrt{1.5}+\sqrt{2})$

$= 0.5(\sqrt{0.5}+1+\sqrt{1.5}+\sqrt{2})$

21. **[B]**

$$\begin{aligned} \cos 2x &= \sin 2x \\ 1 &= \frac{\sin 2x}{\cos 2x} \\ \tan 2x &= 1 \\ 2x &= \frac{\pi}{4} \\ x &= \frac{\pi}{8} \end{aligned}$$

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}(\cos 2x)dx+\int_0^{\frac{\pi}{8}}(\sin 2x)dx$$

22. **[B]**

(5, 1) Hence $b = \dfrac{4}{36}$
(1, 5)
(2, 6)
(6, 2)

(6,3) Hence $a = \dfrac{6}{36}$
(3, 6)
(2, 5)
(5, 2)
(1, 4)
(4, 1)

$$\begin{aligned} \Pr\big((x<2)\setminus(x<4)\big) &= \frac{\frac{16}{36}}{\frac{30}{36}} \\ &= \frac{16}{30} \\ &= \frac{8}{15} \end{aligned}$$

23. **[E]**

5 sufferers

$\Pr(X \geq 3)$

$= {}^5C_3(0.8)^3(0.2)^2 + {}^5C_4(0.8)^4(0.2)^1 + {}^5C_5(0.8)^5(0.2)^0$

$= 1 - ({}^5C_0(0.8)^0(0.2)^5 + {}^5C_1(0.8)^1(0.2)^4 + {}^5C_2(0.8)^2(0.2)^3)$

24. **[A]**

$$\begin{aligned} &= \frac{\left({}^qC_3\right)\left({}^{p-q}C_0\right)}{\left({}^pC_3\right)} \\ &= \frac{q(q-1)(q-2)}{p(p-1)(p-2)} \end{aligned}$$

25. **[E]**

If N is large compared with n, then the binomial distribution can be used to approximate the hypergeometric distribution. The mean of the binomial equals the mean of the hypergeometric. The variance of the binomial is greater than the hypergeometric. Hence:

$$\Pr(X=2) = {}^5C_2\left(\frac{r}{500+r}\right)^2\left(\frac{500}{500+r}\right)^3$$

26. **[B]**

$p = 0.4$

$q = 0.6$

$$\begin{aligned} \Pr(X \geq 3) &= \Pr(X=3)+\Pr(X=4)+\Pr(X=5) \\ &= {}^5C_3(0.4)^3(0.6)^2 + {}^5C_4(0.4)^4(0.6) \\ &\quad + {}^5C_5(0.4)^5 \\ &= 0.2304+0.0768+0.01024 \\ &= 0.3174 \end{aligned}$$

27. **[B]**

$$z = \frac{x-\mu}{\sigma}, \qquad x = 180\text{cm}, \ \mu = 162\text{cm},$$
$$\sigma = \sqrt{64} = 8\text{cm}$$
$$= \frac{180-162}{8}$$
$$= 2.25$$

$$\Pr(X > 180) = 1 - \Pr(X \le 180)$$
$$= 1 - \Pr(Z \le 2.25)$$
$$= 1 - 0.9878$$
$$= 0.0122$$

Short Answers

1.

$$y = \log_e(\cos 2x)$$
$$\frac{dy}{dx} = \frac{-2\sin(2x)}{\cos(2x)}$$
$$= -2\tan(2x)$$ **1M**

$$\int -2\tan(2x)dx = \log_e(\cos 2x) + c$$

where c, is a constant

$$\int \tan(2x)dx = -\frac{1}{2}\log_e(\cos 2x) + c$$ **1A**

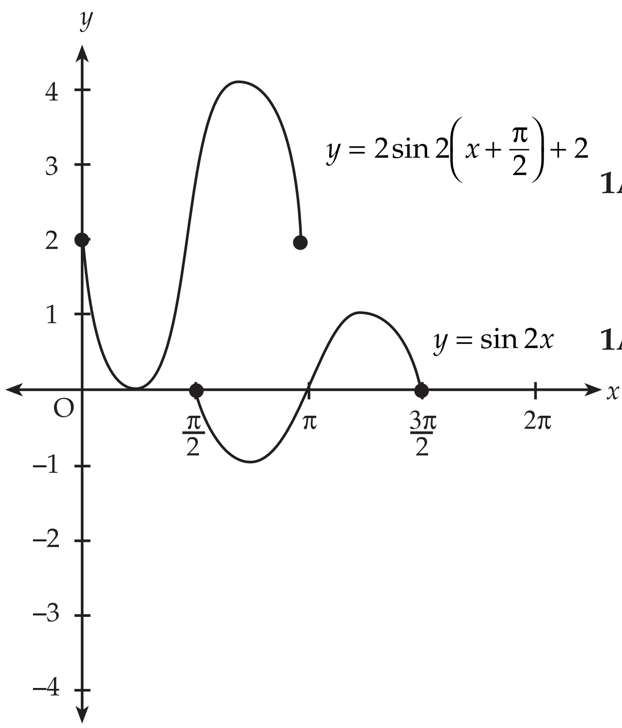
2.

$$f(x) = \sqrt{x} \qquad f'(x) = \frac{1}{2\sqrt{x}}$$
$$f(16) = \sqrt{16} \qquad f'(16) = \frac{1}{2\sqrt{16}}$$
$$= 4 \qquad = \frac{1}{8}$$

Using: $f(x+h) \approx hf'(x) + f(x)$

$$f(15.9) = f(16-0.1)$$
$$\approx -0.1 \times \frac{1}{8} + 4$$ **1M**
$$= 3.9875$$ **1A**

3.



1A

1A

c. $y = 2\left[\sin 2\left(x + \frac{\pi}{2}\right) + 1\right]$

OR

$y = 2\sin 2\left(x + \frac{\pi}{2}\right) + 2$ **1A**

4. a. Coordinates from graphics calculator,
(–4, –10), (2, 8) **2A**

b.

$$\text{Area} = \int_{-4}^{2}\left(g(x) - f(x)\right)dx$$

$$= \int_{-4}^{2}(3x + 2 - x^2 - 5x + 6)dx \quad \textbf{1M}$$

$$= \int_{-4}^{2}(8 - 2x - x^2)dx$$

$$= \left[8x - x^2 - \frac{x^3}{3}\right]_{-4}^{2} \quad \textbf{1M}$$

$$= \left(16 - 4 - \frac{8}{3}\right) - \left(-32 - 16 + \frac{64}{3}\right) \quad \textbf{1M}$$

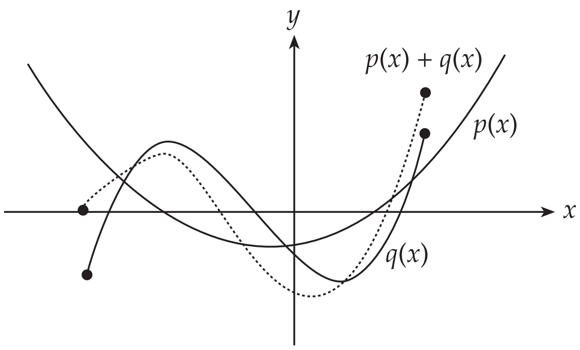
$= 36$ sq units **1A**

5.

Must recognise $m(x)$ only exists where both $p(x)$ and $q(x)$ exist.

1A

Shape correct **1A**



6.

a. [0, 20] **1A**

b. $\frac{dv}{dt} = -9(20 - t)^2$ **1A**

c. i. t=0, $\frac{dv}{dt} = -3600$
Emptying at 3600 l/min. **1A**

ii.

$$\frac{dv}{dt} = -900$$

$$-900 = -9(20 - t)^2$$

$$t = 10 \text{ minutes} \quad \textbf{1A}$$

7. a.

$$Var(X) = E(X)^2 - [E(X)]^2$$

$$= (0.2 + 1.2 + 1.8 + 0.2k^2) - (0.2 + 0.6 + 0.6 + 0.2k)^2$$

$$= (3.2 + 0.2k^2) - (1.4 + 0.2k)^2$$

$$= 3.2 + 0.2k^2 - 1.96 - 0.56k - 0.04k^2$$

$$= 0.16k^2 - 0.56k + 1.24 \quad \textbf{1A}$$

b.

$$0.16k^2 - 0.56k + 1.24 = 7$$

$$0.16k^2 - 0.56k - 5.76 = 0$$

$$k^2 - 3.5k - 36 = 0$$

$$2k^2 - 7k - 72 = 0$$

$$(2k + 9)(k - 8) = 0$$

$$k - 8 = 0$$

$$k = 8$$

$$2k + 9 = 0$$

$$2k = -9$$

$$k = -4.5$$

Reject as $k \in J^{+}$
answer is $k = 8$ **1A**

8.

$$\Pr\left(z > \frac{176 - \mu}{\sigma}\right) = 0.06$$

$$\Pr\left(z \le \frac{176 - \mu}{\sigma}\right) = 0.94$$

Inv Norm (0.94) ≈ 1.55477

$$\frac{176 - \mu}{\sigma} \approx 1.55477 \quad (1) \quad \textbf{1M}$$

$$\Pr\left(z < \frac{148 - \mu}{\sigma}\right) = 0.12$$

Inv Norm (0.12) ≈ -1.17499 (2) **1M**

$$\frac{148 - \mu}{\sigma} \approx -1.17499$$

divide 1 by 2

$$\frac{176 - \mu}{148 - \mu} \approx -\frac{155477}{1.17499}$$

$$176 - \mu \approx -1.3232(148 - \mu)$$

$$-2.3232\mu \approx -371.8636$$

$$\mu \approx 160.05$$

$$\approx 160.1 \text{ cm} \quad \textbf{1A}$$

