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Trial Examination 2001

VCE Mathematical Methods Units 3 & 4

Written Examination 2: Analysis task

Reading time 15 minutes
Writing time 1 hour 30 minutes

Student's Name: ______________________________

Teacher's Name: ______________________________

Structure of Booklet

Number of questions	Number of questions to be answered
4	4

Directions to students

Materials

Question and answer book of 11 pages.
There is a detachable sheet of miscellaneous formulas in the centrefold.
Working space is provided throughout the booklet.
You may bring to the examination up to four pages (two A4 sheets) of pre-written notes.
You may use an approved scientific and/or graphics calculator, ruler, protractor, set square and aids for curve sketching.

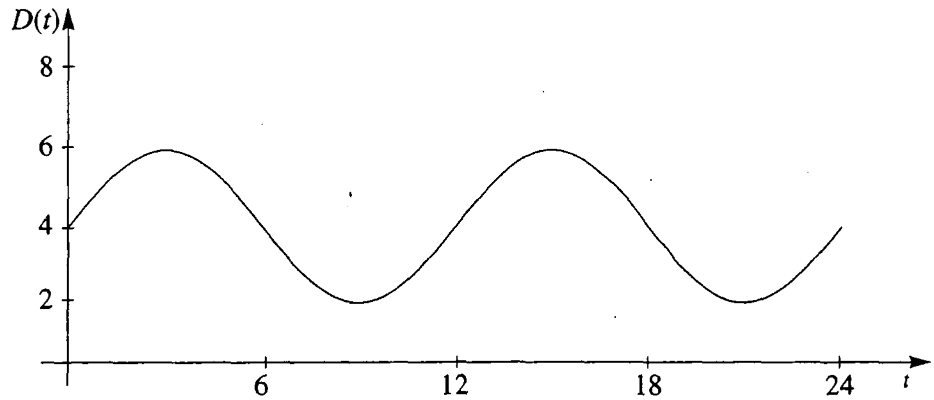
The task

Detach the formula sheet from the centre of this booklet during reading time.
Ensure that you write your student number in the space provided on the front of this booklet.
Answer all questions.
The marks allotted to each part of each question are indicated at the end of each part.
There is a total of 56 marks available for the task.
A decimal approximation will not be accepted if an exact answer is required to a question.
Where an exact answer is required to a question, appropriate working must be shown and calculus must be used to evaluate derivatives and definite integrals.
Unless otherwise indicated, the diagrams in this book are not drawn to scale.
All written responses should be in English.

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2001 VCE Mathematical Methods Examination 2.

Question 1

The depth of water ($D(t)$ metres) in a harbour at t hours after midnight on a certain day is shown on the graph below for $0 \le t \le 24$.



Its equation is of the form $D(t) = a\sin n(t - b) + c$.

a. Write down the values of a, b and c.

3 marks

b. Show that the value of n is $\frac{\pi}{6}$.

1 mark

c. Show your calculations to find the depth of water at 5 a.m.

2 marks

d. A yacht which has an underwater depth of 3 metres wishes to use the harbour. Calculate the period of time in the morning when the yacht is unable to sail in the harbour.

4 marks

A nearby volcanic eruption caused the tide on a different day to follow the equation

$$D(t) = 6e^{-0.1t} + 5\sin\frac{\pi}{6}t + 5$$

where $D(t)$ and t are as defined as before.

e. Complete the table of values below (round your values to two decimal places)

t	$D(t)$
0	
6	
12	

3 marks

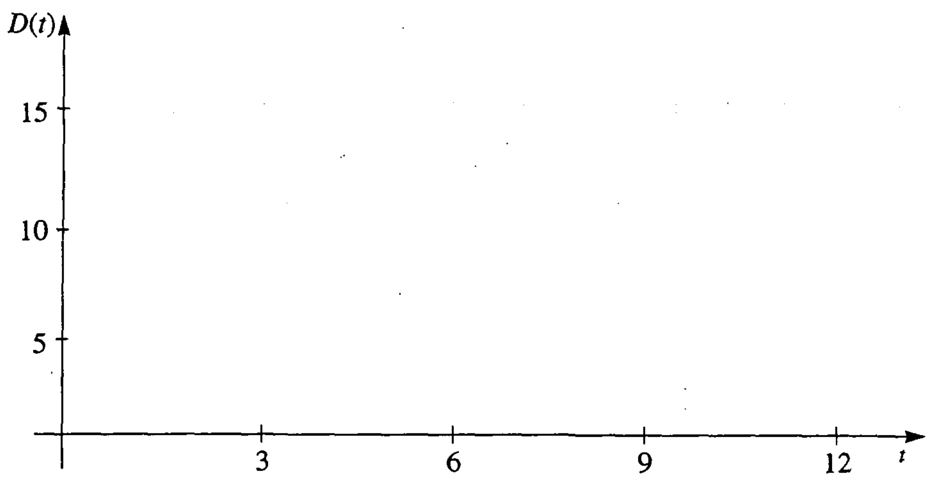
f. Find the coordinates of the first maximum and the first minimum for $t > 0$.

____

____

2 marks

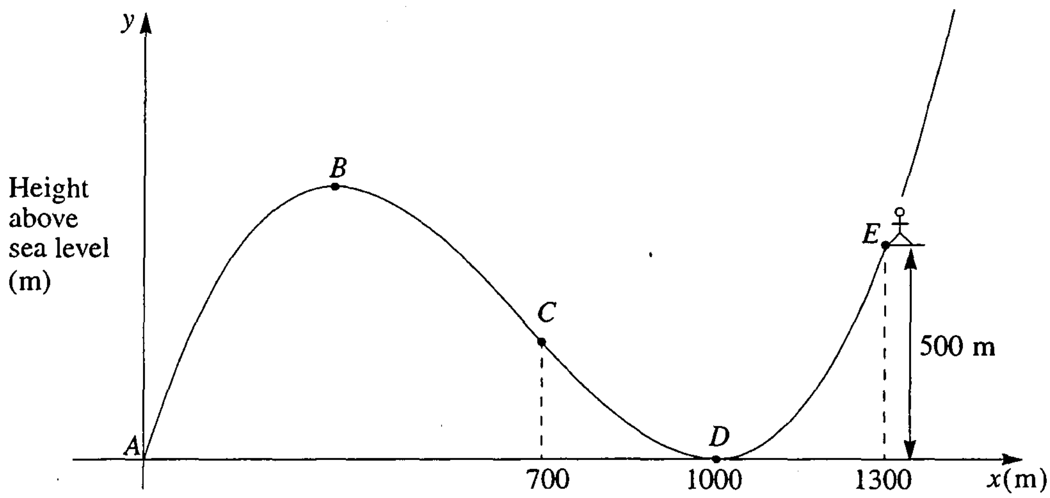
g. Sketch the graph of $D(t)$ for $0 \le t \le 12$, showing each of the points calculated in **e.** and **f.**



2 marks
Total 17 marks

Question 2

A geographical cross-section of part of a mountain range is shown on the set of cartesian axes below.



The general equation for the cross-section is

$$y = Qx(x-p)^2$$

where x is the horizontal distance in metres from A and y is the height above sea level in metres.

a. Deduce the value of p.

1 mark

There is a lookout at E which is situated 1300 m horizontally from A (the origin). It is known to be 500 m above points A and D.

b. Show that the value of Q is 4.3×10^{-6} (when rounded to two significant figures).

2 marks

c. Find an expression for the gradient of the slope at any point.

2 marks

d. Using your answer to part **c.**, find the coordinates of B (rounding your values to the nearest metre).

3 marks

e. A 4 wheel drive vehicle was attempting to drive directly up the mountain from D to B. Calculate the gradient (to three decimal places) of the slope at C which is 700 m horizontally from A.

2 marks

f. Find the horizontal distance (to the nearest metre) at which the slope between B and D is the greatest.

2 marks

Total 12 marks

Question 3

An amusement park offers various rides that differ in the number of occupants.

Given that the X represents the number of occupants on a particular ride, the table below represents the probability distribution.

x	1	2	3	4	5
$\Pr(X = x)$	0.40	0.25	m	0.10	0.10

a. Find the value of m correct to two decimal places.

1 mark

b. Calculate $E(X)$ correct to two decimal places.

2 marks

c. Use your answer for **b.** to determine $\text{Var}(X)$, correct to three decimal places.

3 marks

d. Hence find the standard deviation of X, correct to three decimal places.

1 mark

The park has just established a new ride with different seating requirements. Given that Y represents the number of occupants on this ride and $Y = 2X - 1$,

e. **i.** calculate $E(Y)$, correct to two decimal places.

1 mark

ii. Hence determine $\text{Var}(Y)$, correct to three decimal places.

1 mark

A third ride, where Z represents the number of occupants, is to be considered. The table below represents the binomial probability distribution where n, the number of possible occupants, is 3 and p, the probability of occupancy, is $\frac{1}{3}$.

z	0	1	2	3
$\Pr(Z = z)$	$\frac{8}{27}$	c	d	$\frac{1}{27}$

f. **i.** Find the values of c and d, expressing your answers exactly.

2 marks

ii. Calculate the standard deviation of Z, exactly.

1 mark

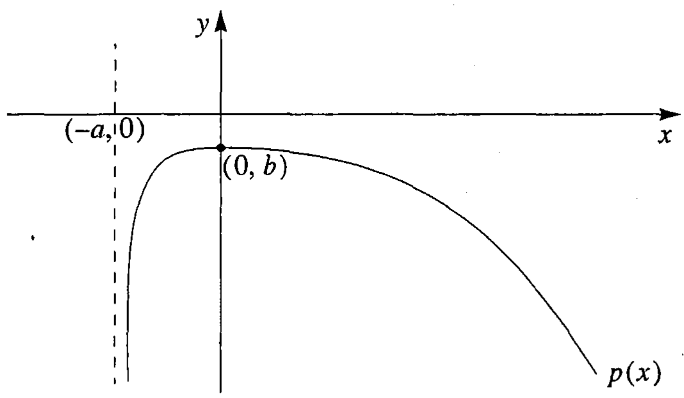
A fourth amusement ride requires exactly three occupants in order to maintain safety standards. A group of 8 children consisting of 4 boys and 4 girls is eagerly waiting their turn on the last ride of the day. An attendant stands by to randomly select a group of three children for this ride.

g. Calculate the probability of the attendant selecting exactly 2 boys in the final group.

1 mark
Total 13 marks

Question 4

The graph of the function $p : (-a, \infty) \to R, p(x) = \frac{-e^{x-1}}{x+1}$ is shown below.



a. Determine the exact values of a and b.

2 marks

b. Show that $p'(x) = \frac{-xe^{x-1}}{(x+1)^2}$

3 marks

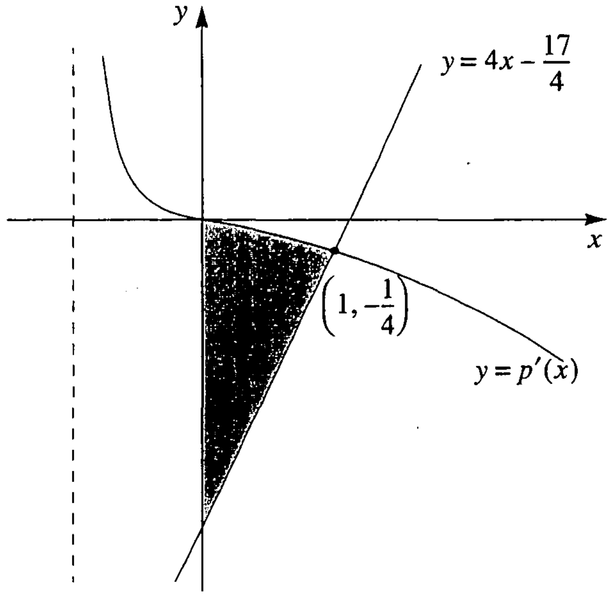
c. Use the expression for $p'(x)$ in **b.** to show that a stationary point occurs at the point $(0, b)$.

2 marks

d. Show that this point is a maximum.

2 marks

e.



The line with equation $y = 4x - \frac{17}{4}$ intersects with $y = p'(x)$ at $\left(1, -\frac{1}{4}\right)$ as shown.

Calculate the **exact** value of the shaded region.

5 marks
Total 14 marks

END OF QUESTION AND ANSWER BOOKLET

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VCE Mathematical Methods Units 3 & 4

Written Examination 2: Analysis task

Suggested Solutions

Question 1

a. $a = 2$, $b = 0$, $c = 4$ [A] [A] [A]

b. Note that the period = 12

$\therefore \frac{2\pi}{n} = 12$

$\therefore n = \frac{2\pi}{12} = \frac{\pi}{6}$ [A]

c. When $t = 5$, $D(t) = 2\sin\frac{5\pi}{6} + 4$ [M]

$= 2 \times \frac{1}{2} + 4$

$= 5$ [A]

d. The yacht must use a depth of at least 3 m, so when $D = 3$ we have

$2\sin\frac{\pi}{6}t + 4 = 3$ [M]

$2\sin\frac{\pi}{6}t = -1$

$\sin\frac{\pi}{6}t = -\frac{1}{2}$ [A]

$\therefore \frac{\pi}{6}t = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$\therefore \frac{\pi}{6}t = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\therefore t = 7, 11$ [A][A]

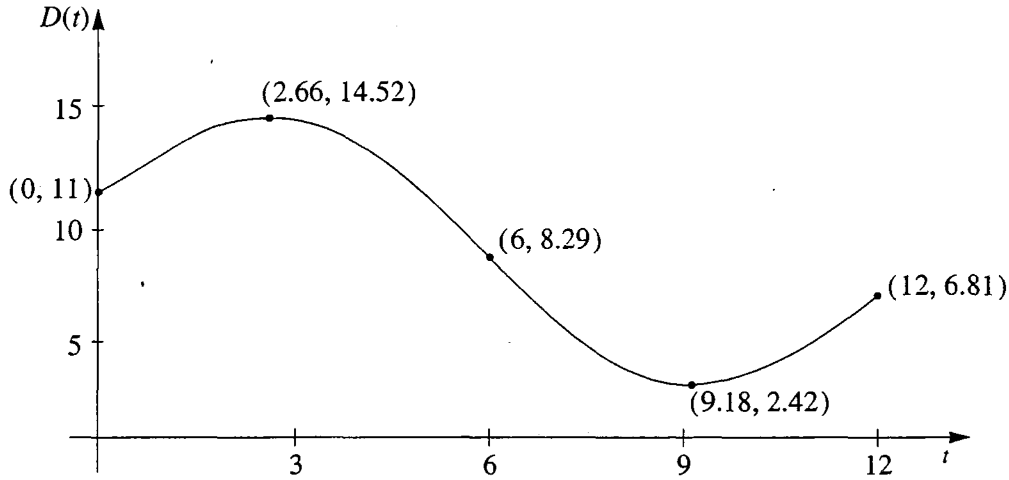
Hence the yacht is unable to sail for 4 hours between 7 a.m. and 11 a.m.

e.

t	$D(t)$		
0	11.00	$6e^{0} + 5\sin 0 + 5$	[A]
6	8.29	$6e^{-0.6} + 5\sin\pi + 5$	[A]
12	6.81	$6e^{-1.2} + 5\sin 2\pi + 5$	[A]

f. Using a graphic calculator or by equating the derivative to zero (not advised):
First maximum = (2.66, 14.52) [A]
First minimum = (9.18, 2.42) [A]

g.



The graph should indicate the points
(0, 11), (2.66, 14.52), (6, 8.29), (9.18, 2.42) and (12, 6.81) [A] [A]

Question 2

a. $p = 1000$ [A]

b. Substitute the point (1300, 500) to obtain $500 = Q1300(300)^2$ [M]

$$\therefore Q = \frac{500}{1300(300)^2}$$

$$= 4.3 \times 10^{-6}$$ [A]

c. If we are assuming x is measured in kilometres, gradient $= \frac{dy}{dx} = 4.3 \times 10^{-6}(3x^2 - 4x + 1)$ [A]

(using either product rule or differentiating the expanded form) [M]

(If assuming x is in metres, gradient $= 4.3 \times 10^{-6}(3x^2 - 4000x + 1000000)$.)

d. At B, $\frac{dy}{dx} = 0$ (using x measured in kilometres)

$\therefore 3x^2 - 4x + 1 = 0$ (since $Q \neq 0$) [M]

$(3x - 1)(x - 1) = 0$

$\therefore x = \frac{1}{3}$ and 1 kilometre

or 333 m and 1000 m [A]

Hence the coordinates of B (using graphic calculator) are (333, 637) [A]

e. At C, $x = 700$

Gradient $= \frac{dy}{dx} = 4.3 \times 10^{-6}(3(700) - 1000)(700 - 1000)$

(by substituting into the factorized form) [M]

$$\frac{dy}{dx} = -1.419$$ [A]

f. For greatest slope, we can differentiate the gradient function and equate to zero, obtaining

$\dfrac{d^2y}{dx^2} = 6x - 4 = 0$ (if x is in kilometres). [M]

$\therefore x = \dfrac{2}{3}$ kilometres or 667 metres [A]

(use of graphic calculator and tracing with derivative function turned on could also be used)

Question 3

a. As $\Sigma \Pr(X = x) = 1$, $m = \Pr(X = 3) = 0.15$ [A]

b. $E(X) = \Sigma x \times p(x)$

$= (1 \times 0.4) + (2 \times 0.25) + (3 \times 0.15) + (4 \times 0.1) + (5 \times 0.1)$ [M]

$= 2.25$ [A]

c. $\text{Var}(X) = E(X^2) + (E(X))^2$

$E(X^2) = (1 \times 0.4) + (4 \times 0.25) + (9 \times 0.15) + (16 \times 0.1) + (25 \times 0.1)$ [M]

$= 0.40 + 1.0 + 1.35 + 1.6 + 2.5$

$= 6.85$ [A]

so $\text{Var}(X) = 6.85 - 2.25^2$

$= 1.7875$

≈ 1.788 [A]

d. $\sigma_x = \sqrt{\text{Var}(X)} = \sqrt{1.7875} = 1.337$ [A]

e. As $Y = 2X - 1$, $E(Y) = E(2X - 1)$

$= 2E(X) - 1$

$= 2 \times 2.25 - 1$

$= 3.50$ [A]

i. $\text{Var}(Y) = \text{Var}(2X - 1)$

$= 4\text{Var}(X)$

$= 4 \times 1.7875$

$= 7.150$ [A]

f.

Z	**0**	**1**	**2**	**3**
$\Pr(Z = z)$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{1}{27}$

i. $c = \Pr(Z = 1) = {}^3C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 = 3 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^2 = \frac{4}{9}$ [A]

$d = \Pr(Z = 2) = {}^3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 = 3 \times \frac{1}{9} \times \frac{2}{3} = \frac{2}{9}$ [A]

ii. $\text{SD}(Z) = \sqrt{\text{Var}(Z)} = \sqrt{npq} = \sqrt{3 \times \frac{1}{3} \times \frac{2}{3}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$ [A]

g. Letting B = "number of boys selected"

then $\Pr(B = 2) = \dfrac{({}^4C_2)({}^4C_1)}{({}^8C_3)} = 0.4286$ [A]

Question 4

a. $x = -a$ is the vertical asymptote.

This is determined by equating the denominator of $p(x)$ to zero, i.e. $x = -1$

$\therefore a = 1$ [A]

$y = b$ is the y intercept.

Substitute $x = 0$ into $p(x)$: $y = \dfrac{-e^{x-1}}{x+1}$, where $x = 0$

$$\therefore y = \frac{-e^{-1}}{1} = \frac{-1}{e}$$

$$\therefore b = \frac{-1}{e}$$ [A]

b. Using the quotient rule: [M]

$$p'(x) = \frac{(x+1)(-e^{x-1}) - (-e^{x-1}) \times 1}{(x+1)^2}$$

$$= \frac{-e^{x-1}[(x+1) - 1]}{(x+1)^2}$$ [M]

$$= \frac{-xe^{x-1}}{(x+1)^2}$$ [A]

c. $p'(x) = \dfrac{-xe^{x-1}}{(x+1)^2} = 0$ at the stationary point [M]

$-xe^{x-1} = 0$ (since $(x+1)^2 \neq 0$)

$\therefore x = 0$ (since $e^{x-1} \neq 0$)

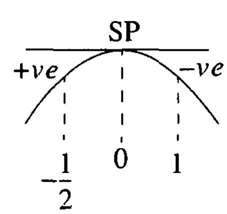
At $x = 0$, $y = b$ (from **a.**)

So a stationary point occurs at $(0, b)$ [A]

d. Either by use of the second derivative, or by testing the gradient of the curve on either side of the stationary point, or by substituting into $p(x)$. For example:

At $x = -\frac{1}{2}$, $p'(x) = \dfrac{\frac{1}{2}e^{(-1/2)-1}}{\left(-\frac{1}{2}+1\right)^2} = \dfrac{\frac{1}{2}e^{-3/2}}{\frac{1}{4}} \approx 0.45$ (positive)

At $x = 1$, $p'(1) = \dfrac{-1e^0}{(1+1)^2} = -\dfrac{1}{4}$ (negative)



[M]

So the stationary point is a maximum. [A]

e. Required area $= \int_0^1 \left[p'(x) - \left(4x - \frac{17}{4}\right)\right]dx$ [A]

$= \int_0^1 \left(\frac{-xe^{x-1}}{(x+1)^2} - 4x + \frac{17}{4}\right)dx$ [A]

$= \left[\frac{-e^{x-1}}{x+1} - 2x^2 + \frac{17}{4}x\right]_0^1$ [M]

$= \left(-\frac{e^0}{2} - 2 + \frac{17}{4}\right) - \left(-\frac{e^{-1}}{1} - 0 + 0\right)$ [M]

$= -\frac{1}{2} - 2 + \frac{17}{4} + \frac{1}{e}$

$= \frac{7}{4} + \frac{1}{e}$ square units [A]