Mathematical Methods (CAS), supplementary questions – extended response solutions and comments

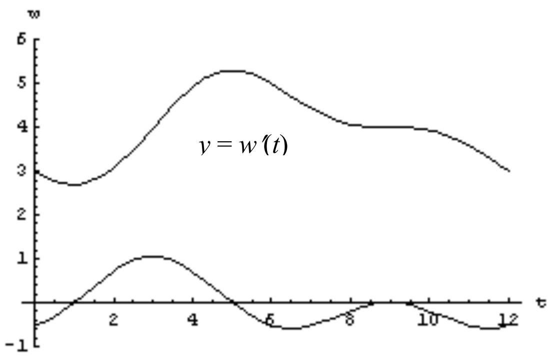
A two-column table format has been used in this section to highlight key aspects of solutions, indicate corresponding mark allocations and provide related comments and advice. The comments and advice in the second column of the table indicate where marks would likely be awarded for student demonstration of reasoning. In general clear demonstration of student reasoning will be necessary to achieve full marks for a question and correct mathematical notation should be used throughout the presentation of solutions. Students are likely to do some questions, or parts of questions, without the use of CAS, and others using CAS. The use of CAS is likely to be particularly helpful for various parts of these supplementary questions.

In the following solutions it may be that one form of an exact solution is given while different CAS will produce other acceptable equivalent forms. For several of the questions it may be advisable for the student to define and store key functions that are to be used in developing solutions. These functions should be explicitly defined in recorded working.

Teachers may wish to incorporate selections of these questions (or similar) in review or practice material for examinations.

Solution	Comments
1. a. $w(0) = 3$ 1 mark	
b. Period of $4 - 0.5\sin\left(\frac{\pi t}{3}\right)$ is 6 Period of $-\cos\left(\frac{\pi t}{6}\right)$ is 12 The lowest common multiple of 6 and 12 is 12 and hence the period of *w* is 12. **Or:** Show that $w(t+12) = w(t)$ Period of $4 - 0.5\sin\left(\frac{\pi t}{3}\right)$ is 6 Period of $-\cos\left(\frac{\pi t}{6}\right)$ is 12 The lowest common multiple of 6 and 12 is 12 and hence the period of *w* is 12. 2 marks	Correct listing of periods of the component functions will be awarded 1 method mark.

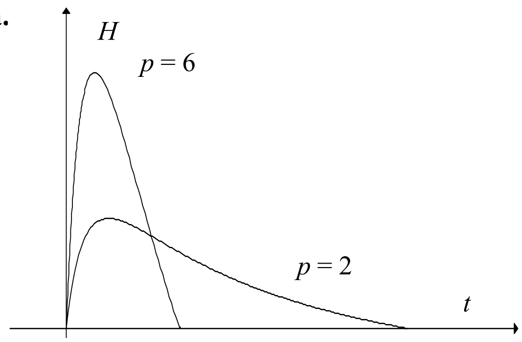
c. $w(t) = 4 - 0.5\sin\left(\frac{\pi t}{3}\right) - \cos\left(\frac{\pi t}{6}\right)$ average value $= \frac{1}{12}\int_0^{12} w(t)dt$ =4 megalitres. 2 marks	Correct integral including limits will be awarded 1 method mark. CAS can be used to find the required value.
d. i. $w'(2) = \frac{(\sqrt{3}+1)\pi}{12}$ 2 marks	CAS can be used to differentiate and evaluate in one step. A mark can be awarded separately for the correct derivative function.
d. ii. Solve $w'(t) = 0$ for t where $w(t) = 4 - 0.5\sin\left(\frac{\pi t}{3}\right) - \cos\left(\frac{\pi t}{6}\right)$ t = 1, 5 or 9 months. 2 marks	1 method mark to be awarded for equating $w'(t) = 0$. Answers from a graph are acceptable.
d. iii. Maximum volume of water is $w(5) = 4 + \frac{3\sqrt{3}}{4}$ megalitres. Minimum volume of water is $w(1) = 4 - \frac{3\sqrt{3}}{4}$ megalitres. 2 marks	Exact answers are required for this question. No marks awarded for approximate answers. CAS can be used to perform the substitution.

e. Correct sketch of $w'(t)$  2 marks	One mark is awarded for correct determination of the points of zero gradient and one mark for the correct shape and relative magnitude of the rate graph.
Total: 13 marks	
2. a. mean $= \int_0^{\infty} \frac{x}{5} e^{-\frac{x}{5}} dx$ $= 5$. 2 marks	Mark awarded for correct integral including limits. A limit form of statement is also correct, such as: $\lim\limits_{k \to \infty} \int_0^k f(x)dx$ CAS can be used to evaluate the integral.
b. Solution of the equation: $\int_0^m \frac{x}{5} e^{-\frac{x}{5}} dx = 0.5$ for m, $m = 5\log_e 2$. 2 marks	Mark awarded for equation, or equivalent demonstration of reasoning. CAS can be used to form expression in terms of m and then solve the corresponding equation.

c. Evaluation of : $$\int_0^{\infty} \frac{1}{5}(x-5)^2 e^{\frac{-x}{5}}\,dx$$ or $$\int_0^{\infty} \frac{1}{5}x^2 e^{\frac{-x}{5}}\,dx - \mu^2$$ $= 25.$ 2 marks	Mark awarded for correct integral including limits. CAS should be used to evaluate integral. The value of the mean can be defined as a constant from **2. a.**
d. $\Pr(X > 5) = \int_5^{\infty} \frac{1}{5} e^{-\frac{x}{5}}\,dx$ $= e^{-1}$. 2 marks	Mark awarded for correct integral including limits. CAS used to evaluate integral.
e. Binomial distribution with $n = 4$ and $p = e^{-1}$. Let Y be the random variable with values, the number of switches which have not failed, then $E(Y) = np = \frac{4}{e}$. 2 marks	Mark awarded for recognition of binomial distribution with value of p taken as answer to **d.**

f. $\Pr(Y \geq 2) = 1 - \Pr(Y \leq 1)$ ≈ 0.4687 2 marks	Mark awarded for the complement being recognised.
Total: 12 marks	
3. a. $f_a(x) = 0$ when $x = a^2$, that is, $c = a^2$. 2 marks	Mark awarded for requiring $f_a(x) = 0$ and a mark awarded for correctly solving the equation.
b. $f_a'(x) = 1 - \dfrac{a}{2\sqrt{x}}$,and $f_a'(x) = 0$ when $x = \dfrac{a^2}{4}$. Decreasing for $(0, \dfrac{a^2}{4})$ and increasing for $(\dfrac{a^2}{4}, \infty)$. 4 marks	Mark awarded for f_a' . Mark awarded for $x = \dfrac{a^2}{4}$. Marks awarded for the correct intervals. CAS can be used to find the derivative and the zero of the derivative.
c. $f_a'(a^2) = \dfrac{1}{2}$, which is independent of a. $y = \dfrac{1}{2}(x - a^2)$ is the required equation. All such tangents are parallel. 3 marks	Mark awarded for gradient. Mark awarded for equation of tangent. Mark awarded for recognition of parallel family of tangents.

d. $f_a(\frac{a^2}{4}) = -\frac{a^2}{4}$, so range of f_a is $[-\frac{a^2}{4}, \infty)$. 2 marks	Marks awarded for left bound and open interval.
e. Area = $-\int_0^{a^2}(x - a\sqrt{x})dx = \frac{a^4}{6}$. 2 marks	Mark awarded for correct integral including limits.
f. i. $b = \frac{a^2}{4}$. 1 mark	
f. ii. $g_a^{-1}(x) = \frac{(\sqrt{4x + a^2} + a)^2}{4}$. 3 marks	Method mark awarded for correct technique for find inverse. Two solutions are obtained if no restriction is entered. Indication of reason for choice of solution is required.
f . iii. domain of g_a^{-1} = range of $g_a = [-\frac{a^2}{4}, \infty)$. 1 mark	
Total: 18 marks	

4. a.  2 marks	
b. $H = 0$ implies $-pt(pt - 12) = 0$. Since $t = 0$ is the start of the process, the end is when $pt - 12 = 0$. So $t = \frac{12}{p}$. 2 marks	Mark awarded for writing equation to be solved. CAS may be used to solve for t in terms of p.
c. i. $\frac{dH}{dt} = \frac{-2p((p+6)t-6)}{(t+1)^3}$ 2 marks	Mark awarded for requiring the derivative equal to 0. CAS may be used to differentiate.
c. ii. $\frac{dH}{dt} = 0$ implies $t = \frac{6}{p+6}$. 2 marks	CAS may be used.

d. When $t = \dfrac{6}{p+6}$, $H = \dfrac{36p}{p+12}$. 2 marks	Mark awarded for substituting for *t.* CAS may be used to substitute.
e. $\dfrac{36p}{p+12} \leq 12$ implies $p \leq 6$. Therefore the greatest value of K is 6. 2 marks	Mark awarded for writing the correct inequality. CAS may be used to solve inequality.
f. i. $10p = 60$. Therefore the reaction must take place at 60º C. 1 mark	
ii. $\dfrac{12}{p} = 2$. Therefore the reaction takes 2 minutes. 1 mark	
Total: 14 marks	

5. a. $\Pr(X \le a) = \int_{30000}^{a} f(t)\,dt$ $= 1 - 9\sqrt{3} \times 10^{10} \times a^{-\frac{5}{2}}$. 2 marks	Mark awarded for the correct integral including limits. CAS may be used to evaluate the integral.
b. Mean $= \int_{30000}^{\infty} xf(x)\,dx$ $= 50\,000$. The mean annual salary is \$50 000. 2 marks	Mark awarded for the correct integral including limits. CAS may be used to evaluate the integral.

c. Solve $\int_{30000}^{m} f(x)dx = 0.5$ for m. This implies $1 - 9\sqrt{3} \times 10^{10} \times m^{-\frac{5}{2}} = 0.5$. Therefore $m = 1875 \times 2^{\frac{22}{5}}$ $\approx 39\ 585$. The median annual salary is \$ 39 585. 3 marks	Mark awarded for the correct equation. CAS may be used to evaluate the integral.
d. $\Pr(X \le 50\ 000) = \int_{30000}^{50000} f(x)dx$ $= 1 - \frac{9\sqrt{15}}{125}$ ≈ 0.7211 Approximately 72% of people in the profession earn less than \$ 50 000 per annually. 2 marks	Mark awarded for the correct integral including limits. CAS may be used to evaluate the integral.

e. $\Pr(X > 45\,000 \mid X > 40\,000)$ $= \dfrac{\Pr(X > 45\,000)}{\Pr(X > 40\,000)}$ = 0.745 correct to three decimal places. 3 marks	Mark awarded for recognition of conditional probability. CAS may be used.
f. This is a binomial distributionwith n = 20 and $p = \dfrac{9\sqrt{15}}{125}$. Let Y be the number of people who earn more than $50 000 annually. $\Pr(Y \geq 2) = 1 - (\Pr(Y = 0) + \Pr(Y = 1))$ $= 0.9874$. 3 marks	A method mark would be awarded for recognition of the binomial distribution, with appropriate parameters. Suitable notation indicating use of built in function for computation is also acceptable.
Total: 15 marks	
6. a. i. $\int_{-\infty}^{\infty} xf(x)\,dx = 0.625$ $\int_0^1 ax^3(b\text{-}x^2)dx = 0.625$ Therefore $\dfrac{a(3b-2)}{12} = 0.625$ and $a = \dfrac{15}{6b-4}$. 3 marks	Mark awarded for integral equated to 0.625

ii. $\int_{-\infty}^{\infty} f(x)\,dx = 1$ $\int_0^1 ax^2(b-x^2)\,dx = 1$ $\frac{a(5b-3)}{15} = 1$. Thus $b = 1$ and $a = \frac{15}{2}$. 3 marks	Mark awarded for correct integral equated to 1. Mark awarded for correct procedure for solving.
b. i. $\Pr(X \le k) = \int_0^k ax^2(b-x^2)\,dx$ $= \frac{k^3(5-3k^2)}{2}$. 2 marks	Mark awarded for correct integral including limits.
b. ii. $\frac{k^3(5-3k^2)}{2} = \frac{17}{64}$ $k = 0.5$. 2 marks	Mark awarded for suitable equation.
b. iii. The median = 0.643 . 2 marks	Mark awarded for correct integral equated to 0.5.

c. $\Pr(X \geq 0.9) = \int_{0.9}^{1} ax^2(b\text{-}x^2)dx$ $= \dfrac{12647}{200000}$. 2 marks	Mark awarded for correct integral including limits.
d. $\Pr(X > 0.8 \mid X > 0.625)$ $= \dfrac{\Pr(X > 0.8)}{\Pr(X > 0.625)}$ $= \dfrac{0.21152}{0.5327}$ (4 decimal places) $= 0.397$ (3 decimal places). 3 marks	Mark awarded for recognition of conditional probability. Mark awarded for correct simplification of conditional probability statement.
Total: 17 marks	