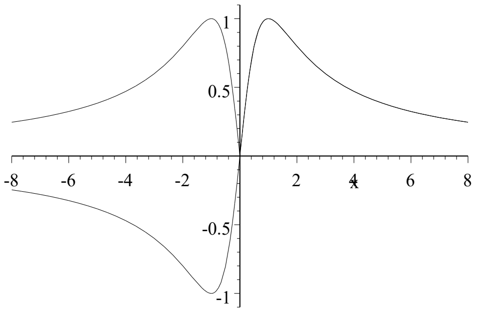
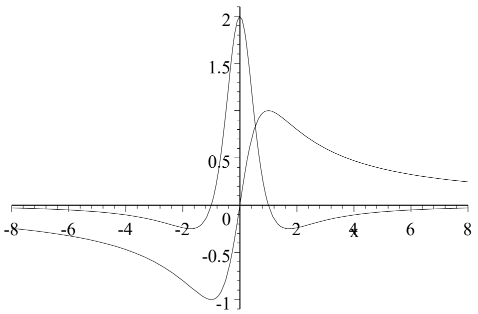
Mathematical Methods (CAS) pilot study: supplementary questions – short answer solutions and comments

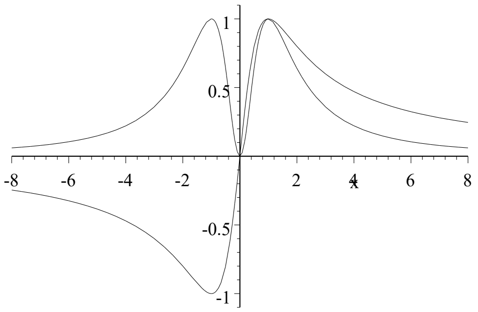
Question 1



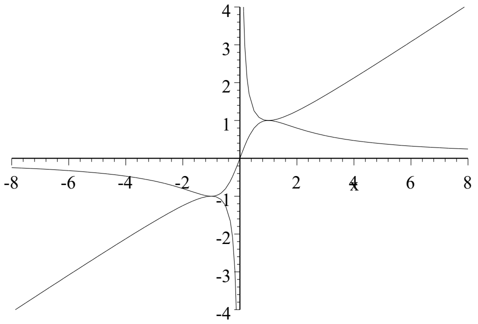
Question 2



Question 3



Question 4



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Question 5

Solve $x^3 + 2x^2 - 5x - 9 = 1$ for x, or $x^3 + 2x^2 - 5x - 10 = 0$ for x. The solutions are -2, $\sqrt{5}$ and $-\sqrt{5}$. Since P is in the first quadrant, its x-coordinate must be $\sqrt{5}$.

Question 6

a. $h'(x) = f(x)\cos x + f'(x)\sin x$

$$h'\left(\frac{\pi}{3}\right) = f(\frac{\pi}{3})\cos(\frac{\pi}{3}) + f'(\frac{\pi}{3})\sin(\frac{\pi}{3}) = \frac{a}{2} + \frac{b\sqrt{3}}{2}$$

b. $h'\left(\frac{\pi}{3}\right) = \frac{1}{2}\log_e(\frac{\pi}{3}) + \frac{3\sqrt{3}}{2\pi}$

Question 7

$f(x) = x^3 + C$, where C is an arbitrary constant. By the chain rule,

$g'(x) = f'(\cos(x))\frac{d}{dx}(\cos(x)) = -f'(\cos(x))\sin(x)$, hence $f'(x) = 3x^2$ and $f(x) = x^3 + C$.

Alternatively, by anti-differentiation, $g(x) = -\int 3\cos^2(x)\sin(x)dx = \cos^3(x) + C$ and $f(x) = x^3 + C$.

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Question 8

a. Let d be the distance from the origin to the vertex of the parabola. Then:

$$d = \sqrt{3^2 + 4^2}$$
$$d = 5$$

Students should identify the location of the turning point by considering the graph of f as a transformation of the graph of $y = x^2$. Students should also recognise the 3, 4, 5 Pythagorean triple.

b. Let d be the distance from the origin to the point (x, y) on the graph, then

$$d = \sqrt{x^2 + y^2}$$

and since the point also lies on the graph, $d = \sqrt{x^2 + ((x-3)^2 + 4)^2}$.

To find the minimum value of d, solve the equation:

$$\frac{d}{dx}\left(\sqrt{x^2 + ((x-3)^2 + 4)^2}\right) = 0$$

for x, that is, solve the equation:

$$\frac{2x^3 - 18x^2 + 63x - 78}{\sqrt{x^4 - 12x^3 + 63x^2 - 156x + 169}} = 0$$

for x to get $x = 2.67$ and, by substitution, $y = 4.11$, correct to two decimal places. Hence the coordinates of P are $(2.67, 4.11)$, correct to two decimal places.

A graph of the function d would be helpful, in particular to check that the solution of the equation $d'(x) = 0$ corresponds to a minimum value of d. There is one exact value real solution to $d'(x) = 0$, however a numerical approximation is likely to be more readily interpreted practically in this context. Students are required to provide the answer correct to two decimal places, so should keep retain sufficient accuracy for the x-value to be able to calculate the corresponding y-value, correct to the required accuracy.

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Question 9

a. $f(x)=\int 2-\cos(x)dx = 2x-\sin(x)+C$, where C is an arbitrary constant, and

$f(\pi)=2\pi-\sin(\pi)+C=0$, so $C=-2\pi$ and hence $f(x)=2x-\sin(x)-2\pi$.

b. $f'(x) = 2-\cos(x)\geq 1$ since $-1\leq\cos(x)\leq 1$ for all $x\in R$. For a stationary point,

$f'(x) = 0$ which is not possible since $f'(x)\geq 1$, for all $x\in R$. Hence f has no stationary points over its domain. Alternatively, students could argue that as $f'(x)\geq 0$ for all $x\in R$, then f is an increasing function over its domain and hence has no stationary points.

Question 10

a. $\frac{d}{dx}(x^3\cos(2x))=3x^2\cos(2x)-2x^3\sin(2x)$

b. Given $u=x^3$, $v=\cos(2x)$, $\frac{du}{dx}=3x^2$ and $\frac{dv}{dx}=-2\sin(2x)$

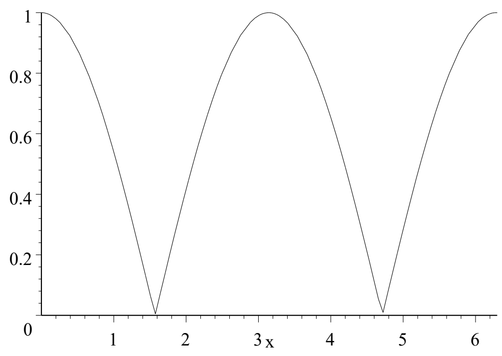
Question 11

a. $\frac{p}{2(200-p)}$, $50<p<200$

b. $p>\frac{400}{3}$

Question 12

a.



b. π

c. Area = $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} |\cos(x)| dx = 2$

Question 13

a. $f(-1) = 0 \quad \Rightarrow \quad a - b + c = -4$

$f(0) = 4 \quad \Rightarrow \quad e = 4$

$f'(1) = -2 \quad \Rightarrow \quad 4a + 3b + 2c = -2$

$f'(0) = 0 \quad \Rightarrow \quad d = 0$

$f(2) = -12 \quad \Rightarrow \quad 4a + 2b + c = -4$

b. $$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 4 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ -2 \\ 0 \\ -4 \end{bmatrix} \quad \text{or} \quad \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 4 & 3 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 & 4 \end{array}\right]$$

c. $f(x) = -2x^4 + 2x^3 + 4$

d. $f'(x) = -8x^3 + 6x^2$

$f'(x) = 0$ when $x = 0$ or $\frac{3}{4}$

Stationary points occur at (0, 4), a stationary point of inflection, and

$\left(\frac{3}{4}, \frac{539}{128}\right)$, a local maximum.

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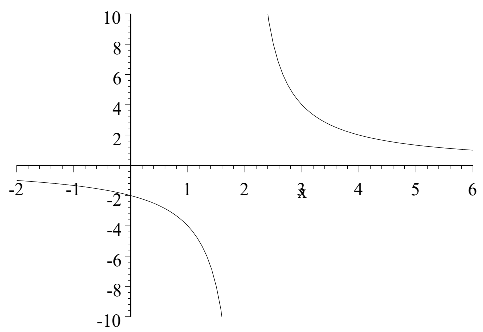
Question 14

a. The transformation is a dilation by a factor of 2 parallel to the *y*-axis. Hence *x* needs to be replaced by $\frac{x}{2}$ in the rule for each function:

i. $y=\left(\frac{x}{2}\right)^2-2=\frac{x^2}{4}-2$

ii. $y=\frac{2}{\frac{x}{2}-1}=\frac{4}{x-2}$

b.



The horizontal asymptote $y = 0$, the vertical asymptote, $x = 2$, and *y*-intercept (0, -2) should be clearly identified, either on the graph or by comment.

Question 15

a. When $k = 1$, the equations are $\begin{matrix}2x-y=1\\ x+y=1\end{matrix}$, and have the solution $x = \frac{2}{3}$ and $y = \frac{1}{3}$.

The solution is the intersection point of the two straight lines, which correspond to the given equations.

b. $x=\frac{2k}{1+2k}$ and $y=\frac{1-2k}{1+2k}$ if $k\neq\frac{-1}{2}$, and if $k = \frac{-1}{2}$, there are no solutions.

c. When $k=\frac{-1}{2}$ the equations correspond to straight lines that are parallel.

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Question 16

Let $q(x) = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$, and so $c = \alpha\beta = -6$, $b = -(\alpha + \beta) = 1$, and hence $q(x) = x2 + x - 6 = (x + 3)(x - 2)$.

Question 17

a. Solve $f'(1) = 2$ for m, $f'(x) = 3x^2 - 12x + m$, $f'(1) = 3 - 12 + m = 2$, and so $m = 11$.

b. Solve $f(3) = 0$ for n, $f(3) = 27 - 54 + 33 + n = 0$, and so $n = -6$.

Alternatively students could divide by $(x - 3)$ set the remainder to zero and solve for n.

Question 18

a. Since at $t = 0$, $h = 60$, so $d = 60$.

$$100a + 10b + c = 16 \qquad h(10) = 220 \qquad (1)$$

$$48a + 8b + c = -12 \qquad h'(4) = -12 \qquad (2)$$

$$4a + 2b + c = 16 \qquad h(2) = 92 \qquad (3)$$

b. $h(t) = t^3 - 12t^2 + 36t + 60$ (This can be obtained by defining the rule for $h(t)$, and solving $h(2) = 92$, $h'(4) = -12$, and $h(10) = 220$ simultaneously. Alternatively, the three equations in a, b and c above could be entered and solved for a, b and c. Alternatively, matrix methods could be used to solve the system of equations for a, b and c.

c. $\frac{1}{10}\int_0^{10}(t^3 - 12t^2 + 36t + 60)dt$

d. $\frac{1}{10}\int_0^{10}(t^3 - 12t^2 + 36t + 60)dt = 90$

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Question 19

a. $f(x) - g(x) = \dfrac{(x-1)^2(3x^2+2x+1)}{12}$

b. $f(x) - g(x) = \dfrac{(x-1)^2(3x^2+2x+1)}{12} \geq 0$

since $(x-1)^2 \geq 0$ and $3x^2+2x+1 = 3(x+\frac{1}{3})^2 + \frac{2}{3} \geq \frac{2}{3} > 0$ and the product of these terms will be non-negative $f(x) \geq g(x)$.

Alternative justifications for $3x^2+2x+1 \geq 0$ could be

- it is an upright parabola with no real roots (since the discriminant $\Delta = \sqrt{-10}$);
- the discriminant $\Delta = \sqrt{-10}$ so no real roots and one point (for example, the point corresponding to $x = 0$) lies above the x-axis;
- its graph is an upright parabola, with minimum value of $\frac{2}{3}$ obtained when $\frac{d}{dx}(3x^2+2x+1) = 6x+2 = 0$. That is, when $x = -\frac{1}{3}$.

Mathematical Methods (CAS) pilot study: supplementary questions – short answer solutions and comments

Question 20

a. Since f is a probability density function for X, $\int_0^4 A(1-x)^2\,dx = 1$, and this can be solved to obtain $A = \dfrac{3}{28}$.

b. $\Pr(X > 2) = \dfrac{3}{28}\int_2^4 (1-x)^2\,dx = 0.929$ or

$1 - \dfrac{3}{28}\int_0^2 (1-x)^2\,dx = 0.929$

c. $\Pr(1 < X < 2) = \dfrac{3}{28}\int_1^2 (1-x)^2\,dx = 0.036$

d. $E(X) = \dfrac{3}{28}\int_0^4 x(1-x)^2\,dx = 3.143$

e. Mode is maximum value of f over $[0,4]$, which occurs when $x = 0$, so is $\dfrac{3}{28} = 0.107$, correct to 3 decimal places.

f. $V(X) = \dfrac{3}{28}\int_0^4 x^2(1-x)^2\,dx \quad -\left(\dfrac{22}{7}\right)^2 = 0.637$, correct to 3 decimal places.

Question 21

a. Probability of getting exactly three red marbles is ${}^nC_3\,p^3(1-p)^{n-3}$,

where p = the number of red marbles divided by the total number of marbles.

b. $n = 4$. Solve $4p^3(1-p) = 0.0988$ for p. Two real solutions, giving $p = 0.3334$ or 0.9732, correct to 4 decimal places.

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Question 22

a. The sum of the probabilities must equal 1, so

$$\frac{1}{2}k^3 + \frac{1}{2}k^3 + \left(\frac{3}{4} - 3k^2\right) + \left(1 - \frac{1}{4}k\right) = 1.$$

Possible solutions are $k = 3, \frac{1}{2}, -\frac{1}{2}$. If $k = 3$, the first two probabilities in the table are greater than 1, and the third is negative, so this is impossible. If $k = -\frac{1}{2}$, first two probabilities in the table are negative and the last is bigger than 1, so this is impossible. Thus $k = \frac{1}{2}$.

b. $\Pr(X) < 2 = \Pr(X = 0) + \Pr(X = 1) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$.

Question 23

a. $\int_{-1}^{1} a(1 - x^2)dx = 1$, so $a = \frac{3}{4}$.

b. Mean is zero by symmetry. Variance = $\int_{-1}^{1} \frac{3}{4}x^2(1 - x^2)dx = 0.200$

c. $\int_{-k}^{k} \frac{3}{4}(1 - x^2)dx = 0.95$ or $\frac{3}{2}\left[x - \frac{x^3}{3}\right]_0^k = 0.95$ and solving for k gives $k = 0.8114$

(correct to 4 decimal places).

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Question 24

a. Mode is the maximum value of $p(t)$, hence it has the value $e^{-1} = 0.3679$

b. Solve $\int_0^k te^{-t}\,dt = 0.5$ for k. $k = 1.6783$

c. $\int_6^\infty te^{-t}\,dt = 7e^{-6} = 0.0174$

Question 25

a. Transition matrix $P = \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}$, initial state $S_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. To find the probabilities for three days later, find $S_3 = P^3S_0$:

$$\begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.364 \\ 0.636 \end{bmatrix}$$

The probability that Mary ate in is 0.636 (correct to 3 decimal places). An alternative approach would be to use a tree diagram.

b. $0.6^3 = 0.216$

c. Solve $PS = S$ for S, where P the transition matrix and S is the matrix of long run probabilities.

If $S = \begin{bmatrix} x \\ y \end{bmatrix}$, then x and y are determined by solving the equations $7x - 4y = 0$ and $x + y = 1$ simultaneously to obtain $S = \begin{bmatrix} \frac{4}{11} \\ \frac{7}{11} \end{bmatrix}$. In the long term, Mary will eat in 63.6% of the time, correct to one decimal place.

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Question 26

a. $\int_0^3 f(y)dy = 1$ since f is a probability density function.

Hence $\int_0^1 ky\,dy + \int_1^3 \frac{k}{2}(3-y)dy = 1$. Solving this for k gives $k = \frac{2}{3}$.

b. Pr $(Y > 1.5) = \int_{1.5}^3 \frac{k}{2}(3-y)dy = 0.375$.

Question 27

a. From the following table, since the third differences are constant, the best model is a cubic polynomial.

t (hours)	N	1st difference	2nd difference	3rd difference
0	9			
1	32	23		
2	75	43	20	
3	144	69	26	6
4	245	101	32	6
5	384	139	38	6
6	567	183	44	6
7	800	233	50	6
8	1089	289	56	6

b. Let the cubic polynomial be of the form $N = at^3 + bt^2 + ct + d$. From the table of differences, $6a = 6$, so $a = 1$. From the table of values, when $t = 0$, N = 9, hence $d = 9$.

When $t = 1$, $a + b + c + \text{d} = 32$, that is, $b + c = 22$. When $t = 2$, $8a + 4b + 2c + d = 75$, that is, $4b + 2c = 58$. Hence $b = 7$, $c = 15$, and the equation of the cubic polynomial is:

$$N = t^3 + 7t^2 + 15t + 9.$$