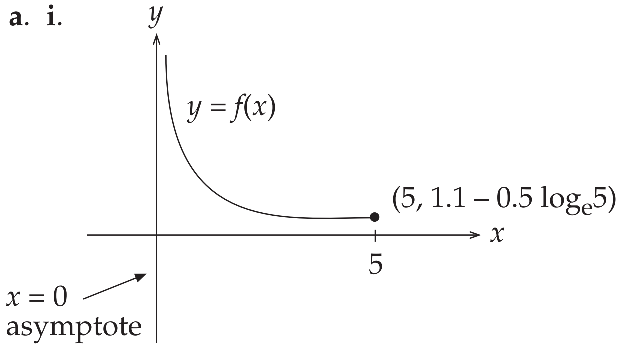
2002 Mathematical Methods
Written examination 2 (analysis task)
Suggested answers and solutions

1. a. i.



ii. The inverse function, f^{-1}, exists because f is a one-to-one function.

iii. To find f^{-1}

$f(x) = 1.1 - 0.5\log_e x$

let $y = 1.1 - 0.5\log_e x$

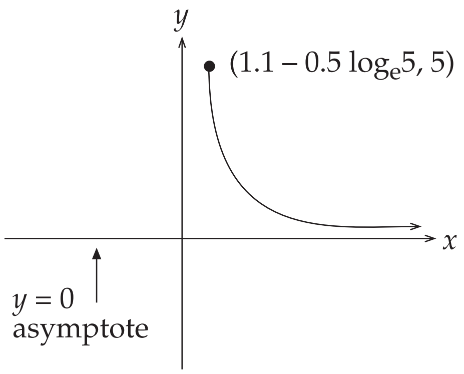
For inverse swap x with y

$$x = 1.1 - 0.5\log_e y$$
$$x - 1.1 = -0.5\log_e y$$
$$0.5\log_e y = 1.1 - x$$
$$\log_e y = 2(1.1 - x)$$
$$y = e^{2(1.1-x)}$$
$$\therefore f^{-1}(x) = e^{2(1.1-x)} = e^{2.2-2x}$$

iv. $\text{dom } f^{-1} = \text{ran } f$

$= [1.1 - 0.5\log_e 5, \infty)$

v.



b. $f(x) = a - b\log_e(x), \quad 0 < x \le 5$

substituting (1, 0.5) gives

$$0.5 = a - b\log_e 1$$
$$0.5 = a - b \times 0$$
$$0.5 = a$$

substituting (1.5, 0.3) gives

$$0.3 = 0.5 - b\log_e 1.5$$
$$b\log_e 1.5 = 0.2$$
$$b = \frac{0.2}{\log_e 1.5}$$

c. $f(x) = 1.1 - 0.5\log_e(x)$

Half the button width is given by

$$f\left(\frac{x}{2}\right) = 1.1 - 0.5\log_e\left(\frac{x}{2}\right)$$
$$= 1.1 - 0.5(\log_e x - \log_e 2)$$
$$= 1.1 - 0.5\log_e x + 0.5\log_e 2$$
$$= f(x) + 0.5\log_e 2$$
$$= f(x) + \log_e 2^{\frac{1}{2}}$$
$$f\left(\frac{x}{2}\right) = f(x) + \log_e \sqrt{2}$$

Hence the time difference is:

$f\left(\frac{x}{2}\right) - f(x) = \log_e \sqrt{2}$ as required to show.

2. a. Let X represent the antenna length of Fhaisi butterflies.

$X \sim N(20, 2^2)$

On the graphics calculator:

press 2nd VARS

2: normalcdf (–1E99, 16, 20, 2)

ENTER

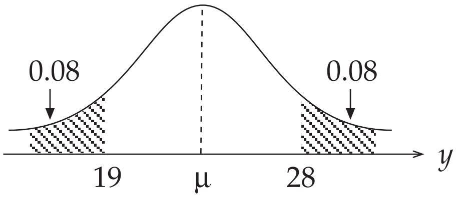
$\Pr(X < 16) = 0.023$ (3 decimal places)

b. $Y \sim N(\mu, \sigma^2)$

Let Y represent the antenna length of Jojo butterflies.

$\Pr(Y < 19) = 0.08$

$\Pr(Y > 28) = 0.08$



By symmetry:

$$\mu = \frac{19 + 28}{2}$$

$\mu = 23.5\text{mm}$

$\Pr(Y < 19) = 0.08$

$$\Pr\left(Z < \frac{19 - 23.5}{\sigma}\right) = 0.08$$

using the graphics calculator
3: invNorm (0.08) ENTER

$$\frac{19 - 23.5}{\sigma} = -1.4051$$

$$\sigma = \frac{19 - 23.5}{-1.4051}$$

$\sigma = 3.2\text{mm}$

OR $\Pr(Y < 19) = 0.08$

$\Pr(Y > 28) = 0.08$

$$\Pr\left(Z < \frac{19 - \mu}{\sigma}\right) = 0.08$$

On graphics calculator:
invNorm(0.08) is –1.4051

$$\frac{19 - \mu}{\sigma} = -1.4051$$

$19 = -1.4051\sigma + \mu$ ①

$$\Pr\left(Z < \frac{19 - \mu}{\sigma}\right) = 0.08$$

$$\Pr\left(Z < \frac{28 - \mu}{\sigma}\right) = 0.92$$

invNorm (0.92) is 1.4051

$$\therefore \frac{28 - \mu}{\sigma} = 1.4051$$

$28 = 1.40507\sigma + \mu$ ②

② + ① gives $47 = 2\mu$

$\therefore \mu = 23.5$ mm

substituting into ① gives

$19 = -1.4051\sigma + 23.5$

$1.4051\sigma = 23.5 - 19$

$$\sigma = \frac{23.5 - 19}{1.4051}$$

$\therefore \sigma \simeq 3.2$ mm

c. Pr(Jojos) = 0.2

Pr(Fhaisis) = 0.8

Let J represent the number of Jojo butterflies.

$X \sim Bi(10, 0.2, 4)$

$$\Pr(J = 4) = {}^{10}C_4 (0.2)^4 (0.8)^6$$

$\simeq 0.088$ (3 decimal places)

d. i.

$$\begin{array}{lll} \text{Fhasis} & 0.5 & \times\ 0.8 \\ \text{Jojo} & 0.1370 & \times\ 0.2 \\ \hline \sum \text{Sum} & 0.427 & \end{array}$$

ii. $\dfrac{0.8 \times 0.5}{0.4274}$

= 0.936 (3 decimal places)

3. a. $y = \frac{1}{2}\left(2x^4 - x^3 - 5x^2 + 3x\right)$

For stationary points $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 4x^3 - \frac{3}{2}x^2 - 5x + \frac{3}{2} = 0$$

b. i. When $x = 1$

$$\frac{dy}{dx} = 4 - \frac{3}{2} - 5 + \frac{3}{2}$$

$= -1$ (gradient of tangent)

gradient of normal is

$$m = \frac{-1}{-1} = 1$$

when $x = 1$

$$y = \frac{1}{2}(2 - 1 - 5 + 3)$$

$$= \frac{1}{2}(-1)$$

$$= -\frac{1}{2}$$

substitute $\left(1, -\frac{1}{2}\right)$, $m = 1$ into

$$y + \frac{1}{2} = 1(x - 1)$$

$$y = x - \frac{3}{2}$$

$\therefore y = x - \frac{3}{2}$ is equation of normal

ii. $y = \frac{1}{2}\left(2x^4 - x^3 - 5x^2 + 3x\right)$

$$y = x - \frac{3}{2}$$

Solve simultaneously to find point(s) of intersection.

$$\frac{1}{2}\left(2x^4 - x^3 - 5x^2 + 3x\right) = x - \frac{3}{2}$$

$$2x^4 - x^3 - 5x^2 + 3x = 2x - 3$$

$$2x^4 - x^3 - 5x^2 + x + 3 = 0$$

$$(x - 1)(x + 1)^2(2x - 3) = 0$$

$$\therefore x = 1, -1 \text{ or } \frac{3}{2}$$

Since $(x + 1)$ is a multiple factor then the line will touch when $x = -1$.

$$y = x - \frac{3}{2}$$

$$= -1 - \frac{3}{2}$$

$$y = -\frac{5}{2} \qquad \left(-1, -\frac{5}{2}\right)$$

Find the points of intersection of the curve and the normal can also be found using the graphics calculator.

The gradient function of the curve is:

$$\frac{dy}{dx} = \frac{1}{2}\left(8x^3 - 3x^2 - 10x + 3\right)$$

When $x = -1$, $\frac{dy}{dx} = 1$

The gradient of the curve at B (-1, -2.5) is the same as the gradient of the line

$y = x - \frac{3}{2}$, the point of intersection, so

$y = x - \frac{3}{2}$ is tangent to the point at B.

c. i. Shaded area

$$= \int_{-1}^{1} \frac{1}{2}\left(2x^4 - x^3 - 5x^2 + 3x\right) - \left(x - \frac{3}{2}\right)dx$$

$$= \int_{-1}^{1}\left(x^4 - \frac{x^3}{2} - \frac{5x^2}{2} + 0.5x + \frac{3}{2}\right)dx$$

ii. $$\text{Area} = \left[\frac{x^5}{5} - \frac{x^4}{8} - \frac{5x^3}{6} + \frac{x^2}{4} + \frac{3x}{2}\right]_{-1}^{1}$$

$$= \left(\frac{1}{5} - \frac{1}{8} - \frac{5}{6} + \frac{1}{4} + \frac{3}{2}\right) - \left(-\frac{1}{5} - \frac{1}{8} + \frac{5}{6} + \frac{1}{4} - \frac{3}{2}\right)$$

$$= \frac{2}{5} - \frac{10}{6} + \frac{6}{2}$$

$= 1.73$ (2 decimal places)

4. $x(t) = 15 + 6\sin\left(\frac{\pi t}{3}\right)$

a. **i**. Maximum occurs when

$$\sin\left(\frac{\pi t}{3}\right) = 1$$

$\therefore x(t) = 15 + 6$

$= 21$ metres

ii. Minimum height occurs when

$$\sin\left(\frac{\pi t}{3}\right) = -1$$

$\therefore x(t) = 15 + 6(-1)$

$= 15 - 6$

$= 9$ metres

$$15 + 6\sin\left(\frac{\pi t}{3}\right) = 9$$

$$6\sin\left(\frac{\pi t}{3}\right) = -6$$

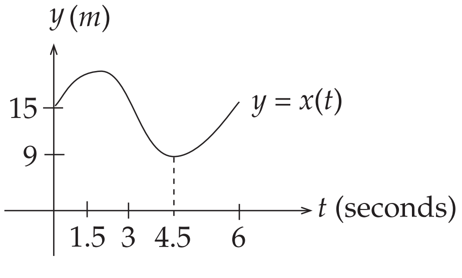
$$\sin\left(\frac{\pi t}{3}\right) = -1$$

$$\frac{\pi t}{3} = \sin^{-1}(-1)$$

$$\frac{\pi t}{3} = \frac{3\pi}{2}$$

$t = \frac{9}{2} = 4.5$ seconds

Or Using graphics calculator:



min (4.5, 9)

b. $y(t) = 15 + e^{0.04t}\sin\left(\frac{\pi t}{3}\right), \quad 0 < t \le 60$

i. Using graphics calculator

$y_1 = 15 + e^{0.04t}\sin\left(\frac{\pi t}{3}\right), \quad 0 < t \le 60$

$y_2 = 6$

2nd CALC 5: Intersect

guess 58 seconds

Platform is first 6m above the ground after 58.03 seconds.

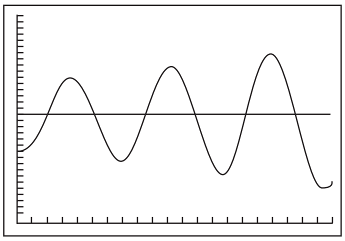
ii. $y_1 = 15 + e^{0.04t}\sin\left(\frac{\pi t}{3}\right)$

$y_2 = 15$

In WINDOW ,

let $X_{min} = 40$ and $X_{max} = 59$

ZOOM 0: zoomfit



from the graph, there are 6 points of intersection from $t = 40$ to 59.

iii. Let $y_1 = 15 + e^{0.04t}\sin\left(\frac{\pi t}{3}\right)$

and $y_2 = 24$

Set WINDOW $X_{min} = 0$, $X_{max} = 60$

ZOOM 0: zoomfit

2nd TRACE 5: Intersect

$t = 55$ seconds (to the nearest second)

c. **i**. $y = 15 + e^{0.04t}\sin\left(\frac{\pi t}{3}\right)$

using the product rule

$$\frac{dy}{dt} = e^{0.04t} \times \frac{\pi}{3}\cos\left(\frac{\pi t}{3}\right) + 0.04e^{0.04t}\sin\left(\frac{\pi t}{3}\right)$$

$$= e^{0.04t}\left(\frac{\pi}{3}\cos\left(\frac{\pi t}{3}\right) + 0.04\sin\left(\frac{\pi t}{3}\right)\right)$$

ii. Platform is closest to the ground when

$$\frac{dy}{dt} = 0$$

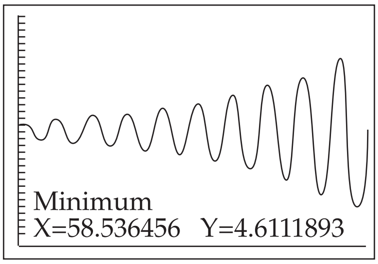
i.e. $e^{0.04t}\left(\frac{\pi}{3}\cos\left(\frac{\pi t}{3}\right) + 0.04\sin\left(\frac{\pi t}{3}\right)\right) = 0$

Using the graphics calculator, enter

$y_1 = 15 + e^{0.04t}\sin\left(\frac{\pi t}{3}\right)$

and find the minimum in the domain [0, 60]

Graphics calculator display

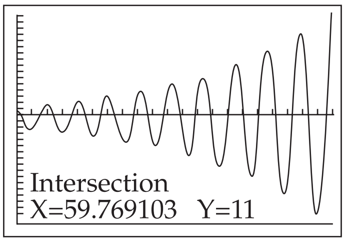


Minimum
X=58.536456 Y=4.6111893

The platform is closest to the ground when t = 58.54 seconds (2 decimal places) and when y = 4.61 metres (2 decimal places)

d. $-11 \le \frac{dy}{dt} \le 11$

$$\underset{\underset{y_1}{\uparrow}}{-11} \le \underset{\underset{y_2}{\uparrow}}{e^{0.04t}\left(\frac{\pi}{3}\cos\left(\frac{\pi t}{3}\right) + 0.04\sin\left(\frac{\pi t}{3}\right)\right)} \le \underset{\underset{y_3}{\uparrow}}{11}$$



Intersection
X=59.769103 Y=11

$X_{min} = 0$
$X_{max} = 60$
Zoomfit

for $t \in (0, 60]$, y_2 is always greater than y_1, and y_2 is less than y_3 for $t \in (0, 59.769]$

so $-11 \le \frac{dy}{dt} \le 11$ for $t \in (0, 59.769]$
(to 3 decimal places)

e. When t = 60, $\frac{dy}{dt} = 11.543443$

$$a\frac{dy}{dt} = \frac{dh}{dt}$$

when t = 60, $\frac{dh}{dt} = 11$

At t = 60

$$a\frac{dy}{dt} = \frac{dh}{dt}$$

$$a \times 11.543443 = 11$$

$$a = \frac{11}{11.543443}$$

$$\therefore a = 0.953$$

$\frac{dh}{dt} \le 11$ for $t \in (0, 60]$ when $a = 0.953$