VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2002

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

									Letter
Figures									
Words									

MATHEMATICAL METHODS

Written examination 2

(Analysis task)

Monday 11 November 2002

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.45 am (1 hour 30 minutes)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
4	4	55

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, up to four pages (two A4 sheets) of pre-written notes (typed or handwritten) and an approved scientific and/or graphics calculator (memory may be retained).
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 12 pages, with a detachable sheet of miscellaneous formulas in the centrefold.
- Working space is provided throughout the book.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other electronic communication devices into the examination room.

Instructions

- Answer **all** questions in the spaces provided.
- A decimal approximation will not be accepted if an exact answer is required to a question.
- Where an exact answer is required to a question, appropriate working must be shown.
- Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

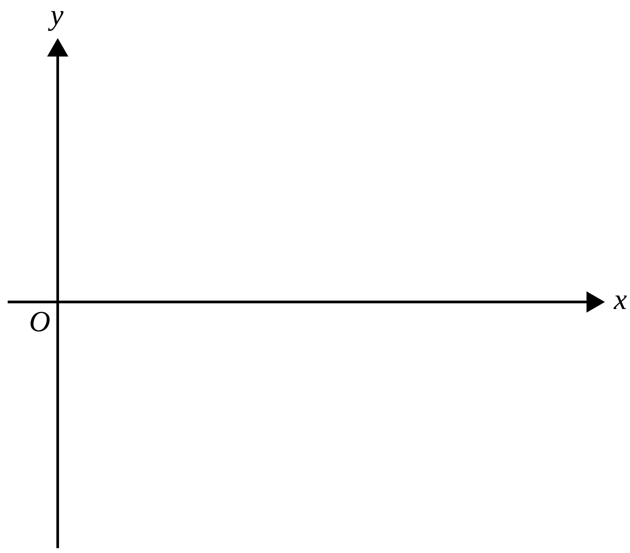
A well-designed computer screen display aims to make it quick and easy for a user to do tasks such as clicking on a screen button. Fitts' Law models the way in which the time taken to move to, and click on, a screen button depends on the distance the mouse is moved and the width of the screen button.

According to Fitts' Law, for a fixed distance travelled by the mouse, the time taken, in seconds, is given by $a - b\log_e(x)$, $0 < x \le 5$, where x cm is the button width and a and b are positive constants for a particular user.

a. Minnie discovers that, for her, $a = 1.1$ and $b = 0.5$.

i. Let $f: (0, 5] \to R$, $f(x) = 1.1 - 0.5\log_e(x)$.

Sketch the graph of $y = f(x)$ on the axes below. Label any asymptote with its equation and any end-point with its exact coordinates.



3 marks

ii. Explain why f^{-1}, the inverse function of f, exists.

1 mark

iii. Find $f^{-1}(x)$, the rule for f^{-1}.

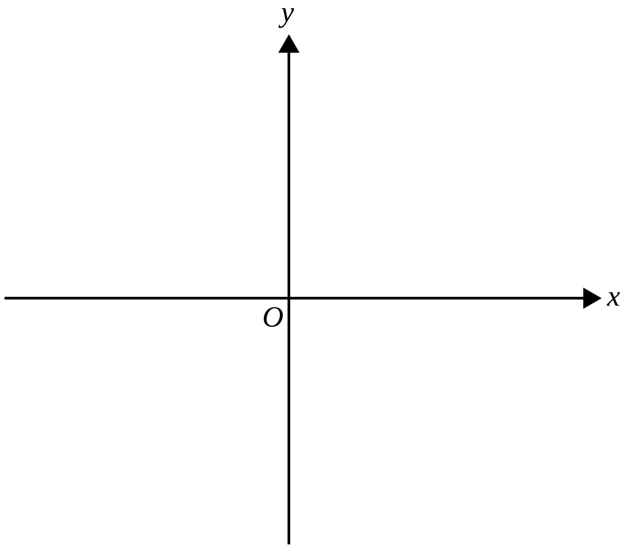
2 marks

Question 1 – continued

iv. State the domain of f^{-1}.

1 mark

v. Sketch the graph of $y = f^{-1}(x)$ on the axes below. Label any asymptote with its equation and any end-point with its exact coordinates.



2 marks

b. Mickey decides to find the values of a and b for his use. He finds that when x is 1, his time is 0.5 seconds, and when x is 1.5, his time is 0.3 seconds.

Find the exact values of a and b for Mickey.

2 marks

c. Show that, when the button width is halved, the time taken by Minnie (for whom $a = 1.1$ and $b = 0.5$) is increased by $\log_e \sqrt{2}$ seconds.

3 marks

Total 14 marks

Working space

Question 2

Emmy is gathering data on two particular species of yellow butterflies, Fhaisi and Jojo, which are very difficult to tell apart. Both species are equally likely to be caught.

One technique for telling the difference between the two species is by measuring the length of their antennas. For Fhaisi butterflies, antenna lengths are normally distributed with a mean of 20 mm and a standard deviation of 2 mm.

a. Find the probability, correct to three decimal places, that a randomly selected Fhaisi butterfly antenna is shorter than 16 mm.

2 marks

b. 8% of Jojo butterfly antennas are shorter than 19 mm and 8% of Jojo butterfly antennas are longer than 28 mm.

Assume that the antenna lengths of Jojo butterflies are also normally distributed.

Find the mean and the standard deviation of antenna lengths of Jojo butterflies, to the nearest 0.1 mm.

4 marks

In the region where Emmy is butterfly hunting, 20% of the yellow butterflies are Jojos and the other 80% are Fhaisis.

c. Find the probability, correct to three decimal places, that a random sample of 10 yellow butterflies from this region will contain exactly 4 Jojo butterflies.

2 marks

d. The probability that a randomly selected Jojo butterfly antenna is shorter than 20 mm is 0.1370.

i. Calculate the proportion, correct to three decimal places, of **all** Jojo and Fhaisi butterfly antennas from this region that are shorter than 20 mm.

2 marks

Emmy examined a single butterfly antenna from this region. It was shorter than 20 mm.

ii. What is the probability, correct to three decimal places, that it is a Fhaisi antenna?

2 marks

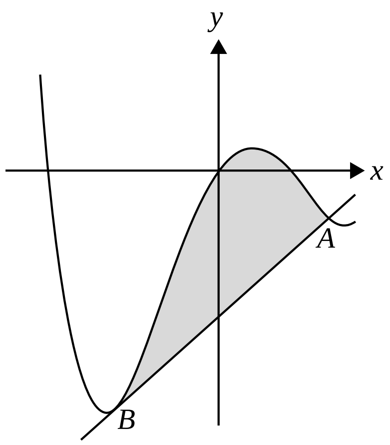
Total 12 marks

Question 3

a. Write down an equation in x, the solutions of which give the x-coordinates of the stationary points of the curve whose equation is $y = \frac{1}{2}(2x^4 - x^3 - 5x^2 + 3x)$.

2 marks

The diagram shows the curve whose equation is $y = \frac{1}{2}(2x^4 - x^3 - 5x^2 + 3x)$ and the normal to the curve at A, where $x = 1$.



b. **i.** Show that the equation of this normal is $y = x - 1.5$.

3 marks

Question 3 – continued

ii. Show that this normal is a tangent to the curve at *B*. Find the exact values of the coordinates of *B*.

4 marks

c. i. Write down a definite integral, the value of which is the area of the shaded region.

2 marks

ii. Find the area of the shaded region, correct to two decimal places.

1 mark

Total 12 marks

Question 4

On an adventure park ride, riders are strapped into seats on a platform which starts 15 metres above the ground and goes up and down repeatedly. The distance, x metres, of the platform above the ground, t seconds after the ride starts, can be modelled by the formula

$$x(t) = 15 + 6\sin\left(\frac{\pi t}{3}\right).$$

a. **i.** According to this model, find the maximum height above the ground reached by the platform.

1 mark

ii. According to this model, how many seconds after the ride starts is the platform first closest to the ground and how high above the ground is it at that time?

2 marks

Tasmania Jones is redesigning the ride so that the platform moves further up and down each cycle.

During the first 60 seconds of the redesigned ride, y metres, the distance of the platform above the ground t seconds after the ride starts, can be modelled by the formula

$$y(t) = 15 + e^{0.04t}\sin\left(\frac{\pi t}{3}\right), \quad 0 < t \le 60.$$

b. **i.** According to this model, the platform is exactly 6 metres above the ground for the first time about 58 seconds into the ride. Find this time correct to two decimal places of a second.

1 mark

ii. According to this model, how many times is the platform exactly 15 metres above the ground from $t = 40$ to $t = 59$?

1 mark

Question 4 – continued

iii. According to this model, find the time from when the ride starts until the platform first reaches 24 metres above the ground. Give your answer correct to the nearest second.

2 marks

c. **i.** Find an expression for $\frac{dy}{dt}$.

2 marks

ii. Hence write down an equation, one solution of which is the value of *t* when the platform is closest to the ground. Find this value of *t*, correct to two decimal places. Also find, according to the model, the distance in metres of the platform above the ground at that time, correct to two decimal places.

3 marks

Safety regulations for the ride require that $-11 \le \frac{dy}{dt} \le 11$.

d. Find the range of values of t, correct to three decimal places, for which $-11 \le \frac{dy}{dt} \le 11$ during the first 60 seconds of the ride.

2 marks

Tasmania can adjust the ride so the model for the distance, in metres, of the platform above the ground, t seconds after the ride starts, becomes

$$h(t) = 15 + ae^{0.04t}\sin\left(\frac{\pi t}{3}\right), \quad 0 < t \le 60, \qquad \text{where } a \text{ is a positive constant.}$$

e. Find, correct to three decimal places, the greatest value of a such that $\frac{dh}{dt}$ is never more than 11 during the first 60 seconds of the ride.

3 marks

Total 17 marks

END OF QUESTION AND ANSWER BOOK

MATHEMATICAL METHODS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time. This formula sheet is provided for your reference.

Mathematical Methods Formulas

Mensuration

area of a trapezium: $\frac{1}{2}(a+b)h$

curved surface area of a cylinder: $2\pi rh$

volume of a cylinder: $\pi r^2 h$

volume of a cone: $\frac{1}{3}\pi r^2 h$

volume of a pyramid: $\frac{1}{3}Ah$

volume of a sphere: $\frac{4}{3}\pi r^3$

area of a triangle: $\frac{1}{2}bc\sin A$

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$

$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$

$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$

$\frac{d}{dx}(\tan(ax)) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$

$\int e^{ax}dx = \frac{1}{a}e^{ax} + c$

$\int \frac{1}{x}dx = \log_e(x) + c$, for $x > 0$

$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$

$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Statistics and Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

mean: $\mu = E(X)$

variance: $\text{var}(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

Discrete distributions			
	$\Pr(X = x)$	mean	variance
general	$p(x)$	$\mu = \Sigma\, x\, p(x)$	$\sigma^2 = \Sigma\,(x-\mu)^2\, p(x)$ $= \Sigma\, x^2 p(x) - \mu^2$
binomial	${}^nC_x\, p^x(1-p)^{n-x}$	np	$np(1-p)$
hypergeometric	$\frac{{}^DC_x\, {}^{N-D}C_{n-x}}{{}^NC_n}$	$n\frac{D}{N}$	$n\frac{D}{N}\left(1-\frac{D}{N}\right)\left(\frac{N-n}{N-1}\right)$
Continuous distributions			
normal	If X is distributed $N(\mu, \sigma^2)$ and $Z = \frac{X-\mu}{\sigma}$, then Z is distributed N(0, 1).		

Table 1 Normal distribution – cdf

x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359	4	8	12	16	20	24	28	32	36
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753	4	8	12	16	20	24	28	32	35
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141	4	8	12	15	19	23	27	31	35
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517	4	8	11	15	19	23	26	30	34
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879	4	7	11	14	18	22	25	29	32
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224	3	7	10	14	17	21	24	27	31
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549	3	6	10	13	16	19	23	26	29
0.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7793	.7823	.7852	3	6	9	12	15	18	21	24	27
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133	3	6	8	11	14	17	19	22	25
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389	3	5	8	10	13	15	18	20	23
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621	2	5	7	9	12	14	16	18	21
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830	2	4	6	8	10	12	14	16	19
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015	2	4	6	7	9	11	13	15	16
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177	2	3	5	6	8	10	11	13	14
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319	1	3	4	6	7	8	10	11	13
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441	1	2	4	5	6	7	8	10	11
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545	1	2	3	4	5	6	7	8	9
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633	1	2	3	3	4	5	6	7	8
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706	1	1	2	3	4	4	5	6	6
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767	1	1	2	2	3	4	4	5	5
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817	0	1	1	2	2	3	3	4	4
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857	0	1	1	2	2	2	3	3	4
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890	0	1	1	1	2	2	2	3	3
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916	0	1	1	1	1	2	2	2	2
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936	0	0	1	1	1	1	1	2	2
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952	0	0	0	1	1	1	1	1	1
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964	0	0	0	0	1	1	1	1	1
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974	0	0	0	0	0	1	1	1	1
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981	0	0	0	0	0	0	0	1	1
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986	0	0	0	0	0	0	0	0	0
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990	0	0	0	0	0	0	0	0	0
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993	0	0	0	0	0	0	0	0	0
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995	0	0	0	0	0	0	0	0	0
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997	0	0	0	0	0	0	0	0	0
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998	0	0	0	0	0	0	0	0	0
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	0	0	0	0	0	0	0	0	0
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0	0	0	0	0	0	0	0	0

END OF FORMULA SHEET