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MATHS METHODS 3 & 4 TRIAL EXAMINATION 1 2003 SOLUTIONS

PART I – Multiple-choice answers

1.	**C**	**10.**	**A**	**19.**	**E**
2.	**E**	**11.**	**E**	**20.**	**B**
3.	**E**	**12.**	**B**	**21.**	**D**
4.	**B**	**13.**	**A**	**22.**	**C**
5.	**C**	**14.**	**D**	**23.**	**B**
6.	**C**	**15.**	**C**	**24.**	**E**
7.	**B**	**16.**	**A**	**25.**	**C**
8.	**B**	**17.**	**D**	**26.**	**D**
9.	**C**	**18.**	**D**	**27.**	**E**

PART I – Multiple-choice solutions

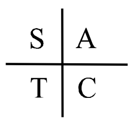
Question 1

The amplitude of the function $y = 2\sin 3\left(x + \frac{\pi}{2}\right)$ is 2 and there is no vertical translation of the basic graph of $y = 2\sin x$. The domain is clearly greater than one complete period so there is no consideration needed about a restricted domain.
So, the range is $[-2,2]$.
Alternatively, just sketch the graph to see the range.
The answer is C.

Question 2

$2\cos(2x) = 1 \qquad 0 \le x \le 2\pi$

So, $\cos(2x) = \frac{1}{2} \qquad$ So, $0 \le 2x \le 4\pi$



$$2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

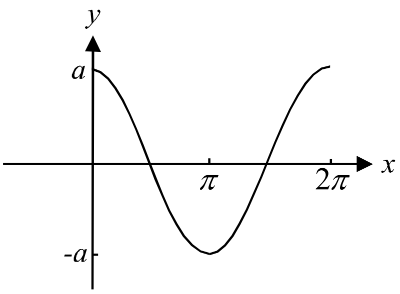
The sum of these solutions is $\frac{24\pi}{6} = 4\pi$.
The answer is E.

Question 3

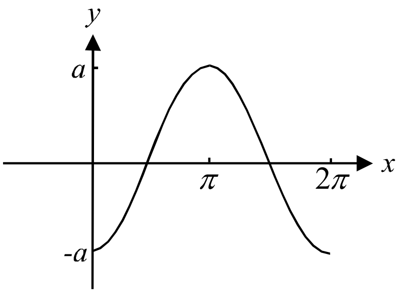
The period of the function is given by $\frac{2\pi}{n}$ where in this case, $n = \frac{\pi}{16}$. So, $2\pi \div \frac{\pi}{16} = 32$.
So, after 32 hours one complete cycle of the drug has been administered.
The answer is E.

Question 4

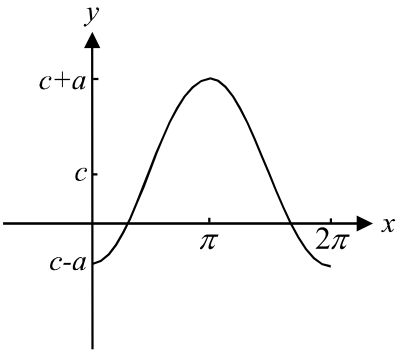
First consider the graph of $y = a\cos(x)$ where a is positive.



Now reflect the graph in the x-axis to obtain $y = -a\cos(x)$.



Now translate the graph c units upwards. Note that since $a > c$ some of the graph will remain below the x-axis.



The only graph that reflects all these features is graph B.
The answer is B.

Question 5

The graph touches the x-axis at $x = a$ and $x = c$. It therefore has "repeated factors" of $(x-a)$ and $(x-c)$. The graph does not cut the x-axis.
The rule could be $y = (x-a)^2(x-c)^2$.
The answer is C.

Question 6

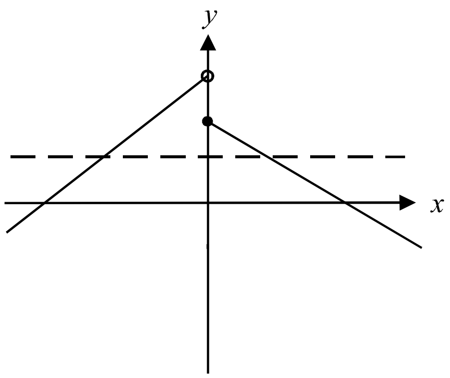
The function $y = \log_e(x + a)$ has a vertical asymptote of $x = -a$.

The function $y = \dfrac{1}{x + a}$ has a vertical asymptote of $x = -a$.

The function $y = \dfrac{1}{x - a} - a$ has a vertical asymptote of $x = a$ as required.

Options D and E do not have a vertical asymptote.
The answer is C.

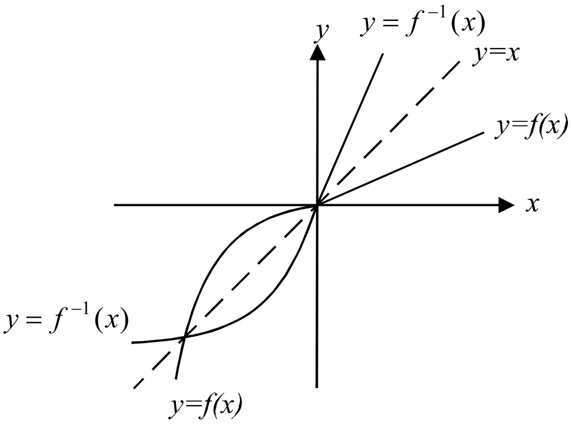
Question 7



If a horizontal line can be drawn, which cuts a function more then once, then that function is many-one. This means that there are many (or more than one!) values of *x* which have the same (one) value of *y*. In this question graph B shows a many-one function.
The answer is B.

Question 8

The graph of the inverse function f^{-1} is obtained by reflecting the graph of $y = f(x)$ in the line $y = x$. This gives us



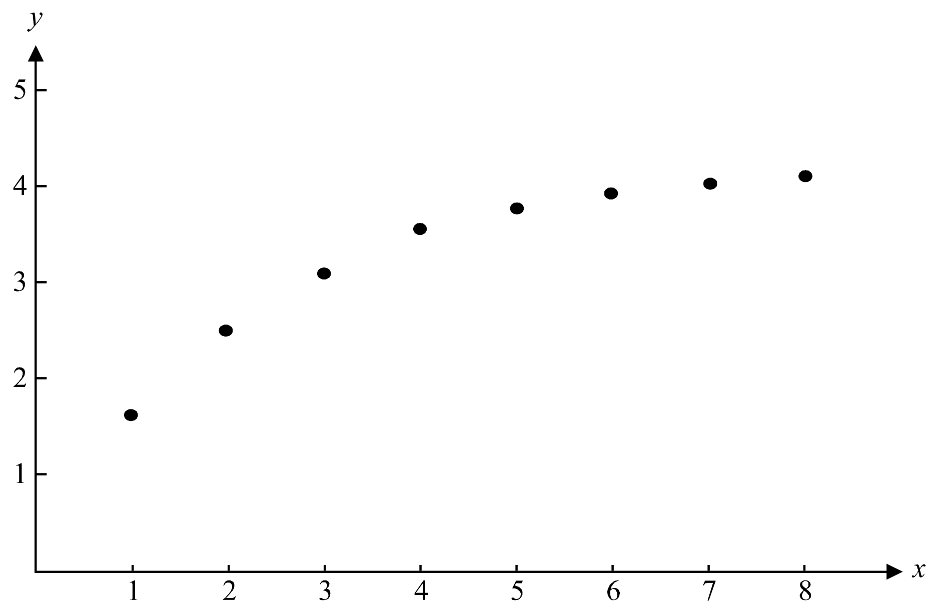
The answer is B.

Question 9

With Pascal's triangle, the 3rd row gives the coefficients of the various x^n terms in the expansion of $(x+1)^2$. The 4th row gives the coefficients of the various x^n terms in the expansion of $(x+1)^3$. And so the pattern continues. To find the coefficients for an expansion of $(x+1)^{10}$ we need to look at the 11th row.
The x^4 term in the expansion of $(x+2)^{10}$ will be $210 \times x^4 \times 2^6$.
So the coefficient is 210×2^6.
The answer is C.

Question 10

Draw a scatterplot of the data on your calculator.



The data could be modelled best by a logarithmic function.
The answer is A.

Question 11

$$5\times 2^{3x}=1$$
$$2^{3x}=\frac{1}{5}$$
$$\log_2\left(\frac{1}{5}\right)=3x$$
$$x=\frac{1}{3}\log_2\left(\frac{1}{5}\right)$$
$$x=\log_2\left(\frac{1}{5}\right)^{\frac{1}{3}}$$
$$x=\log_2\left(\frac{1}{\sqrt[3]{5}}\right)$$

The answer is E.

Question 12

$$e^{\left(\log_e(2x)+2\log_e(x^3)\right)}$$
$$=e^{\left(\log_e(2x)+\log_e\left(x^3\right)^2\right)}$$
$$=e^{\log_e(2x)+\log_e(x^6)}$$
$$=e^{\log_e\left(2x\times x^6\right)}$$
$$=e^{\log_e(2x^7)}$$
$$=2x^7$$

The answer is B.

Question 13

$$y=\frac{1}{\sqrt{1-\cos(x)}}$$
$$=(1-\cos(x))^{\frac{-1}{2}}$$

So, $\dfrac{dy}{dx}=-\dfrac{1}{2}(1-\cos(x))^{\frac{-3}{2}}\times\sin(x)$

$$=\frac{-\sin(x)}{2(1-\cos(x))^{\frac{3}{2}}}$$

The answer is A.

Question 14

Let $y = e^{\tan(2x)}$

$= e^u$ where $u = \tan(2x)$

So, $\dfrac{dy}{du} = e^u$ $\dfrac{du}{dx} = 2\sec^2(2x)$

Now, $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ (Chain rule)

So, $\dfrac{dy}{dx} = e^u \cdot 2\sec^2(2x)$

$= 2e^{\tan(2x)}\sec^2(2x)$

The answer is D.

Question 15

$$y = \log_e\left(2x^2 + 5x\right)$$

The rate of change of y with respect to x is given by $\dfrac{dy}{dx}$.

So, $\dfrac{dy}{dx} = \dfrac{4x+5}{2x^2+5x}$

When $x = 1$,

$\dfrac{dy}{dx} = \dfrac{4+5}{2+5}$

$= \dfrac{9}{7}$

The answer is C.

Question 16

$f'(x) < 0$ means that the gradient of the graph of $y = f(x)$ is negative so that tangents drawn to the graph will slope up to the left(or down to the right).
Such tangents occur for $x \in (-b, c) \cup (d, e)$. Note that at $x = -b, x = c, x = d$ and $x = e$, $f'(x) = 0$. That is, there is a stationary point at these points.
The answer is A.

Question 17

With $x = 5$, $f(x) = f(5)$

Now, $x + h = 4 \cdot 95$

So, $5 + h = 4 \cdot 95$

So, $h = -0 \cdot 05$

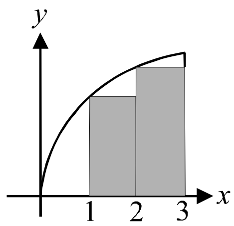
So $f(x+h) \approx f(5) - 0 \cdot 05 f'(5)$

The answer is D.

Question 18

Sketch the graph of $y=\sqrt{9-(x-3)^2}$ on your calculator. We have the upper branch of a circle with radius 3 units and centre at $(3,0)$.

The approximate area required using the left rectangles method is shaded in the diagram and is given by



$$\begin{aligned}
&f(0)\times 1+f(1)\times 1+f(2)\times 1\\
&=0+\sqrt{9-(1-3)^2}+\sqrt{9-(2-3)^2}\\
&=\sqrt{5}+\sqrt{8}
\end{aligned}$$

The answer is D.

Question 19

$$\begin{aligned}
\int -2\sin\left(\frac{x}{2}\right)dx &= -2\int \sin\left(\frac{x}{2}\right)dx\\
&= -2\times -2\cos\left(\frac{x}{2}\right)+c\\
&= 4\cos\left(\frac{x}{2}\right)+c
\end{aligned}$$

An antiderivative is $4\cos\left(\frac{x}{2}\right)$ (that is, the case where $c = 0$).

The answer is E.

Question 20

$$\begin{aligned}
&\int_0^1 (5x-1)^6\,dx\\
&=\left[\frac{1}{35}(5x-1)^7\right]_0^1\\
&=\frac{1}{35}\left(4^7-(-1)^7\right)\\
&=\frac{1}{35}\times 16385\\
&=\frac{3277}{7}
\end{aligned}$$

The answer is B.

Question 21

Now, $\int_a^b (f(x)-g(x))dx = -\int_b^a (f(x)-g(x))dx$.

This rules out options A and B.

Also $\int_a^b (f(x)-g(x))dx$

$= \int_a^b f(x)dx - \int_a^b g(x)dx$

$= -\int_b^a f(x)dx - \int_a^b g(x)dx$

This rules out options C and E.

Now, since $\int_b^b f(x)dx = 0$, option D is correct.

The answer is D.

Question 22

The areas below the x-axis are negative signed areas. To find the total actual area we therefore need to find

$$\int_d^e f(x)dx - \int_c^d f(x)dx + \int_b^c f(x)dx - \int_a^b f(x)dx$$

The answer is C.

Question 23

The term 'discrete' refers to the type of variable that X is rather than the type of probability distribution that X follows. The probability that Kate forgets her text book is always 20% so X cannot be hypergeometric where the probability is different for each trial or in this case for each appearance in class.

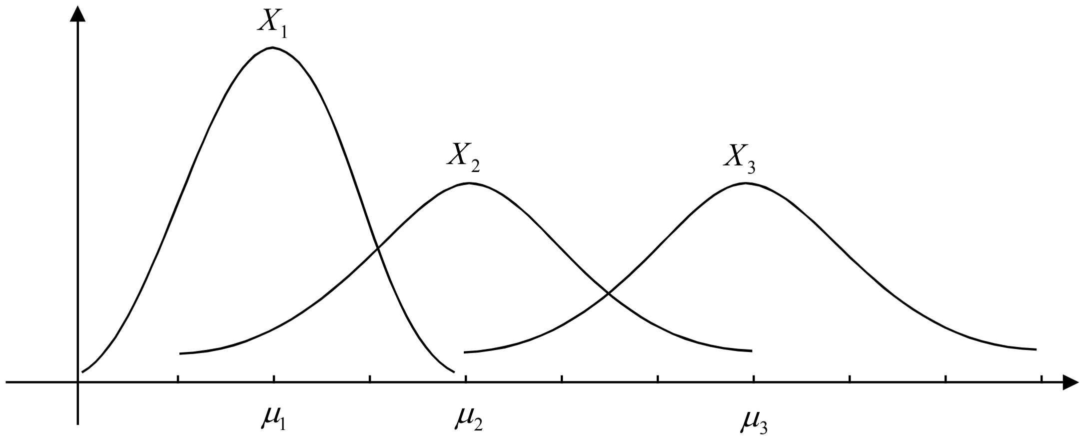
As with option A, the term "continuous" refers to the type of variable that X is, rather than the type of probability distribution that X follows.

The variable X does not follow a normal distribution.

The variable X follows a binomial distribution since there are two possible outcomes at each trial (or lesson) remembering or forgetting, and the probability of forgetting is always the same i.e. 20%.

The answer is B.

Question 24



From the diagram we see that the highest mean is that of the distribution X_3 and the lowest mean is that of distribution X_1. So, options A and B are not true.
The variance of distributions X_2 and X_3 are the same since they are the same shape and they are more spread out than distribution X_1.
So, distribution X_1 has the lowest variance.
The answer is E.

Question 25

The number of black jelly beans eaten by Jane is a hypergeometric random variable since there is no replacement.
Now, $n = 5, D = 10, N = 100$

$$\text{variance} = n\frac{D}{N}\left(1 - \frac{D}{N}\right)\left(\frac{N-n}{N-1}\right) \qquad \text{From the formula sheet}$$

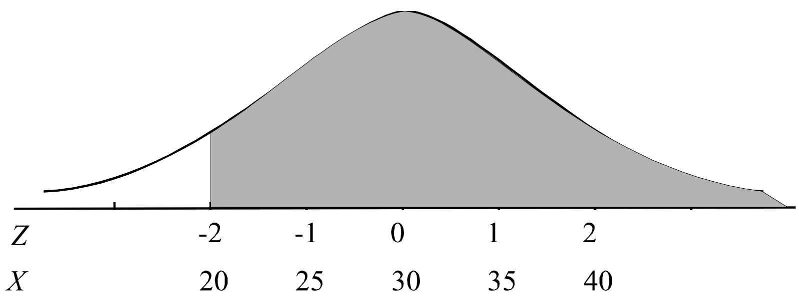
$$= 5 \times \frac{10}{100}\left(1 - \frac{10}{100}\right)\left(\frac{100-5}{100-1}\right)$$

$$= 0 \cdot 432 \text{ (correct to 3 decimal places)}$$

The answer is C.

Question 26

Sketch a diagram.
We see that
$\Pr(Z > -2) = \Pr(X > 20)$
Because of the symmetrical nature of the the normal distribution, this is the same as
$\Pr(X < 40)$
The answer is D.



Question 27

The number of even numbers Mick gets follows a binomial distribution where $n = 6$ and p = probability of throwing an even number. So $p = 0 \cdot 5$.
We require

$$\begin{aligned}\Pr(x \geq 3) &= 1 - \Pr(x < 3)\\ &= 1 - \{\Pr(x = 0) + \Pr(x = 1) + \Pr(x = 2)\}\\ &= 1 - \left\{{}^6C_0(0 \cdot 5)^0(0 \cdot 5)^6 + {}^6C_1(0 \cdot 5)^1(0 \cdot 5)^5 + {}^6C_2(0 \cdot 5)^2(0 \cdot 5)^4\right\}\\ &= 1 - \left\{(0 \cdot 5)^6 + 3(0 \cdot 5)^5 + {}^6C_2(0 \cdot 5)^6\right\}\end{aligned}$$

The answer is E.

PART II - solutions

Question 1

a. Since X is a discrete random variable,

$a + 2a + 3a + 4a = 1$

So, $10a = 1$

$a = \frac{1}{10}$ or $0 \cdot 1$

(1 mark)

b. $E(X) = 12 \times 0 \cdot 1 + 13 \times 0 \cdot 2 + 14 \times 0 \cdot 3 + 15 \times 0 \cdot 4$

$= 14$ players

(1 mark)

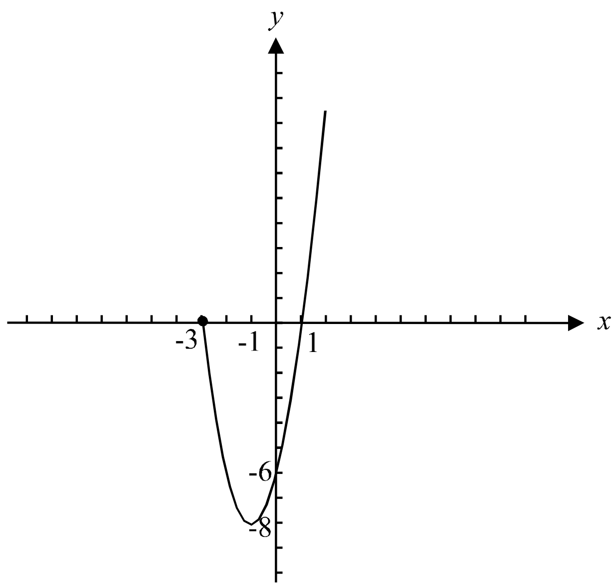
c. $\Pr(X \geq 13) = 0 \cdot 2 + 0 \cdot 3 + 0 \cdot 4$

$= 0 \cdot 9$

(1 mark)

Question 2

a.



Method 1 – algebraically

$$\begin{aligned} f(x) &= 2(x+1)^2 - 8 \\ &= 2(x^2 + 2x + 1) - 8 \\ &= 2x^2 + 4x + 2 - 8 \\ &= 2x^2 + 4x - 6 \\ &= 2(x^2 + 2x - 3) \\ &= 2(x+3)(x-1) \end{aligned}$$

x-intercepts occur at $x = -3$ and $x = 1$. So, the endpoint, which is included, (hence a closed circle) of the graph occurs at $(-3,0)$ since the domain of $f(x)$ is $[-3, \infty)$.

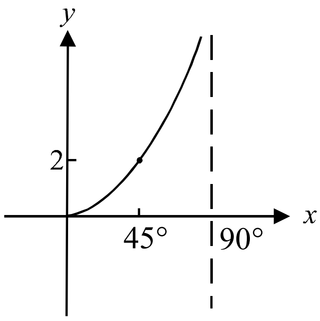
(1 mark) correct graph including endpoint

Method 2 – calculator
Sketch the function. Using a table of values, locate where $x = -3$. This is the endpoint. Sketch the graph including this endpoint.

b. The inverse function $f^{-1}(x)$ will exist if and only if $f(x)$ is one to one. If we can draw a horizontal line through the function $f(x)$ and it cuts the function only once then it is one to one. To achieve this, then the least value of a would be –1 which is the x-coordinate of the turning point of the parabola. **(1 mark)**

Question 3

a.



(1 mark) (need to show asymptote as well as 1 point other than $(0°,0)$)

b. Method 1 – calculator
Set the mode to degrees.
Find the point of intersection.
It is $(26\cdot57°, 1)$ **(2 marks)**
Check from your graph in part **a**. that this is a feasible answer.

Method 2 – algebraically
The point of intersection occurs
when $2\tan(x^\circ) = 1$ **(1 mark)**

$$\tan(x^\circ) = \frac{1}{2}$$

$$x^\circ = 26\cdot57^\circ \text{ (to 2 decimal places)}$$

The point of intersection is $(26\cdot57°,1)$.
(1 mark)
Check from your graph in part **a**. that this is a feasible answer.

Question 4

a. We start with $y = e^x$.

After a dilation by a factor of 2 from the y-axis, we obtain $y = e^{\frac{x}{2}}$.

After a dilation by a factor of 3 from the x-axis, we obtain $y = 3e^{\frac{x}{2}}$.

After a translation of 4 units parallel to the x-axis in the positive direction, we obtain $y = 3e^{\frac{x-4}{2}}$.

After a reflection in the x-axis we obtain $y = -3e^{\frac{1}{2}(x-4)}$.

(1 mark) for obtaining correct function after 2 transformations
(2 marks) for correct function

b. The range of the function is $(-\infty, 0)$.

(1 mark)

Question 5

$$\begin{aligned} y &= \frac{-1}{x+3} - 2 \\ &= -1(x+3)^{-1} - 2 \\ \frac{dy}{dx} &= 1(x+3)^{-2} \times 1 \\ &= \frac{1}{(x+3)^2} \end{aligned}$$

$$\text{At } x = -2, \ \frac{dy}{dx} = \frac{1}{1^2} = 1$$

The gradient of the tangent at $x = -2$ is 1 and so the gradient of the normal at $x = -2$ is $\frac{-1}{1} = -1$. **(1 mark)**

$$\text{When } x = -2, \quad y = \frac{-1}{-2+3} - 2 = -3$$

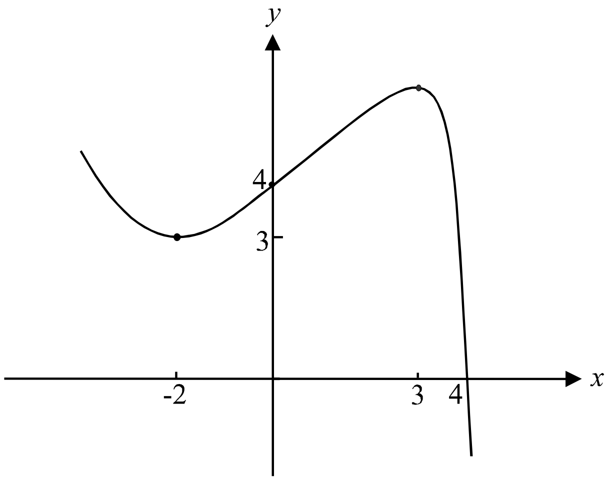
So, the normal through the point $(-2, -3)$ with gradient -1, is given by

$$\begin{aligned} y - -3 &= -1(x - -2) \\ y + 3 &= -1(x+2) \\ y &= -x - 5 \end{aligned}$$

(1 mark)

Question 6

Plot the points $f(-2)=3$ and $f(0)=4$. Note that at $(-2,3)$ there is a stationary point.



Since $f'(x)<0$ for $x\in(-\infty,-2)$, the graph goes up to the left as you move left from $(-2,3)$.
Since $f'(x)>0$ for $x\in(-2,\ 3)$, and $f'(x)<0$ for $x\in(3,\infty)$, there must be a stationary point at $x=3$ ie. $f'(3)=0$.
The average rate of change between $x=0$ and $x=4$ is -1. Now, a straight line with a gradient of -1 passing through the point $(0,4)$ would have to pass through the point $(4,0)$.

Alternatively,
$$\frac{f(0)-f(4)}{0-4}=-1$$
$$\frac{4-f(4)}{-4}=-1$$
$$4-f(4)=4$$
$$-f(4)=0$$
$$f(4)=0$$

So, the graph of *f* passes through $(4,0)$.

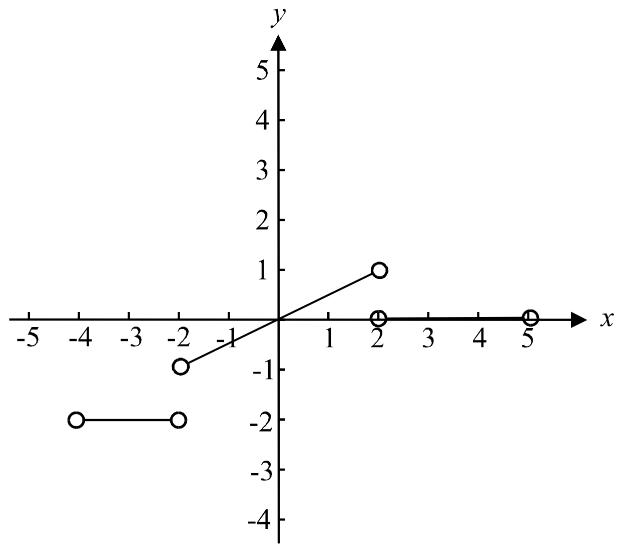
(1 mark) for marking in the points $(-2,3)$ and $(0,4)$
(1 mark) for drawing the correct positive and negative gradient branches
(1 mark) for finding the point $(4,0)$

Question 7

a. $r_f = [-3,-2) \cup [0,4)$.

(1 mark)

b.



(1 mark) for 2 branches with correct endpoints
(2 marks) for 3 branches with correct endpoints

c. $d_{f'} = (-4,-2) \cup (-2,2) \cup (2,5)$

(1 mark)

Question 8

a. $y = (x+2)e^{\frac{x}{2}}$

$$\frac{dy}{dx} = (x+2) \times \frac{1}{2} e^{\frac{x}{2}} + 1 \times e^{\frac{x}{2}} \qquad \text{(Product rule)}$$

$$= \frac{1}{2}(x+2)e^{\frac{x}{2}} + e^{\frac{x}{2}}$$

(1 mark)

b. Hence means use what you have already found.

Now, $\frac{dy}{dx}=\frac{1}{2}(x+2)e^{\frac{x}{2}}+e^{\frac{x}{2}}$ (from part a.)

So, $\int\frac{dy}{dx}dx=\int\frac{1}{2}(x+2)e^{\frac{x}{2}}\,dx+\int e^{\frac{x}{2}}\,dx$ **(1 mark)**

$y=\frac{1}{2}\int(x+2)e^{\frac{x}{2}}\,dx+2e^{\frac{x}{2}}+c$ c is a constant

So, rearranging we have

$\frac{1}{2}\int(x+2)e^{\frac{x}{2}}\,dx=y-2e^{\frac{x}{2}}-c$

$\int(x+2)e^{\frac{x}{2}}\,dx=2y-4e^{\frac{x}{2}}-2c$

$=2(x+2)e^{\frac{x}{2}}-4e^{\frac{x}{2}}-2c$ **(1 mark)** This is an acceptable answer.

$=2xe^{\frac{x}{2}}+4e^{\frac{x}{2}}-4e^{\frac{x}{2}}-2c$

$=2xe^{\frac{x}{2}}-2c$

$=2xe^{\frac{x}{2}}+A$ where $A=-2c$

These last 3 lines are an example of how to "tidy up" further and are included for interest only.

Total 23 marks