Instructions

Answer **all** questions.
A decimal approximation will not be accepted if an exact answer is required to a question.
Where an exact answer is required to a question, appropriate working must be shown.
Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
In questions where more than 1 mark is available, appropriate working must be shown.

Question 1

The rate of flow of water (measured in litres per minute) from a tank when a tap is left on is given by the equation

$$\frac{dV}{dt} = -20\log_e(0.05)t + C \text{ for } 0 < t \leq 240$$

where C is a constant and t is the time in minutes from when the tap is turned on.

a. The rate was measured at exactly 50 litres/min after 20 minutes of flow. Calculate the constant C.

1 mark

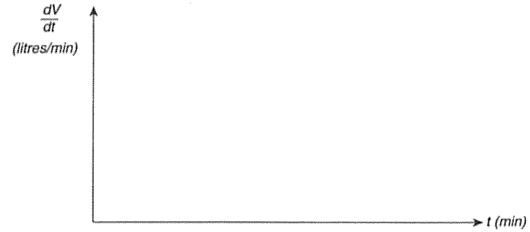
b. Calculate to 3 decimal places the rate of flow when $t = 240$ minutes.

1 mark

c. Calculate the **exact** time when the rate of flow was 40 litres/min.

2 marks

d. Sketch the graph of $\frac{dV}{dt}$ versus t over the specified domain indicating the all important features.



2 marks

e. **i.** Write an equation to calculate the volume of water (V litres) released between the times t_1 minutes and t_2 minutes.

1 mark

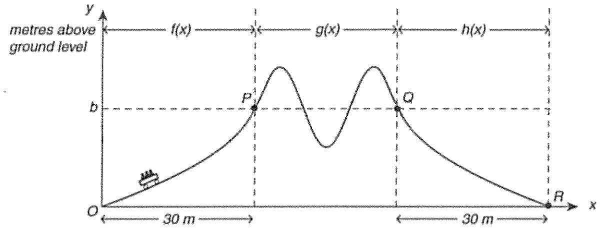
ii. Evaluate this expression to find the volume of water released during the third hour. State your answer to the nearest litre.

2 marks
Total 9 marks

Question 2

Part of a roller coaster track is depicted below. To analyse the equations of the track, the x-axis is aligned with the ground and the y-axis is at point O.

The section of the track for $0 \le x \le 30$ follows the equation $f(x) = e^{0.08x} - 1$.



a. Calculate the value of b (the height of point P) to 3 decimal places.

1 mark

b. Write an expression for the gradient of the tangent at any point along $y = f(x)$.

1 mark

c. Calculate the gradient of the track to 3 decimal places at point P.

1 mark

The middle section of the track from P to Q has the equation $g(x) = A\sin 0.5(x - B) + C$.

d. Write down the values of B and C, giving your answers to 3 decimal places where necessary.

2 marks

e. Write an expression for the gradient of $g(x)$ at any point.

1 mark

f. Given that the gradient of $g(x)$ is equal to the gradient of $f(x)$ at point P (when $x = 30$), evaluate A to 3 decimal places.

2 marks

g. Calculate the period of $g(x)$.

1 mark

h. Q is a point on the downward slope which has a gradient equivalent to the negative of that at P. Write down the x value of point Q to 3 decimal places.

1 mark

i. $h(x)$ can be considered as a reflection of $f(x)$ in the y-axis, followed by a translation in the positive x direction. Write down the equation of $h(x)$.

2 marks

It was required to calculate the cross-sectional area of the entire track from O to R.

j. Write an integral expression using the relevant equations which could be used to calculate this area.

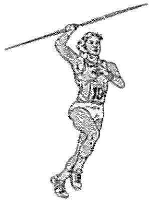
2 marks

k. Evaluate this area to the nearest square metre.

1 mark

Total 15 marks

Question 3



Roshan is a highly competent javelin thrower for his school. He is keen to break his record of 65 metres in the upcoming interschool athletics carnival. Because he has been training so hard his coach deduces that he has an 80% probability of breaking his record with any particular throw. Assuming that each throw is independent, find the probability that in 10 throws

a. exactly 8 throws will pass the record mark. Express your answer to 3 decimal places.

2 marks

b. 8 or more throws pass the record mark. Express your answer to 3 decimal places.

1 mark

c. the first 8 throws will pass the record but the last 2 throws will not. Express your answer as a percentage correct to 1 decimal place.

2 marks

d. that 8 throws pass the record mark, given that he was successful with the last two throws. Express your answer to 3 decimal places.

2 marks

Prior to the carnival, Roshan's training will consist of 250 sets of 10 throws.

Find the expected number of sets in which Roshan is likely to produce

e. **i.** 8 throws that will pass the record mark. Express your answer to 1 decimal place.

1 mark

ii. more than 8 throws that will break the record. Express your answer to 1 decimal place.

1 mark

f. What chance does Roshan have of exactly 8 successful throws in less than 50 of the sets? Express your answer to 4 decimal places.

3 marks

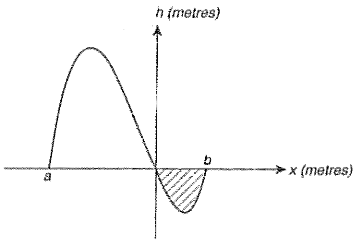
Total 12 marks

Question 4

Kristy-Anne is a local engineer who wishes to build a giant water slide for the Wyndham Community Fun Park. She submits her plans to the local council. These plans show that the slide she wishes to build is closely modelled by the function

$$h(x) = \frac{1}{215}(x^3 + 18x^2 - 360x),\ a \le x \le b$$

where h is the height of the slide in metres in the interval $[a, b]$. The function is drawn on the axes below.



a. Factorise $x^3 + 18x^2 - 360x$ and hence state the values of a and b.

2 marks

b. Using calculus, show that the highest point of the slide, correct to the nearest metre, is 30 metres above the ground.

4 marks

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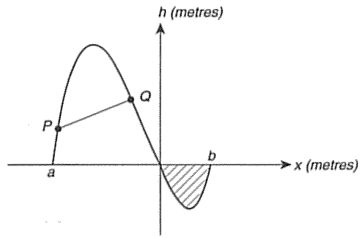
c. Show that the depth of the water at the lowest point of the slide is less than 7 metres.

1 mark

d. What is the average gradient between these two points? Express your answer to the nearest whole number.

2 marks

Kristy-Anne's design is not approved by the council for safety reasons. It is suggested that she reinforce the structure with a strut according the diagram below.



The strut is modelled by the function $g(x) = cx + d$, which is the normal to the curve $y = h(x)$ at Q. The magnitude of the slide's gradient at Q is equal to 2.

e. Find the values of c and d, and hence find a suitable equation for $g(x)$.

4 marks

f. Use calculus to find, correct to the nearest m^2, the area bounded by the slide and the strut PQ.

4 marks

The council further suggests that the structure of the slide be altered so that the gradient of the slope does not exceed 4 at any point. Kristy-Anne decides to scale down her model, ensuring that the foot of the slide remains the same distance from the water's edge. The new model follows the curve of the function

$$f(x) = \frac{1}{N}[x^3 + 18x^2 - 360x]$$

where N is the smallest number that accommodates the council's suggestion.

g. Find the value of N.

2 marks
Total 19 marks

END OF QUESTION AND ANSWER BOOKLET

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Trial Examination 2003

VCE Mathematical Methods
Units 3 & 4

Examination 2: Analysis Task

Suggested Solutions

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Question 1

a. When $\frac{dV}{dt} = 50$ and $t = 20$,

$50 = -20\log_e(0.05 \times 20) + C$

As $\log_e 1 = 0$, $C = 50$ [A]

b. When $t = 240$ minutes,

$\frac{dV}{dt} = -20\log_e(0.05 \times 240) + 50$

$= 0.302$ litres/min [A]

The rate of flow when $t = 240$ minutes is 0.302 litres/min

This could also be done on a graphic calculator by sketching the graph and calculating y when $x = 240$.

c. When $\frac{dV}{dt} = 40$ L/min,

$40 = -20\log_e(0.05t) + 50$

$-10 = -20\log_e(0.05t)$ [M]

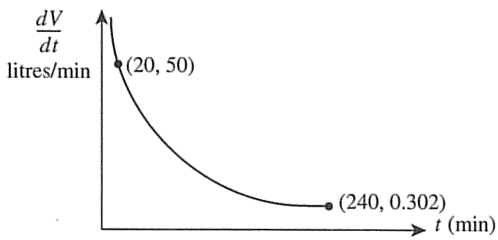
$\frac{1}{2} = \log_e(0.05t)$

$e^{\frac{1}{2}} = 0.05t$

So $t = 20e^{\frac{1}{2}}$ or $20\sqrt{e}$. [A]

The exact time when the rate of flow was 40 litres/min was $20e^{\frac{1}{2}}$ or $20\sqrt{e}$ minutes.

d.



Clearly shown vertical asymptote [A]
Clearly shown end point [A]

e. **i** $V = \int_{t_1}^{t_2} (-20\log_e(0.05t) + 50)dt$ [A]

ii. During the third hour, $t_1 = 120$ and $t_2 = 180$.

Thus $V = \int_{120}^{180} [-20\log e(0.05t) + 50]dt$

Using a graphic calculator, fn int (–20 ln (0.05*x*) + 50, *x*, 120, 180)

or $\int -20\ \ln 0.05x + 50,\ 120,\ 180$ [M]

$= 590.21$

≈ 590 litres. [A]

During the third hour, 590 litres of water was released.

Question 2

a. The height of point $P = f(30)$

$= (e^{0.08(30)} - 1)$

$= 10.023$ [A]

b. The gradient of the tangent $= f'(x)$

$= 0.08e^{0.08x}$ [A]

c. At point *P*, the gradient of the track $= f'(30)$

$= 0.05e^{0.05(30)}$

$= 0.882$ metres/metre [A]

d. This is a sine function translated *B* units to the right and *C* units up, so the coordinates of point *P* represent the constants *B* and *C*.

So $B = 30$ and $C = 10.023$. [A][A]

e. The derivative of $\sin(mx + c)$ is $m\cos(mx + c)$.

Thus $g'(x) = 0.5A\cos(x - 30)$. [A]

f. $f'(30) = g'(30)$ [M]

$0.882 = A \times 0.5 \times \cos 0.5(30 - 30)$

As $\cos 0 = 1$,

$0.882 = A \times 0.5$

$\therefore A = 1.764$ [A]

g. The period of $g(x) = \frac{2\pi}{n}$ (where $n = 0.5$), so the period required is 4π. [A]

h. The *x* value of point *Q* will be $30 + 4\pi + 2\pi$ (as $g(x)$ covers $1\frac{1}{2}$ periods).

Therefore *Q* has an *x* value of 48.850. [A]

i. A reflection in the *y*-axis generates $y = e^{-0.08x} - 1$. [M]

The translation required is 78.850 units to the right. This gives $h(x) = e^{-0.08(x - 78.850)} - 1$. [A]

j. As the area under the exponential components $y = f(x)$ and $y = h(x)$ are equal, an expression could be $A = 2\int_0^{30}(e^{0.08x} - 1)dx + \int_{30}^{48.850}(1.764\sin 0.5(x-30) + 10.023)dx$. [A][A]

k. The area should be evaluated using a graphic calculator.

Alternatively calculus could be used as follows:

$$A = 2\left[\frac{1}{0.08}e^{0.08x} - x\right]_0^{30} + \left[\frac{-1.764}{0.5}\cos 0.5(x-30) + 10.023x\right]_{30}^{48.850}$$

$$= 2\left(\frac{1}{0.08}e^{0.08(30)} - 30 - \frac{1}{0.08}\right) + \left(\frac{-1.764}{0.5}\cos 0.5(18.85) + 10.023(48.85) + \frac{1.764}{0.5} - 10.023 \times 30\right)$$

$$= 2(95.2897) + 195.9895$$

$$= 386.569$$

$$\approx 387 \text{ m}^2$$ [A]

The cross-sectional area to the nearest square metre is 387 m^2.

Question 3

a. This is a binomial experiment where $n = 10$, $p = 0.80$ and X is the number of throws that pass the record mark $X \sim \text{Bi}(n = 10, p = 0.80)$, $\Pr(X = 8) = {}^{10}C_8(0.80)^8(0.20)^2$ [M]

$= 0.302$ [A]

A graphic calculator could also have been used, utilising binompdf(10, 0.8, 8).

b. $\Pr(X \geq 8) = \Pr(X = 8) + \Pr(X = 9) + \Pr(X = 10)$

$= {}^{10}C_8(0.80)^8(0.20)^2 + {}^{10}C_9(0.80)^9(0.20) + (0.80)^{10}$

$= 0.678$ [A]

A graphic calculator could also have been used.

$1 - \Pr((X \leq 7)$

$= 1 - \text{binomcdf}(10, 0.8, 7)$

c. It is known that he has 10 attempts.

Throws 1 to 8 are < 65 m.

Throws 9 and 10 are ≤ 65 m.

Therefore, Pr(the first 8 throws pass and the last 2 do not) $= (0.80)^8(0.20)^2$ [M]

$= 0.00671$

$= 0.7\ \%$ [A]

d. This is a conditional probability.

$$\Pr(8 \text{ pass the record mark} \mid \text{last 2 are successful}) = \Pr\frac{(8 \text{ successful throws with last 2 successful})}{\Pr(\text{last 2 are successful})}$$ [M]

$$= \frac{[{}^{8}C_6(0.8)^2 \times (0.2)^2] \times (0.8)^2}{0.8^2}$$

$= 0.294$ [A]

e. **i** $p = 0.302$, $n = 250$, $q = 0.698$

The expected number of sets in which 8 throws pass the record mark is

$E(x) = 0.302 \times 250$

$= 75.5$ [A]

ii. $p = 0.678$, $n = 250$, $q = 0$

The expected number of sets in which more than 8 throws pass the record mark is

$E(x) = 0.678 \times 250$

$= 169.5$ [A]

f. $\mu = 75.5$, $\sigma = \sqrt{0.302 \times 250 \times 0.698}$

$= 7.259$ [M]

Therefore $\Pr(X < 50) = \Pr\left(Z < \frac{50 - 75.5}{7.259}\right)$

$= \Pr(Z < 8 - 3.513)$ [M]

$1 - \Pr(Z < 3.513) = 1 - 0.9998$

$= 0.0002$ [A]

Question 4

a. $x^3 + 18x^2 - 360x = x(x^2 + 18x - 360)$

$= x(x + 30)(x - 12)$ [A]

The x-intercepts are 0, –30 and 12,

so $a = -30$ and $b = 12$. [A]

b. $h(x) = \frac{1}{215}(x^3 + 18x^2 - 360x)$

$\therefore h'(x) = \frac{1}{215}(3x^2 + 36x - 360)$

$= 0$ at maximum/minimum [M]

$\therefore 3x^2 + 36x - 360 = 0$

$\therefore 3(x^2 + 12x - 120) = 0$

Using a graphic calculator or quadratic formula,

$x = 6.49$ or -18.49 at maximum/minimum. [M]

To find the highest point, draw an appropriate slope diagram.

$x < -18.49$	$h'(x) > 0$	/
$-18.49 < x < 6.49$	$h'(x) > 0$	\
$x > 6.49$	$h'(x) > 0$	/

So, the maximum occurs at $x = -18.49$. [A]

The maximum point of slide is $\frac{1}{215}(-18.49)^3 + 18((-18.49)^2 - 360(-18.49))$

(or by using a graphic calculator) $= 30.1808 \approx 30$ m [A]

The highest point of the slide is 30 m above the ground.

c. From the previous solution and calculations, the minimum occurs at $x = 6.49$

Therefore, the depth of water is $\frac{1}{215}((6.49)^3 + 18(6.49)^2 - 360(6.49))$

(or by use of graphics calculator) = –6.07, which is a depth less than 7 metres. [A]

d. The average gradient between these two points $= \left(\frac{h_2 - h_1}{x_2 - x_1}\right) = \frac{6.07 - 30.18}{6.49 - (-18.49)}$ [M]

$$= -0.965$$
$$\approx -1$$

So, the average gradient is –1 metre/metre [A]

e. The gradient is $h(x)$ at Q is –2. To find Q, solve $h'(x) = -2$.

$$\frac{1}{215}(3x^2 + 36x - 360) = -2$$
$$3x^2 + 36x - 360 = -430$$
$$3x^2 + 36x + 70 = 0$$

Using the quadratic formula or graphic calculator, [M]

$x = -9.559$.

$h(-9.559) = 19.595$, so Q is the point $(-9.559, 19.595)$.

Since $g(x)$ is the normal to $h(x)$ at Q, the gradient of $g(x)$ is $\frac{1}{2}$.

Hence $c = \frac{1}{2}$.

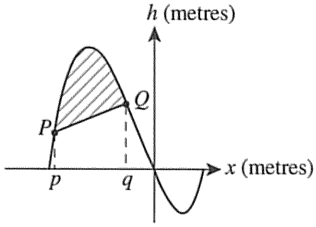
Substitute the gradient and the coordinates of Q into the general linear equation: [M]

$$y - y_1 = m(x - x_1)$$
$$y - 19.595 = \frac{1}{2}(x - 9.559)$$
$$y = \frac{1}{2}x + 24.37$$

hence $d = 24.37$. [A]

A suitable equation for $g(x)$ would be $g(x) = \frac{1}{2}x + 24.37$. [A]

f. The relevant area is the shaded area between $h(x)$ and $g(x)$.



By using a graphics calculator to estimate p and q or by using another relevant method (e.g. equating curves),

$p = -28.03$ and $q = 9.52$ [M][A]

The area would be

$$\int_{-28.03}^{-9.52} \frac{1}{215}(x^3 + 18x - 360x)dx - \int_{-28.03}^{-9.52} \left(\frac{1}{2}x + 24.8\right)dx$$

$$= \frac{3}{215}\left[\frac{1}{4}x^4 + 6x^3 - 180x^2\right]_{-28.03}^{-9.52} - \left[\frac{1}{4}x^2 + 24.28x\right]_{28.03}^{9.52}$$ [M]

$$= \frac{3}{215}\left[\left(\frac{1}{4}(-9.52)^4 + 6(-9.52)^3 - 180(-9.52)^2\right)\right]$$

$$- \left[\left(\frac{1}{4}(-28.03)^4 + 6(-9.52 - 28.03)^3 - 1809.52 - 28.03\right)\right]$$

$$- \left[\left(\frac{1}{4}(-9.52)^2 + 24.28(-9.52)\right) - \left(\frac{1}{4}(-28.03)^2 + 24.28(-28.03)\right)\right]$$

$\approx 468.98 - 281.07$

$= 187.91$

$= 188 \text{ m}^2$

The area bounded by the slide and the strut PQ is 188 m^2.

This can be checked by using the integration feature on the graphic calculator. [M]

g. We want to be sure that the gradient at a, the point of maximum slope and the foot of the slide, is 4.

So $f'(x) = \frac{3}{N}(x^2 + 12x - 120)$ [M]

At a, $x = -30$

$\therefore f'(x) = \frac{3}{N}((-30)^2 + 12(-30) - 120)$

$= \frac{3}{N}(420)$

$= 4$ (at min. value of N)

So $N = 315$ [A]