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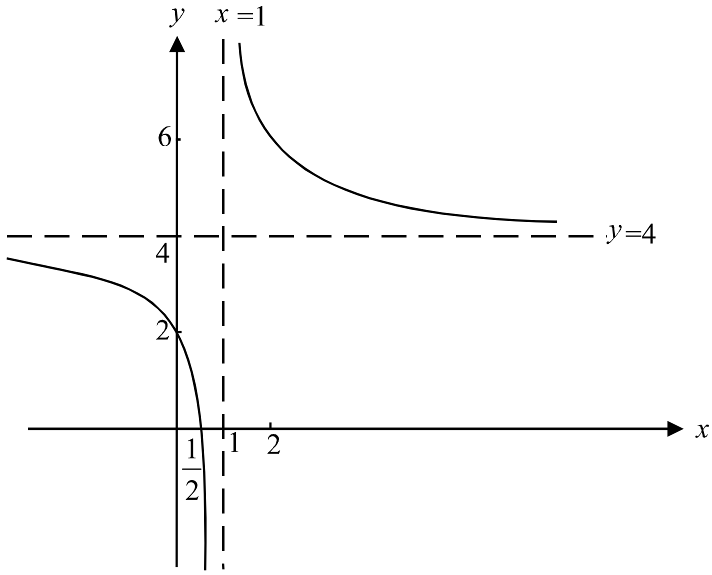
MATHS METHODS 3 & 4
TRIAL EXAMINATION 2
SOLUTIONS
2004

Question 1

a. i.



x-intercepts

$y=0, \quad 0=\frac{-2}{1-x}+4$

$-4(1-x)=-2$

$-4+4x=-2$

$4x=2$

$x=\frac{1}{2}$

y-intercepts

$x=0, \quad y=\frac{-2}{1}+4$

$=2$

(1 mark) correct shape including asymptotes
(1 mark) intercepts

ii. $d=R\setminus\{1\}$

$r=R\setminus\{4\}$

(1 mark)

b. **i.** $y=\frac{-2}{1-x}+4$

$=-2(1-x)^{-1}+4$

$\frac{dy}{dx}=2(1-x)^{-2}\times -1$

$=\frac{-2}{(1-x)^2}$

(1 mark)

ii. From the graph in part **a. i.**, there are no local minima or maxima and no stationary points of inflection, hence there is no solution to the equation $\frac{dy}{dx}=0$.

(1 mark)

c. i. f^{-1} exists because we see from the graph in **a. i.**, that f is a 1:1 function.

(1 mark)

ii. Now, $f(x)=\frac{-2}{1-x}+4$

Let $y=\frac{-2}{1-x}+4$

Swap x and y, so $x=\frac{-2}{1-y}+4$

Rearrange $x-4=\frac{-2}{1-y}$

$$(x-4)(1-y)=-2$$

$$1-y=\frac{-2}{x-4}$$

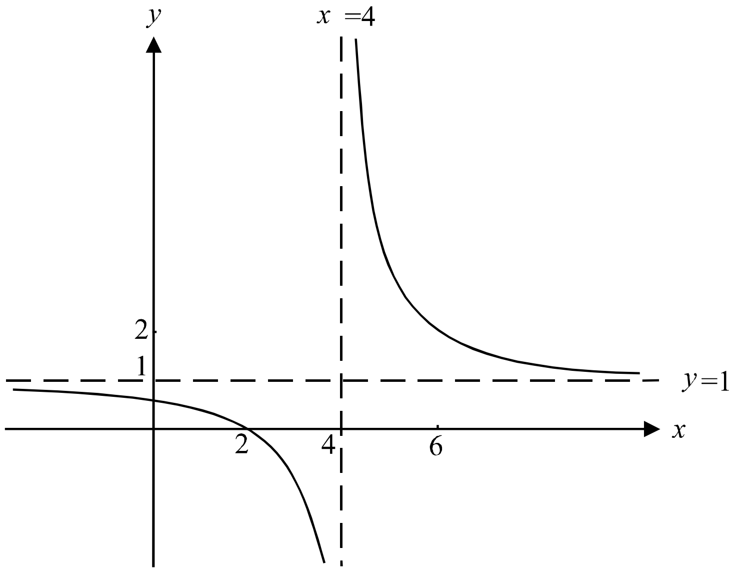
$$-y=\frac{-2}{x-4}-1$$

$$y=\frac{2}{x-4}+1$$

So $f^{-1}(x)=\frac{2}{x-4}+1$ as required.

(1 mark)

iii. Use the graph you have drawn in part **a. i**. and reflect it in the line $y = x$.



The asymptotes for the graph of $y = f(x)$ are $x = 1$ and $y = 4$, therefore for the graph of $y = f^{-1}(x)$, the asymptotes are $y = 1$ and $x = 4$.

On the graph of $y = f(x)$, the graph passed through the points $\left(\frac{1}{2},0\right)$, $(0,2)$ and $(2,6)$.

On the graph of $y = f^{-1}(x)$, the graph passes through the points $\left(0,\frac{1}{2}\right)$, $(2,0)$ and $(6,2)$.

(1 mark) shape
including asymptotes
(1 mark) intercepts

d. To Show. $\int_0^{\frac{1}{2}} f(x)dx = \int_0^{2} f^{-1}(x)dx$

$$\begin{aligned}
\text{L.S.} &= \int_0^{\frac{1}{2}} f(x)dx \\
&= \int_0^{\frac{1}{2}} \left(\frac{-2}{1-x}+4\right)dx \\
&= \int_0^{\frac{1}{2}} \left\{2\left(\frac{-1}{1-x}\right)+4\right\}dx \\
&= \left[2\log_e(1-x)+4x\right]_0^{\frac{1}{2}} \\
&= \left\{\left(2\log_e\left(\frac{1}{2}\right)+2-\left(2\log_e(1)+0\right)\right)\right\} \\
&= 2\log_e\left(\frac{1}{2}\right)+2 \quad \text{since } \log_e(1)=0
\end{aligned}$$

(1 mark)

(1 mark)

$$\begin{aligned}
\text{R.S.} &= \int_0^{2} f^{-1}(x)dx \\
&= \int_0^{2} \left(\frac{2}{x-4}+1\right)dx \\
&= \int_0^{2} \left(2\left(\frac{1}{x-4}\right)+1\right)dx \\
&= \left[2\log_e(x-4)+x\right]_0^{2} \\
&= \left[\log_e(x-4)^2+x\right]_0^{2} \\
&= \left\{\left(\log_e(4)+2\right)-\left(\log_e(16)+0\right)\right\} \\
&= \log_e(4)-\log_e(16)+2 \\
&= \log_e\left(\frac{4}{16}\right)+2 \\
&= \log_e\left(\frac{1}{4}\right)+2 \\
&= \log_e\left(\frac{1}{2}\right)^2+2 \\
&= 2\log_e\left(\frac{1}{2}\right)+2 \\
&= \text{L.S.}
\end{aligned}$$

(1 mark)

(log laws) $n\log_e(m) = \log_e(m^n)$

(1 mark)

(1 mark)

Have shown

e. Method 1

Use your graphics calculator to graph each of the functions $y = f^{-1}(x)$ and $y = f(x)$.
The values of x for which $f(x) = f^{-1}(x)$ occur at $x = 0.44$ and $x = 4.56$ where each number is correct to 2 decimal places.

(2 marks)

Method 2

As the points of intersection must lie on the line $y = x$, find the points of intersection between $y = \frac{2}{x-4} + 1$ and $y = x$.

So, $x = \frac{2}{x-4} + 1$ **(1 mark)**

$$(x-1)(x-4) = 2$$
$$x^2 - 5x + 4 = 2$$
$$x^2 - 5x + 2 = 0$$
$$x = \frac{5 \pm \sqrt{25 - 4 \times 1 \times 2}}{2}$$
$$= \frac{5 \pm \sqrt{17}}{2}$$

The values of x for which $f(x) = f^{-1}(x)$ are $x = \frac{5 \pm \sqrt{17}}{2}$. **(1 mark)**

Method 3

$$f(x) = f^{-1}(x)$$

So, $\frac{-2}{1-x} + 4 = \frac{2}{x-4} + 1$

$$\frac{-2}{1-x} - \frac{2}{x-4} = -3$$
$$\frac{-2(x-4) - 2(1-x)}{(1-x)(x-4)} = -3$$
$$-2x + 8 - 2 + 2x = -3(x - 4 - x^2 + 4x)$$
$$6 = -3(-x^2 + 5x - 4)$$
$$-2 = -x^2 + 5x - 4$$
$$x^2 - 5x + 2 = 0$$ **(1 mark)**
$$x = \frac{5 \pm \sqrt{25 - 4 \times 1 \times 2}}{2}$$
$$x = \frac{5 \pm \sqrt{17}}{2}$$

The values of x for which $f(x) = f^{-1}(x)$ are $x = \frac{5 \pm \sqrt{17}}{2}$.

(1 mark)
Total 16 marks

Question 2

a. The probability that a randomly selected female donor at the facility will have a haemoglobin level that is considered high is 0.1.

(1 mark)

b. This is a binomial distribution because each time we select one of the 10, the probability that they have a level that is considered high is constant i.e. 0.1.

$\Pr(x \geq 2) = 1 - \Pr(x < 2)$ **(1 mark)**

$= 1 - \{\Pr(x = 0) + \Pr(x = 1)\}$

$= 1 - \left({}^{10}C_0 (0 \cdot 1)^0 (0 \cdot 9)^{10} + {}^{10}C_1 (0 \cdot 1)^1 (0 \cdot 9)^9\right)$

$= 0 \cdot 2639$ (to 4 decimal places)

(1 mark)

c. We require x such that

$\Pr(X > x) > 0 \cdot 1$

So $\Pr(X < x) < 0 \cdot 9$

Now $\Pr(z < 1 \cdot 2815) < 0 \cdot 9$

(by reading the table inside out)

So $1 \cdot 2815 = \dfrac{x - 140}{19 \cdot 5}$

(1 mark)

$= 164 \cdot 98925...$

$= 165$ to the nearest whole number

(1 mark)

d.

$\Pr(X < 120) = \Pr(z < -1 \cdot 025641)$

$= \Pr(z > 1 \cdot 025641)$

$= 1 - \Pr(z < 1 \cdot 025641)$

$= 1 - 0 \cdot 8474$

$= 0 \cdot 1526$

$z = \dfrac{120 - 140}{19 \cdot 5}$

$= -1 \cdot 0256$ **(1 mark)**

So 15% (to the nearest whole percent) of female donors at the facility are not allowed to donate because their haemoglobin level is too low.

(1 mark)

e. There are 10% of female donors who have a haemoglobin level that is considered high and from part **d**. there are 15% of female donors who have haemoglobin levels that are too low to donate. So 75% of female donors don't suffer from a haemoglobin level that is considered high or is too low.

(1 mark)

The expected number of female donors in a day in this category is 225.

(1 mark)

f. This is a hypergeometric distribution because there is no replacement.
Now, $N = 12,\ D = 10,\ n = 3$ and $x = 3$. **(1 mark)**

$\Pr(X = 3) = \dfrac{{}^{10}C_3\, {}^{2}C_0}{{}^{12}C_3}$ **(1 mark)**

$= 0.\dot{5}\dot{4}$ **(1 mark)**

Total 12 marks

Question 3

a. $p(x)=\cos(\pi x) \qquad x\in[0,2]$

The amplitude of this function is 1. The normal height of the water is 1 metre.

(1 mark)

b.

i $\text{Area} = -\int_{0.5}^{1.5} \cos(\pi x)\,dx$ **(1 mark)**

$= \frac{-1}{\pi}\left[\sin(\pi x)\right]_{0\cdot 5}^{1\cdot 5}$

$= \frac{-1}{\pi}\left(\sin(1\cdot 5\pi)-\sin(0\cdot 5\pi)\right)$

$= 0\cdot 6366 \quad \text{(to 4 decimal places)}$ **(1 mark)**

ii. Hence the volume of water in a 10 m length of fibreglass channel is

$0\cdot 6366\text{m}^2 \times 10\text{m}$

$= 6\cdot 366\text{m}^3$

$= 6\cdot 366\times 1\,000\,000\text{cm}^3$

$= 6\cdot 366\times 10^6\,\text{mL}$

$= 6366\times 10^3\,\text{mL}$

$= 6366\text{L}$ **(1 mark)**

c. **i.** $p(x)=\cos(\pi x)$

$\frac{dp}{dx} = -\pi\sin(\pi x)$

(1 mark)

ii. The maximum value of $\sin(\pi x)$ is 1 and the minimum value is –1.
Therefore the maximum value of $-\pi\sin(\pi x)$ is π and the minimum value is $-\pi$. **(2 marks)**
Note that the question did not ask for the maximum and minimum points on the fibreglass channel i.e. we were not looking to solve $\frac{dp}{dx}=0$.

iii. When $\frac{dp}{dx}=\pi$, we have

$\pi = -\pi\sin(\pi x), \qquad x\in[0,2]$

So, $\sin(\pi x) = -1$

$\pi x = \frac{3\pi}{2}$

$x = \frac{3}{2}$

(1 mark)

When $\frac{dp}{dx} = -\pi$, we have

$$-\pi = -\pi \sin(\pi x)$$

So, $\sin(\pi x) = 1$

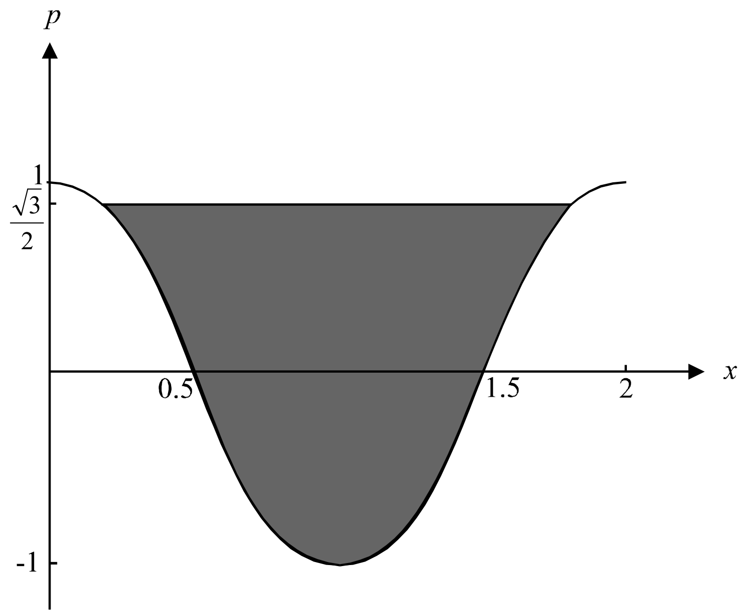
$$\pi x = \frac{\pi}{2}$$

$$x = \frac{1}{2}$$

(1 mark)

So half a metre from either side of the fibreglass channel, the maximum and minimum gradients occur.

d. Sketch a graph.



We need to find the area shaded in the diagram above.
Firstly, find the two values of x where

$$p(x) = \frac{\sqrt{3}}{2}, \qquad x \in [0,2]$$

$$\cos(\pi x) = \frac{\sqrt{3}}{2}$$

$$\pi x = \frac{\pi}{6}, \frac{11\pi}{6}$$

So, $x = \frac{1}{6}, \frac{11}{6}$

So area required $= \int_{\frac{1}{6}}^{\frac{11}{6}} \left(\frac{\sqrt{3}}{2} - \cos(\pi x)\right) dx$ **(1 mark)** correct integrand

(1 mark) correct terminals

$$= \left[\frac{\sqrt{3}}{2}x - \frac{1}{\pi}\sin(\pi x)\right]_{\frac{1}{6}}^{\frac{11}{6}}$$ **(1 mark)**

$$= \left\{\left(\frac{\sqrt{3}}{2} \times \frac{11}{6} - \frac{1}{\pi}\sin\left(\frac{11\pi}{6}\right)\right) - \left(\frac{\sqrt{3}}{12} - \frac{1}{\pi}\sin\left(\frac{\pi}{6}\right)\right)\right\}$$

$$= \frac{11\sqrt{3}}{12} + \frac{1}{2\pi} - \frac{\sqrt{3}}{12} + \frac{1}{2\pi}$$

$$= \frac{5\sqrt{3}}{6} + \frac{1}{\pi} \text{ m}^2$$

(1 mark)
Total 13 marks

Question 4

a. Now, $V(0) = 10$, since the piece of machinery normally has 10 litres of coolant in it.

So $ke^0 + ke^0(0-1)^2 = 10$

$k + k = 10$ since $e^0 = 1$

$k = 5$

(1 mark)

b.

$$V(t) = 5e^{-t} + 5e^{-t}(t-1)^2$$
$$= 5e^{-t}\left(1 + (t-1)^2\right)$$

As $t \to \infty$, $e^t \to \infty$ and hence $\frac{1}{e^t} \to 0$. Since $e^{-t} = \frac{1}{e^t}$, $e^{-t} \to 0$.

So $5e^{-t}\left(1 + (t-1)^2\right) \to 0$ since it doesn't matter how large $\left(1 + (t-1)^2\right)$ becomes if it is being multiplied by a value close to zero then the value of the product of the two approaches zero also. So $V(t) \to 0$

(1 mark)

c. We need to solve $V(t) = 5$

So $5e^{-t} + 5e^{-t}(t-1)^2 = 5$.

Graph the function $y = 5e^{-t} + 5e^{-t}(t-1)^2 - 5$ on your graphics calculator and find the value of *t* when $y = 0$.

Alternatively, find the *t*-coordinate where the functions $y = 5e^{-t} + 5e^{-t}(t-1)^2$ and $y = 5$ intersect. **(1 mark)**

So, $t = 0 \cdot 35$ minutes (correct to 2 decimal places)

(1 mark)

d. Rate of change in first minute

$$= \frac{V(1) - V(0)}{1 - 0}$$

$$= \frac{5e^{-1} + 5e^{-1}(1-1)^2 - \{5e^0 + 5e^0(0-1)^2\}}{1}$$

$$= 5e^{-1} - 10$$

$$= -8 \cdot 1606 \quad \text{(to 4 decimal places)}$$

Rate of change in second minute

$$= \frac{V(2) - V(1)}{2 - 1}$$

$$= \frac{5e^{-2} + 5e^{-2}(2-1)^2 - 5e^{-1}}{1}$$

$$= 10e^{-2} - 5e^{-1}$$

$$= -0 \cdot 4860 \quad \text{(to 4 decimal places)}$$

So the rate of change of V with respect to t is greater in the first minute.
(Note that technically $-0.4860 > -8.1606$ but the negative refers to a decrease in the volume of fluid present and so the fluid is decreasing at a higher rate in the first hour.)

(1 mark) correct rates of change
(1 mark) correct conclusion

e. **i.**

$$V(t) = 5e^{-t} + 5e^{-t}(t-1)^2, \quad t \geq 0$$

$$V'(t) = -5e^{-t} - 5e^{-t}(t-1)^2 + 5e^{-t} \times 2(t-1) \times 1 \quad \text{(product rule)}$$ **(1 mark)**

$$= -5e^{-t}\left(1 + (t-1)^2 - 2(t-1)\right)$$

$$= -5e^{-t}\left(1 + t^2 - 2t + 1 - 2t + 2\right)$$

$$= -5e^{-t}\left(t^2 - 4t + 4\right)$$

$$= -5e^{-t}(t-2)^2$$ **(1 mark)**

A stationary point occurs when

$$V'(t) = 0$$

So $\quad -5e^{-t}(t-2)^2 = 0$

Now, $\quad -5e^{-t} \neq 0$

So, $\quad (t-2)^2 = 0$

$$t = 2$$

A stationary point occurs on the graph of $y = V(t)$ when $t = 2$.

(1 mark)

ii. From the graph it appears that at $x = 2$ there could be a stationary point of inflection. You must confirm that however. Use the first derivative test.

At $t = 1$, $V'(1) = -5e^{-1}(1-2)^2$

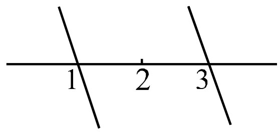
$= -5e^{-1}$

< 0

At $t = 3$, $V'(3) = -5e^{-3}(3-2)^2$

$= -5e^{-3}$

< 0



At $t = 2$, there is a stationary point of inflection.

(1 mark) answer
(1 mark) first derivative test

iii. If an employee momentarily stops the liquid from leaking then the gradient of the graph of $y = V(t)$ will be zero for an instant. This corresponds to the stationary point of inflection on the graph.
Hence when $t = 2$,

$$V(2) = 5e^{-2} + 5e^{-2}(2-1)^2$$
$$= 10e^{-2}$$

Hence the exact volume present is $10e^{-2}$ litres.

(1 mark)

f. From part **e. i.**,

$$V'(t) = -5e^{-t}(t-2)^2$$

Now, $e^{-t} > 0$ for all t,

and $(t-2)^2 > 0$ for all t.

(1 mark)

The product $-5 \times e^{-t} \times (t-2)^2$ must therefore be negative for all *t*.

(1 mark)
Total 14 marks