2004 Mathematical Methods Written Examination 1

Suggested Answers & Solutions

Part 1 (Multiple-choice) Answers

1. C	2. C	3. E	4. C	5. B
6. B	7. B	8. D	9. E	10. B
11. C	12. D	13. B	14. C	15. D
16. C	17. E	18. E	19. A	20. A
21. D	22. B	23. A	24. D	25. C
26. D	27. A			

Part 1 (Multiple-choice) Solutions

Question 1 [C]

Pr(odd) $= \Pr(1) + \Pr(3)$
$= 0.3 + 0.25$
$= 0.55$

3

Question 2 [C]

The Expected Value is found by evaluating $\Sigma w\ \mathrm{p}(w)$.
$= 0 \times 0.2 + 1 \times 0.4 + 2 \times 0.3 + 3 \times 0.1$
$= 0 + 0.4 + 0.6 + 0.3$
$= 1.3$

Question 3 [E]

Consider getting two identical purple socks OR two identical green socks.
There are 10 purple and 6 green socks.

Pr(2 purple) $= \frac{10}{16} \times \frac{9}{15}$ and

Pr(2 green) $= \frac{6}{16} \times \frac{5}{15}$

Now $\frac{90}{240} + \frac{30}{240} + \frac{120}{240} = \frac{1}{2}$

Question 4 [C]

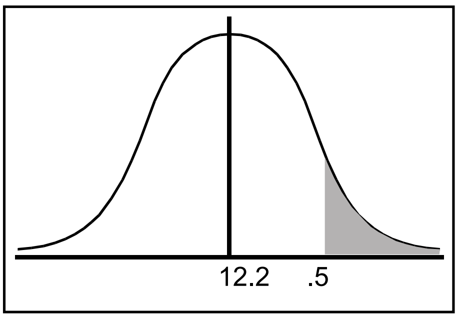
The Mean and Variance of the Binomial Distribution are np and npq respectively.
Mean $= np = 20$, Variance $= npq = 4$
Substituting for np in the variance gives: $20q = 4$ and so $q = 0.2$
Therefore $p = 0.8$
$0.8n = 20$ and so $n = 25$

Question 5 [B]

Convert the raw score to standardised form using $\frac{Z - \mu}{\sigma}$

using $Z = \frac{15 - 12.2}{1.4} = 2$

We want the Pr $(Z > 2)$ so we want $X > 15$.



Question 6 [B]

If the period is $\frac{1}{10}$ then $\frac{2\pi}{n} = \frac{1}{10}$

Therefore $n = 20\pi$
The amplitude is 3 and so the equation is of the form $3 \sin(20\pi t)$.

Question 7 [B]

The period of this rule is $2\pi \div \frac{4\pi}{25}$ which is 12.5 hours. The time between successive low tides is 12.5 hours and so the time between low and high tides is half this time. Therefore the next high tide occurs at 6:15 a.m.

Question 8 [D]

If $0 \le x \le 4\pi$ then $0 \le \frac{x}{2} \le 2\pi$.

Cosine is positive in 1st and 4th quadrants.

Therefore $\frac{x}{2} = \frac{\pi}{6}$ and $2\pi - \frac{\pi}{6}$

Hence $x = \frac{x}{3}$ and $4\pi - \frac{\pi}{3}$

The sum of these two angles is 4π.

Question 9 [E]

$ax^3 - bx = x(ax^2 - b)$

The second factor is the difference of two squares and so the expression can be written as

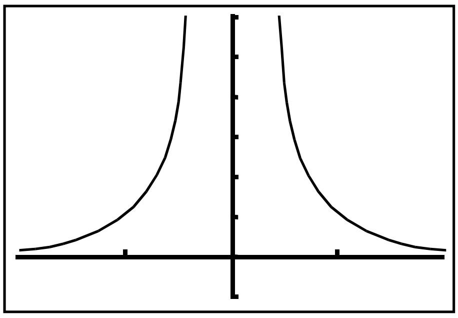
$x[(x(\sqrt{a})^2 - (\sqrt{b})^2]$

$= x[(x\sqrt{a} - \sqrt{b}][x\sqrt{a} + \sqrt{b}]$

This is equivalent to E.

Question 10 [B]

A function has an inverse if it is one-to-one. The only graph that is not one-to-one for the specified domains given is the truncus $g(x) = \frac{1}{x^2}$, as shown.



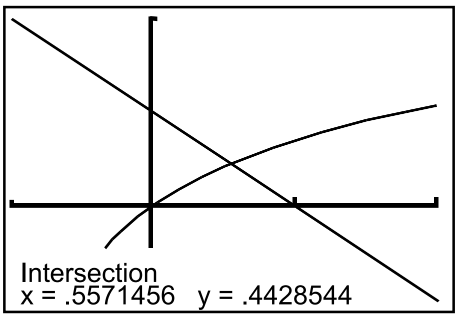
Question 11 [C]

If $5e^{ax} = 2$ then $e^{ax} = \frac{2}{5} = 0.4$

Taking log of both sides gives: $ax = \log_e 0.4$ and so $x = \frac{1}{a} \log_e 0.4$

Question 12 [D]

Finding the point of intersection of $y_1 = \log_e(x+1)$ and $y_2 = 1 - x$ gives



the point above. To two decimal places the closest value of x is 0.56

Question 13 [B]

Q is the x value of the minimum turning point. It is therefore 2.

The Turning Point is (2 ,1) So $y = P(x-2)^2 + 1$.

(5 ,4) is a point on the curve. So $4 = P(5-2)^2 + 1$.

$3 = 9P$

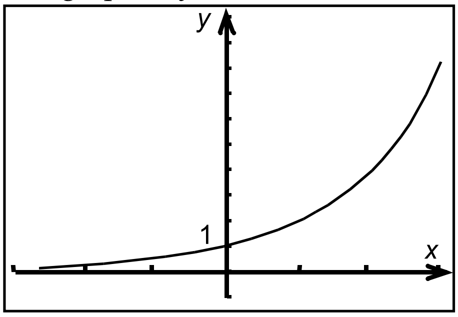
$\Rightarrow P = \frac{1}{3}$

Question 14 [C]

If the cubic polynomial meets the x-axis at $(a, 0)$ and $(b, 0)$ then the factors could be of the form $(x - a)$ and $(x - b)$. Touching the axis at either of these points requires a factor of either $(x - a)^2$ or $(x - b)^2$. The only answer that pairs up one of each of these two alternatives is C.

Question 15 [D]

The graph of $y = 2^x$ is shown below.



One can see that the domain is R , its range is R+, and that it has a horizontal asymptote of $y = 0$. The gradient is always positive as the function is ever-increasing. Substituting $x = 0$ gives $y = 2^0$ which is 1.

Question 16 [C]

If a function is one-to-one and has a horizontal asymptote of $x = 3$ then its inverse function must have a vertical asymptote of $y = 3$ because an inverse function is a mirror image of the function in the line $y = x$.

Question 17 [E]

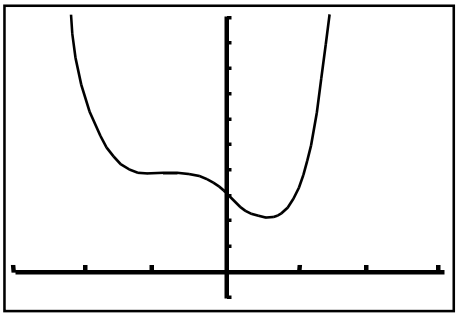
$f(x) = \cos x$ has an amplitude of 1 and a period of 2π.

$g(x) = 5 \cos 3x$ has an amplitude of 5 and a period of $\frac{2\pi}{3}$.

Therefore $f(x)$ has been dilated by a scale factor of 5 from the x-axis and also dilated by a scale factor of 3 from the y-axis.

Question 18 [E]

The graph of $f(x) = (x + 1)^3(x - 1) + 1$ for $-3 \le x \le 3$ is shown. The point of inflexion with a zero gradient can be seen at $x = -1$.



Question 19 [A]

If $y = x^2 + 2x + 4$ then $\frac{dy}{dx} = 2x + 2$. At $x = k$, $\frac{dy}{dx} = 2k + 2$.

Question 20 [A]

At $t = 0$, $N = 1000$
At $t = 10$, $N = 1000e^1$
The average rate of change is found by getting the gradient of the straight line joining these two points.

$\text{m} = \frac{1000e - 1000}{10 - 0} = 171.8281828.$

The closest answer is 172.

Question 21 [D]

$f(x + h) \approx f(x) + h f'(x)$
Let $x = 2$ and $h = -0.2$
Therefore, substituting for x and h in the approximation equation gives
$f(2 - 0.2) \approx f(2) - 0.2 f'(2)$

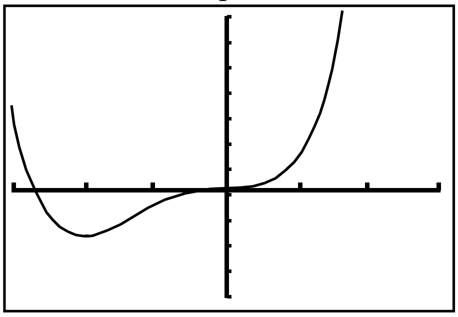
Question 22 [B]

Notice that the gradient of $f(x)$ here is always positive (ever increasing) and flattens out (or tends to zero) as x tends to $-\infty$. Alternative B has these features.

Question 23 [A]

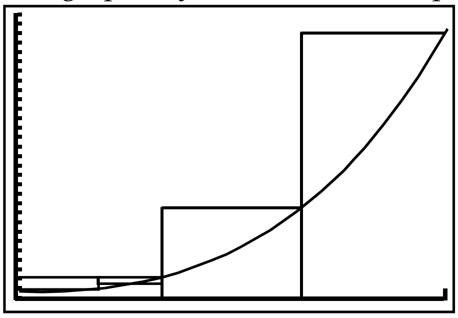
The graph of the cubic function is similar to that with equation $y = x^2(x + 2)$ whose anti-derivative is of the form $\frac{1}{4}x^4 + \frac{2}{3}x^3$. The graph of this anti-derivative is shown below.

Alternative A depicts that shown.



Question 24 [D]

The graph of $y = x^3 + 1$ for the required x values is shown with right-hand rectangles.



The three rectangles (from the right) have areas of 28 (which is $3^3 + 1$), 9 ($2^3 + 1$) and 2 ($1^3 + 1$) respectively. The approximate area is $28 + 9 + 2 = 39$.

Question 25 [C]

The region from $x = a$ to $x = b$ is below the x-axis and so the area will be given by the negative of $\int_a^b f(x)dx$. The rest is above the x-axis and is found by finding $\int_b^c f(x)dx$

Alternative C gives the sum of these two integrals.

Question 26 [D]

The area between two curves is found by obtaining the integral of the upper equation take the lower equation between the two x-values of the point of intersection of the curves and the y-axis. The upper equation here is $y = 2$ and the lower is $y = e^x$.

Point of intersection: $e^x = 2$ so $x = \ln 2$

The area is $\int_0^{\ln 2}(2 - e^x)dx$ which is D.

Question 27 [A]

$$\int_0^{1.5} \frac{1}{2x+1}\,dx = \left[\frac{1}{2}\log_e(2x+1)\right]_0^{1.5}$$

$$= \frac{1}{2}\log_e 4 - \frac{1}{2}\log_e 1$$

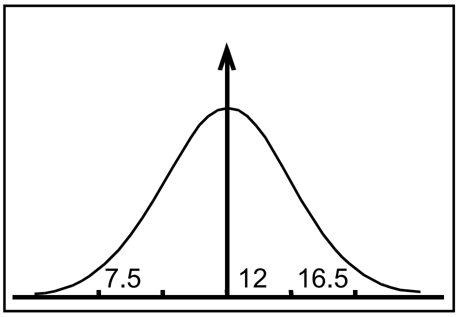
$$= \log_e 2 - 0$$

Therefore $k = 2$.

PART II

Question 1

a. The mean is the axis of symmetry here and occurs at $x = 12$. Each unit is 1.5 wide (the standard deviation) and so the 3rd σ unit on left of mean is at $x = 7.5$ and the 3rd σ unit on right of mean is at $x = 16.5$.



b. Invnorm(0.16, 12, 1.5) = 10.508…
Therefore $k = 10.5$ cm.

Question 2

a. $g(x) = (x - 3)^2$
b. $h(x) = (x - 3)^2 - 1$
c. $k(x) = (2x - 3)^2 - 1$

Question 3

a. Domain of $f = (1, \infty)$. Range of f is R.
b. Because f is a one-to-one function.
c. Switch x for y and vice-versa:

$$x = 0.5 \log_e(y - 1)$$

$$\frac{x}{0.5} = 2x = \log_e(y - 1)$$

$$e^{2x} = y - 1$$

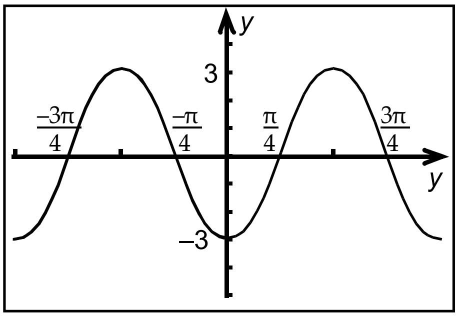
$$y = e^{2x} + 1$$

Therefore $f^{-1}(x) = e^{2x} + 1$

Question 4

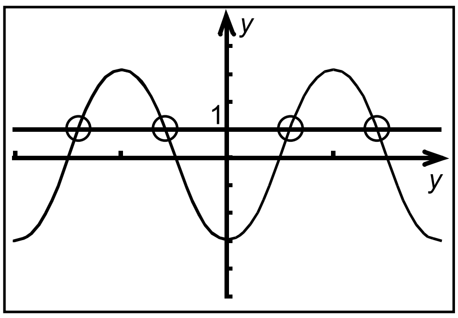
a. The scales on x- and y-axes respectively are $\frac{\pi}{4}$ and 1 as on the examination paper.

The x intercepts are $\frac{-3\pi}{4}, \frac{-\pi}{4}, \frac{\pi}{4}$ and $\frac{3\pi}{4}$ while the y intercept is –3.



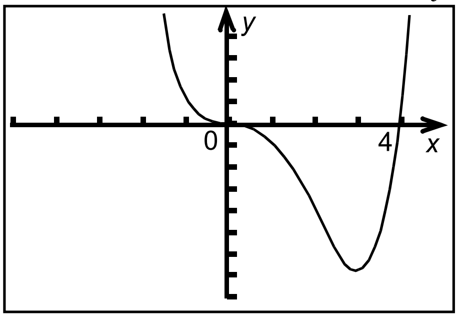
The turning points have co-ordinates $(\frac{-\pi}{2}, 3)$ max , (0, -3) min and $(\frac{\pi}{2}, 3)$ max.

b. Drawing the line $y = 1$ shows 4 points of intersection.



Question 5

a. Window $-5 \le x \le 5$ and $-8 \le y \le 5$.



b. Since the graph has an x-intercept at $x = 4$, this must be the value of b.

Substitute $x = 2$ and $f(2) = -4$ into
$f(x) = ax^3(x-4)$

$-4 = a \times 8 \times -2$ and so $a = \frac{1}{4}$

c. $f(x) = \frac{1}{4}x^3(x-4)$

$= \frac{1}{4}x^4 - x^3$

Therefore $f'(x) = x^3 - 3x^2$

Substitute $x = 4$ to get a gradient of 16.

The equation of the tangent is:

$y - 0 = 16(x-4)$

Answer: $y = 16x - 64$.

Question 6

a. $\Pr(X = 1) = {}^4C_1\, p^1(1-p)^3$

$= 4p(1-p)^3$

b. Using the Product Rule $\frac{dP}{dp}$ is:

$4(1-p)^3 + 4p \times -3(1-p)^2$
$= (1-p)^2[4(1-p) - 12p]$
$= (1-p)^2(4-16p)$
$= 0$ for a maximum value.

Now $p = 1$ is not possible in this context and so the maximum will occur when $p = 0.25$.