2004 Mathematical Methods Written Examination 2

Suggested Solutions

Question 1

a. $f(x) = (x-1)^2(x-2) + 1$

Using the Product Rule with the first function being $(x-1)^2$ and the second $(x-2)$, we see that

$f'(x) = 2(x-1)(x-2) + (x-1)^2$

$= (x-1)[2(x-2) + (x-1)]$

$= (x-1)[2x-4 + x-1]$

$= (x-1)(3x-5)$

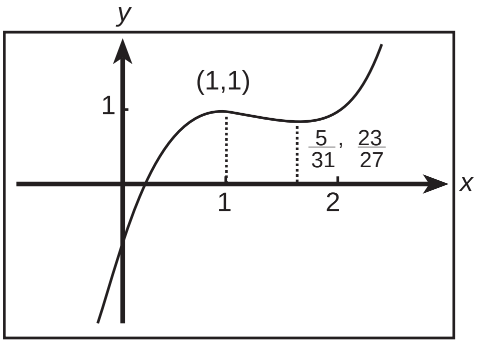
Therefore $u = 3$ and $v = -5$.

b. Now $f'(x) = 0$ for the turning points and so $x - 1 = 0$ or $3x - 5 = 0$.

If $x = 1$ then $y = 1$ so the point (1, 1) is a turning point. Therefore $a = 1$.

If $x = \frac{5}{3}$ then $y = (\frac{5}{3} - 1)^2(\frac{5}{3} - 2) + 1 = \frac{23}{27}$. Therefore $b = \frac{5}{3}$.

c.



If $p = \frac{23}{27}$ which is the y-value of the turning point of $f(x)$ then the graph of $f(x) = p$ would be lowered and have the minimum value on the x-axis.

Therefore one set of solutions is $x < \frac{23}{27}$.

However, the other turning point has a y-value of 1.

If $p > 1$, this also gives one solution only.

d. $f(x) - 1 = (x-1)^2(x-2) + 1 - 1$

$= (x-1)^2(x-2)$

The region bounded by the x-axis and $f(x) - 1$ is given by $\int_1^2 (f(x) - 1)dx$

$= \int_1^2 x^3 - 4x^2 + 5x - 2\, dx$

$= [\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{5}{2}x^2 - 2x]_1^2$

$= (\frac{16}{4} - \frac{32}{3} + 10 - 4) - (\frac{1}{4} - \frac{4}{3} + \frac{5}{2} - 2)$

$= \frac{1}{12}$

The area is therefore $\frac{1}{12}$ or 0.083 correct to 3 decimal places.

e. i. Dilation by a factor of 2 from the y-axis followed by a translation 1 unit down.

ii. $f(\frac{x}{2}) - 1 = (\frac{x}{2} - 1)^2(\frac{x}{2} - 2)$ and so the x-intercepts are $x = 2$ and $x = 4$.

iii. 0.17

f. $f(x + h) - 1 = (x + h - 1)^2(x + h - 2) + 1 - 1$

$= (x + h - 1)^2(x + h - 2)$

If one positive solution only is required for this equation then imagine that the first factor is x^2, i.e. that $h = 1$. There would only be one positive answer, i.e. $x = 2$.

If the graph was translated 2 units to the left then the factors would be $(x + 1)^2x$, which has no positive solutions.

Hence the values that h can take are $1 \le h < 2$.

Question 2

a. i. The sum of the proportions must be 1 for a probability distribution function.
Therefore $4k^2 + 4k + 5k^2 + 2k + k^2 + k + 2k = 1$
Collecting like terms gives $10k^2 + 9k = 1$ which is the same as $10k^2 + 9k - 1 = 0$.

ii. Factorising this equation gives $(10k - 1)(k + 1) = 0$.
Hence $k = 0.1$; it is impossible to have a negative proportion and so $k + 1 = 0$ is discarded.

b. i. This is a Binomial Distribution: $(0.14 + 0.86)^9$
The Expected Value of the Binomial is np which is $0.14 \times 9 = 1.26$

ii. ${}^9C_2\,(0.14)^2(0.86)^7 = 0.245$ to 3 decimal places.

iii. $(0.86)^n \le 0.09$
Take logarithm of both sides gives: $\log_e (0.86)^n \le \log_e 0.09$
$n\log_e (0.86) \le \log_e 0.09$

Dividing both sides of this inequality by $\log_e (0.86)$ which is a negative number

$$n \ge \frac{\log_e 0.09}{\log_e 0.86} \text{ so } n \ge 15.96$$

Answer: $n = 16$

c. Find the Z-score associated with an area of 0.078 to the left of it.
Invnorm (0.078, 0, 1) = –1.419

Now $Z = \frac{X - \mu}{\sigma}$

$-1.419 = \frac{100 - 125}{\sigma}$ and so $-1.419\sigma = -25$

Therefore $\sigma = 17.6$
To the nearest gram $\sigma = 18$.

d. Hypergeometric with N = 15, $n = 3$, D = 4
Probability of at least one defective = 1 – Pr (no defectives)

Pr (0 defectives) $= \frac{({}^4C_0 \times {}^{11}C_3)}{{}^{15}C_3}$

$= 0.3626$

Probability of at least one defective = 1 – 0.3626 = 0.637 to 3 decimal places.

e. Binomial distribution $(0.08 + 0.92)^{12}$
Probability of fewer than 2 defectives = Pr (0 defectives) + Pr (1defective)

$= {}^{12}C_1(0.08)^1(0.92)^{11} + 0.92^{12}$

$= 0.751$ correct to 3 decimal places.

Question 3

a. i. $a + bt \ge 0$ and so $bt \ge -a$.
Dividing both sides by b which is a negative number gives $t \le \frac{-a}{b}$

ii. At $t = 0$ we know that $P(0) = P_0$ and so $P_0 = P_0\, e^{G(0)}$. This means that $e^{G(0)} = 1$.
But $e^0 = 1$ and so , equating indices , G(0) = 0 .

iii. If $G'(t) = a + bt$ then anti-differentiating $G(t) = at + \frac{1}{2}bt^2 + c$ where c is a constant.

But when $t = 0$, G(0) = 0 and so $c = 0$.

Therefore $G(t) = at + \frac{1}{2}bt^2$

b. i. Now $P_0 = 3$ and $G(t) = 0.02t + 0.5 \times -0.0002t^2$ and so $P(t) = 3\, e^{0.02t - 0.0001t^2}$

ii. Since $t \le \frac{-a}{b}$ then $t \le \frac{-0.02}{-0.0002}$ which is 100 years.

iii. $\frac{dP}{dt} = P_0 G'(t)\, eG(t)$

$= 3(0.02 - 0.0002t)e^{0.02t - 0.0001t^2}$

iv. When $t = 20$, $\frac{dP}{dt} = 3\,(0.02 - 0.0002 \times 20)\ e^{0.02 \times 20 - 0.0001 \times 400} = 0.069$

c. i. $\frac{P}{3} = e^{0.01t}$ and so $\log_e \frac{P}{3} = 0.01t$.

Therefore $t = 100 \log_e \frac{P}{3}$.

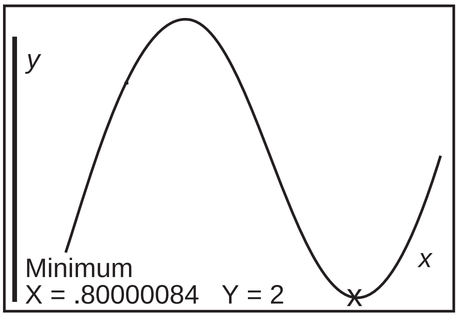
ii. When the population is doubled, $P = 6$. Therefore $t = 100 \log_e 2 = 69.31$
Answer: During the year 2059.

Question 4

a. 122 m

b. 2 m

c. The period is $\frac{2\pi}{2.5\pi}$ which is 0.8 hours. It returns to the lowest point at 1:48 pm.



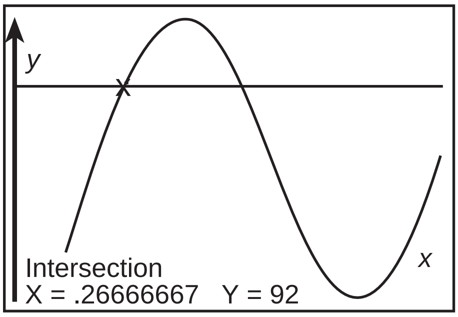
d. i. $92 = 62 + 60 \sin \frac{(5t-1)\pi}{2}$

$30 = 60 \sin \frac{(5t-1)\pi}{2}$

$0.5 = \sin \frac{(5t-1)\pi}{2}$ and so $\frac{(5t-1)\pi}{2} = \frac{\pi}{6}$

$5t - 1 = \frac{1}{3}$ and so $t = \frac{4}{15}$

It first reaches a height of 92 m at 1:16 pm.



ii. The next time it reaches a height of 92 m is when $\frac{(5t-1)}{2} = \frac{5\pi}{6}$

Therefore $5t - 1 = \frac{5}{3}$ and so $t = \frac{8}{15}$

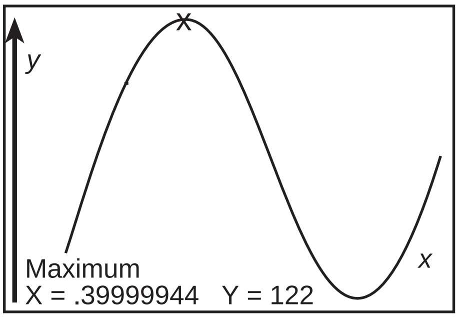
It will be at least 92 m above the ground between $t = \frac{4}{15}$ and $t = \frac{8}{15}$, that is for $\frac{4}{15}$ of an hour which is 16 minutes.

e. i. $h'(t) = 60 \times \cos \frac{(5t-1)\pi}{2}$

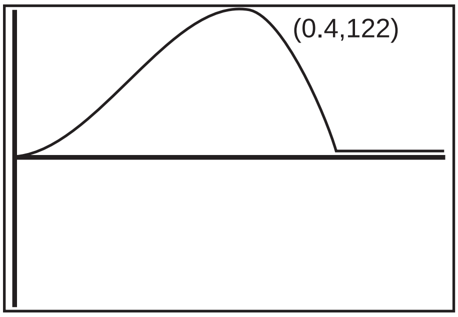
$= 150\pi \cos \frac{(5t-1)\pi}{2}$

ii. When $t = 1$, $h'(1) = 150\pi \cos 2\pi = 150\pi$
To one decimal place this is 471.2 m / h.

f. i. The co-ordinates of the local maximum are (0.4, 122).



ii. The spider's path is made up of three parts:



- For the domain [0, 0.4] it follows the path of the ferris wheel.
- From (0.4, 0.603] the spider follows the addition of ordinates path of the equations for $h(t)$ and $y = -300(t - 0.4)$.
- It then is at ground level and so follows $y = 0$ for $t > 0.603$.

iii. The spider reached the ground in 0.60259334 – 0.4 = 0.20259334 hours, that is 12 minutes (to the nearest minute).