VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2004

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

Figures									Letter
Words									

MATHEMATICAL METHODS

Written examination 2

(Analysis task)

Monday 8 November 2004

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.45 am (1 hour 30 minutes)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
4	4	55

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, up to four pages (two A4 sheets) of pre-written notes (typed or handwritten) and an approved scientific and/or graphics calculator (memory may be retained).
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 15 pages, with a detachable sheet of miscellaneous formulas in the centrefold.
- Working space is provided throughout the book.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other electronic communication devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working must be shown.

Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Consider the function $f: R \to R, f(x) = (x-1)^2(x-2)+1$.

a. If $f'(x) = (x-1)(ux+v)$, where u and v are constants, use calculus to find the values of u and v.

2 marks

b. The coordinates of the turning points of the graph of $y = f(x)$ are $(a, 1)$ and $\left(b, \frac{23}{27}\right)$.
Find the values of a and b.

2 marks

c. Find the real values of p for which the equation $f(x) = p$ has exactly one solution.

2 marks

d. The rule of the function f can also be written as $f(x) = x^3 - 4x^2 + 5x - 1$.

Use calculus to find the area, correct to three decimal places, of the region bounded by the graph of $y = f(x) - 1$ and the x-axis.

3 marks

e. **i.** Describe a sequence of transformations which maps the graph of $y = f(x)$ on to the graph of $y = f\left(\frac{x}{2}\right) - 1$.

ii. Find the x-axis intercepts of the graph of $y = f\left(\frac{x}{2}\right) - 1$.

iii. Use the answers to **d.** and **e.** parts **i.** and **ii.** above to write down the area of the region bounded by the graph of $y = f\left(\frac{x}{2}\right) - 1$ and the x-axis, correct to two decimal places.

2 + 1 + 1 = 4 marks

f. Find the real values of h for which **only one** of the solutions of the equation $f(x + h) = 1$ is positive.

2 marks

Total 15 marks

Question 2

A LUCKYDIPZ is a toy that is packaged inside an egg-shaped chocolate. A certain manufacturer produces 4 different types of LUCKYDIPZ toy – car, ship, plane and ring, in the proportions given in the table below.

car	$4k^2 + 4k$
ship	$5k^2 + 2k$
plane	$k^2 + k$
ring	$2k$

a. **i.** Show that k must be a solution of the quadratic equation $10k^2 + 9k - 1 = 0$

ii. Find the exact value of k.

1 + 1 = 2 marks

b. In response to customer demands, the settings on the machine that produces LUCKYDIPZ have been changed so that now 14% of all LUCKYDIPZ produced contain planes.

A sample of 9 LUCKYDIPZ is randomly selected from the very large number produced by the machine.

i. What is the expected number of LUCKYDIPZ that contain planes in this sample?

ii. What is the probability, correct to three decimal places, that this sample has exactly 2 LUCKYDIPZ that contain planes?

iii. What is the smallest sample size that should be taken so that the probability of selecting **no** LUCKYDIPZ that contains a plane is at most 0.09?

1 + 2 + 2 = 5 marks

c. A LUCKYDIPZ is considered to be defective if it weighs less than 100 grams. The weight of a LUCKYDIPZ is a normally distributed random variable with a mean of 125 grams.

If 7.8 % of all LUCKYDIPZ produced are defective, find, to the nearest gram, the standard deviation of the weight of LUCKYDIPZ.

3 marks

d. Matilda is a quality control engineer who picks up a box of 15 randomly packed LUCKYDIPZ. Unknown to her, there are 4 defective LUCKYDIPZ in the box. She selects 3 LUCKYDIPZ **without** replacement. Find, correct to three decimal places, the probability that at least one of the LUCKYDIPZ Matilda selects is defective.

2 marks

The next day Matilda decides that the machine producing LUCKYDIPZ should be checked each hour. She randomly selects a sample of 12 LUCKYDIPZ as they leave the machine. The machine produces a very large number of LUCKYDIPZ each hour. If the sample contains fewer than 2 defective LUCKYDIPZ, the machine is allowed to run for another hour. If there are 2 or more defective LUCKYDIPZ in the sample, the machine is stopped so that it can be adjusted.

e. What is the probability, correct to three decimal places, that under this testing procedure, the machine is allowed to run for another hour although 8% of LUCKYDIPZ being produced by the machine are defective?

2 marks

Total 14 marks

Question 3

Since the early 1970s, global population growth rates have been declining. The World Bank uses the following model for the population $P(t)$, in thousand millions of a particular group of people (for example, the low income group) at any time after the beginning of 1990.

$$P(t) = P_0 e^{G(t)}$$

where P_0 thousand million is the population of a given group at the beginning of 1990, t is the number of years after the beginning of 1990, and G is a function of t, where

$$G'(t) = a + bt$$

and a and b are real constants.

This model can be applied to various population groups by using appropriate values of P_0, a and b.

a. **i.** This model can be used when $G'(t) \geq 0$.

For a particular group, $a > 0$ and $b < 0$.

Find, **in terms of *a* and *b***, the number of years for which the model can be used for this group.

ii. Show that $G(0) = 0$

iii. Show that $G(t) = at + \frac{1}{2}bt^2$

2 + 2 + 2 = 6 marks

For the low income group, as defined by the World Bank, the population at the beginning of 1990 was three thousand million, with $a = 0.02$ and $b = -0.0002$.

b. **i.** Show that $P(t) = 3e^{(0.02t - 0.0001t^2)}$

ii. What is the greatest number of years for which this model for the low income group can be used?

iii. Using this model, find the rate of change of the population of the low income group with respect to time.

iv. At the beginning of the year 2010, what will be the rate of change of the population (in thousand millions per year) of the low income group? Give your answer correct to three decimal places.

1 + 1 + 1 + 1 = 4 marks

c. For another group, $P = 3e^{0.01t}$, $t \geq 0$, where P thousand million is the population t years after the beginning of 1990.

i. Find a rule for t in terms of P.

ii. Find the year during which the population of this group will be double what its population was at the beginning of the year 1990.

2 + 1 = 3 marks

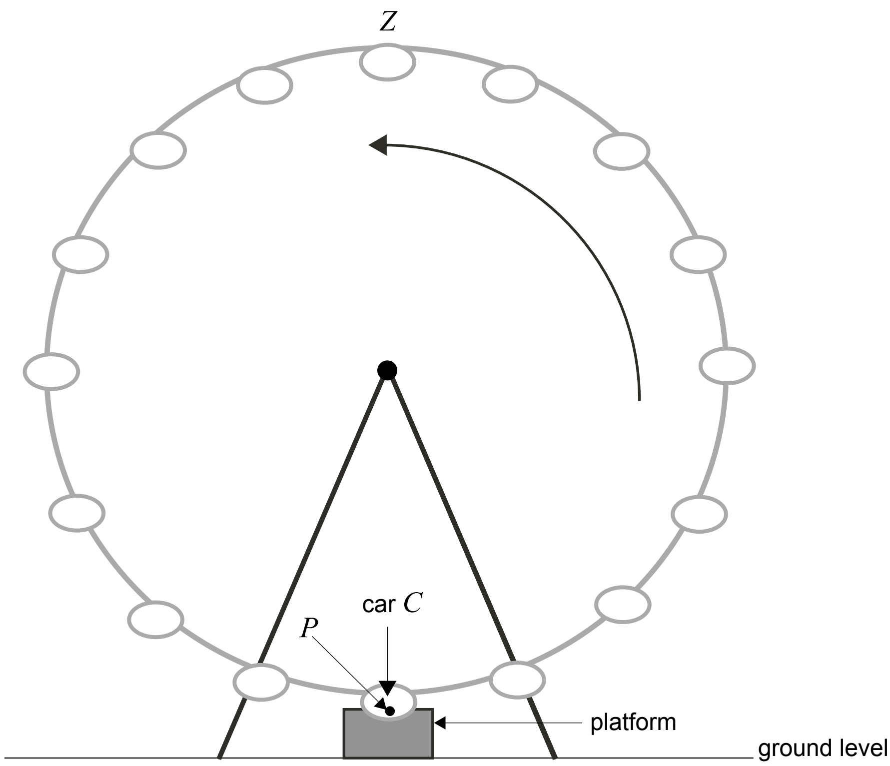
Total 13 marks

Working space

Question 4

A Ferris wheel at a theme park rotates in an anticlockwise direction at a constant rate. People enter the cars of the Ferris wheel from a platform which is above ground level. The Ferris wheel does not stop at any time.

The Ferris wheel has 16 cars, spaced evenly around the circular structure.



A spider attached itself to the point P on the side of car C when the point P was at its lowest point at 1.00 pm.

The height, h metres, of the point P above ground level, at time t hours after 1.00 pm is given by

$$h(t) = 62 + 60 \sin\left(\frac{(5t-1)\pi}{2}\right)$$

a. Write down the maximum height, in metres, of the point P above ground level.

1 mark

b. Write down the minimum height, in metres, of the point P above ground level.

1 mark

c. At what time, after 1.00 pm, does point P first return to its lowest point?

1 mark

d. **i.** Find the time, after 1.00 pm, when P first reaches a height of 92 metres above ground level.

ii. Find the **number of minutes** during one rotation that P is at least 92 metres above ground level.

2 + 1 = 3 marks

e. **i.** Write down an expression, in terms of t, for the rate of change of h with respect to time.

ii. At what rate (in m/h), correct to one decimal place, is h changing when $t = 1$?

1 + 1 = 2 marks

When point P first reaches position Z, that is, its highest point (see diagram page 12), the spider becomes frightened. It drops down from the car on a thread (which remains vertical at all times) at a rate of 5 metres per minute until it reaches the ground.

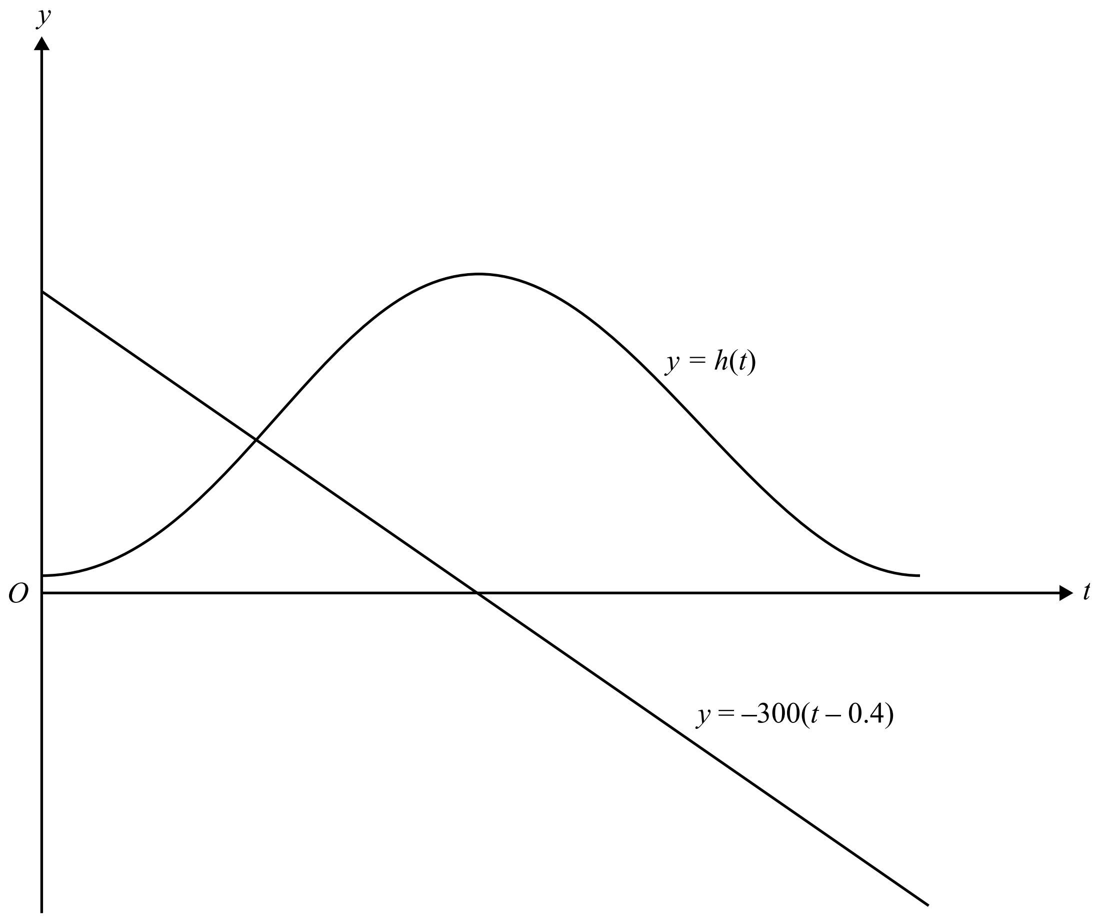
As it drops, the spider's height $s(t)$ metres above ground level at time t (where t is the time in hours after 1.00 pm) is given by

$$s(t) = h(t) - 300(t - 0.4)$$

f. The graph of $y = h(t)$ for the first revolution of the Ferris wheel after 1.00 pm and the graph of $y = -300(t - 0.4)$ are shown together on the axes below.

i. On the diagram label the local maximum point of the graph of $y = h(t)$ with its coordinates.

ii. On the diagram draw a graph which shows the height of the spider above ground level at time t.



iii. Find, to the nearest minute, the time from when the spider leaves car C to when it reaches the ground.

1 + 2 + 2 = 5 marks

Total 13 marks

Working space

END OF QUESTION AND ANSWER BOOK

MATHEMATICAL METHODS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.

Mathematical Methods Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$

$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$

$\int \frac{1}{x} dx = \log_e(x) + c, for\ x > 0$

$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$

$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$

$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$

$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$

$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Statistics and Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

mean: $\mu = E(X)$

variance: $\operatorname{var}(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

Discrete distributions			
	$\Pr(X = x)$	mean	variance
general	$p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$ $= \sum x^2 p(x) - \mu^2$
binomial	${}^nC_x\, p^x(1-p)^{n-x}$	np	$np(1-p)$
hypergeometric	$\frac{{}^DC_x\ {}^{N-D}C_{n-x}}{{}^NC_n}$	$n\frac{D}{N}$	$n\frac{D}{N}\left(1-\frac{D}{N}\right)\left(\frac{N-n}{N-1}\right)$
Continuous distributions			
normal	If X is distributed $N(\mu, \sigma^2)$ and $Z = \frac{X-\mu}{\sigma}$, then Z is distributed $N(0, 1)$.		

Table 1 Normal distribution – cdf

x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359	4	8	12	16	20	24	28	32	36
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753	4	8	12	16	20	24	28	32	35
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141	4	8	12	15	19	23	27	31	35
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517	4	8	11	15	19	23	26	30	34
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879	4	7	11	14	18	22	25	29	32
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224	3	7	10	14	17	21	24	27	31
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549	3	6	10	13	16	19	23	26	29
0.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7793	.7823	.7852	3	6	9	12	15	18	21	24	27
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133	3	6	8	11	14	17	19	22	25
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389	3	5	8	10	13	15	18	20	23
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621	2	5	7	9	12	14	16	18	21
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830	2	4	6	8	10	12	14	16	19
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015	2	4	6	7	9	11	13	15	16
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177	2	3	5	6	8	10	11	13	14
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319	1	3	4	6	7	8	10	11	13
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441	1	2	4	5	6	7	8	10	11
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545	1	2	3	4	5	6	7	8	9
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633	1	2	3	3	4	5	6	7	8
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706	1	1	2	3	4	4	5	6	6
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767	1	1	2	2	3	4	4	5	5
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817	0	1	1	2	2	3	3	4	4
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857	0	1	1	2	2	2	3	3	4
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890	0	1	1	1	2	2	2	3	3
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916	0	1	1	1	1	2	2	2	2
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936	0	0	1	1	1	1	1	2	2
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952	0	0	0	1	1	1	1	1	1
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964	0	0	0	0	1	1	1	1	1
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974	0	0	0	0	0	1	1	1	1
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981	0	0	0	0	0	0	0	1	1
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986	0	0	0	0	0	0	0	0	0
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990	0	0	0	0	0	0	0	0	0
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993	0	0	0	0	0	0	0	0	0
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995	0	0	0	0	0	0	0	0	0
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997	0	0	0	0	0	0	0	0	0
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998	0	0	0	0	0	0	0	0	0
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	0	0	0	0	0	0	0	0	0
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0	0	0	0	0	0	0	0	0

END OF FORMULA SHEET