THE HEFFERNAN GROUP

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MATHS METHODS 3 & 4 TRIAL EXAMINATION 2 SOLUTIONS 2005

Question 1

a. **i.** Since the four supervisors have supervised the draws in the proportions given

$$p - q + p^2 + p^2 + p + 4q = 1$$
$$2p^2 + 2p + 3q = 1$$
$$2p(p+1) = 1 - 3q$$

as required

(1 mark)

ii. Given that $q = \frac{1}{8}$,

$$2p(p+1) = 1 - 3q$$

becomes $2p(p+1) = 1 - \frac{3}{8}$

$$2p^2 + 2p = \frac{5}{8}$$
$$16p^2 + 16p = 5$$
$$16p^2 + 16p - 5 = 0$$
$$(4p+5)(4p-1) = 0$$
$$p = -\frac{5}{4} \quad \text{or} \quad p = \frac{1}{4}$$

Since $p > 0$, $p = \frac{1}{4}$

(1 mark)

So the proportion of nights that George supervised was $p - q = \frac{1}{4} - \frac{1}{8}$

$$= \frac{1}{8}$$

So the probability that George was the supervisor on a particular night was $\frac{1}{8}$.

(1 mark)

b. **i.** There are 25 balls, 13 odd numbered and 12 even numbered balls. The probability that the first one is odd-numbered is $\frac{13}{25}$.

(1 mark)

ii. Method 1

Let O=odd numbered ball

Let N=not an odd numbered ball

$$\Pr(\text{exactly two odd numbered balls chosen})$$
$$= \Pr(O,O,N) + \Pr(O,N,O) + \Pr(N,O,O)$$ **(1 mark)**

$$= \frac{13}{25}\times\frac{12}{24}\times\frac{12}{23} + \frac{13}{25}\times\frac{12}{24}\times\frac{12}{23} + \frac{12}{25}\times\frac{13}{24}\times\frac{12}{23}$$
$$= \frac{1872}{13800}\times 3$$
$$= \frac{234}{575}$$

(1 mark)

Method 2

Use the Hypergeometric distribution formula since there is no replacement.

$$\Pr(\text{exactly two balls are odd numbered})$$
$$= \Pr(X=2)$$
$$= \frac{{}^{D}C_{x}\ {}^{N-D}C_{n-x}}{{}^{N}C_{n}} \quad \text{Hypergeometric Formula}$$
$$= \frac{{}^{13}C_{2}\ {}^{12}C_{1}}{{}^{25}C_{3}}$$ **(1 mark)**

$$= \frac{78\times 12}{2300}$$
$$= \frac{234}{575}$$

(1 mark)

Note that
D = total number of odd numbered balls
N = total number of balls
n = number of balls in the sample
x = number of odd numbered balls in the sample

iii. Use the Hypergeometric distribution formula whereby $D=13,\ x=7,\ N=25$ and $n=15$

$$\Pr(\text{exactly 7 out of the 15 balls chosen are odd numbered})$$
$$= \frac{{}^{D}C_{x}\ {}^{N-D}C_{n-x}}{{}^{N}C_{n}}$$
$$= \frac{{}^{13}C_{7}\ {}^{12}C_{8}}{{}^{25}C_{15}}$$ **(1 mark)**

$$= \frac{1716\times 495}{3268760}$$
$$= 0\cdot 2599 \quad (\text{correct to 4 decimal places})$$

(1 mark)

c. **i.** Because the ball is being replaced, we have a Binomial distribution where $n = 3,\ p = \frac{1}{25}$ and $q = \frac{24}{25}$.

$\Pr(\text{the ball numbered 16 appeared exactly once})$
$= \Pr(X = 1)$
$= {}^nC_x (p)^x (1-p)^{n-x}$
$= {}^3C_1 \left(\frac{1}{25}\right)^1 \left(\frac{24}{25}\right)^2$
$= 0 \cdot 1106$ (correct to 4 decimal places)

(1 mark)

ii. $\Pr(\text{the ball numbered 16 appears at least once})$
$= \Pr(X \geq 1)$
$= 1 - \Pr(X = 0)$ **(1 mark)**
$= 1 - {}^3C_0 \left(\frac{1}{25}\right)^0 \left(\frac{24}{25}\right)^3$
$= 0 \cdot 1153$ (correct to 4 decimal places)

(1 mark)

iii. $\Pr(X = 0) = {}^nC_0 \left(\frac{9}{25}\right)^0 \left(\frac{16}{25}\right)^n$ (since there are 9 balls numbered with a single digit)

$0 \cdot 044 = 1 \times 1 \times \left(\frac{16}{25}\right)^n$

So $0 \cdot 044 = (0 \cdot 64)^n$ **(1 mark)**

$\log_{10}(0 \cdot 044) = \log_{10}(0 \cdot 64)^n$
$\log_{10}(0 \cdot 044) = n \log_{10}(0 \cdot 64)$
$n = \frac{\log_{10}(0 \cdot 044)}{\log_{10}(0 \cdot 64)}$
$= 6 \cdot 999...$
$= 7$ (closest whole number)

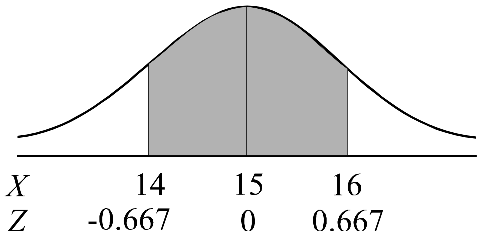
(1 mark)

d. We have a Normal distribution with $\mu = 15$ and $\sigma = 1 \cdot 5$

Method 1 Using the normal distribution cdf table

$$\begin{aligned}
&\Pr(14 < X < 16)\\
&= \Pr(X < 16) - \Pr(X < 14) \quad \textbf{(1 mark)}\\
&= \Pr(z < 0 \cdot 667) - (1 - \Pr(z < 0 \cdot 667))\\
&= \Pr(z < 0 \cdot 667) - 1 + \Pr(z < 0 \cdot 667)\\
&= 2 \times \Pr(z < 0 \cdot 667) - 1\\
&= 2 \times 0 \cdot 7477 - 1\\
&= 0 \cdot 4954
\end{aligned}$$

So 49.5% (to 1 decimal place) of balls supplied by the manufacturer are actually used.



$$z = \frac{16 - 15}{1 \cdot 5}$$
$$= 0 \cdot 667 \text{ to 3 dec places}$$
$$z = \frac{14 - 15}{1 \cdot 5}$$
$$= -0 \cdot 667 \text{ to 3 dec places}$$

Method 2 Using a graphics calculator

$\Pr(14 < X < 16) = 0.4950$ (using –0.6666666 and 0.6666666 as the lower and upper bounds respectively)
So 49.5% (to 1 decimal place) of balls supplied by the manufacturer are actually used.

(2 marks)

Total 15 marks

Question 2

a. The minimum distance of the shelf from the floor is $120 - 25 = 95\text{cm}$ because the minimum value that $\sin(\frac{\pi t}{300})$ can have is –1. Hence, $120 + 25 \times -1 = 95\text{cm}$.

(1 mark)

b. The maximum distance of the shelf from the floor similarly is $120 + 25 = 145\text{cm}$.
So the vertical distance that the shelf moves through is $145 - 95 = 50\text{cm}$.

(1 mark)

c.

$$\begin{aligned}
d(75) &= 120 + 25\sin(\frac{75\pi}{300})\\
&= 120 + 25\sin(\frac{\pi}{4})\\
&= 120 + \frac{25\sqrt{2}}{2}
\end{aligned}$$

At $t = 75$ seconds, the shelf is $\left(120 + \frac{25\sqrt{2}}{2}\right)$cm above the floor.

(1 mark)

d. Sketch the graphs of $d = 120 + 25\sin(\frac{\pi t}{300})$ and $d = 100$.

Note that the first 10 minutes of movement corresponds to between $t = 0$ and $t = 600$ secs. Make sure that your window extends to $X = 600$.

Find the two points of intersection that occur in this domain.

They are $(388 \cdot 55, 100)$ and $(511 \cdot 45, 100)$.

The shelf is exactly 1 metre above the floor at $t = 388 \cdot 55$ and at $t = 511 \cdot 45$ (to 2 decimal places).

(2 marks)

e. The rate at which the distance of the shelf above the floor is changing is given by $d'(t)$.

Now $d(t) = 120 + 25\sin(\frac{\pi t}{300})$

So $d'(t) = 25 \times \frac{\pi}{300}\cos(\frac{\pi t}{300})$

$= \frac{\pi}{12}\cos(\frac{\pi t}{300})$

(1 mark)

(Note that if $y = 25\sin(\frac{\pi t}{300})$

then, $y = 25\sin(u)$ where $u = \frac{\pi t}{300}$

$\frac{dy}{du} = 25\cos(u)$ $\frac{du}{dt} = \frac{\pi}{300}$

so, $\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}$ Chain rule

$= 25\cos(u) \cdot \frac{\pi}{300}$

$= \frac{25\pi}{300}\cos(\frac{\pi t}{300})$

$= \frac{\pi}{12}\cos(\frac{\pi t}{300}))$

At $t = 50$,

$d'(50) = \frac{\pi}{12}\cos(\frac{\pi \times 50}{300})$

$= 0 \cdot 23 \text{ cm/sec}$

(1 mark)

f. From part e., $d'(t) = \frac{\pi}{12}\cos(\frac{\pi t}{300})$.

Graph this function.

This function intersects the horizontal axis at $t = 150$ and $t = 450$. Between these two times the rate of change of the height of the shelf above the floor is negative; that is, $d'(t) < 0$ for $150 < t < 450$.

(2 marks)

g. The two shelves are at the same height above the floor when $h(t)=h_1(t)$;
that is, when

$$120+25\sin(\frac{\pi t}{300})=120+50\cos(\frac{\pi t}{300})$$

$$25\sin(\frac{\pi t}{300})=50\cos(\frac{\pi t}{300})$$ **(1 mark)**

$$\frac{\sin(\frac{\pi t}{300})}{\cos(\frac{\pi t}{300})}=\frac{50}{25}$$

$$\tan(\frac{\pi t}{300})=2$$ **(1 mark)**

$$\frac{\pi t}{300}=1.1071...,\pi+1.1071...,\ 2\pi+1.1071...,\ ...$$

$$t=105.7249...,\ 405.7249...,\ 705.7249...,\ ...$$

S	A
T	C

Over the domain $t\in[0,600]$, we have

$$t=105.7249...,\ 405.7249...$$

So the two shelves are at the same height at $t=105\cdot 72$ and $t=405\cdot 72$ (to 2 decimal places).

(1 mark)
Total 12 marks

Question 3

a. Use a graphics calculator to sketch the function and calculate the x-intercept. The answer is $x=1\cdot 27$ (correct to 2 decimal places).

(1 mark)

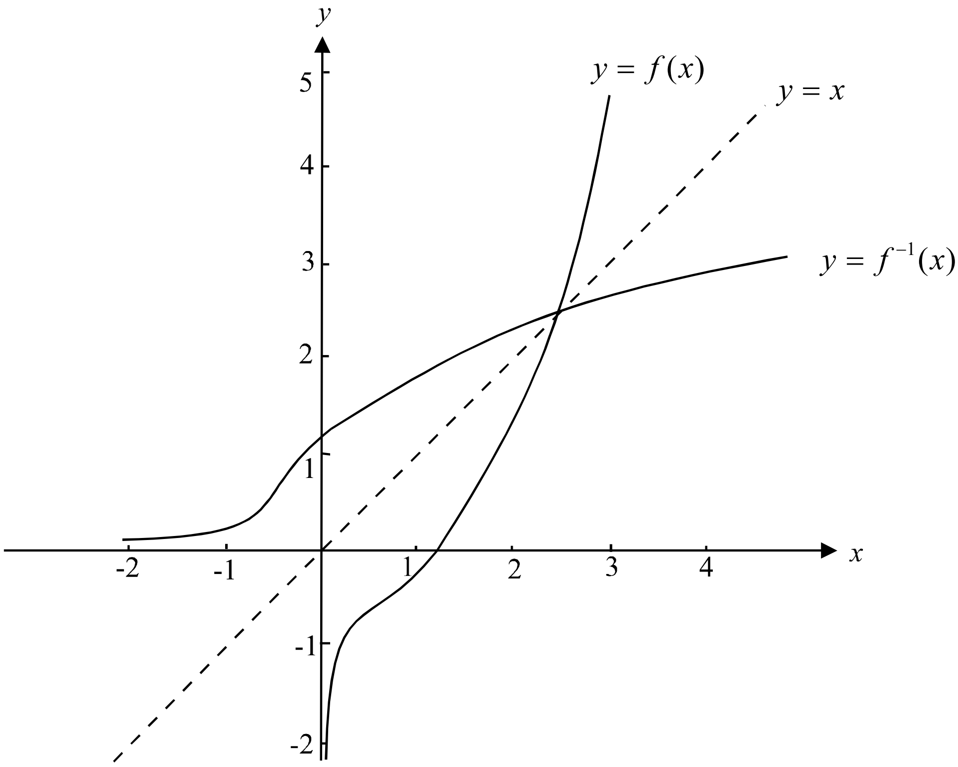
b. The function $f(x)=\log_e(2x)-2x+x^2$ is not defined for $x\le 0$ because $\log_e(2x)$ is not defined for $x\le 0$.

(1 mark)

c. The inverse function $f^{-1}(x)$ exists because as we see from the graph, the graph of $y=f(x)$ is 1:1.

(1 mark)

d.



(1 mark)

e. Since the graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ in the line $y = x$, the y-intercept of the graph of $y = f^{-1}(x)$ is the x-intercept of the graph of $y = f(x)$ which is 1.27 (correct to 2 decimal places) from part a.

(1 mark)

f. $f(x) = \log_e(2x) - 2x + x^2, \quad x > 0$

$$f'(x) = \frac{2}{2x} - 2 + 2x$$
$$= \frac{1}{x} - 2 + 2x$$

A stationary point occurs when $f'(x) = 0$

Let $\frac{1}{x} - 2 + 2x = 0$ **(1 mark)**

$1 - 2x + 2x^2 = 0, \; x \neq 0$

Now, $\Delta = b^2 - 4ac$
$= (-2)^2 - 4 \times 2 \times 1$
$= -4$

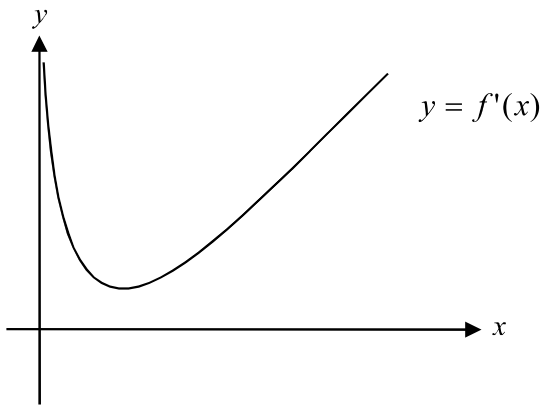
Since $\Delta < 0$, there are no solutions to the equation $2x^2 - 2x + 1 = 0$ and hence the graph of $y = f(x)$ has no stationary points.

(1 mark)

g. The rate of change of $f(x)$ with respect to x is given by $f'(x)$.

From part f. $f'(x)=\frac{1}{x}-2+2x$.

A sketch of $y=f'(x)$ is shown below.



(1 mark)

From the graph we see that the function $f'(x)$ is always positive so the rate of change of $f(x)$ with respect to x is always positive.

(1 mark)

h.
$$f(x)=\log_e(2x)-2x+x^2, \quad x>0$$
$$f'(x)=\frac{1}{x}-2+2x \qquad \text{from part f.}$$

Now, $f'(3)=\frac{1}{3}-2+2\times 3$
$$=\frac{13}{3}$$

So the gradient of the tangent to $y=f(x)$ at $x=3$ is $\frac{13}{3}$.

Therefore the gradient of the normal to $y=f(x)$ at $x=3$ is $\frac{-3}{13}$.

(1 mark)

Also $f(3)=\log_e(6)-2\times 3+3^2$
$$=\log_e(6)+3$$

The equation of the normal to $f(x)$ at $x=3$ is given by
$$y-y_1=m(x-x_1)$$

where $m=\frac{-3}{13}$ and $(x_1,y_1)=(3,\log_e(6)+3)$

So $y-(\log_e(6)+3)=\frac{-3}{13}(x-3)$ is the equation of the normal.

(1 mark)

Simplifying we obtain:

$$13y - 13(\log_e(6) + 3) = -3(x - 3)$$
$$13y - 13\log_e(6) - 39 = -3x + 9$$
$$13y + 3x = 13\log_e(6) + 48$$

To find the *x* intercept let $y = 0$.

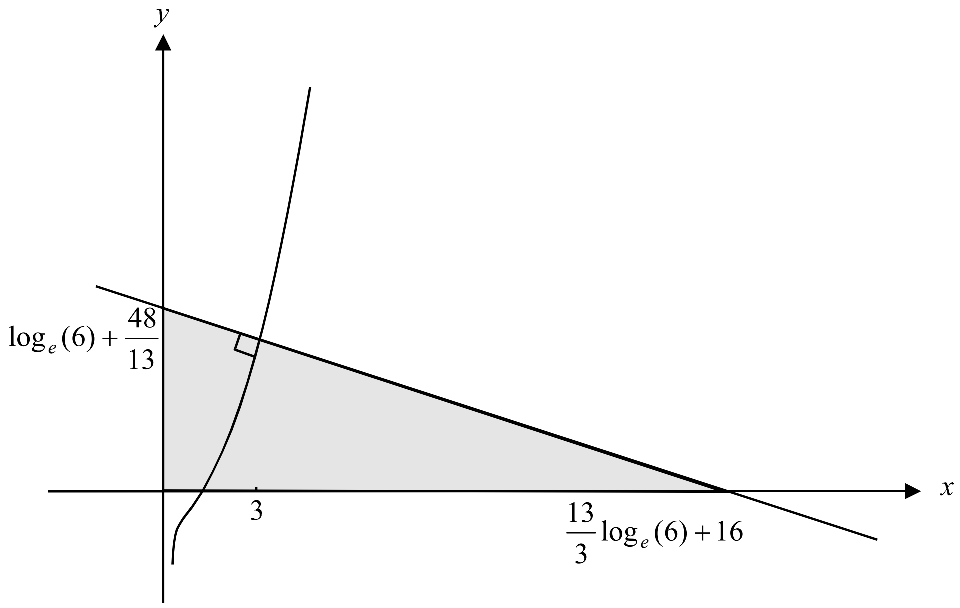
So, $3x = 13\log_e(6) + 48$

$$x = \frac{13}{3}\log_e(6) + 16$$

To find the *y* intercept let $x = 0$

$$13y = 13\log_e(6) + 48$$
$$y = \log_e(6) + \frac{48}{13}$$

(1 mark)



The area required is shaded in the diagram above and is that of a triangle with base of width $\frac{13}{3}\log_e(6) + 16$ and height of $\log_e(6) + \frac{48}{13}$.

(1 mark)

$$\text{Area required} = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times \left(\frac{13}{3}\log_e(6) + 16\right) \times \left(\log_e(6) + \frac{48}{13}\right)$$

So, $a = \frac{13}{3}$, $b = 16$ and $c = \frac{48}{13}$.

(1 mark)
Total 14 marks

Question 4

a. $N(t)=50\times e^{(t^2-qt+1)}, \quad t\geq 0, q\geq 0$

When $t=0$, $\quad N(t)=50\times e^{(0-0+1)}$

$=50e$

$=136$ to the nearest whole number

(1 mark)

b. According to this model the population of a colony can never equal zero because $N(t)=50\times e^{(t^2-qt+1)}$ and $50\neq 0$ and $e^{(t^2-qt+1)}\neq 0$ so $N(t)\neq 0$. That is, there is no number that you can raise e to, in order to equal zero.

(1 mark)

c. Let $q=0$ for a particular colony.

So, $\quad N(t)=50\times e^{(t^2-qt+1)}$

becomes $N(t)=50\times e^{(t^2+1)}$

Let $\quad y=50\times e^{(t^2+1)}$

Now $\quad y=50\times e^{(u)}$ where $\quad u=t^2+1$

$\frac{dy}{du}=50\times e^{(u)}$ and $\frac{du}{dt}=2t$

$\frac{dy}{dt}=\frac{dy}{du}.\frac{du}{dt}$ Chain rule

So $\quad \frac{dy}{dt}=50\times e^{(u)}\cdot 2t$

$=100t\times e^{(t^2+1)}$ **(1 mark)**

Now $t\geq 0$ and $e^{t^2+1}>0$ so $100t\times e^{t^2+1}>0$ and therefore $\frac{dy}{dt}>0$ so the population is increasing.

(1 mark)

d. Now $\quad N(t)=50\times e^{(t^2-qt+1)}$

Let $\quad y=50\times e^{(t^2-qt+1)}$

$y=50\times e^{u}$ where $u=t^2-qt+1$

$\frac{dy}{du}=50\times e^{u}$ $\quad \frac{du}{dt}=2t-q$

$\frac{dy}{dt}=\frac{dy}{du}\cdot\frac{du}{dt}$

$\frac{dy}{dt}=50\times e^{u}\times(2t-q)$

$=50(2t-q)e^{(t^2-qt+1)}$ **(1 mark)**

Now, a minimum occurs when $\frac{dy}{dt}=0$.

So, $\quad 50(2t-q)e^{(t^2-qt+1)}=0$

Now $50\neq 0$ and $e^{(t^2-qt+1)}\neq 0$

So $\quad 2t-q=0$

(1 mark)

$t=\frac{q}{2}$ $\quad$ – $\quad$ (*)

Also, the minimum that occurs is 50.

So $\quad N(t)=50\times e^{t^2-qt+1}$

becomes $\quad 50=50\times e^{t^2-qt+1}$

$1=e^{t^2-qt+1}$

$e^0=e^{t^2-qt+1}$

So $\quad t^2-qt+1=0$

(1 mark)

From (*), $t=\frac{q}{2}$

So, $\quad \frac{q^2}{4}-\frac{q^2}{2}+1=0$

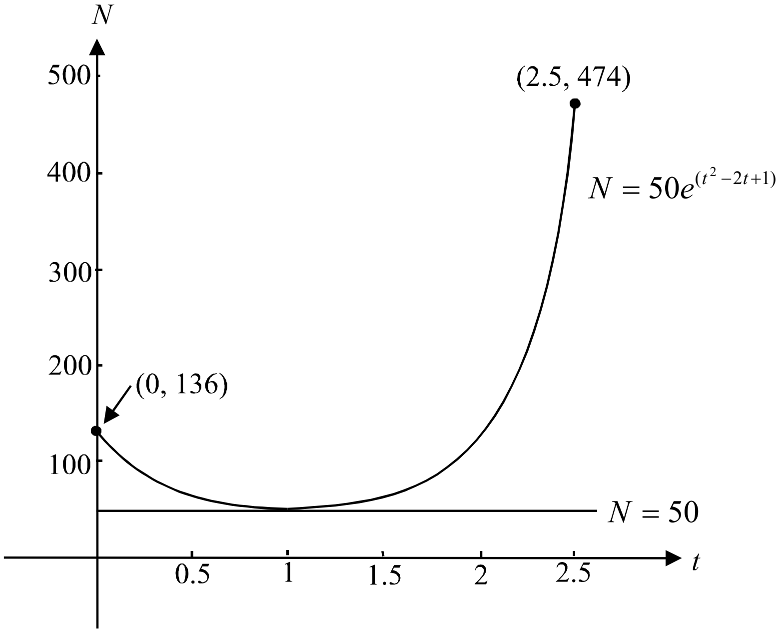
$1-\frac{q^2}{4}=0$

$q^2=4$

$q=2$ since $q\geq 0$

(1 mark)

e.



(2 marks)

f. $\quad N=50e^{\left(t^2-qt+1\right)}$ and $N=50$

The graphs of these functions intersect when

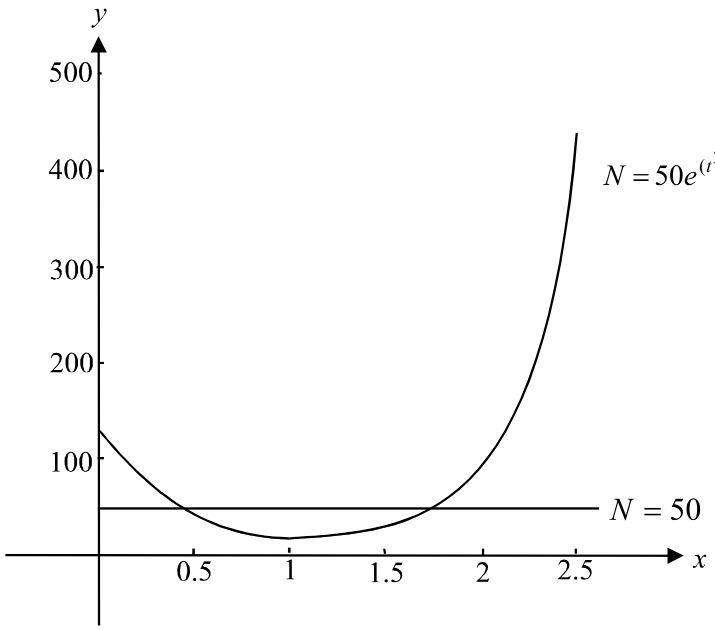
$50e^{\left(t^2-qt+1\right)}=50$ **(1 mark)**

$e^{\left(t^2-qt+1\right)}=1$

$e^{\left(t^2-qt+1\right)}=e^0$ **(1 mark)**

$t^2-qt+1=0$

For 2 points of intersection, for example as shown in the diagram below, $\Delta > 0$.

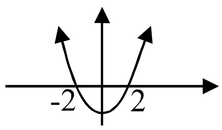


So $\Delta = q^2 - 4\times 1\times 1 > 0$

$$q^2 - 4 > 0$$

(1 mark)

Consider the graph of $y = q^2 - 4$



We want the values of q for which $y = q^2 - 4 > 0$ i.e. we want $q < -2$ or $q > 2$.
So the values of q for which the population of a mice colony would drop below 50 are $q > 2$ since $q \geq 0$.

(1 mark)

(Note: given that $t^2 - qt + 1 = 0$, if there is one point of intersection,

then $\Delta = 0$

$$q^2 - 4\times 1\times 1 = 0$$

$$q^2 - 4 = 0$$

$$(q-2)(q+2) = 0$$

$$q = +2 \text{ or } q = -2$$

but $q \geq 0$ so $q = 2$

this confirms our answer to part d.)

Total 14 marks