# UNIT 4 MATHEMATICAL METHODS 2005
# WRITTEN EXAMINATION 2 - SOLUTIONS

**QUESTION 1**

**a.** (i) Using the Pythagorean Theorem:

$$d^2 + w^2 = 15^2$$
$$d^2 = 225 - w^2$$

$$S = kd^2 w = k(225 - w^2)w = k(225w - w^3)$$

(ii) $d^2 = 225 - w^2$

$d = \pm\sqrt{225 - w^2}$ As dimensions must be positive in value, $d \neq -\sqrt{225 - w^2}$

$\therefore d = \sqrt{225 - w^2}$

$\therefore w^2 < 225$

$w < \pm 15$

As dimensions must be positive in value, $0 < w(cm) < 15$.

**b.** (i) $S = k(225w - w^3) = 225wk - kw^3$

As $k > 0$, and $0 < w(cm) < 15$, the graph of $S$ lies entirely above the $w$ axis for all accepted values of $w$.

$$\int_0^5 k(225w - w^3)\, dw = 425$$

$$k\int_0^5 (225w - w^3)\, dw = 425$$

$$k\left[\frac{225w^2}{2} - \frac{w^4}{4}\right]_0^5 = 425$$

$$k\left(\frac{225(25)}{2} - \frac{(5)^4}{4}\right) - (0) = 425$$

$$k\left(\frac{10625}{4}\right) = 425$$

$$\therefore k = \frac{4}{25}$$

(ii) $S = \frac{4}{25}(225w - w^3)$

A maximum can occur at an end point of a domain or a stationary point.
Using the calculator – the following points can be obtained:

When $w = 0$, $S = 0$.
When $w = 7$, $S = 197.12$.
Stationary point occurs at $(8.660, 207.846)$ - which lies outside the possible domain.

Therefore, maximum strength obtained when $w = 7$ cm.

Therefore, $d = \sqrt{225 - w^2} = \sqrt{225 - 49} = \sqrt{176} = \sqrt{16 \times 11} = 4\sqrt{11}$ cm

**c.** (i) $C(\$) = 0.2 \times Area\ Face$

$C(\$) = 0.2(dw) = 0.2w\sqrt{225 - w^2}$

(ii) $C = 0.2w(225 - w^2)^{1/2}$

Product Rule: $\frac{dC}{dw} = 0.2w \times \left(\frac{1}{2} \times -2w \times (225 - w^2)^{-1/2}\right) + (225 - w^2)^{1/2} \times 0.2$

$= \frac{-0.2w^2}{\sqrt{225 - w^2}} + 0.2\sqrt{225 - w^2}$

Let $\frac{dC}{dw} = 0$

$$\frac{-0.2w^2}{\sqrt{225 - w^2}} + 0.2\sqrt{225 - w^2} = 0$$

$$\frac{0.2w^2}{\sqrt{225 - w^2}} = 0.2\sqrt{225 - w^2}$$

$$0.2w^2 = 0.2\left(\sqrt{225 - w^2}\right)^2$$

$w^2 = 225 - w^2$
$2w^2 = 225$

$$w = \sqrt{\frac{225}{2}} = \frac{15}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{15\sqrt{2}}{2}\ cm$$

Maximum cost occurs when $w = \frac{15\sqrt{2}}{2}\ cm$ and is equal to $\$22.50$

**QUESTION 2**

**a.** (i) $\Pr(tuna\ \ from\ Boat\ 1) = \dfrac{1800}{3000} = \dfrac{3}{5}$

(ii) Let $X =$ Number of tuna

$E(X) = \dfrac{3}{5} \times 500 = 300$

**b.** (i) $\Pr(Boat\ 1 / tuna) = \dfrac{\Pr(Boat\ 1 \cap tuna)}{\Pr(tuna)} = \dfrac{1800/5000}{2600/5000} = \dfrac{1800}{2600} = \dfrac{9}{13}$

Alternatively – the probability may be obtained directly from the given table.

$\dfrac{Number\ tuna\ caught\ by\ Boat 1}{Total\ number\ tuna\ caught} = \dfrac{1800}{2600} = \dfrac{9}{13}$

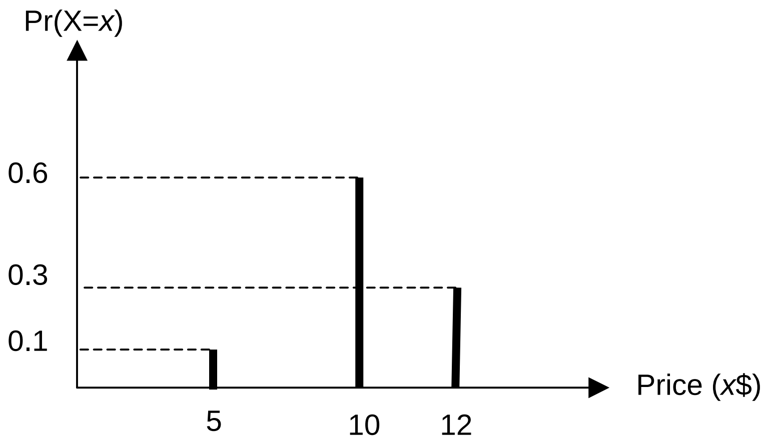
(ii) $\Pr(one\ is\ a\ tuna) = \Pr(tuna\ \ from\ Boat\ 1\ AND\ not\ \ from\ Boat\ 2)$ OR
$\Pr(tuna\ \ from\ Boat\ 2\ AND\ not\ \ from\ Boat\ 1)$

$= \left(\dfrac{1800}{3000} \times \dfrac{1200}{2000}\right) + \left(\dfrac{800}{2000} \times \dfrac{1200}{3000}\right)$

$= 0.36 + 0.16 = 0.520$

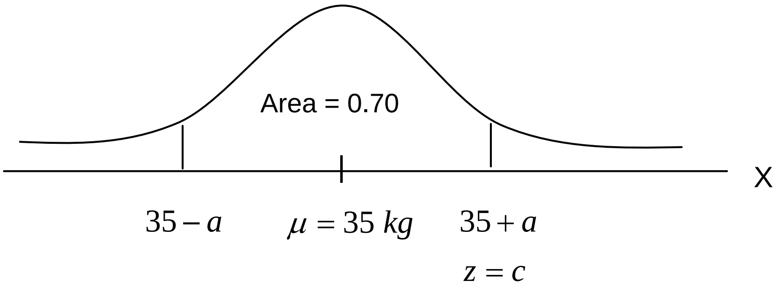
**c.** (i)

| | Tuna | Salmon | Other |
|---|---|---|---|
| Number of fish | 1800 | 900 | 300 |
| Price per kilo (*x*$) | 10 | 12 | 5 |
| Probability (Pr(X=*x*) | $\dfrac{1800}{3000} = 0.6$ | $\dfrac{900}{3000} = 0.3$ | $\dfrac{300}{3000} = 0.1$ |




(ii) $E(\$) = (10 \times 0.6) + (12 \times 0.3) + (5 \times 0.1) = \$10.10\ /\ kg$

**d.** (i)




$\Pr(X < 35 + a) = 0.85$
$\Pr(z < c) = 0.85$
$invnorm(0.85, 0, 1) = 1.0364$

$\therefore c = 1.0364$

$$\frac{X - \mu}{\sigma} = 1.0364$$

$$\frac{35 + a - 35}{\sigma} = 1.0364$$

$a = 1.0364\sigma$

(ii) Binomial.

$X$ = *Number acceptable fish*
$p$ = *probability acceptable* $= 0.70$
$n = 5$

Find $\Pr(X > 2) = 1 - \Pr(X = 0) - \Pr(X = 1) - \Pr(X = 2)$

$$= 1 - \Pr(X \le 2) = 1 - binomcdf(5, 0.7, 2) = 0.837$$

**e.** (i) Let $X$ = number of acceptable fish

$p = p$

$n = 5$

$\Pr(X = 2) = \binom{5}{2}(p)^2(1-p)^3 = 10p^2(1-p)^3$

(ii) $P'(p) = \left[10p^2 \times -3(1-p)^2\right] + \left[(1-p)^3 \times 20p\right]$

$$= -30p^2(1-p)^2 + 20p(1-p)^3$$
$$= 10p(1-p)^2\left[-3p + 2(1-p)\right]$$
$$= 10p(1-p)^2\left[-3p + 2 - 2p)\right]$$
$$= 10p(1-p)^2(2-5p)$$

Let $P'(p) = 0$

$10p(1-p)^2(2-5p) = 0$

$p = 0, 1, \frac{2}{5}$

As $0 < p < 1$, $p = \frac{2}{5}$.

**QUESTION 3**

**a.** (i) As there is no reflection in the X axis, $a$ represents the amplitude.

Amplitude $= \frac{1}{2} \times$ Distance between maximum and minimum values $= \frac{1}{2} \times 8 = 4\ m$

(ii) Period = 12 metres

$$\frac{2\pi}{b} = 12$$

$$b = \frac{\pi}{6}$$

(iii) $h(x) = 4\cos\left(\frac{\pi x}{6}\right) + c$

As curve passes through the point $(0, 6)$: $6 = 4\cos(0) + c \quad \therefore c = 2$.

**b.** Find $x$ when $h = 2(\sqrt{2} + 1)$

$$4\cos\left(\frac{\pi x}{6}\right) + 2 = 2(\sqrt{2} + 1)$$

$$\cos\left(\frac{\pi x}{6}\right) = \frac{\sqrt{2}}{2}$$

1st Quadrant angle: $Cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

Solutions to lie in Quadrants 1 and 4.

$\left(\frac{\pi x}{6}\right) = \frac{\pi}{4}$ or $\left(\frac{\pi x}{6}\right) = 2\pi - \frac{\pi}{4}$

$x = \frac{3}{2}, \frac{21}{2}$ metres

**c.** $f(x+h) \approx f(x) + hf'(x)$

Let $x = 3$
$x + h = 2.9$
$\therefore h = -0.1$

$f(2.9) \approx f(3) - 0.1f'(3)$

$f(x) = 4\cos\left(\frac{\pi x}{6}\right) + 2 \quad \therefore \quad f(3) = 2$

$f'(x) = -\frac{2\pi}{3}\sin\left(\frac{\pi x}{6}\right) \quad f'(3) = -2.0944$

As $f(2.9) \approx f(3) - 0.1f'(3)$
$f(2.9) \approx 2 - 0.1(-2.0944) \approx 2.20922$

Change in height $=$ *final height* $-$ *initial height*
$2 - 2.20922 = -0.20922 = -0.209$ metres.

i.e. the height has decreased by $0.209$ metres.

**d.** (i) To join functions smoothly, the gradients at the point of contact must be equal.

Ramps meet at $(6, -2)$

$h'(x) = -\frac{2\pi}{3}\sin\left(\frac{\pi x}{6}\right)$

$h'(0) = -\frac{2\pi}{3}\sin(0) = 0$

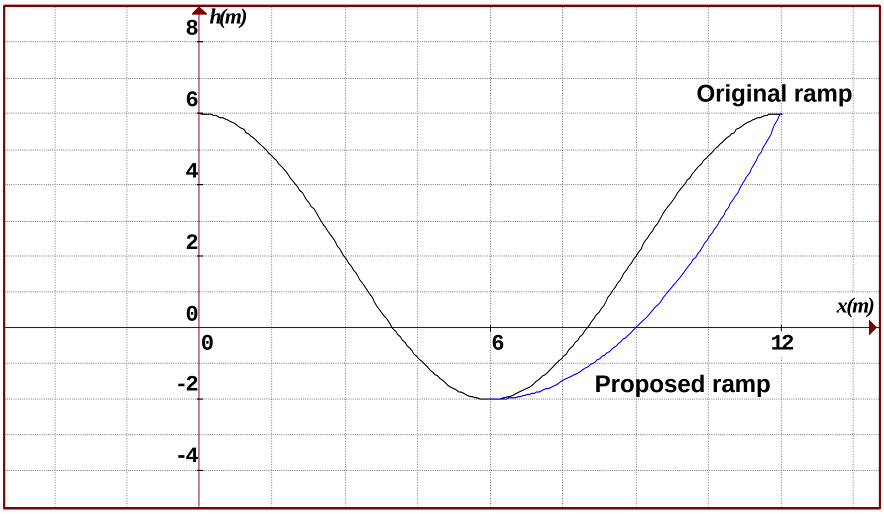
$g(x) = \frac{2}{9}(x-6)^2 - 2$

$g'(x) = \frac{4}{9}(x-6)$

$g'(6) = \frac{4}{9}(6-6) = 0$

As gradients are the same at the point of contact, join is smooth.

(ii)




**Find points of Intersection:** Coordinates can only be found using the INTERSECT function of the graphics calculator (or via iteration).

Point of Intersection is $(12, 6)$.

$$\int_6^{12} 4\cos\left(\frac{\pi x}{6}\right)+2-\left(\frac{2}{9}(x-6)^2-2\right)dx$$

$$\int_6^{12} 4\cos\left(\frac{\pi x}{6}\right)+2-\frac{2}{9}(x-6)^2+2\,dx$$

$$\int_6^{12} 4\cos\left(\frac{\pi x}{6}\right)-\frac{2}{9}(x-6)^2+4\,dx$$

$$\left[\frac{24}{\pi}\sin\left(\frac{\pi x}{6}\right)-\frac{2(x-6)^3}{27}+4x\right]_6^{12}$$

$$=32-24=8\,units^2$$

**QUESTION 4**

**a.** (i) $y = A\log_e(t+b)$

When $t=0$, $y=0$

$A\log_e(b)=0$

$\log_e(b)=0$
$\therefore b=1$

(ii) $y = A\log_e(t+1)$

When $t=8$, $y=50$
$A\log_e(8+1)=50$
$A\log_e(9)=50$
$A=\dfrac{50}{\log_e 9}$

**b.** (i) Find $y$ when $t=15$.

$$y=\left(\frac{50}{\log_e 9}\right)\log_e(t+1)$$

$$y=\left(\frac{50}{\log_e 9}\right)\log_e(15+1)=\left(\frac{50}{\log_e 9}\right)\log_e(16) \text{ units}$$

(ii) $\left(\dfrac{50}{\log_e 9}\right)\log_e(t+1)=\dfrac{1}{\log_e 9}$

$\log_e(t+1)=\dfrac{1}{50}$

$e^{1/50}=t+1$

$t=e^{1/50}-1\ \mu\text{sec}$

**c.** (i) $y=\left(\dfrac{50}{\log_e 9}\right)\log_e(t+1)$

$$t=\left(\frac{50}{\log_e 9}\right)\log_e(y+1)$$

$$\frac{t\log_e 9}{50}=\log_e(y+1)$$

$$e^{\frac{t}{50}\log_e 9}=y+1$$

$$y=f^{-1}(t)=e^{\frac{t}{50}\log_e 9}-1=e^{\log_e 9^{t/50}}-1$$

$$y=f^{-1}(t)=9^{t/50}-1$$

Alternatively: $\dfrac{t}{50}\log_e 9=\log_e(y+1)$

$$\log_e 9^{t/50}=\log_e(y+1)$$

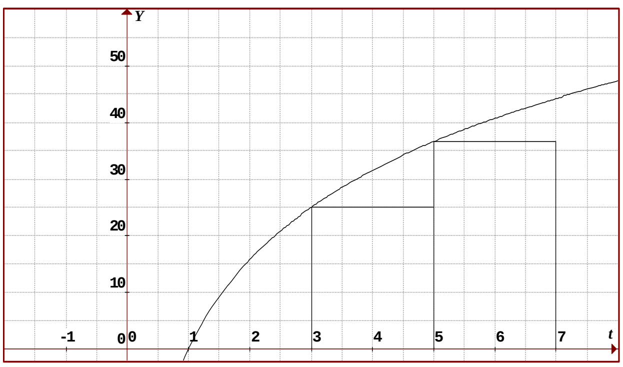
$$9^{t/50}=y+1$$

$$y=f^{-1}(t)=9^{t/50}-1$$

(ii) Range $f^{-1}=$ Domain $f(t):(0,\infty)$. Note restriction on domain due to the fact that time cannot be negative.

**d.** (i) Translation of 1 unit in the positive direction, parallel to the $t$ axis.

(ii)




Area = $(0\times 2)+\left(\dfrac{50\log_e 3}{\log_e 9}\times 2\right)+\left(\dfrac{50\log_e 5}{\log_e 9}\times 2\right)=123.259\ units^2$

**e.** (i) $y = t\log_e t$

Product Rule: $\dfrac{dy}{dt} = \left(t \times \dfrac{1}{t}\right) + \log_e t = 1 + \log_e t$

(ii) $\int (1 + \log_e t)\,dt = t\log_e t + c$

$\int (1)\,dt + \int (\log_e t)\,dt = t\log_e t + c$

$\int (\log_e t)\,dt = t\log_e t - \int 1\,dt + c$

$= t\log_e t - t + d$

(iii) Area $= \int_1^7 (A\log_e t)\,dt = A\int_1^7 (\log_e t)\,dt$

$= A[t\log_e t - t]_1^7$
$= A(6.6214 - -1) = 7.6214A$

As $A = \dfrac{50}{\log_e 9}$, Area = $\dfrac{50}{\log_e 9} \times 7.6214 = 173.432\ units^2$