2005 Mathematical Methods (CAS) GA 2: Examination 1

GENERAL COMMENTS

The number of students who sat this examination in 2005 was 333, compared with 389 in 2004. Marks ranged from 5 to the maximum score of 50. Student responses showed that the paper was accessible and that it provided the opportunity for students to demonstrate what they knew. Of the whole cohort, 30 students scored 90% or more of the available marks, and 204 scored 50% or more of the available marks. The mean score for the paper was 28.6, with a mean of 17.9 (out of 27) for the multiple-choice section and a mean of 10.7 (out of 23) for the short-answer section. The median score for the paper was 29 marks. There was a slight decrease in both the mean and median scores compared to 2004.

Overall, the symbolic facility of CAS was used well. There was no discernable advantage seen by the assessors of one CAS over another, although in some cases techniques for dealing with the mathematics involved varied according to the CAS. There were many very good responses to the questions in Part II and several students were able to work through the questions completely and obtain full marks (or close to it) for this section. There was little evidence to suggest that some students had run out of time and were therefore not able to make a reasonable attempt at Part II.

Students should be familiar with the examination instructions and realise that:

- a decimal approximation will not be accepted if an exact answer is required to a question
- in questions where more than one mark is available, appropriate working must be shown.

Areas of strength and weakness

Strengths

Common areas of strength were:

- finding the inverse function of a function
- sketching and interpreting graphs
- interpreting and determining the mean of a discrete probability distribution given in the form of a table
- numerical solutions of equations.

Weaknesses

Common areas of weakness were:

- determining when a system of equations has infinitely many solutions
- functional notation
- transformations of a function using matrix notation
- derivative of the absolute value function.

SPECIFIC INFORMATION

Part I – Multiple-choice

The table below indicates the percentage of students who chose each option. The correct answer is indicated by shading.

Question	% A	% B	% C	% D	% E	% No Answer
1	2	0	1	89	8	0
2	4	90	4	2	0	0
3	2	4	7	65	22	0
4	7	7	14	6	65	0
5	2	5	21	68	3	0
6	19	12	12	17	38	1
7	2	2	14	8	74	0
8	1	2	8	89	0	0
9	11	13	49	11	15	1
10	86	2	2	0	10	0
11	4	7	3	69	17	0
12	8	13	11	50	19	0
13	53	14	8	11	14	0
14	4	4	86	2	4	0
15	35	14	14	15	21	1

Question	% A	% B	% C	% D	% E	% No Answer
16	95	1	1	2	2	0
17	2	91	2	1	4	0
18	4	2	83	8	3	0
19	69	16	8	4	3	0
20	2	4	66	19	10	0
21	39	45	10	3	3	0
22	16	5	38	18	22	0
23	4	71	6	15	4	0
24	8	65	6	14	8	0
25	4	8	15	5	68	0
26	7	78	3	7	5	1
27	21	33	29	12	5	0

Multiple-choice questions 1, 2, 8, 10, 14, 16 and 17 were answered correctly by more than 85% of students. The multiple-choice questions that were not well answered (with less than 50% of students obtaining the correct answer) were 6, 9, 15, 21, 22 and 27.

In Question 6, students were given two equations in two unknowns and told that they had infinitely many solutions. The two equations were the equations of straight lines, so will have infinitely many solutions when one equation is a scalar multiple of the other, that is, when $c = 2a$ and $d = 2b$.

In Question 15, students were required to find the image of a curve after the application of a transformation given in matrix form. There were two simple ways to do this. The first method was to recognise that the sequence of transformations is a left shift by 2 units and a dilation by a factor of $\frac{1}{2}$ from the y-axis. So under the left shift, $y = x^3$ has an image $y = (x + 2)^3$, and then after the dilation, the image will be $y = (2x + 2)^3$. Alternatively, let $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0.5(x-2) \\ y \end{bmatrix}$. Then $x = 2x' + 2,\ y = y'$ and $y = x^3$ becomes $y' = (2x' + 2)^3$, so the image is $y = (2x + 2)^3$.

In Question 21, students were required to find the derivative of the absolute value of a function. This is often most easily done by hand. Since $|\cos(x)| = \cos(x)$ if $\cos(x) \ge 0$ and $-\cos(x)$ if $\cos(x) < 0$, then for the interval in question, $\frac{\pi}{2} < x < \frac{3\pi}{2}$, $|\cos(x)| = -\cos(x)$. The derivative with respect to x on this interval is $\sin(x)$. If CAS technology is used, the derivative of $|\cos(x)|$ may be given as $-\text{sign}(\cos(x))\sin(x)$. Students should be familiar with the *sign* or signum function, where $sign(x) = \begin{cases} 1 & if \quad x \ge 0 \\ -1 & if \quad x < 0 \end{cases}$.

In Question 22, students were required to find the derivative of $f(e^{2x})$ with respect to x, and the chain rule needed to be used. The derivative of $f(u)$ with respect to u will be $f'(u)$, and so if $u(x) = e^{2x}$, the derivative of $f(u)$ with respect to x will be $u'(x)f'(u) = 2e^{2x} f'(e^{2x})$.

To answer Question 27 correctly, students needed to know the properties of integrals, and be able to calculate the area of a region under a curve (or at least know **how** to do this). For the given function G, it is known that $G(0) = 0$. For $t \in (0, a]$, $G(t)$ will be the area under the curve $y = f(x)$ bounded by $x = 0$ and $x = t$. For $t \in [-a, 0)$, $G(t) = \int_0^t f(x)dx = -\int_t^0 f(x)dx$. Now, since $t < 0$, $\int_t^0 f(x)dx$ corresponds to the area between the graph of $y = f(x)$ and the x-axis, bounded by $x = t$ and by $x = 0$, this area will be negative. Hence for $t \in [-a, 0)$, $G(t) > 0$.

Part II – Short-answer

Question 1

Marks	0	1	2	3	4	Average
%	28	25	7	3	37	**2.0**

Correct response:
mean = 50 (by symmetry)

Let Z be the random variable with standard normal probability distribution. Find z_o, such that $\Pr(Z < z_o) = 0.7$. Hence $z_o = 0.5244$. Then $0.5244 = \frac{(55-50)}{\sigma}$ where σ is the standard deviation, and so $\sigma = 9.5$.

Students should have been able to determine that the mean was 50 by simply drawing a diagram or interpreting the given information correctly. Some students used 0.3 or 0.7 as standard normal scores, and some also used 0.6179 ($\Pr(Z < 0.3)$) or 0.3821 for standard normal scores. Some students gave clearly incorrect values for both the mean and standard deviation (for example, mean = 10 and standard deviation = 40); some consideration of the context should have shown that these values were not reasonable. Some students tried to use the binomial distribution, although the question clearly stated that the distribution was normal.

Question 2

2a.

Marks	0	1	2	Average
%	50	35	15	**0.7**

Correct response:

$f(x) \geq 0$ for all $x \in R$ since $a > 0$, **and** $\int_a^{\infty} \frac{a}{x^2}dx = -\left[\frac{a}{x}\right]_a^{\infty} = 1$

This question was very poorly done for a basic explanation of why f was a probability density function. Some students gave the second property but failed to mention the first. Students should be familiar with the definition of a probability density function.

2b.

Marks	0	1	2	Average
%	55	6	39	**0.9**

Correct response:

$\Pr(X > 2a) = \int_{2a}^{\infty} \frac{a}{x^2}dx = -\left[\frac{a}{x}\right]_{2a}^{\infty} = \frac{1}{2}$

Some students calculated $\int_a^{2a} \frac{a}{x^2}dx = -\left[\frac{a}{x}\right]_a^{2a} = \frac{1}{2}$ but then failed to conclude that $\Pr(X > 2a) = 1 - \Pr(X < 2a)$. Some students took a particular value for a, and evaluated the integral, but then failed to generalise for arbitrary a. Other students incorrectly wrote expressions such as $\Pr(X > 2a) = \int_{2a}^{100} \frac{a}{x^2}dx$; this may have been **how** they evaluated the integral using their CAS, but they need to write down the correct mathematical **formulation**.

2c.

Marks	0	1	Average
%	67	33	**0.4**

Correct response:
median = $2a$

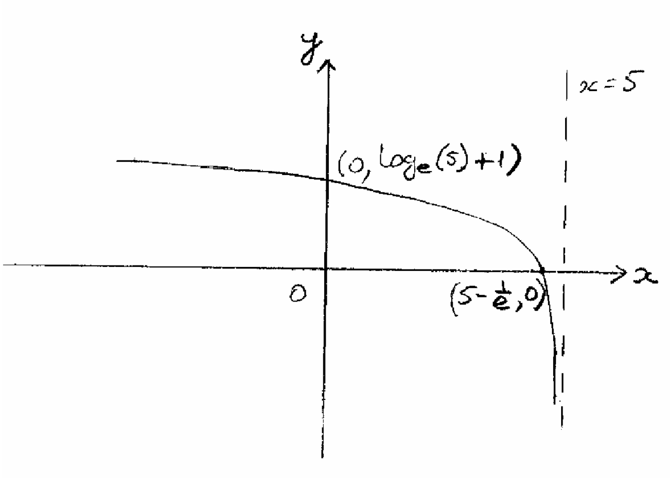
Some students who had correctly calculated the value in part b. above were then unable to write this down correctly.

Question 3

3a.

Marks	0	1	2	3	Average
%	14	21	22	43	**2.0**

Correct response:



The graph on a calculator screen is of limited use for graphs such as these. Many students did not give exact intercepts. Some tried to 'cheat' by finding numerical approximations and then writing these as fractions (which were incorrect). Many students failed to identify the vertical asymptote at all, while some identified a horizontal asymptote. If they failed to identify the vertical asymptote, the right-hand end of the graph frequently ended abruptly on or close to the *x*-axis. Many students were also unable to represent asymptotic behaviour correctly. A few students still wrote a circle where their graph ended on the asymptote – this should not be encouraged, as students will lose marks for doing it.

3b.

Marks	0	1	2	Average
%	16	12	73	**1.6**

Correct response:

$x = \log_e(5 - y) + 1$

$x - 1 = \log_e(5 - y)$

$e^{x-1} = (5 - y)$

$y = 5 - e^{x-1}$

Then $f^{-1}(x) = 5 - e^{x-1}$

This was generally done fairly well. Some students clearly did the middle steps using CAS.

Question 4

4a.

Marks	0	1	Average
%	77	23	**0.3**

Correct response:

The coordinates of the midpoint of AC are $\left(\frac{-p+p}{2}, \frac{f(-p)+f(p)}{2}\right)$ or $(0, f(0))$. Hence $f(0) = \frac{f(-p)+f(p)}{2}$.

Alternatively, students could equate the slopes of AB and BC, giving $\frac{f(-p)-f(0)}{-p-0} = \frac{f(0)-f(p)}{0-p}$, which can be simplified to $f(0) = \frac{f(-p)+f(p)}{2}$.

This question was not done very well. It appeared that students were unable to work with the information given to them. Some students gave other reasons which were correct. Some students incorrectly wrote $f(0) = 0$ here, which was not indicated in the graph.

4bi.

Marks	0	1	2	3	4	Average
%	47	21	10	1	21	**1.3**

Correct response:

Since $f(x) = ax^3 + bx^2 + cx + d$

$f(0) = d,$

$f(1) = a + b + c + d,$ and

$f(-1) = -a + b - c + d,$

then $f(-1) + f(1) = 2b + 2d$, or

$$\frac{f(-1) + f(1)}{2} = b + d$$

$$= b + f(0).$$

But $\dfrac{f(-1) + f(1)}{2} = f(0)$ (from above), so $b = 0$.

This question was very poorly done. Many students who attempted this were incorrect from the beginning, writing things like $f(0) = 0$, $f(1) = 1$ and $f(-1) = -1$.

4bii.

Marks	0	1	Average
%	78	22	**0.2**

Correct response:

$f(-x) + f(x) = 2bx^2 + 2d$ (from b.i. above), but $b = 0$ and $d = f(0)$, so $f(0) = \dfrac{f(-x) + f(x)}{2}$

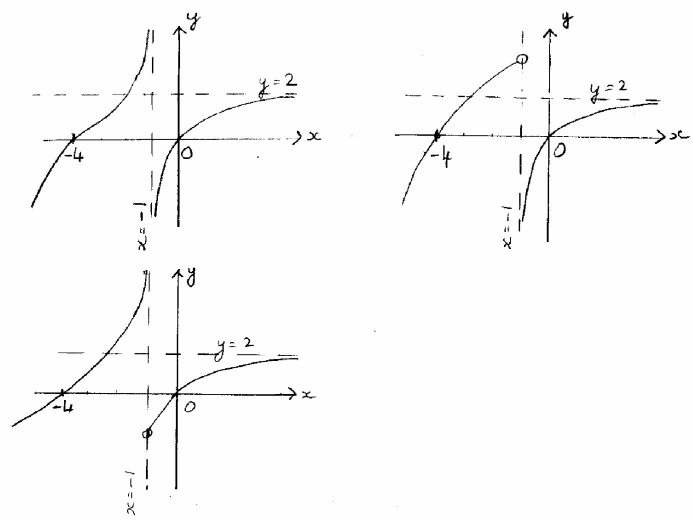
This question was also poorly done.

Question 5

Marks	0	1	2	3	Average
%	9	20	59	12	**1.8**

Correct response:

There were many possible graphs that fit the criteria given. The following are examples of some acceptable ones. Note that for a graph to have a vertical asymptote, it only needs to be asymptotic on one side.



Generally students were able to indicate the asymptotes correctly and draw an appropriate right branch. However, unsurprisingly, students had great trouble trying to determine the nature of the left branch, with the horizontal asymptote causing problems. Many shapes were possible for the left branch particularly. To make it pass through the point $(-4, 0)$ students frequently drew a branch constrained by the asymptotes for which the gradient was negative throughout. Some students drew a left branch with a positive slope, but left it hanging in midair as they did not know what to do about going through the horizontal asymptote. Some students did a hook turn near the horizontal asymptote so the graph was also asymptotic to the line $y = 2$ (but no longer the graph of a function). Some students were careless and failed to ensure the right branch passed through the origin. Some students did not seem to understand that the graph should pass through both the origin and $(-4, 0)$.