Mathematical Methods 3/4 Trial Exam 2 Solutions 2006
Free download and print from www.itute.com

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
E	E	D	D	E	A	D	C	A	A	C

12	13	14	15	16	17	18	19	20	21	22
A	B	D	A	D	C	D	C	B	D	D

Q1 $e^{2x+2}=e^x$, $e^{2x+2}-e^x=0$, $e^x\left(e^{x+2}-1\right)=0$.
Since $e^x\neq 0$, $\therefore e^{x+2}-1=0$, $e^{x+2}=1$, $x+2=0$, $x=-2$.

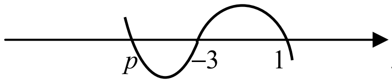
Q2 $3\cos(7x)+1$ is an even function. If $x=a$ and $x=b$ are the first two positive solutions to $3\cos(7x)+1=0$, then $x=-a$ and $x=-b$ are the first two negative solutions. Hence the sum = 0.

Q3 $\log_2\left(4a^p\right)=\log_2 4+\log_2 a^p=2+\frac{\log_a a^p}{\log_a 2}=2+\frac{p}{\log_a 2}$.

Q4 For $f(x)$ to be defined, $(x+1)^2>0$, $\therefore x\neq -1$.

Q5 $|2x-1|<1$ is equivalent to $(2x-1)^2<1$, $(2x-1)^2-1<0$, $[(2x-1)-1][(2x-1)+1]<0$, $\therefore 4x(x-1)<0$, $\therefore 0<x<1$.

Q6



Q7 From graph, $b=-\frac{5}{2}$, $c=2$, $\therefore y=a\left(x-\frac{5}{2}\right)^2+2$.

The graph passes through $(0,0)$, $\therefore 0=a\left(0-\frac{5}{2}\right)^2+2$,

$\therefore a=-\frac{8}{25}$.

Q8 Transformation of $y=|x|$: From graph, $y=a|x-p|+3$.
The graph passes through $(0,0)$, $\therefore 0=a|-p|+3$, $\therefore ap+3=0$, $a=-\frac{3}{p}$.

Hence $y=-\frac{3|x-p|}{p}+3=3\left(1-\frac{|x-p|}{p}\right)=3\left(1-\frac{|p-x|}{p}\right)$.

Q9 Any relation has an inverse.

Q10 $f(x)\to f\left(x+\frac{1}{2}\right)\to f\left(x+\frac{1}{2}\right)-\frac{1}{4}\to -\left[f\left(x+\frac{1}{2}\right)-\frac{1}{4}\right]$,

$\therefore g(x)=-f\left(x+\frac{1}{2}\right)+\frac{1}{4}=-\left[-\left(x+\frac{1}{2}\right)^2+\left(x+\frac{1}{2}\right)\right]+\frac{1}{4}=x^2$.

Q11 Use graphics calculator to display $y=x+\sin\left(\frac{\pi x}{2}\right)$. In the interval $[0,4]$, the local minimum value is 1.7895 and the local maximum value is 2.2105. $\therefore x+\sin\left(\frac{\pi x}{2}\right)-c=0$ will have more than one solution if $1.8<c<2.2$.

Q12 Use graphics calculator to display $N=5\times 2^{0.1t}$, determine $\frac{dN}{dt}$ at $t=10$. $\frac{dN}{dt}\approx 0.7$.

Q13 At 6.00 am, $t=6$, $h=1.5+0.6\cos\pi=0.9$.
At 8.00 am, $t=8$, $h=1.5+0.6\cos\frac{8\pi}{6}=1.2$.

Average rate $=\frac{1.2-0.9}{8-6}=0.150$.

Q14 Total area $=-\int_a^b\left(f(x)-g(x)\right)dx+\int_b^c\left(f(x)-g(x)\right)dx$
$=\int_b^a\left(f(x)-g(x)\right)dx+\int_b^c\left(f(x)-g(x)\right)dx$.

Q15 $\int_0^1 2\left(x-f(x)\right)dx=2\int_0^1\left(x-f(x)\right)dx=2\left(\int_0^1 x\,dx-\int_0^1 f(x)dx\right)$
$=2\left[\frac{x^2}{2}\right]_0^1-2\left[F(x)\right]_0^1=1-2\left(F(1)-F(0)\right)=1-2F(1)+2F(0)$.

Q16 For $\pi<x<3\pi$, $f(x)=\left|\cos\left(\frac{x}{2}\right)\right|=-\cos\left(\frac{x}{2}\right)$,

$f'(x)=\frac{1}{2}\sin\left(\frac{x}{2}\right)$, $\therefore f'(a)=\frac{1}{2}\sin\left(\frac{a}{2}\right)$.

Q17 Let $f(x)=\sqrt{x}$, $f'(x)=\frac{1}{2\sqrt{x}}$, $a=16$, $h=-1$.

$\sqrt{15}=\sqrt{16+^-1}\approx\sqrt{16}+^-1\times\frac{1}{2\sqrt{16}}=4-0.125=3.875$.

Q18 Check the gradient of the curve. As $x\to-\infty$, $f'(x)\to 0$. As $x\to\infty$, $f'(x)\to 0$. Gradient is always negative. Slope is steepest (most negative) at $x=0$.

Q19 Graph becomes more symmetrical as n increases. Graph becomes more asymmetrical if p increases or decreases past 0.5.

Q20 $\Pr(X \le 2 \mid X \ge 1) = \dfrac{\Pr(X \le 2 \cap X \ge 1)}{\Pr(X \ge 1)}$

$= \dfrac{\Pr(X = 1) + \Pr(X = 2)}{\Pr(X \ge 1)} = \dfrac{0.7}{0.9} = \dfrac{7}{9}$.

Q21
$\Pr(X > \mu + 8) = \Pr(X > \mu + 2\sigma) = \Pr(Z > 2) = 1 - \Pr(Z < 2)$.

Q22 $\int_1^2 k \sin(\pi x) dx = 1$, $\left[\dfrac{-k\cos(\pi x)}{\pi}\right]_1^2 = 1$,

$\dfrac{-k\cos(2\pi)}{\pi} - \dfrac{-k\cos(\pi)}{\pi} = 1$, $\dfrac{-2k}{\pi} = 1$, $\therefore k = -\dfrac{\pi}{2}$.

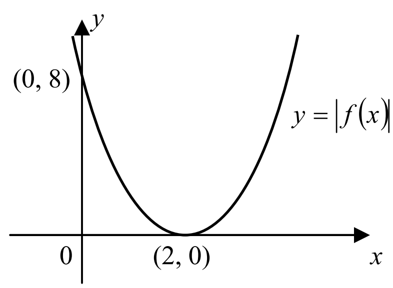
SECTION 2

Q1a. $f(x) = (x+b)^3 + c = x^3 + 3bx^2 + 3b^2x + b^3 + c$
$= x^3 - 6x^2 + 12x + p$, $\therefore 3b = -6$ and $b^3 + c = p$,
$\therefore b = -2$ and $c = p + 8$.

Q1b. $x^3 - 6x^2 + 12x + p = 0$, $\therefore (x-2)^3 + p + 8 = 0$,
$(x-2)^3 = -(p+8)$, $\therefore x - 2 = \sqrt[3]{-(p+8)} = -\sqrt[3]{p+8}$,
$x = 2 - \sqrt[3]{p+8}$, which is defined for all real *p*.

Q1ci. For $f(x) = (x-2)^3 + p + 8$ to have a stationary point on the *x*-axis, $p + 8 = 0$, $p = -8$.

Q1cii.

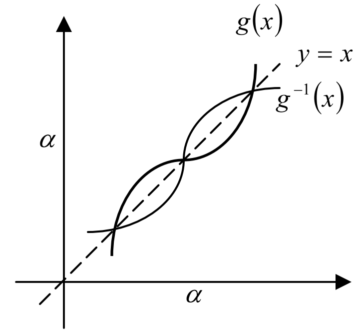


Q1d. Since $f(x) = (x+b)^3 + c$,
$\therefore f(x-b) = ((x-b)+b)^3 + c = x^3 + c$,
$\therefore f(x-b) - c = x^3$.
Compare with $f(x+u) + v = x^3$, $u = -b = 2$ and
$v = -c = -p - 8$.

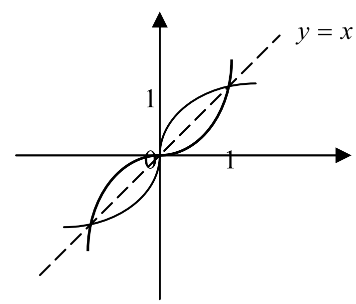
Q1ei. For $p = -7$, $f(x) = x^3 - 6x^2 + 12x + p = (x-2)^3 + 1$.
Equation of function *f*: $y = (x-2)^3 + 1$.
Equation of function f^{-1}: $x = (y-2)^3 + 1$. Express *y* as the subject of the equation, $x - 1 = (y-2)^3$, $y - 2 = \sqrt[3]{x-1}$,
$y = \sqrt[3]{x-1} + 2$. $\therefore f^{-1}(x) = \sqrt[3]{x-1} + 2$.

Q1eii. $y = \sqrt[3]{x-1} + 2 = (x-1)^{\frac{1}{3}} + 2$,
$\dfrac{dy}{dx} = \dfrac{1}{3}(x-1)^{-\frac{2}{3}} = \dfrac{1}{3(x-1)^{\frac{2}{3}}}$. Maximal domain is $R \setminus \{1\}$.

Q1f. The graphs of $g(x) = (x-\alpha)^3 + \alpha$ and $g^{-1}(x)$ are shown below.



The total area of the enclosed regions is the same as the total area enclosed after vertical and horizontal translations by *α*.



Total area $= 4 \times \int_0^1 (x - x^3) dx = 4\left[\dfrac{x^2}{2} - \dfrac{x^4}{4}\right]_0^1 = 1$.

Q2a. $P(t) = Ae^{-at}$. At $t = 0$, $P(0) = Ae^0 = A$.

Q2bi. $P(t) = Ae^{-at}$, $\dfrac{dP}{dt} = -aAe^{-at} = -aP$, $\therefore \dfrac{dP}{dt} \propto P$.

Q2bii. Since $\dfrac{dP}{dt} \propto P$, $\therefore \dfrac{dP}{dt}$ is halved when *P* is halved, i.e. $P = \dfrac{1}{2}A$. $\therefore \dfrac{1}{2}A = Ae^{-at}$, $e^{-at} = \dfrac{1}{2}$, $e^{at} = 2$, $at = \log_e 2$,
$t = \dfrac{\log_e 2}{a}$.

Q2ci. $D(t) = P(0) - P(t) = A - P(t)$.

Q2cii. $A = D(t) + P(t)$, $\dfrac{A}{P(t)} = \dfrac{D(t) + P(t)}{P(t)}$, $\dfrac{A}{P(t)} = \dfrac{D(t)}{P(t)} + 1$,
$e^{at} = \dfrac{D(t)}{P(t)} + 1$, $\therefore at = \log_e\left(\dfrac{D(t)}{P(t)} + 1\right)$, $t = \dfrac{1}{a}\log_e\left(\dfrac{D(t)}{P(t)} + 1\right)$.

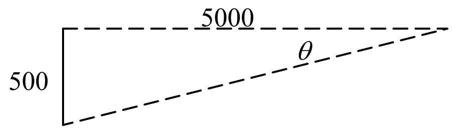
Q2di.

$$t = \frac{1}{a}\log_e\left(\frac{D(t)}{P(t)} + 1\right) = \frac{1}{1.39\times10^{-11}}\log_e(0.0196+1) = 1.40\times10^9$$

Q2dii. Let $r = \frac{D(t)}{P(t)}$, $t = \frac{1}{a}\log_e(r+1)$, $\frac{dt}{dr} = \frac{1}{a(r+1)}$.

$$\Delta t \approx \frac{dt}{dr}\Delta r = \frac{\Delta r}{a(r+1)} = \frac{0.00130}{1.39\times10^{-11}(0.0196+1)} = 9.17\times10^7 .$$

Q3ai.



$\tan\theta = \frac{500}{5000}$, $\theta = 0.100$. Lower bound for θ is -0.100 .

Q3aii. $\frac{h-500}{5000} = \tan\theta$, $\therefore h = 5000\tan\theta + 500$.

Q3b. $h = 5000\tan\theta + 500$, $\frac{dh}{d\theta} = 5000\sec^2\theta = \frac{5000}{\cos^2\theta}$

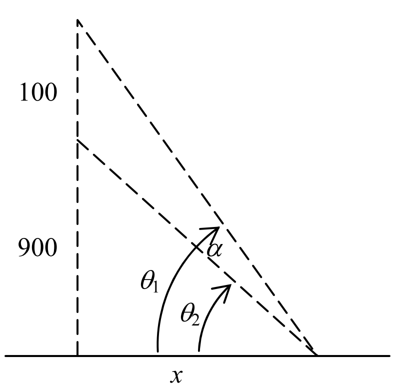
Related rates: $\frac{dh}{dt} = \frac{dh}{d\theta}\times\frac{d\theta}{dt}$, $\therefore \frac{dh}{dt} = \frac{5000}{\cos^2\theta}\times\frac{d\theta}{dt}$.

Hence $\frac{d\theta}{dt} = \frac{\cos^2\theta}{5000}\times\frac{dh}{dt}$.

Q3c. If $\frac{dh}{dt}$ is constant, then $\frac{d\theta}{dt} \propto \cos^2\theta$.

Since $-0.100 \le \theta < \frac{\pi}{2}$, $\therefore \cos^2\theta$ and hence $\frac{d\theta}{dt}$ is maximum when $\theta = 0$.

Q3di.



$\tan\theta_1 = \frac{1000}{x}$, $\theta_1 = \tan^{-1}\left(\frac{1000}{x}\right)$.

$\tan\theta_2 = \frac{900}{x}$, $\theta_2 = \tan^{-1}\left(\frac{900}{x}\right)$.

$\therefore \alpha = \theta_1 - \theta_2 = \tan^{-1}\left(\frac{1000}{x}\right) - \tan^{-1}\left(\frac{900}{x}\right)$.

Q3dii. Use graphics calculator to sketch $\alpha = \tan^{-1}\left(\frac{1000}{x}\right) - \tan^{-1}\left(\frac{900}{x}\right)$. Find x where maximum α occurs, $x = 949$ m.

Q3diii. $\alpha = 0.052656^c = 0.052656\times\frac{180^o}{\pi} = 3.02^o$.

Q3div. For $x \ge 2000$, α is maximum at $x = 2000$,

$\alpha = 0.040794\times\frac{180^o}{\pi} = 2.34^o$.

Q4a.

$\Pr(4.95 \le L \le 5.05) = \text{normalcdf}(4.95,5.05,5.00,0.02) = 0.988$

Q4b. $\Pr(3.92 \le d \le 4.08) = \int_{3.92}^{4.08} 750(d-3.9)(4.1-d)\,dd$

= 0.944 (by graphics calculator)

Q4c. Proportion acceptable $= 0.988\times0.944 = 0.933$,

$\therefore$ proportion unacceptable $= 1 - 0.933 = 0.067$.

Q4d.

	L	L'	
d	0.933	0.011	0.944
d'	0.055	0.001	0.056
	0.988	0.012	1

Required proportion $= \frac{0.055}{0.067} = 0.821$.

Q4e. $95\%\times20 = 19$.

Binomial distribution: $n = 20$, $p = 0.933$, $x \ge 19$,

$\Pr(X \ge 19) = \Pr(X = 19) + \Pr(X = 20) = 0.3588 + 0.2498 = 0.609$

Q4f. Pr(second inspection) $= 0.609\times0.3 + 0.391\times0.9 = 0.535$

Pr(no second inspection) $= 1 - 0.535 = 0.465$.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors