Trial Examination 2006

VCE Mathematical Methods Units 3 & 4

Written Examination 2

Suggested Solutions

SECTION 1

1	A	**B**	C	D	E
2	A	B	C	D	**E**
3	A	B	C	**D**	E
4	**A**	B	C	D	E
5	A	B	C	D	**E**
6	A	B	**C**	D	E
7	A	**B**	C	D	E
8	A	B	C	D	**E**
9	A	B	C	**D**	E
10	A	B	**C**	D	E
11	A	B	C	D	**E**

12	A	B	C	**D**	E
13	A	**B**	C	D	E
14	**A**	B	C	D	E
15	A	B	C	**D**	E
16	A	B	**C**	D	E
17	A	B	C	**D**	E
18	A	B	C	D	**E**
19	A	B	C	**D**	E
20	A	B	**C**	D	E
21	A	B	**C**	D	E
22	**A**	B	C	D	E

Question 1

$$d = 12 + 3\cos\left(\frac{\pi}{12}t\right)$$

$$= 12 + 3\cos\left(\frac{\pi}{12} \times 15\right) \quad \text{when } t = 15$$

$$= 12 + 3\cos\left(\frac{5\pi}{4}\right)$$

$$\approx 9.88$$

So the depth is about 9.88 m.

Answer B

Question 2

$$\sin(3x) = \frac{8}{16} \qquad 0 \le x \le \pi \Rightarrow 0 \le 3x \le 3\pi$$

$$\sin(3x) = \frac{1}{2}$$

$$3x = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$$

$$3x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}$$

$$\text{sum of answers} = \frac{36\pi}{18}$$

$$= 2\pi$$

Answer E

Question 3

$y = a\sin(b(x - c))$

amplitude = 2, period = $\frac{2\pi}{3}$

$\therefore a = \pm 2, b = 3$

The options with $a = 2$ are not correct as they would have a cycle beginning at $x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$.

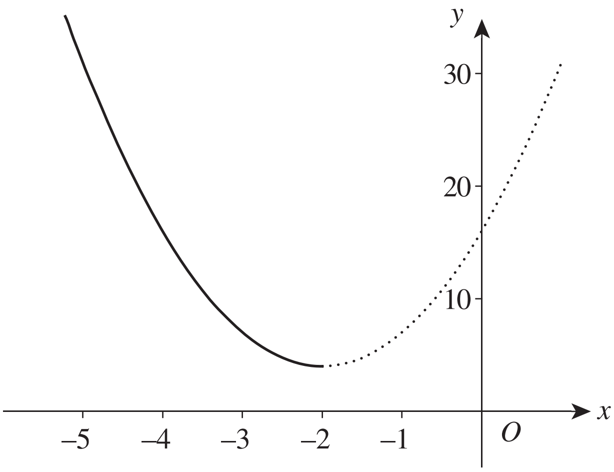
Answer **D** will be a vertically reflected sin graph with a cycle beginning at $x = \frac{\pi}{4}$, which is suitable.

Answer D

Question 4

The function will have an inverse if it is one-to-one.

$\therefore a \leq -2$

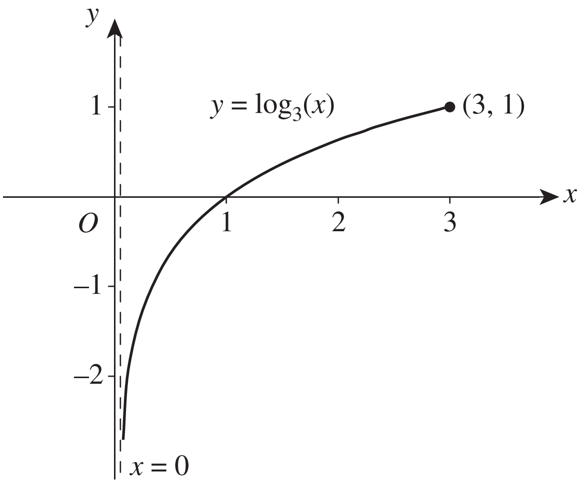


Answer A

Question 5

$f(x) = \log_3(x),\ x \in (0, 3]$

$f(3) = 1$



The range of f is $(-\infty, 1]$.

Answer E

Question 6

$(x-a)^3(x^2+b)(x^3+c) = 0$

$\therefore\ x = a, \sqrt[3]{-c}$

$x^2 + b = 0$ has no real solutions when $b > 0$.

$\therefore$ two real solutions

Answer C

Question 7

$$f(g(x)) = \sqrt{9 - (\sqrt{x})^2}$$
$$= \sqrt{9 - x}$$

$\therefore$ require $\text{ran}_g \subseteq \text{dom}_f$

$\text{ran}_g \subseteq [-3, 3]$

$\therefore -3 \le g(x) \le 3$

$0 \le \sqrt{x} \le 3$

$0 \le x \le 9$ (which is a subset of dom_g)

$\therefore x \in [0, 9]$

Answer B

Question 8

$y_1 = (x + 1)^2$

$y_2 = -(x + 1)^2$ reflection in the x-axis

$y_3 = -(x + 1)^2 - 2$ translation by 2 units in the negative y-direction

$y_4 = 3(-(x + 1)^2 - 2)$ dilation by a factor of 3 from the x-axis

$= -3(x + 1)^2 - 6$

Answer E

Question 9

$16^x - 4^{x+1} = 32$

$4^{2x} - 4 \times 4^x = 32$

$(4^x)^2 - 4(4^x) - 32 = 0$

$(4^x + 4)(4^x - 8) = 0$

$4^x + 4 = 0$ $\quad$ $4^x - 8 = 0$

$4^x = -4$ $\quad$ $4^x = 8$

no solution $\quad$ $2^{2x} = 2^3$

$2x = 3$

$x = 1.5$

Answer D

Question 10

$$4\log_2(x) + 4\log_2(\sqrt{x}) - \log_2(x^3) = -6$$
$$\log_2(x^4) + \log_2(x^2) - \log_2(x^3) = -6$$
$$\log_2\left(\frac{x^6}{x^3}\right) = -6$$
$$\log_2(x^3) = -6$$
$$x^3 = 2^{-6}$$
$$= \frac{1}{64}$$
$$x = \frac{1}{4} \text{ (which is a suitable solution)}$$

Answer C

Question 11

$$f(x) = e^{-2x}\sin(x)$$
$$f'(x) = -2e^{-2x}\sin(x) + e^{-2x}\cos(x)$$

Answer E

Question 12

$$y = \log_e\left(\frac{1}{\tan(x)}\right)$$
$$\frac{dy}{dx} = \frac{\frac{d}{dx}\left(\frac{1}{\tan(x)}\right)}{\left(\frac{1}{\tan(x)}\right)}$$

Now $\frac{1}{\tan(x)} = (\tan(x))^{-1}$

$$\frac{d}{dx}\left(\frac{1}{\tan(x)}\right) = -(\tan(x))^{-2}\sec^2(x)$$
$$= -\frac{\sec^2(x)}{\tan^2(x)}$$
$$= -\frac{1}{\sin^2(x)}$$
$$\frac{dy}{dx} = \frac{-\frac{1}{\sin^2(x)}}{\frac{\cos(x)}{\sin(x)}}$$
$$= -\frac{1}{\sin^2(x)} \times \frac{\sin(x)}{\cos(x)}$$
$$= \frac{-1}{\sin(x)\cos(x)}$$

Answer D

Question 13

$$y = \frac{20p}{(1-2p)^4}$$

$$\frac{dy}{dp} = \frac{20(1-2p)^4 - 20p \times 4(1-2p)^3(-2)}{(1-2p)^8}$$

$$= \frac{20(1-2p)^3[(1-2p)+8p]}{(1-2p)^8}$$

$$= \frac{20(1+6p)}{(1-2p)^5}$$

Answer B

Question 14

$$y = 3x^3 - 6x^2$$

$$\frac{dy}{dx} = 9x^2 - 12x$$

when $x = 1$, $y = -3$

and $m_T = -3$

$$\therefore m_N = \frac{1}{3}$$

$$y - (-3) = \frac{1}{3}(x-1)$$

$$3y + 9 = x - 1$$

$$3y = x - 10$$

Answer A

Question 15

$$f(0) = (0+1)e^0$$

$$= 1$$

$$f(2) = (2+1)e^4$$

$$= 3e^4$$

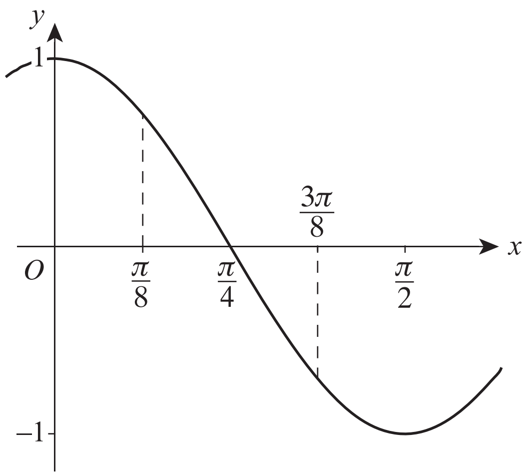
average rate of change $= \dfrac{f(2) - f(0)}{2 - 0}$

$$= \frac{3e^4 - 1}{2}$$

Answer D

Question 16

$\int_{\frac{\pi}{8}}^{a} \cos(2x)dx = 0 \Rightarrow$ that the area bounded by the curve $y = \cos(2x)$, the line $x = \frac{\pi}{8}$ and the x-axis will be equal to the area bounded by the curve, the line $x = a$ and the x-axis.



$a = \frac{3\pi}{8}$ by symmetry.

Answer C

Question 17

$$\begin{aligned}\text{area} &\approx w(f(2) + f(3) + f(4)) \\ &= 1(\sqrt{2} + 1 + \sqrt{3} + 1 + \sqrt{4} + 1) \\ &= 5 + \sqrt{2} + \sqrt{3}\end{aligned}$$

Answer D

Question 18

$$\begin{aligned}\text{It is possible that } f'(x) &= k(x+2)(x-1) \\ &= k(x^2 + x - 2)\end{aligned}$$

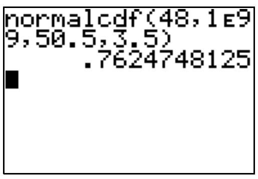
$\therefore$ it is possible that $f(x) = k\left(\frac{x^3}{3} + \frac{x^2}{2} - 2x + c\right)$

Answer E

Question 19

$$X \sim \text{N}(50.5, 3.5^2)$$

$\Pr(X > 48) \approx 0.762$

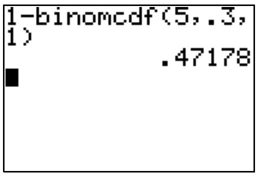


Answer D

Question 20

$$X \sim \text{Bi}(5, 0.3)$$

$$\Pr(X \ge 2) = 1 - \Pr(X \le 1)$$
$$\approx 0.472$$

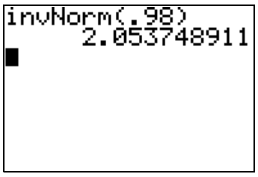


Answer C

Question 21

$X \sim \text{N}(997, \sigma^2)$, $Z \sim \text{N}(0, 1)$

$$\Pr(Z < z) = 0.98$$
$$z \approx 2.0537$$
$$z = \frac{x - \mu}{\sigma}$$
$$\sigma = \frac{x - \mu}{z}$$
$$= \frac{1005 - 997}{2.0537}$$
$$= 3.9$$



Answer C

Question 22

$$\text{require} \int_1^a \frac{2}{x} = 1$$

$$2[\log_e(x)]_1^a = 1, x > 0$$
$$2[\log_e(a) - \log_e(1)] = 1$$
$$2\log_e(a) = 1$$
$$\log_e(a) = \frac{1}{2}$$
$$a = e^{\frac{1}{2}}$$
$$= \sqrt{e}$$

Answer A

SECTION 2

Question 1

a. **i.** $c = -1$ (from horizontal asymptote) A1

ii.

$$y = \frac{a}{(x+b)^2} - 1$$

$$f(0) = 3 \;\therefore\; 3 = \frac{a}{b^2} - 1$$

$$\frac{a}{b^2} = 4$$

$$a = 4b^2$$ A1

$$f(2) = 0 \;\therefore\; 0 = \frac{a}{(2+b)^2} - 1$$

$$\frac{a}{(2+b)^2} = 1$$

$$a = (2+b)^2$$ A1

$$\therefore\; 4b^2 = (2+b)^2$$

$$4b^2 = 4 + 4b + b^2$$

$$3b^2 - 4b - 4 = 0$$

$$(3b+2)(b-2) = 0$$ M1

$$b = 2 \text{ as } b > 0$$

$$\text{and } a = 4b^2 = 4(2)^2 = 16$$

(for a and b) A1

b.

$$f: y = \frac{16}{(x+2)^2} - 1 \qquad \text{dom}_f: [0, \infty), \text{ran}_f: (-1, 3]$$

$$f^{-1}: x = \frac{16}{(y+2)^2} - 1$$ M1

$$\frac{16}{(y+2)^2} = x + 1$$

$$(y+2)^2 = \frac{16}{x+1}$$

$$y + 2 = \pm\sqrt{\frac{16}{x+1}}$$ M1

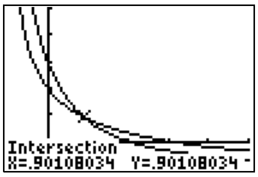
$$y = -2 \pm \sqrt{\frac{16}{x+1}}$$

$$\text{but } y > 0 \;\therefore\; y = -2 + \sqrt{\frac{16}{x+1}} = -2 + \frac{4}{\sqrt{x+1}}$$

$$f^{-1}(x) = -2 + \frac{4}{\sqrt{x+1}}$$ A1

$\text{dom}_{f^{-1}}: (-1, 3]$, $\text{ran}_{f^{-1}}: [0, \infty)$ A1

c. $x = 0.9011$ A1



d. $\text{area} = \int_{0}^{0.9011} \left(-2 + \frac{4}{\sqrt{x+1}}\right)dx + \int_{0.9011}^{2} \left(\frac{16}{(x+2)^2} - 1\right)dx$ M1, A1

$= \int_{0}^{0.9011} \left(-2 + 4(x+1)^{-\frac{1}{2}}\right)dx + \int_{0.9011}^{2} (16(x+2)^{-2} - 1)dx$

$= \left[-2x + 2 \times 4(x+1)^{\frac{1}{2}}\right]_{0}^{0.9011} + [-16(x+2)^{-1} - x]_{0.9011}^{2}$ A1

$= \left(-2(0.9011) + 8(1.9011)^{\frac{1}{2}}\right) - \left(-2(0) + 8(1)^{\frac{1}{2}}\right)$

$+ (-16(4)^{-1} - 2) - (-16(2.9011)^{-1} - 0.9011)$

$\approx 1.64 \text{ cm}^2$ A1

Question 2

a. $A = 20$ A1

$B = 25$ A1

$T = 100 = \frac{2\pi}{n}$

$\therefore\ n = \frac{\pi}{50}$ A1

b. $f(x) = 20\cos\left(\frac{\pi x}{50}\right) + 25$

$f'(x) = -\frac{\pi}{50} \times 20\sin\left(\frac{\pi x}{50}\right)$

$= -\frac{2\pi}{5}\sin\left(\frac{\pi x}{50}\right)$ A1

c. **i.** $r = f(64)$

$= 20\cos\left(\frac{\pi(64)}{50}\right) + 25$

≈ 12.2515 A1

ii. $f'(64) = -\frac{2\pi}{5}\sin\left(\frac{\pi(64)}{50}\right)$

≈ 0.9683 A1

d. $f'(x) = C\dfrac{D}{D(x-50)}$ $\quad 64 < x < p$

$= \dfrac{C}{x-50}$ M1

$\dfrac{C}{64-50} = 0.9683$ (smooth join when $x = 64$)

$C = 13.5556$ A1

also $r = C\log_e(D(64-50))$ (smooth join when $x = 64$)

$\log_e(14D) = \dfrac{r}{C}$ A1

$14D = e^{\frac{r}{C}}$

$D = \dfrac{1}{14}e^{\frac{r}{C}}$

≈ 0.1764 A1

e. Gradient of logarithmic curve at p is 0.4.

$\therefore \dfrac{C}{p-50} = 0.4$ M1

$p - 50 = \dfrac{C}{0.4}$

$p = \dfrac{C}{0.4} + 50$

≈ 83.8889

≈ 84 A1

$q = C\log_e(D(p-50))$

≈ 24.2351

≈ 24 A1

f. When $y = 0$, $0 = p + q - x$

$\therefore x = p + q$

≈ 108 A1

Question 3

a. $V_{\text{cone}} = \frac{1}{3}\pi r^2 h$

$4\pi = \frac{1}{3}\pi r^2 h$

$r^2 h = 12$

$h = \frac{12}{r^2}$ A1

b. $A = \pi r\sqrt{r^2 + h^2}$

$= \pi r\sqrt{r^2 + \left(\frac{12}{r^2}\right)^2}$ A1

$= \pi r\sqrt{r^2 + \frac{144}{r^4}}$

$= \pi r\sqrt{\frac{r^6 + 144}{r^4}}$ M1

$= \pi r \times \frac{\sqrt{r^6 + 144}}{r^2}$

$= \frac{\pi\sqrt{r^6 + 144}}{r}$ A1

c. $A = \frac{\pi(r^6 + 144)^{\frac{1}{2}}}{r}$

$\therefore \frac{dA}{dr} = \frac{\pi\left[\frac{1}{2}(r^6 + 144)^{-\frac{1}{2}} \times 6r^5 \times r - (r^6 + 144)^{\frac{1}{2}}\right]}{r^2}$ M1

$= \frac{\pi\left[\frac{3r^6}{\sqrt{r^6 + 144}} - \sqrt{r^6 + 144}\right]}{r^2}$

$= \frac{\pi\left[\frac{3r^6 - (r^6 + 144)}{\sqrt{r^6 + 144}}\right]}{r^2}$ M1

$= \frac{\pi(2r^6 - 144)}{r^2\sqrt{r^6 + 144}}$ A1

$= \frac{2\pi(r^6 - 72)}{r^2\sqrt{r^6 + 144}}$

d. $\frac{dA}{dr} = 0$

$r^6 - 72 = 0$ M1

$r^6 = 72$

$r = \sqrt[6]{72}$

≈ 2.04 A1

$h = \frac{12}{(\sqrt[6]{72})^2}$

≈ 2.88

So the dimensions are height 2.88 m and radius 2.04 m. A1

e. **i.** $V = \frac{1}{3}\pi R^2 H$ and $R = \frac{2}{3}H$ by similar triangles

$\therefore\ V = \frac{1}{3}\pi\left(\frac{2H}{3}\right)^2 H$

$= \frac{4}{27}\pi H^3$ A1

ii. $\frac{dV}{dt} = -0.5\sqrt{H}$

$\frac{dH}{dt} = \frac{dV}{dt} \times \frac{dH}{dV}$ (so $\frac{dH}{dV}$ is required)

$\frac{dV}{dH} = \frac{4}{9}\pi H^2$

$\therefore\ \frac{dH}{dV} = \frac{9}{4\pi H^2}$

$\therefore\ \frac{dH}{dt} = -0.5\sqrt{H} \times \frac{9}{4\pi H^2}$ M1, A1

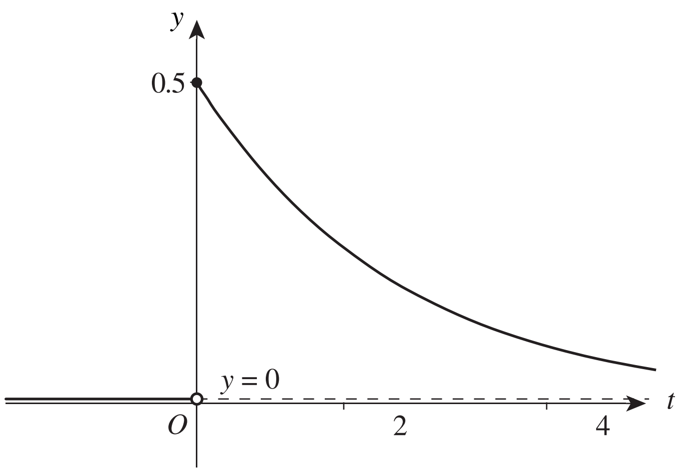
when $H = 1.5$, $\frac{dH}{dt} = -0.5\sqrt{1.5} \times \frac{9}{4\pi(1.5)^2}$

≈ -0.19

So the height is changing at –0.19 m/min (or decreasing at 0.19 m/min). A1

Question 4

a.



Correct intercept and shape A1
Correct asymptote A1
Zero elsewhere A1

b. require m such that $\int_0^m \frac{1}{2}e^{-\frac{t}{2}}dt = \frac{1}{2}$ M1

$$\left[-e^{-\frac{t}{2}}\right]_0^m = \frac{1}{2}$$

$$-e^{-\frac{m}{2}} + e^0 = \frac{1}{2}$$

$$-e^{-\frac{m}{2}} + 1 = \frac{1}{2}$$

$$-e^{-\frac{m}{2}} = -\frac{1}{2}$$

$$e^{-\frac{m}{2}} = \frac{1}{2}$$ A1

$$-\frac{m}{2} = \log_e\left(\frac{1}{2}\right)$$

$$m = -2\log_e\left(\frac{1}{2}\right)$$

$$= 2\log_e(2)$$

$$= \log_e(4)$$ (or equivalent) A1

c. $\Pr(T \geq 2) = 1 - \Pr(T \leq 2)$

$$= 1 - \int_0^2 \frac{1}{2}e^{-\frac{t}{2}}dt$$ M1

$$= 1 - \left[-e^{-\frac{t}{2}}\right]_0^2$$ A1

$$= 1 + \left[e^{-\frac{t}{2}}\right]_0^2$$

$$= 1 + (e^{-1} - e^0)$$

$$= 1 + \frac{1}{e} - 1$$

$$= \frac{1}{e}$$ A1

d. $\Pr(T < 2) = 1 - \Pr(T \geq 2)$

$$= 1 - \frac{1}{e}$$ A1

Let Y equal the number of time intervals that are less than 2 minutes.

$$Y \sim \text{Bi}\left(5, 1 - \frac{1}{e}\right)$$

$$\Pr(Y = 3) = \binom{5}{3}\left(1 - \frac{1}{e}\right)^3\left(\frac{1}{e}\right)^2$$ M1

$$\approx 0.3418$$ A1

e. $\Pr(T \geq 3 \mid T \geq 2) = \dfrac{\Pr(T \geq 3 \cap T \geq 2)}{\Pr(T \geq 2)}$ M1

$= \dfrac{\Pr(T \geq 3)}{\Pr(T \geq 2)}$ A1

$\Pr(T \geq 3) = 1 + \left[e^{-\frac{t}{2}}\right]_0^3$ (similar to part **c.**)

$= 1 + e^{-\frac{3}{2}}$ A1

$\therefore \Pr(T \geq 3 \mid T \geq 2) = \dfrac{e^{-\frac{3}{2}}}{e^{-1}}$

$= e^{-\frac{1}{2}}$

$= \dfrac{1}{\sqrt{e}}$ A1

