VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2006

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

Letter

Figures									
Words									

MATHEMATICAL METHODS (CAS)

Written examination 2

Monday 6 November 2006

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer based CAS, their full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 21 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Answer sheet for multiple-choice questions.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1

Instructions for Section I

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Question 1

The graph with equation $y = x^2$ is translated 3 units down and 2 units to the right.

The resulting graph has equation

A. $y = (x - 3)^2 + 2$

B. $y = (x - 2)^2 + 3$

C. $y = (x - 2)^2 - 3$

D. $y = (x + 2)^2 - 3$

E. $y = (x + 2)^2 + 3$

Question 2

The smallest positive value of x for which $\tan(2x) = 1$ is

A. 0

B. $\frac{\pi}{8}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{2}$

E. π

Question 3

The range of the function $f: [-2, 7) \to R,\ f(x) = 5 - x$ is

A. $(-2, 7]$

B. $[-2, 7)$

C. $(-2, \infty)$

D. $(-2, 7)$

E. R

Question 4

For the system of simultaneous linear equations

$$z = 3$$
$$x + y = 5$$
$$x - y = 1$$

an equivalent matrix equation is

A. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$

C. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$

Question 5

A bag contains three white balls and seven yellow balls. Three balls are drawn, one at a time, from the bag, without replacement.

The probability that they are all yellow is

A. $\frac{3}{500}$

B. $\frac{27}{1000}$

C. $\frac{21}{100}$

D. $\frac{7}{24}$

E. $\frac{243}{1000}$

Question 6

The function $f: [a, \infty) \rightarrow R$, with rule $f(x) = \log_e (x^4)$, will have an inverse function if

A. $a < 0$

B. $a \leq -1$

C. $a \leq 1$

D. $a > 0$

E. $a \geq -1$

Question 7

The function g has rule $g(x) = \log_e |x - b|$, where b is a real constant.

The maximal domain of g is

A. R^+

B. $R \backslash \{b\}$

C. R

D. $(-b, b)$

E. (b, ∞)

Question 8

The average value of the function $y = \cos(x)$ over the interval $[0, \frac{\pi}{2}]$ is

A. $\frac{1}{\pi}$

B. $\frac{\pi}{4}$

C. 0.5

D. $\frac{2}{\pi}$

E. $\frac{\pi}{2}$

Question 9

If $y = 3a^{2x} + b$, then x is equal to

A. $\frac{1}{6}\log_a(y-b)$

B. $\frac{1}{2}\log_a\left(\frac{y-b}{3}\right)$

C. $\frac{1}{6}\log_a\left(\frac{y}{b}\right)$

D. $\frac{1}{2}\log_a\left(\frac{3y}{b}\right)$

E. $\frac{1}{3}\log_{2a}(y-b)$

Question 10

The radius of a sphere is increasing at a rate of 3 cm/min.

When the radius is 6 cm, the rate of increase, in cm^3/min, of the volume of the sphere is

A. 432π

B. 48π

C. 144π

D. 108π

E. 16π

Question 11

The value(s) of k for which $|2k + 1| = k + 1$ are

A. 0 only

B. $-\frac{2}{3}$ only

C. 0 or $\frac{2}{3}$

D. 0 or $-\frac{2}{3}$

E. $-\frac{1}{2}$ or 0

Question 12

A fair coin is tossed 10 times.

The probability, correct to four decimal places, of getting 8 or more heads is

A. 0.0039

B. 0.0107

C. 0.0547

D. 0.9453

E. 0.9893

Question 13

The transformation $T: R^2 \to R^2$, which maps the curve with equation $y = \log_e(x)$ to the curve with equation $y = \log_e(2x-4) + 3$, could have rule

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

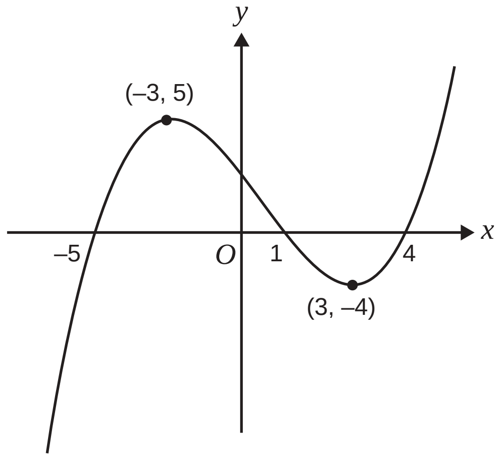
B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Question 14



For the graph of $y = f(x)$ shown above, $f'(x)$ is negative when

A. $-3 < x < 3$

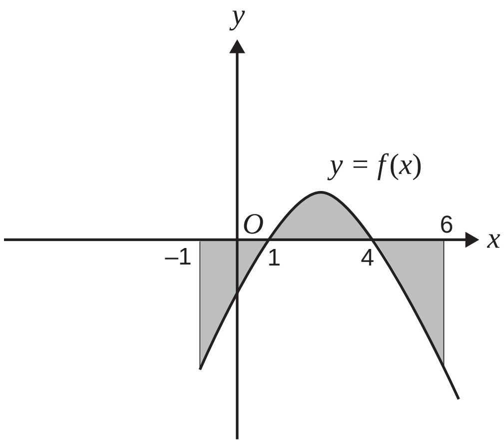
B. $-3 \le x \le 3$

C. $x < -3$ or $x > 3$

D. $x \le -3$ or $x \ge 3$

E. $-5 < x < 1$ or $x > 4$

Question 15



The total area of the shaded regions in the diagram is given by

A. $\int_{-1}^{6} f(x)dx$

B. $-\int_{-1}^{0} f(x)dx + \int_{0}^{6} f(x)dx$

C. $\int_{1}^{4} f(x)dx + 2\int_{4}^{6} f(x)dx$

D. $-\int_{-1}^{1} f(x)dx + \int_{1}^{4} f(x)dx - \int_{4}^{6} f(x)dx$

E. $-\int_{1}^{-1} f(x)dx + \int_{1}^{4} f(x)dx - \int_{6}^{4} f(x)dx$

Question 16

Let $f'(x) = g'(x) + 3$, $f(0) = 2$ and $g(0) = 1$.

Then $f(x)$ is given by

A. $f(x) = g(x) + 3x + 1$

B. $f(x) = g'(x) + 3$

C. $f(x) = g(x) + 3x$

D. $f(x) = 1$

E. $f(x) = g(x) + 3$

Question 17

The function f satisfies the functional equation $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ where x and y are any non-zero real numbers.

A possible rule for the function is

A. $f(x) = \log_e |x|$

B. $f(x) = \frac{1}{x}$

C. $f(x) = 2^x$

D. $f(x) = 2x$

E. $f(x) = \sin(2x)$

Question 18

The discrete random variable X has the following probability distribution.

X	-1	0	1
$\Pr(X = x)$	a	b	0.4

If the mean of X is 0.3 then

A. $a = 0.3$ and $b = 0.3$

B. $a = 0.2$ and $b = 0.4$

C. $a = 0.4$ and $b = 0.2$

D. $a = 0.1$ and $b = 0.5$

E. $a = 0.7$ and $b = 0.3$

Question 19

The simultaneous linear equations $(m - 2)x + 3y = 6$ and $2x + (m - 3)y = m - 1$ have **no solution** for

A. $m \in R \setminus \{0, 5\}$

B. $m \in R \setminus \{0\}$

C. $m \in R \setminus \{6\}$

D. $m = 5$

E. $m = 0$

Question 20

Let $f: R \rightarrow R$ be a differentiable function.

Then for all $x \in R$, the derivative of $f(\sin(4x))$ with respect to x is equal to

A. $4\cos(4x)\, f'(x)$

B. $\sin(4x)\, f'(x)$

C. $f'(\sin(4x))$

D. $4 f'(\sin(4x))$

E. $4\cos(4x)\, f'(\sin(4x))$

Question 21

The times (in minutes) taken for students to complete a university test are normally distributed with a mean of 200 minutes and standard deviation 10 minutes.

The proportion of students who complete the test in less than 208 minutes is closest to

A. 0.200

B. 0.212

C. 0.758

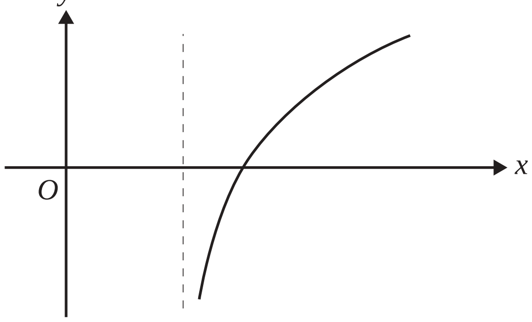
D. 0.788

E. 0.800

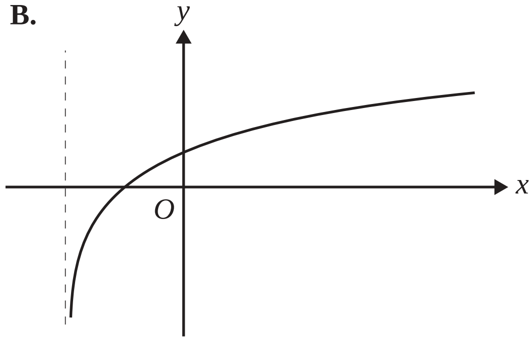
Question 22

Which one of the following could be the graph of $y = a \log_e (x - b)$ where $a < 0$ and $b > 0$?

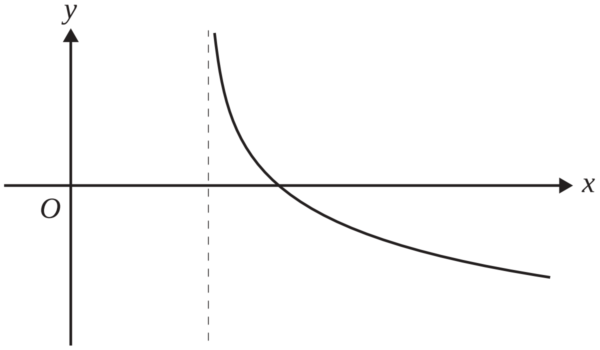
A.



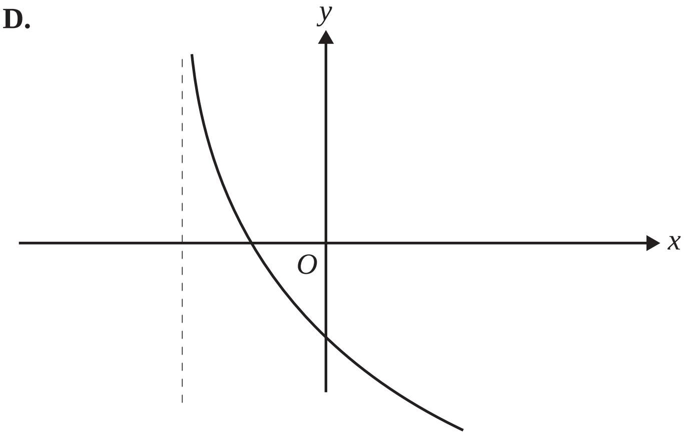
B.



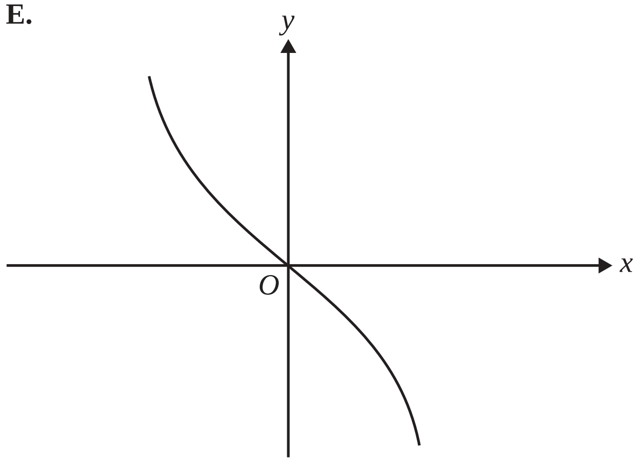
C.



D.



E.



END OF SECTION 1
TURN OVER

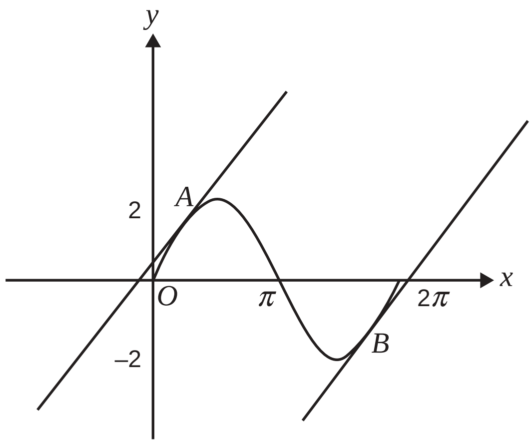
SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Consider the function $f:[0, 2\pi] \to R$, $f(x) = 2\sin(x)$. The graph of f is shown below, with tangents drawn at points A and B.



a. **i.** Find $f'(x)$.

ii. Find the maximum and minimum values of $|f'(x)|$.

1 + 2 = 3 marks

b. **i.** The gradient of the curve with equation $y = f(x)$, when $x = \frac{\pi}{3}$, is 1. Find the **other** value of x for which the gradient of the curve, with equation $y = f(x)$, is 1. (The exact value must be given.)

ii. Find the equation of the tangent to the curve at $x = \frac{\pi}{3}$. (Exact values must be given.)

iii. Find the axes intercepts of the tangent found in **b. ii.** (Exact values must be given.)

1 + 2 + 3 = 6 marks

c. The two tangents to the curve at points A and B have gradient 1. A translation of m units in the positive direction of the x-axis takes the tangent at A to the tangent at B. Find the exact value of m.

2 marks

d. Let $h: R \rightarrow R$, $h(x) = 2\,|\sin(x)|$. Find the general solution, for x, of the equation $h(x) = 1$.

2 marks

Total 13 marks

Question 2

Each night Kim goes to the gym or the pool. If she goes to the gym one night, the probability she goes to the pool the next night is 0.4, and if she goes to the pool one night, the probability she goes to the gym the next night is 0.7.

a. Suppose she goes to the gym one Monday night.

i. What is the probability that she goes to the pool on each of the next three nights? (The exact value must be given.)

ii. What is the probability that she goes to the pool on exactly two of the next three nights? (The exact value must be given.)

1 + 2 = 3 marks

b. In the long term, what proportion of nights does she go to the pool? (Answer correct to three decimal places.)

1 mark

When Kim goes to the gym, the time, T hours, that she spends working out is a continuous random variable with probability density function given by

$$f(t) = \begin{cases} 4t^3 - 24t^2 + 44t - 24 & \text{if} \quad 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

c. Sketch the graph of $y = f(t)$ on the axes below. Label any stationary points with their coordinates, correct to two decimal places.

3 marks

d. What is the probability, correct to three decimal places, that she spends less than 75 minutes working out when she goes to the gym?

2 marks

e. What is the probability, correct to two decimal places, that she spends more than 75 minutes working out on 4 out of the 5 next times she goes to the gym?

2 marks

f. Find the median time, to the nearest minute, that she spends working out in the gym.

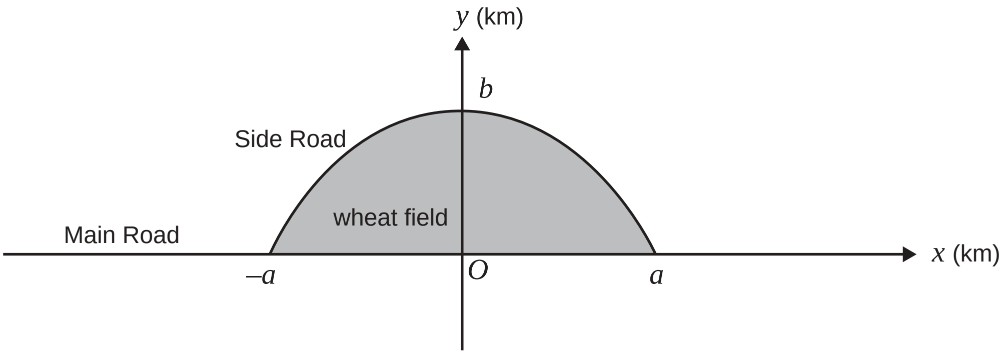
3 marks

Total 14 marks

Question 3

Tasmania Jones' wheat field lies between two roads as shown in the diagram below.

Main Road lies along the x-axis and Side Road lies along the curve with equation $y = 3 - e^x - e^{-x}$.



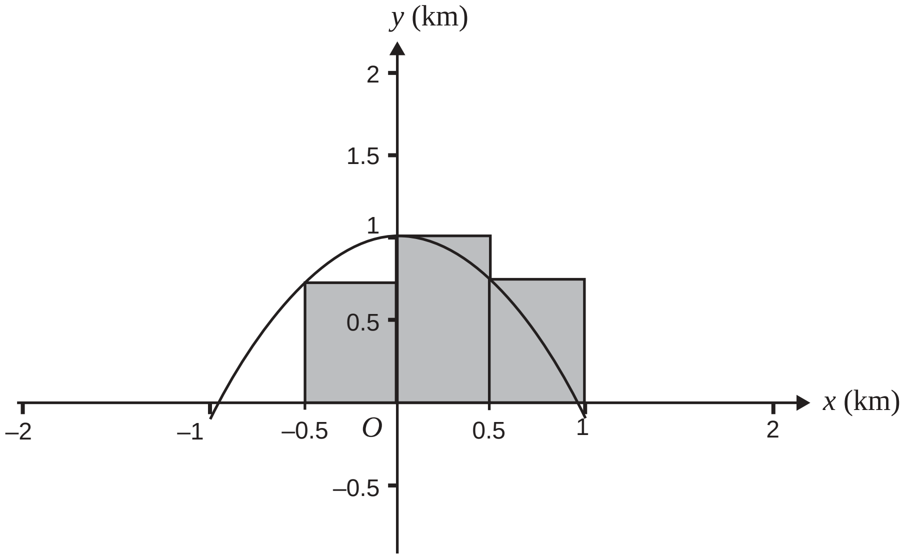
a. The y-axis intercept of the graph representing Side Road is b.
Show that $b = 1$.

1 mark

b. Find the exact value of a.

1 mark

c. Since a is close to 1, Tasmania finds an approximation to the area of the wheat field by using rectangles of width 0.5 km, as shown on the following diagram.



i. Complete the table of values for y, where $y = 3 - e^x - e^{-x}$, giving values correct to two decimal places.

x	–0.5	0	0.5
y			

ii. Use the table to find Tasmania's approximation to the area of the wheat field, measured in square kilometres, correct to one decimal place.

iii. Tasmania uses this approximation to the area to estimate the value of the wheat in his field at harvest time. He estimates that he will obtain w kg of wheat from each square kilometre of field. The current price paid to growers is $\$m$ per kg of wheat. Write a formula for his estimated value, $\$V$, of the wheat in his field.

1 + 2 + 1 = 4 marks

d. Tasmania Jones decides to find another approximation to the area of the wheat field. He approximates the curve representing Side Road with a parabola which passes through the points (0, 1), (1, 0) and (–1, 0). He finds the area enclosed by the parabola and the x-axis as an approximation to the area of his wheat field.

i. Find the equation of this parabola.

ii. Find the area enclosed by the parabola and the x-axis, giving your answer correct to two decimal places.

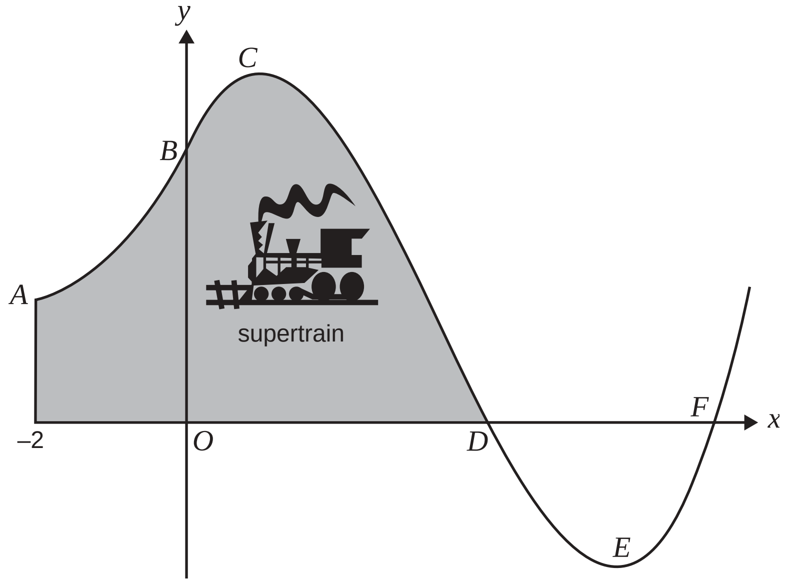
1 + 2 = 3 marks

e. Find the values of k, where k is a positive real number, for which the equation $3 - ke^{x} - e^{-x} = 0$ has one or more solutions for x.

4 marks

Total 13 marks

Question 4



A part of the track for Tim's model train follows the curve passing through *A*, *B*, *C*, *D*, *E* and *F* shown above. Tim has designed it by putting axes on the drawing as shown. The track is made up of two curves, one to the left of the *y*-axis and the other to the right.

B is the point (0, 7).

The curve from *B* to *F* is part of the graph of $f(x) = px^3 + qx^2 + rx + s$ where p, q, r and s are constants and $f'(0) = 4.25$.

a. **i.** Show that $s = 7$.

ii. Show that $r = 4.25$.

1 + 1 = 2 marks

The furthest point reached by the track in the positive y direction occurs when $x = 1$. Assume $p > 0$.

b. **i.** Use this information to find q in terms of p.

ii. Find $f(1)$ in terms of p.

iii. Find the value of a in terms of p for which $f'(a) = 0$ where $a > 1$.

iv. If $a = \frac{17}{3}$, show that $p = 0.25$ and $q = -2.5$.

2 + 1 + 1 + 2 = 6 marks

For the following assume $f(x) = 0.25x^3 - 2.5x^2 + 4.25x + 7$.

c. Find the exact coordinates of D and F.

2 marks

d. Find the greatest distance that the track is from the x-axis, when it is below the x-axis, correct to two decimal places.

1 mark

The curve from A to B is part of the graph with equation $g(x) = \dfrac{a}{1-bx}$, where a and b are positive real constants.

The track passes smoothly from one section of the track to the other at B (that is, the gradients of the curves are equal at B).

e. Find the exact values of a and b.

3 marks

f. Find the area of the shaded section bounded by the track between $x = -2$ and D, correct to two decimal places.

4 marks

Total 18 marks

END OF QUESTION AND ANSWER BOOK

MATHEMATICAL METHODS AND MATHEMATICAL METHODS (CAS)

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.

This formula sheet is provided for your reference.

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Mathematical Methods and Mathematical Methods CAS Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$

$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$

$\int \frac{1}{x} dx = \log_e |x| + c$

$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$

$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$

$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$

$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$

$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

mean: $\mu = E(X)$

variance: $\text{var}(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

probability distribution		**mean**	**variance**
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

END OF FORMULA SHEET