2007 VCAA Mathematical Methods Exam 2 Solutions

SECTION 1: Multiple-choice questions

1	2	3	4	5	6	7	8	9	10	11
A	D	A	C	D	B	A	B	E	E	E

12	13	14	15	16	17	18	19	20	21	22
C	D	A	B	E	C	D	A	A	A	E

Q1 $f(x)=6-2x$.
$12=6-2x$, $x=-3$.
$-4=6-2x$, $x=5$.
$\therefore D$ is $[-3,5]$

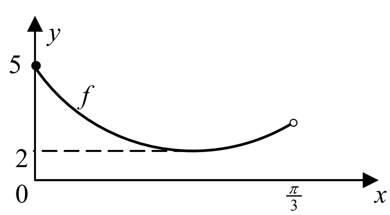
Q2 $f(g(x))=e^{2g(x)+3}=e^{2x^2+4x-3}$

Q3 $\int\left(\frac{1}{x^2}-\frac{1}{\cos^2 x}\right)dx=\int\left(\frac{1}{x^2}-\sec^2 x\right)dx$
$=-\frac{1}{x}-\tan x$

Q4 $f(x)=x^3-\sqrt{x+1}$,
$f(0)=0^3-\sqrt{0+1}=-1$,
$f(3)=3^3-\sqrt{3+1}=25$.
Average rate $=\frac{25-(-1)}{3-0}=\frac{26}{3}$.

Q5 $\int\left(\sin 2x+24x^3\right)dx=-\frac{1}{2}\cos 2x+6x^4+c$

Q6 By Graphics calculator.



The range of *f* is $[2,5]$.

Q7 $\Pr(X<10.5)=\Pr(X<11-2\times 0.25)=\Pr(X<\mu-2\sigma)$
$=\Pr(Z<-2)=\Pr(Z>2)$

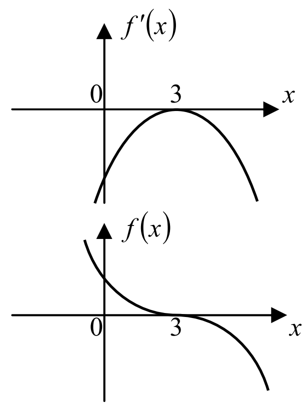
Q8 $f'(x)$ is undefined at $2x+4=0$, i.e. $x=-2$.
$\therefore f'(x)$ is discontinuous at $x=-2$.

Q9 $k=\int_{-2}^{-1}\frac{1}{x}dx=-\int_{1}^{2}\frac{1}{x}dx=-[\log_e x]_1^2=-\log_e 2=\log_e\left(\frac{1}{2}\right)$.
$\therefore e^k=\frac{1}{2}$.

Q10 $f(x)$ is an increasing function. $f(0)=-2$, $f(1)=e^2-3$.
The range of $f(x)$ is $[0,e^2-3)$.
$\therefore$ the domain of $f^{-1}(x)$ is $[0,e^2-3)$.
Let $y=e^{2x}-3$ be the equation of $f(x)$.
$y+3=e^{2x}$, $\therefore x=\frac{1}{2}\log_e(y+3)$
$\therefore$ the equation of $f^{-1}(x)$ is $y=\frac{1}{2}\log_e(x+3)$.

Q11 $\left(e^{2x}\right)^2-5\left(e^{2x}\right)+4=0$,
$\left(e^{2x}-4\right)\left(e^{2x}-1\right)=0$.
$e^{2x}=1$, $x=0$,
or $e^{2x}=4$, $2x=\log_e 4$, $x=\log_e 2$.
Solution set is $\{0,\log_e 2\}$.

Q12



Q13 By graphics calculator.
Local maximum at $x=-5$, local minimum at $x=\frac{1}{2}$.
Gradient is negative for $x\in\left(-5,\frac{1}{2}\right)$.

Q14 $f(x)=\log_e|x-3|+6$ is defined for $x\neq 3$.
Maximal domain is $R\setminus\{3\}$.

Q15
$y=3x^{\frac{5}{2}}\rightarrow y=-3x^{\frac{5}{2}}\rightarrow y=-3(x-3)^{\frac{5}{2}}\rightarrow y=-3(x-3)^{\frac{5}{2}}-4$.

Q16 $f(x)=(x-a)^2 g(x)$,
$f'(x)=2(x-a)g(x)+(x-a)^2 g'(x)$
$f'(x)=(x-a)\left(2g(x)+(x-a)g'(x)\right)$.

Q17 $E(X)=\int_0^2 x\left(\frac{x}{2}\right)dx=\int_0^2 \frac{x^2}{2}dx=\left[\frac{x^3}{6}\right]_0^2=\frac{4}{3}$.

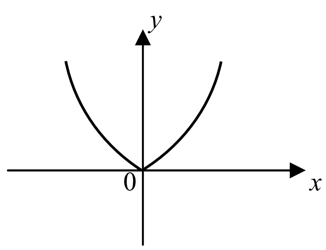
Q18 $\Pr(X<x)=0.35$, $x=invNorm(0.35,130,2.7)=129$.

Q19 $a+b=1-(0.2+0.2+0.3)=0.3$.
$E(X)=0a+2\times0.2+4\times0.2+6\times0.3+8b=5$,
$\therefore 8b=2$, $b=0.25$ and $a=0.05$.

Q20 $\tan^2\frac{\theta}{3}=1$ and $\theta\in[0,2\pi]$.
$\therefore \tan\frac{\theta}{3}=1$, $\frac{\theta}{3}=\frac{\pi}{4}$, $\theta=\frac{3\pi}{4}$.
Note: When $\therefore \tan\frac{\theta}{3}=-1$, $\frac{\theta}{3}=\frac{3\pi}{4}$, $\theta=\frac{9\pi}{4}\notin[0,2\pi]$.

Q21 $\cos^2 x+2\cos x=0$, $\cos x(\cos x+2)=0$.
Since $\cos x+2\neq0$, $\therefore \cos x=0$.

Q22 Let $f(x)=x(x-1)$ and $g(x)=-|x|$.
$y=f(g(x))=g(x)(g(x)-1)=-|x|(-|x|-1)=|x|^2+|x|$
Graphics calculator:



SECTION 2:

Q1a $V=\pi r^2h$, $1000=\pi r^2h$, $h=\frac{1000}{\pi r^2}$.

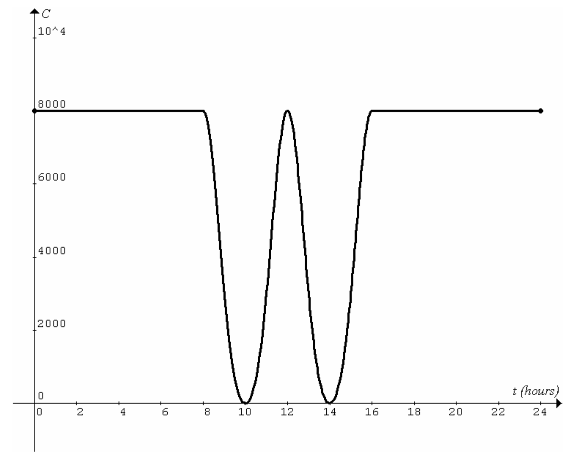
Q1b Area of top plus bottom $=2\times\pi r^2$.
Area of curved surface $=2\pi rh=2\pi r\left(\frac{1000}{\pi r^2}\right)=\frac{2000}{r}$.
$\therefore A=\frac{2000}{r}+2\pi r^2$ cm^2.

Q1c $\frac{dA}{dr}=-\frac{2000}{r^2}+4\pi r$. Let $\frac{dA}{dr}=0$.
$-\frac{2000}{r^2}+4\pi r=0$, $r^3=\frac{500}{\pi}$, $r=\left(\frac{500}{\pi}\right)^{\frac{1}{3}}$.

Q1d $A_{\min}=\frac{2000}{\left(\frac{500}{\pi}\right)^{\frac{1}{3}}}+2\pi\left(\frac{500}{\pi}\right)^{\frac{2}{3}}=553.58$ cm^2.

Q2a When $t=8$, $C=1000(\cos0+2)^2-1000=8000$.
When $t=16$, $C=1000(\cos4\pi+2)^2-1000=8000$.
$\therefore m=8000$ for $C(t)$ to be continuous.

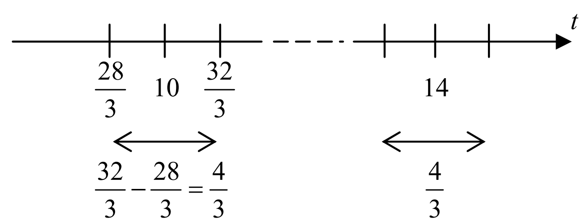
Q2b Use graphics calculator to sketch the middle section.



Q2c $C_{\min}=0$ when $t=10$ or 14.

Q2d Use graphics calculator to find the first intersection of the middle section and the horizontal line $C=1250$.
$t=\frac{28}{3}$ hours after midnight, i.e. 9.20 am.

Q2e



Total length of time $=2\times\frac{4}{3}=\frac{8}{3}$ hours.

Q2fi $T(x)=p(q^x-1)$,
$T(1)=p(q-1)=5$, $T(2)=p(q^2-1)=12.5$.
$\therefore \frac{T(2)}{T(1)}=\frac{p(q-1)(q+1)}{p(q-1)}=q+1$,
i.e. $\frac{12.5}{5}=q+1$, $q=1.5$. $\therefore p=\frac{5}{q-1}=10$.

Q2fii Hence $T(x)=10(1.5^x-1)$,
$T(4)=10(1.5^4-1)=40.625$ minutes.

Q2g
Required time $=40.625+19+\frac{1}{2}\times40.625=79.9375$ minutes.
Available time $=\frac{4}{3}$ hours $=80$ minutes.
Spare time $=80-79.9375=0.0625$ minutes $=3.75$ s

Q3a $g(x)=2(x^3-6x^2+8x)$, $g'(x)=2(3x^2-12x+8)$.

Let $g'(x)=0$, $3x^2-12x+8=0$,

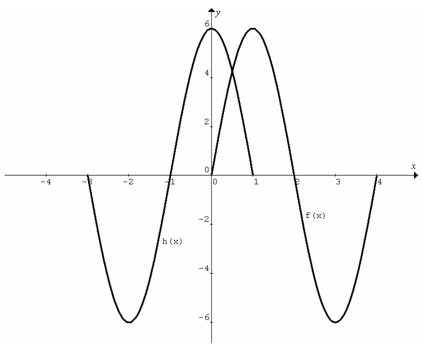
$\therefore x=\dfrac{6\pm2\sqrt{3}}{3}$.

$g(x)$ is maximum at $x=\dfrac{6-2\sqrt{3}}{3}$.

Q3b Shaded area =

$$2\left\{\int_0^1\left[2(x^3-6x^2+8x)-6\sin\left(\frac{\pi x}{2}\right)\right]dx+\int_1^2\left[6\sin\left(\frac{\pi x}{2}\right)-2(x^3-6x^2+8x)\right]dx\right\}.$$

Q3ci Reflect $f(x)$ in the x-axis, then translate to the left by 3 units.



Q3cii Translate $g(x)$ to the left by 3 units to obtain a new cubic function $j(x)=g(x+3)=2(x+3)(x+3-2)(x+3-4)$

$=2(x+3)(x+1)(x-1)$.

$j:[-3,1]\rightarrow R,\ j(x)=2(x+3)(x+1)(x-1)$.

Another one is $k:[-3,1]\rightarrow R, k(x)=-2(x+3)(x+1)(x-1)$, which is the reflection of function *j* in the *x*-axis.

Q4a $h(x)=2-e^{-x}$ is an increasing function. $h(0)=2-e^0=1$, $h(2)=2-e^{-2}$. $\therefore$ the range of function *h* is $[1,2-e^{-2}]$.

Q4bi The domain of h^{-1} is $[1,2-e^{-2}]$.

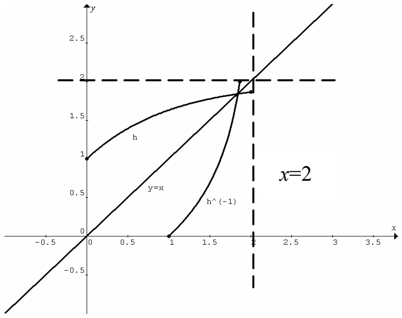
Let $y=2-e^{-x}$ be the equation of *h*.

$e^{-x}=2-y$, $-x=\log_e(2-y)$, $x=-\log_e(2-y)$.

$\therefore$ the equation of h^{-1} is $y=-\log_e(2-x)$ or $\log_e\left(\dfrac{1}{2-x}\right)$.

$\therefore h^{-1}:[1,2-e^{-2}]\rightarrow R$, $h^{-1}(x)=\log_e\left(\dfrac{1}{2-x}\right)$.

Q4bii



Q4c $y=x$ and $y=2-e^{-x}$.

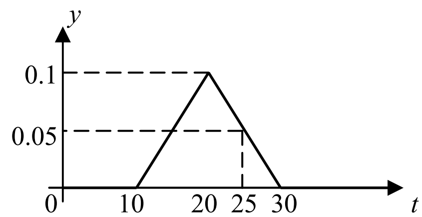
By graphics calculator, $x=y=1.8414$

Point of intersection is $(1.84,1.84)$.

Q4d Area $=2\times\int_0^{1.8414}(2-e^{-x}-x)dx$

$=2\left[2x+e^{-x}-\dfrac{x^2}{2}\right]_0^{1.8414}=2.29$ square units.

Q5a



Q5b When $t=25$, $y=0.05$.

$\Pr(T<25)=1-\Pr(T>25)=1-\dfrac{1}{2}(30-25)0.05=0.875=\dfrac{7}{8}$.

Q5c $\Pr(T\le15\,|\,T\le25)=\dfrac{\Pr(T\le15)}{\Pr(T\le25)}=\dfrac{\Pr(T>25)}{\Pr(T\le25)}$

$=\dfrac{\frac{1}{8}}{\frac{7}{8}}=\dfrac{1}{7}$.

Q5d Binomial: $n=6$, $p=\dfrac{7}{8}$.

$\Pr(X\ge4)=1-\Pr(X\le3)=1-binomcdf(6,0.875,3)=0.9709$.

Q5e Binomial: $n=6$, $\Pr(T<b)=p$.

$Q=\Pr(X=3\cup X=4)=\Pr(X=3)+\Pr(X=4)$

$={}^6C_3p^3(1-p)^3+{}^6C_4p^4(1-p)^2$

$=20p^3(1-p)^3+15p^4(1-p)^2$

$=5p^3(1-p)^2[4(1-p)+3p]$

$=5p^3(1-p)^2(4-p)$.

Q5fi By graphics calculator: $Q_{\max}=0.5887$ when $p=0.5858$.

Q5fii $\Pr(T<b)=0.5858$, $\therefore\Pr(T>b)=1-0.5858=0.4142$.

$\therefore\int_b^{30}\dfrac{1}{100}(30-t)dt=0.4142$, $\left[-\dfrac{(30-t)^2}{200}\right]_b^{30}=0.4142$.

$\therefore\dfrac{(30-b)^2}{200}=0.4142$, where $20<b<30$.

$\therefore b=20.9$.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors