Mathematical Methods Exam 2: SOLUTIONS

Solutions to Multiple Choice Questions

1. B 2. B 3. D 4. E 5. D 6. B 7. A 8. A 9. C
10. C 11. D 12. C 13. D 14. A 15. D 16. E
17. A 18. B 19. D 20. E 21. C 22. A

Question 1

$f(g(x)) = \frac{2}{x^2} + 1$. The range is $(1, \infty)$. **B**

Question 2

$d_g = d_h \cap d_f$ which is (1, 2).

$g(x) = (x-1)^2(x+2)(1-x)$

$= -(x-1)^3(x+2)$ **B**

Question 3

- a reflection in the x-axis: $f(x) \to -f(x)$
- dilation by a scale factor of 3 from the x-axis: $-f(x) \to -3f(x)$
- dilation by a scale factor of 2 from the y-axis $-3f(x) \to -3f\left(\frac{1}{2}x\right)$
- translation 5 units to the left

$-3f\left(\frac{1}{2}x\right) \to -3f\left(\frac{1}{2}(x+5)\right)$ **D**

Question 4

The graph has the shape of the graph with rule $y = (ax)^{\frac{4}{3}} + 1$ or $y = (bx)^2 + 1$, where a and b are real constants. When $x = 2$, $y = 2$. Thus, $y = x^2 + 1$ and $y = x^{\frac{4}{3}} + 1$ are not possible. $y = \left(\frac{x}{2}\right)^{\frac{4}{3}} + 1$. **E**

Question 5

At points of intersection,

$\sin(2x) = \sqrt{3}\cos(2x)$

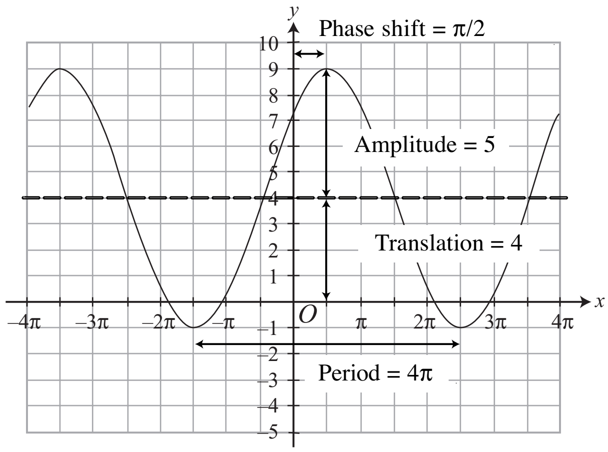
$\frac{\sin(2x)}{\cos(2x)} = \sqrt{3}$

$\tan(2x) = \sqrt{3}$

$2x = -\frac{2\pi}{3}, \frac{\pi}{3}$

$x = -\frac{\pi}{3}, \frac{\pi}{6}$ **D**

Question 6



Amplitude = 5

Period = 4π

Rule is of the form $y = a\cos(n(x - \varepsilon)) + k$

$a = 5,\ \varepsilon = \frac{\pi}{2},\ k = 4$

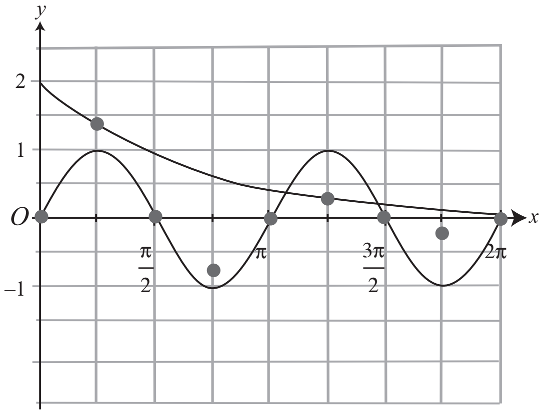
Period $= \frac{2\pi}{n}$, therefore $n = \frac{2\pi}{4\pi} = \frac{1}{2}$

Rule is $y = 5\cos\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right) + 4$

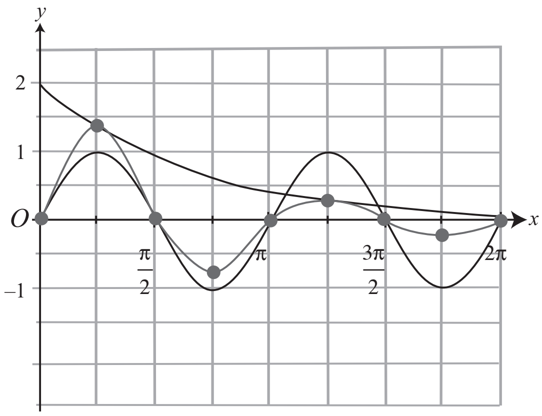
or $y = 5\cos\left(\frac{x}{2} - \frac{\pi}{4}\right) + 4$ **B**

Question 7

When graphing product of functions, the key points occur when either function has the value of ±1 or 0, as illustrated in the diagram below. Mark these key points.



Then "join the dots".



A

Question 8

$g(x) = 3(e^{2(x-1)} + 3)$

$= 3e^{2(x-1)} + 9$

The graph of g has an asymptote at $y = 9$. Thus the graph of the inverse of g has an asymptote at $x = 9$. **A**

Question 9

The graph of $y = e^x$ has been reflected in both x and y axes, and then translated up.

Reject option B because it does not show the translation up. None of the other options indicate the possibility of a translation parallel to the x-axis.

Ignoring the magnitude of any dilations (because there is no scale on the axes), the transformed graph will be of the form $y = -e^{-x} + k$. Since $b < 0$, require $y = -ae^{bx} + k$. **C**

Question 10

$\log_2(x + 2) + \log_2(x - 1) = 2$

$\log_2((x + 2)(x - 1)) = 2$

$(x + 2)(x - 1) = 4$

$x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 2$

$x = 2$ only as $x > 1$ **C**

Question 11

The coordinates of the turning point of $y = 2(x - 3)^6 + 4$ are (3, 4). The tangent is $y = 4$. **D**

Question 12

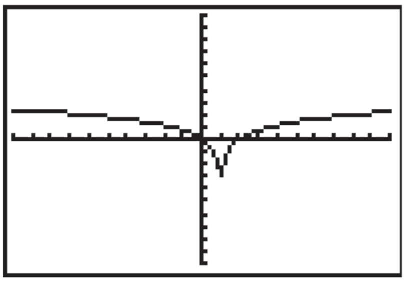
The average rate of change $= \dfrac{f(1) - f(-2)}{1 - -2}$

$= \dfrac{\sqrt{1} - \sqrt{4}}{3}$

$= -\dfrac{1}{3}$ **C**

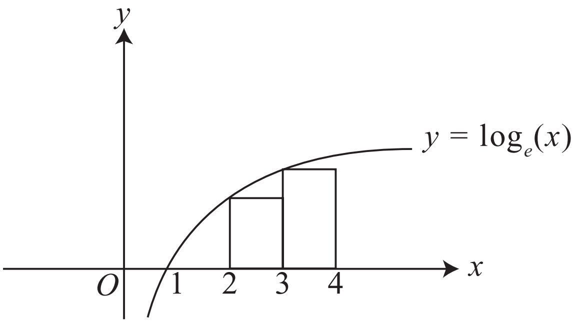
Question 13

The graph of $y = \log_e |x-1|$ is shown below.



$\frac{d}{dx}\log_e |x-1| = \frac{1}{x-1}$ for $x \in R \setminus \{1\}$ **D**

Question 14

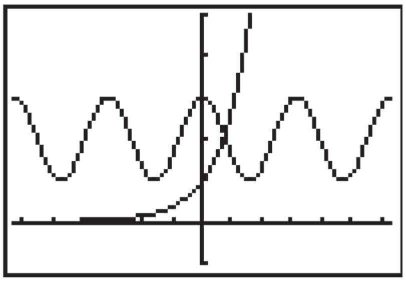


The area of the rectangles = $\log_e(2) + \log_e(3)$

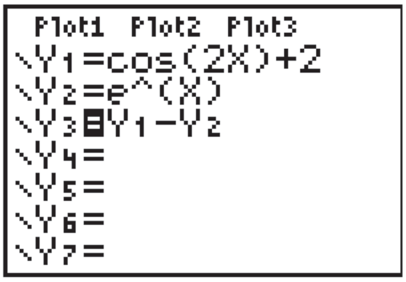
$= \log_e(6)$ **A**

Question 15

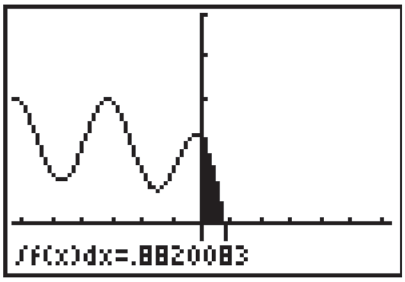
Sketch both curves and find the intersection.



Graph $f(x) - g(x)$



Find $\int_0^{0.7387} (f(x) - g(x))dx$



t

D

Question 16

$$\int_1^k (\log_e(x)+1)\,dx = 1$$

$$[x\log_e(x) - x + x]_1^k = 1$$

$$[x\log_e(x)]_1^k = 1$$

$k\log_e(k) - 1 \times 0 = 1$

$k\log_e(k) = 1$ **E**

Question 17

f has a minimum turning point at $x = -1$ and a stationary point of inflection at $x = 2$. **A**

Question 18

Let X denote the number of children with blue eyes. $X \sim Bi\left(n = 3, p = \frac{1}{4}\right)$

$$\Pr(X \geq 1) = 1 - \Pr(X = 0)$$
$$= 1 - \left(\frac{3}{4}\right)^3$$
$$= 1 - \left(\frac{27}{64}\right)$$
$$= \frac{37}{64}$$

B

Question 19

$p(x) = \dfrac{x^2+1}{10}$, for $x \in \{-1, 0, 1, 2\}$

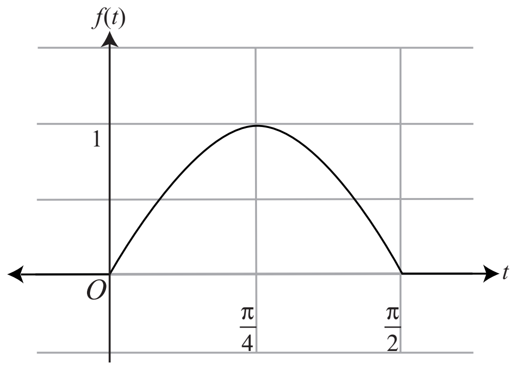
$p(-1) = 0.2$, $p(0) = 0.1$, $p(1) = 0.2$, $p(2) = 0.5$

$$E(X) = \sum x\,p(x)$$
$$= -1\times 0.2 + 0\times 0.1 + 1\times 0.2 + 2\times 0.5$$
$$= 1$$

D

Question 20

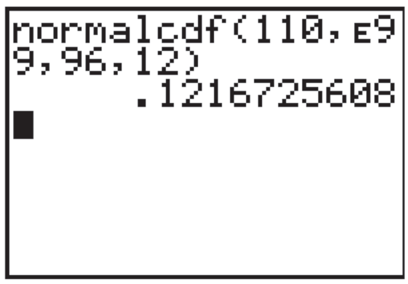
The mode is the value of t that gives the maximum value of $f(t)$.



By symmetry, this occurs at $\frac{\pi}{4}$. **E**

Question 21

$T \sim N(\mu = 96, \sigma^2 = 12^2)$



$\Pr(X > 110) = 0.1217$

Approximately 12% exceeded the limit. **C**

Question 22

$\Pr(SSSM) = \frac{6}{12}\times\frac{5}{11}\times\frac{4}{10}\times\frac{6}{9} = \frac{2}{33}$ **A**

Mathematical Methods Exam 2: SOLUTIONS

Section 2: Extended answers

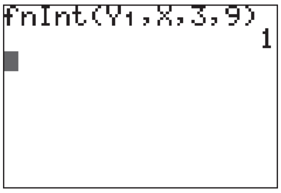
Question 1

a. For a probability density function,

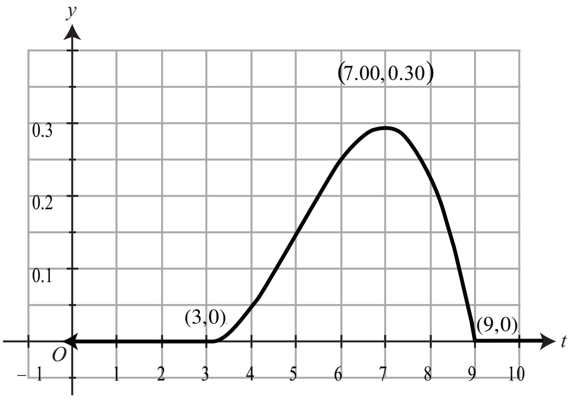
$\int_{-\infty}^{\infty} v(t)\,dt = 1$. Therefore,

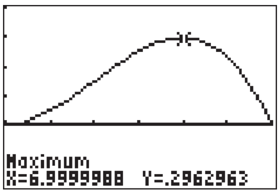
$$\int_3^9 -\frac{1}{108}\left(t^3 - 15t^2 + 63t - 81\right)dt = 1$$

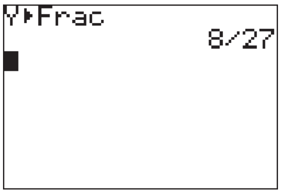
Showing integral with correct terminals **1M**
(*dt* must be shown to get the mark)



b.







Correct shape and skew **1A**
Correct domain and axes intercepts **1A**
Correct stationary point **1A**

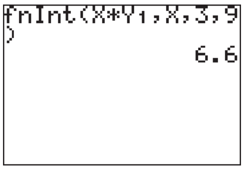
c. From the graph, the mode is 7, as this is the value of *t* at which the maximum value occurs. **1 M**
(Consequential from part b)

d. $\mu = \int_{-\infty}^{\infty} t\,v(t)\,dt$. Therefore,

$$\mu = -\frac{1}{108}\int_3^9 \left(t\left(t^3 - 15t^2 + 63t - 81\right)\right)dt \quad \textbf{1M}$$

$$= \frac{33}{5} \text{ or } 6.6$$

The mean incubation period is 6.6 days. **1A**



e. $X \sim N(5,\sigma^2)$, $Z \sim N(0,1)$

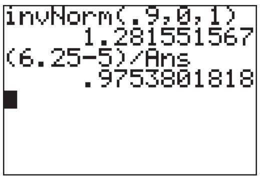
$\Pr(Z < z) = 0.9$ **1M**

$z = 1.28155$

$$z = \frac{x-\mu}{\sigma}$$

$$1.28155 = \frac{6.25-5}{\sigma}$$

$\sigma = 0.98$



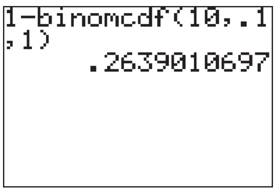
The standard deviation is 0.98 hours **1A**

f. Let Y be the number of patients for whom the response time to the drug is more than 6 hours 15 minutes.

$Y \sim Bi(10, 0.1)$ **1M**

$$\Pr\left(Y \geq 2\right) = \sum_{n=2}^{10} {}^{10}C_n \left(0.9\right)^n \left(0.1\right)^{10-n}$$

$= 0.264$ **1A**



g. $Y \sim Bi(k, 0.1)$

Consider the case where there is 95% chance.

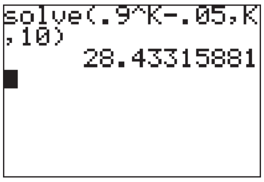
$\Pr(Y \geq 1) = 0.95$

$1 - \Pr(Y = 0) = 0.95$ **1M**

$1 - 0.9^k = 0.95$

$0.9^k = 0.05$

$k \approx 28.43$ (using numerical solve/ solver)



Round up because $k \in Z^+$ and the probability must be at least 95%.

The least value of k is 29 patients. **1A**

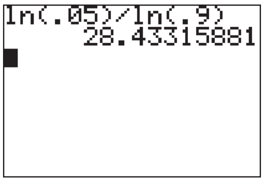
Alternative 1 to using numerical solve/ solver.

$0.9^k = 0.05$

$\log_e\left(0.9^k\right) = \log_e\left(0.05\right)$

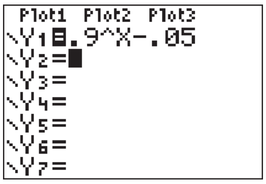
$k\log_e\left(0.9\right) = \log_e\left(0.05\right)$

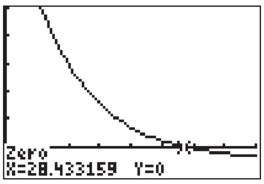
$k = \dfrac{\log_e\left(0.05\right)}{\log_e\left(0.9\right)} \approx 28.43$



However, this involves more steps and still requires the use of a calculator.

Alternative 2: graphical approach.

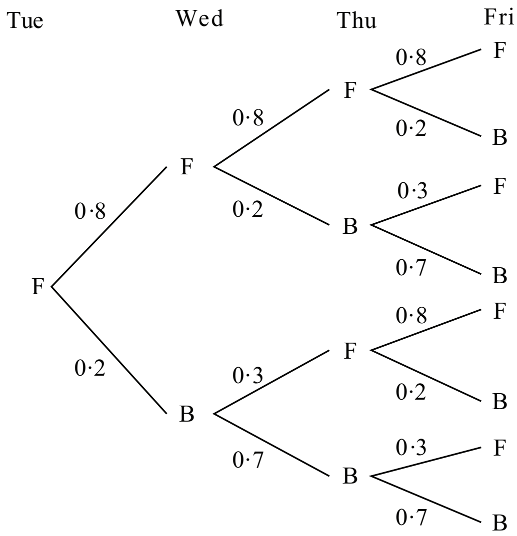




$0.9k - 0.05 = 0$

$k \approx 28.43$

h. **1M**



$\Pr(FFFB) = 0.8^2 \times 0.2 = 0.128$

$\Pr(FFBB) = 0.8 \times 0.2 \times 0.7 = 0.112$

$\Pr(FBFB) = 0.2 \times 0.3 \times 0.2 = 0.012$

$\Pr(FBBB) = 0.2 \times 0.7^2 = 0.098$ **1M**

$0.128 + 0.112 + 0.012 + 0.098 = 0.35$

The probability of Fluaway on Tuesday and Blugone on Friday is 0.35. **1A**

Question 2

a. $f(x) = A(x^{\frac{4}{3}} - 5)$

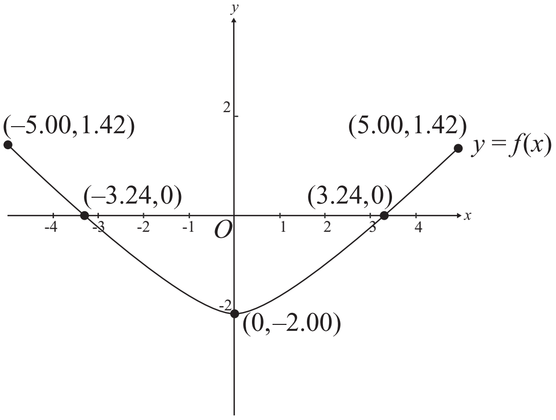
$-2 = A(0-5)$ **1M**

$A = \frac{2}{5}$

b. endpoints (–5.00,1.42) and (5.00,1.42) **1A**

turning point (0,–2.00) and correct shape **1A**

x-intercepts (–3.24,0) and (3.24,0) **1A**



c. Max depth = 2 + 1.42

= 3.42 m **1A**

d. i. $A = -2 \times \frac{2}{5}\int_0^{5^{\frac{3}{4}}} (x^{\frac{4}{3}} - 5)dx$ **1M**

$= -\frac{4}{5}\left[\frac{3}{7}x^{\frac{7}{3}} - 5x\right]_0^{5^{\frac{3}{4}}}$ **1M**

$= -\frac{4}{5}\left(\left(\frac{3}{7}(5^{\frac{3}{4}})^{\frac{7}{3}} - 5^{\frac{7}{4}}\right) - (0)\right)$ **1M**

$= 7.643\text{ m}^2$ correct to 3 decimal places **1A**

ii. $V = 1000A$

$= 7643\text{ m}^3$ **1A**

e. i. $V = 1000A$

$= 1000(\frac{1}{2}\text{ base} \times \text{height})$ **1M**

$= 2000 \times 5^{\frac{3}{4}}$

$= 6687.40\text{ m}^3$ **1A**

ii. $\frac{dV}{dt} = \frac{dh}{dt} \times \frac{dV}{dh}$ **1A**

$\frac{dV}{dt} = -0.02 \times \frac{dV}{dh}$

$V = 1000\frac{1}{2}bh$

Using similar triangles:

$\frac{\frac{1}{2}b}{h} = \frac{5^{\frac{3}{4}}}{2}$

$\frac{1}{2}b = \frac{5^{\frac{3}{4}}}{2}h$ **1M**

$V = 500 \times 5^{\frac{3}{4}}h^2$ **1A**

$\frac{dV}{dh} = 1000 \times 5^{\frac{3}{4}}h$

$\frac{dV}{dt} = -0.02 \times 1000 \times 5^{\frac{3}{4}}$

$= -20 \times 5^{\frac{3}{4}}h\text{ m}^3/\text{h}$ **1A**

The rate at which water is decreasing is $20 \times 5^{\frac{3}{4}}h\text{ m}^3/\text{h}$

Question 3

a. $12 = A \times B^2 \quad \ldots(1)$

$48 = A \times B^4 \quad \ldots(2)$

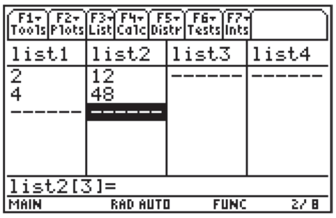
Divide (2) by (1)

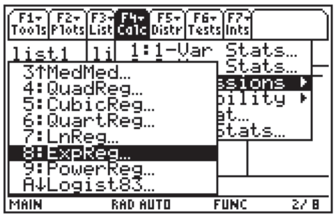
$4 = B^2$

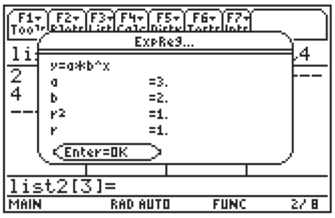
$B = 2$, as $B > 0$ **1A**

$A = \frac{12}{4} = 3$ **1A**

OR use ExpReg on the calculator.







b. Dilation by a factor of $\frac{1}{3}$ from the y-axis:

$y_1 = 3 \times 2^{3x}$ **1A**

Translation of $\frac{4}{3}$ units parallel to the x-axis:

$y_2 = 3 \times 2^{(3x-4)} = 3 \times 2^{\left(3(x-\frac{4}{3})\right)}$

Translation of 1 unit parallel to the y-axis:

$g_2(x) = 3 \times 2^{(3x-4)} + 1 = 3 \times 2^{\left(3(x-\frac{4}{3})\right)} + 1$ **1A**

Range $(1, \infty)$ **1A**

c. i. $g_3(x) = 3 \times 2^{(2x+1)} - 1$

Let $y = 3 \times 2^{(2x+1)} - 1$

Inverse: Swap x and y

$x = 3 \times 2^{(2y+1)} - 1$ **1M**

$2^{2y+1} = \frac{x+1}{3}$

$2y + 1 = \log_2(\frac{x+1}{3})$

$g_3^{-1}(x) = \frac{1}{2}\log_2(\frac{x+1}{3}) - \frac{1}{2}$ **1M**

$= \frac{1}{2}\log_2(\frac{x+1}{6})$

ii. Dilation by a factor of $\frac{1}{2}$ from the x-axis

Dilation by a factor of 6 from the y-axis. **1A**

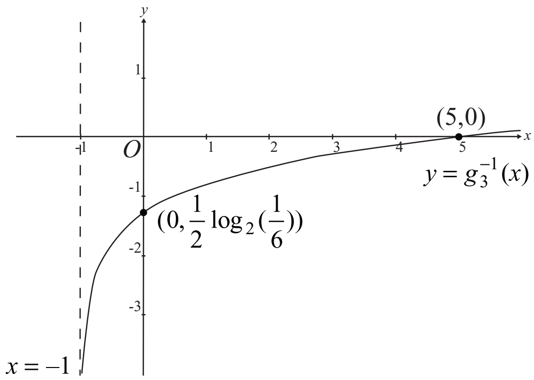
Translation of 1 unit to the left **1A**

iii. Asymptote: $x = -1$ and shape **1A**

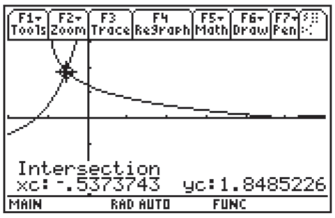
Coordinates of the x-intercept: (5, 0)

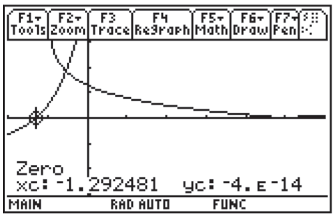
Coordinates of the y-intercept:

$(0, \frac{1}{2}\log_2(\frac{1}{6}))$ **1A**

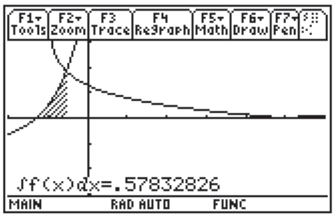


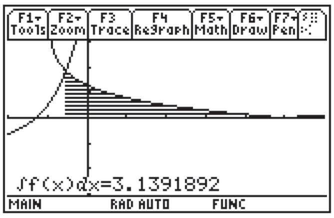
iv. Sketch both graphs and find the intersection and x-intercepts.





$$A = \int_{-1.292481}^{-0.53737} g_3(x)dx + \int_{-0.53737}^{5} \left| g_3^{-1}(x) \right| dx$$ **1M**





$A \approx 0.578328 + 3.139189$

$= 3.7$ units^2 correct to one decimal place **1A**

Question 4

a. By Pythagoras' theorem

$x^2 + y^2 = 50^2$

$y = \sqrt{2500 - x^2}$ (reject negative solution) **1A**

b. Highest position when $x = 0$.
Endpoint will be (0, 50). $y = 50$ cm **1A**

Lowest position when $x = \pm 30$. The corresponding y-axis value is

$y = \sqrt{2500 - (30)^2} = 40$ or

$y = \sqrt{2500 - (-30)^2} = 40$

$y = 40$ cm **1A**

c. Period along the x-axis $= \dfrac{2\pi}{\pi/8} = 16$ sec **1A**

d. $30\cos\left(\dfrac{\pi}{8}t\right) = 15$

$\cos\left(\dfrac{\pi}{8}t\right) = \dfrac{1}{2}$

$\left(\dfrac{\pi}{8}t\right) = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

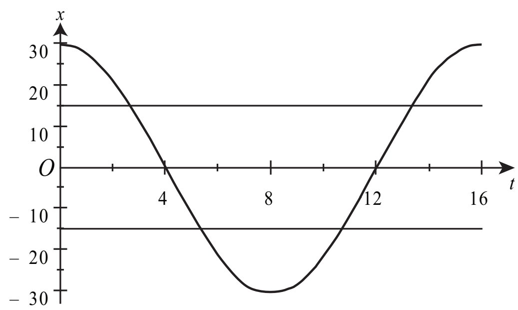
$t = \dfrac{8}{3}, \dfrac{40}{3}$ **1M**

$30\cos\left(\dfrac{\pi}{8}t\right) = -15$

$\cos\left(\dfrac{\pi}{8}t\right) = -\dfrac{1}{2}$

$\left(\dfrac{\pi}{8}t\right) = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

$t = \dfrac{16}{3}, \dfrac{32}{3}$ **1M**



Time for which is given by

$\left(\dfrac{16}{3} - \dfrac{8}{3}\right) + \left(\dfrac{40}{3} - \dfrac{32}{3}\right) = \dfrac{16}{3}$ seconds **1A**

e. $\frac{dx}{dt} = -\frac{15\pi}{4}\sin\left(\frac{\pi}{8}t\right)$ **1M**

When $x = 15$, $t = \frac{8}{3}, \frac{40}{3}, \ldots$ (from part **d.**)

$$x'\left(\frac{8}{3}\right) = -\frac{15\pi}{4}\sin\left(\frac{\pi}{8}\times\frac{8}{3}\right)$$
$$= -\frac{15\pi}{4}\sin\left(\frac{\pi}{3}\right)$$
$$= -\frac{15\pi}{4}\times\frac{\sqrt{3}}{2}$$
$$= -\frac{15\sqrt{3}\pi}{8}$$ **1M**

Speed is the magnitude of $x'(t)$

Speed = $\frac{15\sqrt{3}\pi}{8}$ cm/s **1A**

f. $x'(t) = -\frac{15\pi}{4}\sin\left(\frac{\pi}{8}t\right)$

The maximum speed occurs where

$\frac{d}{dt}\left(x'(t)\right) = 0$ **1M**

$$\frac{d}{dt}\left(-\frac{15\pi}{4}\sin\left(\frac{\pi}{8}t\right)\right) = 0$$
$$-\frac{15\pi}{4}\times\frac{\pi}{8}\cos\left(\frac{\pi}{8}t\right) = 0$$
$$\cos\left(\frac{\pi}{8}t\right) = 0$$
$$\frac{\pi}{8}t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$t = \frac{\pi}{2}\times\frac{8}{\pi}, \frac{3\pi}{2}\times\frac{8}{\pi}, \ldots$ **1M**

$t = 4, 12, \ldots$

(This is also evident from the graph of x v. t, which show the maximum gradient at $t = 4, 12, \ldots$)

$$x'(4) = -\frac{15\pi}{4}\sin\left(\frac{\pi}{2}\right) = -\frac{15\pi}{4}$$
$$x'(12) = -\frac{15\pi}{4}\sin\left(\frac{3\pi}{2}\right) = \frac{15\pi}{4}$$

The speed is the magnitude of $x'(t)$.

Hence the maximum speed is $\frac{15\pi}{4}$ cm/s. **1A**