Neap:

Trial Examination 2007

# VCE Mathematical Methods Units 3 & 4

Written Examination 2

## Suggested Solutions

### SECTION 1

| | |
|---|---|
| 1 | D |
| 2 | E |
| 3 | E |
| 4 | B |
| 5 | A |
| 6 | D |
| 7 | B |
| 8 | C |
| 9 | C |
| 10 | A |
| 11 | D |
| 12 | C |
| 13 | D |
| 14 | C |
| 15 | B |
| 16 | D |
| 17 | A |
| 18 | D |
| 19 | B |
| 20 | E |
| 21 | D |
| 22 | E |

## SECTION 1

### Question 1

$y = (2x + 4)^2$

$\quad = (2(x + 2))^2$

$\quad = 4(x + 2)^2$

$y = a(x - h)^2 + k$

$a = 4 \Rightarrow$ a dilation from the $x$-axis by a factor of 4.

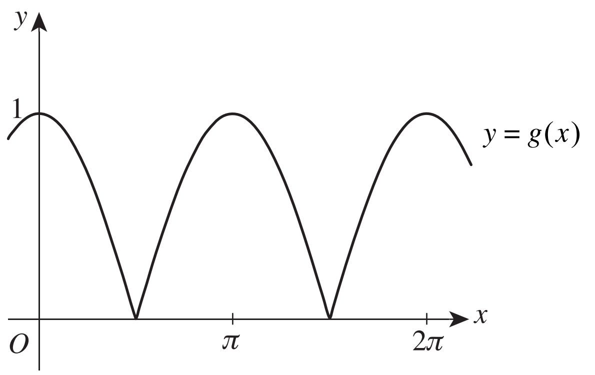
$h = -2 \Rightarrow$ a translation of 2 units to the left.

**Answer D**

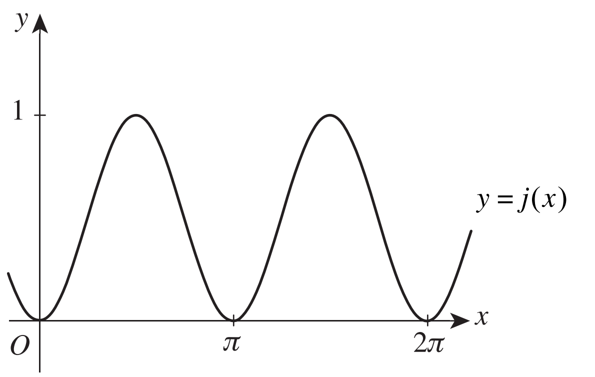
### Question 2

$f(x) = \sin\left(\frac{x}{2} + \pi\right)$ has a period of $2\pi$.

$g(x) = |\cos(x)|$ has a period of $\pi$, as may be seen from the graph below.




$j(x) = \sin^2(x)$ also has a period of $\pi$, as may be seen from the graph below.

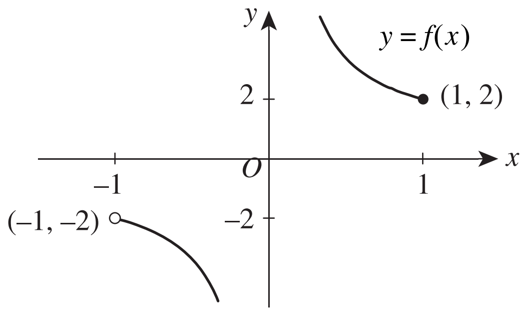



$k(x) = 1 - \pi\cos\left(2\left(\frac{\pi}{8} - x\right)\right)$ has a period of $\frac{2\pi}{2} = \pi$.

Thus each of the three functions $g$, $j$ and $k$ has a period of $\pi$.

**Answer E**

**Question 3**




From the graph of $f(x)$ above it can be seen that its range is $R\backslash[-2, 2)$.

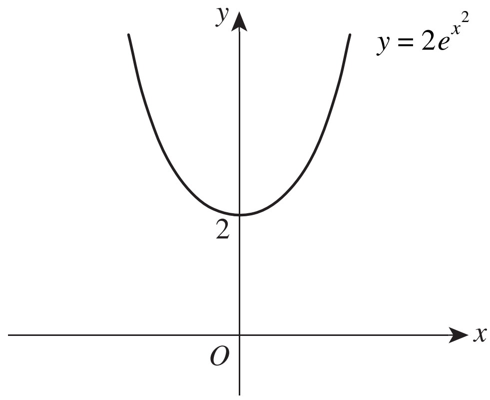
**Answer E**

**Question 4**

$$\log_e\left(\frac{1}{m^2 - n^2}\right) = \log_e(m^2 - n^2)^{-1}$$
$$= -\log_e(m^2 - n^2)$$
$$= -\log_e((m - n)(m + n))$$
$$= -\log_e(m - n) - \log_e(m + n)$$

**Answer B**

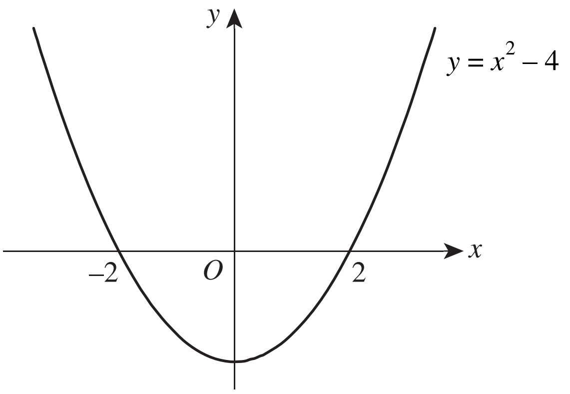
**Question 5**




For an inverse function to exist *f* must be a one-to-one function over the domain $(-\infty, a)$.
Hence **A** is the only option that can be correct.

**Answer A**

**Question 6**




$\sqrt{x^2 - 4} \neq 0$

$\Rightarrow x^2 - 4 > 0$

$\Rightarrow x < -2$ or $x > 2$

**Answer D**

**Question 7**

$y = \log_e|x|$ becomes $y = \log_e|3x|$ after a dilation of a factor of $\frac{1}{3}$ from the $y$-axis.

$y = \log_e|3x|$ becomes $y = \log_e|3(x+3)|$ after a translation of –3 units parallel to the $x$-axis.

$y = \log_e|3(x+3)|$ becomes $y = \log_e|3(-x+3)| = \log_e|-3x+9|$ after a reflection in the $y$-axis.

**Answer B**

**Question 8**

$$f(x) = e^{\frac{1}{2}x}$$

$$f'(x) = \frac{1}{2}e^{\frac{1}{2}x}$$

$$\therefore f'(2) = \frac{1}{2}e^{\frac{1}{2} \times 2}$$

$$= \frac{1}{2}e$$

$$\log_e(f'(2)) = \log_e\left(\frac{1}{2}e\right)$$

$$= \log_e\left(\frac{1}{2}\right) + \log_e(e)$$

$$= -\log_e(2) + 1$$

$$= 1 - \log_e(2)$$

**Answer C**

**Question 9**

If $p(x)$ is divisible by $2x + 2$ (i.e. $2(x+1)$), then $p(-1) = 0$.

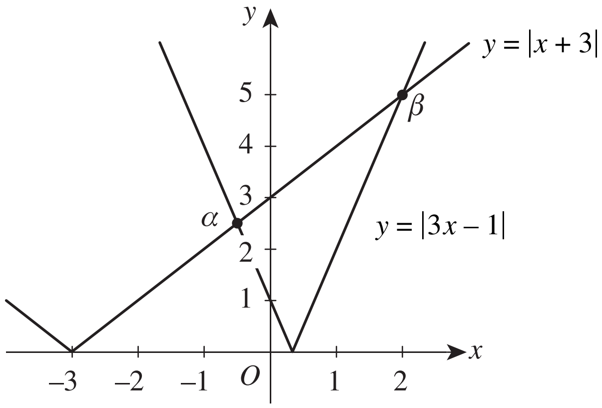
$$\therefore -(-1)^9 - 2(-1)^7 + a(-1)^2 + 1 = 0$$

$$1 + 2 + a + 1 = 0$$

$$a = -4$$

**Answer C**

**Question 10**




$\alpha: x + 3 = -(3x - 1)$

$x + 3 = -3x + 1$

$4x = -2$

$x = -\frac{1}{2}$

$\beta: x + 3 = 3x - 1$

$-2x = -4$

$x = 2$

So $|3x - 1| \geq |x + 3|$ when $x \leq -\frac{1}{2}$ or $x \geq 2$.

The points of intersection can also be found using a graphics calculator.

**Answer A**

**Question 11**

$\Pr(\text{even}) = 0.5$

$X:$ number of even numbers recorded in 8 rolls

$X \sim \text{Bi}(8, 0.5)$

$\text{E}(X) = 8 \times 0.5$

$= 4$

$\Pr(X \leq 4) = \binom{8}{0}(0.5)^4(0.5)^4 + \binom{8}{1}(0.5)^8 + \binom{8}{2}(0.5)^8 + \binom{8}{3}(0.5)^8 + \binom{8}{4}(0.5)^8$

$= 0.6367$

Or, using a graphics calculator, the answer may be found from bincdf(8, 0.5, 4).

**Answer D**

**Question 12**

$\text{ran } f = R = \text{dom } f^{-1}$

To find the rule for $f^{-1}$:

$x = \log_e \sqrt{-y + 2}$

$e^x = \sqrt{-y + 2}$

$(e^x)^2 = -y + 2$

$y = -e^{2x} + 2$

$\therefore f^{-1}: R \to R, \text{ where } f^{-1}(x) = -e^{2x} + 2$

**Answer C**

**Question 13**

The graphs in I could show $f(x) = \log_e(x)$ so that $f'(x) = \frac{1}{x}$, which is as shown.

The graphs in II could show $f(x) = |x|$ so that $f'(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$

The graph of $f'$ shown looks like this except that it should not be defined at $x = 0$.

The graphs in III could be $f(x) = e^{-x} - c$ so that $f'(x) = -e^{-x}$, which is as shown.
Hence I and III are valid.
**Answer D**

**Question 14**

Option C would only necessarily be true if *f* were symmetrical about the *y*-axis. All the other options are true given the conditions stated in the question.
**Answer C**

**Question 15**

We require the area bounded by the curve $y = f(x)$ and the horizontal line $y = d$ between $x = a$ and $x = b$.

Using the rule that area $= \int (\text{upper function} - \text{lower function})dx$, the area is given by $\int_a^b (d - f(x))dx$.

$$\begin{aligned} \text{area} &= \int_a^b d\,dx - \int_a^b f(x)dx \\ &= [d \times x]_a^b - \int_a^b f(x)dx \\ &= d(b-a) - \int_a^b f(x)dx \end{aligned}$$

**Answer B**

**Question 16**

$$\begin{aligned} \int_2^3 \frac{3}{(x-1)(x+2)}dx &= \int_2^3 \left(\frac{1}{x-1} - \frac{1}{x+2}\right)dx \\ &= [\log_e|x-1| - \log_e|x+2|]_2^3 \\ &= \left[\log_e\left|\frac{x-1}{x+2}\right|\right]_2^3 \\ &= \log_e\left(\frac{2}{5}\right) - \log_e\left(\frac{1}{4}\right) \\ &= \log_e\left(\frac{2}{5} \times \frac{4}{1}\right) \\ &= \log_e\left(\frac{8}{5}\right) \end{aligned}$$

**Answer D**

**Question 17**

We require $\frac{d}{dx}\left(\frac{uv}{w}\right) = \frac{d}{dx}\left(\frac{u}{w} \times v\right)$.

Applying the product rule, we have $\frac{u}{w}\frac{d}{dx}(v) + v\frac{d}{dx}\left(\frac{u}{w}\right)$.

Applying the quotient rule to the second term, we have

$$\frac{u}{w}v' + v\left(\frac{wu' - uw'}{w^2}\right) = \frac{uv'}{w} + \frac{vwu'}{w^2} - \frac{uvw'}{w^2}$$

$$= \frac{vwu' + uwv' - uvw'}{w^2}$$

**Answer A**

**Question 18**

$$p + 2p + p + 0.2 = 1$$

$$4p = 0.8$$

$$p = 0.2$$

$$\begin{aligned} \mathrm{E}(X) &= 0 \times p + 1 \times 2p + 2 \times p + 4 \times 0.2 \\ &= 4p + 0.8 \\ &= 4 \times 0.2 + 0.8 \\ &= 1.6 \end{aligned}$$

**Answer D**

**Question 19**

$$\int_0^6 A\sin\left(\frac{\pi}{6}x\right)dx = 1$$

$$A\left[-\frac{6}{\pi}\cos\left(\frac{\pi}{6}x\right)\right]_0^6 = 1$$

$$A\left[-\frac{6}{\pi}\cos(\pi) - \left(-\frac{6}{\pi}\cos(0)\right)\right] = 1$$

$$A\left(\frac{6}{\pi} + \frac{6}{\pi}\right) = 1$$

$$A\left(\frac{12}{\pi}\right) = 1$$

$$A = \frac{\pi}{12}$$

**Answer B**

**Question 20**

Let $f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ so that $f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$.

Now $f(48.5) = f(49 - 0.5) \approx f(49) - 0.5f'(49)$

$$f(49) = \frac{1}{\sqrt{49}} = \frac{1}{7}$$

$$f'(49) = -\frac{1}{2(49)^{\frac{3}{2}}} = -\frac{1}{2(7)^3}$$

$$\begin{aligned} f(48.5) &\approx \frac{1}{7} - \frac{1}{2}\left(-\frac{1}{2(7)^3}\right) \\ &= \frac{1}{7} + \frac{1}{4} \times \frac{1}{7^3} \\ &= \frac{1}{7}\left(1 + \frac{1}{4 \times 7^2}\right) \\ &= \frac{1}{7}\left(1 + \frac{1}{4 \times 49}\right) \end{aligned}$$

**Answer E**

**Question 21**

$X \sim \mathrm{N}(\mu, 2)$

$\Pr(X \le 510) = 0.95$

$$\begin{aligned} z = \frac{510 - \mu}{2} &= \text{invNorm}(0.95) \\ 510 - \mu &= 2 \times 1.645 \\ \mu &= 506.7 \\ &= 507 \text{ to the nearest mL} \end{aligned}$$

**Answer D**

**Question 22**

$g(f(x)) = g(a^x)$

Let $y = g(a^x)$

We require $\frac{dy}{dx}$, which represents $g'(f(x))$

Now let $a^x = e^{kx}$: $\log_e(a^x) = \log_e(e^{kx}) = kx$

$$x\log_e(a) = kx$$

$$k = \log_e(a)$$

So $y = g(e^{x\log_e(a)})$

Now let $u = e^{x\log_e(a)}$, so $y = g(u)$ and $\frac{dy}{du} = g'(u)$.

$$\frac{du}{dx} = \log_e(a) \times e^{x\log_e(a)} = \log_e(a) \times a^x$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= g'(u) \times \log_e(a) \times a^x$$

$$= a^x\log_e(a)g'(e^{x\log_e(a)})$$

$$= a^x\log_e(a)g'(a^x)$$

**Answer E**

## SECTION 2

### Question 1

**a.** **i.** $f(2\pi) = 2\pi - \cos(4\pi)$

$= 2\pi - 1$

The coordinates are $(2\pi, 2\pi - 1)$. A1

**ii.** $(0.515, 0)$, using graphics calculator. A1

**b.** $f'(x) = 1 + 2\sin(2x)$ A1

NB: $x \in [0, 2\pi]$

$1 + 2\sin(2x) = 0$ for stationary points

$\sin(2x) = -\frac{1}{2}$

$2x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$ M1

$x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

For a maximum, $x = \frac{7\pi}{12}$ or $\frac{19\pi}{12}$. A1

**c.** **i.** $x - \cos(2x) = x + 1$

$\cos(2x) = -1$

$2x = \pi, 3\pi$

$x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ A1

**ii.** $g'(x) = 1$ A1

$f'(x) = 1 + 2\sin(2x)$

At $x = \frac{\pi}{2}$, $f'\left(\frac{\pi}{2}\right) = 1 + 2\sin(\pi)$ M1

$= 1$

$= g'(x)$

At $x = \frac{3\pi}{2}$, $f'\left(\frac{3\pi}{2}\right) = 1 + 2\sin(3\pi)$

$= 1$

$= g'(x)$

$g(x)$ touches $f(x)$ at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. A1

**iii.** $c = -1$ A1

**Question 2**

**a.** **i.** $\Pr(X \geq 3) = 0.8413$
cdfnorm(3, 999, 4, 1)
84.1% A1

**ii.** $(0.8413)^5$ A1
$= 0.422$ A1

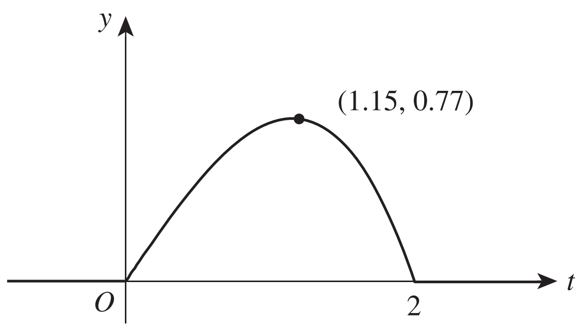
**b.** Let $T$ be the time spent completing security clearances.
$T \sim \mathrm{N}(\mu, 6)$
$\Pr(T \geq 30) = 0.1$
$\Pr(T < 30) = 0.9$ M1
$\Pr\left(z < \frac{30-\mu}{6}\right) = 0.9$
$\frac{30-\mu}{6} = 1.2816$ M1
$\mu = 22.31$ minutes
$= 22$ minutes and 19 seconds A1

**c.** **i.** Let $X$ be the number of hits in 6 shots.
$X \sim \mathrm{Bi}(6, 0.85)$ M1
$\Pr(X \geq 5) = \binom{6}{5}0.85^5 \times 0.15 + \binom{6}{6}0.85^6$
or 1 – bincdf(6, 0.85, 4)
$= 0.77648$ A1
$= 0.776$ (to three decimal places)

**ii.** $\mathrm{E}(X) = np$
$= 6 \times 0.85$
$= 5.1$ A1

**iii.** Let $Y$ be the number of satisfactory magazines.
$Y \sim \mathrm{Bi}(10, 0.77648)$
$\mathrm{E}(Y) = 10 \times 0.77648$
$= 7.76$ A1

**d.** **i.**




Correct curve for $t \in [0, 2]$ A1
Correct curve $(y = 0)$ for $t < 0$ and $t > 2$ A1
Correct coordinates of maximum point A1

**ii.** 1.15 hours = 1 hour and 9 minutes A1

**iii.** $\Pr(0.75 < T < 1.5) = \int_{0.75}^{1.5} \frac{1}{4}t(4 - t^2)\,dt$ M1
$= 0.547$, using graphics calculator A1

**Question 3**

**a.** If the parabola is to touch the $x$-axis at $x = 2$, then $x^2 + px + q = (x - 2)^2$. M1

$x^2 + px + q = x^2 - 4x + 4$

So $p = -4$, $q = 4$. A1

**b.** As $y = x^2 - 4x + 4 + k$, $\frac{dy}{dx} = 2x - 4$.

The gradient of the line $y = 2x$ is 2, and so $2x - 4 = 2$. M1

Hence $x = 3$. A1

At $x = 3$, $y = 6$. Substituting these values into the original equation gives

$3^2 - 4 \times 3 + 4 + k = 6$

$k = 5$ A1

**c.** The equation $x^2 + px + q = x$ must have two solutions.

This means that the discriminant of the equation $x^2 + (p - 1)x + q = 0$ must be positive. M1

Let $a = 1$, $b = p - 1$ and $c = q$.

Then $(p - 1)^2 - 4 \times 1 \times q > 0$.

$(p - 1)^2 > 4q$, as required. A1

**d.** $x^2 + px + q = 2x$

$x^2 + (p - 2)x + q = 0$ M1

$x = \frac{2 - p \pm \sqrt{(p - 2)^2 - 4q}}{2} = \frac{2 - p \pm \sqrt{p^2 - 4p + 4 - 4q}}{2}$ A1

**e.** **i.** $L_1 = \frac{1 - p - \sqrt{p^2 - 2p + 1 - 4q}}{2} - \frac{2 - p - \sqrt{p^2 - 4p + 4 - 4q}}{2}$ M1

$= \frac{1}{2}(-1 + \sqrt{p^2 - 4p + 4 - 4q} - \sqrt{p^2 - 2p + 1 - 4q})$ A1

$L_2 = \frac{2 - p + \sqrt{p^2 - 4p + 4 - 4q}}{2} - \frac{1 - p - \sqrt{p^2 - 2p + 1 - 4q}}{2}$

$= \frac{1}{2}(1 + \sqrt{p^2 - 4p + 4 - 4q} - \sqrt{p^2 - 2p + 1 - 4q})$ A1

**ii.** $|L_1 - L_2| = \frac{1}{2} \times 2 = 1$

So $|L_1 - L_2|$ is constant. A1

**f.** The pattern in the solutions for the parabola and each line is very similar to that in **e.** above.

For example, $ax^2 + px + q = 2x$

$$ax^2 + (p-2)x + q = 0$$

$$x = \frac{2 - p \pm \sqrt{(p-2)^2 - 4q}}{2a}$$

$$= \frac{2 - p \pm \sqrt{p^2 - 4p + 4 - 4q}}{2a}$$

Similarly, for the parabola intersecting the line $y = x$, we get

$$x = \frac{1 - p \pm \sqrt{p^2 - 2p + 1 - 4q}}{2a}$$ M1

Notice that the only change in the solutions is that the denominator is now $2a$.

Hence $L_3 = \frac{1 - p - \sqrt{p^2 - 2p + 1 - 4q}}{2a} - \frac{2 - p - \sqrt{p^2 - 4p + 4 - 4q}}{2a}$

$$= \frac{1}{2a}(-1 + \sqrt{p^2 - 4p + 4 - 4q} - \sqrt{p^2 - 2p + 1 - 4q})$$

$$L_4 = \frac{2 - p + \sqrt{p^2 - 4p + 4 - 4q}}{2a} - \frac{1 - p - \sqrt{p^2 - 2p + 1 - 4q}}{2a}$$ M1

$$= \frac{1}{2a}(1 + \sqrt{p^2 - 4p + 4 - 4q} - \sqrt{p^2 - 2p + 1 - 4q})$$

Hence $|L_3 - L_4| = 2 \times \frac{1}{2a}$, as $a > 0$

$$= \frac{1}{a}$$ A1

**Question 4**

**a.** **i.** 15.3 cm A1

**ii.** $C(x) = We^{-bx}$

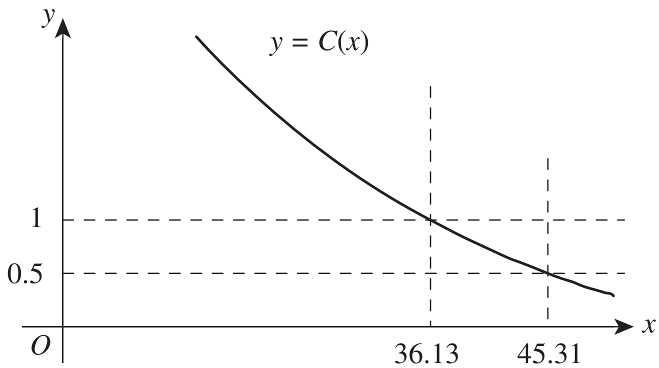
$$4.93 = 15.3e^{-15b}$$

$$e^{-15b} = \frac{4.93}{15.3}$$

$$b = -\frac{1}{15}\log_e\left(\frac{4.93}{15.3}\right)$$

$$= 0.0755$$ A1

**iii.**




Using a graphics calculator, we find that the compacted layers will be between 0.5 and 1 cm in thickness at depths between

36.13 cm and A1

45.31 cm. A1

**b.** Let $x = \dfrac{\log_e(2)}{b}$.

$C = We^{-b\frac{\log_e(2)}{b}}$

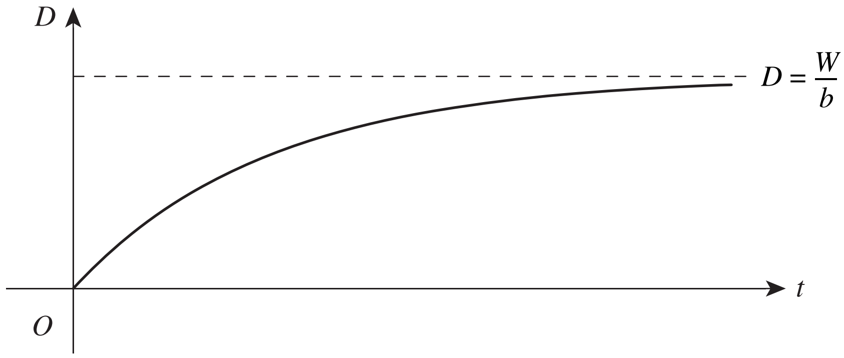
$= We^{-\log_e(2)}$

$= We^{\log_e(2^{-1})}$ M1

$= W \times 2^{-1}$

$= \dfrac{1}{2}W$ A1

**c.**




Curve with asymptotic behaviour A1
Curve passes through the origin A1
Correct equation of asymptote A1

**d.** **i.** $36.2 = \dfrac{2.06}{b}(1 - e^{-20b})$

Using a graphics calculator,

$y_1 = 36.2x$

$y_2 = 2.06(1 - e^{-20x})$ M1

Solving gives $x = 0.01323$.

$\therefore b = 0.01323$ A1

**ii.** $D = \dfrac{2.06}{0.01323}(1 - e^{-0.01323t})$

As $t \to \infty$ M1

$D \to \dfrac{2.06}{0.01323}$

$D \to 156$ cm A1

**iii.** $\dfrac{dD}{dt} = 156(0.0132e^{-0.01323t})$ A1

$= 2.06e^{-0.01323t}$

$1.3 = 2.06e^{-0.01323t}$ M1

$t = -\dfrac{1}{0.01323}\log_e\left(\dfrac{1.3}{2.06}\right)$

$= 34.87$

As $t = 0$ at the end of 2005, then $t = 35$ at the end of 1970.

So $t = 34.87$ at a point during 1971.
Hence the layer is from 1971. A1