THE SCHOOL FOR EXCELLENCE
UNIT 3 & 4 MATHEMATICAL METHODS 2007
COMPLIMENTARY WRITTEN EXAMINATION 1 - SOLUTIONS

QUESTION 1

a. $f(x) = (x^2 + 2x + 2)e^{-x}$

$f'(x) = (2x+2)e^{-x} - e^{-x}(x^2 + 2x + 2)$ M1

$= e^{-x}(2x + 2 - x^2 - 2x - 2)$

$= -x^2 e^{-x}$

$= -\dfrac{x^2}{e^x}$ A1

b. Stationary points occur when $f'(x) = 0$.

$-\dfrac{x^2}{e^x} = 0$

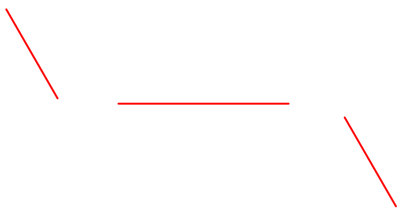
$\therefore x = 0$

$f(0) = 2$

$\therefore$The coordinates of the stationary point are $(0, 2)$. A1

c.

$x < 0$	$x = 0$	$x > 0$
eg. $f'(-1) = -\dfrac{1}{e^{-1}}$ $\therefore f'(x) < 0$	$f'(0) = 0$	eg. $f'(2) = -\dfrac{4}{e^2}$ $\therefore f'(x) < 0$

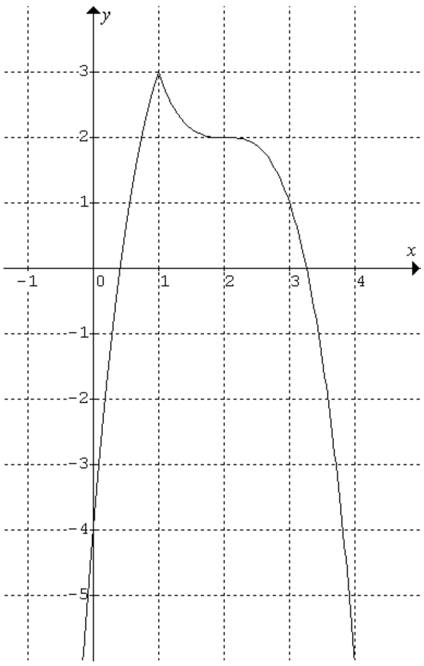


$\therefore$ The point $(0, 2)$ is a stationary point of inflexion. A1

QUESTION 2

a. $f(x) = 3 - \left|(x-2)^3 + 1\right|$

$$= \begin{cases} 4 + (x-2)^3, x \le 1 \\ 2 - (x-2)^3, x > 1 \end{cases}$$ M1



A1

b. $f'(x) = \begin{cases} 3(x-2)^2, x < 1 \\ -3(x-2)^2, x > 1 \end{cases}$ M1

Note: The derivative does not exist at sharp corners i.e. at $x = 1$. A1

c. (i) Since $x = \frac{1}{2} < 1$, use $f'(x) = 3(x-2)^2$:

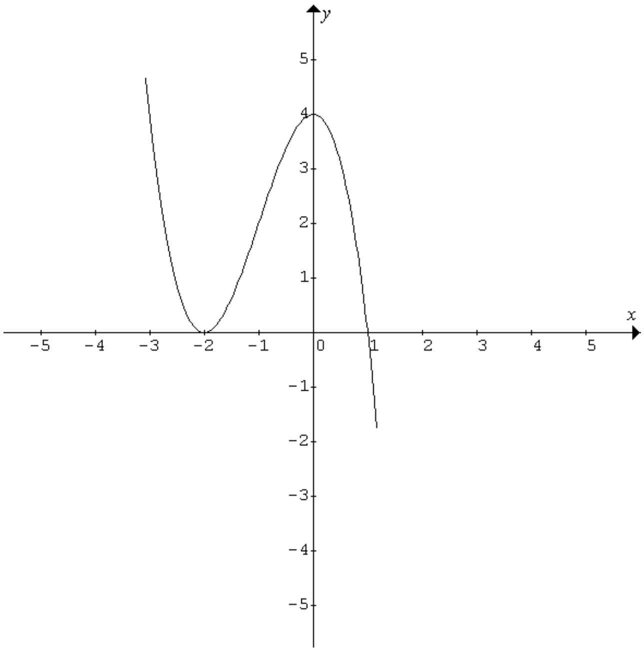
$f'(\tfrac{1}{2}) = 3(\tfrac{1}{2} - 2)^2 = \tfrac{27}{4}$ A1

(ii) $f'(1)$ does not exist A1

$f'(x)$ is undefined at $x = 1$ as the graph has a cusp at this point.

QUESTION 3

a.



Shape M1
Turning points and axes intercepts M1

b. Model: $y = a(x+2)^2(x-1)$

The $(x+2)^2$ comes from there being an *x*-intercept and also a turning point at $x = -2$.
The $(x-1)$ comes from there being an *x*-intercept at $x = 1$.

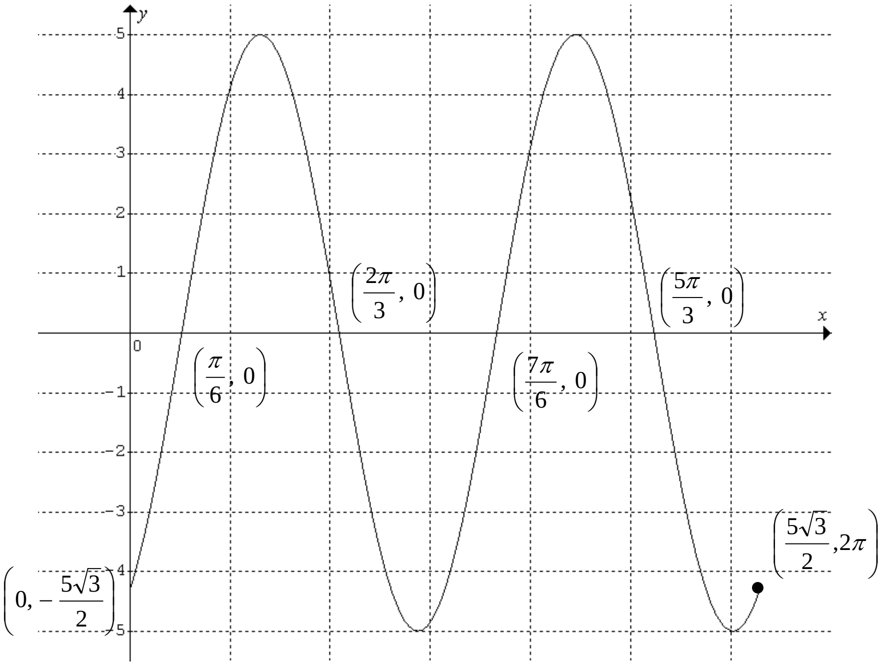
As $f(0) = 4$:

Substitute $(0, 4)$ into the model:

$$4 = a(0+2)^2(0-1)$$
$$\therefore a = -1$$

Therefore: $a = -1,\ b = 2$ and $c = -1$.

QUESTION 4



Shape	M1
Axes intercepts	M2
Endpoints	M1

X intercepts, Let $y = 0$:

$$y = 5\sin 2\left(x - \frac{\pi}{6}\right)$$

$$5\sin 2\left(x - \frac{\pi}{6}\right) = 0$$

$$\sin 2\left(x - \frac{\pi}{6}\right) = 0$$

$$2\left(x - \frac{\pi}{6}\right) = 0 \quad \text{or} \quad 2\left(x - \frac{\pi}{6}\right) = \pi$$

$$x = \frac{\pi}{6}, \frac{2\pi}{3}$$

Add period to each solution: $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

Y intercepts, Let $x = 0$:

$$y = -\frac{5\sqrt{3}}{2}$$

QUESTION 5

a. $\int_{-1}^{1}(x^4 - 2x^3 + 1 + a)dx = \frac{2}{5}$

$\left[\frac{x^5}{5} - \frac{x^4}{2} + (1+a)x\right]_{-1}^{1} = \frac{2}{5}$ M1

$\left(\frac{1}{5} - \frac{1}{2} + 1 + a\right) - \left(-\frac{1}{5} - \frac{1}{2} - 1 - a\right) = \frac{2}{5}$

$\frac{2}{5} + 2 + 2a = \frac{2}{5}$

$a = -1$ A1

b. $\int_{-1}^{0}(x^4 - 2x^3)dx - \int_{0}^{1}(x^4 - 2x^3)dx$ M1

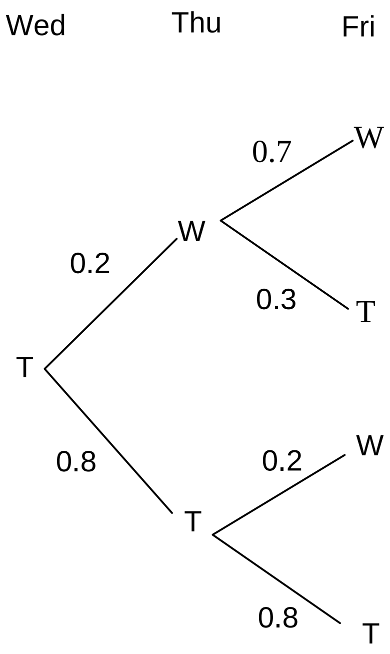
$= \left[\frac{x^5}{5} - \frac{x^4}{2}\right]_{-1}^{0} - \left[\frac{x^5}{5} - \frac{x^4}{2}\right]_{0}^{1}$

$= 0 - \left(-\frac{1}{5} - \frac{1}{2}\right) - \left(\left(\frac{1}{5} - \frac{1}{2}\right) - 0\right)$

$= 1$ square units A1

QUESTION 6

a. Let T = Tram
Let W = Walk



M2

b. $\Pr(\textit{Walks to school on Friday}) = (0.2)(0.7) + (0.8)(0.2) = 0.30$ A1

QUESTION 7

a. $\Pr(X > 36) = 1 - \Pr(X < 36)$

$$= 1 - \Pr\left(Z < \frac{36-22}{7}\right)$$

$$= 1 - \Pr(Z < 2)$$

$$= 1 - 0.98$$

$$= 0.02$$ A1

b. $\Pr(X < 8 | X < 22) = \dfrac{\Pr(X < 8 \ \ and \ \ X < 22)}{\Pr(X < 22)}$

$$= \frac{\Pr(X < 8)}{\Pr(X < 22)}$$

$$= \frac{\Pr\left(Z < \frac{8-22}{7}\right)}{\Pr\left(Z < \frac{22-22}{7}\right)}$$

$$= \frac{\Pr(Z < -2)}{\Pr(Z < 0)}$$ M1

$$= \frac{1 - \Pr(Z < 2)}{\Pr(Z < 0)}$$

$$= \frac{0.02}{0.5} = 0.04$$ A1

QUESTION 8

$$\frac{dA}{dt} = 15\, cm^2 / min$$

Find $\dfrac{dr}{dt}$ when $r = 5\, cm$.

$A = \pi r^2 \qquad \therefore \dfrac{dA}{dr} = 2\pi r$

$$\therefore \frac{dr}{dA} = \frac{1}{2\pi r}$$ M1

Chain rule: $\dfrac{dr}{dt} = \dfrac{dA}{dt} \times \dfrac{dr}{dA}$ M1

$$= 15 \times \frac{1}{2\pi r} = \frac{15}{2\pi r}.$$

When $r = 5$, $\dfrac{dr}{dt} = \dfrac{15}{10\pi} = \dfrac{3}{2\pi}\ cm/\min$ A1

QUESTION 9

a. (i) Let $y = f^{-1}(x)$.

Then: $x = \frac{1}{2}\log_e(y-1)$ M1

$\therefore 2x = \log_e(y-1)$

$\therefore y - 1 = e^{2x}$

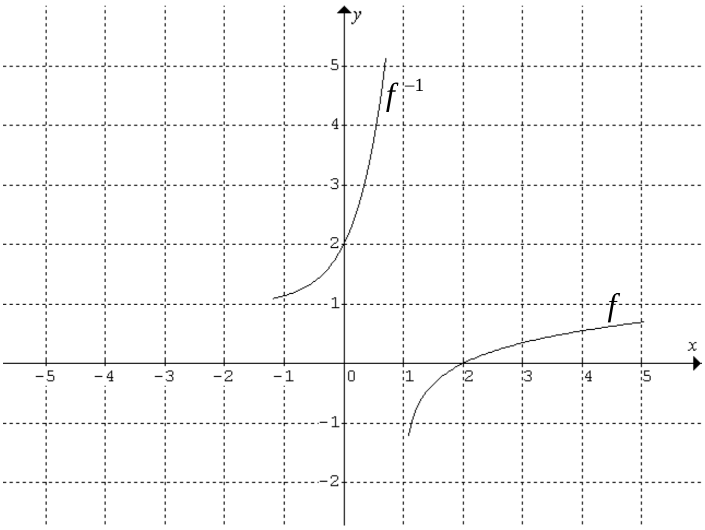
$\therefore y = e^{2x} + 1$

$\therefore f^{-1}(x) = e^{2x} + 1$ A1

(ii) $dom\ f^{-1} = \text{ran } f = R$

$ran^{-1} f = \text{dom } f = (1, \infty)$ A1

b.



A2

QUESTION 10

a. $k\int_{-a}^{a}(a-x)\,dx=1$ M1

$\therefore k\left[ax-\frac{x^2}{2}\right]_{-a}^{a}=1$

$\therefore k\left(a^2-\frac{a^2}{2}-\left(-a^2-\frac{a^2}{2}\right)\right)=1$

$\therefore k\left(2a^2\right)=1$

$\therefore k=\frac{1}{2a^2}$ M1

b. $E(X)=\frac{1}{2a^2}\int_{-a}^{a}x(a-x)\,dx$

$\therefore -2=\frac{1}{2a^2}\int_{-a}^{a}x(a-x)\,dx$ M1

$\therefore -2=\frac{1}{2a^2}\left[\frac{ax^2}{2}-\frac{x^3}{3}\right]_{-a}^{a}$

$\therefore -2=\frac{1}{2a^2}\left(\frac{a^3}{2}-\frac{a^3}{3}-\left(\frac{a^3}{2}+\frac{a^3}{3}\right)\right)$

$\therefore -2=-\frac{1}{2a^2}\left(\frac{2a^3}{3}\right)$

$\therefore -2=-\frac{a}{3}$

$\therefore a=6$ M1