Test 5

Section A: Technology free. 52 marks
Section B: CAS technology assumed. 38 marks
Suggested time: 90 minutes

Section A: Short answer and extended response questions. Technology free.

Specific instructions to students

- Answer **all** questions in the spaces provided.
- A decimal approximation will not be accepted if an **exact** answer is required to a question.
- In questions where more than one mark is available, appropriate working **must** be shown.

QUESTION 1 **Total 5 marks**

a Let $T = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$ be a transition matrix and $S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ be an initial state matrix. Find the following:

i S_1 **2 marks**

$S_1 = TS_0$

$= \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 0.8 \times 1 + 0.3 \times 0 \\ 0.2 \times 1 + 0.7 \times 0 \end{bmatrix}$

$= \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$

ii S_2 **1 mark**

$S_2 = TS_1$

$= \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$

$= \begin{bmatrix} 0.8 \times 0.8 + 0.3 \times 0.2 \\ 0.2 \times 0.8 + 0.7 \times 0.2 \end{bmatrix}$

$= \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$

b Suppose that the probability a particular search engine on the Internet is accessed depends on whether it was accessed the previous day. The probability that the search engine is accessed today is 0.65 if it was accessed yesterday and 0.18 if it was not accessed yesterday.

i Write a transition matrix to represent the probabilities of accessing the search engine. **1 mark**

$\begin{bmatrix} 0.65 & 0.18 \\ 0.35 & 0.82 \end{bmatrix}$

ii If a person accessed the search engine on Monday, write a matrix expression to represent the probability that the search engine is accessed on Wednesday. (Do not evaluate.) **1 mark**

$\begin{bmatrix} 0.65 & 0.18 \\ 0.35 & 0.82 \end{bmatrix}\begin{bmatrix} 0.65 & 0.18 \\ 0.35 & 0.82 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

QUESTION 2 **Total 9 marks**

a Three girls and two boys are to receive prizes.

i How many different ways can they be seated in a row on stage to receive their prizes? **1 mark**

Any of the 5 can sit in the first seat, then any 4 in the second seat . . . $= 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

ii If the girls and boys were to sit in alternate seats, how many possible arrangements would be possible? **1 mark**

Arrangement must be girl, boy, girl, boy, girl
$= 3 \times 2 \times 2 \times 1 \times 1 = 12$ ways.

b Evaluate the following.

i $\frac{10!}{7!} \times 2$ **2 marks**

$\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \times 2$

$= 10 \times 9 \times 8 \times 2$

$= 1440$

ii $\frac{4!}{0!}$ **1 mark**

$\frac{4 \times 3 \times 2 \times 1}{1}$

$= 24$

iii ${}^7C_3$ **1 mark**

$\frac{7!}{3! \times 4!}$

$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1}$

$= 35$

c There are 8 horses in a race. **1 mark**

i In how many ways can the first 3 places be filled?

The first place can be filled by 8 horses, the second place by 7 and the third place by 6.

$8 \times 7 \times 6 = 336$

ii If 1 horse does not finish the race, in how many ways can the first 3 places be filled now? **2 marks**

Since only 7 finish the race, $7 \times 6 \times 5 = 210$.

QUESTION 3 **Total 9 marks**

a In how many ways can a committee of 4 students be selected from a group of 10? **2 marks**

$$\binom{10}{4} = \frac{10!}{4! \times 6!}$$
$$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1}$$
$$= 210$$

b If Mary, one of the students in the group, must be included, how many committees are now possible? **2 marks**

Since Mary is included, this leaves 3 students to be selected from 9 students.

$$\binom{9}{3} = \frac{9!}{3! \times 6!}$$
$$= \frac{9 \times 8 \times 7}{3 \times 2 \times 1}$$
$$= 84$$

c A cricket team is to be selected from a squad consisting of 7 batsmen, 6 bowlers, 2 all-rounders and 2 wicket keepers.

i A team is to contain 4 batsmen, 5 bowlers, 1 all-rounder and 1 wicket keeper. How many teams are possible? **3 marks**

Four batsmen are selected from seven batsmen, five bowlers are selected from six bowlers, one all-rounder is selected from two all-rounders and one wicket keeper is selected from two wicket keepers.

$$\binom{7}{4} \times \binom{6}{5} \times \binom{2}{1} \times \binom{2}{1}$$
$$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times 6 \times 2 \times 2$$
$$= 35 \times 12 \times 2$$
$$= 840$$

ii If the captain is a batsman and the vice captain is a wicket keeper, how many teams are now possible? **2 marks**

The captain and vice captain must be included in the team; this leaves 3 batsmen to be selected from 6 batsmen, 5 bowlers from 6 bowlers and 1 all-rounder from 2 all-rounders.

$$\binom{6}{3} \times 6 \times 2 \times 1$$
$$= \frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times 12$$
$$= 240$$

QUESTION 4 **Total 6 marks**

A committee of 4 is to be selected from a group of 10 people which has Anne and Peter as members of the group.

a How many committees are possible if any of the 10 can be selected? **1 mark**

$$\binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1}$$
$$= 210$$

b If Anne and Peter must be on the committee, how many groups are now possible? **1 mark**

Since Anne and Peter must be on the committee, this leaves 2 places on the committee to be filled from 8 people $= \binom{8}{2} = \frac{8 \times 7}{2 \times 1}$
$= 28$

c A committee is selected at random. What is the probability Anne and Peter are on the committee? **2 marks**

$$\Pr(\text{Anne and Peter}) = \frac{28}{210} = \frac{2}{15}$$

d Find the probability that a committee selected at random does not contain Anne and Peter. **2 marks**

$$\Pr(\text{not Anne and Peter}) = 1 - \Pr(\text{Anne and Peter})$$
$$= 1 - \frac{2}{15}$$
$$= \frac{13}{15}$$

QUESTION 5 **Total 13 marks**

a Express $\frac{1-2x}{1-x}$ in the form $\frac{a}{x-1} + b$, where $a = -1$ and $b = 2$. **3 marks**

$$\frac{1-2x}{1-x} = \frac{a}{1-x} + b$$
$$1 - 2x = a + b(1 - x)$$
$$= a + b - bx$$

equate coefficients

$b = 2$

$a + b = 1$

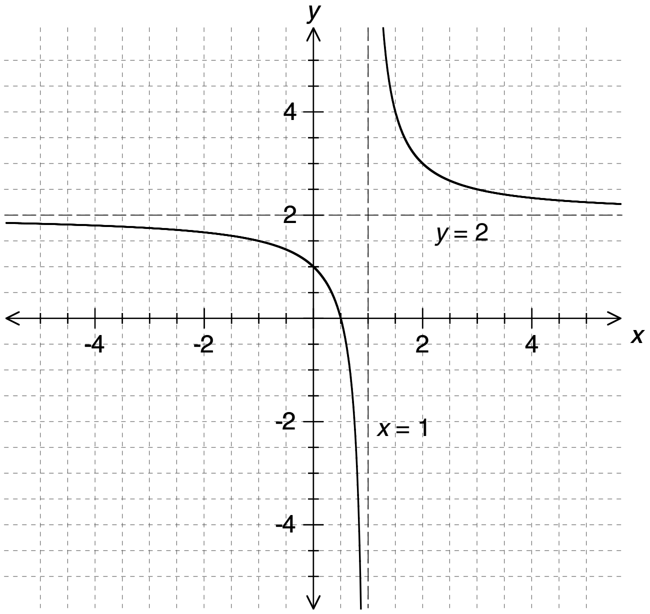
$a + 2 = 1 \Rightarrow a = -1$

thus $\frac{1-2x}{1-x} = \frac{-1}{1-x} + 2$

b Show that $\frac{1-2x}{1-x}$ can be written as $\frac{1}{x-1} + 2$. **1 mark**

$$\frac{1-2x}{1-x} = \frac{-1}{1-x} + 2$$
$$= -\frac{1}{1-x} + 2$$
$$= \frac{1}{-(1-x)} + 2$$
$$= \frac{1}{x-1} + 2$$

c Hence, sketch the graph of $f(x) = \frac{1-2x}{1-x}$, labelling any asymptotes and x and y intercepts. **4 marks**



d If $f(x)$ is reflected in the line $y = x$ to give the graph $g(x) = \frac{px+q}{rx+s}$, find the values of p, q, r and s. **5 marks**

$g(x)$ is the inverse function of $f(x)$.

$$x = \frac{1-2y}{1-y}$$

$$x = \frac{2y-1}{y-1}$$

$$x(y-1) = 2y-1$$

$$xy - x = 2y - 1$$

$$xy - 2y = x - 1$$

$$y(x-2) = x - 1$$

$$y = \frac{x-1}{x-2}$$

Thus $g(x) = \frac{x-1}{x-2}$ and $p = r = 1$, $q = -1$, $s = -2$

QUESTION 6 **Total 10 marks**

a There is a 0.9 chance that a car will pull into a service station in the next 15 minutes. What is the probability that:

i three cars will pull into the service station in the next 15 minutes? **1 mark**

$(0.9)^3 = 0.729$

ii no cars will pull into the service station in the next 15 minutes? **2 marks**

$(0.1)^3 = 0.001$

b $\Pr(A) = 0.65$, $\Pr(A \cup B) = 0.95$ and $\Pr(B) = 0.45$.

i Find $\Pr(A \cap B)$. **2 marks**

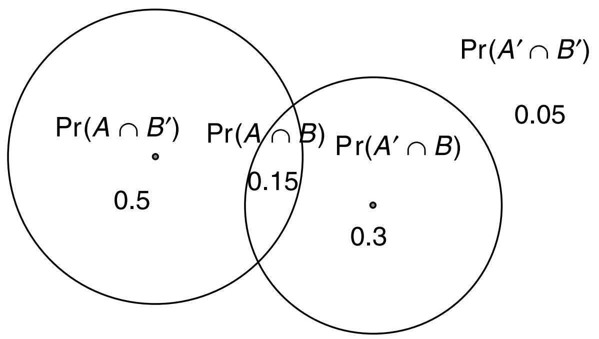
$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$0.95 = 0.65 + 0.45 - \Pr(A \cap B)$$

$$\Pr(A \cap B) = 1.1 - 0.95$$

$$= 0.15$$

ii Draw a Venn diagram of the probabilities between A and B. **3 marks**



iii Find $\Pr(A' \cup B')$. **2 marks**

$$\Pr(A' \cup B') = \Pr(A') + \Pr(B') - \Pr(A' \cap B')$$

$$= 0.35 + 0.55 - 0.05 = 0.85$$

or

$$1 - \Pr(A \cap B) = 1 - 0.15 = 0.85$$

Section B: Multiple-choice questions. CAS technology assumed.

Specific instructions to students

- A correct answer scores 1, and an incorrect answer scores 0.
- Marks are not deducted for incorrect answers.
- No marks are given if more than one answer is given.
- Choose the alternative which most correctly answers the question and mark your choice on the multiple-choice answer section at the bottom of each page, as shown in the example below.

1 A B C D E **USE PENCIL ONLY**

- Use pencil only.

QUESTION 7

In how many ways can 7 books be arranged on a shelf?

A 7

B ${}^7C_1$

C $\frac{7!}{1!}$

D $\frac{7!}{1! \times 6!}$

E $\binom{7}{2}$

QUESTION 8
How many different ways can 3 fiction and 4 non-fiction books be arranged on a shelf, if the fiction books must come first?

A 2
B 12
C 24
D 144
E 5040

QUESTION 9
In how many ways can 3 tunes be selected from 15 tunes?

A 3
B 3!
C $\frac{15}{3!}$
D $\frac{15!}{3!}$
E $\frac{15!}{3! \times 12!}$

QUESTION 10
Five letters of the word DREAMING are arranged in a row. What is the probability that the letters end in a vowel? (Vowels are A, E, I, O, U.)

A $\frac{1}{5}$
B $\frac{5}{8}$
C $\frac{3}{8}$
D $\frac{3!}{8!}$
E $\frac{5!}{8!}$

QUESTION 11
A box of magazines contains 6 sports magazines and 5 computer magazines. Four books are selected at random. What is the probability that 2 sports and 2 computer magazines are selected?

A $\frac{5}{11}$
B $\frac{4}{11}$
C $\frac{4!}{11!}$
D $\frac{4!}{6! \times 5!}$
E $2 \times \frac{4!}{11!}$

ONE ANSWER PER LINE **USE PENCIL ONLY**

7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E

Section B: Extended response questions. CAS technology assumed.

Specific instructions to students

- Answer **all** questions in the spaces provided.
- In questions where more than one mark is available, appropriate working **must** be shown.

QUESTION 12 **Total 16 marks**

a If $\frac{n!}{(n-1)!} = 182$, evaluate n. **2 marks**

$n = 182$

b Write the sixth line of Pascal's triangle. **1 mark**

1, 6, 15, 20, 15, 6, 1

i From the answer, show that ${}^6C_2 = {}^6C_4$. **1 mark**

$1\ {}^6C_0,\ 6\ {}^6C_1,\ 15\ {}^6C_2,\ 20\ {}^6C_3,\ 15\ {}^6C_4,\ 6\ {}^6C_5,\ 1\ {}^6C_6$

ii Verify this using the general formula for ${}^nC_r$. **1 mark**

$${}^6C_2 = \frac{6!}{2! \times 4!} = \frac{6!}{4! \times 2!} = {}^6C_4$$

c A manufacturer sells and services a particular make of car in two nearby towns, A and B. A car purchased at town A has a 0.25 probability that its next service will be at town B, while a car purchased at town B has a 0.1 probability that its next service will be at town A. It is assumed that cars are only serviced in either of these towns.

i Determine the transition matrix, T, which can represent this information. **1 mark**

$$T = \begin{bmatrix} 0.75 & 0.1 \\ 0.25 & 0.9 \end{bmatrix}$$

Cars are serviced every 6 months and 3000 cars and 2000 cars were sold in town A and town B respectively over the last 6 months.

ii Estimate the number of these cars serviced in towns A and B after 12 months. **2 marks**

The estimate is given by T^2S where $S = \begin{bmatrix} 3000 \\ 2000 \end{bmatrix}$.

$$T^2S = \begin{bmatrix} 0.75 & 0.1 \\ 0.25 & 0.9 \end{bmatrix}^2 \begin{bmatrix} 3000 \\ 2000 \end{bmatrix}$$
$$= \begin{bmatrix} 2093 \\ 2907 \end{bmatrix}$$

Note: the larger number has been rounded down. Also accept 2908.

iii Estimate the number of these particular cars that will be serviced in town A and town B in the long term. **2 marks**

The estimate is given by $T^{20}S$. (Any other large number will do.)

$$T^{20}S = \begin{bmatrix} 0.75 & 0.1 \\ 0.25 & 0.9 \end{bmatrix}^{20} \begin{bmatrix} 3000 \\ 2000 \end{bmatrix}$$
$$= \begin{bmatrix} 1429 \\ 3571 \end{bmatrix}$$

d From a group of 15 women and 13 men, a committee of 3 is to be formed. What is the probability the committee contains:

i at least one man? **4 marks**

The committee can contain 2 women and 1 man $= \binom{15}{2} \times \binom{13}{1} = 1365$,

Or 1 woman and 2 men $= \binom{15}{1} \times \binom{13}{2} = 1170$

Or 0 women and 3 men $= \binom{15}{0} \times \binom{13}{3} = 286$

Total number of possibilities $= 1365 + 1170 + 286 = 2821$

A committee of three from 28 people $= \binom{28}{3} = 3276$

Pr(at least 2 men) $= \frac{2821}{3276} = \frac{31}{36} \approx 0.8611$

ii exactly 2 men, given that at least 1 man is chosen? **2 marks**

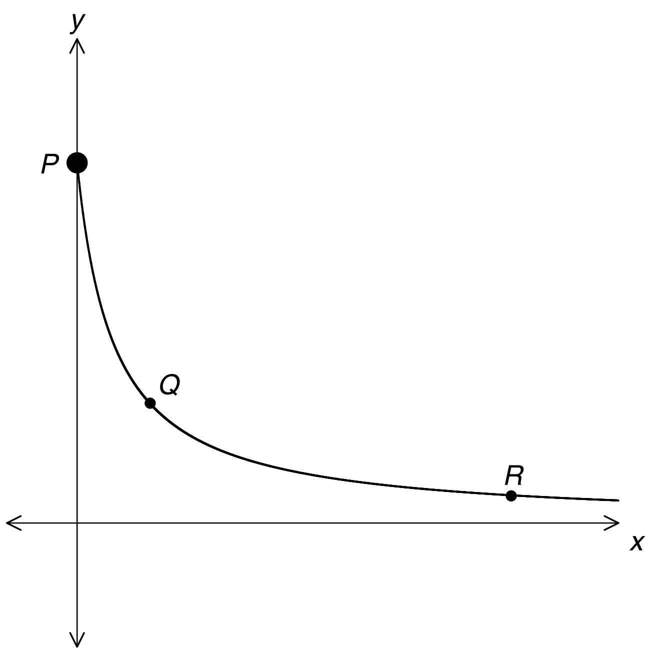
Pr(exactly two men|at least one man)

$$= \frac{\text{Pr(exactly two men} \cap \text{at least one man)}}{\text{Pr(at least one man)}}$$

$$= \frac{\frac{1170}{3276}}{\frac{31}{36}} = \frac{90}{217} \approx 0.4147$$

QUESTION 13 **Total 9 marks**

The figure shows part of the graph of $y = \frac{4}{x+1}$.



a Find the average rate of change from P, the y intercept, to Q, where $x = 3$. **2 marks**

The coordinates of P are (0, 4). The y value when $x = 3$ is $\frac{4}{3+1} = 1$. The coordinates of Q are (3, 1).
Average rate of change from P to Q

$$= \frac{1-4}{3-0} = -\frac{3}{3} = -1$$

b If R is the point at which $x = a$, find the average rate of change from P to R. **2 marks**

The y value for $R = \frac{4}{a+1}$. Average rate of change from P to $R = \dfrac{\frac{4}{a+1} - 4}{a - 0} = -\frac{4}{a+1}$ (use CAS for this simplification.)

c Find the average rate of change from P to R when $a = \{10, 1000, 10^6\}$. Give answers correct to four decimal places. **1 mark**

CAS: −0.3636, −0.0040, 0.0000

d Hence, estimate the average rate of change from P to R as a approaches infinity. **1 mark**

Average rate of change approaches zero.

e A particle moves in a straight line so that its distance from a fixed point O is given by $x = \frac{4}{t+1}$, $t > 0$, where x metres is the distance at time t seconds.

i What is the average speed for the first 2 seconds? **2 marks**

The average speed

$$= \frac{\frac{4}{3} - 4}{2 - 0} = \frac{-\frac{8}{3}}{2} = -\frac{4}{3} = 1\frac{1}{3}\text{ m/s.}$$

ii What is the long-term average speed of the particle? **1 mark**

Zero m/s

QUESTION 14 **Total 8 marks**

$P(4, 6)$, $R(2, 0)$ and $Q(1, a)$ are the vertices of a right-angled triangle with the right angle at the x axis.

a Find the gradient of the side PR. **1 mark**

$$m_{PR} = \frac{6-0}{4-2} = 3$$

b Hence, state the gradient of the side QR. **1 mark**

Since $QR \perp PR$, $m_{QR} = -\frac{1}{3}$

c Write the gradient of the side QR in terms of a. Hence, show that the coordinates of Q are $\left(1, \frac{1}{3}\right)$. **3 marks**

$m_{QR} = \frac{a-0}{1-2} = -a.$

Thus:

$-a = -\frac{1}{3}$

$a = \frac{1}{3}.$

The coordinates of Q are $\left(1, \frac{1}{3}\right)$.

d Find the lengths of each of the 3 sides of the triangle and hence confirm Pythagoras' theorem. **3 marks**

$$d(PQ) = \sqrt{(4-1)^2 + \left(6-\frac{1}{3}\right)^2} = \frac{1}{3}\sqrt{370}$$

$$\Rightarrow (d(PQ))^2 = \frac{370}{9}$$

$$d(QR) = \sqrt{(2-1)^2 + \left(0-\frac{1}{3}\right)^2} = \frac{1}{3}\sqrt{10}$$

$$\Rightarrow (d(QR))^2 = \frac{10}{9}$$

$$d(PR) = \sqrt{(4-2)^2 + (6-0)^2} = 2\sqrt{10}$$

$$\Rightarrow (d(PR))^2 = 40$$

Thus $(d(PQ))^2 = (d(QR))^2 + (d(PR))^2$