THE HEFFERNAN GROUP

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MATHS METHODS 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2008

Question 1

$$f(x) = \frac{1}{x} + 2x \text{ and } g(x) = x - 1$$
$$f(g(x)) = f(x-1)$$
$$= \frac{1}{x-1} + 2(x-1)$$

(1 mark)

Question 2

a.

$$2e^{(x+1)} = 6$$
$$e^{(x+1)} = 3$$
$$\log_e(3) = x + 1$$
$$x = \log_e(3) - 1$$

(1 mark)

b.

$$2\log_5(x) - \log_5(3x) = 1, \quad x > 0$$
$$\log_5(x^2) - \log_5(3x) = 1$$
$$\log_5\left(\frac{x^2}{3x}\right) = 1$$

(1 mark)

$$\log_5\left(\frac{x}{3}\right) = 1, \quad \text{since } x \neq 0$$
$$5^1 = \frac{x}{3}$$
$$x = 15$$

(1 mark)

Question 3

a.

$$y=\frac{x^3-1}{e^{2x}}$$

$$\frac{dy}{dx}=\frac{e^{2x}\times 3x^2-\left(x^3-1\right)\times 2e^{2x}}{\left(e^{2x}\right)^2}$$ (Quotient rule)

(1 mark) –note that because the question is worth 1 mark and the question hasn't asked for the simplification, the following lines are not required but are shown here to give an example of how you would go on to simplify if you were asked to

$$=\frac{e^{2x}\left(3x^2-2\left(x^3-1\right)\right)}{e^{4x}}$$

$$=\frac{e^{2x}\left(3x^2-2x^3+2\right)}{e^{4x}}$$

$$=\frac{3x^2-2x^3+2}{e^{2x}}$$

b. $h(x)=e^{\tan(x)}$

$h'(x)=\sec^2(x)e^{\tan(x)}$ (Chain rule) **(1 mark)** correct derivative

(Alternatively, let $y=e^{\tan(x)}$

and let $u=\tan(x)$

$$\frac{du}{dx}=\sec^2(x)$$

so $y=e^u$

$$\frac{dy}{du}=e^u$$

Now, $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ (Chain rule)

$$=e^u\cdot\sec^2(x)$$

$$=\sec^2(x)e^{\tan(x)}\;)$$

So $h'\left(\frac{\pi}{3}\right)=\sec^2\left(\frac{\pi}{3}\right)e^{\tan\left(\frac{\pi}{3}\right)}$

$$=\frac{1}{\cos^2\left(\frac{\pi}{3}\right)}\cdot e^{\sqrt{3}}$$

$$=\frac{1}{\left(\frac{1}{2}\right)^2}\cdot e^{\sqrt{3}}$$

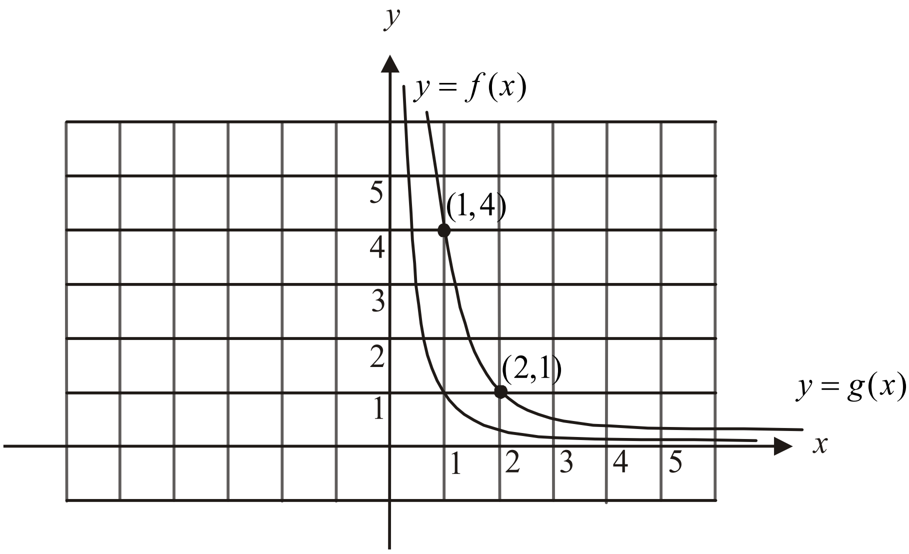
$$=\frac{1}{\frac{1}{4}}\cdot e^{\sqrt{3}}$$

$$=4e^{\sqrt{3}}$$

(1 mark) correct answer

Question 4

a. **i.**

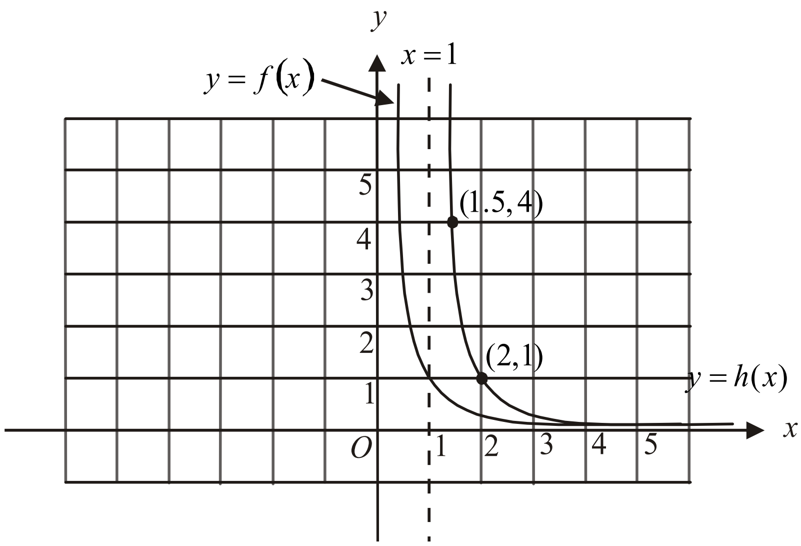


(1 mark) – correct graph of $y=g(x)$ and labelling of two points

ii. Since the graph of $f(x)=\dfrac{1}{x^2}$ has been dilated by a factor of 2 from the y-axis, we replace the x with $\dfrac{x}{2}$ to obtain the rule for $y=g(x)$.

$$g(x)=\frac{1}{\left(\frac{x}{2}\right)^2}=\frac{4}{x^2}$$ **(1 mark)**

b. **i.**



(1 mark) correct graph including 2 labelled points

ii. Equation of vertical asymptote is $x=1$. **(1 mark)**

iii. Equation of horizontal asymptote is $y=0$. **(1 mark)**

iv. $h(x)=\dfrac{1}{(x-1)^2}$ **(1 mark)**

Question 5

This is a binomial distribution with $p = 0 \cdot 9$ and $n = 3$.
Let X represent the number of females offered a scholarship this year.

$$\begin{aligned}\Pr(X < 2) &= \Pr(X = 0) + \Pr(X = 1)\\ &= {}^3C_0(0 \cdot 9)^0(0 \cdot 1)^3 + {}^3C_1(0 \cdot 9)^1(0 \cdot 1)^2\\ &= (0 \cdot 1)^3 + 3 \times 0 \cdot 9 \times 0 \cdot 01\\ &= 0 \cdot 001 + 0 \cdot 027\\ &= 0 \cdot 028\end{aligned}$$

(1 mark) (second line)

(1 mark)

Question 6

X	0	1	2	5	10
$\Pr(X = x)$	0.1	a	0.4	b	0.2

a. Since X is a discrete random variable,

$0 \cdot 1 + a + 0 \cdot 4 + b + 0 \cdot 2 = 1$

$a + b = 0 \cdot 3 \quad -(1)$

Also, since $E(X) = 3 \cdot 5$,

$0 \times 0 \cdot 1 + 1 \times a + 2 \times 0 \cdot 4 + 5 \times b + 10 \times 0 \cdot 2 = 3 \cdot 5$

$a + 5b = 0 \cdot 7 \quad -(2)$

$(2)-(1)$ gives $4b = 0 \cdot 4$

$b = 0 \cdot 1$

In (1) gives $a = 0 \cdot 2$

(1 mark) correct method attempted
(1 mark) correct answers

b. The median of X is 2 because 50% of the distribution is greater than 2 and 50% is less than 2.

(1 mark)

c. $\Pr(0,0) + \Pr(1,1) + \Pr(2,2) + \Pr(5,5) + \Pr(10,10)$

$= 0 \cdot 1^2 + 0 \cdot 2^2 + 0 \cdot 4^2 + 0 \cdot 1^2 + 0 \cdot 2^2$

$= 0 \cdot 01 + 0 \cdot 04 + 0 \cdot 16 + 0 \cdot 01 + 0 \cdot 04$

$= 0 \cdot 26$

(1 mark)

Question 7

a.

$$\cos\left(\frac{\pi x}{8}\right)-\frac{1}{\sqrt{2}}=0$$

$$\cos\left(\frac{\pi x}{8}\right)=\frac{1}{\sqrt{2}}$$

S	A
T	C

$$0\le x\le 20$$

$$0\le\frac{\pi x}{8}\le\frac{20\pi}{8}$$

$$0\le\frac{\pi x}{8}\le\frac{5\pi}{2}$$

$$\frac{\pi x}{8}=\frac{\pi}{4},\frac{7\pi}{4},\frac{9\pi}{4}$$ **(1 mark)**

$$x=2,14,18$$ **(1 mark)**

b. Method 1

The graph of $g(x)=\cos\left(\frac{\pi x}{8}\right)$ has **maxima** (for $x\ge 0$) at $x=0,\ x=16,\ x=32$ and so on because the period of the graph is given by $2\pi\div\frac{\pi}{8}=16$.

The graph of $y=h(x)$ is the graph of $y=g(x)$ that has been reflected in the x-axis translated 4 units to the left, and translated 1 unit down. **(1 mark)**

The graph of $h(x)=-\cos\left(\frac{\pi}{8}(x+4)\right)-1$ will therefore have **minima** at $x=-4$, $x=12$ and $x=28$. For $x\in[0,20]$ the required value of x is 12. **(1 mark)**

Method 2

The minimum occurs when $h(x)=-2$ because the amplitude is 1 and there is a translation of 1 unit down.

$$-\cos\left(\frac{\pi}{8}(x+4)\right)-1=-2$$

$$\cos\left(\frac{\pi}{8}(x+4)\right)=1$$

$$\frac{\pi}{8}(x+4)=\ldots-2\pi,0,2\pi,4\pi,\ldots$$

$$x+4=\ldots-16,0,16,32\ldots$$

$$x=\ldots-20,-4,12,28,\ldots$$ **(1 mark)**

For $x\in[0,20]$, $x=12$. **(1 mark)**

Method 3

$$h(x)=-\cos\left(\frac{\pi}{8}(x+4)\right)-1$$

$$h'(x)=\frac{\pi}{8}\left(\sin\left(\frac{\pi}{8}(x+4)\right)\right)=0$$ **(1 mark)**

$\sin\left(\frac{\pi}{8}(x+4)\right)=0$ for a maximun or minimum

$$\frac{\pi}{8}(x+4)=\ldots-\pi,0,\pi,2\pi,3\pi,\ldots$$

$$x+4=\ldots-8,0,8,16,24,\ldots$$

$$x=\ldots-12,-4,4,12,20,\ldots$$

At $x=4$ and at $x=20$ there is a maximum.
At $x=12$, there is a minimum. **(1 mark)**

Question 8

Let $f(x)=\cos(2x)$,

$$f'(x)=-2\sin(2x)$$

$$f'\left(\frac{\pi}{6}\right)=-2\sin\left(\frac{\pi}{3}\right)$$

$$=-2\times\frac{\sqrt{3}}{2}$$

$$=-\sqrt{3}$$

So the gradient of the tangent at $x=\frac{\pi}{6}$ is $-\sqrt{3}$. **(1 mark)**

Now $f\left(\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{3}\right)$

$$=\frac{1}{2}$$

The tangent passes through the point $\left(\frac{\pi}{6},\frac{1}{2}\right)$.

Equation of tangent

$$y-\frac{1}{2}=-\sqrt{3}\left(x-\frac{\pi}{6}\right)$$

$$y=-\sqrt{3}x+\frac{\sqrt{3}\pi}{6}+\frac{1}{2}$$

(1 mark)

Question 9

a. $\frac{d}{dx}(\log_e(\sin(x))=\frac{\cos(x)}{\sin(x)}$

(1 mark)

b. Hence $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{3\cos(x)}{\sin(x)}\,dx=3\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\cos(x)}{\sin(x)}\,dx$

$$=3[\log_e(\sin(x))]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$ **(1 mark)**

$$=3\left\{\log_e\left(\sin\left(\frac{\pi}{2}\right)\right)-\log_e\left(\sin\left(\frac{\pi}{6}\right)\right)\right\}$$

$$=3\left\{\log_e(1)-\log_e\left(\frac{1}{2}\right)\right\}$$

$$=3\left\{0-\log_e\left(\frac{1}{2}\right)\right\}$$

$$=-3\log_e\left(\frac{1}{2}\right)$$ **(1 mark)**

$$=-3\log_e\left(2^{-1}\right)$$

$$=3\log_e(2)$$

Question 10

The graphs of $y = f(x)$ and $y = g(x)$ intersect when

$$f(x) = g(x)$$
$$\frac{1}{x} = \frac{x^2}{8}$$
$$8 = x^3$$
$$x = 2$$

The graphs intersect when $x = 2$. **(1 mark)**

$$\text{Area} = \int_1^2 \left(f(x) - g(x)\right)dx$$ **(1 mark)**

$$= \int_1^2 \left(\frac{1}{x} - \frac{x^2}{8}\right)dx$$

$$= \left[\log_e|x| - \frac{x^3}{24}\right]_1^2$$ **(1 mark)**

$$= \left(\log_e(2) - \frac{8}{24}\right) - \left(\log_e(1) - \frac{1}{24}\right)$$

$$= \log_e(2) - \frac{8}{24} - 0 + \frac{1}{24}$$

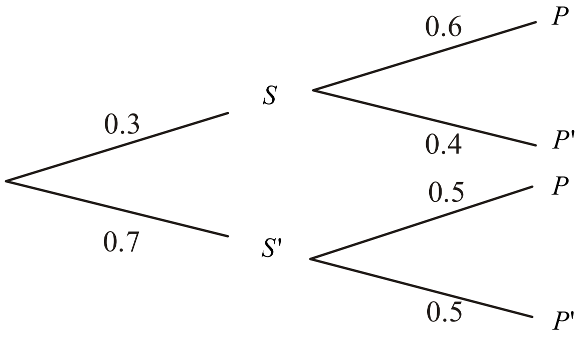
$$= \log_e(2) - \frac{7}{24} \text{ square units.}$$

(1 mark)

Question 11

a. Let $\Pr(\text{Harry sleeps well}) = S$
$\Pr(\text{Harry doesn't sleep well}) = S'$
$\Pr(\text{Harry plays well}) = P$
$\Pr(\text{Harry doesn't play well}) = P'$

Using a tree diagram,



(1 mark)

$$\begin{aligned}\Pr(P) &= \Pr(S \cap P) + \Pr(S' \cap P)\\ &= 0\cdot 3\times 0\cdot 6 + 0\cdot 7\times 0\cdot 5\\ &= 0\cdot 18 + 0\cdot 35\\ &= 0\cdot 53\end{aligned}$$

(1 mark)

b.

$$\Pr(S|P') = \frac{\Pr(S \cap P')}{\Pr(P')}$$ **(1 mark)**

$$\begin{aligned}&= \frac{0\cdot 3\times 0\cdot 4}{0\cdot 3\times 0\cdot 4 + 0\cdot 7\times 0\cdot 5}\\ &= \frac{0\cdot 12}{0\cdot 12 + 0\cdot 35}\\ &= \frac{0.12}{0.47}\\ &= \frac{12}{47}\end{aligned}$$

(1 mark)

Question 12

a. To show: $f(1) = g(1)$; that is the graphs intersect at the point where $x = 1$.

$$\begin{aligned} LS &= f(1) \\ &= (1-1)^2 \\ &= 0 \\ RS &= g(1) \\ &= \log_e(1) \\ &= 0 \\ &= LS \end{aligned}$$

So $a = 1$ as required.

(1 mark)

b. From the graph, $g(x) > f(x)$ for $x \in [1, b]$.

Let $h(x) = g(x) - f(x)$

$$h(x) = \log_e(x) - (x-1)^2$$

$$h'(x) = \frac{1}{x} - 2(x-1)$$

(1 mark)

$h(x)$ is a max/min when $h'(x) = 0$.

$$\frac{1}{x} - 2(x-1) = 0$$

$$\begin{aligned} \frac{1}{x} &= 2(x-1) \\ 1 &= 2x(x-1) \\ 0 &= 2x^2 - 2x - 1 \\ x &= \frac{2 \pm \sqrt{4 - 4 \times 2 \times -1}}{4} \\ &= \frac{2 \pm \sqrt{12}}{4} \\ &= \frac{2 \pm 2\sqrt{3}}{4} \\ &= \frac{1 \pm \sqrt{3}}{2} \end{aligned}$$

(1 mark)

Now $x = \dfrac{1-\sqrt{3}}{2}$

< 0

Since this is outside the domain of $x \in [1, b]$ we reject this value of x.

So the maximum occurs at $x = \dfrac{1+\sqrt{3}}{2}$.

(1 mark) including the rejection of $x = \dfrac{1-\sqrt{3}}{2}$

Total 40 marks

