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Trial Exam Paper

2008

MATHEMATICAL METHODS/ MATHEMATICAL METHODS (CAS)

Written examination 1

Worked solutions

PLEASE NOTE: The written examinations for Mathematical Methods 1 and Mathematical Methods 1 (CAS) are identical.

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details.

Question 1

a. Let $f(x) = e^{2x} \sin x$. Find $f'(x)$.

2 marks

Solution:

Using the product and chain rule

$$f'(x) = 2e^{2x} \sin x + e^{2x} \cos x$$
$$= e^{2x}(2\sin x + \cos x)$$

Mark allocation

- 1 method mark for using the product rule
- 1 answer mark for correct answer

b. If $f:(-\infty,3) \to R$, is such that $f'(x) = \frac{1}{x-3}$ and $f(2) = 4$. Find the rule for $f(x)$.

2 marks

Solution:

Antidifferentiate to obtain

$$f(x) = \int \frac{1}{x-3} dx$$
$$= \log_e |x-3| + c$$

(Note: be careful to include the absolute value signs!)

Then substitute when $x = 2$, $f(x) = 4$ to obtain

$$4 = \log_e |2-3| + c$$
$$4 = \log_e 1 + c$$
$$4 = 0 + c$$
$$c = 4$$

so $f(x) = \log_e |x-3| + 4$

Mark allocation

- 1 mark for correct antiderivative
- 1 mark for correct value of c

Question 2

a. Let $f(x) = 2x - 5$ and $g(x) = \sin x$. Evaluate $f(g(\frac{\pi}{6}))$.

2 marks

Solution:

First find $f(g(x))$.

$f(g(x)) = 2\sin x - 5$

then evaluate when $x = \frac{\pi}{6}$

to obtain

$$f(g(\frac{\pi}{6})) = 2\sin\frac{\pi}{6} - 5$$
$$= 2 \times \frac{1}{2} - 5$$
$$= -4$$

Mark allocation

- 1 mark for finding $f(g(x))$
- 1 mark for correct value of $f(g(\frac{\pi}{6}))$

b. The equation $y = 4x^3 + 8x^2 - 11x + 3$ can be written in the form $y = (x+3)(ax-b)^2$ where $a,b>0$. State the values of a and b.

2 marks

Solution:

By inspection $a^2x^3 = 4x^3$ so $a = \pm 2$

and $3b^2 = 3$ so $b = \pm 1$. As $a,b > 0$ $a = 2, b = 1$.

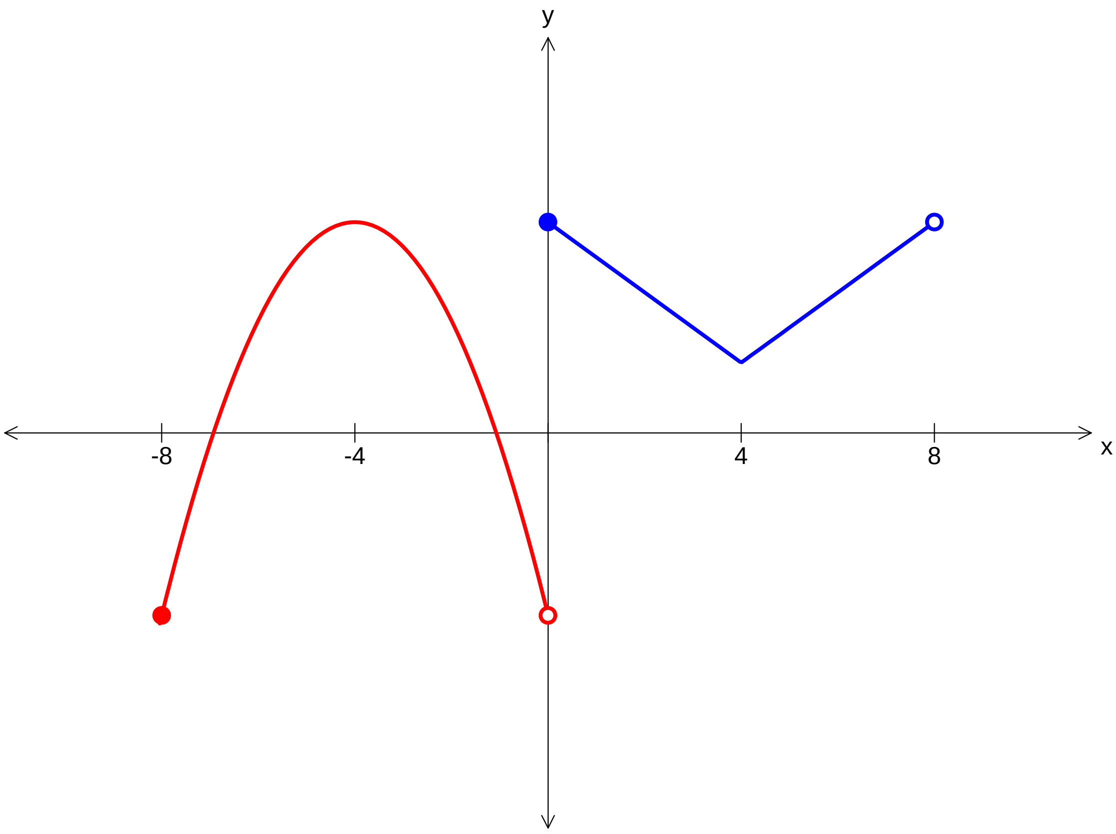
Mark allocation

- 1 mark for correct value of a
- 1 mark for correct value of b

Other methods can also be used such as expanding the brackets on the left hand side or factorising the cubic on the right hand side.

Question 3

The diagram shows the graph of a function with domain $[-8,8)$.



a. For what values of x is the graph of the function continuous?

1 mark

Solution:

The function is not continuous for $x = 0$. So the function is continuous everywhere in the domain of $[-8,8)$ except $x = 0$ i.e. $x \in [-8,8) \setminus \{0\}$.

Mark allocation

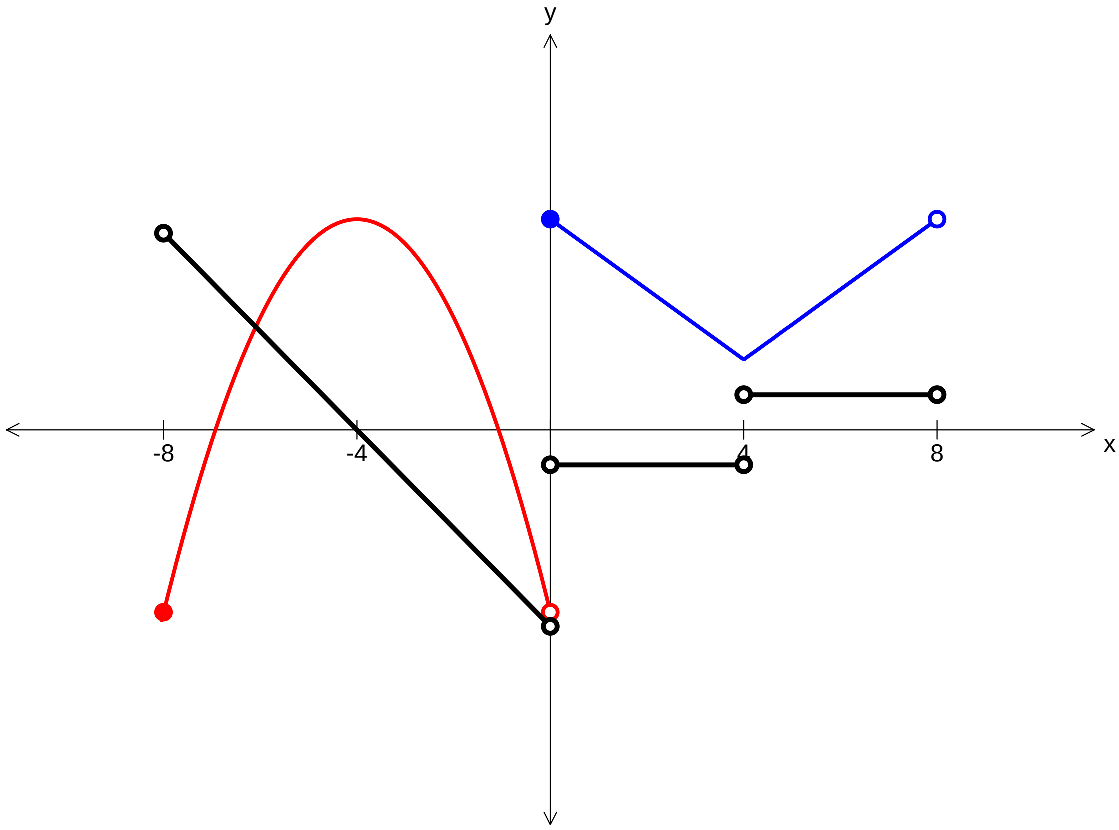
- 1 mark for correct answer

Note: answer can be written in any form. An alternative form is $[-8,0) \cup (0,8)$.

b. For the graph shown above, sketch on the same set of axes the graph of the derivative function.

2 marks

Solution:



Note: graphs are not differentiable at the endpoints of any section!

Mark allocation

- 1 mark for either section of the graph correct—must have correct endpoints
- 2 marks for both sections correct.

c. Write down the domain of the derivative function.

1 mark

Solution:

The graph is not differentiable at the endpoints, at the vertex point of the absolute value function or at any point that is not continuous. So the domain of the derivative function is $(-8,0)\cup(0,4)\cup(4,8)$

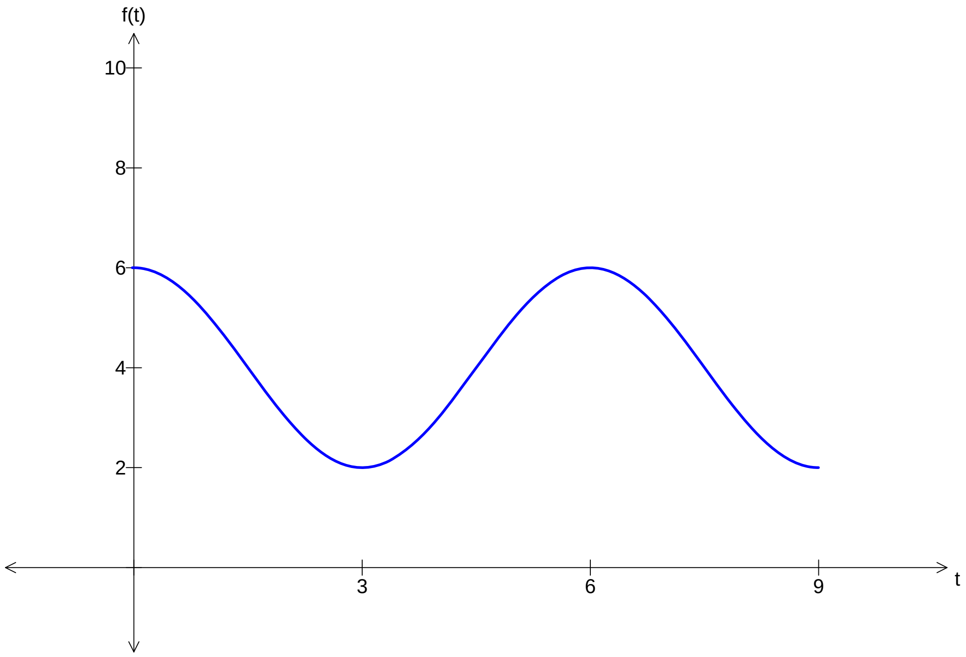
Mark allocation

- 1 mark for correct answer

Question 4

The graph shown is that of a function with the rule

$f:[0,9]\to R,\;\; f(t)=A\cos(nt)+\,b.$



a. Find A, n and b.

3 marks

Solution:

Graph is centred around $f(t)=4$ so $b=4$.

Amplitude is 2 so $A=2$

graph completes 1.5 cycles in 9 units so 1 cycle is completed in 6 units $\Rightarrow \dfrac{2\pi}{n}=6$

rearranging gives

$$2\pi = 6n$$

$$n=\frac{2\pi}{6}$$

$$n=\frac{\pi}{3}$$

Mark allocation

- 1 mark each for correct A, n and b.

b. Solve the equation $f(t)=3$ for $t\in[0,9]$.

2 marks

Solution:

Set $2\cos\frac{\pi t}{3}+4=3$

This gives—

$$2\cos\frac{\pi t}{3}=-1$$

$\cos\frac{\pi t}{3}=\frac{-1}{2}$, the first quadrant angle is $\frac{\pi}{3}$

and we are looking for an angle in 2nd and 3rd quadrant for $0\le t\le 9$

ie $0\le\frac{\pi t}{3}\le 3\pi$

$$\frac{\pi t}{3}=\pi-\frac{\pi}{3},\pi+\frac{\pi}{3},3\pi-\frac{\pi}{3}$$

$$\frac{\pi t}{3}=\frac{2\pi}{3},\frac{4\pi}{3},\frac{8\pi}{3}$$

$$t=2,4,8$$

Mark allocation

- 1 mark for setting $f(t)=3$
- 1 mark for the 3 correct solutions.

c. Let $g:[-1,8]\to R,\ g(t)=3f(t+1)-1$.

Find the smallest value of p such that $g(t)=p$ has exactly 2 solutions.

1 mark

Solution:

The graph of $g(t)$ is the graph of $f(t)$ with the y- values tripled and shifted to the left 1 unit and down 1 unit. This means the lowest point of the graph was 2 and is now $2\times3-1$. So if $p=5$ the line $y=p$ would touch the graph of $g(t)$ exactly twice in the domain.

f(t) = 2cosπt/3 + 4

f(t+1)=2cosπ/3(t+1) + 4

3f(t+1) =6cosπ/3(t+1) + 12

g(t) = 3f(t+1) – 1 =6cosπ/3(t+1) + 11

This has a minimum value of 5 twice in the given domain

Mark allocation

- 1 mark for $p=5$.

Question 5

If $f(x)=\log_e(x^2+1)$, then $f'(x)=\dfrac{2x}{x^2+1}$.

Use this fact to find an antiderivative of $\dfrac{(x+1)^2}{x^2+1}$.

2 marks

Solution:

$$\int \frac{(x+1)^2}{x^2+1}dx$$

$$=\int \frac{(x^2+2x+1)}{x^2+1}dx \text{ by expanding the brackets}$$

$$=\int \frac{x^2+1}{x^2+1}dx+\int \frac{2x}{x^2+1}dx \text{ by separating the terms}$$

$$=\int 1dx+\int \frac{2x}{x^2+1}dx$$

$$=x+\log_e(x^2+1)+c \text{ using result given}$$

Mark allocation

- 1 mark for expanding the bracket
- 1 mark for correct answer.

Question 6

A binomial random variable has a mean of 12 and a variance of 9.

Find the parameters *n* and *p*.

2 marks

Solution:

With a binomial random variable the mean is given as $\mu=np$ and the variance is $\sigma^2=npq$.
So

$$np=12 \quad -(1)$$

$$\text{and } npq=9 \quad -(2)$$

$$\Rightarrow 12q=9 \text{ substituting eqn (1) into eqn (2)}$$

$$\Rightarrow q=\frac{9}{12}=\frac{3}{4}$$

and so $p=\dfrac{1}{4}$.

Substituting into equation (1) gives

$$\frac{1}{4}n=12$$

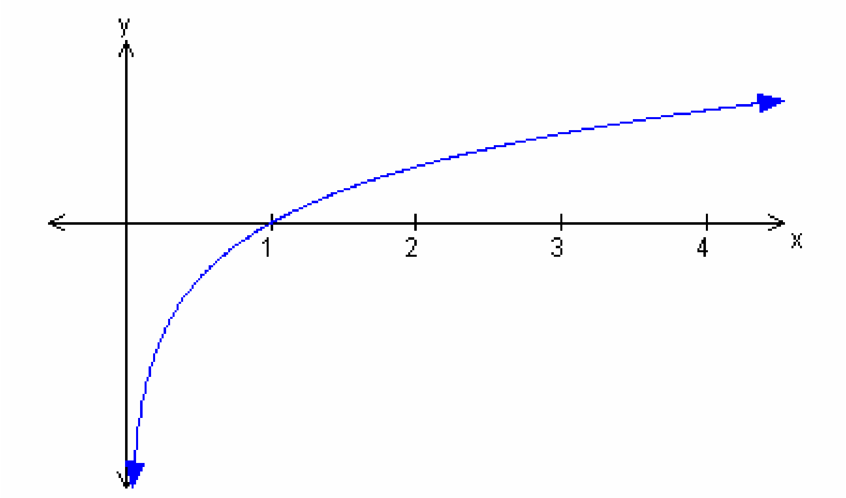
so $n=48$

Mark allocation

- 1 mark for correct value of *p*
- 1 mark for correct value of *n*.

Question 7

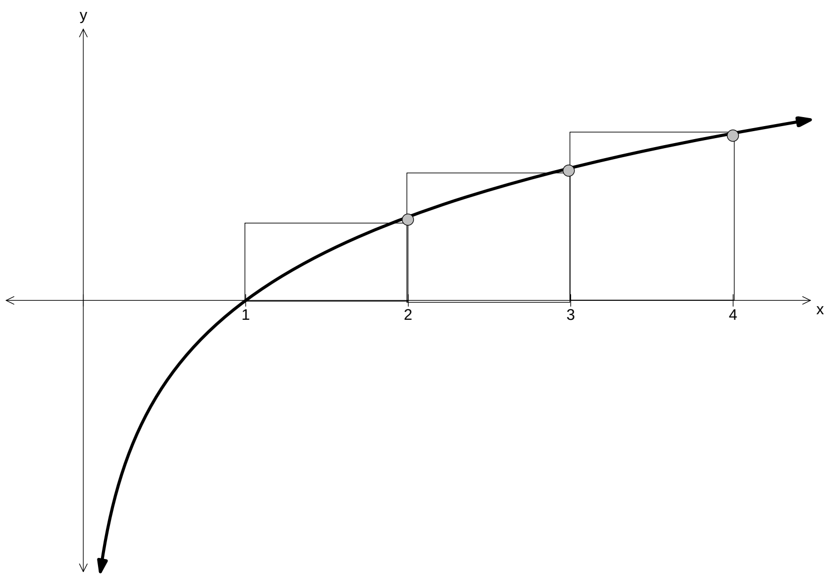
The given graph is of the function $y = \log_e(x)$



Using three right rectangles, the approximate area bounded by this function and the x axis between $x = 1$ and $x = 4$ is given by $\log_e B$. Find the value of *B*.

2 marks

Solution:



Using right rectangles the rectangles are anchored to the graph at $x = 2, x = 3, x = 4$. The widths of the rectangles are 1 unit and the heights are determined by the height of the function at $x = 2, x = 3, x = 4$.

Question 7 – continued
TURN OVER

So the area equals—

$$A = 1\times f(2) + 1\times f(3) + 1\times f(4)$$
$$= 1\times \log_e 2 + 1\times \log_e 3 + 1\times \log_e 4$$
$$= \log_e(2\times 3\times 4) \text{ using a log law}$$
$$= \log_e 24$$

$$so\ B = 24$$

Mark allocation

- 1 mark for heights of $\log_e 2, \log_e 3, \log_e 4$
- 1 mark for correct value of B.

Question 8

Two events A and B, from a given event space, are such that
$\Pr(A\cap B) = 0.18,\ \Pr(A) = x$ *and* $\Pr(B) = 2x$,

a. find x if $\Pr(A\cup B) = 0.78$.

2 marks

Solution:

$$\Pr(A) + \Pr(B) - \Pr(A\cap B) = \Pr(A\cup B) \text{ universal law for probability}$$
$$\Rightarrow x + 2x - 0.18 = 0.78$$
$$3x = 0.96$$
$$x = 0.32$$

Mark allocation

- 1 mark for setting up correct equation
- 1 mark for correct answer

b. find x if A and B are independent.

2 marks

Solution:

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$ if A and B are independent

$\Rightarrow\ 0.18 = x \times 2x$

$0.18 = 2x^2$

$0.09 = x^2$

$\pm 0.3 = x$

$x = 0.3\ \ as\ \ x > 0$

Mark allocation

- 1 mark for statement indicating A and B are independent
- 1 mark for answer

Question 9

a. The graph of g is obtained from the graph of the function f with rule $f(x) = e^{x+2}$ by a translation of $+3$ units parallel to the x-axis. Write down the rule for g.

1 mark

Solution:

Graph is moved to the right 3 units so replace x with x-*3* to get

$g(x) = e^{(x-3)+2} = e^{x-1}$

Mark allocation

- 1 mark for correct answer

b. The graph of h is obtained from the graph of g by a reflection in the y-axis. Write down the rule for h.

1 mark

Solution:

Graph is reflected in y-axis so replace x with $-x$ to get

$h(x) = e^{(-x-1)}$

Mark allocation

- 1 mark for answer—note this would be marked consequentially from previous answer—use previous answer (even if wrong) and place a negative sign in front of the x.

Question 9 – continued
TURN OVER

c. The graph of *k* is obtained from the graph of *h* by a dilation by a scale factor of $\frac{1}{3}$ from the *y*-axis. Write down the rule for *k*.

1 mark
1+1+1 = 3 marks

Solution:

Dilation of a factor $\frac{1}{3}$ from the *y*-axis means in the equation *x* is replaced by *3x*.

This gives $k(x) = e^{(-3x-1)}$

Mark allocation

- 1 mark for the answer—again mark consequentially from part b.

Question 10

The random variable *X* has the following probability distribution.

x	0	1	2
$\Pr(X = x)$	a	0.4	b

If the mean of X is 1.3, find the value of *a* and *b*.

3 marks

Solution:

$$\mu = E(X) = \sum x\,p(x)$$
$$= 0 \times a + 1 \times 0.4 + 2 \times b$$
$$= 0.4 + 2b$$
$$\Rightarrow 0.4 + 2b = 1.3$$
$$2b = 0.9$$
$$b = 0.45$$

$$and \sum p(x) = 1$$
$$so\ a + 0.4 + 0.45 = 1$$
$$a + 0.85 = 1$$
$$a = 0.15$$

Mark allocation

- 1 mark for setting up equation for μ
- 1 mark for value of b
- 1 mark for value of a.

Question 11

Find the exact area bounded by the curves $y = e^{-2x}$, $y = e^{-4x}$, the y-axis and the line $x = 1$.

3 marks

Solution:

$$Area = \int_0^1 (e^{-2x} - e^{-4x})dx$$

$$= \left[-\frac{1}{2}e^{-2x} + \frac{1}{4}e^{-4x} \right]_0^1$$

$$= (-\frac{1}{2}e^{-2} + \frac{1}{4}e^{-4}) - (-\frac{1}{2} + \frac{1}{4})$$

$$= \frac{1}{4} - \frac{1}{2}e^{-2} + \frac{1}{4}e^{-4} \ sq\ units$$

Mark allocation

- 1 mark for setting up integral—must have correct terminals, correct top and bottom curve and *dx*.
- 1 mark for correct antidifferentiation.
- 1 mark for correct answer.

CONTINUED
PLEASE TURN OVER

Question 12

Find the value of *k* for which $y = \frac{-1}{4}x + k$ is a normal to $y = 2x^2 - 8x$.

3 marks

Solution:

$$\text{gradient of line is } -\frac{1}{4}$$
$$\Rightarrow \text{gradient of normal is } 4$$
$$\text{so we want } \frac{dy}{dx} = 4$$
$$\frac{dy}{dx} = 4x - 8$$
$$\Rightarrow 4x - 8 = 4$$
$$4x = 12$$
$$x = 3$$
$$\text{if } x = 3 \text{ then } y = 2(3)^2 - 8(3) \text{ (using equation of curve)}$$
$$\text{gives } y = -6$$
$$\text{using the point } (3,-6) \text{ in equation } y = -\frac{1}{4}x + k$$
$$-6 = -\frac{1}{4} \times 3 + k$$
$$-6 = -\frac{3}{4} + k$$
$$k = -5\frac{1}{4}$$

Mark allocation

- 1 mark for recognising gradient of normal is 4
- 1 mark for finding the point (3, -6)
- 1 mark for correct value of *k*.

END OF SOLUTIONS BOOK