Year 2008

VCE

Mathematical Methods CAS

Trial Examination 2

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Victorian Certificate of Education
2008

STUDENT NUMBER

									Letter
Figures									
Words									

MATHEMATICAL METHODS CAS

Trial Written Examination 2

Reading time: 15 minutes
Total writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved **graphics** calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer booklet of 28 pages with a detachable sheet of miscellaneous formulas at the end of this booklet.
- Answer sheet for multiple choice questions.

Instructions

- Detach the formula sheet from the end of this book during reading time.
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, and sign your name in the space provided to verify this.
- All written responses must be in English.

At the end of the examination

- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1

Instructions for Section I

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
A correct answer scores 1 mark, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No mark will be given if more than one answer is completed for any question.

Question 1

The average value of the function $y = 3\sin^2(2x)$ over $0 \le x \le \frac{\pi}{2}$ is

A. 0.00472

B. 0.0019

C. 1.5

D. 4.5

E. $\frac{3\pi}{4}$

Question 2

The gradient of the curve $y = \left|\sin\left(x^2\right)\right|$ at $x = k$, is given by

A. $\lim\limits_{h \to 0} \frac{\left|\sin\left(k^2 + h\right) - \sin\left(k^2\right)\right|}{h}$

B. $\lim\limits_{h \to 0} \frac{\left|\sin\left(k^2 + h^2\right)\right| - \left|\sin\left(k^2\right)\right|}{h}$

C. $\lim\limits_{h \to 0} \frac{\left|\cos\left(k^2 + h^2\right) - \cos\left(k^2\right)\right|}{h}$

D. $\lim\limits_{h \to 0} \frac{\left|\cos(k + h)^2\right| - \left|\cos\left(k^2\right)\right|}{h}$

E. $\lim\limits_{h \to 0} \frac{\left|\sin\left((k + h)^2\right)\right| - \left|\sin\left(k^2\right)\right|}{h}$

Question 3

The function $f:[0,c] \to R$ has the rule $f(x) = c - 2x$, where c is a non-zero real constant. The inverse function is

A. $f^{-1}: R \to R$ where $f^{-1}(x) = \dfrac{1}{c-2x}$

B. $f^{-1}: \left[0, \dfrac{1}{c}\right] \to R$ where $f^{-1}(x) = c - \dfrac{1}{2x}$

C. $f^{-1}: \left[0, \dfrac{1}{c}\right] \to R$ where $f^{-1}(x) = \dfrac{1}{c-2x}$

D. $f^{-1}: [-c, c] \to R$ where $f^{-1}(x) = \dfrac{x-c}{2}$

E. $f^{-1}: [-c, c] \to R$ where $f^{-1}(x) = \dfrac{c-x}{2}$

Question 4

The graph of $y = \sqrt{x+3}$ is reflected in the y-axis, then translated 2 units to the left, and 3 units up. The equation of the new graph is

A. $y = \sqrt{1-x} + 3$

B. $y = \sqrt{5-x} + 3$

C. $y = -\sqrt{x+5} + 3$

D. $y = -\sqrt{5-x} - 3$

E. $y = -\sqrt{x+1} - 3$

Question 5

The number of frogs N, in a colony, varies with time according to the rule $N(t) = 20e^{0.2t}$, where t is the time in months, and $t \geq 0$. The average rate of change in the number of frogs over the first three months is closest to

A. 36.4

B. 12.1

C. 5.5

D. 7.3

E. 2.4

Question 6

If $f(x)=x^3$ $g(x)=\cos(x)$ and $h(x)=\sqrt{x}$ then $\cos^3(\sqrt{x})$ is equal to

A. $g(f(h(x)))$

B. $f(g(h(x)))$

C. $h(g(f(x)))$

D. $f(h(g(x)))$

E. $g(h(f(x)))$

Question 7

Using a linear approximation, with $f(x)=\sqrt[3]{x}$, then $\sqrt[3]{-63.5}$ is

A. $f(64)-0.5f'(64)$

B. $f(-64)+0.5f'(-64)$

C. $f(-64)-0.5f'(-64)$

D. $f(4)+0.5f'(4)$

E. $f(-4)-0.5f'(-4)$

Question 8

If $f(x)$ and $g(x)$ are two differentiable functions with

$f'(x)=\frac{d}{dx}(f(x))$ and $g'(x)=\frac{d}{dx}(g(x))$, then $\frac{d}{dx}(f(g(x)))$ is equal to

A. $f'(g(x))$

B. $f'(g'(x))$

C. $f'(g(x))+f(g'(x))$

D. $g'(x)f'(g(x))$

E. $f'(x)g'(x)$

Question 9

Which of the following is **false** for the graph of the function

$f:[0,\pi]\to R$ where $f(x)=300\tan(3x)$?

A. The graph crosses the x-axis at $x=0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

B. The graph has asymptotes at $x=\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

C. The range is $[-300, 300]$

D. The graph has three cycles in $[0,\pi]$

E. The graph has a domain of $[0,\pi]$

Question 10

If a is a non-zero real constant, then the range of the function

$f:[0,a]\to R$, $f(x)=1-a\sin\left(\frac{\pi x}{a}\right)$ is

A. $[1-a, a]$

B. $[1, 1-a]$

C. $[1-a, 1+a]$

D. $[-a, 1]$

E. $[0, 2]$

Question 11

For the graph of $y=x^4-4x^2$, the subset for which the gradient is negative is given by

A. $(-2, 2)$

B. $(-\infty,-2)\cup(2,\infty)$

C. $(-\infty,-1.414)\cup(0,1.414)$

D. $(-\sqrt{2},0)\cup(\sqrt{2},\infty)$

E. $(-\infty,-\sqrt{2})\cup(0,\sqrt{2})$

Question 12

Two simultaneous linear equations are $2x-3y=q$ and $px+6y=10$.
Which of the following statements is **false?**

A. If $p=-4$ and $q \in R$ there is no unique solution.

B. If $p \neq -4$ and $q \in R$ there is a unique solution.

C. If $p=-4$ and $q=-5$ then there is an infinite number of solutions.

D. If $p=-4$ and $q \neq -5$ then there is more than one solution.

E. If $p=-4$ and $q \neq -5$ then there is no solution.

Question 13

If $\int_0^a f(x)\,dx = A$, then $\int_0^a (1-f(x))\,dx$ is equal to

A. $x-A$

B. $x+A$

C. $a-A$

D. $A-a$

E. $a+A$

Question 14

A certain curve has its gradient given by $4\sin(2x)$. If the curve crosses the *x*-axis at $x=\frac{5\pi}{3}$, then it crosses the *y*-axis at

A. -4

B. 4

C. -3

D. -1

E. $-\sqrt{3}$

Question 15

Consider the graphs of $y = kx$ and $y = x^2 + bx + c^2$ where *b*, *c* and *k* are all real numbers. Which of the following statements is **false?**

A. If $k = b + 2c$, the graph of $y = kx$ is a tangent to the graph of $y = x^2 + bx + c^2$.

B. If $k = b - 2c$, the graph of $y = kx$ touches the graph of $y = x^2 + bx + c^2$.

C. If $b + 2c < k < b - 2c$, the graph of $y = kx$ does not intersect the graph of $y = x^2 + bx + c^2$.

D. If $k > b + 2c$ the graph of $y = kx$ intersects the graph of $y = x^2 + bx + c^2$ at two distinct points.

E. If $k < b - 2c$ the graph of $y = kx$ intersects the graph of $y = x^2 + bx + c^2$ at two distinct points

Question 16

Let $f : R \to R$ be a differentiable function such that

- $f'(x) = 0$ at $x = -1$ and $x = 1$
- $f'(x) < 0$ for $x < -1$ and $-1 < x < 1$
- $f'(x) > 0$ for $x > 1$

Then which of the following is most correct?

A. The graph has a stationary point of inflexion at $x = -1$ and a maximum at $x = 1$.

B. The graph has a stationary point of inflexion at $x = -1$ and a minimum at $x = 1$.

C. The graph has a stationary point of inflexion at $x = 1$ and a maximum at $x = -1$.

D. The graph has a maximum at $x = -1$ and a minimum at $x = 1$.

E. The graph has a minimum at $x = -1$ and a maximum at $x = 1$.

Question 17

For two events *A* and *B*, $\Pr(A) = \Pr(B) = p$ where $0 < p < 1$
Which of the following statements is **false?**

A. If $\Pr(A \cap B) = \dfrac{p}{2}$, then $\Pr(\overline{A} \cap \overline{B}) = \dfrac{2-3p}{2}$

B. If *A* and *B* are mutually exclusive, then $\Pr(A \cup B) = 2p$

C. If *A* and *B* are independent, then $\Pr(A \cup B) = p(2-p)$

D. If *A* and *B* are independent, then $\Pr(A/B) = p$

E. If *A* and *B* are mutually exclusive, then $\Pr(A/B) = \dfrac{1}{2}$

Question 18

The discrete random variable *X* has the following probability distribution where $0 < a < 1$ and $0 < b < 1$.

X	-1	1
$\Pr(X = x)$	a	b

$VAR(X)$ is equal to

A. $b^2 - a^2$

B. $(b-a)^2$

C. 0

D. $4a(1-a)$

E. $2a(1-a)$

Question 19

The continuous random variable X has a probability density function given by

$$f(x)=\begin{cases} k(a-x) & \text{for } 0\le x\le a \\ k(a+x) & \text{for } -a\le x\le 0 \\ 0 & \text{elsewhere}\end{cases}$$

If $a>0$ and $k>0$, then

A. $k=\dfrac{1}{a^2}$

B. $k=\dfrac{2}{a^2}$

C. $k=\dfrac{1}{2a^2}$

D. $k=\dfrac{1}{2a}$

E. $k=\dfrac{2}{a}$

Question 20

If Z has the standard normal distribution and $\Pr(|Z|<c)=a$, where $0<c<3$ and $0<a<1$, then $\Pr(Z\ge -c)$ is equal to

A. $\dfrac{1+a}{2}$

B. $\dfrac{1-a}{2}$

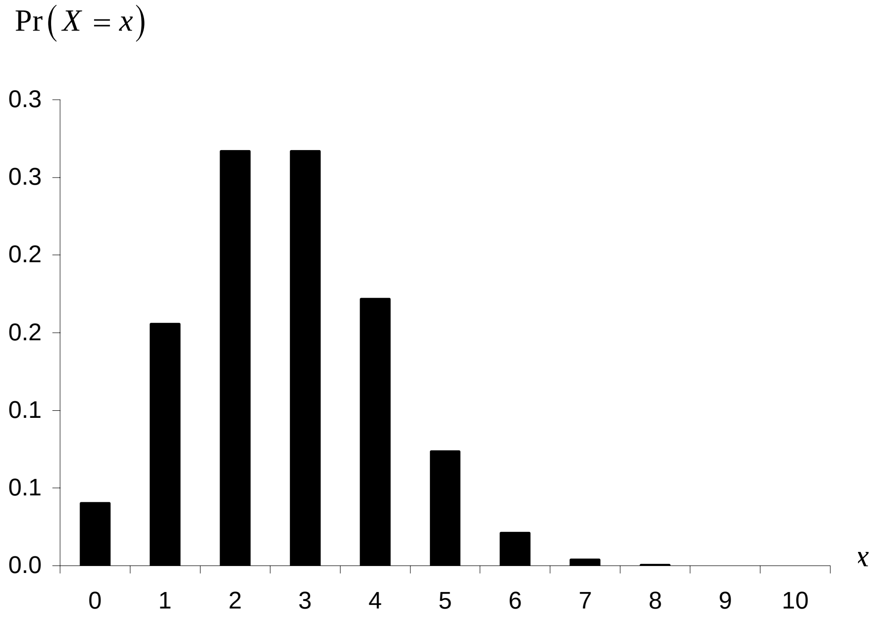
C. $\dfrac{1+2a}{2}$

D. $\dfrac{1-2a}{2}$

E. $1-\dfrac{a}{2}$

Question 21

The probability distribution of a binomial random variable X is shown graphically below. If the number of trials is n and $n = 10$ and p is the probability of a success on any one trial, then the most likely value for p is equal to



A. 0.3

B. 0.4

C. 0.5

D. 0.6

E. 0.7

Question 22

A box contains b black balls and r red balls. Two balls are drawn from the box at random without replacement. The probability that one is red and one is black, is given by

A. $\dfrac{br}{(b+r)^2}$

B. $\dfrac{2br}{(b+r)^2}$

C. $\dfrac{br}{(b+r)(b+r-1)}$

D. $\dfrac{2br}{(b+r)(b+r-1)}$

E. $\dfrac{2br}{(b+r-1)^2}$

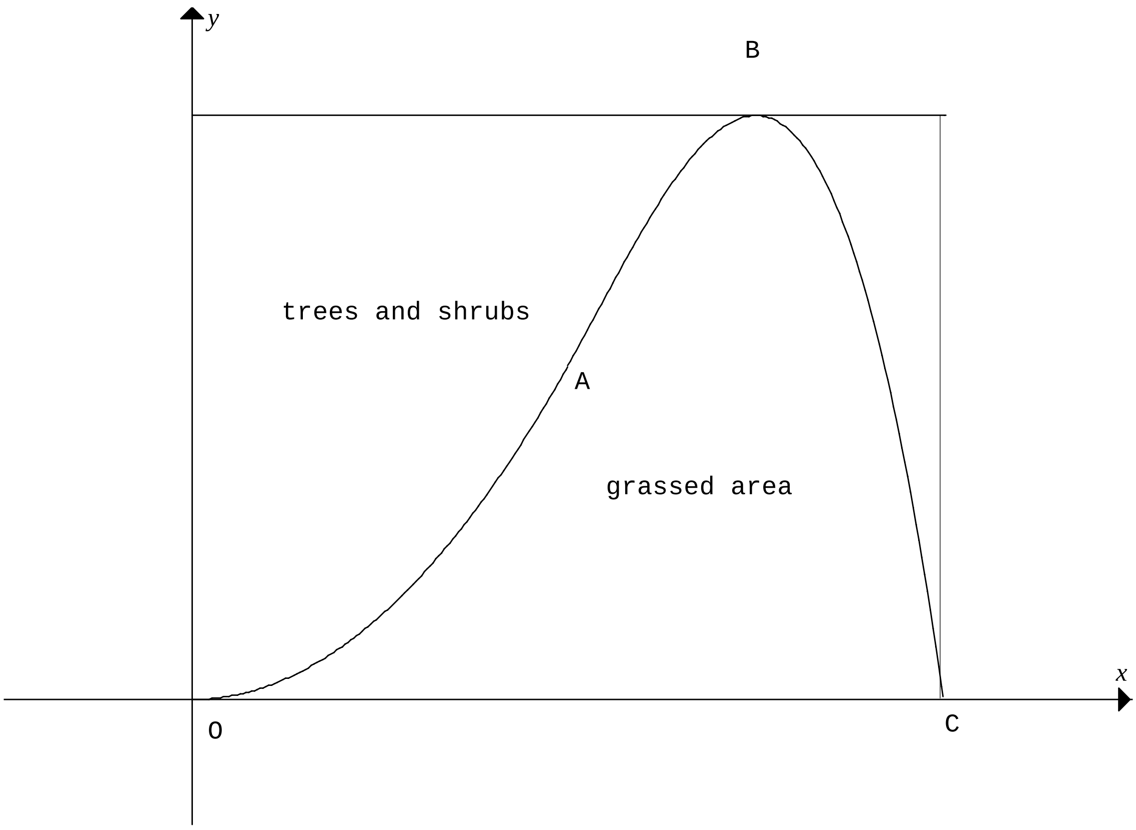
END OF SECTION 1

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Question 1



Jim is a keen gardener and is working on the plan for the landscape of a section of his new back garden. This section of land is a rectangular block starting from the back line of his house, which lies along the x-axis and the left side of his fence which is the y-axis. The right side of this section of garden is the line through $x = 4$ at the point C. Distances are measured in metres and the origin O is shown. The boundary, separating the grassed area (the area between the curve and the x-axis) from the trees and shrubs, is made of two curves, by the function

$$f(x) = \begin{cases} x^2 & \text{for } 0 \le x \le 2 \\ ax^3 + bx^2 + cx + d & \text{for } 2 \le x \le 4 \end{cases}$$

where a, b c and d are constants.

The point B is halfway between the points A and C and is the furthest point reached by boundary and the garden block as measured in the positive *y*-direction. Jim likes this design as the join at A, $x = 2$ where the two curves meet, is smooth.

a. Use the above information to write down four equations which could be used to find the values of *a*, *b* *c* and *d*.

4 marks

b. Solve the equations to show that $a = -2$, $b = 13$, $c = -24$ and $d = 16$.

2 marks

c. Jim wants to find an approximation to the area of the grassed region by using rectangles of width one metre. Show that if he uses either left or right (end-point) rectangles he obtains the same value. Using this approximation, what is the ratio of the grassed area to the area of the trees and shrubs?

3 marks

d. **i.** Write an expression for the total area of the grassed region using definite integrals.

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ii. Find the exact area of the grassed region.

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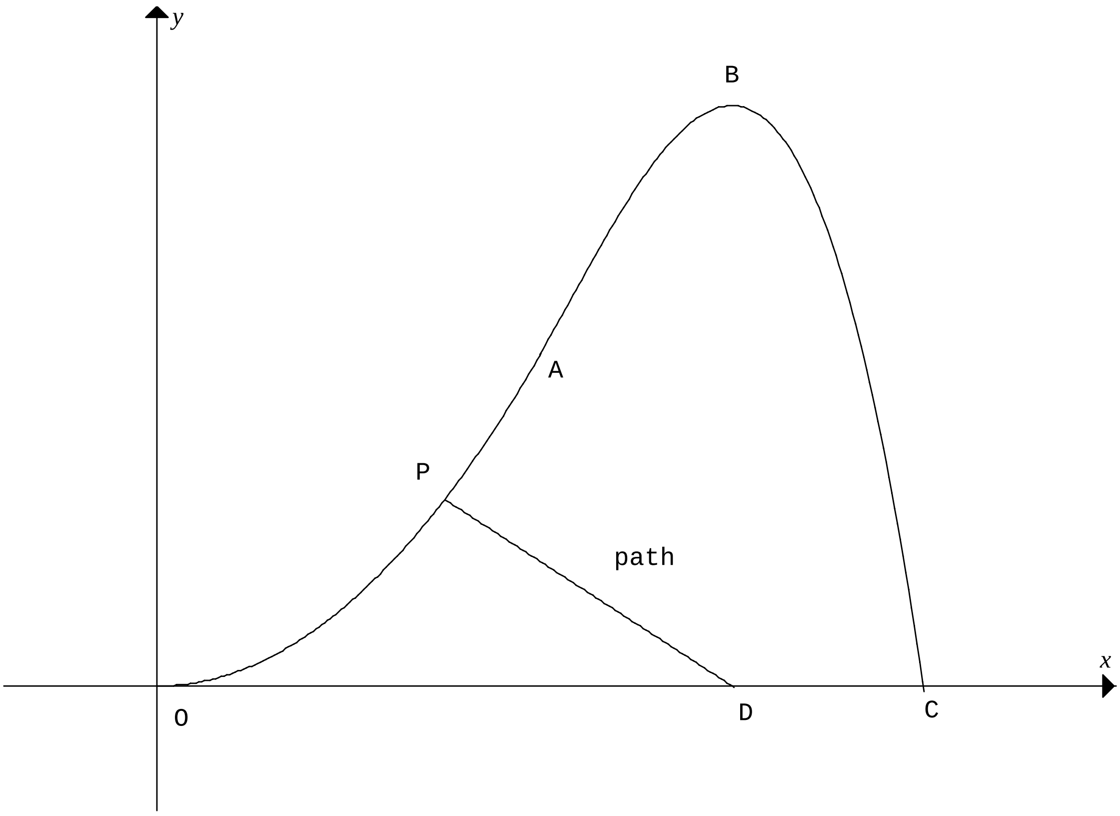
1 + 1 = 2 marks



e. Jim's back door to his house is located at a point D, directly in line with the point B. He now decides to lay stepping stones, in a straight line path from his door to meet a point P on the boundary of the grassed region, as shown above on page 16.

i. If the *x*-coordinate of P is p and $0 < p < 2$, find an expression in terms of p, for the length of the path from his back door to the boundary.

ii. If Jim wants this path to be as short as possible, find the value of p and the exact length of the path. (You are not required to justify the nature of the stationary point.)

2 + 3 = 5 marks
Total 16 marks

Question 2

Consider the function $f:[0,4\pi]\to R,\ f(x)=2x+8\sin\left(\frac{x}{2}\right)$.

a. **i.** Find $f'(x)$

ii. Find the coordinates on the graph of f where the gradient is a maximum.

1 + 2 = 3 marks

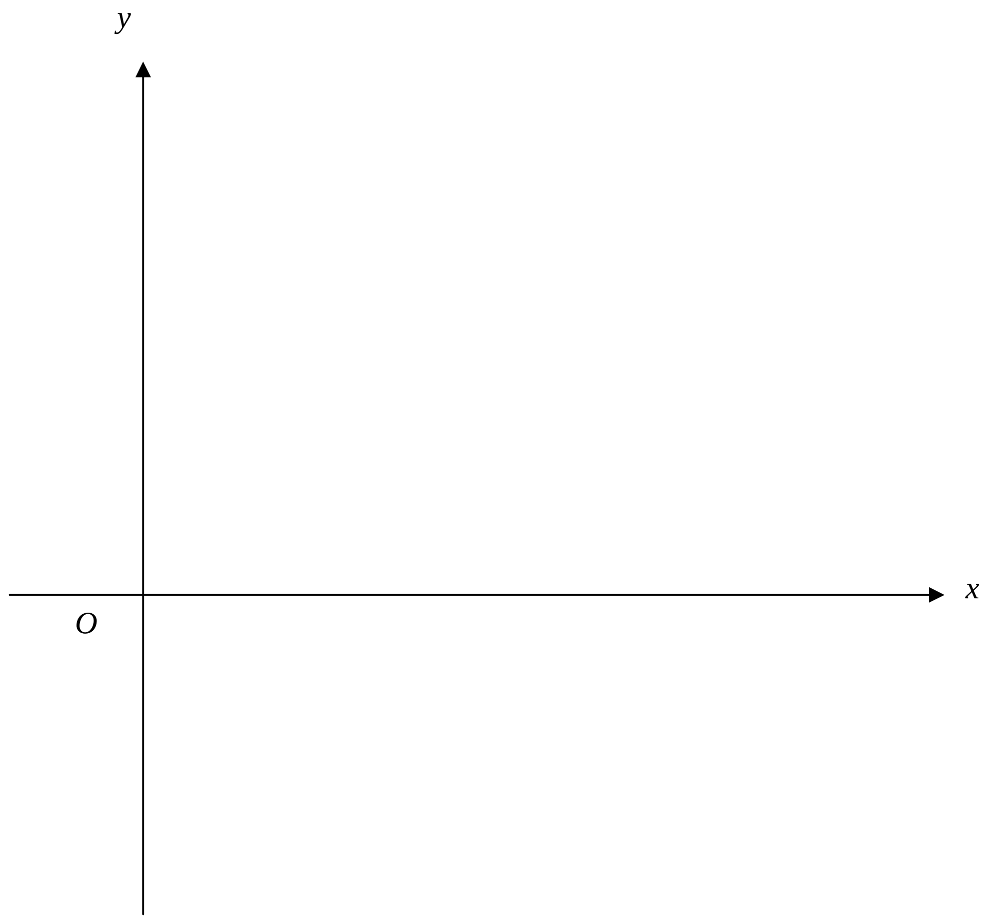
b. Find the coordinates of all the turning points on the graph of f.
(The exact values must be given. You are not required to justify the nature of the turning points.)

2 marks

c. Find the equation of the tangent to the curve $y = f(x)$ at the point where $x = 2\pi$.
(Exact values must be given.)

2 marks

d. Sketch the graph of $y = f(x)$ on the axes below, along with the tangent found in **c.** Clearly label the scale.



2 marks

e. Let $h: R \to R$, $h(x) = 2x + 8\sin\left(\frac{x}{2}\right)$.

Find the general solution of $h'(x) = 0$

1 mark

Total 10 marks

Question 3

The continuous random variable T, the time in years of the lifetime of one type of a car battery, has a probability density function f with the rule

$$f(t) = \begin{cases} \dfrac{81}{2(2t+1)^3} & \text{if } 1 \le t \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a. Sketch the graph of $y = f(t)$ on the axes provided, clearly labelling the scale.

1 mark

b. Find, the exact probability that a car battery lasts longer than three years if has lasted for at least two years.

3 marks

c. A household has four cars with each car having this type of car battery. Find the probability, correct to three decimal places, that, in at least one of the cars, the battery lasts longer than two years.

2 marks

d. Find, correct to two decimal places, the expected lifetime in years of this type of car battery.

2 marks

e. Find correct to one decimal place, the median time in years for the lifetime of this car battery.

3 marks

f. Another more expensive type of car battery, follows a normal distribution and it is found that 18% of these last for a period of less than 25 months, while only 4% last longer than 57 months. Find the mean and standard deviation of the lifetime of this battery, giving your answers to the nearest month.

4 marks
Total 15 marks

Question 4

Given the function $f : D \to R$, $f(x) = 3\log_e\left(1+\frac{x}{2}\right)$,

a. Find D, which is the maximal domain of the function f.

1 mark

b. Show that the graph of f has no turning points.

1 mark

c. State a sequence of three transformations including scale factors, which takes the graph of $y = \log_e(x)$ to the graph of f.

3 marks

d. For $f(x) = 3\log_e\left(1+\frac{x}{2}\right)$

i. If $f(u-2)+f(v-2)=f(auv+b)$, where u and v are positive real numbers, find the values of a and b.

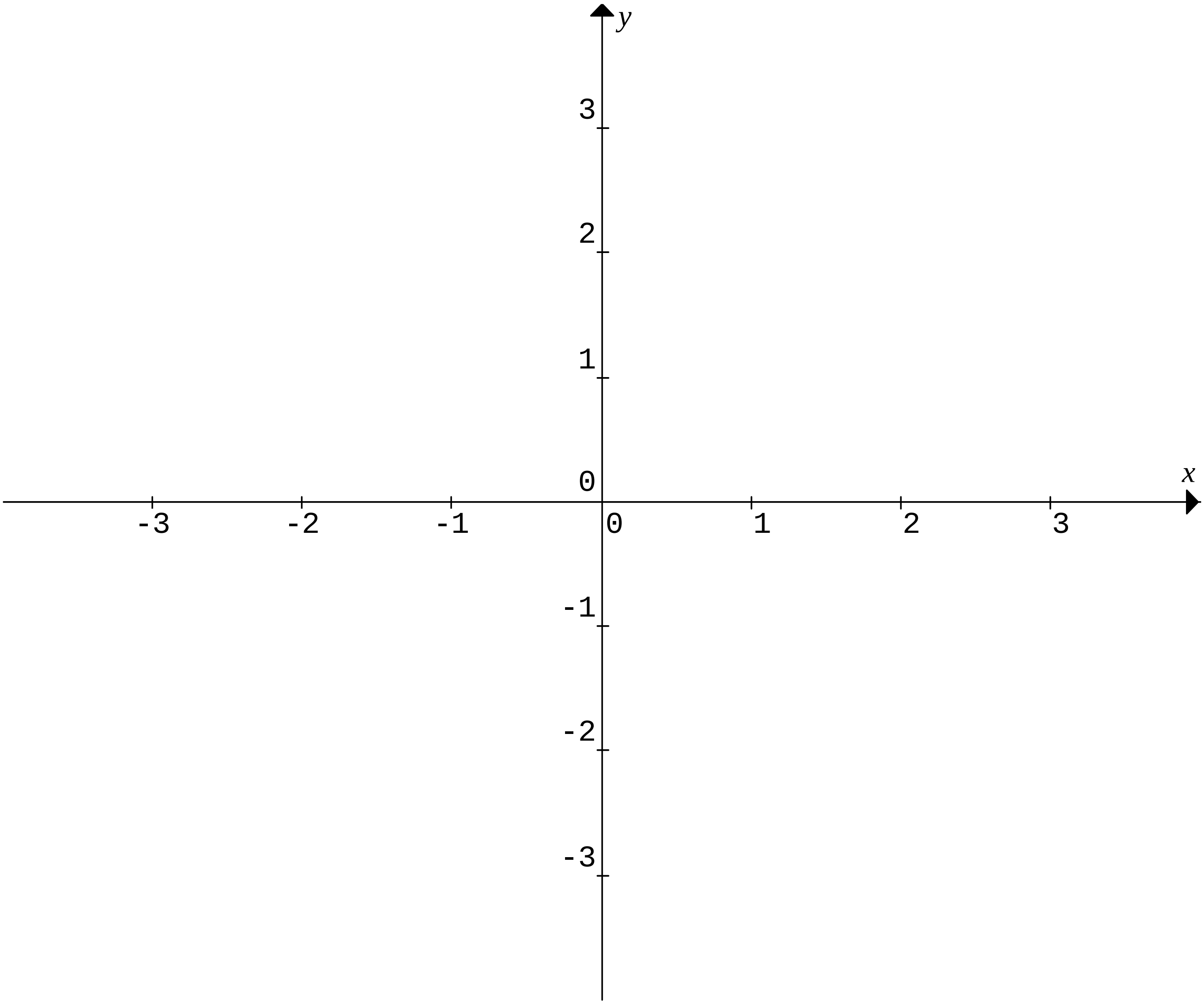
ii. For what values of u does $f(u)+f(-u)=f\left(-\frac{u^2}{2}\right)$ hold?

2 + 2 = 4 marks

e. Find the inverse function f^{-1}.

2 marks

f. Sketch the graphs of f and f^{-1} on the axes below, clearly labelling the graphs, stating any axial intercepts and giving the equations of all asymptotes.



2 marks

g. One of the coordinates of the points of intersection between the graphs of f and f^{-1}, is given by (p, p) where $p > 0$.

i. Find the value of p, correct to three decimal places.

ii. Show and explain why the area of the region enclosed by the graph of f, the x-axis and the line $x = p$ is given by $p^2 - p$.

1 + 3 = 4 marks
Total 17 marks

END OF EXAMINATION

MATHEMATICAL METHODS CAS

Written examination 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time. This formula sheet is provided for your reference.

Mathematical Methods and CAS Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of triangle:	$\frac{1}{2}bc\sin(A)$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c\ ,\ n \neq -1$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$$

$$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$$

$$\int \frac{1}{x} dx = \log_e|x| + c$$

$$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$$

$$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$$

$$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + h\,f'(x)$

Probability

$$\Pr(A) = 1 - \Pr(A')$$

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$\Pr(A/B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Mean: $\mu = E(X)$

variance: $\mathrm{var}(X) = \sigma^2 = E\left((X-\mu)^2\right) = E\left(X^2\right) - \mu^2$

probability distribution		mean	variance
discrete	$\Pr(X = x) = p(x)$	$\mu = E(X)$	$\sigma^2 = \sum (x-\mu)^2\,p(x)$
Continuous	$\Pr(a < X < b) = \int_a^b f(x)\,dx$	$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\,f(x)\,dx$

ANSWER SHEET

STUDENT NUMBER

Figures										Letter
Words										

SIGNATURE ____

SECTION 1

1	**A**	**B**	**C**	**D**	**E**
2	**A**	**B**	**C**	**D**	**E**
3	**A**	**B**	**C**	**D**	**E**
4	**A**	**B**	**C**	**D**	**E**
5	**A**	**B**	**C**	**D**	**E**
6	**A**	**B**	**C**	**D**	**E**
7	**A**	**B**	**C**	**D**	**E**
8	**A**	**B**	**C**	**D**	**E**
9	**A**	**B**	**C**	**D**	**E**
10	**A**	**B**	**C**	**D**	**E**
11	**A**	**B**	**C**	**D**	**E**
12	**A**	**B**	**C**	**D**	**E**
13	**A**	**B**	**C**	**D**	**E**
14	**A**	**B**	**C**	**D**	**E**
15	**A**	**B**	**C**	**D**	**E**
16	**A**	**B**	**C**	**D**	**E**
17	**A**	**B**	**C**	**D**	**E**
18	**A**	**B**	**C**	**D**	**E**
19	**A**	**B**	**C**	**D**	**E**
20	**A**	**B**	**C**	**D**	**E**
21	**A**	**B**	**C**	**D**	**E**
22	**A**	**B**	**C**	**D**	**E**