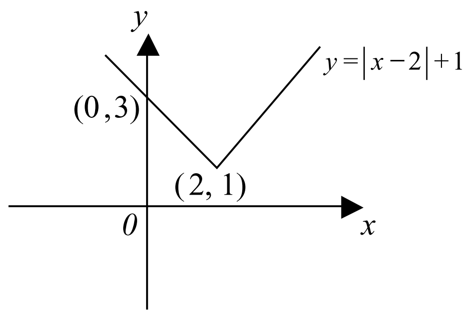
The Mathematical Association of Victoria
MATHEMATICAL METHODS AND MATHEMATICAL METHODS(CAS)
2008 Trial written examination 1 – Solutions

Question 1

a.



Shape **[1A]**

Cusp and intercept with correct coordinates. **[1A]**

b. $|x-2|+1=2$

$x-2+1=2$

$x=3$ **[1A]**

OR

$-x+2+1=2$

$x=1$ **[1A]**

c. f is not a one to one function.

Hence f^{-1} does not exist. **[1A]**

Question 2

a. $3\log_2(2x) - 2\log_2(x) = 1$

$\log_2(2x)^3 - \log_2(x)^2 = \log_2(2)$

$\log_2\left(\dfrac{8x^3}{x^2}\right) = \log_2(2)$ **[1M]**

$\log_2(8x) = \log_2(2)$

$8x = 2$

$x = \dfrac{1}{4}$ **[1A]**

b. $A(t) = 8e^{-\frac{t}{10}}$

When the area is halved,

$4 = 8e^{-\frac{t}{10}}$

$e^{-\frac{t}{10}} = \dfrac{1}{2}$ **[1M]**

$-\dfrac{t}{10} = \log_e\left(\dfrac{1}{2}\right)$

$t = -10\log_e\left(\dfrac{1}{2}\right)$ **[1A]**

Alternatively
$t = 10\log_e(2)$

Question 3

a. $y = \frac{\tan(x)}{x}$

Apply the quotient rule. **[1M]**

Let $u = \tan(x)$ and $v = x$

$\frac{du}{dx} = \sec^2(x)$ and $\frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{x\sec^2(x) - \tan(x)}{x^2}$$ **[1A]**

(or an equivalent form of the above)

b. $f(x) = \sin(x)$ and $g(x) = \log_e(x)$

$f(g(x)) = \sin(\log_e(x))$ **[1A]**

Apply the chain rule

$$\frac{d}{dx}\left(f\left(g(x)\right)\right) = \cos\left(\log_e(x)\right) \times \frac{1}{x}$$

$$\begin{aligned}\frac{d}{dx}\left(f\left(g(1)\right)\right) &= \cos\left(\log_e(1)\right)\\ &= \cos(0)\\ &= 1\end{aligned}$$ **[1A]**

Question 4

a. $f:[0, a] \rightarrow R, f(x) = 2\cos(\pi x) - 1$

Period $= \dfrac{2\pi}{\pi} = 2$

To cover one period the domain must be [0, 2].

Hence $a = 2$. **[1M]**

b. $f(2x) = 0$

$2\cos(\pi x) - 1 = 0$

$\cos(2\pi x) = \dfrac{1}{2}$

$2\pi x = \dfrac{\pi}{3} \text{ or } \dfrac{5\pi}{3} \text{ or } \dfrac{7\pi}{3} \text{ or } \dfrac{11\pi}{3} \text{ or } \dfrac{13\pi}{3} \ldots$

$x = \dfrac{1}{6} \text{ or } \dfrac{5}{6} \text{ or } \dfrac{7}{6} \text{ or } \dfrac{11}{6} \text{ or } \dfrac{13}{6} \ldots$ **[1M]**

Since $x \in [0, 2]$,

$x = \dfrac{1}{6} \text{ or } \dfrac{5}{6} \text{ or } \dfrac{7}{6} \text{ or } \dfrac{11}{6}$ **[1A]**

Question 5

a. Let X be the number of students using a Tasmania Implements calculator.

$$X \sim Bi\left(3, \frac{3}{4}\right)$$

$$\Pr\left(X=2\right) = {}^{3}C_{2}\left(\frac{3}{4}\right)^{2}\left(\frac{1}{4}\right)$$ **[1M]**

$$\Pr\left(X=2\right) = {}^{3}C_{2}\left(\frac{3}{4}\right)^{2}\left(\frac{1}{4}\right)$$

$$= 3 \times \frac{9}{16} \times \frac{1}{4}$$

$$= \frac{27}{64}$$ **[1A]**

b. $$\Pr(X=2 \mid X \geq 1) = \frac{\Pr\left(X=2 \cap X \geq 1\right)}{\Pr\left(X \geq 1\right)}$$

$$= \frac{\Pr\left(X=2\right)}{1-\Pr\left(X=0\right)}$$ **[1M]**

$$\Pr(X=2 \mid X \geq 1) = \frac{\frac{27}{64}}{1-\left(\frac{1}{4}\right)^{3}}$$

$$= \frac{\frac{27}{64}}{\frac{63}{64}}$$

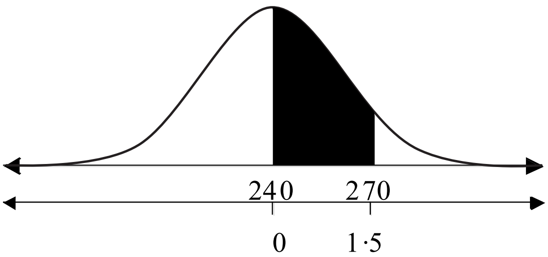
$$= \frac{27}{63} = \frac{3}{7}$$ **[1A]**

Question 6

a. $\Pr(240 < X < 270) = \Pr(0 < Z < 1.5)$ **[1M]**

$= 0.5 - 0.067$

$= 0.433$ **[1A]**



b. $\int_0^1 \left(x^{\frac{4}{3}}\right) dx = \left[\frac{3}{7}x^{\frac{7}{3}}\right]_0^1$ **[1M]**

$= \frac{3}{7}$

For a probability density function,

$\int_{-\infty}^{\infty} f(x)\,dx = 1$. Therefore

$\int_1^a (x)dx = \frac{4}{7}$ **[1M]**

$\left[\frac{x^2}{2}\right]_1^a = \frac{4}{7}$

$\frac{a^2}{2} - \frac{1}{2} = \frac{4}{7}$ **[1M]**

$a^2 = \frac{15}{7}$

$a = \sqrt{\frac{15}{7}}$ **[1A]**

Question 7

a. $V = \frac{1}{3}\pi r^2 h$

However, $r = \frac{h}{2}$

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$ … **equation 1**

When $h = 6$ m

$V = \frac{1}{3}\pi\left(\frac{6}{2}\right)^2 \times 6$

Cancel as appropriate to get
$V = 18\pi \text{ m}^3$ **[1A]**

b. $\frac{dV}{dt} = 2 \text{ m}^3\text{/s}$ and require $\frac{dh}{dt}$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ … **equation 2** **[1M]**

From **equation 1** in **a.** above,

$V = \frac{1}{12}\pi h^3$, therefore

$\frac{dV}{dh} = \frac{1}{4}\pi h^2$ … **equation 3**

Substitute **equation 3** in **equation 2**

$2 = \frac{1}{4}\pi h^2 \times \frac{dh}{dt}$ **[1M]**

$\frac{dh}{dt} = \frac{8}{\pi h^2}$

When $h = 6$ m

$\frac{dh}{dt} = \frac{8}{36\pi} \text{ m/s} = \frac{2}{9\pi} \text{ m/s}$ **[1A]**

Question 8

a. $f(x) = (\sin(x))^3$

Apply the chain rule

$f'(x) = 3(\sin(x))^2 \times \cos(x)$ **[1M]**

$f'(x) = 3\sin^2(x)\cos(x)$ **[1A]**

b. i. $\frac{d}{dx}\left(\sin^3(x)\right) = 3\cos(x)\sin^2(x)$

Take $\int dx$ of both sides of the equation

$\sin^3(x) + c_1 = 3\int\left(\cos(x)\sin^2(x)\right)dx$

$\int\left(\cos(x)\sin^2(x)\right)dx = \frac{1}{3}\sin^3(x) + c$ **[1A]**

(c and c_1 are real constants)

ii. The gradient of the tangent is given by

$$f'\left(\frac{\pi}{4}\right) = 3\sin^2\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)$$

$$= 3 \times \left(\frac{1}{\sqrt{2}}\right)^2 \times \frac{1}{\sqrt{2}}$$

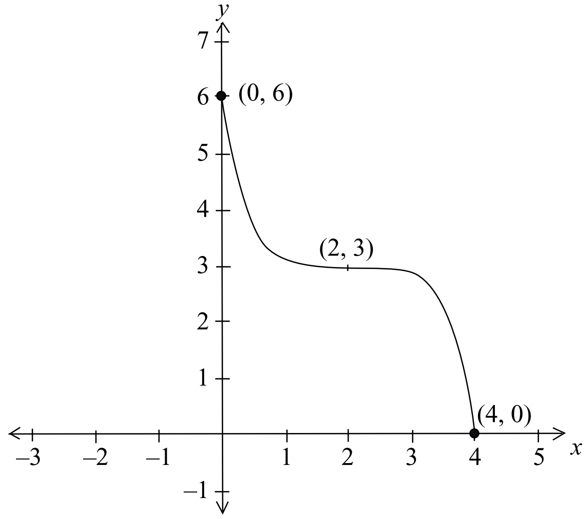
$$= \frac{3}{2\sqrt{2}}$$ **[1M]**

Gradient of normal $= -\dfrac{1}{f'\left(\frac{\pi}{4}\right)}$

$$= -\frac{2\sqrt{2}}{3}$$ **[1A]**

Question 9

a. $y = -3\left(\frac{1}{2}x - 1\right)^5 + 3$



Correct shape with (2, 3) **[1A]**

Correct intercepts (0, 6) and (4, 0) and closed circles **[1A]**

b. $\int_0^4 \left(-3(\frac{1}{2}x - 1)^5 + 3\right) dx$ **[1M]**

$= \left[-\left(\frac{1}{2}x - 1\right)^6 + 3x\right]_0^4$ **[1M]**

$= -1 + 12 + 1$

$= 12 \text{ units}^2$ **[1A]**

END OF SOLUTIONS