The Mathematical Association of Victoria
MATHEMATICAL METHODS (CAS)
2008 Trial written examination 2 – Solutions – Multiple choice

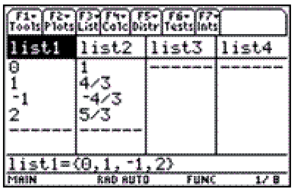
SECTION 1
Solutions

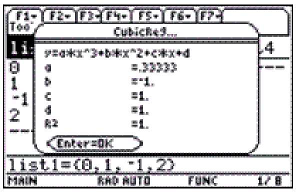
1. D	2. E	3. A	4. C	5. A	6. B
7. E	8. C	9. E	10. B	11. B	12. A
13. D	14. E	15. B	16. B	17. A	18. C
19. A	20. B	21. B	22. C		

Question 1

Answer D

Enter data into the calculator and choose cubic regression.





The exact answer is required.

$y = \frac{1}{3}x^3 - x^2 + x + 1$

$3y = x^3 - 3x^2 + 3x + 3$

Question 2

Answer E

$h(x)$ has factors x, x,$(x - b)$ and $(x - c)$.

The coefficient of x^4 is negative.

Since $a > 0$, the rule is $h(x) = -ax^2(x - b)(x - c)$.

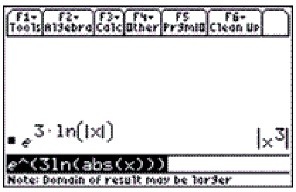
Question 3

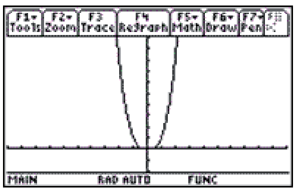
Answer A

$f(g(x)) = e^{3\log_e|x|} = e^{\log_e|x^3|} = |x^3|$

The domain of $f(g(x))$ is the same as the domain of $g(x)$: $R\backslash\{0\}$.

The range is R^+.





Question 4 **Answer C**

$f : (-\infty, 3] \to R$ where $f(x) = (x-3)^{\frac{2}{3}} + 1$

Let $y = (x-3)^{\frac{2}{3}} + 1$

Inverse swap x and y

$x = (y-3)^{\frac{2}{3}} + 1$

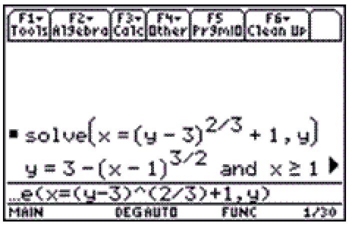
$(y-3)^{\frac{2}{3}} = x - 1$

$y - 3 = -(x-1)^{\frac{3}{2}}$, take the negative square root

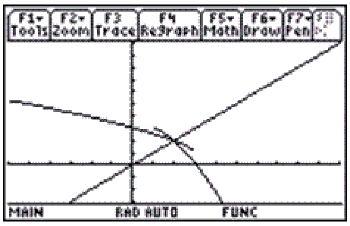
$y = 3 - (x-1)^{\frac{3}{2}}$

OR

Solve $x = (y-3)^{\frac{2}{3}} + 1$ for y on the CAS



$f : [1, \infty) \to R$, where $f^{-1}(x) = 3 - (x-1)^{\frac{3}{2}}$



Question 5 **Answer A**

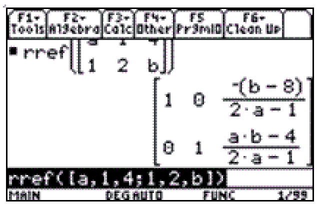
Solve the matrix equation to obtain the result

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{8-b}{2a-1} \\ \frac{ab-4}{2a-1} \end{bmatrix}$$

OR

Determinant = $2a - 1 \neq 0$

Hence there will be a unique solution for $a \in R \setminus \left\{\frac{1}{2}\right\}$ and $b \in R$



Note also that if:

$a = \frac{1}{2}$ and $b = 8$, there are infinite solutions (the solutions are identical)

$a = \frac{1}{2}$ and $b \in R \setminus \{8\}$, there is no solution (the solutions are inconsistent)

Question 6 **Answer B**

$-h(2x-4) = -h(2(x-2))$

The transformation from $h(x)$ to $-h(2(x-2))$ involves

A reflection in the x-axis, a dilation by a scale factor of $\frac{1}{2}$ from the y-axis and a translation 2units right.

Question 7 **Answer E**

The rule will be of the form $f(x) = a\sin\left(n(x - \varepsilon)\right) + k$.

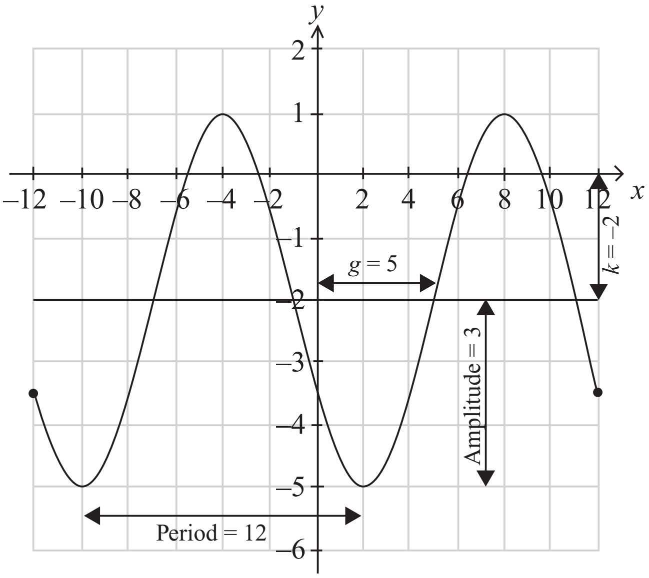
Amplitude = 3, therefore $a = 3$.
Translation down 2 units, therefore $k = -2$.
Phase shift, $\varepsilon = 5$.

$$\text{Period} = \frac{2\pi}{n}$$

$$12 = \frac{2\pi}{n}$$

$$n = \frac{2\pi}{12}$$

$$n = \frac{\pi}{6}$$

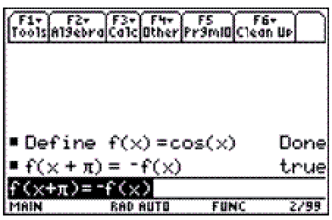


Question 8 **Answer C**

$\cos(x + \pi) = -\cos(x)$ (this can be seen from a unit circle or the graph of $y = \cos(x)$).

Hence $f(x + \pi) = -f(x)$

This can be verified using a CAS.



Question 9 **Answer E**

Let x' and y' be the images of x and y, respectively, under T. The transformation of $y = e^x$ to $y = e^{-2(x+1)}$ involves:

$x = -2(x' + 1)$ and $y = y'$
$x' + 1 = -\tfrac{1}{2}x$
$x' = -\tfrac{1}{2}x - 1$ and $y' = y$

Question 10 **Answer B**

$e^{2x} + be^x + 1 = 0$
Let $e^x = p$
$p^2 + bp + 1 = 0$
$\Delta < 0$ for no real solutions
$b^2 - 4 = 0$

Hence $\{b : -2 < b < 2\}$

Also there is no solution at $b = 2$ as
$e^{2x} + 2e^x + 1 = (e^x + 1)^2 = 0$
$e^x \neq -1$

Thus $\{b : -2 < b \leq 2\}$

Question 11 **Answer B**

$$f(x) = \begin{cases} x^2 + 4x + a & \text{for } x \geq 0 \\ bx + c & \text{for } x < 0 \end{cases}$$

For $f(x)$ to be differentiable at $x = 0$, $f(x)$ has to be continuous at $x = 0$.

$x^2 + 4x + a = bx + c$
Hence at $x = 0$, $a = c$

Also $\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x)$
$2x + 4 = b$
At $x = 0$ $b = 4$

Question 12 **Answer A**

$$f(x+h) \approx hf'(x) + f(x)$$
$$f(x+h) - f(x) \approx hf'(x)$$
$$\approx -0.1 \times \frac{2}{2x+1}$$
$$\approx -0.1 \times \frac{2}{3} = -\frac{1}{15}$$

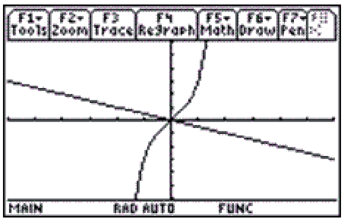
Question 13 **Answer D**

$y = x^5 + 2x$

$\frac{dy}{dx} = 5x^4 + 2 = 2$ at $x = 0$.

The equation of the normal is

$y = -\frac{1}{2}x.$



Question 14 **Answer E**

$\frac{x^2}{y} = 1, y = x^2$

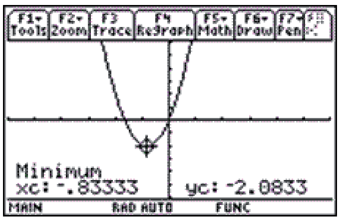
Substitute $y = x^2$ into p

$p = 5x + 3y = 5x + 3x^2$

$\frac{dp}{dx} = 5 + 6x = 0$ or $x = -\frac{b}{2a} = -\frac{5}{6}$

$x = -\frac{5}{6}$

p has a minimum value when $x = -\frac{5}{6}$



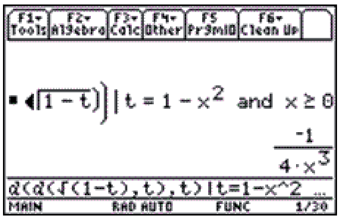
Question 15 **Answer B**

$x = \sqrt{(1-t)}$

$v = \dfrac{dx}{dt} = -\dfrac{1}{2\sqrt{(1-t)}}$

$a = \dfrac{dv}{dt} = -\dfrac{1}{4(1-t)^{\frac{3}{2}}}$

$a = -\dfrac{1}{4x^3}$



Question 16 **Answer B**

average value $= \dfrac{1}{b-a}\int_a^b g(x)\,dx$

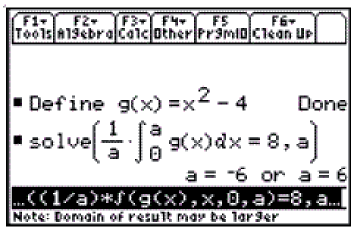
Apply the formula to this case

$8 = \dfrac{1}{a-0}\int_0^a \left(x^2-4\right)dx$

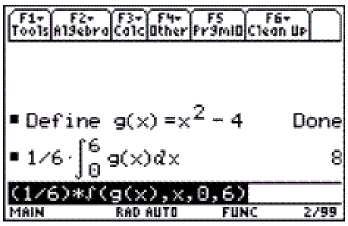
Solving for a gives the result

$a = 6$

Reject the negative solution because the domain of g requires that $a > 0$.



OR

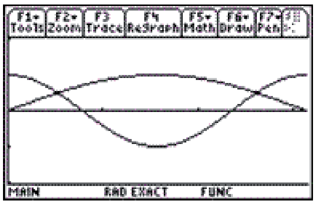


Question 17 **Answer A**

$y = \sin(x)$ is the top curve and $y = \cos(2x)$ is the bottom curve.

$\sin(\frac{\pi}{6}) = \cos(\frac{\pi}{3})$ and $\sin(\frac{5\pi}{6}) = \cos(\frac{5\pi}{3})$

$$\text{Area} = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\sin(x) - \cos(2x)\right)dx$$

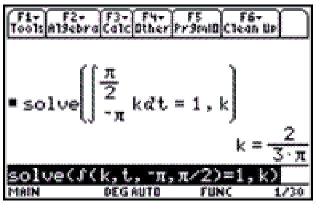


Question 18 **Answer C**

For a probability density function, $\int_{-\infty}^{\infty} f\left(t\right)dt = 1$.

Solve $\int_{-\pi}^{\frac{\pi}{2}} (k)dt = 1$ for k using CAS

$$k = \frac{2}{3\pi}$$



OR

$$0 + \int_{-\pi}^{\pi/2} k\,dt = 1$$

$$\left[kt\right]_{-\pi}^{\pi/2} = 1$$

$$\left[\frac{\pi}{2}k - \left(-\pi k\right)\right] = 1$$

$$\frac{3\pi}{2}k = 1$$

$$k = \frac{2}{3\pi}$$

Question 19

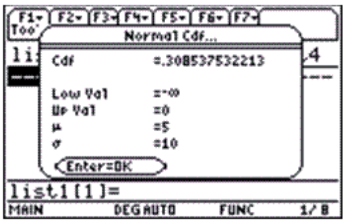
Answer A

Let X be the percentage increase in the value of stocks.

$X \sim N(\mu = 5, \sigma^2 = 10^2)$

Stocks decrease in value if $X < 0$.

$\Pr(X < 0) \approx 0.31$



Question 20

Answer B

This is a problem of selection without replacement. Let S denote dialling Stewart's phone number.

$$\Pr(S', S', S) = \frac{7}{8} \times \frac{6}{7} \times \frac{1}{6}$$

Note from this pattern that the probability that the first, second, third, …, eighth number dialled is Stewart's number, is $\frac{1}{8}$ each time.

Question 21 **Answer B**

$\sum p(x) = 1$

$h + h^2 + \frac{1}{2} - h = 1$

$h^2 = \frac{1}{2}$

$h = \frac{1}{\sqrt{2}}$ (Reject negative solution as $h \in [0,1]$))

$\mu = \sum x\, p(x)$

$= 0 + \left(1 \times \left(\frac{1}{\sqrt{2}}\right)^2\right) + \left(\sqrt{2} \times \left(\frac{1}{2} - \frac{1}{\sqrt{2}}\right)\right)$

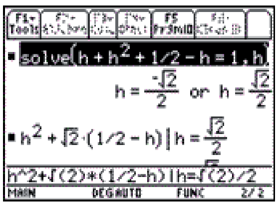
$= \frac{1}{2} + \frac{\sqrt{2}}{2} - 1$

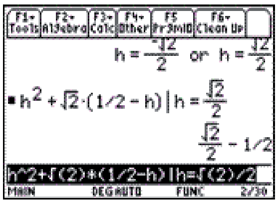
$= \frac{\sqrt{2}}{2} - \frac{1}{2}$

$= \frac{\sqrt{2} - 1}{2}$

OR

Use CAS



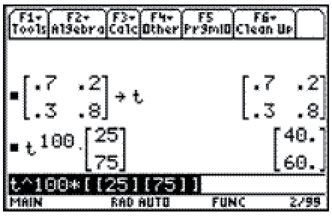


Question 22 **Answer C**

Method 1

Consider, say, the 100th state.

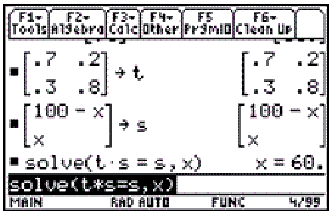
$S_{100} = T^{100} S_0 = \begin{bmatrix} 40 \\ 60 \end{bmatrix}$



Method 2

The cardinal number of elements in this case is 100. The steady state matrix will therefore be of the form $\begin{bmatrix} 100-x \\ x \end{bmatrix}$. When the steady-state has been reached, $T \times \begin{bmatrix} 100-x \\ x \end{bmatrix} = \begin{bmatrix} 100-x \\ x \end{bmatrix}$.

Solving the matrix equation for x gives the result $x = 60$, $100 - x = 40$



Method 3

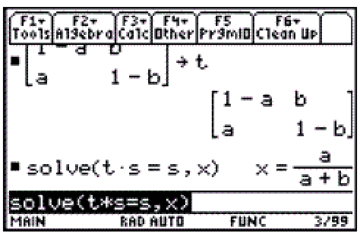
If Method 2 above is applied to a general transition matrix, $T = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$ and the steady state matrix is written as the proportions

$\begin{bmatrix} 1-x \\ x \end{bmatrix}$, the solution is

$x = \dfrac{a}{a+b} = \dfrac{0.3}{0.3+0.2} = 0.6$.

As a **proportion**, the steady-state matrix is $\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$. However, there are 100 elements.

Using the cardinal numbers, the steady-state matrix is $\begin{bmatrix} 40 \\ 60 \end{bmatrix}$.



The Mathematical Association of Victoria
MATHEMATICAL METHODS (CAS)
2008 Trial written examination 2: Solutions – Extended Answer Questions

SECTION 2
Question 1

a. Solve $h^2 + \left(\frac{b}{2}\right)^2 = b^2$ for h

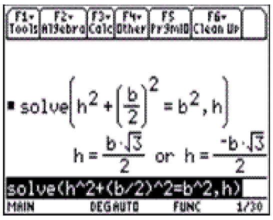
$h = \frac{\sqrt{3}b}{2}$ **[1M]**

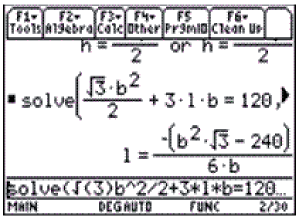
SA = area of the 2 triangles + 3 rectangles

$bh + 3lb = 120$

$\frac{\sqrt{3}b^2}{2} + 3lb = 120$ **[1M]**

$l = \frac{240 - \sqrt{3}b^2}{6b}, \; 3 \le b \le 9$

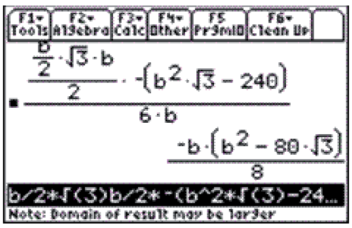




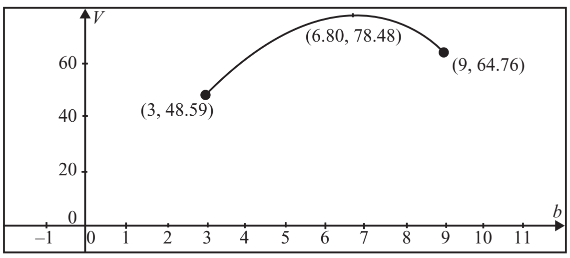
b. V = area of the triangle × length **[1M]**

$= \frac{b}{2} \times \frac{\sqrt{3}b}{2} \times \frac{240 - \sqrt{3}b^2}{6b}$ **[1M]**

$= \frac{b(80\sqrt{3} - b^2)}{8}, \; 3 \le b \le 9$ as required



c.



Correct shape **[1A]**

Correct turning point (6.80, 78.48) **[1A]**

Correct end points with closed circles (3, 48.59) and (9, 64.76) **[1A]**

d. The maximum occurs at the turning point not an end point.

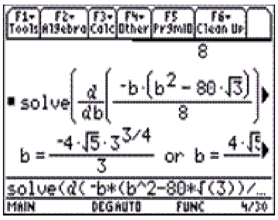
$$\frac{dV}{db} = 10\sqrt{3} - \frac{3b^2}{8}$$

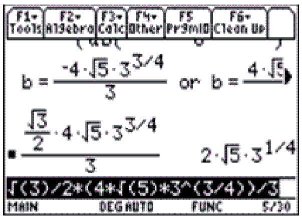
Solve $10\sqrt{3} - \frac{3b^2}{8} = 0$ for b **[1M]**

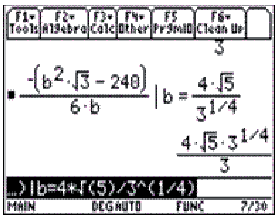
$b = \frac{4\sqrt{5}}{\sqrt[4]{3}}$ as req **[1A]**

$h = \frac{\sqrt{3}}{2}b = 2\sqrt{5}\sqrt[4]{3}\,\text{cm}$ **[1A]**

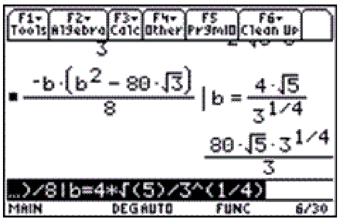
$l = \frac{240 - \sqrt{3}b^2}{6b} = \frac{4\sqrt{5}}{3^{\frac{3}{4}}}\text{cm}$ **[1A]**





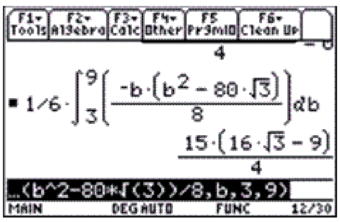


e. $V = \frac{b(80\sqrt{3} - b^2)}{8} = \frac{80\sqrt{5}}{3^{\frac{3}{4}}}\text{cm}^3$ **[1A]**



f. Average value $= \frac{1}{9-3}\int_3^9 \left(\frac{b(80\sqrt{3} - b^2)}{8}\right)db$ **[1M]**

$= \frac{15(16\sqrt{3} - 9)}{4}\text{cm}^3$ **[1A]**



Question 2

a. $N(t) = 200e^{-mt}$

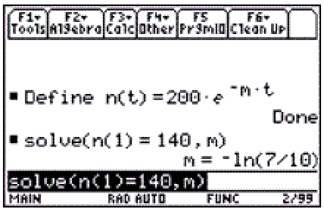
Solve for m, $N(1) = 140$

$140 = 200\, e^{-m}$ **[1M]**

$$m = -\log_e\left(\frac{7}{10}\right)$$

$$m = \log_e\left(\frac{10}{7}\right)\text{, as required.}$$ **[1M]**

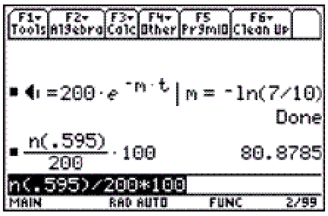
Checking with CAS



b. percentage $= \dfrac{n(0.595)}{200} \times \dfrac{100}{1}$ **[1M]**

$$= 100 \times e^{-\log_e\left(\frac{10}{7}\right)\times 0.595}$$

$= 81\%$ **[1A]**

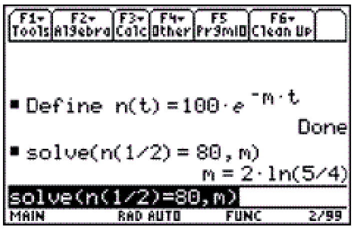


c. $N(t) = 100e^{-mt}$

To find m, solve $N\left(\frac{1}{2}\right) = 80$

$80 = 100e^{-\frac{1}{2}m}$

$m = 2\log_e\left(\frac{5}{4}\right)$ **[1M]**



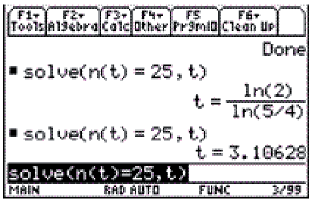
To find when 75% of words are replaced,

$25 = 100e^{-2\log_e\left(\frac{5}{4}\right)\times t}$

$t = \dfrac{\log_e(2)}{\log_e\left(\frac{5}{4}\right)}$ **[1M]**

$t = 3.106$

It will take 3106 years for 75% replacement. **[1A]**



d. To find a, $B(0) = 15$

$$\frac{a}{1+3e^{0}} = 15$$

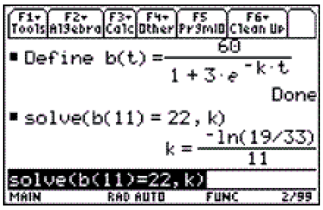
$a = 15 \times 4$ **[1M]**

$B(11) = 22$

$$\frac{60}{1+3e^{-k\times 11}} = 22$$

Solve for k

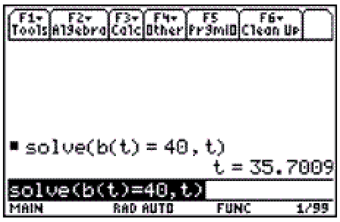
$$k = -\frac{1}{11}\log_e\left(\frac{19}{33}\right) = \frac{1}{11}\log_e\left(\frac{33}{19}\right)$$ **[1M]**



Solve for t, $B(t) = 40$

$t \approx 35.70$

The buzz word count achieved at 36 months. **[1A]**

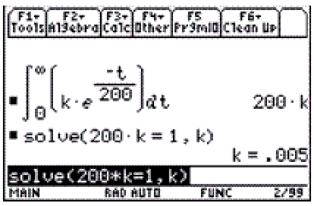


e. For a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Solve for k, $\int_{0}^{\infty} k\,e^{-\frac{t}{200}}\,dt = 1$

$k = 0.005$, as required **[1M]**

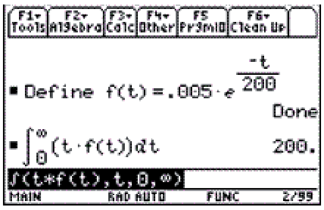


f. $\mu = \int_{-\infty}^{\infty} x\, f\left(x\right) dx$ for a probability density function

$\mu = 0.005\int_{0}^{\infty}\left(t\, e^{-\frac{t}{200}}\right) dt$ **[1M]**

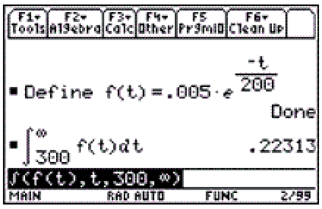
$= 200$

The mean 200 years. **[1A]**



g. $\Pr(T > 300) = 0.005 \int_{300}^{\infty} e^{-\frac{t}{200}} dt$ **[1M]**

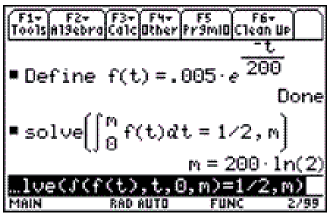
$\Pr(T > 300) = 0.223$ **[1A]**



h. To find the median, *m*, solve

$0.005\int_{0}^{m} e^{-\frac{t}{200}} dx = \frac{1}{2}$ **[1M]**

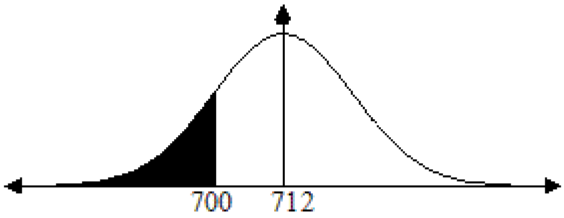
$m = 200\log_e(2)$ **[1A]**



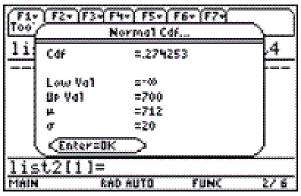
Question 3

a. $X \sim N(\mu = 712, \sigma^2 = 20^2)$

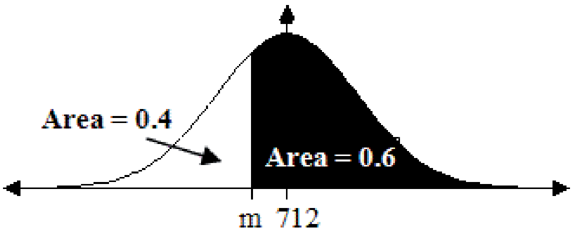
$\Pr(X < 700) = 0.274$ **[1M]**



The proportion of cartons less than 700 g is 0.274. **[1A]**



b. $X \sim N(\mu = 712, \sigma^2 = 20^2)$

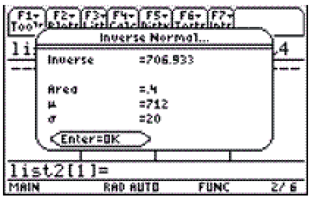


$\Pr(X > m) = 0.6$

$\Pr(X \leq m) = 0.4$ **[1M]**

$m = 706.9$

60% of cartons exceed 706.9 grams. **[1A]**

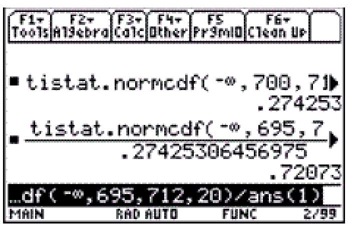


c. $\Pr\left(X < 695 | X < 700\right)$

$$= \frac{\Pr\left(X < 695 \cup X < 700\right)}{\Pr\left(X < 700\right)}$$

$$= \frac{\Pr\left(X < 695\right)}{\Pr\left(X < 700\right)}$$ **[1M]**

$\Pr(X < 695 | X < 700) = 0.721$ **[1A]**

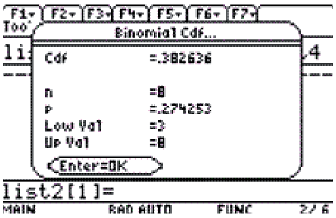


d. Let Y be the number of underweight cartons.

$Y \sim Bi(8, 0.274)$ **[1M]**

(The probability of a carton being underweight was found in part **a.** to be 0.274)

$\Pr(Y \geq 3) = 0.383$ **[1A]**



e. The transition matrix is as follows:

	Large today	Medium today
Large Tomorrow	0.45	0.75
Medium Tomorrow	0.55	0.25

The probability that Jamie orders medium-sized today and large-sized tomorrow is 0.75. **[1A]**

f. i. Pr(L,L,M) = 0.75 × 0.45 × 0.55

Pr (L,L,M) = 0.1856 **[1A]**

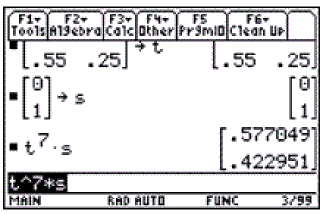
ii. The n^{th} state is given by $S_n = T^n \times S_0$

This Tuesday, $S_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{matrix} \leftarrow \text{Large} \\ \leftarrow \text{Medium} \end{matrix}$

Next Tuesday, $S_7 = T^7 \times S_0$

$S_7 = \begin{bmatrix} 0.5770 \\ 0.4230 \end{bmatrix}$ **[1M]**

The probability of large-sized is 0.5770. **[1A]**



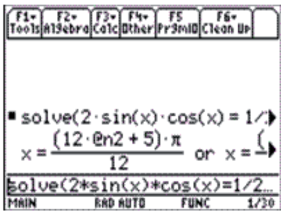
Question 4

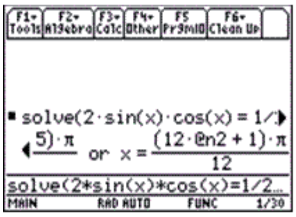
a. $2\sin(x)\cos(x)=\frac{1}{2}$

Using CAS

$x=\frac{(12n+5)\pi}{12}$ **[1A]**

$x=\frac{(12n+1)\pi}{12}$ where $n \in Z$ **[1A]**





OR

$2\sin(x)\cos(x)=\frac{1}{2}$

$\sin(2x)=\frac{1}{2}$

$x=\ldots.\frac{\pi}{12},\frac{5\pi}{12}\ldots.$

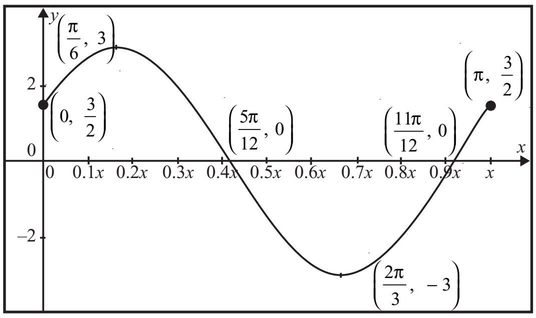
$x=\frac{\pi}{12}+n\pi$ or $x=\frac{5\pi}{12}+n\pi$, where $n \in Z$

b. i. A dilation of a factor of 3 from the x-axis **[1A]**

and a translation of $\frac{\pi}{12}$ units to the left. **[1A]**

(The order does not matter.)

ii.



Correct shape **[1A]**

Correct coordinates for the x-intercepts

$\left(\frac{5\pi}{12}, 0\right)$ and $\left(\frac{11\pi}{12}, 0\right)$ **[1A]**

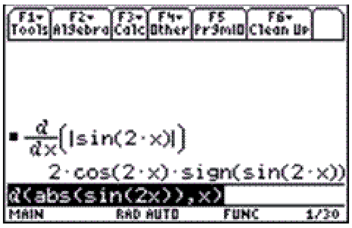
Correct end points with closed circles

$\left(0, \frac{3}{2}\right)$ and $\left(\pi, \frac{3}{2}\right)$ **[1A]**

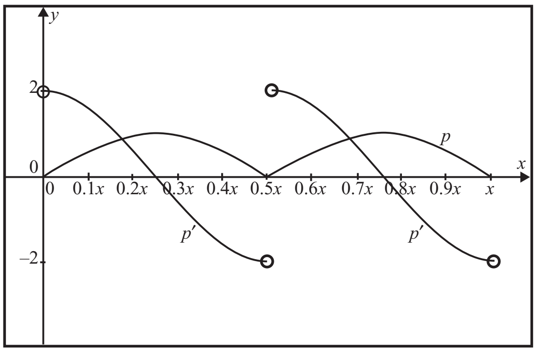
Correct turning points

$\left(\frac{\pi}{6}, 3\right)$ and $\left(\frac{2\pi}{3}, -3\right)$ **[1A]**

c. i. $p'(x)=\begin{cases} 2\cos(2x) \text{ if } \sin(2x)\geq 0 \\ -2\cos(2x) \text{ if } \sin(2x)<0 \end{cases}$ **[2A]**



ii.

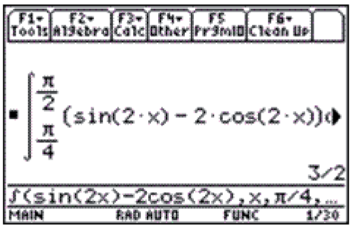


Correct graph for p **[1A]**
Correct graph for p' **[1A]**

x-intercepts of p': $\frac{\pi}{4}, \frac{3\pi}{4}$
Open circles **[1A]**

iii. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\sin(2x)-2\cos(2x)\right)dx$ **[1M]**

$= 1.5$ unit^2 **[1A]**



END OF SOLUTIONS